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1 Basic

1.1 Default Code [37e06b]

```
#include <bits/stdc++.h>
using namespace std;

#define iter(v) v.begin(),v.end()
#define SZ(v) int(v.size())
#define pb emplace_back
#define ff first
#define ss second

using ll = long long;
using pii = pair<int, int>;
using pll = pair<ll, ll>;

#ifdef zisk
void debug(){cerr << "\n";}
template<class T, class ... U>
void debug(T a, U ... b){cerr << a << " ", debug(b...);}
#endif
template<class T> void pary(T l, T r){
    while (l != r) cerr << *l << " ", l++;
    cerr << "\n";
}
```

```
}
#else
#define debug(...) void()
#define pary(...) void()
#endif

template<class A, class B>
ostream& operator<<(ostream& o, pair<A,B> p)
{ return o << '(' << p.ff << ',' << p.ss << ')'; }

int main(){
    ios_base::sync_with_stdio(0); cin.tie(0);
}

1.2 .vimrc [07b138]

sy on
se nu rnu bs=2 sw=4 ts=4 hls ls=2 si acd bo=all mouse=a
map <F9> :w<bar>!g++ "% " -o %:r -std=c++17 -Wall -
    Wextra -Wshadow -O2 -Dzisk -g -fsanitize=undefined,
    address<CR>
map <F8> :!./%:r<CR>
inoremap {<CR> {<CR>><ESC>ko
ca Hash w !cpp -dD -P -fpreprocessed \\\ tr -d '[:space
    :]' \\\ md5sum \\\ cut -c-6
# -D_GLIBCXX_ASSERTIONS, -D_GLIBCXX_DEBUG

1.3 Fast IO [c7ddf]

// from JAW
inline int my_getchar() {
    const int N = 1<<20;
    static char buf[N];
    static char *p = buf, *end = buf;
    if(p == end) {
        if((end = buf + fread(buf, 1, N, stdin)) == buf)
            return EOF;
        p = buf;
    }
    return *p++;
}

inline int readint(int &x) {
    static char c, neg;
    while((c = my_getchar()) < '-') {
        if(c == EOF) return 0;
    }
    neg = (c == '-') ? -1 : 1;
    x = (neg == 1) ? c - '0' : 0;
    while((c = my_getchar()) >= '0') x = (x << 3) + (x <<
        1) + (c - '0');
    x *= neg;
    return 1;
}

const int kBufSize = 524288;
char inbuf[kBufSize];
char buf_[kBufSize]; size_t size_;
inline void Flush_() { write(1, buf_, size_); size_ =
    0; }
inline void CheckFlush_(size_t sz) { if (sz + size_ >
    kBufSize) Flush_(); }

inline void PutInt(int a) {
    static char tmp[22] = "01234567890123456789\n";
    CheckFlush_(10);
    if(a < 0){
        *(buf_ + size_) = '-';
        a = ~a + 1;
        size_++;
    }
    int tail = 20;
    if (!a) {
        tmp[--tail] = '0';
    } else {
        for (; a; a /= 10) tmp[--tail] = (a % 10) ^ '0';
    }
    memcpy(buf_ + size_, tmp + tail, 21 - tail);
    size_ += 21 - tail;
}

int main(){
    Flush_();
}
```

```
    return 0;
}
```

1.4 Random [4cf9ed]

```
mt19937 rng(chrono::system_clock::now().
    time_since_epoch().count());
```

1.5 Checker [9c99bd]

```
#!/usr/bin/env bash
set -e
while ;; do
    python3 gen.py > test.txt
    diff <(. /a.exe < test.txt) <(. /b.exe < test.txt)
done
```

1.6 PBDS Tree [7c702a]

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
using Tree = tree<int, null_type, less<>, rb_tree_tag,
    tree_order_statistics_update>;
// .find_by_order(x)
// .order_of_key(x)
```

1.7 Pragma [6006f6]

```
#pragma GCC optimize("Ofast,no-stack-protector")
#pragma GCC optimize("no-math-errno,unroll-loops")
#pragma GCC target("sse,sse2,sse3,ssse3,sse4")
#pragma GCC target("popcnt,abm,mmx,avx,arch=skylake")
__builtin_ia32_ldmxcsr(__builtin_ia32_stmxcsr())|0x8040)
```

1.8 SVG Writer [7adcc8]

```
class SVG {
    void p(string_view s) { o << s; }
    void p(string_view s, auto v, auto... vs) {
        auto i = s.find('$');
        o << s.substr(0, i) << v, p(s.substr(i + 1), vs...)
        ;
    }
    ofstream o; string c = "red";
public: // SVG svg("test.svg", 0, 0, 100, 100)
    SVG(auto f, auto x1, auto y1, auto x2, auto y2) : o(f)
    {
        p("<svg xmlns='http://www.w3.org/2000/svg' "
            "viewBox='$ $ $ $'\n"
            "<style>{*stroke-width:0.5%;}</style>\n",
            x1, -y2, x2 - x1, y2 - y1); }
    ~SVG() { p("</svg>\n"); }
    void color(string nc) { c = nc; }
    void line(auto x1, auto y1, auto x2, auto y2) {
        p("<line x1='$' y1='$' x2='$' y2='$' stroke='$' />\n",
            x1, -y1, x2, -y2, c); }
    void circle(auto x, auto y, auto r) {
        p("<circle cx='$' cy='$' r='$' stroke='$' "
            "fill='none' />\n", x, -y, r, c); }
    void text(auto x, auto y, string s, int w = 12) {
        p("<text x='$' y='$' font-size='$px'>$</text>\n",
            x, -y, w, s); }
};
```

2 Data Structure

2.1 Heavy-Light Decomposition [f2dbca]

```
struct HLD{ // 1-based
    int n, ts = 0; // ord is 1-based
    vector<vector<int>> g;
    vector<int> par, top, down, ord, dpt, sub;
    explicit HLD(int _n): n(_n), g(n + 1),
        par(n + 1), top(n + 1), down(n + 1),
        ord(n + 1), dpt(n + 1), sub(n + 1) {}
    void add_edge(int u, int v){ g[u].pb(v); g[v].pb(u);
    }
    void dfs(int now, int p){
        par[now] = p; sub[now] = 1;
        for(int i : g[now]){
            if(i == p) continue;
            dpt[i] = dpt[now] + 1;
            dfs(i, now);
            sub[now] += sub[i];
        }
    }
};
```

```
        if(sub[i] > sub[down[now]]) down[now] = i;
    }
}

void cut(int now, int t){
    top[now] = t; ord[now] = ++ts;
    if(!down[now]) return;
    cut(down[now], t);
    for(int i : g[now]){
        if(i != par[now] && i != down[now])
            cut(i, i);
    }
}

void build(){ dfs(1, 1), cut(1, 1); }
int query(int a, int b){
    int ta = top[a], tb = top[b];
    while(ta != tb){
        if(dpt[ta] > dpt[tb]) swap(ta, tb), swap(a, b);
        // ord[tb], ord[b]
        tb = top[b = par[tb]];
    }
    if(ord[a] > ord[b]) swap(a, b);
    // ord[a], ord[b]
    return a; // lca
}
};

2.2 Link Cut Tree [cf4f34]

// 1-based
template <typename Val, typename SVal> struct LCT {
    struct node {
        int pa, ch[2]; bool rev; int size;
        Val v, sum, rsum; SVal sv, sub, vir;
        node() : pa{0}, ch{0, 0}, rev{false}, size{1}, v{},
            sum{}, rsum{}, sv{}, sub{}, vir{} {}
    };
    #define cur o[u]
    #define lc cur.ch[0]
    #define rc cur.ch[1]
    vector<node> o;
    bool is_root(int u) const {
        return o[cur.pa].ch[0] != u && o[cur.pa].ch[1] != u;
    }
    bool is_rch(int u) const {
        return o[cur.pa].ch[1] == u && !is_root(u);
    }
    void down(int u) {
        for (int c : {lc, rc}) if (c) {
            if (cur.rev) set_rev(c);
        }
        cur.rev = false;
    }
    void up(int u) {
        cur.sum = o[lc].sum + cur.v + o[rc].sum;
        cur.rsum = o[rc].rsum + cur.v + o[lc].rsum;
        cur.sub = cur.vir + o[lc].sub + o[rc].sub + cur.sv;
        cur.size = o[lc].size + o[rc].size + 1;
    }
    void set_rev(int u) {
        swap(lc, rc), swap(cur.sum, cur.rsum);
        cur.rev ^= 1;
    }
    /* --- */
    void rotate(int u) {
        int f = cur.pa, g = o[f].pa, l = is_rch(u);
        if (cur.ch[l ^ 1]) o[cur.ch[l ^ 1]].pa = f;
        if (not is_root(f)) o[g].ch[is_rch(f)] = u;
        o[f].ch[l] = cur.ch[l ^ 1], cur.ch[l ^ 1] = f;
        cur.pa = g, o[f].pa = u; up(f);
    }
    void splay(int u) {
        vector<int> stk = {u};
        while (not is_root(stk.back()))
            stk.push_back(o[stk.back()].pa);
        while (not stk.empty())
            down(stk.back()), stk.pop_back();
        for (int f = cur.pa; not is_root(u); f = cur.pa) {
            if (lis_root(f))
                rotate(is_rch(u) == is_rch(f) ? f : u);
            rotate(u);
        }
        up(u);
    }
    void access(int x) {
        for (int u = x, last = 0; u; u = cur.pa) {

```

```

    splay(u);
    cur.vir = cur.vir + o[rc].sub - o[last].sub;
    rc = last; up(last = u);
}
splay(x);
}
int find_root(int u) {
    int la = 0;
    for (access(u); u; u = lc) down(la = u);
    return la;
}
void split(int x, int y) { chroot(x); access(y); }
void chroot(int u) { access(u); set_rev(u); }
/* --- */
LCT(int n = 0) : o(n + 1) { o[0].size = 0; }
void set_val(int u, const Val &v) {
    splay(u); cur.v = v; up(u); }
void set_sval(int u, const SVal &v) {
    access(u); cur.sv = v; up(u); }
Val query(int x, int y) {
    split(x, y); return o[y].sum; }
SVal subtree(int p, int u) {
    chroot(p); access(u); return cur.vir + cur.sv; }
bool connected(int u, int v) {
    return find_root(u) == find_root(v); }
void link(int x, int y) {
    chroot(x); access(y);
    o[y].vir = o[y].vir + o[x].sub;
    up(o[x].pa = y);
}
void cut(int x, int y) {
    split(x, y); o[y].ch[0] = o[x].pa = 0; up(y); }
#undef cur
#undef lc
#undef rc
};

```

2.3 Treap [2ac37e]

```

mt19937 rng(880301);
struct node {
    ll data; int sz;
    node *l, *r;
    node(ll k = 0) : data(k), sz(1), l(0), r(0) {}
    void up() {
        sz = 1;
        if (l) sz += l->sz;
        if (r) sz += r->sz;
    }
    void down() {}
};
node pool[1000010]; int pool_cnt = 0;
node *newnode(ll k) { return &(pool[pool_cnt++] = node(k)); }
int sz(node *a) { return a ? a->sz : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (int(rng() % (sz(a) + sz(b))) < sz(a))
        return a->down(), a->r = merge(a->r, b), a->up(),
            a;
    return b->down(), b->l = merge(a, b->l), b->up(), b;
}
// a: key <= k, b: key > k
void split(node *o, node *&a, node *&b, ll k) {
    if (!o) return a = b = 0, void();
    o->down();
    if (o->data <= k)
        a = o, split(o->r, a->r, b, k), a->up();
    else b = o, split(o->l, a, b->l, k), b->up();
}
// a: size k, b: size n - k
void split2(node *o, node *&a, node *&b, int k) {
    if (sz(o) <= k) return a = o, b = 0, void();
    o->down();
    if (sz(o->l) + 1 <= k)
        a = o, split2(o->r, a->r, b, k - sz(o->l) - 1);
    else b = o, split2(o->l, a, b->l, k);
    o->up();
}
node *kth(node *o, ll k) { // 1-based
    if (k <= sz(o->l)) return kth(o->l, k);
    if (k == sz(o->l) + 1) return o;
    return kth(o->r, k - sz(o->l) - 1);
}

```

```

}
int Rank(node *o, ll key) { // num of key < key
    if (!o) return 0;
    if (o->data < key)
        return sz(o->l) + 1 + Rank(o->r, key);
    else return Rank(o->l, key);
}
bool erase(node *&o, ll k) {
    if (!o) return 0;
    if (o->data == k) {
        node *t = o;
        o->down(), o = merge(o->l, o->r);
        return 1;
    }
    node *&t = k < o->data ? o->l : o->r;
    return erase(t, k) ? o->up(), 1 : 0;
}
void insert(node *&o, ll k) {
    node *a, *b;
    split(o, a, b, k),
        o = merge(a, merge(new node(k), b));
}
tuple<node*, node*, node*> interval(node *&o, int l,
    int r) { // 1-based
    node *a, *b, *c; // b: [l, r]
    split2(o, a, b, l - 1), split2(b, b, c, r - l + 1);
    return make_tuple(a, b, c);
}

```

2.4 KD Tree [375ca2]

```

namespace kdt {
    int root, lc[maxn], rc[maxn], xl[maxn], xr[maxn],
        yl[maxn], yr[maxn];
    point p[maxn];
    int build(int l, int r, int dep = 0) {
        if (l == r) return -1;
        function<bool(const point &, const point &)> f =
            [dep](const point &a, const point &b) {
                if (dep & 1) return a.x < b.x;
                else return a.y < b.y;
            };
        int m = (l + r) >> 1;
        nth_element(p + l, p + m, p + r, f);
        xl[m] = xr[m] = p[m].x;
        yl[m] = yr[m] = p[m].y;
        lc[m] = build(l, m, dep + 1);
        if (~lc[m]) {
            xl[m] = min(xl[m], xl[lc[m]]);
            xr[m] = max(xr[m], xr[lc[m]]);
            yl[m] = min(yl[m], yl[lc[m]]);
            yr[m] = max(yr[m], yr[lc[m]]);
        }
        rc[m] = build(m + 1, r, dep + 1);
        if (~rc[m]) {
            xl[m] = min(xl[m], xl[rc[m]]);
            xr[m] = max(xr[m], xr[rc[m]]);
            yl[m] = min(yl[m], yl[rc[m]]);
            yr[m] = max(yr[m], yr[rc[m]]);
        }
        return m;
    }
    bool bound(const point &q, int o, long long d) {
        double ds = sqrt(d + 1.0);
        if (q.x < xl[o] - ds || q.x > xr[o] + ds ||
            q.y < yl[o] - ds || q.y > yr[o] + ds)
            return false;
        return true;
    }
    long long dist(const point &a, const point &b) {
        return (a.x - b.x) * 1ll * (a.x - b.x) +
            (a.y - b.y) * 1ll * (a.y - b.y);
    }
    void dfs(
        const point &q, long long &d, int o, int dep = 0)
        {
            if (!bound(q, o, d)) return;
            long long cd = dist(p[o], q);
            if (cd != 0) d = min(d, cd);
            if ((dep & 1) && q.x < p[o].x ||
                !(dep & 1) && q.y < p[o].y) {
                if (~lc[o]) dfs(q, d, lc[o], dep + 1);
                if (~rc[o]) dfs(q, d, rc[o], dep + 1);
            }
        }
}

```

```

    } else {
        if (~rc[o]) dfs(q, d, rc[o], dep + 1);
        if (~lc[o]) dfs(q, d, lc[o], dep + 1);
    }
}
void init(const vector<point> &v) {
    for (int i = 0; i < v.size(); ++i) p[i] = v[i];
    root = build(0, v.size());
}
long long nearest(const point &q) {
    long long res = 1e18;
    dfs(q, res, root);
    return res;
}
} // namespace kdt

```

2.5 Leftist Tree [e91538]

```

struct node {
    ll v, data, sz, sum;
    node *l, *r;
    node(ll k)
        : v(0), data(k), sz(1), l(0), r(0), sum(k) {}
};
ll sz(node *p) { return p ? p->sz : 0; }
ll V(node *p) { return p ? p->v : -1; }
ll sum(node *p) { return p ? p->sum : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (a->data < b->data) swap(a, b);
    a->r = merge(a->r, b);
    if (V(a->r) > V(a->l)) swap(a->r, a->l);
    a->v = V(a->r) + 1, a->sz = sz(a->l) + sz(a->r) + 1;
    a->sum = sum(a->l) + sum(a->r) + a->data;
    return a;
}
void pop(node *&o) {
    node *tmp = o;
    o = merge(o->l, o->r);
    delete tmp;
}

```

2.6 Convex 1D/1D [79e448]

```

template<class T>
struct DynamicHull {
    struct seg { int x, l, r; };
    T f; int C; deque<seg> dq; // range: 1~C
    explicit DynamicHull(T _f, int _C): f(_f), C(_C) {}
    // max t s.t. f(x, t) >= f(y, t), x < y, maintain max
    int intersect(int x, int y) {
        int l = 0, r = C + 1;
        while (l + 1 < r) {
            int mid = (l + r) / 2;
            if (f(x, mid) >= f(y, mid)) l = mid;
            else r = mid;
        }
        return l;
    }
    void push_back(int x) {
        for (int i; !dq.empty() &&
            (i = dq.back().l, f(dq.back().x, i) < f(x, i));
            )
            dq.pop_back();
        if (dq.empty()) return dq.pb(seg({x, 1, C})), void();
        dq.back().r = intersect(dq.back().x, x);
        dq.pb(seg({x, dq.back().l + 1, C}));
    }
    int query(int x) {
        while (dq.front().r < x) dq.pop_front();
        return dq.front().x;
    }
};

```

3 Flow & Matching

3.1 Dinic [801a71]

```

struct Dinic { // 0-based, O(V^2E), unit flow: O(min(V
    ^{2/3}E, E^{3/2})), bipartite matching: O(sqrt(V)E)
    struct edge {
        ll to, cap, flow, rev;

```

