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Literature Research of Financial Engineering

original paper: Hedging With Linear Regressions and Neural Networks

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Highlights of our study

- We study neural networks as nonparametric estimation tools for the hedging of options.
- We design a network, named HedgeNet, that directly outputs a hedging strategy.
- The network reduces the hedging error significantly.
- A similar and even more outstanding benefit arises by simple linear regressions that incorporate the leverage effect.

- Different practitioners will adopt different treatment methods for their assets based on their own utility. For an instance, if the financial entity was on the "buy-side," taking on short positions in options to collect the volatility risk premium, and interested in maximizing the Sharpe ratio of her position. This entity would then try to hedge the exposure to the price movements in the underlying by trading it.

- The mark-to-market accounting convention requires a good control of the hedging error for short periods, even when onsidering long-dated options.
- Beginning with Hutchinson, Lo, and Poggio (1994) and Malliaris and Salchenberger (1993), artificial neural networks (ANNs) are being proposed as a nonparametric tool for the risk management of options.

- Supplementary explanation: hedging error. The BS model is universally recognized as correct. Black-Scholes performs the best if implied volatility is plugged in. On one hand, since here we hedge only discretely, using the BS Delta leads to an error even if the data are simulated from the Black-Scholes model. On the other hand, the practitioner lacks perfect knowledge about the true market model but who is able to partially access the market prices.

In the literature, other volatilities, such as historical volatility estimates or GARCH predicted volatilities have been used. The hedger would conduct a data-driven, nonparametric regression strategy in discrete time, based on the price data available to the hedger. We evaluate the hedging error of a historical delta-hedging strategy - a regression strategy based on the discrete observations.

Since the regression-based strategy is free of parametric model assumptions about the securities, it is robust to misspecifi-cation of models and their parameters. Thus it is expected to serve as a benchmark for evaluating various parametric strategies; any existing parametric strategies would be justifiable only if they "outperform" this benchmark strategy when there is uncertainty about the true model.

- The variance of the hedged portfolio is approximated by the MSHE. Let's assume a situation. The mark-to-market accounting convention requires a good control of the hedging error for short periods, even when considering long-dated options. To reduce the variance of her portfolio the operator is allowed to buy or sell the underlying.

Today, she sells the option, say at price C_0. She is now allowed to buy delta shares of the underlying at price S_0 and $C_0 - \delta$ units of the risk-free asset. Today's portfolio value equals $V_0 = 0$. Tomorrow The portfolio value is then given by

$$V_1^\delta = \delta S_1 + (1 + r_{\text{ovn}} \Delta t)(C_0 - \delta S_0) - C_1$$

Since Δt is small and presumably the expected return on the risky asset happens to be equal to the risk-free return, $\mathrm{var}[V_1^\delta] = \mathrm{E}[\left(V_1^\delta\right)^2]$, which is called mean squared hedging error (MSHE), namely

$$E\left[(\delta S_1 + (1 + r_{\text{overnight}}\Delta t)(C_0 - \delta S_0) - C_1)^2\right]$$

Hedged positions. Black-Scholes Delta (BS Delta): $\delta_{\mathrm{BS}} = N(d_1)$ where N denotes the cumulative normal distribution function. It is derived from the BS model That means

$$\delta_{\mathrm{BS}} = f_{\mathrm{BS}(M,\sigma_{\mathrm{impl}}\sqrt{\tau})}$$

Here the moneyness $M = \frac{S_0}{K}$ and the square root of total implied variance $\sigma_{\mathrm{impl}}\sqrt{\tau})$.

