

# Contents

<b>1 Basic</b>	<b>1</b>	<b>5 Graph</b>	<b>12</b>
1.1 Default Code	1	5.1 BCC	12
1.2 .vimrc	1	5.2 SCC	12
1.3 Fast IO	1	5.3 2-SAT	13
1.4 Random	1	5.4 Dominator Tree	13
1.5 PBDS Tree	1	5.5 Virtual Tree	13
1.6 Pragma	2	5.6 Fast DMST	13
1.7 SVG Writer	2	5.7 Vizing	13
<b>2 Data Structure</b>	<b>2</b>	5.8 Maximum Clique	14
2.1 Heavy-Light Decomposition	2	5.9 Number of Maximal Clique	14
2.2 Link Cut Tree	2	5.10 Minimum Mean Cycle	14
2.3 Treap	3	5.11 Minimum Steiner Tree	15
2.4 KD Tree	3	5.12 Count Cycles	15
2.5 Leftist Tree	3	<b>6 Math</b>	<b>15</b>
2.6 Convex 1D/1D	4	6.1 Extended Euclidean Algorithm	15
2.7 Dynamic Convex Hull	4	6.2 Floor & Ceil	15
<b>3 Flow &amp; Matching</b>	<b>4</b>	6.3 Legendre	15
3.1 Dinic	4	6.4 Simplex	16
3.2 Bounded Flow	4	6.5 Simplex Construction	16
3.3 MCMF	5	6.6 DiscreteLog	16
3.4 Min Cost Circulation	5	6.7 Miller Rabin & Pollard Rho	16
3.5 Gomory Hu	5	6.8 XOR Basis	17
3.6 Stoer Wagner Algorithm	5	6.9 Linear Equation	17
3.7 Bipartite Matching	6	6.10 Chinese Remainder Theorem	17
3.8 Kuhn Munkres Algorithm	6	6.11 Sqrt Decomposition	17
3.9 Max Simple Graph Matching	6	6.12 Floor Sum	17
3.10 Stable Marriage	7	<b>7 Polynomial</b>	<b>18</b>
3.11 Flow Model	7	7.1 FWHT	18
<b>4 Geometry</b>	<b>7</b>	7.2 FFT	18
4.1 Geometry Template	7	7.3 NTT	18
4.2 Polar Angle Comparator	8	7.4 Polynomial Operation	18
4.3 Minkowski Sum	8	7.5 Generating Function	20
4.4 Intersection of Circle and Convex Polygon	8	Ordinary Generating Function	20
4.5 Intersection of Circles	8	Exponential Generating Function	20
4.6 Tangent Line of Circles	8	7.6 Bostan Mori	20
4.7 Intersection of Line and Convex Polygon	8	<b>8 String</b>	<b>21</b>
4.8 Intersection of Line and Circle	9	8.1 KMP Algorithm	21
4.9 Point in Circle	9	8.2 Manacher Algorithm	21
4.10 Point in Convex	9	8.3 Lyndon Factorization	21
4.11 Half Plane Intersection	9	8.4 Suffix Array	21
4.12 Minimum Enclosing Circle	9	8.5 Suffix Automaton	21
4.13 3D Point	9	8.6 Z-value Algorithm	22
4.14 ConvexHull3D	10	8.7 Main Lorentz	22
4.15 Delaunay Triangulation	10	8.8 AC Automaton	22
4.16 Voronoi Diagram	11	8.9 Palindrome Automaton	23
4.17 Polygon Union	11	<b>9 Misc</b>	<b>23</b>
4.18 Tangent Point to Convex Hull	11	9.1 Cyclic Ternary Search	23
4.19 Heart	11	9.2 Matroid	23
4.20 Rotating Sweep Line	11	9.3 Simulate Annealing	23
4.21 Vector In Poly	11	9.4 Binary Search On Fraction	23
4.22 Convex Hull DP	12	9.5 Min Plus Convolution	23
4.23 Calculate Points in Triangle	12	<b>10 Notes</b>	<b>23</b>
		10.1 Geometry	23
		Rotation Matrix	23
		Triangles	24
		Quadrilaterals	24
		Spherical coordinates	24
		Green's Theorem	24
		Point-Line Duality	24
		10.2 Trigonometry	24
		10.3 Calculus	24
		10.4 Sum & Series	24
		10.5 Misc	25
		10.6 Number	25

## 1 Basic

### 1.1 Default Code [2d2229]

```
#include <bits/stdc++.h>
using namespace std;

#define iter(v) v.begin(),v.end()
#define SZ(v) int(v.size())
#define pb emplace_back
#define ff first
#define ss second

using ll = long long;
using pii = pair<int, int>;
using pll = pair<ll, ll>;

#ifdef zisk
void debug(){cerr << "\n";}
template<class T, class ... U>
```

```
void debug(T a, U ... b){cerr << a << " ", debug(b...);}
template<class T> void pary(T l, T r){
    while (l != r) cerr << *l << " ", l++;
    cerr << "\n";
}
#else
#define debug(...) void()
#define pary(...) void()
#endif

template<class A, class B>
ostream& operator<<(ostream& o, pair<A,B> p)
{ return o << '(' << p.ff << ',' << p.ss << ')'; }
```

```
int main(){
    ios_base::sync_with_stdio(false); cin.tie(0);
}
```

### 1.2 .vimrc [476d29]

```
sy on
se nu rnu bs=2 sw=4 ts=4 hls ls=2 si acd bo=all mouse=a
map <F9> :w<bar>!g++ "% " -o %:r -std=c++17 -Wall -
Wextra -Wshadow -O2 -Dzisk -g -fsanitize=undefined,
address<CR>
map <F8> :!./%:r<CR>
inoremap {<CR> {<CR><ESC>ko
ca Hash w !cpp -dD -P -fpreprocessed \ | tr -d '[:space
:]' \ | md5sum \ | cut -c-6
inoremap fj <ESC>
vnoremap fj <ESC>
" -D_GLIBCXX_ASSERTIONS, -D_GLIBCXX_DEBUG
```

### 1.3 Fast IO [4f6f0e]

```
char readchar() {
    const int N = 1<<20;
    static char buf[N];
    static char *p = buf, *end = buf;
    if(p == end) {
        if((end = buf + fread(buf, 1, N, stdin)) == buf)
            return EOF;
        p = buf;
    }
    return *p++;
}

const int buf_size = 524288;
struct Writer {
    char buf[buf_size]; int size = 0, ret;
    void flush() { ret = write(1, buf, size); size = 0; }
    void _flush(int sz) { if (sz + size > buf_size) flush(); }
    void write_char(char c) { _flush(1); buf[size++] = c; }
    void write_int(int x) {
        const int len = 20;
        _flush(len); int ptr = 0;
        if (x < 0) buf[size++] = '-', x = -x;
        if (x == 0) buf[size + (ptr++)] = '0';
        else for (; x; x /= 10) buf[size + (ptr++)] = '0' +
            x % 10;
        reverse(buf + size, buf + size + ptr);
        size += ptr;
    }
}; // remember to call flush
```

### 1.4 Random [4cf9ed]

```
mt19937 rng(chrono::system_clock::now().
    time_since_epoch().count());
```

### 1.5 PBDS Tree [7c702a]

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
using Tree = tree<int, null_type, less<, rb_tree_tag,
    tree_order_node_statistics_update>;
// .find_by_order(x)
// .order_of_key(x)
```

## 1.6 Pragma [6006f6]

```
#pragma GCC optimize("Ofast,no-stack-protector")
#pragma GCC optimize("no-math-errno,unroll-loops")
#pragma GCC target("sse,sse2,sse3,ssse3,sse4")
#pragma GCC target("popcnt,abm,mmx,avx,arch=skylake")
__builtin_ia32_ldmxcsr(__builtin_ia32_stmxcsr())|0x8040
```

## 1.7 SVG Writer [7adcc8]

```
class SVG {
    void p(string_view s) { o << s; }
    void p(string_view s, auto v, auto... vs) {
        auto i = s.find('$');
        o << s.substr(0, i) << v, p(s.substr(i + 1), vs...)
    }
    ofstream o; string c = "red";
public: // SVG svg("test.svg", 0, 0, 100, 100)
    SVG(auto f, auto x1, auto y1, auto x2, auto y2) : o(f) {
        p("<svg xmlns='http://www.w3.org/2000/svg' "
          "viewBox='$ $ $ $'>\n"
          "<style>{*stroke-width:0.5%;}</style>\n",
          x1, -y2, x2 - x1, y2 - y1); }
    ~SVG() { p("</svg>\n"); }
    void color(string nc) { c = nc; }
    void line(auto x1, auto y1, auto x2, auto y2) {
        p("<line x1='$' y1='$' x2='$' y2='$' stroke='$'>\n"
          "x1, -y1, x2, -y2, c); }
    void circle(auto x, auto y, auto r) {
        p("<circle cx='$' cy='$' r='$' stroke='$' "
          "fill='none'>\n", x, -y, r, c); }
    void text(auto x, auto y, string s, int w = 12) {
        p("<text x='$' y='$' font-size='$px'>$</text>\n",
          x, -y, w, s); }
};
```

# 2 Data Structure

## 2.1 Heavy-Light Decomposition [f2dbca]