```

};
int n, s, t;
vector<vector<edge>> g;
vector<int> dis, ind;

void init(int _n) {
    n = _n;
    g.assign(n, vector<edge>());
}
void reset() {
    for (int i = 0; i < n; ++i)
        for (auto &j : g[i]) j.flow = 0;
}
void add_edge(int u, int v, ll cap) {
    g[u].pb(edge{v, cap, 0, SZ(g[v]))});
    g[v].pb(edge{u, 0, 0, SZ(g[u]) - 1});
    //change g[v] to cap for undirected graphs
}
bool bfs() {
    dis.assign(n, -1);
    queue<int> q;
    q.push(s), dis[s] = 0;
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (auto &e : g[cur]) {
            if (dis[e.to] == -1 && e.flow != e.cap) {
                q.push(e.to);
                dis[e.to] = dis[cur] + 1;
            }
        }
    }
    return dis[t] != -1;
}
ll dfs(int u, ll cap) {
    if (u == t || !cap) return cap;
    for (int &i = ind[u]; i < SZ(g[u]); ++i) {
        edge &e = g[u][i];
        if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
            ll df = dfs(e.to, min(e.cap - e.flow, cap));
            if (df) {
                e.flow += df;
                g[e.to][e.rev].flow -= df;
                return df;
            }
        }
    }
    dis[u] = -1;
    return 0;
}
ll maxflow(int _s, int _t) {
    s = _s; t = _t;
    ll flow = 0, df;
    while (bfs()) {
        ind.assign(n, 0);
        while ((df = dfs(s, INF))) flow += df;
    }
    return flow;
}
}

```

3.2 Bounded Flow [758826]

```

struct BoundedFlow : Dinic {
    vector<ll> tot;
    void init(int _n) {
        Dinic::init(_n + 2);
        tot.assign(n, 0);
    }
    void add_edge(int u, int v, ll lcap, ll rcap) {
        tot[u] -= lcap, tot[v] += lcap;
        g[u].pb(edge{v, rcap, lcap, SZ(g[v]))});
        g[v].pb(edge{u, 0, 0, SZ(g[u]) - 1});
    }
    bool feasible() {
        ll sum = 0;
        int vs = n - 2, vt = n - 1;
        for (int i = 0; i < n - 2; ++i)
            if (tot[i] > 0)
                add_edge(vs, i, 0, tot[i]), sum += tot[i];
            else if (tot[i] < 0) add_edge(i, vt, 0, -tot[i]);
        if (sum != maxflow(vs, vt)) sum = -1;
        for (int i = 0; i < n - 2; ++i)

```

```

    if(tot[i] > 0)
        g[vs].pop_back(), g[i].pop_back();
    else if(tot[i] < 0)
        g[i].pop_back(), g[vt].pop_back();
    return sum != -1;
}
ll boundedflow(int _s, int _t) {
    add_edge(_t, _s, 0, INF);
    if(!feasible()) return -1;
    ll x = g[_t].back().flow;
    g[_t].pop_back(), g[_s].pop_back();
    return x - maxflow(_t, _s); // min
    //return x + maxflow(_s, _t); // max
}
};

```

3.3 MCMF [8d3644]

```

struct MCMF { //  $\theta$ -based,  $O(SPFA * |f|)$ 
    struct edge {
        ll from, to, cap, flow, cost, rev;
    };
    int n;
    int s, t; ll mx;
    //mx: maximum amount of flow
    vector<vector<edge*>> g;
    vector<ll> dis, up;
    bool BellmanFord(ll &flow, ll &cost) {
        vector<edge*> past(n);
        vector<int> inq(n);
        dis.assign(n, INF); up.assign(n, 0);
        queue<int> q;
        q.push(s), inq[s] = 1;
        up[s] = mx - flow, past[s] = 0, dis[s] = 0;
        while (!q.empty()) {
            int u = q.front();
            q.pop(), inq[u] = 0;
            if (!up[u]) continue;
            for (auto &e : g[u])
                if (e.flow != e.cap &&
                    dis[e.to] > dis[u] + e.cost) {
                    dis[e.to] = dis[u] + e.cost, past[e.to] = &e;
                    up[e.to] = min(up[u], e.cap - e.flow);
                    if (!inq[e.to]) inq[e.to] = 1, q.push(e.to);
                }
        }
        if (dis[t] == INF) return 0;
        flow += up[t], cost += up[t] * dis[t];
        for (ll i = t; past[i]; i = past[i]->from) {
            auto &e = *past[i];
            e.flow += up[t], g[e.to][e.rev].flow -= up[t];
        }
        return 1;
    }
    pll MinCostMaxFlow(int _s, int _t) {
        s = _s, t = _t;
        ll flow = 0, cost = 0;
        while (BellmanFord(flow, cost));
        return pll(flow, cost);
    }
    void init(int _n, ll _mx) {
        n = _n, mx = _mx;
        g.assign(n, vector<edge*>());
    }
    void add_edge(int a, int b, ll cap, ll cost) {
        g[a].pb(edge{a, b, cap, 0, cost, SZ(g[b])});
        g[b].pb(edge{b, a, 0, 0, -cost, SZ(g[a]) - 1});
    }
};

```

3.4 Min Cost Circulation [47cf18]

```

struct MinCostCirculation { //  $\theta$ -based,  $O(VE * E \log C)$ 
    struct edge {
        ll from, to, cap, fcap, flow, cost, rev;
    };
    int n;
    vector<edge*> past;
    vector<vector<edge*>> g;
    vector<ll> dis;
    void BellmanFord(int s) {
        vector<int> inq(n);
        dis.assign(n, INF);
    }
};

```

```

queue<int> q;
auto relax = [&](int u, ll d, edge *e) {
    if (dis[u] > d) {
        dis[u] = d, past[u] = e;
        if (!inq[u]) inq[u] = 1, q.push(u);
    }
};
relax(s, 0, 0);
while (!q.empty()) {
    int u = q.front();
    q.pop(), inq[u] = 0;
    for (auto &e : g[u])
        if (e.cap > e.flow)
            relax(e.to, dis[u] + e.cost, &e);
}
}
void try_edge(edge &cur) {
    if (cur.cap > cur.flow) return ++cur.cap, void();
    BellmanFord(cur.to);
    if (dis[cur.from] + cur.cost < 0) {
        ++cur.flow, --g[cur.to][cur.rev].flow;
        for (int i = cur.from; past[i]; i = past[i]->from) {
            auto &e = *past[i];
            ++e.flow, --g[e.to][e.rev].flow;
        }
    }
    ++cur.cap;
}
}
void solve(int mxlg) { // mxlg >= Log(max cap)
    for (int b = mxlg; b >= 0; --b) {
        for (int i = 0; i < n; ++i)
            for (auto &e : g[i])
                e.cap *= 2, e.flow *= 2;
        for (int i = 0; i < n; ++i)
            for (auto &e : g[i])
                if (e.fcap >> b & 1)
                    try_edge(e);
    }
}
}
void init(int _n) {
    n = _n;
    past.assign(n, nullptr);
    g.assign(n, vector<edge*>());
}
}
void add_edge(ll a, ll b, ll cap, ll cost) {
    g[a].pb(edge{a, b, 0, cap, 0, cost, SZ(g[b]) + (a
        == b)});
    g[b].pb(edge{b, a, 0, 0, 0, -cost, SZ(g[a]) - 1});
}
};

```

3.5 Gomory Hu [82d968]

```

void GomoryHu(Dinic &flow) { //  $\theta$ -based
    int n = flow.n;
    vector<int> par(n);
    for (int i = 1; i < n; ++i) {
        flow.reset();
        add_edge(i, par[i], flow.maxflow(i, par[i]));
        for (int j = i + 1; j < n; ++j)
            if (par[j] == par[i] && ~flow.dis[j])
                par[j] = i;
    }
}

```

3.6 Stoer Wagner Algorithm [a9917b]

```

struct StoerWagner { //  $\theta$ -based,  $O(V^3)$ 
    int n;
    vector<int> vis, del;
    vector<ll> wei;
    vector<vector<ll>> edge;
    void init(int _n) {
        n = _n;
        del.assign(n, 0);
        edge.assign(n, vector<ll>(n));
    }
    void add_edge(int u, int v, ll w) {
        edge[u][v] += w, edge[v][u] += w;
    }
    void search(int &s, int &t) {
        vis.assign(n, 0); wei.assign(n, 0);
    }
};

```



```

s = t = -1;
while (1) {
    ll mx = -1, cur = 0;
    for (int i = 0; i < n; ++i)
        if (!del[i] && !vis[i] && mx < wei[i])
            cur = i, mx = wei[i];
    if (mx == -1) break;
    vis[cur] = 1, s = t, t = cur;
    for (int i = 0; i < n; ++i)
        if (!vis[i] && !del[i]) wei[i] += edge[cur][i];
}
}
ll solve() {
    ll ret = INF;
    for (int i = 0, x=0, y=0; i < n-1; ++i) {
        search(x, y), ret = min(ret, wei[y]), del[y] = 1;
        for (int j = 0; j < n; ++j)
            edge[x][j] = (edge[j][x] + edge[y][j]);
    }
    return ret;
}
};

```

3.7 Bipartite Matching [013c49]

*//min vertex cover: take all unmatched vertices in L
and find alternating tree,
//ans is not reached in L + reached in R
// O(VE)*

```

int n; // 1-based, max matching
int mx[maxn], my[maxn];
bool adj[maxn][maxn], vis[maxn];
bool dfs(int u) {
    if (vis[u]) return 0;
    vis[u] = 1;
    for (int v = 1; v <= n; v++) {
        if (!adj[u][v]) continue;
        if (!my[v] || (my[v] && dfs(my[v]))) {
            mx[u] = v, my[v] = u;
            return 1;
        }
    }
    return 0;
}
// O(E sqrt(V)), O(E Log V) for random sparse graphs
struct BipartiteMatching { // 0-based
    int nl, nr;
    vector<int> mx, my, dis, cur;
    vector<vector<int>> g;
    bool dfs(int u) {
        for (int &i = cur[u]; i < SZ(g[u]); ++i) {
            int e = g[u][i];
            if (!my[e] || (dis[my[e]] == dis[u] + 1 && dfs(my[e])))
                return mx[my[e] = u] = e, 1;
        }
        dis[u] = -1;
        return 0;
    }
    bool bfs() {
        int ret = 0;
        queue<int> q;
        dis.assign(nl, -1);
        for (int i = 0; i < nl; ++i)
            if (!mx[i]) q.push(i), dis[i] = 0;
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            for (int e : g[u])
                if (!my[e]) ret = 1;
                else if (!dis[my[e]]) {
                    q.push(my[e]);
                    dis[my[e]] = dis[u] + 1;
                }
        }
        return ret;
    }
    int matching() {
        int ret = 0;
        mx.assign(nl, -1); my.assign(nr, -1);
        while (bfs()) {
            cur.assign(nl, 0);
            for (int i = 0; i < nl; ++i)

```

```

        if (!mx[i] && dfs(i)) ++ret;
    }
    return ret;
}
void add_edge(int s, int t) { g[s].pb(t); }
void init(int _nl, int _nr) {
    nl = _nl, nr = _nr;
    g.assign(nl, vector<int>());
}
};

```

3.8 Kuhn Munkres Algorithm [683e0a]

```

struct KM { // 0-based, maximum matching, O(V^3)
    int n, ql, qr;
    vector<vector<ll>> w;
    vector<ll> hl, hr, slk;
    vector<int> fl, fr, pre, qu, vl, vr;
    void init(int _n) {
        n = _n;
        // -INF for perfect matching
        w.assign(n, vector<ll>(n, 0));
        pre.assign(n, 0);
        qu.assign(n, 0);
    }
    void add_edge(int a, int b, ll wei) {
        w[a][b] = wei;
    }
    bool check(int x) {
        if (vl[x] = 1, ~fl[x])
            return (vr[qu[qr++]] = fl[x]) = 1;
        while (~x) swap(x, fr[fl[x] = pre[x]]);
        return 0;
    }
    void bfs(int s) {
        slk.assign(n, INF); vl.assign(n, 0); vr.assign(n, 0);
        ql = qr = 0, qu[qr++] = s, vr[s] = 1;
        for (ll d;;) {
            while (ql < qr)
                for (int x = 0, y = qu[ql++]; x < n; ++x)
                    if (!vl[x] && slk[x] >= (d = hl[x] + hr[y] - w[x][y])) {
                        if (pre[x] = y, d) slk[x] = d;
                        else if (!check(x)) return;
                    }
            d = INF;
            for (int x = 0; x < n; ++x)
                if (!vl[x] && d > slk[x]) d = slk[x];
            for (int x = 0; x < n; ++x) {
                if (vl[x]) hl[x] += d;
                else slk[x] -= d;
                if (vr[x]) hr[x] -= d;
            }
            for (int x = 0; x < n; ++x)
                if (!vl[x] && !slk[x] && !check(x)) return;
        }
    }
    ll solve() {
        fl.assign(n, -1); fr.assign(n, -1); hl.assign(n, 0);
        hr.assign(n, 0);
        for (int i = 0; i < n; ++i)
            hl[i] = *max_element(iter(w[i]));
        for (int i = 0; i < n; ++i) bfs(i);
        ll res = 0;
        for (int i = 0; i < n; ++i) res += w[i][fl[i]];
        return res;
    }
};

```

3.9 Max Simple Graph Matching [907d7c]

```

struct Matching { // 0-based, O(V^3)
    queue<int> q; int n;
    vector<int> fa, s, vis, pre, match;
    vector<vector<int>> g;
    int Find(int u) {
        return u == fa[u] ? u : fa[u] = Find(fa[u]);
    }
    int LCA(int x, int y) {
        static int tk = 0; tk++; x = Find(x); y = Find(y);
        for (; swap(x, y);) if (x != n) {
            if (vis[x] == tk) return x;
            vis[x] = tk;

```

```

    x = Find(pre[match[x]]);
}
}
void Blossom(int x, int y, int l) {
    for (; Find(x) != l; x = pre[y]) {
        pre[x] = y, y = match[x];
        if (s[y] == 1) q.push(y), s[y] = 0;
        for (int z: {x, y}) if (fa[z] == z) fa[z] = l;
    }
}
bool Bfs(int r) {
    iota(iter(fa), 0); fill(iter(s), -1);
    q = queue<int>(); q.push(r); s[r] = 0;
    for (; !q.empty(); q.pop()) {
        for (int x = q.front(); int u : g[x])
            if (s[u] == -1) {
                if (pre[u] == x, s[u] == 1, match[u] == n) {
                    for (int a = u, b = x, last;
                        b != n; a = last, b = pre[a])
                        last = match[b], match[b] = a, match[a] = b;
                    return true;
                }
                q.push(match[u]); s[match[u]] = 0;
            }
        else if (!s[u] && Find(u) != Find(x)) {
            int l = LCA(u, x);
            Blossom(x, u, l); Blossom(u, x, l);
        }
    }
    return false;
}
Matching(int _n) : n(_n), fa(n + 1), s(n + 1), vis(n
    + 1), pre(n + 1, n), match(n + 1, n), g(n) {}
void add_edge(int u, int v)
{ g[u].pb(v), g[v].pb(u); }
int solve() {
    int ans = 0;
    for (int x = 0; x < n; ++x)
        if (match[x] == n) ans += Bfs(x);
    return ans;
} // match[x] == n means not matched
};

```

3.10 Stable Marriage

- 1: Initialize $m \in M$ and $w \in W$ to free
- 2: **while** \exists free man m who has a woman w to propose to **do**
- 3: $w \leftarrow$ first woman on m 's list to whom m has not yet proposed
- 4: **if** \exists some pair (m', w) **then**
- 5: **if** w prefers m to m' **then**
- 6: $m' \leftarrow$ free
- 7: $(m, w) \leftarrow$ engaged
- 8: **end if**
- 9: **else**
- 10: $(m, w) \leftarrow$ engaged
- 11: **end if**
- 12: **end while**

3.11 Flow Model

- Maximum/Minimum flow with lower bound / Circulation problem
 1. Construct super source S and sink T .
 2. For each edge (x, y, l, u) , connect $x \rightarrow y$ with capacity $u - l$.
 3. For each vertex v , denote by $in(v)$ the difference between the sum of incoming lower bounds and the sum of outgoing lower bounds.
 4. If $in(v) > 0$, connect $S \rightarrow v$ with capacity $in(v)$, otherwise, connect $v \rightarrow T$ with capacity $-in(v)$.
 - To maximize, connect $t \rightarrow s$ with capacity ∞ (skip this in circulation problem), and let f be the maximum flow from S to T . If $f \neq \sum_{v \in V, in(v) > 0} in(v)$, there's no solution. Otherwise, the maximum flow from s to t is the answer.
 - To minimize, let f be the maximum flow from S to T . Connect $t \rightarrow s$ with capacity ∞ and let the flow from S to T be f' . If $f + f' \neq \sum_{v \in V, in(v) > 0} in(v)$, there's no solution. Otherwise, f' is the answer.
 5. The solution of each edge e is $l_e + f_e$, where f_e corresponds to the flow of edge e on the graph.
- Construct minimum vertex cover from maximum matching M on bipartite graph (X, Y)
 1. Redirect every edge: $y \rightarrow x$ if $(x, y) \in M$, $x \rightarrow y$ otherwise.
 2. DFS from unmatched vertices in X .
 3. $x \in X$ is chosen iff x is unvisited.
 4. $y \in Y$ is chosen iff y is visited.
- Minimum cost cyclic flow

1. Construct super source S and sink T
2. For each edge (x, y, c) , connect $x \rightarrow y$ with $(cost, cap) = (c, 1)$ if $c > 0$, otherwise connect $y \rightarrow x$ with $(cost, cap) = (-c, 1)$
3. For each edge with $c < 0$, sum these cost as K , then increase $d(y)$ by 1, decrease $d(x)$ by 1
4. For each vertex v with $d(v) > 0$, connect $S \rightarrow v$ with $(cost, cap) = (0, d(v))$
5. For each vertex v with $d(v) < 0$, connect $v \rightarrow T$ with $(cost, cap) = (0, -d(v))$
6. Flow from S to T , the answer is the cost of the flow $C + K$
- Maximum density induced subgraph
 1. Binary search on answer, suppose we're checking answer T
 2. Construct a max flow model, let K be the sum of all weights
 3. Connect source $s \rightarrow v, v \in G$ with capacity K
 4. For each edge (u, v, w) in G , connect $u \rightarrow v$ and $v \rightarrow u$ with capacity w
 5. For $v \in G$, connect it with sink $v \rightarrow t$ with capacity $K + 2T - (\sum_{e \in E(v)} w(e)) - 2w(v)$
 6. T is a valid answer if the maximum flow $f < K|V|$
- Minimum weight edge cover
 1. Let $w'(u, v) = w(u, v) - \mu(u) - \mu(v)$, where $\mu(v)$ is the cost of the cheapest edge incident to v .
 2. Find the minimum weight matching M with w' . The answer is $\sum \mu(v) + w'(M)$.
- Project selection problem
 1. If $p_v > 0$, create edge (s, v) with capacity p_v ; otherwise, create edge (v, t) with capacity $-p_v$.
 2. Create edge (u, v) with capacity w with w being the cost of choosing u without choosing v .
 3. The mincut is equivalent to the maximum profit of a subset of projects.
- Dual of minimum cost maximum flow
 1. Capacity c_{uv} , Flow f_{uv} , Cost w_{uv} , Required Flow difference for vertex b_u .
 2. If all w_{uv} are integers, then optimal solution can happen when all p_u are integers.

$$\begin{aligned}
 & \min \sum_{uv} w_{uv} f_{uv} \\
 & -f_{uv} \geq -c_{uv} \Leftrightarrow \min \sum_u b_u p_u + \sum_{uv} c_{uv} \max(0, p_v - p_u - w_{uv}) \\
 & \sum_v f_{vu} - \sum_v f_{uv} = -b_u \quad p_u \geq 0
 \end{aligned}$$

4 Geometry

4.1 Geometry Template [86f0f1]