Similarly, we get

$$\delta_{\text{ANN}} = f_{\text{ANN}(M,\sigma_{\text{impl}}\sqrt{\tau})}$$

$$\delta_{\text{LN}} = f_{\text{LN}(M,\sigma_{\text{impl}}\sqrt{\tau})}$$

- We introduce linear regression models that lead to hedging ratios that are linear in several option sensitivities, which are motivated by the *leverage effect*. (Plus: In the original paper, the author used the linear regression model as a reference benchmark, but I have presented the experimental results comprehensively here)

Delta-only
Vega-only
Gamma-only
Vanna-only
Delta-Gamma
Delta-Vega
Delta-Vanna
Delta-Vega-Gamma
Delta-Vega-Vanna
Delta-Gamma-Vanna
Delta-Vega-Gamma-Vanna
Hull-White
Hull-White-relaxed

Figure 1: possible_LR_model

布莱克—斯科尔斯—默顿方程(4.5.14)满足终值条件(4.5.15)以及边界条件(4.5.17)和(4.5.18)的解为：

$$c(t,x)=xN(d_+(T-t,x))-Ke^{-r(T-t)}N(d_-(T-t,x)),\quad 0\leqslant t<T, x>0 \tag{4.5.19}$$

其中：

$$d_\pm(\tau,x)=\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K}+\left(r\pm\frac{\sigma^2}{2}\right)\tau\right] \tag{4.5.20}$$

并且 N 是累积标准正态分布

$$N(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{z^2}{2}}dz=\frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty}e^{-\frac{z^2}{2}}dz \tag{4.5.21}$$

我们有时记为：

$$\mathrm{BSM}(\tau,x;K,r,\sigma)=xN(d_+(\tau,x))-Ke^{-r\tau}N(d_-(\tau,x)) \tag{4.5.22}$$

并且称 $\mathrm{BSM}(\tau,x;K,r,\sigma)$ 为布莱克—斯科尔斯—默顿函数。在此公式中，τ 和 x 分别表示离到期日尚余的时间和当前股价。参数 K，r 和 σ 分别是敲定价格、利率和股价波动率。

Figure 2: Solutions_to_the_BSM_equation

式（4. 5. 19）中函数 $c(t,x)$ 关于各变量的导数称为希腊字母，其中：

delta:

$$c_x(t,x) = N\big(d_+(T-t,x)\big), \tag{4.5.23}$$

theta:

$$c_t(t,x) = -rKe^{-r(T-t)}N\big(d_-(T-t,x)\big) - \frac{\sigma x}{2\sqrt{T-t}}N'\big(d_+(T-t,x)\big). \tag{4.5.24}$$

gamma:

$$c_{xx}(t,x) = N'\big(d_+(T-t,x)\big)\frac{\partial}{\partial x}d_+(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'\big(d_+(T-t,x)\big). \tag{4.5.25}$$

Vega: 关于波动率的偏导数

Rho: 关于无风险利率的偏导数

Figure 3: Greek_alphabet

- ANN. An ANN is a composition of simple elements called neurons, which maps input features to outputs. Such an ANN then forms a directed, weighted graph. Due to different motivations, We consider three different feature sets for the trainable part of HedgeNet(the ANN we design here). They are

Artificial Neural Network Models

$$\mathrm{ANN}(M; \sigma_{\mathrm{impl}}\sqrt{\tau})$$

$$\mathrm{ANN}(\delta_{\mathrm{BS}}; V_{\mathrm{BS}}; \tau)$$

$$\mathrm{ANN}(\delta_{\mathrm{BS}}; V_{\mathrm{BS}}; \mathrm{Va}_{\mathrm{BS}}; \tau)$$

Here, δ_{BS}denotes δ_{BS}; V_{BS} denotes Vega, the sensitivity of the option price with respect to the implied volatility; $\mathrm{Va}_{\mathrm{BS}}$ denotes Vanna namely the sensitivity of Delta with respect to volatility.

Like this, we output the Hedging Ratio δ directly.
And we show the MSHE of different models.