```
struct HLD{ // 1-based
    int n, ts = 0; // ord is 1-based
    vector<vector<int>> g;
    vector<int> par, top, down, ord, dpt, sub;
    explicit HLD(int _n): n(_n), g(n + 1),
    par(n + 1), top(n + 1), down(n + 1),
    ord(n + 1), dpt(n + 1), sub(n + 1) {}
    void add_edge(int u, int v){ g[u].pb(v); g[v].pb(u); }
    void dfs(int now, int p){
        par[now] = p; sub[now] = 1;
        for(int i : g[now]){
            if(i == p) continue;
            dpt[i] = dpt[now] + 1;
            dfs(i, now);
            sub[now] += sub[i];
            if(sub[i] > sub[down[now]]) down[now] = i;
        }
    }
    void cut(int now, int t){
        top[now] = t; ord[now] = ++ts;
        if(!down[now]) return;
        cut(down[now], t);
        for(int i : g[now]){
            if(i != par[now] && i != down[now])
                cut(i, i);
        }
    }
    void build(){ dfs(1, 1), cut(1, 1); }
    int query(int a, int b){
        int ta = top[a], tb = top[b];
        while(ta != tb){
            if(dpt[ta] > dpt[tb]) swap(ta, tb), swap(a, b);
            // ord[ta], ord[b]
            tb = top[b = par[tb]];
        }
        if(ord[a] > ord[b]) swap(a, b);
        // ord[a], ord[b]
        return a; // lca
    }
};
```

## 2.2 Link Cut Tree [cf4f34]

```
// 1-based
template <typename Val, typename SVal> struct LCT {
    struct node {
        int pa, ch[2]; bool rev; int size;
        Val v, sum, rsum; SVal sv, sub, vir;
        node() : pa{0}, ch{0, 0}, rev{false}, size{1}, v{},
            sum{}, rsum{}, sv{}, sub{}, vir{} {}
    };
    #define cur o[u]
    #define lc cur.ch[0]
    #define rc cur.ch[1]
    vector<node> o;
    bool is_root(int u) const {
        return o[cur.pa].ch[0] != u && o[cur.pa].ch[1] != u; }
    bool is_rch(int u) const {
        return o[cur.pa].ch[1] == u && !is_root(u); }
    void down(int u) {
        for (int c : {lc, rc}) if (c) {
            if (cur.rev) set_rev(c);
        }
        cur.rev = false;
    }
    void up(int u) {
        cur.sum = o[lc].sum + cur.v + o[rc].sum;
        cur.rsum = o[rc].rsum + cur.v + o[lc].rsum;
        cur.sub = cur.vir + o[lc].sub + o[rc].sub + cur.sv;
        cur.size = o[lc].size + o[rc].size + 1;
    }
    void set_rev(int u) {
        swap(lc, rc), swap(cur.sum, cur.rsum);
        cur.rev ^= 1;
    }
    /* --- */
    void rotate(int u) {
        int f = cur.pa, g = o[f].pa, l = is_rch(u);
        if (cur.ch[l ^ 1]) o[cur.ch[l ^ 1]].pa = f;
        if (not is_root(f)) o[g].ch[is_rch(f)] = u;
        o[f].ch[l] = cur.ch[l ^ 1], cur.ch[l ^ 1] = f;
        cur.pa = g, o[f].pa = u; up(f);
    }
    void splay(int u) {
        vector<int> stk = {u};
        while (not is_root(stk.back()))
            stk.push_back(o[stk.back()].pa);
        while (not stk.empty())
            down(stk.back()), stk.pop_back();
        for (int f = cur.pa; not is_root(u); f = cur.pa) {
            if (!is_root(f))
                rotate(is_rch(u) == is_rch(f) ? f : u);
            rotate(u);
        }
        up(u);
    }
    void access(int x) {
        for (int u = x, last = 0; u; u = cur.pa) {
            splay(u);
            cur.vir = cur.vir + o[rc].sub - o[last].sub;
            rc = last; up(last = u);
        }
        splay(x);
    }
    int find_root(int u) {
        int la = 0;
        for (access(u); u; u = lc) down(la = u);
        return la;
    }
    void split(int x, int y) { chroot(x); access(y); }
    void chroot(int u) { access(u); set_rev(u); }
    /* --- */
    LCT(int n = 0) : o(n + 1) { o[0].size = 0; }
    void set_val(int u, const Val &v) {
        splay(u); cur.v = v; up(u); }
    void set_sval(int u, const SVal &v) {
        access(u); cur.sv = v; up(u); }
    Val query(int x, int y) {
        split(x, y); return o[y].sum; }
    SVal subtree(int p, int u) {
        chroot(p); access(u); return cur.vir + cur.sv; }
    bool connected(int u, int v) {
        return find_root(u) == find_root(v); }
};
```

```

void link(int x, int y) {
    chroot(x); access(y);
    o[y].vir = o[y].vir + o[x].sub;
    up(o[x].pa = y);
}
void cut(int x, int y) {
    split(x, y); o[y].ch[0] = o[x].pa = 0; up(y); }
#undef cur
#undef lc
#undef rc
};

2.3 Treap [2ac37e]
mt19937 rng(880301);
struct node {
    ll data; int sz;
    node *l, *r;
    node(ll k = 0) : data(k), sz(1), l(0), r(0) {}
    void up() {
        sz = 1;
        if (l) sz += l->sz;
        if (r) sz += r->sz;
    }
    void down() {}
};
node pool[1000010]; int pool_cnt = 0;
node *newnode(ll k) { return &(pool[pool_cnt++] = node(k)); }
int sz(node *a) { return a ? a->sz : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (int(rng() % (sz(a) + sz(b))) < sz(a))
        return a->down(), a->r = merge(a->r, b), a->up(),
            a;
    return b->down(), b->l = merge(a, b->l), b->up(), b;
}
// a: key <= k, b: key > k
void split(node *o, node *&a, node *&b, ll k) {
    if (!o) return a = b = 0, void();
    o->down();
    if (o->data <= k)
        a = o, split(o->r, a->r, b, k), a->up();
    else b = o, split(o->l, a, b->l, k), b->up();
}
// a: size k, b: size n - k
void split2(node *o, node *&a, node *&b, int k) {
    if (sz(o) <= k) return a = o, b = 0, void();
    o->down();
    if (sz(o->l) + 1 <= k)
        a = o, split2(o->r, a->r, b, k - sz(o->l) - 1);
    else b = o, split2(o->l, a, b->l, k);
    o->up();
}
node *kth(node *o, ll k) { // 1-based
    if (k <= sz(o->l)) return kth(o->l, k);
    if (k == sz(o->l) + 1) return o;
    return kth(o->r, k - sz(o->l) - 1);
}
int Rank(node *o, ll key) { // num of key < key
    if (!o) return 0;
    if (o->data < key)
        return sz(o->l) + 1 + Rank(o->r, key);
    else return Rank(o->l, key);
}
bool erase(node *&o, ll k) {
    if (!o) return 0;
    if (o->data == k) {
        node *t = o;
        o->down(), o = merge(o->l, o->r);
        return 1;
    }
    node *&t = k < o->data ? o->l : o->r;
    return erase(t, k) ? o->up(), 1 : 0;
}
void insert(node *&o, ll k) {
    node *a, *b;
    split(o, a, b, k),
        o = merge(a, merge(new node(k), b));
}
tuple<node*, node*, node*> interval(node *&o, int l,
    int r) { // 1-based
    node *a, *b, *c; // b: [l, r]

```

```

    split2(o, a, b, l - 1), split2(b, b, c, r - l + 1);
    return make_tuple(a, b, c);
}