```

using ld = ll;
using pdd = pair<ld, ld>;
#define X first
#define Y second
// ld eps = 1e-7;

pdd operator+(pdd a, pdd b)
{ return {a.X + b.X, a.Y + b.Y}; }
pdd operator-(pdd a, pdd b)
{ return {a.X - b.X, a.Y - b.Y}; }
pdd operator*(ld i, pdd v)
{ return {i * v.X, i * v.Y}; }
pdd operator*(pdd v, ld i)
{ return {i * v.X, i * v.Y}; }
pdd operator/(pdd v, ld i)
{ return {v.X / i, v.Y / i}; }
ld dot(pdd a, pdd b)
{ return a.X * b.X + a.Y * b.Y; }
ld cross(pdd a, pdd b)
{ return a.X * b.Y - a.Y * b.X; }
ld abs2(pdd v)
{ return v.X * v.X + v.Y * v.Y; }
ld abs(pdd v)
{ return sqrt(abs2(v)); }
int sgn(ld v)
{ return v > 0 ? 1 : (v < 0 ? -1 : 0); }
// int sgn(ld v){ return v > eps ? 1 : (v < -eps ? -1 : 0); }

int ori(pdd a, pdd b, pdd c)
{ return sgn(cross(b - a, c - a)); }
bool collinearity(pdd a, pdd b, pdd c)
{ return ori(a, b, c) == 0; }
bool btw(pdd p, pdd a, pdd b)
{ return collinearity(p, a, b) && sgn(dot(a - p, b - p)) <= 0; }

bool seg_intersect(pdd p1, pdd p2, pdd p3, pdd p4){

```

```

if(btw(p1, p3, p4) || btw(p2, p3, p4) || btw(p3, p1,
    p2) || btw(p4, p1, p2))
    return true;
return ori(p1, p2, p3) * ori(p1, p2, p4) < 0 &&
    ori(p3, p4, p1) * ori(p3, p4, p2) < 0;
}
pdd intersect(pdd p1, pdd p2, pdd p3, pdd p4){
    ld a123 = cross(p2 - p1, p3 - p1);
    ld a124 = cross(p2 - p1, p4 - p1);
    return (p4 * a123 - p3 * a124) / (a123 - a124);
}
pdd perp(pdd p1)
{ return pdd(-p1.Y, p1.X); }
pdd projection(pdd p1, pdd p2, pdd p3)
{ return p1 + (p2 - p1) * dot(p3 - p1, p2 - p1) / abs2(
    p2 - p1); }
pdd reflection(pdd p1, pdd p2, pdd p3)
{ return p3 + perp(p2 - p1) * cross(p3 - p1, p2 - p1) /
    abs2(p2 - p1) * 2; }
pdd linearTransformation(pdd p0, pdd p1, pdd q0, pdd q1
    , pdd r) {
    pdd dp = p1 - p0, dq = q1 - q0, num(cross(dp, dq),
        dot(dp, dq));
    return q0 + pdd(cross(r - p0, num), dot(r - p0, num))
        / abs2(dp);
} // from line p0--p1 to q0--q1, apply to r

```

4.2 Polar Angle Comparator [808e89]

```

// -1: a // b (if same), 0/1: a < b
int cmp(p1 a, p1 b, bool same = true){
#define is_neg(k) (sgn(k.Y) < 0 || (sgn(k.Y) == 0 &&
    sgn(k.X) < 0))
    int A = is_neg(a), B = is_neg(b);
    if(A != B)
        return A < B;
    if(sgn(cross(a, b)) == 0)
        return same ? abs2(a) < abs2(b) : -1;
    return sgn(cross(a, b)) > 0;
}

```

4.3 Minkowski Sum [98abff]

```

void reorder_poly(vector<pdd>& pnts){
    int mn = 0;
    for(int i = 1; i < (int)pnts.size(); i++){
        if(pnts[i].Y < pnts[mn].Y || (pnts[i].Y == pnts[mn].Y
            && pnts[i].X < pnts[mn].X))
            mn = i;
        rotate(pnts.begin(), pnts.begin() + mn, pnts.end());
    }
}

vector<pdd> minkowski(vector<pdd> P, vector<pdd> Q){
    reorder_poly(P);
    reorder_poly(Q);
    int psz = P.size();
    int qsz = Q.size();
    P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
    vector<pdd> ans;
    int i = 0, j = 0;
    while(i < psz || j < qsz){
        ans.pb(P[i] + Q[j]);
        int t = sgn(cross(P[i + 1] - P[i], Q[j + 1] - Q[j]));
        if(t >= 0) i++;
        if(t <= 0) j++;
    }
    return ans;
}

```

4.4 Intersection of Circle and Convex Polygon [63653d]

```

double _area(pdd pa, pdd pb, double r){
    if(abs(pa)<abs(pb)) swap(pa, pb);
    if(abs(pb)<eps) return 0;
    double S, h, theta;
    double a=abs(pb),b=abs(pa),c=abs(pb-pa);
    double cosB = dot(pb,pb-pa) / a / c, B = acos(cosB);
    double cosC = dot(pa,pb) / a / b, C = acos(cosC);
    if(a > r){
        S = (C/2)*r*r;
        h = a*b*sin(C)/c;
    }
}

```

```

if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt
    (r*r-h*h));
}
else if(b > r){
    theta = PI - B - asin(sin(B)/r*a);
    S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
}
else S = .5*sin(C)*a*b;
return S;
}

double areaPolyCircle(const vector<pdd> poly,const pdd
    &o,const double r){
    double S=0;
    for(int i=0;i<SZ(poly);++i)
        S+=_area(poly[i]-o,poly[(i+1)%SZ(poly)]-o,r)*ori(0,
            poly[i],poly[(i+1)%SZ(poly)]);
    return fabs(S);
}

```

4.5 Intersection of Circles [f7a2fe]

```

bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
    pdd o1 = a.O, o2 = b.O;
    double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d =
        sqrt(d2);
    if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2)
        return 0;
    pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1
        * r1) / (2 * d2));
    double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 +
        r2 - d) * (-r1 + r2 + d));
    pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2);
    p1 = u + v, p2 = u - v;
    return 1;
}

```

4.6 Tangent Line of Circles [c51d90]

```

vector<Line> CCTang(const Cir& c1, const Cir& c2,
    int sign1 ){
    vector<Line> ret;
    double d_sq = abs2(c1.O - c2.O);
    if (sgn(d_sq) == 0) return ret;
    double d = sqrt(d_sq);
    pdd v = (c2.O - c1.O) / d;
    double c = (c1.R - sign1 * c2.R) / d; // cos t
    if (c * c > 1) return ret;
    double h = sqrt(max(0.0, 1.0 - c * c)); // sin t
    for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
        pdd n = pdd(v.X * c - sign2 * h * v.Y,
            v.Y * c + sign2 * h * v.X);
        pdd p1 = c1.O + n * c1.R;
        pdd p2 = c2.O + n * (c2.R * sign1);
        if (sgn(p1.X - p2.X) == 0 and
            sgn(p1.Y - p2.Y) == 0)
            p2 = p1 + perp(c2.O - c1.O);
        ret.pb(Line(p1, p2));
    }
    return ret;
}

```

4.7 Intersection of Line and Convex Polygon [157258]

```

int TangentDir(vector<p11> &C, p11 dir) {
    return cyc_tsearch(SZ(C), [&](int a, int b) {
        return cross(dir, C[a]) > cross(dir, C[b]);
    });
}

#define cml(i) sign(cross(C[i] - a, b - a))
pii lineHull(p11 a, p11 b, vector<p11> &C) {
    int A = TangentDir(C, a - b);
    int B = TangentDir(C, b - a);
    int n = SZ(C);
    if (cml(A) < 0 || cml(B) > 0)
        return pii(-1, -1); // no collision
    auto gao = [&](int l, int r) {
        for (int t = l; (l + 1) % n != r; ) {
            int m = ((l + r + (l < r ? 0 : n)) / 2) % n;
            (cml(m) == cml(t) ? l : r) = m;
        }
        return (l + !cml(r)) % n;
    };
}

```



```

};
pii res = pii(gao(B, A), gao(A, B)); // (i, j)
if (res.X == res.Y) // touching the corner i
    return pii(res.X, -1);
if (!cmpL(res.X) && !cmpL(res.Y)) // along side i, i
    +1
    switch ((res.X - res.Y + n + 1) % n) {
        case 0: return pii(res.X, res.X);
        case 2: return pii(res.Y, res.Y);
    }
/* crossing sides (i, i+1) and (j, j+1)
crossing corner i is treated as side (i, i+1)
returned in the same order as the line hits the
convex */
return res;
} // convex cut: (r, L]

```

4.8 Intersection of Line and Circle [9183db]

```

vector<pdd> circleLineIntersection(pdd c, double r, pdd
a, pdd b) {
    pdd p = a + (b - a) * dot(c - a, b - a) / abs2(b - a)
    ;
    double s = cross(b - a, c - a), h2 = r * r - s * s /
    abs2(b - a);
    if (sgn(h2) < 0) return {};
    if (sgn(h2) == 0) return {p};
    pdd h = (b - a) / abs(b - a) * sqrt(h2);
    return {p - h, p + h};
}

```

4.9 Point in Circle [ecf954]

```

// return q's relation with circumcircle of tri(p[0],p
[1],p[2])
bool in_cc(const array<p11, 3> &p, p11 q) {
    __int128 det = 0;
    for (int i = 0; i < 3; ++i)
        det += __int128(abs2(p[i]) - abs2(q)) * cross(p[(i
        + 1) % 3] - q, p[(i + 2) % 3] - q);
    return det > 0; // in: >0, on: =0, out: <0
}

```

4.10 Point in Convex [f86640]

```

bool PointInConvex(const vector<p11> &C, p11 p, bool
strict = true) {
    int a = 1, b = SZ(C) - 1, r = !strict;
    if (SZ(C) == 0) return false;
    if (SZ(C) < 3) return r && btw(C[0], C.back(), p);
    if (ori(C[0], C[a], C[b]) > 0) swap(a, b);
    if (ori(C[0], C[a], p) >= r || ori(C[0], C[b], p) <=
    -r)
        return false;
    while (abs(a - b) > 1) {
        int c = (a + b) / 2;
        (ori(C[0], C[c], p) > 0 ? b : a) = c;
    }
    return ori(C[a], C[b], p) < r;
}

```

4.11 Half Plane Intersection [dfb833]

```

// from 8BQube
p11 area_pair(Line a, Line b)
{ return p11(cross(a.Y - a.X, b.X - a.X), cross(a.Y - a
.X, b.Y - a.X)); }
bool isin(Line l0, Line l1, Line l2) {
    // Check inter(l1, l2) strictly in l0
    auto [a02X, a02Y] = area_pair(l0, l2);
    auto [a12X, a12Y] = area_pair(l1, l2);
    if (a12X - a12Y < 0) a12X *= -1, a12Y *= -1;
    return (__int128) a02Y * a12X - (__int128) a02X *
    a12Y > 0; // C^4
}
/* Having solution, check size > 2 */
/* ---^--- Line.X ---^--- Line.Y ---^--- */
vector<Line> halfPlaneInter(vector<Line> arr) {
    sort(iter(arr), [&](Line a, Line b) -> int {
        if (cmp(a.Y - a.X, b.Y - b.X, 0) != -1)
            return cmp(a.Y - a.X, b.Y - b.X, 0);
        return ori(a.X, a.Y, b.Y) < 0;
    });
    deque<Line> dq(1, arr[0]);

```

```

for (auto p : arr) {
    if (cmp(dq.back().Y - dq.back().X, p.Y - p.X, 0) ==
    -1)
        continue;
    while (SZ(dq) >= 2 && !isin(p, dq[SZ(dq) - 2], dq.
    back()))
        dq.pop_back();
    while (SZ(dq) >= 2 && !isin(p, dq[0], dq[1]))
        dq.pop_front();
    dq.pb(p);
}
while (SZ(dq) >= 3 && !isin(dq[0], dq[SZ(dq) - 2], dq
.back()))
    dq.pop_back();
while (SZ(dq) >= 3 && !isin(dq.back(), dq[0], dq[1]))
    dq.pop_front();
return vector<Line>(iter(dq));
}

```

4.12 Minimum Enclosing Circle [5af6d5]

```

using ld = long double;
pair<pdd, ld> circumcenter(pdd a, pdd b, pdd c);
pair<pdd, ld> MinimumEnclosingCircle(vector<pdd> &pts){
    random_shuffle(iter(pts));
    pdd c = pts[0];
    ld r = 0;
    for(int i = 1; i < SZ(pts); i++){
        if(abs(pts[i] - c) <= r) continue;
        c = pts[i]; r = 0;
        for(int j = 0; j < i; j++){
            if(abs(pts[j] - c) <= r) continue;
            c = (pts[i] + pts[j]) / 2;
            r = abs(pts[i] - c);
            for(int k = 0; k < j; k++){
                if(abs(pts[k] - c) > r)
                    tie(c, r) = circumcenter(pts[i], pts[j], pts[
                    k]);
            }
        }
    }
    return {c, r};
}

```

4.13 3D Point [badbbd]

```

// Copy from 8BQube
struct Point {
    double x, y, z;
    Point(double _x = 0, double _y = 0, double _z = 0): x
    (_x), y(_y), z(_z){}
    Point(pdd p) { x = p.X, y = p.Y, z = abs2(p); }
};
Point operator-(Point p1, Point p2)
{ return Point(p1.x - p2.x, p1.y - p2.y, p1.z - p2.z);
}
Point operator+(Point p1, Point p2)
{ return Point(p1.x + p2.x, p1.y + p2.y, p1.z + p2.z);
}
Point operator*(Point p1, double v)
{ return Point(p1.x * v, p1.y * v, p1.z * v); }
Point operator/(Point p1, double v)
{ return Point(p1.x / v, p1.y / v, p1.z / v); }
Point cross(Point p1, Point p2)
{ return Point(p1.y * p2.z - p1.z * p2.y, p1.z * p2.x -
    p1.x * p2.z, p1.x * p2.y - p1.y * p2.x); }
double dot(Point p1, Point p2)
{ return p1.x * p2.x + p1.y * p2.y + p1.z * p2.z; }
double abs(Point a)
{ return sqrt(dot(a, a)); }
Point cross3(Point a, Point b, Point c)
{ return cross(b - a, c - a); }
double area(Point a, Point b, Point c)
{ return abs(cross3(a, b, c)); }
double volume(Point a, Point b, Point c, Point d)
{ return dot(cross3(a, b, c), d - a); }
//Azimuthal angle (longitude) to x-axis in interval [-
pi, pi]
double phi(Point p) { return atan2(p.y, p.x); }
//Zenith angle (latitude) to the z-axis in interval [0,
pi]
double theta(Point p) { return atan2(sqrt(p.x * p.x + p
.y * p.y), p.z); }

```

```

Point masscenter(Point a, Point b, Point c, Point d)
{ return (a + b + c + d) / 4; }
pdd proj(Point a, Point b, Point c, Point u) {
// proj. u to the plane of a, b, and c
Point e1 = b - a;
Point e2 = c - a;
e1 = e1 / abs(e1);
e2 = e2 - e1 * dot(e2, e1);
e2 = e2 / abs(e2);
Point p = u - a;
return pdd(dot(p, e1), dot(p, e2));
}
Point rotate_around(Point p, double angle, Point axis)
{
double s = sin(angle), c = cos(angle);
Point u = axis / abs(axis);
return u * dot(u, p) * (1 - c) + p * c + cross(u, p)
    * s;
}

```

4.14 ConvexHull3D [156311]

