Sheet 1 Numerik 1 Witt, Borowitz

Σ	1	2	3	4

1) Consider the following system:

$$v^{(iv)}(x) - a(x)\,u'(x) = f(x) \qquad (\mathrm{I})$$

$$u''(x) + b(x)\,v(x) = g(x) \qquad (\mathrm{II})$$

Transform this system onto a system consisting of ODEs with order 1.

Answer: Let $v_0 := v(x), \dots, v_3 := v'''(x)$,
$u_0 := u(x),\ u_1 := u'(x)$

$$(\mathrm{I}')\quad v^{(iv)}(x) = a(x)u'(x) + f(x)$$

$$(\mathrm{II}')\quad u''(x) = -b(x)v(x) + g(x)$$

Lemma 1.2.4 $\Rightarrow$

$$\begin{bmatrix} v_0' \\ v_1' \\ v_2' \\ v_3' \\ u_0' \\ u_1' \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ a(x)\,u_1(x) + f(x) \\ u_1 \\ -b(x)\,v_0(x) + g(x) \end{bmatrix}$$

Problem 1.2 (**Matrixvalued Mappings**, 4 points)
Let $A \in \mathbb{C}^{n\times n}$. We define the exponential function by

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} A^k.$$

Assume absolute convergence.

a) Let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix. Calculate e^D.

Calculation. Let $D = \mathrm{diag}(d_1, \dots, d_n)$ where $d_1, \dots, d_n \in \mathbb{C}$. Then obviously we have $D^k = \mathrm{diag}(d_1^k, \dots, d_n^k)$ for all $k \in \mathbb{N}$. Therefore:

$$e^D = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} D^k = \lim_{m\to\infty} \sum_{k=0}^{m} \mathrm{diag}\left(\frac{d_1^k}{k!}, \dots, \frac{d_n^k}{k!}\right) = \lim_{m\to\infty} \mathrm{diag}\left(\sum_{k=0}^{m} \frac{d_1^k}{k!}, \dots, \sum_{k=0}^{m} \frac{d_n^k}{k!}\right)$$

Since $\lim_{m\to\infty} \sum_{k=0}^{m} \frac{d_i^k}{k!} = e^{d_i}$, $\forall i = 1, \dots, n$ we obtain $e^D = \mathrm{diag}(e^{d_1}, \dots, e^{d_n})$.

b) Let $A \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ invertible. Show that $e^{SAS^{-1}} = Se^AS^{-1}$.

Proof. We have for all $k \in \mathbb{N}$ that:

$$(SAS^{-1})^k = \underbrace{(SAS^{-1})(SAS^{-1}) \cdots (SAS^{-1})(SAS^{-1})}_{k\text{-times}} = SA\underbrace{S^{-1}S}_{=I_n}AS \cdots SA\underbrace{S^{-1}S}_{=I_n}AS^{-1}$$

$= SA^kS^{-1}$, since all intermediate SS^{-1} cancel out. Therefore

$$e^{SAS^{-1}} = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!}(SAS^{-1})^k = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} SA^kS^{-1} = \lim_{m\to\infty} \left(S\left[\sum_{k=0}^{m} \frac{1}{k!} A^k\right] S^{-1}\right)$$

Since we have absolute convergence of $\sum_{k=0}^{\infty} A^k$ we can compute:

$$e^{SAS^{-1}} = \lim_{m\to\infty} \left(S\left[\sum_{k=0}^{m} \frac{1}{k!} A^k\right] S^{-1}\right) = S\left[\lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} A^k\right] S^{-1} = Se^AS^{-1}$$

$\square$

c) Consider the following IVP for vector-valued functions $u : [0, T] \to \mathbb{C}^n$

$$u' = Au,$$
$$u(0) = u_0.$$

Assume that $A \in \mathbb{C}^{n\times n}$ is diagonalisable. Show that $u(t) = e^{At}u_0$ is a solution.

Proof. First of all we have
$u(0) = e^{0\cdot A} u_0 = I_n u_0 = u_0$, where we used the convention $A^0 = I_n$, for all $A \in \mathbb{C}^{n\times n}$.
Secondly, since A is diagonisable $\Rightarrow \exists S \in GL_n(\mathbb{C})$: $D := \mathrm{diag}(a_1, \dots, a_n) = S^{-1}AS$, for $a_1, \dots, a_n \in \mathbb{C}$.
Therefore:

$$u(t) = e^{At} u_0 = e^{(SAS^{-1})t} u_0 = e^{SDtS^{-1}} u_0 = Se^{Dt}S^{-1} u_0$$

Since we have $\frac{d}{dt} e^{Dt} = De^{Dt}$, we get

$$\frac{d}{dt} u(t) = \frac{d}{dt}\left(Se^{Dt}S^{-1}\right)u_0 = S\frac{d}{dt}\left(e^{Dt}\right)S^{-1}u_0 = SDe^{Dt}S^{-1} u_0$$

$$\overset{D = S^{-1}AS}{=} \underbrace{S S^{-1}}_{= I_n} A S e^{Dt} S^{-1} u_0 = A(S e^{Dt} S^{-1}) u_0 = A e^{S Dt S^{-1}} u_0 = A e^{At} u_0 = A u(t)$$

Hence $u(t) = e^{At} u_0$ is a solution to the given IVP. $\square$

Problem 1.3 (**Grönwall's inequality**, 4 Points)
Proof the differential formulation of Grönwall's inequality:
Let I denote an Intervall of the form $[a, c]$ or $[a, \infty)$, with $c > a$. Let $w(t)$ and $b(t)$ be continuous functions. Furthermore, let $w(t)$ be differentiable and satisfy the differential inequality

$$w'(t) \leqslant b(t) w(t).$$

Then w is bounded:

$$w(t) \leqslant w(a) \exp\left(\int_a^t b(s) \mathrm{d}s\right), \qquad \forall t \in I.$$

Proof. Set $v(t) := \exp\left(\int_a^t b(s)\, ds\right), \ \forall t \in I$

v ist diff'able $\Longrightarrow$

$$v'(t) = \frac{d}{dt}\left(\int_a^t b(s)\, ds\right) v(t) = b(t) v(t), \quad t \in I^\circ$$

We notice that

1. $v(a) = \exp\left(\int_a^a b(s)\, ds\right) = \exp(0) = 1$
2. $\exp(x) > 0, \ \forall x \in \mathbb{R}$

$\Rightarrow \quad v(t) > 0, \forall t \in I$

Especially $v(t) \neq 0, \ \forall t \in I.$

$$\frac{d}{dt} \frac{w(t)}{v(t)} = \frac{w'(t) v(t) - w(t) v'(t)}{v(t)^2}$$

$$= \frac{w'(t) v(t) - w(t) b(t) v(t)}{v(t)^2} = \frac{w'(t) - w(t) b(t)}{v(t)}$$

$$\underset{\substack{w'(t) \leq b(t) w(t) \\ v(t) > 0}}{\leq} 0, \quad t \in I^\circ$$

$\Rightarrow \frac{d}{dt} \frac{\omega(t)}{v(t)} \leq 0, \ \forall t \in I^\circ$. Thus, $\frac{w(t)}{v(t)}$ is monotonously decreasing.

$$\Rightarrow \frac{\omega(t)}{v(t)} \leq \frac{\omega(a)}{v(a)} = \frac{\omega(a)}{1} = \omega(a), \quad \forall I \ni t \geq a$$

$$\Rightarrow \omega(t) \leq \omega(a) v(t) = \omega(a) \exp\left(\int_a^t b(s) ds\right), \ t \in I.$$

$\square$