2.4 KD Tree [375ca2]
namespace kdt {
    int root, lc[maxn], rc[maxn], xl[maxn], xr[maxn],
        yl[maxn], yr[maxn];
    point p[maxn];
    int build(int l, int r, int dep = 0) {
        if (l == r) return -1;
        function<bool(const point &, const point &)> f =
            [dep](const point &a, const point &b) {
                if (dep & 1) return a.x < b.x;
                else return a.y < b.y;
            };
        int m = (l + r) >> 1;
        nth_element(p + l, p + m, p + r, f);
        xl[m] = xr[m] = p[m].x;
        yl[m] = yr[m] = p[m].y;
        lc[m] = build(l, m, dep + 1);
        if (~lc[m]) {
            xl[m] = min(xl[m], xl[lc[m]]);
            xr[m] = max(xr[m], xr[lc[m]]);
            yl[m] = min(yl[m], yl[lc[m]]);
            yr[m] = max(yr[m], yr[lc[m]]);
        }
        rc[m] = build(m + 1, r, dep + 1);
        if (~rc[m]) {
            xl[m] = min(xl[m], xl[rc[m]]);
            xr[m] = max(xr[m], xr[rc[m]]);
            yl[m] = min(yl[m], yl[rc[m]]);
            yr[m] = max(yr[m], yr[rc[m]]);
        }
        return m;
    }
    bool bound(const point &q, int o, long long d) {
        double ds = sqrt(d + 1.0);
        if (q.x < xl[o] - ds || q.x > xr[o] + ds ||
            q.y < yl[o] - ds || q.y > yr[o] + ds)
            return false;
        return true;
    }
    long long dist(const point &a, const point &b) {
        return (a.x - b.x) * 1ll * (a.x - b.x) +
            (a.y - b.y) * 1ll * (a.y - b.y);
    }
    void dfs(
        const point &q, long long &d, int o, int dep = 0)
    {
        if (!bound(q, o, d)) return;
        long long cd = dist(p[o], q);
        if (cd != 0) d = min(d, cd);
        if ((dep & 1) && q.x < p[o].x ||
            !(dep & 1) && q.y < p[o].y) {
            if (~lc[o]) dfs(q, d, lc[o], dep + 1);
            if (~rc[o]) dfs(q, d, rc[o], dep + 1);
        } else {
            if (~rc[o]) dfs(q, d, rc[o], dep + 1);
            if (~lc[o]) dfs(q, d, lc[o], dep + 1);
        }
    }
    void init(const vector<point> &v) {
        for (int i = 0; i < v.size(); ++i) p[i] = v[i];
        root = build(0, v.size());
    }
    long long nearest(const point &q) {
        long long res = 1e18;
        dfs(q, res, root);
        return res;
    }
} // namespace kdt

2.5 Leftist Tree [e91538]
struct node {
    ll v, data, sz, sum;
    node *l, *r;
    node(ll k)
        : v(0), data(k), sz(1), l(0), r(0), sum(k) {}
};
ll sz(node *p) { return p ? p->sz : 0; }

```

```

11 V(node *p) { return p ? p->v : -1; }
11 sum(node *p) { return p ? p->sum : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (a->data < b->data) swap(a, b);
    a->r = merge(a->r, b);
    if (V(a->r) > V(a->l)) swap(a->r, a->l);
    a->v = V(a->r) + 1, a->sz = sz(a->l) + sz(a->r) + 1;
    a->sum = sum(a->l) + sum(a->r) + a->data;
    return a;
}
void pop(node *&o) {
    node *tmp = o;
    o = merge(o->l, o->r);
    delete tmp;
}

```

## 2.6 Convex 1D/1D [a449dd]

```

template<class T>
struct DynamicHull {
    struct seg { int x, l, r; };
    T f; int C; deque<seg> dq; // range: 1~C
    explicit DynamicHull(T _f, int _C): f(_f), C(_C) {}
    // max t s.t. f(x, t) >= f(y, t), x < y, maintain max
    int intersect(int x, int y) {
        int l = 0, r = C + 1;
        while (l + 1 < r) {
            int mid = (l + r) / 2;
            if (f(x, mid) >= f(y, mid)) l = mid;
            else r = mid;
        }
        return l;
    }
    void push_back(int x) {
        for (int i; !dq.empty() &&
            (i = dq.back().l, f(dq.back().x, i) < f(x, i));
            )
            dq.pop_back();
        if (dq.empty()) return dq.pb(seg({x, 1, C})), void();
        dq.back().r = intersect(dq.back().x, x);
        if (dq.back().r + 1 <= C) dq.pb(seg({x, dq.back().r + 1, C}));
    }
    int query(int x) {
        while (dq.front().r < x) dq.pop_front();
        return dq.front().x;
    }
};

```

## 2.7 Dynamic Convex Hull [b45ebc]

```

// only works for integer coordinates!! maintain max
struct Line {
    mutable ll a, b, p;
    bool operator<(const Line &rhs) const { return a < rhs.a; }
    bool operator<(ll x) const { return p < x; }
};
struct DynamicHull : multiset<Line, less<>> {
    static const ll kInf = 1e18;
    bool isect(iterator x, iterator y) {
        if (y == end()) { x->p = kInf; return 0; }
        if (x->a == y->a) x->p = x->b > y->b ? kInf : -kInf;
        else x->p = iceil(y->b - x->b, x->a - y->a);
        return x->p >= y->p;
    }
    void addline(ll a, ll b) {
        auto z = insert({a, b, 0}); y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    }
    ll query(ll x) {
        auto l = *lower_bound(x);
        return l.a * x + l.b;
    }
};

```

## 3 Flow & Matching

### 3.1 Dinic [801a71]

```

struct Dinic { // 0-based, O(V^2E), unit flow: O(min(V
    ^{2/3}E, E^{3/2})), bipartite matching: O(sqrt(V)E)
    struct edge {
        ll to, cap, flow, rev;
    };

    int n, s, t;
    vector<vector<edge>> g;
    vector<int> dis, ind;

    void init(int _n) {
        n = _n;
        g.assign(n, vector<edge>());
    }
    void reset() {
        for (int i = 0; i < n; ++i)
            for (auto &j : g[i]) j.flow = 0;
    }
    void add_edge(int u, int v, ll cap) {
        g[u].pb(edge{v, cap, 0, SZ(g[v]}));
        g[v].pb(edge{u, 0, 0, SZ(g[u]} - 1));
        //change g[v] to cap for undirected graphs
    }
    bool bfs() {
        dis.assign(n, -1);
        queue<int> q;
        q.push(s), dis[s] = 0;
        while (!q.empty()) {
            int cur = q.front(); q.pop();
            for (auto &e : g[cur]) {
                if (dis[e.to] == -1 && e.flow != e.cap) {
                    q.push(e.to);
                    dis[e.to] = dis[cur] + 1;
                }
            }
        }
        return dis[t] != -1;
    }
    ll dfs(int u, ll cap) {
        if (u == t || !cap) return cap;
        for (int &i = ind[u]; i < SZ(g[u]); ++i) {
            edge &e = g[u][i];
            if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
                ll df = dfs(e.to, min(e.cap - e.flow, cap));
                if (df) {
                    e.flow += df;
                    g[e.to][e.rev].flow -= df;
                    return df;
                }
            }
        }
        dis[u] = -1;
        return 0;
    }
    ll maxflow(int _s, int _t) {
        s = _s; t = _t;
        ll flow = 0, df;
        while (bfs()) {
            ind.assign(n, 0);
            while ((df = dfs(s, INF))) flow += df;
        }
        return flow;
    }
};

```

### 3.2 Bounded Flow [758826]

```

struct BoundedFlow : Dinic {
    vector<ll> tot;
    void init(int _n) {
        Dinic::init(_n + 2);
        tot.assign(n, 0);
    }
    void add_edge(int u, int v, ll lcap, ll rcap) {
        tot[u] -= lcap, tot[v] += lcap;
        g[u].pb(edge{v, rcap, lcap, SZ(g[v]}));
        g[v].pb(edge{u, 0, 0, SZ(g[u]} - 1));
    }
    bool feasible() {

```

```

    ll sum = 0;
    int vs = n - 2, vt = n - 1;
    for(int i = 0; i < n - 2; ++i)
        if(tot[i] > 0)
            add_edge(vs, i, 0, tot[i]), sum += tot[i];
        else if(tot[i] < 0) add_edge(i, vt, 0, -tot[i]);
    if(sum != maxflow(vs, vt)) sum = -1;
    for(int i = 0; i < n - 2; i++)
        if(tot[i] > 0)
            g[vs].pop_back(), g[i].pop_back();
        else if(tot[i] < 0)
            g[i].pop_back(), g[vt].pop_back();
    return sum != -1;
}
ll boundedflow(int _s, int _t) {
    add_edge(_t, _s, 0, INF);
    if(!feasible()) return -1;
    ll x = g[_t].back().flow;
    g[_t].pop_back(), g[_s].pop_back();
    return x - maxflow(_t, _s); // min
    //return x + maxflow(_s, _t); // max
}
};

```

### 3.3 MCMF [671e14]