```

struct convex_hull_3D {
struct Face {
int a, b, c;
Face(int ta, int tb, int tc): a(ta), b(tb), c(tc) {}
}; // return the faces with pt indexes
vector<Face> res;
vector<Point> P;
convex_hull_3D(const vector<Point> &_P): res(), P(_P) {
// all points coplanar case will WA, O(n^2)
int n = SZ(P);
if (n <= 2) return; // be careful about edge case
// ensure first 4 points are not coplanar
swap(P[1], *find_if(iter(P), [&](auto p) { return sgn
    (abs2(P[0] - p)) != 0; }));
swap(P[2], *find_if(iter(P), [&](auto p) { return sgn
    (abs2(cross3(p, P[0], P[1]))) != 0; }));
swap(P[3], *find_if(iter(P), [&](auto p) { return sgn
    (volume(P[0], P[1], P[2], p)) != 0; }));
vector<vector<int>> flag(n, vector<int>(n));
res.emplace_back(0, 1, 2); res.emplace_back(2, 1, 0);
for (int i = 3; i < n; ++i) {
vector<Face> next;
for (auto f : res) {
int d = sgn(volume(P[f.a], P[f.b], P[f.c], P[i]));
if (d <= 0) next.pb(f);
int ff = (d > 0) - (d < 0);
flag[f.a][f.b] = flag[f.b][f.c] = flag[f.c][f.a]
    = ff;
}
for (auto f : res) {
auto F = [&](int x, int y) {
if (flag[x][y] > 0 && flag[y][x] <= 0)
next.emplace_back(x, y, i);
};
F(f.a, f.b); F(f.b, f.c); F(f.c, f.a);
}
res = next;
}
}
bool same(Face s, Face t) {
if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.a])) != 0)
return 0;
if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.b])) != 0)
return 0;
if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.c])) != 0)
return 0;
return 1;
}
int polygon_face_num() {
int ans = 0;
for (int i = 0; i < SZ(res); ++i)
ans += none_of(res.begin(), res.begin() + i, [&](
    Face g) { return same(res[i], g); });
return ans;
}
double get_volume() {
double ans = 0;
for (auto f : res)
ans += volume(Point(0, 0, 0), P[f.a], P[f.b], P[f.c
]);
}

```

```

return fabs(ans / 6);
}
double get_dis(Point p, Face f) {
Point p1 = P[f.a], p2 = P[f.b], p3 = P[f.c];
double a = (p2.y - p1.y) * (p3.z - p1.z) - (p2.z - p1
    .z) * (p3.y - p1.y);
double b = (p2.z - p1.z) * (p3.x - p1.x) - (p2.x - p1
    .x) * (p3.z - p1.z);
double c = (p2.x - p1.x) * (p3.y - p1.y) - (p2.y - p1
    .y) * (p3.x - p1.x);
double d = 0 - (a * p1.x + b * p1.y + c * p1.z);
return fabs(a * p.x + b * p.y + c * p.z + d) / sqrt(a
    * a + b * b + c * c);
}
};
// n^2 delaunay: facets with negative z normal of
// convexhull of (x, y, x^2 + y^2), use a pseudo-point
// (0, 0, inf) to avoid degenerate case

```

4.15 Delaunay Triangulation [982e64]

```

/* Delaunay Triangulation:
Given a sets of points on 2D plane, find a
triangulation such that no points will strictly
inside circumcircle of any triangle. */
struct Edge {
int id; // oidx[id]
list<Edge>::iterator twin;
Edge(int _id = 0): id(_id) {}
};
struct Delaunay { // 0-base
int n, oidx[N];
list<Edge> head[N]; // result udir. graph
pll p[N];
void init(int _n, pll _p[]) {
n = _n, iota(oidx, oidx + n, 0);
for (int i = 0; i < n; ++i) head[i].clear();
sort(oidx, oidx + n, [&](int a, int b)
{ return _p[a] < _p[b]; });
for (int i = 0; i < n; ++i) p[i] = _p[oidx[i]];
divide(0, n - 1);
}
void addEdge(int u, int v) {
head[u].push_front(Edge(v));
head[v].push_front(Edge(u));
head[u].begin()->twin = head[v].begin();
head[v].begin()->twin = head[u].begin();
}
void divide(int l, int r) {
if (l == r) return;
if (l + 1 == r) return addEdge(l, l + 1);
int mid = (l + r) >> 1, nw[2] = {l, r};
divide(l, mid), divide(mid + 1, r);
auto gao = [&](int t) {
pll pt[2] = {p[nw[0]], p[nw[1]]};
for (auto it : head[nw[t]]) {
int v = ori(pt[1], pt[0], p[it.id]);
if (v > 0 || (v == 0 && abs2(pt[t ^ 1] - p[it.
    id]) < abs2(pt[1] - pt[0])))
return nw[t] = it.id, true;
}
return false;
};
while (gao(0) || gao(1));
addEdge(nw[0], nw[1]); // add tangent
while (true) {
pll pt[2] = {p[nw[0]], p[nw[1]]};
int ch = -1, sd = 0;
for (int t = 0; t < 2; ++t)
for (auto it : head[nw[t]])
if (ori(pt[0], pt[1], p[it.id]) > 0 && (
    ch == -1 || in_cc({pt[0], pt[1], p[ch
    ]}, p[it.id])))
ch = it.id, sd = t;
if (ch == -1) break; // upper common tangent
for (auto it = head[nw[sd]].begin(); it != head[
    nw[sd]].end(); )
if (seg_strict_intersect(pt[sd], p[it->id], pt[
    sd ^ 1], p[ch]))
head[it->id].erase(it->twin), head[nw[sd]].
    erase(it++);
else ++it;
nw[sd] = ch, addEdge(nw[0], nw[1]);
}
}

```

```

    }
}
} tool;

```

4.16 Voronoi Diagram [da0c5e]

```

// all coord. is even, you may want to call
// halfPlaneInter after then
vector<vector<Line>> vec;
void build_voronoi_line(int n, pll *arr) {
    tool.init(n, arr); // Delaunay
    vec.clear(), vec.resize(n);
    for (int i = 0; i < n; ++i)
        for (auto e : tool.head[i]) {
            int u = tool.oidx[i], v = tool.oidx[e.id];
            pll m = (arr[v] + arr[u]) / 2LL, d = perp(arr[v]
                - arr[u]);
            vec[u].pb(Line(m, m + d));
        }
}

```

4.17 Polygon Union [9fbf66]

```

// from 8BQube
ld rat(pll a, pll b) {
    return sgn(b.X) ? (ld)a.X / b.X : (ld)a.Y / b.Y;
} // all poly. should be ccw
ld polyUnion(vector<vector<pll>> &poly) {
    ld res = 0;
    for (auto &p : poly)
        for (int a = 0; a < SZ(p); ++a) {
            pll A = p[a], B = p[(a + 1) % SZ(p)];
            vector<pair<ld, int>> segs = {{0, 0}, {1, 0}};
            for (auto &q : poly) {
                if (&p == &q) continue;
                for (int b = 0; b < SZ(q); ++b) {
                    pll C = q[b], D = q[(b + 1) % SZ(q)];
                    int sc = ori(A, B, C), sd = ori(A, B, D);
                    if (sc != sd && min(sc, sd) < 0) {
                        ld sa = cross(D - C, A - C), sb = cross(D -
                            C, B - C);
                        segs.pb(sa / (sa - sb), sgn(sc - sd));
                    }
                    if (!sc && !sd && &q < &p && sgn(dot(B - A, D
                        - C)) > 0) {
                        segs.pb(rat(C - A, B - A), 1);
                        segs.pb(rat(D - A, B - A), -1);
                    }
                }
            }
            sort(iter(segs));
            for (auto &s : segs) s.X = clamp(s.X, 0.0, 1.0);
            ld sum = 0;
            int cnt = segs[0].second;
            for (int j = 1; j < SZ(segs); ++j) {
                if (!cnt) sum += segs[j].X - segs[j - 1].X;
                cnt += segs[j].Y;
            }
            res += cross(A, B) * sum;
        }
    return res / 2;
}

```

4.18 Tangent Point to Convex Hull [523bc1]

```

// from 8BQube
/* The point should be strictly out of hull
   return arbitrary point on the tangent line */
pii get_tangent(vector<pll> &C, pll p) {
    auto gao = [&](int s) {
        return cyc_tsearch(SZ(C), [&](int x, int y)
            { return ori(p, C[x], C[y]) == s; });
    };
    return pii(gao(1), gao(-1));
} // return (a, b), ori(p, C[a], C[b]) >= 0

```

4.19 Heart [082d19]

```

pdd circenter(pdd p0, pdd p1, pdd p2) { // radius = abs
    (center)
    p1 = p1 - p0, p2 = p2 - p0;
    double x1 = p1.X, y1 = p1.Y, x2 = p2.X, y2 = p2.Y;
    double m = 2. * (x1 * y2 - y1 * x2);
    pdd center;

```

```

    center.X = (x1 * x1 * y2 - x2 * x2 * y1 + y1 * y2 * (
        y1 - y2)) / m;
    center.Y = (x1 * x2 * (x2 - x1) - y1 * y1 * x2 + x1 *
        y2 * y2) / m;
    return center + p0;
}
pdd incenter(pdd p1, pdd p2, pdd p3) { // radius = area
    / s * 2
    double a = abs(p2 - p3), b = abs(p1 - p3), c = abs(p1
        - p2);
    double s = a + b + c;
    return (a * p1 + b * p2 + c * p3) / s;
}
pdd masscenter(pdd p1, pdd p2, pdd p3)
{ return (p1 + p2 + p3) / 3; }
pdd orthcenter(pdd p1, pdd p2, pdd p3)
{ return masscenter(p1, p2, p3) * 3 - circenter(p1, p2,
    p3) * 2; }

```

4.20 Rotating Sweep Line [f5f689]

```

struct Event {
    pll d; int u, v;
    bool operator<(const Event &b) const {
        int ret = cmp(d, b.d, false);
        return ret == -1 ? false : ret; } // no tie-break
};
void rotatingSweepLine(const vector<pll> &p) {
    const int n = SZ(p);
    vector<Event> e; e.reserve(n * (n - 1));
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++) // pos[i] < pos[j] when
            // the event occurs
            if (i != j) e.pb(p[j] - p[i], i, j);
    sort(iter(e));
    vector<int> ord(n), pos(n);
    iota(iter(ord), 0);
    sort(iter(ord), [&](int i, int j) { // initial order
        return p[i].Y != p[j].Y ? p[i].Y < p[j].Y : p[i].
            X < p[j].X; });
    for (int i = 0; i < n; i++) pos[ord[i]] = i;
    // initialize
    for (int i = 0, j = 0; i < SZ(e); i = j) {
        // do something
        vector<pii> tmp;
        for (; j < SZ(e) && !(e[i] < e[j]); j++)
            tmp.pb(pii(e[j].u, e[j].v));
        sort(iter(tmp), [&](pii x, pii y) {
            return pii(pos[x.ff], pos[x.ss]) < pii(pos[y.ff
                ], pos[y.ss]); });
        for (auto [x, y] : tmp) // pos[x] + 1 == pos[y]
            tie(ord[pos[x]], ord[pos[y]], pos[x], pos[y]) =
                make_tuple(ord[pos[y]], ord[pos[x]], pos[y],
                    pos[x]);
    }
}

```

4.21 Vector In Poly [c6d0fa]

```

// ori(a, b, c) >= 0, valid: "strict" angle from a-b to
// a-c
bool btwangle(pll a, pll b, pll c, pll p, int strict) {
    return ori(a, b, p) >= strict && ori(a, p, c) >=
        strict;
}
// whether vector{cur, p} in counter-clockwise order
// prv, cur, nxt
bool inside(pll prv, pll cur, pll nxt, pll p, int
    strict) {
    if (ori(cur, nxt, prv) >= 0)
        return btwangle(cur, nxt, prv, p, strict);
    return !btwangle(cur, prv, nxt, p, !strict);
}

```

5 Graph

5.1 BCC [d04ebe]

```

struct BCC { // 0-based, allow multi edges but not allow
    loops
    int n, m, cnt = 0;
    // n:|V|, m:|E|, cnt:#bcc
    // bcc i : vertices bcc_v[i] and edges bcc_e[i]

```

```

vector<vector<int>> bcc_v, bcc_e;
vector<vector<pii>> g; // original graph
vector<pii> edges; // 0-based
BCC(int _n, vector<pii> _edges):
    n(_n), m(SZ(_edges)), g(_n), edges(_edges){
    for(int i = 0; i < m; i++){
        auto [u, v] = edges[i];
        g[u].pb(pii(v, i)); g[v].pb(pii(u, i));
    }
}
void make_bcc(){ bcc_v.pb(); bcc_e.pb(); cnt++; }
// modify these if you need more information
void add_v(int v){ bcc_v.back().pb(v); }
void add_e(int e){ bcc_e.back().pb(e); }
void build(){
    vector<int> in(n, -1), low(n, -1), stk;
    vector<vector<int>> up(n);
    int ts = 0;
    auto _dfs = [&](auto dfs, int now, int par, int pe)
        -> void{
        if(pe != -1) up[now].pb(pe);
        in[now] = low[now] = ts++;
        stk.pb(now);
        for(auto [v, e] : g[now]){
            if(e == pe) continue;
            if(in[v] != -1){
                if(in[v] < in[now]) up[now].pb(e);
                low[now] = min(low[now], in[v]);
                continue;
            }
            dfs(dfs, v, now, e);
            low[now] = min(low[now], low[v]);
        }
        if((now != par && low[now] >= in[par]) || (now ==
            par && SZ(g[now]) == 0)){
            make_bcc();
            for(int v = stk.back(); v = stk.back()){
                stk.pop_back(), add_v(v);
                for(int e : up[v]) add_e(e);
                if(v == now) break;
            }
            if(now != par) add_v(par);
        }
    };
    for(int i = 0; i < n; i++)
        if(in[i] == -1) _dfs(_dfs, i, i, -1);
}
};

```

5.2 SCC [2c9a01]

```

struct SCC{ // 0-based, output reversed topo order
    int n, cnt = 0;
    vector<vector<int>> g;
    vector<int> sccid;
    explicit SCC(int _n): n(_n), g(_n), sccid(n, -1) {}
    void add_edge(int u, int v){
        g[u].pb(v);
    }
    void build(){
        vector<int> in(n, -1), low(n), stk;
        vector<bool> instk(n);
        int ts = 0;
        auto dfs1 = [&](auto dfs, int now) -> void{
            stk.pb(now); instk[now] = true;
            in[now] = low[now] = ts++;
            for(int i : g[now]){
                if(in[i] == -1)
                    dfs(dfs, i), low[now] = min(low[now], low[i]);
                else if(instk[i] && in[i] < in[now])
                    low[now] = min(low[now], in[i]);
            }
            if(low[now] == in[now]){
                for(; stk.back() != now; stk.pop_back())
                    sccid[stk.back()] = cnt, instk[stk.back()] = false;
                sccid[now] = cnt++, instk[now] = false, stk.
                    pop_back();
            }
        };
        for(int i = 0; i < n; i++)
            if(in[i] == -1) dfs1(dfs1, i);
    }
};

```

```

}
};

```

5.3 2-SAT [0686a5]

```

struct SAT { // 0-based
    int n;
    vector<bool> istrue;
    SCC scc;
    SAT(int _n): n(_n), istrue(n + n), scc(n + n) {}
    int neg(int a) {
        return a >= n ? a - n : a + n;
    }
    void add_clause(int a, int b) {
        scc.add_edge(neg(a), b), scc.add_edge(neg(b), a);
    }
    bool solve() {
        scc.build();
        for (int i = 0; i < n; ++i) {
            if (scc.sccid[i] == scc.sccid[i + n]) return
                false;
            istrue[i] = scc.sccid[i] < scc.sccid[i + n];
            istrue[i + n] = !istrue[i];
        }
        return true;
    }
};

```

5.4 Dominator Tree [2da9bb]

```

struct Dominator {
    int n;
    vector<vector<int>> g, r, rdom; int tk;
    vector<int> dfn, rev, fa, sdom, dom, val, rp;
    Dominator(int _n) : n(_n), g(_n), r(_n), rdom(_n), tk(0)
        {
            dfn = rev = fa = sdom = dom =
                val = rp = vector<int>(n, -1);
        }
    void add_edge(int x, int y) { g[x].push_back(y); }
    void dfs(int x) {
        rev[dfn[x] = tk] = x;
        fa[tk] = sdom[tk] = val[tk] = tk; tk++;
        for (int u : g[x]) {
            if (dfn[u] == -1) dfs(u), rp[dfn[u]] = dfn[x];
            r[dfn[u]].push_back(dfn[x]);
        }
    }
    void merge(int x, int y) { fa[x] = y; }
    int find(int x, int c = 0) {
        if (fa[x] == x) return c ? -1 : x;
        if (int p = find(fa[x], 1); p != -1) {
            if (sdom[val[x]] > sdom[val[fa[x]]])
                val[x] = val[fa[x]];
            fa[x] = p;
            return c ? p : val[x];
        } else return c ? fa[x] : val[x];
    }
    vector<int> build(int s) {
        // return the father of each node in dominator tree
        dfs(s); // p[i] = -2 if i is unreachable, par[s] =
            -1
        for (int i = tk - 1; i >= 0; --i) {
            for (int u : r[i])
                sdom[i] = min(sdom[i], sdom[find(u)]);
            if (i) rdom[sdom[i]].push_back(i);
            for (int u : rdom[i]) {
                int p = find(u);
                dom[u] = (sdom[p] == i ? i : p);
            }
            if (i) merge(i, rp[i]);
        }
        vector<int> p(n, -2); p[s] = -1;
        for (int i = 1; i < tk; ++i)
            if (sdom[i] != dom[i]) dom[i] = dom[dom[i]];
        for (int i = 1; i < tk; ++i)
            p[rev[i]] = rev[dom[i]];
        return p;
    }
};

```

5.5 Virtual Tree [f977f8]

// copy from 8BQube