Output and Result

		1 day			2 days		
		Calls	Puts	Both	Calls	Puts	Both
	Zero hedge	4.01	4.78	4.54	8.31	9.73	9.29
	BS Delta	0.687	0.655	0.665	1.58	1.54	1.55
Regressions	Delta-only	−21.3	−14.8	−16.9	−16.3	−12.8	−13.9
Regressions	Vega-only	−13.7	−11.7	−12.3	−10.4	−10.1	−10.2
Regressions	Gamma-only	−15.5	−10.1	−11.8	−14.5	−11.2	−12.2
Regressions	Vanna-only	−12.4	−12.6	−12.5	−10.6	−13.0	−12.2
Regressions	Delta-Gamma	−21.6	−14.8	−17.0	−17.1	−13.1	−14.4
Regressions	Delta-Vega	−21.4	−14.9	−17.0	−16.4	−12.8	−13.9
Regressions	Delta-Vanna	−22.6	**−16.6**	**−18.5**	**−17.7**	**−15.4**	**−16.1**
Regressions	Delta-Vega-Gamma	−21.5	−14.8	−17.0	−16.8	−13.5	−14.5
Regressions	Delta-Vega-Vanna	**−23.0**	**−16.6**	**−18.7**	**−18.1**	**−15.4**	**−16.2**
Regressions	Delta-Gamma-Vanna	−22.6	**−16.6**	**−18.5**	**−17.7**	**−15.2**	**−16.0**
Regressions	Delta-Vega-Gamma-Vanna	**−22.9**	**−16.4**	**−18.5**	−17.4	**−14.9**	**−15.7**
Regressions	Hull-White	**−23.1**	**−16.9**	**−18.9**	**−17.8**	**−14.5**	**−15.5**
Regressions	Relaxed Hull-White	**−23.2**	**−16.9**	**−18.9**	**−18.3**	**−14.6**	**−15.8**
ANNs	M; $\sigma_{\text{impl}}\sqrt{\tau}$	−22.3	−15.6	−17.7	−17.1	−10.9	−12.8
ANNs	Δ_{BS}; $\mathcal{V}_{\text{BS}}$; τ	**−23.4**	**−16.9**	**−18.9**	**−18.6**	−12.9	−14.7
ANNs	Δ_{BS}; $\mathcal{V}_{\text{BS}}$; Va_{BS}; τ	−21.9	−14.4	−16.8	−12.5	−12.9	−12.8

Figure 4: Performance of the linear regressions and ANNs on the S&P 500 dataset. the unit of line 1 and the ones from line 3 on are percentage.

We now present the results on the performance of the various statistical hedging models in terms of MSHE reduction. As a quick summary, the hedging ratios of the ANNs do not outperform the linear regression models.

Output and Result

On the S&P 500 dataset, the Hull-White and Delta-Vega-Vanna regressions tend to perform the best, with Hull-White better on the one-day hedging period, and the Delta-Vega-Vanna regression better on the two-day period. On the Euro Stoxx 50 dataset, the Delta-Vega-Gamma-Vanna regression tends to perform the best.

Conclusion

In this work, we consider the problem of hedging an option over one period. We consider statistical, regression-type hedging ratios (in contrast to model-implied hedging ratios). To study whether the option sensitivities already capture the relevant nonlinearities we develop a suitable ANN architecture.

Conclusion

Experiments involving both quoted prices (S&P 500 options) and high-frequency tick data (Euro Stoxx 50 options) show that the ANNs perform roughly as well (but not better) as the sensitivity-based linear regression models. However, the ANNs are not able to find additional nonlinear features.

Hence, option sensitivities by themselves (in particular, Delta, Vega, and Vanna) in combination with a linear regression are sufficient for a good hedging performance.

Conclusion

The linear regression models improve the hedging performance (in terms of MSHE) of the BS Delta by about 15-20% in real-world datasets. An explanation is the leverage effect that allows the partial hedging of changes in the implied volatility by using the underlying.

Conclusion

As a rule of thumb, historical data seem to imply that calls should be hedged with about 0.9 δ_{BS} and puts with about 1.1 δ_{BS}. With the presence of sufficient historic data, we recommend to follow a hedging strategy obtained from a linear regression on the BS Delta, BS Vega, BS Vanna, and possibly the BS Gamma.

References

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