```

struct MCMF { // 0-based
    struct Edge {
        ll from, to, cap, flow, cost, rev;
    };
    int n, s, t;
    vector<vector<Edge>> g;
    vector<Edge*> past;
    vector<ll> dis, up, pot;
    explicit MCMF(int _n): n(_n), g(n), past(n), dis(n),
        up(n), pot(n) {}
    void add_edge(ll a, ll b, ll cap, ll cost) {
        g[a].pb(Edge{a, b, cap, 0, cost, SZ(g[b])});
        g[b].pb(Edge{b, a, 0, 0, -cost, SZ(g[a]) - 1});
    }
    bool BellmanFord() {
        vector<bool> inq(n);
        fill(iter(dis), INF);
        queue<int> q;
        auto relax = [&](int u, ll d, ll cap, Edge *e) {
            if (cap > 0 && dis[u] > d) {
                dis[u] = d, up[u] = cap, past[u] = e;
                if (!inq[u]) inq[u] = 1, q.push(u);
            }
        };
        relax(s, 0, INF, 0);
        while (!q.empty()) {
            int u = q.front();
            q.pop(), inq[u] = 0;
            for (auto &e : g[u]) {
                ll d2 = dis[u] + e.cost + pot[u] - pot[e.to];
                relax(e.to, d2, min(up[u], e.cap - e.flow), &e);
            }
        }
        return dis[t] != INF;
    }
    pair<ll, ll> solve(int _s, int _t, bool neg = true) {
        s = _s, t = _t; ll flow = 0, cost = 0;
        if (neg) BellmanFord(), pot = dis;
        for (; BellmanFord(); pot = dis) {
            for (int i = 0; i < n; ++i)
                if (dis[i] != INF) dis[i] += pot[i] - pot[s];
            flow += up[t], cost += up[t] * dis[t];
            for (int i = t; past[i]; i = past[i]->from) {
                auto &e = *past[i];
                e.flow += up[t], g[e.to][e.rev].flow -= up[t];
            }
        }
        return {flow, cost};
    }
};

```

### 3.4 Min Cost Circulation [47cf18]

```

struct MinCostCirculation { // 0-based, O(VE * ElogC)
    struct edge {
        ll from, to, cap, fcap, flow, cost, rev;
    };
};

```

```

};
int n;
vector<edge*> past;
vector<vector<edge>> g;
vector<ll> dis;
void BellmanFord(int s) {
    vector<int> inq(n);
    dis.assign(n, INF);
    queue<int> q;
    auto relax = [&](int u, ll d, edge *e) {
        if (dis[u] > d) {
            dis[u] = d, past[u] = e;
            if (!inq[u]) inq[u] = 1, q.push(u);
        }
    };
    relax(s, 0, 0);
    while (!q.empty()) {
        int u = q.front();
        q.pop(), inq[u] = 0;
        for (auto &e : g[u])
            if (e.cap > e.flow)
                relax(e.to, dis[u] + e.cost, &e);
    }
}
void try_edge(edge &cur) {
    if (cur.cap > cur.flow) return ++cur.cap, void();
    BellmanFord(cur.to);
    if (dis[cur.from] + cur.cost < 0) {
        ++cur.flow, --g[cur.to][cur.rev].flow;
        for (int i = cur.from; past[i]; i = past[i]->from) {
            auto &e = *past[i];
            ++e.flow, --g[e.to][e.rev].flow;
        }
    }
    ++cur.cap;
}
void solve(int mxlg) { // mxlg >= Log(max cap)
    for (int b = mxlg; b >= 0; --b) {
        for (int i = 0; i < n; ++i)
            for (auto &e : g[i])
                e.cap *= 2, e.flow *= 2;
        for (int i = 0; i < n; ++i)
            for (auto &e : g[i])
                if (e.fcap >> b & 1)
                    try_edge(e);
    }
}
void init(int _n) {
    n = _n;
    past.assign(n, nullptr);
    g.assign(n, vector<edge>());
}
void add_edge(ll a, ll b, ll cap, ll cost) {
    g[a].pb(edge{a, b, 0, cap, 0, cost, SZ(g[b]) + (a
        == b)});
    g[b].pb(edge{b, a, 0, 0, 0, -cost, SZ(g[a]) - 1});
}
};

```

### 3.5 Gomory Hu [82d968]

```

void GomoryHu(Dinic &flow) { // 0-based
    int n = flow.n;
    vector<int> par(n);
    for (int i = 1; i < n; ++i) {
        flow.reset();
        add_edge(i, par[i], flow.maxflow(i, par[i]));
        for (int j = i + 1; j < n; ++j)
            if (par[j] == par[i] && ~flow.dis[j])
                par[j] = i;
    }
}

```

### 3.6 Stoer Wagner Algorithm [a9917b]

```

struct StoerWagner { // 0-based, O(V^3)
    int n;
    vector<int> vis, del;
    vector<ll> wei;
    vector<vector<ll>> edge;
    void init(int _n) {
        n = _n;
    }
};

```



```

    del.assign(n, 0);
    edge.assign(n, vector<ll>(n));
}
void add_edge(int u, int v, ll w) {
    edge[u][v] += w, edge[v][u] += w;
}
void search(int &s, int &t) {
    vis.assign(n, 0); wei.assign(n, 0);
    s = t = -1;
    while (1) {
        ll mx = -1, cur = 0;
        for (int i = 0; i < n; ++i)
            if (!del[i] && !vis[i] && mx < wei[i])
                cur = i, mx = wei[i];
        if (mx == -1) break;
        vis[cur] = 1, s = t, t = cur;
        for (int i = 0; i < n; ++i)
            if (!vis[i] && !del[i]) wei[i] += edge[cur][i];
    }
}
ll solve() {
    ll ret = INF;
    for (int i = 0, x=0, y=0; i < n-1; ++i) {
        search(x, y), ret = min(ret, wei[y]), del[y] = 1;
        for (int j = 0; j < n; ++j)
            edge[x][j] = (edge[j][x] += edge[y][j]);
    }
    return ret;
}
};

```

### 3.7 Bipartite Matching [013c49]

*//min vertex cover: take all unmatched vertices in L  
and find alternating tree,  
//ans is not reached in L + reached in R  
// O(VE)*

*int n; // 1-based, max matching*  
*int mx[maxn], my[maxn];*  
*bool adj[maxn][maxn], vis[maxn];*  
*bool dfs(int u) {*

```

    if (vis[u]) return 0;
    vis[u] = 1;
    for (int v = 1; v <= n; v++) {
        if (!adj[u][v]) continue;
        if (!my[v] || (my[v] && dfs(my[v]))) {
            mx[u] = v, my[v] = u;
            return 1;
        }
    }
    return 0;
}

```

*// O(E sqrt(V)), O(E log V) for random sparse graphs*

```

struct BipartiteMatching { // 0-based
    int nl, nr;
    vector<int> mx, my, dis, cur;
    vector<vector<int>> g;
    bool dfs(int u) {
        for (int &i = cur[u]; i < SZ(g[u]); ++i) {
            int e = g[u][i];
            if (!my[e] || (dis[my[e]] == dis[u] + 1 && dfs(
                my[e])))
                return mx[my[e] = u] = e, 1;
        }
        dis[u] = -1;
        return 0;
    }
    bool bfs() {
        int ret = 0;
        queue<int> q;
        dis.assign(nl, -1);
        for (int i = 0; i < nl; ++i)
            if (!mx[i]) q.push(i), dis[i] = 0;
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            for (int e : g[u])
                if (!my[e]) ret = 1;
                else if (!dis[my[e]]) {
                    q.push(my[e]);
                    dis[my[e]] = dis[u] + 1;
                }
        }
    }
}

```

```

    return ret;
}
int matching() {
    int ret = 0;
    mx.assign(nl, -1); my.assign(nr, -1);
    while (bfs()) {
        cur.assign(nl, 0);
        for (int i = 0; i < nl; ++i)
            if (!mx[i] && dfs(i)) ++ret;
    }
    return ret;
}
void add_edge(int s, int t) { g[s].pb(t); }
void init(int _nl, int _nr) {
    nl = _nl, nr = _nr;
    g.assign(nl, vector<int>());
}
};

```

### 3.8 Kuhn Munkres Algorithm [683e0a]