```
vector<int> vG[N];
int top, st[N];
int vrt = -1;

void insert(int u) {
    if (top == -1) return st[++top] = vrt = u, void();
    int p = LCA(st[top], u);
    if (dep[vrt] > dep[p]) vrt = p;
    if (p == st[top]) return st[++top] = u, void();
    while (top >= 1 && dep[st[top - 1]] >= dep[p])
        vG[st[top - 1]].pb(st[top]), --top;
    if (st[top] != p)
        vG[p].pb(st[top]), --top, st[++top] = p;
    st[++top] = u;
}

void reset(int u) {
    for (int i : vG[u]) reset(i);
    vG[u].clear();
}

void solve(vector<int> &v) {
    top = -1;
    sort(ALL(v), [&](int a, int b) { return dfn[a] < dfn[b]; });
    for (int i : v) insert(i);
    while (top > 0) vG[st[top - 1]].pb(st[top]), --top;
    // do something
    reset(vrt);
}
```

5.6 Fast DMST [7b274d]

```
struct E { int s, t; ll w; }; // 0-base
struct PQ {
    struct P {
        ll v; int i;
        bool operator>(const P &b) const { return v > b.v; }
    };
    priority_queue<P, vector<P>, greater<>> pq; ll tag;
    // min heap
    void push(P p) { p.v -= tag; pq.emplace(p); }
    P top() { P p = pq.top(); p.v += tag; return p; }
    void join(PQ &b) {
        if (pq.size() < b.pq.size())
            swap(pq, b.pq), swap(tag, b.tag);
        while (!b.pq.empty()) push(b.top()), b.pq.pop();
    }
}; // O(E log^2 V), use Leftist tree for O(E log V)
vector<int> dmst(const vector<E> &e, int n, int root) {
    vector<PQ> h(n * 2);
    for (int i = 0; i < int(e.size()); ++i)
        h[e[i].t].push({e[i].w, i});
    vector<int> a(n * 2); iota(iter(a), 0);
    vector<int> v(n * 2, -1), pa(n * 2, -1), r(n * 2);
    auto o = [&](auto Y, int x) -> int {
        return x == a[x] ? x : a[x] = Y(Y, a[x]);
    };
    auto S = [&](int i) { return o(o, e[i].s); };
    int pc = v[root] = n;
    for (int i = 0; i < n; ++i) if (v[i] == -1)
        for (int p = i; v[p] < 0 || v[p] == i; p = S(r[p])) {
            if (v[p] == i)
                for (int q = pc++; p != q; p = S(r[p])) {
                    h[p].tag -= h[p].top().v; h[q].join(h[p]);
                    pa[p] = a[p] = q;
                }
            while (S(h[p].top().i) == p) h[p].pq.pop();
            v[p] = i; r[p] = h[p].top().i;
        }
    vector<int> ans;
    for (int i = pc - 1; i >= 0; i--) if (v[i] != n) {
        for (int f = e[r[i]].t; f != -1 && v[f] != n; f = pa[f])
            v[f] = n;
        ans.push_back(r[i]);
    }
    return ans; // default minimize, returns edgeid array
}
```

5.7 Vizing [f4ebad]

// find D+1 edge coloring of a graph with max deg D

```
struct vizing { // returns edge coloring in adjacent
    matrix G. 1 - based
    const int N = 105;
    int C[N][N], G[N][N], X[N], vst[N], n; // ans: G[i][j]
}

void init(int _n) { n = _n; // n = |V|+1
    for (int i = 0; i <= n; ++i)
        for (int j = 0; j <= n; ++j)
            C[i][j] = G[i][j] = 0;
}

void solve(vector<pii> &E) {
    auto update = [&](int u)
    { for (X[u] = 1; C[u][X[u]]; ++X[u]); };
    auto color = [&](int u, int v, int c) {
        int p = G[u][v];
        G[u][v] = G[v][u] = c;
        C[u][c] = v, C[v][c] = u;
        C[u][p] = C[v][p] = 0;
        if (p) X[u] = X[v] = p;
        else update(u), update(v);
        return p;
    };
    auto flip = [&](int u, int c1, int c2) {
        int p = C[u][c1];
        swap(C[u][c1], C[u][c2]);
        if (p) G[u][p] = G[p][u] = c2;
        if (!C[u][c1]) X[u] = c1;
        if (!C[u][c2]) X[u] = c2;
        return p;
    };
    fill_n(X + 1, n, 1);
    for (int t = 0; t < SZ(E); ++t) {
        int u = E[t].X, v0 = E[t].Y, v = v0, c0 = X[u], c
            = c0, d;
        vector<pii> L;
        fill_n(vst + 1, n, 0);
        while (!G[u][v0]) {
            L.emplace_back(v, d = X[v]);
            if (!C[v][c]) for (int a = SZ(L) - 1; a >= 0;
                --a) c = color(u, L[a].X, c);
            else if (!C[u][d]) for (int a = SZ(L) - 1; a >=
                0; --a) color(u, L[a].X, L[a].Y);
            else if (vst[d]) break;
            else vst[d] = 1, v = C[u][d];
        }
        if (!G[u][v0]) {
            for (; v; v = flip(v, c, d), swap(c, d));
            if (int a; C[u][c0]) {
                for (a = SZ(L) - 2; a >= 0 && L[a].Y != c; --
                    a);
                for (; a >= 0; --a) color(u, L[a].X, L[a].Y);
            }
            else --t;
        }
    }
}
```

5.8 Maximum Clique [d50aa9]

```
struct MaxClique { // fast when N <= 100
    bitset<N> G[N], cs[N];
    int ans, sol[N], q, cur[N], d[N], n;
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) G[i].reset();
    }
    void add_edge(int u, int v) {
        G[u][v] = G[v][u] = 1;
    }
    void pre_dfs(vector<int> &r, int l, bitset<N> mask) {
        if (l < 4) {
            for (int i : r) d[i] = (G[i] & mask).count();
            sort(ALL(r), [&](int x, int y) { return d[x] > d[
                y]; });
        }
        vector<int> c(SZ(r));
        int lft = max(ans - q + 1, 1), rgt = 1, tp = 0;
        cs[1].reset(), cs[2].reset();
        for (int p : r) {
            int k = 1;
            while ((cs[k] & G[p]).any()) ++k;
            if (k > rgt) cs[++rgt + 1].reset();
        }
    }
}
```



```

    cs[k][p] = 1;
    if (k < lft) r[tp++] = p;
}
for (int k = lft; k <= rgt; ++k)
    for (int p = cs[k]._Find_first(); p < N; p = cs[k]._Find_next(p))
        r[tp] = p, c[tp] = k, ++tp;
dfs(r, c, l + 1, mask);
}
void dfs(vector<int> &r, vector<int> &c, int l,
    bitset<N> mask) {
    while (!r.empty()) {
        int p = r.back();
        r.pop_back(), mask[p] = 0;
        if (q + c.back() <= ans) return;
        cur[q++] = p;
        vector<int> nr;
        for (int i : r) if (G[p][i]) nr.pb(i);
        if (!nr.empty()) pre_dfs(nr, l, mask & G[p]);
        else if (q > ans) ans = q, copy_n(cur, q, sol);
        c.pop_back(), --q;
    }
}
int solve() {
    vector<int> r(n);
    ans = q = 0, iota(ALL(r), 0);
    pre_dfs(r, 0, bitset<N>(string(n, '1')));
    return ans;
}
};

```

5.9 Number of Maximal Clique [11fa26]

```

struct BronKerbosch { // 1-base
    int n, a[N], g[N][N];
    int S, all[N][N], some[N][N], none[N][N];
    void init(int _n) {
        n = _n;
        for (int i = 1; i <= n; ++i)
            for (int j = 1; j <= n; ++j) g[i][j] = 0;
    }
    void add_edge(int u, int v) {
        g[u][v] = g[v][u] = 1;
    }
    void dfs(int d, int an, int sn, int nn) {
        if (S > 1000) return; // pruning
        if (sn == 0 && nn == 0) ++S;
        int u = some[d][0];
        for (int i = 0; i < sn; ++i) {
            int v = some[d][i];
            if (g[u][v]) continue;
            int tsu = 0, tnn = 0;
            copy_n(all[d], an, all[d + 1]);
            all[d + 1][an] = v;
            for (int j = 0; j < sn; ++j)
                if (g[v][some[d][j]])
                    some[d + 1][tsu++] = some[d][j];
            for (int j = 0; j < nn; ++j)
                if (g[v][none[d][j]])
                    none[d + 1][tnn++] = none[d][j];
            dfs(d + 1, an + 1, tsu, tnn);
            some[d][i] = 0, none[d][nn++] = v;
        }
    }
    int solve() {
        iota(some[0], some[0] + n, 1);
        S = 0, dfs(0, 0, n, 0);
        return S;
    }
};

```

5.10 Minimum Mean Cycle [3e5d2b]

```

// from 8BQube
ll road[N][N]; // input here
struct MinimumMeanCycle {
    ll dp[N + 5][N], n;
    pll solve() {
        ll a = -1, b = -1, L = n + 1;
        for (int i = 2; i <= L; ++i)
            for (int k = 0; k < n; ++k)
                for (int j = 0; j < n; ++j)
                    dp[i][j] =

```

```

        min(dp[i - 1][k] + road[k][j], dp[i][j]);
    for (int i = 0; i < n; ++i) {
        if (dp[L][i] >= INF) continue;
        ll ta = 0, tb = 1;
        for (int j = 1; j < n; ++j)
            if (dp[j][i] < INF &&
                ta * (L - j) < (dp[L][i] - dp[j][i]) * tb)
                ta = dp[L][i] - dp[j][i], tb = L - j;
        if (ta == 0) continue;
        if (a == -1 || a * tb > ta * b) a = ta, b = tb;
    }
    if (a != -1) {
        ll g = __gcd(a, b);
        return pll(a / g, b / g);
    }
    return pll(-1LL, -1LL);
}
void init(int _n) {
    n = _n;
    for (int i = 0; i < n; ++i)
        for (int j = 0; j < n; ++j) dp[i + 2][j] = INF;
}
};

```

5.11 Minimum Steiner Tree [21acea]

```

// from 8BQube
//  $O(V \cdot 3^T + V^2 \cdot 2^T)$ 
struct SteinerTree { // 0-base
    static const int T = 10, N = 105, INF = 1e9;
    int n, dst[N][N], dp[1 << T][N], tdst[N];
    int vcost[N]; // the cost of vertices
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) {
            for (int j = 0; j < n; ++j) dst[i][j] = INF;
            dst[i][i] = vcost[i] = 0;
        }
    }
    void add_edge(int ui, int vi, int wi) {
        dst[ui][vi] = min(dst[ui][vi], wi);
    }
    void shortest_path() {
        for (int k = 0; k < n; ++k)
            for (int i = 0; i < n; ++i)
                for (int j = 0; j < n; ++j)
                    dst[i][j] =
                        min(dst[i][j], dst[i][k] + dst[k][j]);
    }
    int solve(const vector<int> &ter) {
        shortest_path();
        int t = SZ(ter);
        for (int i = 0; i < (1 << t); ++i)
            for (int j = 0; j < n; ++j) dp[i][j] = INF;
        for (int i = 0; i < n; ++i) dp[0][i] = vcost[i];
        for (int msk = 1; msk < (1 << t); ++msk) {
            if (!(msk & (msk - 1))) {
                int who = __lg(msk);
                for (int i = 0; i < n; ++i)
                    dp[msk][i] =
                        vcost[ter[who]] + dst[ter[who]][i];
            }
            for (int i = 0; i < n; ++i)
                for (int submsk = (msk - 1) & msk; submsk; submsk = (submsk - 1) & msk)
                    dp[msk][i] = min(dp[msk][i],
                        dp[submsk][i] + dp[msk ^ submsk][i] -
                        vcost[i]);
            for (int i = 0; i < n; ++i) {
                tdst[i] = INF;
                for (int j = 0; j < n; ++j)
                    tdst[i] =
                        min(tdst[i], dp[msk][j] + dst[j][i]);
            }
            for (int i = 0; i < n; ++i) dp[msk][i] = tdst[i];
        }
        int ans = INF;
        for (int i = 0; i < n; ++i)
            ans = min(ans, dp[(1 << t) - 1][i]);
        return ans;
    }
};

```

6 Math

6.1 Extended Euclidean Algorithm [c51ae9]

```
// ax+ny = 1, ax+ny == ax == 1 (mod n)
void extgcd(ll x, ll y, ll &g, ll &a, ll &b) {
    if (y == 0) g=x, a=1, b=0;
    else extgcd(y, x%y, g, b, a), b-=(x/y)*a;
}
```

6.2 Floor & Ceil [134881]

```
ll ifloor(ll a, ll b){
    return a / b - (a % b && (a < 0) ^ (b < 0));
}
ll iceil(ll a, ll b){
    return a / b + (a % b && (a < 0) ^ (b > 0));
}
```

6.3 Legendre [4e4b23]

```
// the Jacobi symbol is a generalization of the
// Legendre symbol,
// such that the bottom doesn't need to be prime.
// (n/p) -> same as Legendre
// (n/ab) = (n/a)(n/b)
// work with Long Long
```

```
int Jacobi(int a, int m) {
    int s = 1;
    for (; m > 1; ) {
        a %= m;
        if (a == 0) return 0;
        const int r = __builtin_ctz(a);
        if ((r & 1) && ((m + 2) & 4)) s = -s;
        a >>= r;
        if (a & m & 2) s = -s;
        swap(a, m);
    }
    return s;
}

// 0: a == 0
// -1: a isn't a quad res of p
// else: return X with X^2 % p == a
// doesn't work with Long Long
int QuadraticResidue(int a, int p) {
    if (p == 2) return a & 1;
    if (int jc = Jacobi(a, p); jc <= 0) return jc;
    int b, d;
    for (; ; ) {
        b = rand() % p;
        d = (1LL * b * b + p - a) % p;
        if (Jacobi(d, p) == -1) break;
    }
    int f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
    for (int e = (1LL + p) >> 1; e >>= 1; ) {
        if (e & 1) {
            tmp = (1LL * g0 * f0 + 1LL * d * (1LL * g1 * f1 % p)) % p;
            g1 = (1LL * g0 * f1 + 1LL * g1 * f0) % p;
            g0 = tmp;
        }
        tmp = (1LL * f0 * f0 + 1LL * d * (1LL * f1 * f1 % p)) % p;
        f1 = (2LL * f0 * f1) % p;
        f0 = tmp;
    }
    return g0;
}
```

6.4 Simplex [aa7741]

```
// maximize c^T x
// subject to Ax <= b, x >= 0
// and stores the solution;
typedef long double T; // Long double, Rational, double
// + mod<P>...
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-9, inf = 1/.0;
#define ltj(X) if(s == -1 || mp(X[j], N[j]) < mp(X[s], N[s])) s=j
#define rep(i, l, n) for(int i = l; i < n; i++)
```

```
struct LPSolver {
    int m, n;
    vector<int> N, B;
    vvd D;

    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(SZ(b)), n(SZ(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
        rep(i, 0, m) rep(j, 0, n) D[i][j] = A[i][j];
        rep(i, 0, m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
        rep(j, 0, n) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m+1][n] = 1;
    }

    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
        rep(i, 0, m+2) if (i != r && abs(D[i][s]) > eps) {
            T *b = D[i].data(), inv2 = b[s] * inv;
            rep(j, 0, n+2) b[j] -= a[j] * inv2;
            b[s] = a[s] * inv2;
        }
        rep(j, 0, n+2) if (j != s) D[r][j] *= inv;
        rep(i, 0, m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]);
    }

    bool simplex(int phase) {
        int x = m + phase - 1;
        for (; ; ) {
            int s = -1;
            rep(j, 0, n+1) if (N[j] != -phase) ltj(D[x]);
            if (D[x][s] >= -eps) return true;
            int r = -1;
            rep(i, 0, m) {
                if (D[i][s] <= eps) continue;
                if (r == -1 || mp(D[i][n+1] / D[i][s], B[i]) < mp(D[r][n+1] / D[r][s], B[r])) r = i;
            }
            if (r == -1) return false;
            pivot(r, s);
        }
    }

    T solve(vd &x) {
        int r = 0;
        rep(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
        if (D[r][n+1] < -eps) {
            pivot(r, n);
            if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
        }
        rep(i, 0, m) if (B[i] == -1) {
            int s = 0;
            rep(j, 1, n+1) ltj(D[i]);
            pivot(i, s);
        }
    }

    bool ok = simplex(1); x = vd(n);
    rep(i, 0, m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
};
```

6.5 Floor Sum [e82b4d]

```
// from 8BQube
ll floor_sum(ll n, ll m, ll a, ll b) {
    assert(m);
    if (m < 0) return -floor_sum(n, -m, a, b-m-1);
    ll ans = 0;
    if (a >= m)
        ans += (n - 1) * n * (a / m) / 2, a %= m;
    if (b >= m)
        ans += n * (b / m), b %= m;
    ll y_max = (a * n + b) / m, x_max = (y_max * m - b);
    if (y_max == 0) return ans;
    ans += (n - (x_max + a - 1) / a) * y_max;
    ans += floor_sum(y_max, a, m, (a - x_max % a) % a);
    return ans;
} // sum^{n-1}_0 floor((a * i + b) / m) in Log(n + m + a + b)
```

6.6 DiscreteLog [da27bf]

