Group and Surface

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Fundamental Group of Surface

Path $\Rightarrow$ Loop $\Rightarrow$ Surface

Path function

$$f: \quad [0,1] \mapsto X$$

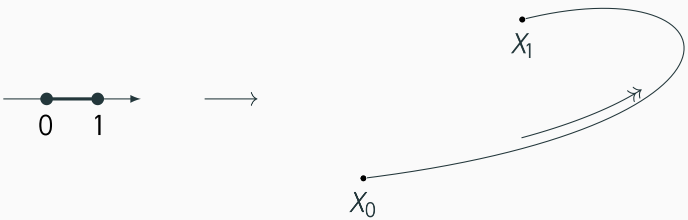

Figure 1: Path function

Definition

f and f' are two continuous maps. If there is a continuous function H that

$$H : \quad X \times [0,1] \to Y$$

$$H(x,0) = f(x) \quad \textbf{and} \quad H(x,1) = f'(x)$$

f is then homotopic to f', denoted as $f \simeq f'$

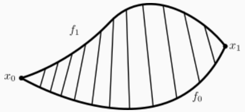

Figure 2: Straight line homotopy

Homotopy

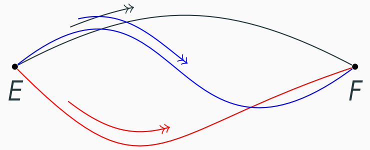

Figure 3: Homotopy equivalent class

Equivalence class: $[f]$

Group

Definition

Group is a set G with an operation $*$ that satisfies:

- There exists an identity element $e \in G$.
- If $\alpha \in G$, then its inverse $\alpha^{-1} \in G$.
- Associativity

e.g

$\mathbf{N}$ is not a group under $+$, but $\mathbf{Z}$ is.

Now we generate a free group using homotopy equivalence class under the operation $*$

$$[a] * [b] = [a * b]$$

$$G = \prod^{*} G_\alpha$$

$$G_\alpha = \prod^{*} [a_\alpha] \quad \textbf{or} \quad G_\alpha \supseteq \{x * y \mid x, y \in [\alpha_i]\}$$

Figure 4: $f * f'$

Fundamental Group

The easiest homotopy group:

$$\pi_1(X, x_0)$$

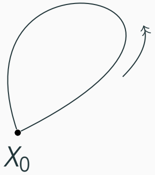

Figure 5: Fundamental Group

Fundamental Group of Surface

Polygonal region scheme

Assign each edge with a label. Default direction is set as from p_{k-1} to p_k.

Labelling scheme:

$$w = (a_1)^{\epsilon_1} * (a_2)^{\epsilon_2} * (a_3)^{\epsilon_3} * (a_4)^{\epsilon_4} * \dots * (a_n)^{\epsilon_n}$$

where $\epsilon = \pm 1$

Note: This is an element of the free group generated by homoptopic equivalence class!

Fundamental Group of Surface

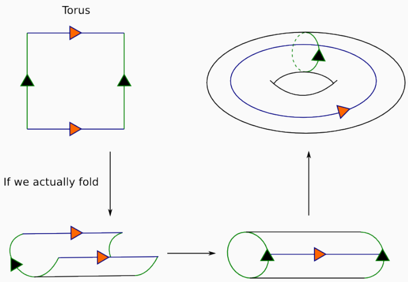

Figure 6: torus

$$w = a_1 \cdot b_1^{-1} \cdot a_2^{-1} \cdot b_2 \textbf{ and } a_1 \sim a_2, b_1 \sim b_2$$

Fundamental Group of Surface

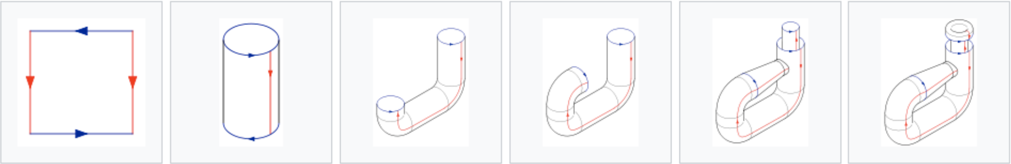

Figure 7: Klein Bottle

$$w = a_1^{-1} \cdot b_1 \cdot a_2^{-1} \cdot b_2^{-1} \text{ and } a_1 \sim a_2, b_1 \sim b_2$$

References

James R. Munkres.
Topology. Second Edition.
Massachusetts Institute of Technology, Prentice Hall, 2000.