```

struct KM { // 0-based, maximum matching, O(V^3)
    int n, ql, qr;
    vector<vector<ll>> w;
    vector<ll> hl, hr, slk;
    vector<int> fl, fr, pre, qu, vl, vr;
    void init(int _n) {
        n = _n;
        // -INF for perfect matching
        w.assign(n, vector<ll>(n, 0));
        pre.assign(n, 0);
        qu.assign(n, 0);
    }
    void add_edge(int a, int b, ll wei) {
        w[a][b] = wei;
    }
    bool check(int x) {
        if (vl[x] = 1, ~fl[x])
            return (vr[qu[qr++]] = fl[x]) = 1;
        while (~x) swap(x, fr[fl[x] = pre[x]]);
        return 0;
    }
    void bfs(int s) {
        slk.assign(n, INF); vl.assign(n, 0); vr.assign(n, 0);
        ql = qr = 0, qu[qr++] = s, vr[s] = 1;
        for (ll d;;) {
            while (ql < qr)
                for (int x = 0, y = qu[ql++]; x < n; ++x)
                    if (!vl[x] && slk[x] >= (d = hl[x] + hr[y] -
                        w[x][y])) {
                        if (pre[x] = y, d) slk[x] = d;
                        else if (!check(x)) return;
                    }
            d = INF;
            for (int x = 0; x < n; ++x)
                if (!vl[x] && d > slk[x]) d = slk[x];
            for (int x = 0; x < n; ++x) {
                if (vl[x]) hl[x] += d;
                else slk[x] -= d;
                if (vr[x]) hr[x] -= d;
            }
            for (int x = 0; x < n; ++x)
                if (!vl[x] && !slk[x] && !check(x)) return;
        }
    }
    ll solve() {
        fl.assign(n, -1); fr.assign(n, -1); hl.assign(n, 0);
        hr.assign(n, 0);
        for (int i = 0; i < n; ++i)
            hl[i] = *max_element(iter(w[i]));
        for (int i = 0; i < n; ++i) bfs(i);
        ll res = 0;
        for (int i = 0; i < n; ++i) res += w[i][fl[i]];
        return res;
    }
}

```

### 3.9 Max Simple Graph Matching [907d7c]

```

struct Matching { // 0-based, O(V^3)
    queue<int> q; int n;
    vector<int> fa, s, vis, pre, match;

```

```

vector<vector<int>> g;
int Find(int u)
{ return u == fa[u] ? u : fa[u] = Find(fa[u]); }
int LCA(int x, int y) {
    static int tk = 0; tk++; x = Find(x); y = Find(y);
    for (; swap(x, y)) if (x != n) {
        if (vis[x] == tk) return x;
        vis[x] = tk;
        x = Find(pre[match[x]]);
    }
}
void Blossom(int x, int y, int l) {
    for (; Find(x) != l; x = pre[y]) {
        pre[x] = y, y = match[x];
        if (s[y] == 1) q.push(y), s[y] = 0;
        for (int z: {x, y}) if (fa[z] == z) fa[z] = l;
    }
}
bool Bfs(int r) {
    iota(iter(fa), 0); fill(iter(s), -1);
    q = queue<int>(); q.push(r); s[r] = 0;
    for (; !q.empty(); q.pop()) {
        for (int x = q.front(); int u : g[x])
            if (s[u] == -1) {
                if (pre[u] = x, s[u] = 1, match[u] == n) {
                    for (int a = u, b = x, last;
                        b != n; a = last, b = pre[a])
                        last = match[b], match[b] = a, match[a] = b;
                    return true;
                }
                q.push(match[u]); s[match[u]] = 0;
            }
        else if (!s[u] && Find(u) != Find(x)) {
            int l = LCA(u, x);
            Blossom(x, u, l); Blossom(u, x, l);
        }
    }
    return false;
}
Matching(int _n) : n(_n), fa(n + 1), s(n + 1), vis(n + 1), pre(n + 1, n), match(n + 1, n), g(n) {}
void add_edge(int u, int v)
{ g[u].pb(v), g[v].pb(u); }
int solve() {
    int ans = 0;
    for (int x = 0; x < n; ++x)
        if (match[x] == n) ans += Bfs(x);
    return ans;
} // match[x] == n means not matched
};

```

### 3.10 Stable Marriage

- 1: Initialize  $m \in M$  and  $w \in W$  to free
- 2: while  $\exists$  free man  $m$  who has a woman  $w$  to propose to do
- 3:  $w \leftarrow$  first woman on  $m$ 's list to whom  $m$  has not yet proposed
- 4: if  $\exists$  some pair  $(m', w)$  then
- 5: if  $w$  prefers  $m$  to  $m'$  then
- 6:  $m' \leftarrow$  free
- 7:  $(m, w) \leftarrow$  engaged
- 8: end if
- 9: else
- 10:  $(m, w) \leftarrow$  engaged
- 11: end if
- 12: end while

### 3.11 Flow Model

- Maximum/Minimum flow with lower bound / Circulation problem
  1. Construct super source  $S$  and sink  $T$ .
  2. For each edge  $(x, y, l, u)$ , connect  $x \rightarrow y$  with capacity  $u - l$ .
  3. For each vertex  $v$ , denote by  $in(v)$  the difference between the sum of incoming lower bounds and the sum of outgoing lower bounds.
  4. If  $in(v) > 0$ , connect  $S \rightarrow v$  with capacity  $in(v)$ , otherwise, connect  $v \rightarrow T$  with capacity  $-in(v)$ .
    - To maximize, connect  $t \rightarrow s$  with capacity  $\infty$  (skip this in circulation problem), and let  $f$  be the maximum flow from  $S$  to  $T$ . If  $f \neq \sum_{v \in V, in(v) > 0} in(v)$ , there's no solution. Otherwise, the maximum flow from  $s$  to  $t$  is the answer.
    - To minimize, let  $f$  be the maximum flow from  $S$  to  $T$ . Connect  $t \rightarrow s$  with capacity  $\infty$  and let the flow from  $S$  to  $T$  be  $f'$ . If  $f + f' \neq \sum_{v \in V, in(v) > 0} in(v)$ , there's no solution. Otherwise,  $f'$  is the answer.
  5. The solution of each edge  $e$  is  $l_e + f_e$ , where  $f_e$  corresponds to the flow of edge  $e$  on the graph.

- Construct minimum vertex cover from maximum matching  $M$  on bipartite graph  $(X, Y)$ 
  1. Redirect every edge:  $y \rightarrow x$  if  $(x, y) \in M$ ,  $x \rightarrow y$  otherwise.
  2. DFS from unmatched vertices in  $X$ .
  3.  $x \in X$  is chosen iff  $x$  is unvisited.
  4.  $y \in Y$  is chosen iff  $y$  is visited.
- Minimum cost cyclic flow
  1. Construct super source  $S$  and sink  $T$
  2. For each edge  $(x, y, c)$ , connect  $x \rightarrow y$  with  $(cost, cap) = (c, 1)$  if  $c > 0$ , otherwise connect  $y \rightarrow x$  with  $(cost, cap) = (-c, 1)$
  3. For each edge with  $c < 0$ , sum these cost as  $K$ , then increase  $d(y)$  by 1, decrease  $d(x)$  by 1
  4. For each vertex  $v$  with  $d(v) > 0$ , connect  $S \rightarrow v$  with  $(cost, cap) = (0, d(v))$
  5. For each vertex  $v$  with  $d(v) < 0$ , connect  $v \rightarrow T$  with  $(cost, cap) = (0, -d(v))$
  6. Flow from  $S$  to  $T$ , the answer is the cost of the flow  $C + K$
- Maximum density induced subgraph
  1. Binary search on answer, suppose we're checking answer  $T$
  2. Construct a max flow model, let  $K$  be the sum of all weights
  3. Connect source  $s \rightarrow v, v \in G$  with capacity  $K$
  4. For each edge  $(u, v, w)$  in  $G$ , connect  $u \rightarrow v$  and  $v \rightarrow u$  with capacity  $w$
  5. For  $v \in G$ , connect it with sink  $v \rightarrow t$  with capacity  $K + 2T - (\sum_{e \in E(v)} w(e) - 2w(v))$
  6.  $T$  is a valid answer if the maximum flow  $f < K|V|$
- Minimum weight edge cover
  1. Let  $w'(u, v) = w(u, v) - \mu(u) - \mu(v)$ , where  $\mu(v)$  is the cost of the cheapest edge incident to  $v$ .
  2. Find the minimum weight matching  $M$  with  $w'$ . The answer is  $\sum \mu(v) + w'(M)$ .
- Project selection problem
  1. If  $p_v > 0$ , create edge  $(s, v)$  with capacity  $p_v$ ; otherwise, create edge  $(v, t)$  with capacity  $-p_v$ .
  2. Create edge  $(u, v)$  with capacity  $w$  with  $w$  being the cost of choosing  $u$  without choosing  $v$ .
  3. The mincut is equivalent to the maximum profit of a subset of projects.
- Dual of minimum cost maximum flow
  1. Capacity  $c_{uv}$ , Flow  $f_{uv}$ , Cost  $w_{uv}$ , Required Flow difference for vertex  $b_u$ .
  2. If all  $w_{uv}$  are integers, then optimal solution can happen when all  $p_u$  are integers.

$$\begin{aligned}
 & \min \sum_{uv} w_{uv} f_{uv} \\
 & -f_{uv} \geq -c_{uv} \Leftrightarrow \min \sum_u b_u p_u + \sum_{uv} c_{uv} \max(0, p_v - p_u - w_{uv}) \\
 & \sum_v f_{vu} - \sum_v f_{uv} = -b_u \quad p_u \geq 0
 \end{aligned}$$

## 4 Geometry

### 4.1 Geometry Template [86f0f1]