```
int DiscreteLog(int s, int x, int y, int m) {
    constexpr int kStep = 32000;
    unordered_map<int, int> p;
    int b = 1;
    for (int i = 0; i < kStep; ++i) {
        p[y] = i;
        y = 1LL * y * x % m;
        b = 1LL * b * x % m;
    }
    for (int i = 0; i < m + 10; i += kStep) {
        s = 1LL * s * b % m;
        if (p.find(s) != p.end()) return i + kStep - p[s];
    }
    return -1;
}

int DiscreteLog(int x, int y, int m) {
    if (m == 1) return 0;
    int s = 1;
    for (int i = 0; i < 100; ++i) {
        if (s == y) return i;
        s = 1LL * s * x % m;
    }
    if (s == y) return 100;
    int p = 100 + DiscreteLog(s, x, y, m);
    if (fpow(x, p, m) != y) return -1;
    return p; //returns: x^p = y (mod m)
}
```

6.7 Miller Rabin & Pollard Rho [d3ecd2]

```
// n < 4,759,123,141      3 : 2, 7, 61
// n < 1,122,004,669,633 4 : 2, 13, 23, 1662803
// n < 3,474,749,660,383 6 : primes <= 13
// n < 2^64              7 :
// 2, 325, 9375, 28178, 450775, 9780504, 1795265022
ll mul(ll a, ll b, ll n){
    return (__int128)a * b % n;
}

bool Miller_Rabin(ll a, ll n) {
    if ((a = a % n) == 0) return 1;
    if (n % 2 == 0) return n == 2;
    ll tmp = (n - 1) / ((n - 1) & (1 - n));
    ll t = __lg(((n - 1) & (1 - n))), x = 1;
    for (; tmp; tmp >>= 1, a = mul(a, a, n))
        if (tmp & 1) x = mul(x, a, n);
    if (x == 1 || x == n - 1) return 1;
    while (--t)
        if ((x = mul(x, x, n)) == n - 1) return 1;
    return 0;
}

bool prime(ll n){
    vector<ll> tmp = {2, 325, 9375, 28178, 450775,
        9780504, 1795265022};
    for(ll i : tmp)
        if(!Miller_Rabin(i, n)) return false;
    return true;
}

map<ll, int> cnt;
void PollardRho(ll n) {
    if (n == 1) return;
    if (prime(n)) return ++cnt[n], void();
    if (n % 2 == 0) return PollardRho(n / 2), ++cnt[2],
        void();
    ll x = 2, y = 2, d = 1, p = 1;
#define f(x, n, p) ((mul(x, x, n) + p) % n)
    while (true) {
        if (d != n && d != 1) {
            PollardRho(n / d);
            PollardRho(d);
            return;
        }
        if (d == n) ++p;
        x = f(x, n, p), y = f(f(y, n, p), n, p);
        d = gcd(abs(x - y), n);
    }
}
```

6.8 XOR Basis [006505]

```
const int digit = 60; // [0, 2^digit)
struct Basis{
```

```
int total = 0, rank = 0;
vector<ll> b;
Basis(): b(digit) {}
bool add(ll v){ // Gauss Jordan Elimination
    total++;
    for(int i = digit - 1; i >= 0; i--){
        if(!(1LL << i & v)) continue;
        if(b[i] != 0){
            v ^= b[i];
            continue;
        }
        for(int j = 0; j < i; j++){
            if(1LL << j & v) v ^= b[j];
        }
        for(int j = i + 1; j < digit; j++){
            if(1LL << i & b[j]) b[j] ^= v;
        }
        b[i] = v;
        rank++;
        return true;
    }
    return false;
}

ll getMax(ll x = 0){
    for(ll i : b) x = max(x, x ^ i);
    return x;
}

ll getMin(ll x = 0){
    for(ll i : b) x = min(x, x ^ i);
    return x;
}

bool can(ll x){
    return getMin(x) == 0;
}

ll kth(ll k){ // kth smallest, 0-indexed
    vector<ll> tmp;
    for(ll i : b) if(i) tmp.pb(i);
    ll ans = 0;
    for(int i = 0; i < SZ(tmp); i++){
        if(1LL << i & k) ans ^= tmp[i];
    }
    return ans;
}
```

6.9 Linear Equation [ab487b]

```
vector<int> RREF(vector<vector<ll>> &mat){
    int N = mat.size(), M = mat[0].size();
    int rk = 0;
    vector<int> cols;
    for (int i = 0; i < M; i++) {
        int cnt = -1;
        for (int j = N - 1; j >= rk; j--)
            if(mat[j][i] != 0) cnt = j;
        if(cnt == -1) continue;
        swap(mat[rk], mat[cnt]);
        ll lead = mat[rk][i];
        for (int j = 0; j < M; j++) mat[rk][j] = mat[rk][j] *
            modinv(lead) % mod;
        for (int j = 0; j < N; j++) {
            if(j == rk) continue;
            ll tmp = mat[j][i];
            for (int k = 0; k < M; k++)
                mat[j][k] = (mat[j][k] - mat[rk][k] * tmp % mod
                    + mod) % mod;
        }
        cols.pb(i);
        rk++;
    }
    return cols;
}

struct LinearEquation{
    bool ok;
    vector<ll> par; //particular solution (Ax = b)
    vector<vector<ll>> homo; //homogenous (Ax = 0)
    vector<vector<ll>> rref;
    //first M columns are matrix A
    //last column of eq is vector b
    void solve(const vector<vector<ll>> &eq){
        int M = (int)eq[0].size() - 1;
        rref = eq;
        auto piv = RREF(rref);
        int rk = piv.size();
        if(piv.size() && piv.back() == M){
            ok = 0; return;
        }
```

```

    }
    ok = 1;
    par.resize(M);
    vector<bool> ispiv(M);
    for (int i = 0; i < rk; i++) {
        par[piv[i]] = rref[i][M];
        ispiv[piv[i]] = 1;
    }
    for (int i = 0; i < M; i++) {
        if (ispiv[i]) continue;
        vector<ll> h(M);
        h[i] = 1;
        for (int j = 0; j < rk; j++) h[piv[j]] = rref[j][i]
            ? mod-rref[j][i] : 0;
        homo.pb(h);
    }
}
};

```

6.10 Chinese Remainder Theorem [6ef4a3]

```

pll solve_crt(ll x1, ll m1, ll x2, ll m2){
    ll g = gcd(m1, m2);
    if ((x2 - x1) % g) return {0, 0}; // no sol
    m1 /= g; m2 /= g;
    ll _, p, q;
    extgcd(m1, m2, _, p, q); // p <= C
    ll lcm = m1 * m2 * g;
    ll res = ((__int128)p * (x2 - x1) % lcm * m1 % lcm +
        x1) % lcm;
    // be careful with overflow, C^3
    return {(res + lcm) % lcm, lcm}; // (x, m)
}

```

6.11 Sqrt Decomposition [8d7bc0]

```

// for all i in [l, r], floor(n / i) = x
for(int l = 1, r; l <= n; l = r + 1){
    int x = ifloor(n, l);
    r = ifloor(n, x);
}
// for all i in [l, r], ceil(n / i) = x
for(int l = 1, r = n; r >= 1; r = l - 1){
    int x = iceil(n, r);
    l = iceil(n, x);
}

```

7 Misc

7.1 Cyclic Ternary Search [9017cc]

```

/* bool pred(int a, int b);
f(0) ~ f(n - 1) is a cyclic-shift U-function
return idx s.t. pred(x, idx) is false forall x*/
int cyc_tsearch(int n, auto pred) {
    if (n == 1) return 0;
    int l = 0, r = n; bool rv = pred(1, 0);
    while (r - l > 1) {
        int m = (l + r) / 2;
        if (pred(0, m) ? rv : pred(m, (m + 1) % n)) r = m;
        else l = m;
    }
    return pred(1, r % n) ? 1 : r % n;
}

```

7.2 Matroid

我們稱一個二元組 $M = (E, \mathcal{I})$ 為一個擬陣，其中 $\mathcal{I} \subseteq 2^E$ 為 E 的子集所形成的非空集合，若：

- 若 $S \in \mathcal{I}$ 以及 $S' \subseteq S$ ，則 $S' \in \mathcal{I}$
- 對於 $S_1, S_2 \in \mathcal{I}$ 滿足 $|S_1| < |S_2|$ ，存在 $e \in S_2 \setminus S_1$ 使得 $S_1 \cup \{e\} \in \mathcal{I}$

除此之外，我們有以下的定義：

- 位於 \mathcal{I} 中的集合我們稱之為獨立集 (independent set)，反之不在 \mathcal{I} 中的我們稱為相依集 (dependent set)
- 極大的獨立集為基底 (base)、極小的相依集為迴路 (circuit)
- 一個集合 Y 的秩 (rank) $r(Y)$ 為該集合中最大的獨立子集，也就是 $r(Y) = \max\{|X| \mid X \subseteq Y \text{ 且 } X \in \mathcal{I}\}$

性質：

1. $X \subseteq Y \wedge Y \in \mathcal{I} \implies X \in \mathcal{I}$
2. $X \subseteq Y \wedge X \notin \mathcal{I} \implies Y \notin \mathcal{I}$
3. 若 B 與 B' 皆是基底且 $B \subseteq B'$ ，則 $B = B'$
若 C 與 C' 皆是迴路且 $C \subseteq C'$ ，則 $C = C'$
4. $e \in E \wedge X \subseteq E \implies r(X) \leq r(X \cup \{e\}) \leq r(X) + 1$ i.e. 加入一個元素後秩不會降底，最多增加 1

5. $\forall Y \subseteq E, \exists X \subseteq Y, r(X) = |X| = r(Y)$

一些等價的性質：

1. 對於所有 $X \subseteq E$ ， X 的極大獨立子集都有相同的大小
2. 對於 $B_1, B_2 \in \mathcal{B} \wedge B_1 \neq B_2$ ，對於所有 $e_1 \in B_1 \setminus B_2$ ，存在 $e_2 \in B_2 \setminus B_1$ 使得 $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$
3. 對於 $X, Y \in \mathcal{I}$ 且 $|X| < |Y|$ ，存在 $e \in Y \setminus X$ 使得 $X \cup \{e\} \in \mathcal{B}$
4. 如果 $r(X \cup \{e_1\}) = r(X \cup \{e_2\}) = r(X)$ ，則 $r(X \cup \{e_1, e_2\}) = r(X)$ 。
如果 $r(X \cup \{e\}) = r(X)$ 對於所有 $e \in E'$ 都成立，則 $r(X \cup E') = r(X)$ 。

擬陣交

Data: 兩個擬陣 $M_1 = (E, \mathcal{I}_1)$ 以及 $M_2 = (E, \mathcal{I}_2)$

Result: I 為最大的位於 $\mathcal{I}_1 \cap \mathcal{I}_2$ 中的獨立集

$I \leftarrow \emptyset$

$X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$

$X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$

while $X_1 \neq \emptyset$ 且 $X_2 \neq \emptyset$ do

if $e \in X_1 \cap X_2$ then

$I \leftarrow I \cup \{e\}$

else

構造交換圖 $\mathcal{D}_{M_1, M_2}(I)$

在交換圖上找到一條 X_1 到 X_2 且沒有捷徑的路徑 P

$I \leftarrow I \Delta P$

end if

$X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$

$X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$

end while

8 Polynomial

8.1 FWHT [c9cdb6]

```

/* x: a[j], y: a[j + (L >> 1)]
or: (y += x * op), and: (x += y * op)
xor: (x, y = (x + y) * op, (x - y) * op)
invop: or, and, xor = -1, -1, 1/2 */
void fwt(int *a, int n, int op) { //or
    for (int L = 2; L <= n; L <= 1)
        for (int i = 0; i < n; i += L)
            for (int j = i; j < i + (L >> 1); ++j)
                a[j + (L >> 1)] += a[j] * op;
}
const int N = 21;
int f[N][1 << N], g[N][1 << N], h[N][1 << N], ct[1 << N];
void subset_convolution(int *a, int *b, int *c, int L)
{
    // c_k = \sum_{i | j = k, i & j = 0} a_i * b_j
    int n = 1 << L;
    for (int i = 1; i < n; ++i)
        ct[i] = ct[i & (i - 1)] + 1;
    for (int i = 0; i < n; ++i)
        f[ct[i]][i] = a[i], g[ct[i]][i] = b[i];
    for (int i = 0; i <= L; ++i)
        fwt(f[i], n, 1), fwt(g[i], n, 1);
    for (int i = 0; i <= L; ++i)
        for (int j = 0; j <= i; ++j)
            for (int x = 0; x < n; ++x)
                h[i][x] += f[j][x] * g[i - j][x];
    for (int i = 0; i <= L; ++i) fwt(h[i], n, -1);
    for (int i = 0; i < n; ++i) c[i] = h[ct[i]][i];
}

```

8.2 FFT [13ec2f]

// Errichto: FFT for double works when the result < 1e15, and < 1e18 with long double

```

using val_t = complex<double>;
template<int MAXN>
struct FFT {
    const double PI = acos(-1);
    val_t w[MAXN];
    FFT() {
        for (int i = 0; i < MAXN; ++i) {
            double arg = 2 * PI * i / MAXN;
            w[i] = val_t(cos(arg), sin(arg));
        }
    }
    void bitrev(vector<val_t> &a, int n) //same as NTT
    void trans(vector<val_t> &a, int n, bool inv = false)
    {
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;

```

```

    for (int i = 0; i < n; i += L) {
        for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
            val_t tmp = a[j + dl] * (inv ? conj(w[x]) : w[x]);
            a[j + dl] = a[j] - tmp;
            a[j] += tmp;
        }
    }
}
if (inv) {
    for (int i = 0; i < n; ++i) a[i] /= n;
}
}
//multiplying two polynomials A * B:
//fft.trans(A, siz, 0), fft.trans(B, siz, 0):
//A[i] *= B[i], fft.trans(A, siz, 1);
};

```

8.3 NTT [bf683f]

```

//((2^16)+1, 65537, 3
//7*17*(2^23)+1, 998244353, 3
//1255*(2^20)+1, 1315962881, 3
//51*(2^25)+1, 1711276033, 29
// only works when sz(A) + sz(B) - 1 <= MAXN
template<int MAXN, ll P, ll RT> //MAXN must be 2^k
struct NTT {
    ll w[MAXN];
    ll mpow(ll a, ll n);
    ll minv(ll a) { return mpow(a, P - 2); }
    NTT() {
        ll dw = mpow(RT, (P - 1) / MAXN);
        w[0] = 1;
        for (int i = 1; i < MAXN; ++i) w[i] = w[i - 1] * dw % P;
    }
    void bitrev(vector<ll> &a, int n) {
        int i = 0;
        for (int j = 1; j < n - 1; ++j) {
            for (int k = n >> 1; (i ^= k) < k; k >>= 1);
            if (j < i) swap(a[i], a[j]);
        }
    }
    void operator()(vector<ll> &a, int n, bool inv = false) { //0 <= a[i] < P
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
                for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
                    ll tmp = a[j + dl] * w[x] % P;
                    if ((a[j + dl] = a[j] - tmp) < 0) a[j + dl] += P;
                    if ((a[j] += tmp) >= P) a[j] -= P;
                }
            }
        }
        if (inv) {
            reverse(a.begin() + 1, a.begin() + n);
            ll invn = minv(n);
            for (int i = 0; i < n; ++i) a[i] = a[i] * invn % P;
        }
    }
};

```

8.4 Polynomial Operation [77a8a8]

```

// Copy from 8BQube
#define fi(s, n) for (int i = (int)(s); i < (int)(n); ++i)
#define neg(x) (x ? P - x : 0)
#define V (*this)
template<int MAXN, ll P, ll RT> // MAXN = 2^k
struct Poly : vector<ll> { // coefficients in [0, P)
    using vector<ll>::vector;
    static inline NTT<MAXN, P, RT> ntt;
    int n() const { return (int)size(); } // n() >= 1
    Poly(const Poly &p, int m) : vector<ll>(m) { copy_n(p.data(), min(p.n(), m), data()); }
};

```

```

Poly &irev() { return reverse(data(), data() + n()), V; }
Poly &isz(int m) { return resize(m), V; }
static ll minv(ll x) { return ntt.minv(x); }
Poly &iadd(const Poly &rhs) { // n() == rhs.n()
    fi(0, n()) if ((V[i] += rhs[i]) >= P) V[i] -= P;
    return V;
}
Poly &imul(ll k) {
    fi(0, n()) V[i] = V[i] * k % P;
    return V;
}
Poly Mul(const Poly &rhs) const {
    int m = 1;
    while (m < n() + rhs.n() - 1) m <= 1;
    assert(m <= MAXN);
    Poly X(V, m), Y(rhs, m);
    ntt(X, m), ntt(Y, m);
    fi(0, m) X[i] = X[i] * Y[i] % P;
    ntt(X, m, true);
    return X.isz(n() + rhs.n() - 1);
}
Poly Inv() const { // V[0] != 0, 2*sz<=MAXN
    if (n() == 1) return {minv(V[0])};
    int m = 1;
    while (m < n() * 2) m <= 1;
    assert(m <= MAXN);
    Poly Xi = Poly(V, (n() + 1) / 2).Inv().isz(m);
    Poly Y(V, m);
    ntt(Xi, m), ntt(Y, m);
    fi(0, m) {
        Xi[i] *= (2 - Xi[i] * Y[i]) % P;
        if ((Xi[i] %= P) < 0) Xi[i] += P;
    }
    ntt(Xi, m, true);
    return Xi.isz(n());
}
Poly &shift_inplace(const ll &c) { // 2 * sz <= MAXN
    int n = V.n();
    vector<ll> fc(n), ifc(n);
    fc[0] = ifc[0] = 1;
    for (int i = 1; i < n; i++) {
        fc[i] = fc[i - 1] * i % P;
        ifc[i] = minv(fc[i]);
    }
    for (int i = 0; i < n; i++) V[i] = V[i] * fc[i] % P;
    Poly g(n);
    ll cp = 1;
    for (int i = 0; i < n; i++) g[i] = cp * ifc[i] % P, cp = cp * c % P;
    V = V.irev().Mul(g).isz(n).irev();
    for (int i = 0; i < n; i++) V[i] = V[i] * ifc[i] % P;
    return V;
}
Poly shift(const ll &c) const { return Poly(V).shift_inplace(c); }
Poly _Sqrt() const { // Jacobi(V[0], P) = 1
    if (n() == 1) return {QuadraticResidue(V[0], P)};
    Poly X = Poly(V, (n() + 1) / 2)._Sqrt().isz(n());
    return X.iadd(Mul(X.Inv()).isz(n())).imul(P / 2 + 1);
}
Poly Sqrt() const { // 2 * sz <= MAXN
    Poly a;
    bool has = 0;
    for (int i = 0; i < n(); i++) {
        if (V[i]) has = 1;
        if (has) a.push_back(V[i]);
    }
    if (!has) return V;
    if ((n() + a.n()) % 2 || Jacobi(a[0], P) != 1) {
        return Poly();
    }
    a = a.isz((n() + a.n()) / 2)._Sqrt();
    int sz = a.n();
    a.isz(n());
    rotate(a.begin(), a.begin() + sz, a.end());
    return a;
}

```



