Cubical Set

Honours Research Presentation

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Introduction

Definition

Elementary interval:
If $l \in \mathbb{R}$, a closed interval in $\mathbb{R}$ that has the form of $\mathtt{I} = [l, l+1]$ or $[l]$ is called elementary interval.

- $[l, l+1]$ is called non degenerated component
- $[l]$ is called degenerated component

Introduction

Definition

Elementary Cube:
If Q is a finite product of elementary intervals, namely,

$$Q = \mathrm{I}_1 \times \mathrm{I}_2 \times \mathrm{I}_3 \ldots \times \mathrm{I}_d,$$

Q is then an elementary cube.

- d is called the embedding number. $emb(Q) = d$
- The dimension of Q is the number of non degenerated components in Q.
- K_n^d means the collection that contains every elementary cubes in a space that has the dimension n and embedding number d.

Introduction

Definition

K-chain:

K-chain is the sum of the span of numerous elementary cubes.

Or, let $\alpha_1, \alpha_2...$ be every $\mathbb{R}$, k-chain for $Q_1, Q_2...$ is

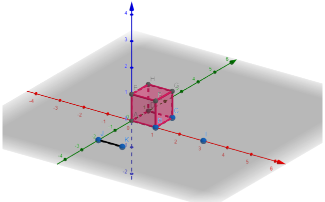
$$C = \sum \alpha_i \cdot \widehat{Q}_i$$

Method

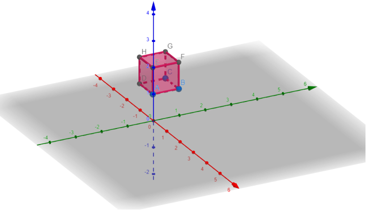
- Find the product of k-chains
- Find the boundary operator of k-chain

Method

From intuition, what would the product between these two spaces be?



$$C_1 = \alpha_1 \cdot [0,1] \times \widehat{[0,1]} \times [0,1] + \alpha_2 \cdot [0,1] \times \widehat{[-2]} + \alpha_3 \cdot \widehat{[3]}$$



$$C_2 = \alpha_4 \cdot [-1,0] \times \widehat{[0,1]} \times [1,2]$$

Method

Inspired by box product, it should be the sum of every combination between the elementary cubes of two chains, so

$$C_1 \times C_2 = \sum \alpha_i \cdot \alpha_j \; \widehat{Q_i \times Q_j}$$

Method

Let c_1 induced by K_K and c_2 induced by K'_K. Cubical product is defined as

$$c_1 \diamond c_2 = \sum_{p \in K_K, Q \in K'_K} < c_1, \widehat{P} >< c_2, \widehat{Q} > \widehat{P \times Q}$$

where,

$$< c_1, c_2 >= \sum \alpha_i \beta_i$$

Method

Now, for boundary. To find boundary generally, we want an operator(∂_s) that input a Q with dimension s and return a Q' that has dimension $s-1$.

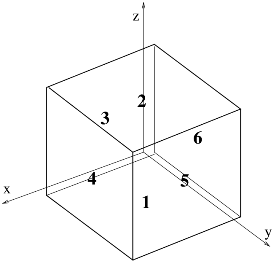
Some facts we know are:

- $$\partial_1 : [l, l+1] \mapsto [l+1] - [l]$$
  $$[l] \mapsto 0$$
- For Q that $dim(Q) > 1$, its each non degenerated interval will undergo ∂_1, degenerated interval will simply disappear.
- The operator needs to have an order.

Method

e.g

A simple cube of $[0,1] \times [0,1] \times [0,1]$ will become $(A_4 - A_6) + (A_5 - A_3) + (A_2 - A_1)$



Conclusion

Recursively defined,

$$\partial_k \colon K_k^d \longrightarrow K_{k-1}^d$$

For $d > 1$ let $I = I_1(Q)$, $P = I_2(Q) \times ... \times I_d(Q)$

$$\partial_k \widehat{Q} \;=\; \partial_{k1} \widehat{I} \diamond \widehat{P} + (-1)^{k_1} \widehat{I} \diamond \partial_{k2} \widehat{P}$$

where $k_1 = dim(I)$, $k_2 = dim(P)$

Absolutely defined, let $\{U_1, U_2 ...\}$ be the non degenerated interval, it is mixed with many degenerated ones

$$\partial \widehat{Q} \;=\; \sum \pm \partial \widehat{U}_i \diamond \{\widehat{Q_{Ui}}\}$$

where $\widehat{Q_{Ui}}$ is the interval list after $\widehat{U}_i$

References

Kaczynski Mischaikow Mrozek (2004)
Computational Homology, Springer New York, NY. Published: 01 December 2010