```

using ld = ll;
using pdd = pair<ld, ld>;
#define X first
#define Y second
// ld eps = 1e-7;

pdd operator+(pdd a, pdd b)
{ return {a.X + b.X, a.Y + b.Y}; }
pdd operator-(pdd a, pdd b)
{ return {a.X - b.X, a.Y - b.Y}; }
pdd operator*(ld i, pdd v)
{ return {i * v.X, i * v.Y}; }
pdd operator*(pdd v, ld i)
{ return {i * v.X, i * v.Y}; }
pdd operator/(pdd v, ld i)
{ return {v.X / i, v.Y / i}; }
ld dot(pdd a, pdd b)
{ return a.X * b.X + a.Y * b.Y; }
ld cross(pdd a, pdd b)
{ return a.X * b.Y - a.Y * b.X; }
ld abs2(pdd v)
{ return v.X * v.X + v.Y * v.Y; }
ld abs(pdd v)
{ return sqrt(abs2(v)); }
int sgn(ld v)
{ return v > 0 ? 1 : (v < 0 ? -1 : 0); }
// int sgn(ld v){ return v > eps ? 1 : (v < -eps ? -1 : 0); }
int ori(pdd a, pdd b, pdd c)
{ return sgn(cross(b - a, c - a)); }

```

```

bool collinearity(pdd a, pdd b, pdd c)
{ return ori(a, b, c) == 0; }
bool btw(pdd p, pdd a, pdd b)
{ return collinearity(p, a, b) && sgn(dot(a - p, b - p))
  <= 0; }

bool seg_intersect(pdd p1, pdd p2, pdd p3, pdd p4){
  if(btw(p1, p3, p4) || btw(p2, p3, p4) || btw(p3, p1,
    p2) || btw(p4, p1, p2))
    return true;
  return ori(p1, p2, p3) * ori(p1, p2, p4) < 0 &&
    ori(p3, p4, p1) * ori(p3, p4, p2) < 0;
}

pdd intersect(pdd p1, pdd p2, pdd p3, pdd p4){
  ld a123 = cross(p2 - p1, p3 - p1);
  ld a124 = cross(p2 - p1, p4 - p1);
  return (p4 * a123 - p3 * a124) / (a123 - a124);
}

pdd perp(pdd p1)
{ return pdd(-p1.Y, p1.X); }
pdd projection(pdd p1, pdd p2, pdd p3)
{ return p1 + (p2 - p1) * dot(p3 - p1, p2 - p1) / abs2(
  p2 - p1); }
pdd reflection(pdd p1, pdd p2, pdd p3)
{ return p3 + perp(p2 - p1) * cross(p3 - p1, p2 - p1) /
  abs2(p2 - p1) * 2; }
pdd linearTransformation(pdd p0, pdd p1, pdd q0, pdd q1
  , pdd r) {
  pdd dp = p1 - p0, dq = q1 - q0, num(cross(dp, dq),
    dot(dp, dq));
  return q0 + pdd(cross(r - p0, num), dot(r - p0, num))
    / abs2(dp);
} // from line p0--p1 to q0--q1, apply to r

```

## 4.2 Polar Angle Comparator [808e89]

```

// -1: a // b (if same), 0/1: a < b
int cmp(p11 a, p11 b, bool same = true){
#define is_neg(k) (sgn(k.Y) < 0 || (sgn(k.Y) == 0 &&
  sgn(k.X) < 0))
  int A = is_neg(a), B = is_neg(b);
  if(A != B)
    return A < B;
  if(sgn(cross(a, b)) == 0)
    return same ? abs2(a) < abs2(b) : -1;
  return sgn(cross(a, b)) > 0;
}

```

## 4.3 Minkowski Sum [98abff]

```

void reorder_poly(vector<pdd>& pnts){
  int mn = 0;
  for(int i = 1; i < (int)pnts.size(); i++){
    if(pnts[i].Y < pnts[mn].Y || (pnts[i].Y == pnts[mn].Y
      && pnts[i].X < pnts[mn].X))
      mn = i;
  }
  rotate(pnts.begin(), pnts.begin() + mn, pnts.end());
}

vector<pdd> minkowski(vector<pdd> P, vector<pdd> Q){
  reorder_poly(P);
  reorder_poly(Q);
  int psz = P.size();
  int qsz = Q.size();
  P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
  vector<pdd> ans;
  int i = 0, j = 0;
  while(i < psz || j < qsz){
    ans.pb(P[i] + Q[j]);
    int t = sgn(cross(P[i + 1] - P[i], Q[j + 1] - Q[j]));
    if(t >= 0) i++;
    if(t <= 0) j++;
  }
  return ans;
}

```

## 4.4 Intersection of Circle and Convex Polygon [63653d]

```

double _area(pdd pa, pdd pb, double r){
  if(abs(pa)<abs(pb)) swap(pa, pb);
  if(abs(pb)<eps) return 0;

```

```

double S, h, theta;
double a=abs(pb),b=abs(pa),c=abs(pb-pa);
double cosB = dot(pb,pb-pa) / a / c, B = acos(cosB);
double cosC = dot(pa,pb) / a / b, C = acos(cosC);
if(a > r){
  S = (C/2)*r*r;
  h = a*b*sin(C)/c;
  if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt(
    r*r-h*h));
}
else if(b > r){
  theta = PI - B - asin(sin(B)/r*a);
  S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
}
else S = .5*sin(C)*a*b;
return S;
}

double areaPolyCircle(const vector<pdd> poly,const pdd
  &O,const double r){
  double S=0;
  for(int i=0;i<SZ(poly);++i)
    S+=_area(poly[i]-O,poly[(i+1)%SZ(poly)]-O,r)*ori(0,
      poly[i],poly[(i+1)%SZ(poly)]);
  return fabs(S);
}

```

## 4.5 Intersection of Circles [f7a2fe]

```

bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
  pdd o1 = a.O, o2 = b.O;
  double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d =
    sqrt(d2);
  if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2)
    return 0;
  pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1
    * r1) / (2 * d2));
  double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 +
    r2 - d) * (-r1 + r2 + d));
  pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2)
    ;
  p1 = u + v, p2 = u - v;
  return 1;
}

```

## 4.6 Tangent Line of Circles [c51d90]

```

vector<Line> CCTang(const Cir& c1, const Cir& c2,
  int sign1){
  vector<Line> ret;
  double d_sq = abs2(c1.O - c2.O);
  if (sgn(d_sq) == 0) return ret;
  double d = sqrt(d_sq);
  pdd v = (c2.O - c1.O) / d;
  double c = (c1.R - sign1 * c2.R) / d; // cos t
  if (c * c > 1) return ret;
  double h = sqrt(max(0.0, 1.0 - c * c)); // sin t
  for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
    pdd n = pdd(v.X * c - sign2 * h * v.Y,
      v.Y * c + sign2 * h * v.X);
    pdd p1 = c1.O + n * c1.R;
    pdd p2 = c2.O + n * (c2.R * sign1);
    if (sgn(p1.X - p2.X) == 0 and
      sgn(p1.Y - p2.Y) == 0)
      p2 = p1 + perp(c2.O - c1.O);
    ret.pb(Line(p1, p2));
  }
  return ret;
}

```

## 4.7 Intersection of Line and Convex Polygon [157258]

```

int TangentDir(vector<p11> &C, p11 dir) {
  return cyc_tsearch(SZ(C), [&](int a, int b) {
    return cross(dir, C[a]) > cross(dir, C[b]);
  });
}

#define cml(i) sign(cross(C[i] - a, b - a))
pii lineHull(p11 a, p11 b, vector<p11> &C) {
  int A = TangentDir(C, a - b);
  int B = TangentDir(C, b - a);
  int n = SZ(C);
  if (cml(A) < 0 || cml(B) > 0)

```