```

pair<Poly, Poly> DivMod(const Poly &rhs) const { //
    (rhs.back()) != 0
    if (n() < rhs.n()) return {{0}, V};
    const int m = n() - rhs.n() + 1;
    Poly X(rhs);
    X.irev().isz(m);
    Poly Y(V);
    Y.irev().isz(m);
    Poly Q = Y.Mul(X.Inv()).isz(m).irev();
    X = rhs.Mul(Q), Y = V;
    fi(0, n()) if ((Y[i] - X[i]) < 0) Y[i] += P;
    return {Q, Y.isz(max(1, rhs.n() - 1))};
}
Poly Dx() const {
    Poly ret(n() - 1);
    fi(0, ret.n()) ret[i] = (i + 1) * V[i + 1] % P;
    return ret.isz(max(1, ret.n()));
}
Poly Sx() const {
    Poly ret(n() + 1);
    fi(0, n()) ret[i + 1] = minv(i + 1) * V[i] % P;
    return ret;
}
Poly _tmul(int nn, const Poly &rhs) const {
    Poly Y = Mul(rhs).isz(n()) + nn - 1;
    return Poly(Y.data() + n() - 1, Y.data() + Y.n());
}
vector<ll> _eval(const vector<ll> &x, const vector<
    Poly> &up) const {
    const int m = (int)x.size();
    if (!m) return {};
    vector<Poly> down(m * 2);
    // down[1] = DivMod(up[1]).second;
    // fi(2, m * 2) down[i] = down[i / 2].DivMod(up[i])
    // .second;
    down[1] = Poly(up[1]).irev().isz(n()).Inv().irev().
        _tmul(m, V);
    fi(2, m * 2) down[i] = up[i ^ 1]._tmul(up[i].n() -
        1, down[i / 2]);
    vector<ll> y(m);
    fi(0, m) y[i] = down[m + i][0];
    return y;
}
static vector<Poly> _tree1(const vector<ll> &x) {
    const int m = (int)x.size();
    vector<Poly> up(m * 2);
    fi(0, m) up[m + i] = {neg(x[i]), 1};
    for (int i = m - 1; i > 0; --i) up[i] = up[i * 2].
        Mul(up[i * 2 + 1]);
    return up;
}
vector<ll> Eval(const vector<ll> &x) const { // 1e5,
    1s
    auto up = _tree1(x);
    return _eval(x, up);
}
static Poly Interpolate(const vector<ll> &x, const
    vector<ll> &y) { // 1e5, 1.4s
    const int m = (int)x.size();
    vector<Poly> up = _tree1(x), down(m * 2);
    vector<ll> z = up[1].Dx()._eval(x, up);
    fi(0, m) z[i] = y[i] * minv(z[i]) % P;
    fi(0, m) down[m + i] = {z[i]};
    for (int i = m - 1; i > 0; --i)
        down[i] = down[i * 2].Mul(up[i * 2 + 1]).iadd(
            down[i * 2 + 1].Mul(up[i * 2]));
    return down[1];
}
Poly Ln() const { // V[0] == 1, 2*sz<=MAXN
    return Dx().Mul(Inv()).Sx().isz(n());
}
Poly Exp() const { // V[0] == 0, 2*sz<=MAXN
    if (n() == 1) return {1};
    Poly X = Poly(V, (n() + 1) / 2).Exp().isz(n());
    Poly Y = X.Ln();
    Y[0] = P - 1;
    fi(0, n()) if ((Y[i] = V[i] - Y[i]) < 0) Y[i] += P;
    return X.Mul(Y).isz(n());
}
// M := P(P - 1). If k >= M, k := k % M + M.
Poly Pow(ll k) const { // 2*sz<=MAXN
    int nz = 0;

```

```

    while (nz < n() && !V[nz]) ++nz;
    if (nz * min(k, (ll)n()) >= n()) return Poly(n());
    if (!k) return Poly(Poly{1}, n());
    Poly X(data() + nz, data() + nz + n() - nz * k);
    const ll c = ntt.mpow(X[0], k % (P - 1));
    return X.Ln().imul(k % P).Exp().imul(c).irev().isz(
        n()).irev();
}
// sum_j w_j [x^j] f(x^i) for i in [0, m]
Poly power_projection(Poly wt, int m) { // 4*sz <=
    MAXN!
    assert(n() == wt.n());
    if (!n()) {
        return Poly(m + 1, 0);
    }
    if (V[0] != 0) {
        ll c = V[0];
        V[0] = 0;
        Poly A = V.power_projection(wt, m);
        fi(0, m + 1) A[i] = A[i] * fac[i] % P; //
            factorial
        Poly B(m + 1);
        ll pow = 1;
        fi(0, m + 1) B[i] = pow * ifac[i] % P, pow = pow
            * c % P; // inv. of fac
        A = A.Mul(B).isz(m + 1);
        fi(0, m + 1) A[i] = A[i] * fac[i] % P;
        return A;
    }
}
int n = 1;
while (n < V.n()) n *= 2;
isz(n), wt.isz(n).irev();
int k = 1;
Poly p(wt, 2 * n), q(V, 2 * n);
q.imul(P - 1);

while (n > 1) {
    Poly r(2 * n * k);
    fi(0, 2 * n * k) r[i] = (i % 2 == 0 ? q[i] : neg(
        q[i]));
    Poly pq = p.Mul(r).isz(4 * n * k);
    Poly qq = q.Mul(r).isz(4 * n * k);
    fi(0, 2 * n * k) {
        pq[2 * n * k + i] += p[i];
        qq[2 * n * k + i] += q[i] + r[i];
        pq[2 * n * k + i] %= P;
        qq[2 * n * k + i] %= P;
    }
    fill(p.begin(), p.end(), 0);
    fill(q.begin(), q.end(), 0);
    for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
        p[n * j + i] = pq[(2 * n) * j + (2 * i + 1)];
        q[n * j + i] = qq[(2 * n) * j + (2 * i + 0)];
    }
    n /= 2, k *= 2;
}
Poly ans(k);
fi(0, k) ans[i] = p[2 * i];
return ans.irev().isz(m + 1);
}
Poly FPSInv() {
    const int n = V.n() - 1;
    if (n == -1) return {};
    assert(V[0] == 0);
    if (n == 0) return V;
    assert(V[1] != 0);
    ll c = V[1], ic = minv(c);
    imul(ic);
    Poly wt(n + 1);
    wt[n] = 1;

    Poly A = V.power_projection(wt, n);
    Poly g(n);
    fi(1, n + 1) g[n - i] = n * A[i] % P * minv(i) % P;
    g = g.Pow(neg(minv(n)));
    g.insert(g.begin(), 0);

    ll pow = 1;
    fi(0, g.n()) g[i] = g[i] * pow % P, pow = pow * ic
        % P;
    return g;
}

```

```

}
Poly TMul(const Poly &rhs) const { // this[i] - rhs[j]
    j = k;
    return Poly(*this).irev().Mul(rhs).isz(n()).irev();
}
Poly FPScomp(Poly g) { // solves V(g(x))
    auto rec = [&](auto &rec, int n, int k, Poly Q) ->
        Poly {
            if (n == 1) {
                Poly p(2 * k);
                irev();
                fi(0, k) p[2 * i] = V[i];
                return p;
            }
            Poly R(2 * n * k);
            fi(0, 2 * n * k) R[i] = (i % 2 == 0 ? Q[i] : neg(
                Q[i]));
            Poly QQ = Q.Mul(R).isz(4 * n * k);
            fi(0, 2 * n * k) {
                QQ[2 * n * k + i] += Q[i] + R[i];
                QQ[2 * n * k + i] %= P;
            }
            Poly nxt_Q(2 * n * k);
            for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
                nxt_Q[n * j + i] = QQ[(2 * n) * j + (2 * i + 0)];
            }
            Poly nxt_p = rec(rec, n / 2, k * 2, nxt_Q);
            Poly pq(4 * n * k);
            for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
                pq[(2 * n) * j + (2 * i + 1)] += nxt_p[n * j + i];
                pq[(2 * n) * j + (2 * i + 1)] %= P;
            }
            Poly p(2 * n * k);
            fi(0, 2 * n * k) p[i] = (p[i] + pq[2 * n * k + i]) % P;
            pq.pop_back();
            Poly x = pq.TMul(R);
            fi(0, 2 * n * k) p[i] = (p[i] + x[i]) % P;
            return p;
        };
    int sz = 1;
    while(sz < n() || sz < g.n()) sz <= 1;
    return isz(sz), rec(rec, sz, 1, g.imul(P-1).isz(2 * sz)).isz(sz).irev();
}
};
#undef fi
#undef V
#undef neg
using Poly_t = Poly<1 << 19, 998244353, 3>;

```

8.5 Generating Function

Ordinary Generating Function

- $C(x) = A(rx)$: $c_n = r^n a_n$ 的一般生成函數。
- $C(x) = A(x) + B(x)$: $c_n = a_n + b_n$ 的一般生成函數。
- $C(x) = A(x)B(x)$: $c_n = \sum_{i=0}^n a_i b_{n-i}$ 的一般生成函數。
- $C(x) = A(x)^k$: $c_n = \sum_{i_1+i_2+\dots+i_k=n} a_{i_1} a_{i_2} \dots a_{i_k}$ 的一般生成函數。
- $C(x) = xA(x)'$: $c_n = na_n$ 的一般生成函數。
- $C(x) = \frac{A(x)}{1-x}$: $c_n = \sum_{i=0}^n a_i$ 的一般生成函數。
- $C(x) = A(1) + x \frac{A(1)-A(x)}{1-x}$: $c_n = \sum_{i=n}^{\infty} a_i$ 的一般生成函數。

常用展開式

- $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$
- $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$, $\binom{a}{n} = \frac{a(a-1)(a-2)\dots(a-n+1)}{n!}$

常見生函

- 卡特蘭數: $f(x) = \frac{1-\sqrt{1-4x}}{2x}$

Exponential Generating Function

a_0, a_1, \dots 的指數生成函數:

$$\hat{A}(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \dots$$

- $\hat{C}(x) = \hat{A}(x) + \hat{B}(x)$: $c_n = a_n + b_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}^{(k)}(x)$: $c_n = a_{n+k}$ 的指數生成函數
- $\hat{C}(x) = x\hat{A}(x)$: $c_n = na_n$ 的指數生成函數

- $\hat{C}(x) = \hat{A}(x)\hat{B}(x)$: $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)^k$: $c_n = \sum_{i_1+i_2+\dots+i_k=n} \binom{n}{i_1, i_2, \dots, i_k} a_{i_1} a_{i_2} \dots a_{i_k}$ 的指數生成函數
- $\hat{C}(x) = \exp(A(x))$: 假設 $A(x)$ 是一個分量 (component) 的生成函數, 那麼 $\hat{C}(x)$ 是將 n 個有編號的東西分成若干個分量的指數生成函數

Lagrange's Inversion Formula

如果 F 跟 G 互反, 則有 $F(0), G(0) = 0, F'(0), G'(0) \neq 0$ 。若 H 為任意 FPS, 則

$$n[x^n]G(x) = [x^{n-1}] \frac{1}{(F(x)/x)^n}$$

$$n[x^n]H(G(x)) = [x^{n-1}] H'(x) \frac{1}{(F(x)/x)^n}$$

8.6 Bostan Mori [41c3bc]

```

const ll mod = 998244353;
NTT<262144, mod, 3> ntt;
// Finds the k-th coefficient of P / Q in O(d log d log k)
// size of NTT has to > 2 * d
ll BostanMori(vector<ll> P, vector<ll> Q, long long k)
{
    int d = max((int)P.size(), (int)Q.size() - 1);
    vector M = {P, Q};
    M[0].resize(d, 0);
    M[1].resize(d + 1, 0);
    int sz = (2 * d + 1 == 1 ? 2 : (1 << (__lg(2 * d) + 1)));
    vector<ll> Qn(sz);
    vector N(2, vector<ll>(sz));
    while(k) {
        fill(iter(Qn), 0);
        for(int i = 0; i < d + 1; i++){
            Qn[i] = M[1][i] * ((i & 1) ? -1 : 1);
            if(Qn[i] < 0) Qn[i] += mod;
        }
        ntt(Qn, sz, false);

        ll t[2] = {k & 1, 0};
        for(int i = 0; i < 2; i++){
            fill(iter(N[i]), 0);
            copy(iter(M[i]), N[i].begin());
            ntt(N[i], sz, false);
            for(int j = 0; j < sz; j++)
                N[i][j] = N[i][j] * Qn[j] % mod;
            ntt(N[i], sz, true);
            for(int j = t[i]; j < 2 * siz(M[i]); j += 2){
                M[i][j >> 1] = N[i][j];
            }
        }
        k >>= 1;
    }
    return M[0][0] * ntt.minv(M[1][0]) % mod;
}

ll LinearRecursion(vector<ll> a, vector<ll> c, ll k) {
    // a_n = \sum_{j=1}^d c_j a_{n-j}
    int d = siz(a);
    int sz = (2 * d + 1 == 1 ? 2 : (1 << (__lg(2 * d) + 1)));

    c[0] = mod - 1;
    for(ll &i : c) i = i ? mod - i : 0;

    auto A = a; A.resize(sz);
    auto C = c; C.resize(sz);
    ntt(A, sz, false), ntt(C, sz, false);
    for(int i = 0; i < sz; i++) A[i] = A[i] * C[i] % mod;
    ntt(A, sz, true);
    A.resize(d);

    return BostanMori(A, c, k);
}

```

9 String

9.1 KMP Algorithm [c8b75f]

```

// 0-based
// fail[i] = max k < i s.t. s[0..k] = s[i-k..i]
vector<int> kmp_build_fail(const string &s){
    int n = SZ(s);

```

```

vector<int> fail(n, -1);
int cur = -1;
for(int i = 1; i < n; i++){
    while(cur != -1 && s[cur + 1] != s[i])
        cur = fail[cur];
    if(s[cur + 1] == s[i])
        cur++;
    fail[i] = cur;
}
return fail;
}

void kmp_match(const string &s, const vector<int> &fail
, const string &t){
    int cur = -1;
    int n = SZ(s), m = SZ(t);
    for(int i = 0; i < m; i++){
        while(cur != -1 && (cur + 1 == n || s[cur + 1] != t
[i]))
            cur = fail[cur];
        if(cur + 1 < n && s[cur + 1] == t[i])
            cur++;
        // cur = max k s.t. s[0..k] = t[i-k..i]
    }
}

```

9.2 Manacher Algorithm [caf0f4]

```

/* center i: radius z[i * 2 + 1] / 2
   center i, i + 1: radius z[i * 2 + 2] / 2
   both aba, abba have radius 2 */
vector<int> manacher(const string &tmp){ // 0-based
    string s = "%";
    int l = 0, r = 0;
    for(char c : tmp) s += c, s += '%';
    vector<int> z(SZ(s));
    for(int i = 0; i < SZ(s); i++){
        z[i] = r > i ? min(z[2 * l - i], r - i) : 1;
        while(i - z[i] >= 0 && i + z[i] < SZ(s)
            && s[i + z[i]] == s[i - z[i]])
            ++z[i];
        if(z[i] + i > r) r = z[i] + i, l = i;
    }
    return z;
}

```

9.3 Lyndon Factorization [7c612b]

```

// partition s = w[0] + w[1] + ... + w[k-1],
// w[0] >= w[1] >= ... >= w[k-1]
// each w[i] strictly smaller than all its suffix
void duval(const string &s, vector<pii> &w){
    for(int n = (int)s.size(), i = 0, j, k; i < n; ) {
        for(j = i + 1, k = i; j < n && s[k] <= s[j]; j++)
            k = (s[k] < s[j] ? i : k + 1);
        // if (i < n / 2 && j >= n / 2) {
        //     for min cyclic shift, call duval(s + s)
        //     then here s.substr(i, n / 2) is min cyclic shift
        // }
        for(; i <= k; i += j - k)
            w.pb(pii(i, j - k)); // s.substr(l, len)
    }
}

```

9.4 Suffix Array [cd67ea]

```

struct SuffixArray {
    vector<int> sa, lcp, rank; // lcp[i] is lcp of sa[i]
    // and sa[i-1]
                                // sa[0] = s.size()
                                // character should be 1-
                                // based
    SuffixArray(string& s, int lim=256) { // or
        basic_string<int>
        int n = s.size() + 1, k = 0, a, b;
        vector<int> x(n, 0), y(n), ws(max(n, lim));
        rank.assign(n, 0);
        for(int i = 0; i < n - 1; i++) x[i] = s[i];
        sa = lcp = y, iota(sa.begin(), sa.end(), 0);
        for(int j = 0, p = 0; p < n; j = max(1, j * 2),
            lim = p) {
            p = j, iota(y.begin(), y.end(), n - j);
            for(int i = 0; i < n; i++)

```

```

                if (sa[i] >= j) y[p++] = sa[i] - j;
            for(int &i : ws) i = 0;
            for(int i = 0; i < n; i++) ws[x[i]]++;
            for(int i = 1; i < lim; i++) ws[i] += ws[i - 1];
            for(int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            for(int i = 1; i < n; i++){
                a = sa[i - 1], b = sa[i];
                x[b] = (y[a] == y[b] && y[a + j] == y[b + j]) ?
                    p - 1 : p++;
            }
        }
        for(int i = 1; i < n; i++) rank[sa[i]] = i;
        for(int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for(k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};

```

9.5 Suffix Automaton [016373]

```

struct exSAM {
    const int CNUM = 26;
    // Len: maxlength, link: fail link
    // LenSorted: topo order, cnt: occur
    vector<int> len, link, lenSorted, cnt;
    vector<vector<int>> next;
    int total = 0;
    int newnode() {
        return total++;
    }
    void init(int n) { // total number of characters
        len.assign(2 * n, 0); link.assign(2 * n, 0);
        lenSorted.assign(2 * n, 0); cnt.assign(2 * n, 0);
        next.assign(2 * n, vector<int>(CNUM));
        newnode(), link[0] = -1;
    }
    int insertSAM(int last, int c) {
        // not exSAM: cur = newnode(), p = last
        int cur = next[last][c];
        len[cur] = len[last] + 1;
        int p = link[last];
        while(p != -1 && !next[p][c])
            next[p][c] = cur, p = link[p];
        if(p == -1) return link[cur] = 0, cur;
        int q = next[p][c];
        if(len[p] + 1 == len[q]) return link[cur] = q, cur;
        int clone = newnode();
        for(int i = 0; i < CNUM; ++i)
            next[clone][i] = len[next[q][i]] ? next[q][i] :
                0;
        len[clone] = len[p] + 1;
        while(p != -1 && next[p][c] == q)
            next[p][c] = clone, p = link[p];
        link[link[cur] = clone] = link[q];
        link[q] = clone;
        return cur;
    }
    void insert(const string &s) {
        int cur = 0;
        for(auto ch : s) {
            int &nxt = next[cur][int(ch - 'a')];
            if(!nxt) nxt = newnode();
            cnt[cur = nxt] += 1;
        }
    }
    void build() {
        queue<int> q;
        q.push(0);
        while(!q.empty()) {
            int cur = q.front();
            q.pop();
            for(int i = 0; i < CNUM; ++i)
                if(next[cur][i])
                    q.push(insertSAM(cur, i));
        }
        vector<int> lc(total);
        for(int i = 1; i < total; ++i) ++lc[len[i]];
        partial_sum(iter(lc), lc.begin());
        for(int i = 1; i < total; ++i) lenSorted[--lc[len[
i]]] = i;
    }
};

```

```

void solve() {
    for (int i = total - 2; i >= 0; --i)
        cnt[link[lenSorted[i]]] += cnt[lenSorted[i]];
}
};

```

9.6 Z-value Algorithm [488d87]

```

// z[i] = max k s.t. s[0..k-1] = s[i..i+k-1]
// i.e. Length of longest common prefix
// z[0] = 0
vector<int> z_function(const string &s){
    int n = s.size();
    vector<int> z(n);
    for(int i = 1, l = 0, r = 0; i < n; i++){
        if(i <= r) z[i] = min(r - i + 1, z[i - l]);
        while(i + z[i] < n && s[z[i]] == s[i + z[i]])
            z[i]++;
        if(i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
    return z;
}

```

9.7 Main Lorentz [fcfb8f]

```

struct Rep{ int minl, maxl, len; };
vector<Rep> rep; // 0-base
// p \in [minl, maxl] => s[p, p + i) = s[p + i, p + 2i)
void main_lorentz(const string &s, int sft = 0) {
    const int n = s.size();
    if (n == 1) return;
    const int nu = n / 2, nv = n - nu;
    const string u = s.substr(0, nu), v = s.substr(nu,
        ru(u.rbegin(), u.rend()), rv(v.rbegin(), v.rend()
        ));
    main_lorentz(u, sft), main_lorentz(v, sft + nu);
    const auto z1 = z_function(ru), z2 = z_function(v + '
        #' + u),
        z3 = z_function(ru + '#' + rv), z4 =
            z_function(v);
    auto get_z = [](const vector<int> &z, int i) {
        return (0 <= i and i < (int)z.size()) ? z[i] : 0;
    };
    auto add_rep = [&](bool left, int c, int l, int k1,
        int k2) {
        const int L = max(1, l - k2), R = min(l - left, k1)
        ;
        if (L > R) return;
        if (left) rep.emplace_back(Rep({sft + c - R, sft +
            c - L, l}));
        else rep.emplace_back(Rep({sft + c - R - 1 + 1, sft
            + c - L - 1 + 1, l}));
    };
    for (int cnt = 0; cnt < n; cnt++) {
        int l, k1, k2;
        if (cnt < nu) {
            l = nu - cnt;
            k1 = get_z(z1, nu - cnt);
            k2 = get_z(z2, nv + 1 + cnt);
        } else {
            l = cnt - nu + 1;
            k1 = get_z(z3, nu + 1 + nv - 1 - (cnt - nu));
            k2 = get_z(z4, (cnt - nu) + 1);
        }
        if (k1 + k2 >= 1)
            add_rep(cnt < nu, cnt, l, k1, k2);
    }
}

```

9.8 AC Automaton [e925e5]

```

const int SIGMA = 26;
struct AC_Automaton {
    // child: trie, next: automaton
    vector<vector<int>> child, next;
    vector<int> fail, cnt, ord;
    int total = 0;
    int newnode() {
        return total++;
    }
    void init(int len) { // len >= 1 + total len
        child.assign(len, vector<int>(SIGMA, -1));

```

```

        next.assign(len, vector<int>(26, -1));
        fail.assign(len, -1); cnt.assign(len, 0);
        ord.clear();
        newnode();
    }
    int input(string &s) {
        int cur = 0;
        for (char c : s) {
            if (child[cur][c - 'A'] == -1)
                child[cur][c - 'A'] = newnode();
            cur = child[cur][c - 'A'];
        }
        return cur; // return the end node of string
    }
    void make_fl() {
        queue<int> q;
        q.push(0), fail[0] = -1;
        while(!q.empty()) {
            int R = q.front();
            q.pop(); ord.pb(R);
            for (int i = 0; i < SIGMA; i++)
                if (child[R][i] != -1) {
                    int X = next[R][i] = child[R][i], Z = fail[R]
                    ;
                    while (Z != -1 && child[Z][i] == -1)
                        Z = fail[Z];
                    fail[X] = Z != -1 ? child[Z][i] : 0;
                    q.push(X);
                }
            else next[R][i] = R ? next[fail[R]][i] : 0;
        }
    }
    void solve() {
        for (int i : ord | views::reverse)
            cnt[fail[i]] += cnt[i];
    }
};

```

9.9 Palindrome Automaton [8a071b]

```

struct PalindromicTree {
    struct node {
        int nxt[26], fail, len; // num = depth of fail link
        int cnt, num; // cnt = occur, num = #pal_suffix of
            this node
        node(int l = 0) : nxt{}, fail(0), len(l), cnt(0), num
            (0) {}
    };
    vector<node> st; vector<int> s; int last, n;
    void init() {
        st.clear(); s.clear(); last = 1; n = 0;
        st.pb(0); st.pb(-1);
        st[0].fail = 1; s.pb(-1);
    }
    int getFail(int x) {
        while (s[n - st[x].len - 1] != s[n]) x = st[x].fail
        ;
        return x;
    }
    void add(int c) {
        s.pb(c - 'a'); ++n;
        int cur = getFail(last);
        if (!st[cur].nxt[c]) {
            int now = SZ(st);
            st.pb(st[cur].len + 2);
            st[now].fail = st[getFail(st[cur].fail)].nxt[c];
            st[cur].nxt[c] = now;
            st[now].num = st[st[now].fail].num + 1;
        }
        last = st[cur].nxt[c]; ++st[last].cnt;
    }
    void dpcnt() {
        for(int i = SZ(st) - 1; i >= 0; i--){
            auto nd = st[i];
            st[nd.fail].cnt += nd.cnt;
        }
    }
    int size() { return (int)st.size() - 2; }
};

```

10 Notes

10.1 Geometry

Rotation Matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- rotate 90° : $(x, y) \rightarrow (-y, x)$
- rotate -90° : $(x, y) \rightarrow (y, -x)$

Triangles

Side lengths: a, b, c

$$\text{Semiperimeter: } p = \frac{a+b+c}{2}$$

$$\text{Area: } A = \sqrt{p(p-a)(p-b)(p-c)}$$

$$\text{Circumradius: } R = \frac{abc}{4A}$$

$$\text{Inradius: } r = \frac{A}{p}$$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$$\text{Length of bisector (divides angles in two): } s_a = \sqrt{bc \left(1 - \left(\frac{a}{b+c} \right)^2 \right)}$$

$$\text{Law of sines: } \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$$

$$\text{Law of cosines: } a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$\text{Law of tangents: } \frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$$

Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

Spherical coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \text{atan2}(y, x) \end{aligned}$$

Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{L^+} (P dx + Q dy)$$

$$\text{Area} = \frac{1}{2} \oint_L x dy - y dx$$

Circular sector:

$$x = x_0 + r \cos \theta$$

$$y = y_0 + r \sin \theta$$

$$\begin{aligned} A &= r \int_{\alpha}^{\beta} (x_0 + \cos \theta) \cos \theta + (y_0 + \sin \theta) \sin \theta d\theta \\ &= r(r\theta + x_0 \sin \theta - y_0 \cos \theta)|_{\alpha}^{\beta} \end{aligned}$$

Point-Line Duality

$$p = (a, b) \leftrightarrow p^* : y = ax - b$$

- $p \in l \iff l^* \in p^*$
- p_1, p_2, p_3 are collinear $\iff p_1^*, p_2^*, p_3^*$ intersect at a point
- p lies above $l \iff l^*$ lies above p^*
- lower convex hull \leftrightarrow upper envelope

10.2 Trigonometry

$$\begin{aligned} \sinh x &= \frac{1}{2}(e^x - e^{-x}) & \cosh x &= \frac{1}{2}(e^x + e^{-x}) \\ \sin n\pi &= 0 & \cos n\pi &= (-1)^n \end{aligned}$$

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sin(2\alpha) &= 2 \cos \alpha \sin \alpha \\ \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha \end{aligned}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$(V + W) \tan(\alpha - \beta)/2 = (V - W) \tan(\alpha + \beta)/2$$

where V, W are lengths of sides opposite angles α, β .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}$, $\phi = \text{atan2}(b, a)$.

10.3 Calculus

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln |\cos ax|}{a}$$

$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \text{erf}(x)$$

$$\int x e^{ax} = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int \sin^2(x) = \frac{x}{2} - \frac{1}{4} \sin 2x$$

$$\int \sin^3 x = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x$$

$$\int \cos^2(x) = \frac{x}{2} + \frac{1}{4} \sin 2x$$

$$\int \cos^3 x = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x$$

$$\int x \sin x = \sin x - x \cos x$$

$$\int x \cos x = \cos x + x \sin x$$

$$\int x e^x = e^x(x - 1)$$

$$\int x^2 e^x = e^x(x^2 - 2x + 2)$$

$$\int x^2 \sin x = 2x \sin x - (x^2 - 2) \cos x$$

$$\int x^2 \cos x = 2x \cos x + (x^2 - 2) \sin x$$

$$\int e^x \sin x = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^x \cos x = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int x e^x \sin x = \frac{1}{2} e^x (x \sin x - x \cos x + \cos x)$$

$$\int x e^x \cos x = \frac{1}{2} e^x (x \sin x + x \cos x - \sin x)$$

10.4 Sum & Series

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

10.5 Misc

• Cramer's rule

$$\begin{aligned} ax + by &= e & x &= \frac{ed - bf}{ad - bc} \\ cx + dy &= f & y &= \frac{af - ec}{ad - bc} \end{aligned}$$

• Vandermonde's Identity

$$C(n+m, k) = \sum_{i=0}^k C(n, i)C(m, k-i)$$

• Kirchhoff's Theorem

Denote L be a $n \times n$ matrix as the Laplacian matrix of graph G , where $L_{ii} = d(i)$, $L_{ij} = -c$ where c is the number of edge (i, j) in G .

- The number of undirected spanning in G is $|\det(\tilde{L}_{11})|$.
- The number of directed spanning tree rooted at r in G is $|\det(\tilde{L}_{rr})|$.

• Tutte's Matrix

Let D be a $n \times n$ matrix, where $d_{ij} = x_{ij}$ (x_{ij} is chosen uniformly at random) if $i < j$ and $(i, j) \in E$, otherwise $d_{ij} = -d_{ji}$. $\frac{\text{rank}(D)}{2}$ is the maximum matching on G .

• Cayley's Formula

- Given a degree sequence d_1, d_2, \dots, d_n for each *labeled* vertices, there are $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\dots(d_n-1)!}$ spanning trees.
- Let $T_{n,k}$ be the number of *labeled* forests on n vertices with k components, such that vertex $1, 2, \dots, k$ belong to different components. Then $T_{n,k} = kn^{n-k-1}$.

• Erdős-Gallai theorem

A sequence of nonnegative integers $d_1 \geq \dots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only

if $d_1 + \dots + d_n$ is even and $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k)$ holds

for every $1 \leq k \leq n$.

• Gale-Ryser theorem

A pair of sequences of nonnegative integers $a_1 \geq \dots \geq a_n$ and b_1, \dots, b_n

is bigraphic if and only if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^n \min(b_i, k)$

holds for every $1 \leq k \leq n$.

• Fulkerson-Chen-Anstee theorem

A sequence $(a_1, b_1), \dots, (a_n, b_n)$ of nonnegative integer pairs with $a_1 \geq$

$\dots \geq a_n$ is digraphic if and only if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^n \min(b_i, k)$

$1) + \sum_{i=k+1}^n \min(b_i, k)$ holds for every $1 \leq k \leq n$.

• Pick's theorem

For simple polygon, when points are all integer, we have

$$A = \#\{\text{lattice points in the interior}\} + \frac{\#\{\text{lattice points on the boundary}\}}{2} - 1.$$

• Möbius inversion formula

- $f(n) = \sum_{d|n} g(d) \Leftrightarrow g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$
- $f(n) = \sum_{n|d} g(d) \Leftrightarrow g(n) = \sum_{n|d} \mu(\frac{d}{n}) f(d)$

• Spherical cap

- A portion of a sphere cut off by a plane.
- r : sphere radius, a : radius of the base of the cap, h : height of the cap, θ : $\arcsin(a/r)$.
- Volume $= \pi h^2(3r - h)/3 = \pi h(3a^2 + h^2)/6 = \pi r^3(2 + \cos \theta)(1 - \cos \theta)^2/3$.
- Area $= 2\pi r h = \pi(a^2 + h^2) = 2\pi r^2(1 - \cos \theta)$.

• Lagrange multiplier

- Optimize $f(x_1, \dots, x_n)$ when k constraints $g_i(x_1, \dots, x_n) = 0$.
- Lagrangian function $\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) = f(x_1, \dots, x_n) - \sum_{i=1}^k \lambda_i g_i(x_1, \dots, x_n)$.
- The solution corresponding to the original constrained optimization is always a saddle point of the Lagrangian function.

• Nearest points of two skew lines

- Line 1 : $v_1 = p_1 + t_1 d_1$
- Line 2 : $v_2 = p_2 + t_2 d_2$

- $n = d_1 \times d_2$
- $n_1 = d_1 \times n$
- $n_2 = d_2 \times n$
- $c_1 = p_1 + \frac{(p_2 - p_1) \cdot n_2}{d_1 \cdot n_2} d_1$
- $c_2 = p_2 + \frac{(p_1 - p_2) \cdot n_1}{d_2 \cdot n_1} d_2$

• Bernoulli numbers

$$B_0 = 1, B_1^\pm = \pm \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0$$

$$\sum_{j=0}^m \binom{m+1}{j} B_j = 0, \text{EGF is } B(x) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

$$S_m(n) = \sum_{k=1}^n k^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k^+ n^{m+1-k}$$

• Stirling numbers of the second kind Partitions of n distinct elements into exactly k groups.

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), S(n, 1) = S(n, n) = 1$$

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$$

$$x^n = \sum_{i=0}^n S(n, i) (x)_i$$

• Pentagonal number theorem

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2} \right)$$

• Catalan numbers

$$C_n^{(k)} = \frac{1}{(k-1)n+1} \binom{kn}{n}$$

$$C^{(k)}(x) = 1 + x[C^{(k)}(x)]^k$$

• Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t.

$$\pi(j) \geq j, k \text{ j:s s.t. } \pi(j) > j.$$

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

10.6 Number

• Some prime numbers:

12721, 13331, 14341, 75577, 123457, 222557, 556679, 999983,
1097774749, 1076767633, 100102021, 999997771, 1001010013,
1000512343, 987654361, 999991231, 999888733, 98789101,
987777733, 999991921, 1010101333, 1010102101, 1000000000039,
1000000000000037, 2305843009213693951, 4611686018427387847,
9223372036854775783, 18446744073709551557

• Number of partitions of n :

n	2	3	4	5	6	7	8	9	20	30	40	50	100
$p(n)$	2	3	5	7	11	15	22	30	627	5604	4e4	2e5	2e8

• Maximum number of divisors:

n	100	1e3	1e6	1e9	1e12	1e15	1e18
$d(n)$	12	32	240	1344	6720	26880	103680

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\binom{2n}{n}$	2	6	20	70	252	924	3432	12870	48620	184756	7e5	2e6	1e7	4e7	1.5e8

• Number of ways to partition a set of n labeled elements:

n	2	3	4	5	6	7	8	9	10	11	12	13
B_n	2	5	15	52	203	877	4140	21147	115975	7e5	4e6	3e7

• Fibonacci numbers:

n	1	2	3	4	5	31	45	88
F_n	1	1	3	5	8	1346269	1e9	1e18