```

    return pii(-1, -1); // no collision
    auto gao = [&](int l, int r) {
        for (int t = l; (l + 1) % n != r; ) {
            int m = ((l + r + (l < r ? 0 : n)) / 2) % n;
            (cmpL(m) == cmpL(t) ? l : r) = m;
        }
        return (l + !cmpL(r)) % n;
    };
    pii res = pii(gao(B, A), gao(A, B)); // (i, j)
    if (res.X == res.Y) // touching the corner i
        return pii(res.X, -1);
    if (!cmpL(res.X) && !cmpL(res.Y)) // along side i, i
        +1
        switch ((res.X - res.Y + n + 1) % n) {
            case 0: return pii(res.X, res.X);
            case 2: return pii(res.Y, res.Y);
        }
    /* crossing sides (i, i+1) and (j, j+1)
    crossing corner i is treated as side (i, i+1)
    returned in the same order as the line hits the
    convex */
    return res;
} // convex cut: (r, l]

```

#### 4.8 Intersection of Line and Circle [9183db]

```

vector<pdd> circleLineIntersection(pdd c, double r, pdd
a, pdd b) {
    pdd p = a + (b - a) * dot(c - a, b - a) / abs2(b - a)
    ;
    double s = cross(b - a, c - a), h2 = r * r - s * s /
    abs2(b - a);
    if (sgn(h2) < 0) return {};
    if (sgn(h2) == 0) return {p};
    pdd h = (b - a) / abs(b - a) * sqrt(h2);
    return {p - h, p + h};
}

```

#### 4.9 Point in Circle [ecf954]

```

// return q's relation with circumcircle of tri(p[0],p
[1],p[2])
bool in_cc(const array<p11, 3> &p, p11 q) {
    __int128 det = 0;
    for (int i = 0; i < 3; ++i)
        det += __int128(abs2(p[i]) - abs2(q)) * cross(p[(i
        + 1) % 3] - q, p[(i + 2) % 3] - q);
    return det > 0; // in: >0, on: =0, out: <0
}

```

#### 4.10 Point in Convex [f86640]

```

bool PointInConvex(const vector<p11> &C, p11 p, bool
strict = true) {
    int a = 1, b = SZ(C) - 1, r = !strict;
    if (SZ(C) == 0) return false;
    if (SZ(C) < 3) return r && btw(C[0], C.back(), p);
    if (ori(C[0], C[a], C[b]) > 0) swap(a, b);
    if (ori(C[0], C[a], p) >= r || ori(C[0], C[b], p) <=
    -r)
        return false;
    while (abs(a - b) > 1) {
        int c = (a + b) / 2;
        (ori(C[0], C[c], p) > 0 ? b : a) = c;
    }
    return ori(C[a], C[b], p) < r;
}

```

#### 4.11 Half Plane Intersection [dfb833]

```

// from 8BQube
p11 area_pair(Line a, Line b)
{ return p11(cross(a.Y - a.X, b.X - a.X), cross(a.Y - a
.X, b.Y - a.X)); }
bool isin(Line l0, Line l1, Line l2) {
    // Check inter(l1, l2) strictly in l0
    auto [a02X, a02Y] = area_pair(l0, l2);
    auto [a12X, a12Y] = area_pair(l1, l2);
    if (a12X - a12Y < 0) a12X *= -1, a12Y *= -1;
    return (__int128) a02Y * a12X - (__int128) a02X *
    a12Y > 0; // C^4
}
/* Having solution, check size > 2 */
/* --^-- Line.X--^-- Line.Y --^-- */

```

```

vector<Line> halfPlaneInter(vector<Line> arr) {
    sort(iter(arr), [&](Line a, Line b) -> int {
        if (cmp(a.Y - a.X, b.Y - b.X, 0) != -1)
            return cmp(a.Y - a.X, b.Y - b.X, 0);
        return ori(a.X, a.Y, b.Y) < 0;
    });
    deque<Line> dq(1, arr[0]);
    for (auto p : arr) {
        if (cmp(dq.back().Y - dq.back().X, p.Y - p.X, 0) ==
        -1)
            continue;
        while (SZ(dq) >= 2 && !isin(p, dq[SZ(dq) - 2], dq.
        back()))
            dq.pop_back();
        while (SZ(dq) >= 2 && !isin(p, dq[0], dq[1]))
            dq.pop_front();
        dq.pb(p);
    }
    while (SZ(dq) >= 3 && !isin(dq[0], dq[SZ(dq) - 2], dq
    .back()))
        dq.pop_back();
    while (SZ(dq) >= 3 && !isin(dq.back(), dq[0], dq[1]))
        dq.pop_front();
    return vector<Line>(iter(dq));
}

```

#### 4.12 Minimum Enclosing Circle [5af6d5]

```

using ld = long double;
pair<pdd, ld> circumcenter(pdd a, pdd b, pdd c);
pair<pdd, ld> MinimumEnclosingCircle(vector<pdd> &pts){
    random_shuffle(iter(pts));
    pdd c = pts[0];
    ld r = 0;
    for(int i = 1; i < SZ(pts); i++){
        if(abs(pts[i] - c) <= r) continue;
        c = pts[i]; r = 0;
        for(int j = 0; j < i; j++){
            if(abs(pts[j] - c) <= r) continue;
            c = (pts[i] + pts[j]) / 2;
            r = abs(pts[i] - c);
            for(int k = 0; k < j; k++){
                if(abs(pts[k] - c) > r)
                    tie(c, r) = circumcenter(pts[i], pts[j], pts[
                    k]);
            }
        }
    }
    return {c, r};
}

```

#### 4.13 3D Point [badbbd]

```

// Copy from 8BQube
struct Point {
    double x, y, z;
    Point(double _x = 0, double _y = 0, double _z = 0): x
    (_x), y(_y), z(_z){}
    Point(pdd p) { x = p.X, y = p.Y, z = abs2(p); }
};
Point operator-(Point p1, Point p2)
{ return Point(p1.x - p2.x, p1.y - p2.y, p1.z - p2.z);
}
Point operator+(Point p1, Point p2)
{ return Point(p1.x + p2.x, p1.y + p2.y, p1.z + p2.z);
}
Point operator*(Point p1, double v)
{ return Point(p1.x * v, p1.y * v, p1.z * v); }
Point operator/(Point p1, double v)
{ return Point(p1.x / v, p1.y / v, p1.z / v); }
Point cross(Point p1, Point p2)
{ return Point(p1.y * p2.z - p1.z * p2.y, p1.z * p2.x -
    p1.x * p2.z, p1.x * p2.y - p1.y * p2.x); }
double dot(Point p1, Point p2)
{ return p1.x * p2.x + p1.y * p2.y + p1.z * p2.z; }
double abs(Point a)
{ return sqrt(dot(a, a)); }
Point cross3(Point a, Point b, Point c)
{ return cross(b - a, c - a); }
double area(Point a, Point b, Point c)
{ return abs(cross3(a, b, c)); }
double volume(Point a, Point b, Point c, Point d)
{ return dot(cross3(a, b, c), d - a); }

```

```
//Azimuthal angle (Longitude) to x-axis in interval [-pi, pi]
double phi(Point p) { return atan2(p.y, p.x); }
//Zenith angle (Latitude) to the z-axis in interval [0, pi]
double theta(Point p) { return atan2(sqrt(p.x * p.x + p.y * p.y), p.z); }
Point masscenter(Point a, Point b, Point c, Point d)
{ return (a + b + c + d) / 4; }
pdd proj(Point a, Point b, Point c, Point u) {
// proj. u to the plane of a, b, and c
Point e1 = b - a;
Point e2 = c - a;
e1 = e1 / abs(e1);
e2 = e2 - e1 * dot(e2, e1);
e2 = e2 / abs(e2);
Point p = u - a;
return pdd(dot(p, e1), dot(p, e2));
}
Point rotate_around(Point p, double angle, Point axis)
{
double s = sin(angle), c = cos(angle);
Point u = axis / abs(axis);
return u * dot(u, p) * (1 - c) + p * c + cross(u, p) * s;
}
```

#### 4.14 ConvexHull3D [156311]

```
struct convex_hull_3D {
struct Face {
int a, b, c;
Face(int ta, int tb, int tc): a(ta), b(tb), c(tc) {}
}; // return the faces with pt indexes
vector<Face> res;
vector<Point> P;
convex_hull_3D(const vector<Point> &_P): res(), P(_P) {
// all points coplanar case will WA, O(n^2)
int n = SZ(P);
if (n <= 2) return; // be careful about edge case
// ensure first 4 points are not coplanar
swap(P[1], *find_if(iter(P), [&](auto p) { return sgn(abs2(P[0] - p)) != 0; }));
swap(P[2], *find_if(iter(P), [&](auto p) { return sgn(abs2(cross3(p, P[0], P[1]))) != 0; }));
swap(P[3], *find_if(iter(P), [&](auto p) { return sgn(volume(P[0], P[1], P[2], p)) != 0; }));
vector<vector<int>> flag(n, vector<int>(n));
res.emplace_back(0, 1, 2); res.emplace_back(2, 1, 0);
for (int i = 3; i < n; ++i) {
vector<Face> next;
for (auto f : res) {
int d = sgn(volume(P[f.a], P[f.b], P[f.c], P[i]));
if (d <= 0) next.pb(f);
int ff = (d > 0) - (d < 0);
flag[f.a][f.b] = flag[f.b][f.c] = flag[f.c][f.a] = ff;
}
for (auto f : res) {
auto F = [&](int x, int y) {
if (flag[x][y] > 0 && flag[y][x] <= 0)
next.emplace_back(x, y, i);
};
F(f.a, f.b); F(f.b, f.c); F(f.c, f.a);
}
res = next;
}
}
bool same(Face s, Face t) {
if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.a])) != 0) return 0;
if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.b])) != 0) return 0;
if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.c])) != 0) return 0;
return 1;
}
int polygon_face_num() {
int ans = 0;
for (int i = 0; i < SZ(res); ++i)
ans += none_of(res.begin(), res.begin() + i, [&](Face g) { return same(res[i], g); });
}
```

```
return ans;
}
double get_volume() {
double ans = 0;
for (auto f : res)
ans += volume(Point(0, 0, 0), P[f.a], P[f.b], P[f.c]);
return fabs(ans / 6);
}
double get_dis(Point p, Face f) {
Point p1 = P[f.a], p2 = P[f.b], p3 = P[f.c];
double a = (p2.y - p1.y) * (p3.z - p1.z) - (p2.z - p1.z) * (p3.y - p1.y);
double b = (p2.z - p1.z) * (p3.x - p1.x) - (p2.x - p1.x) * (p3.z - p1.z);
double c = (p2.x - p1.x) * (p3.y - p1.y) - (p2.y - p1.y) * (p3.x - p1.x);
double d = 0 - (a * p1.x + b * p1.y + c * p1.z);
return fabs(a * p.x + b * p.y + c * p.z + d) / sqrt(a * a + b * b + c * c);
}
};
// n^2 delaunay: facets with negative z normal of
// convexhull of (x, y, x^2 + y^2), use a pseudo-point
// (0, 0, inf) to avoid degenerate case
```

#### 4.15 Delaunay Triangulation [6a9916]

```
/* Delaunay Triangulation:
Given a sets of points on 2D plane, find a
triangulation such that no points will strictly
inside circumcircle of any triangle. */
struct Edge {
int id; // oidx[id]
list<Edge>::iterator twin;
Edge(int _id = 0): id(_id) {}
};
struct Delaunay { // 0-base
int n;
vector<int> oidx;
vector<list<Edge>> head; // result udir. graph
vector<p11> p;
Delaunay(int _n, vector<p11> _p): n(_n), oidx(n),
head(n), p(n) {
iota(iter(oidx), 0);
for (int i = 0; i < n; ++i) head[i].clear();
sort(iter(oidx), [&](int a, int b) {
return _p[a] < _p[b]; });
for (int i = 0; i < n; ++i) p[i] = _p[oidx[i]];
divide(0, n - 1);
}
void addEdge(int u, int v) {
head[u].push_front(Edge(v));
head[v].push_front(Edge(u));
head[u].begin()->twin = head[v].begin();
head[v].begin()->twin = head[u].begin();
}
void divide(int l, int r) {
if (l == r) return;
if (l + 1 == r) return addEdge(l, l + 1);
int mid = (l + r) >> 1, nw[2] = {l, r};
divide(l, mid), divide(mid + 1, r);
auto gao = [&](int t) {
p11 pt[2] = {p[nw[0]], p[nw[1]]};
for (auto it : head[nw[t]]) {
int v = ori(pt[1], pt[0], p[it.id]);
if (v > 0 || (v == 0 && abs2(pt[t ^ 1] - p[it.id]) < abs2(pt[1] - pt[0])))
return nw[t] = it.id, true;
}
return false;
};
while (gao(0) || gao(1));
addEdge(nw[0], nw[1]); // add tangent
while (true) {
p11 pt[2] = {p[nw[0]], p[nw[1]]};
int ch = -1, sd = 0;
for (int t = 0; t < 2; ++t)
for (auto it : head[nw[t]])
if (ori(pt[0], pt[1], p[it.id]) > 0 && (ch == -1 || in_cc({pt[0], pt[1], p[ch]}, p[it.id])))
ch = it.id, sd = t;
}
```

```

    if (ch == -1) break; // upper common tangent
    for (auto it = head[nw[sd]].begin(); it != head[
        nw[sd]].end(); )
        if (seg_strict_intersect(pt[sd], p[it->id], pt[
            sd ^ 1], p[ch]))
            head[it->id].erase(it->twin), head[nw[sd]].
                erase(it++);
        else ++it;
    nw[sd] = ch, addEdge(nw[0], nw[1]);
}
};

```

#### 4.16 Voronoi Diagram [e4f408]

```

// all coord. is even, you may want to call
// halfPlaneInter after then
vector<vector<Line>> vec;
void build_voronoi_line(int n, vector<pll> &pts) {
    Delaunay tool(n, pts); // Delaunay
    vec.clear(), vec.resize(n);
    for (int i = 0; i < n; ++i)
        for (auto e : tool.head[i]) {
            int u = tool.oidx[i], v = tool.oidx[e.id];
            pll m = (pts[v] + pts[u]) / 2LL, d = perp(pts[v]
                - pts[u]);
            vec[u].pb(Line(m, m + d));
        }
}

```

#### 4.17 Polygon Union [9fbf66]

```

// from 8BQube
ld rat(pll a, pll b) {
    return sgn(b.X) ? (ld)a.X / b.X : (ld)a.Y / b.Y;
} // all poly. should be ccw
ld polyUnion(vector<vector<pll>> &poly) {
    ld res = 0;
    for (auto &p : poly)
        for (int a = 0; a < SZ(p); ++a) {
            pll A = p[a], B = p[(a + 1) % SZ(p)];
            vector<pair<ld, int>> segs = {{0, 0}, {1, 0}};
            for (auto &q : poly) {
                if (&p == &q) continue;
                for (int b = 0; b < SZ(q); ++b) {
                    pll C = q[b], D = q[(b + 1) % SZ(q)];
                    int sc = ori(A, B, C), sd = ori(A, B, D);
                    if (sc != sd && min(sc, sd) < 0) {
                        ld sa = cross(D - C, A - C), sb = cross(D -
                            C, B - C);
                        segs.pb(sa / (sa - sb), sgn(sc - sd));
                    }
                    if (!sc && !sd && &q < &p && sgn(dot(B - A, D
                        - C)) > 0) {
                        segs.pb(rat(C - A, B - A), 1);
                        segs.pb(rat(D - A, B - A), -1);
                    }
                }
            }
            sort(iter(segs));
            for (auto &s : segs) s.X = clamp(s.X, 0.0, 1.0);
            ld sum = 0;
            int cnt = segs[0].second;
            for (int j = 1; j < SZ(segs); ++j) {
                if (!cnt) sum += segs[j].X - segs[j - 1].X;
                cnt += segs[j].Y;
            }
            res += cross(A, B) * sum;
        }
    return res / 2;
}

```

#### 4.18 Tangent Point to Convex Hull [523bc1]

```

// from 8BQube
/* The point should be strictly out of hull
   return arbitrary point on the tangent line */
pii get_tangent(vector<pll> &C, pll p) {
    auto gao = [&](int s) {
        return cyc_tsearch(SZ(C), [&](int x, int y)
            { return ori(p, C[x], C[y]) == s; });
    };
    return pii(gao(1), gao(-1));
} // return (a,b), ori(p, C[a], C[b]) >= 0

```

#### 4.19 Heart [082d19]

```

pdd circenter(pdd p0, pdd p1, pdd p2) { // radius = abs
    (center)
    p1 = p1 - p0, p2 = p2 - p0;
    double x1 = p1.X, y1 = p1.Y, x2 = p2.X, y2 = p2.Y;
    double m = 2. * (x1 * y2 - y1 * x2);
    pdd center;
    center.X = (x1 * x1 * y2 - x2 * x2 * y1 + y1 * y2 * (
        y1 - y2)) / m;
    center.Y = (x1 * x2 * (x2 - x1) - y1 * y1 * x2 + x1 *
        y2 * y2) / m;
    return center + p0;
}
pdd incenter(pdd p1, pdd p2, pdd p3) { // radius = area
    / s * 2
    double a = abs(p2 - p3), b = abs(p1 - p3), c = abs(p1
        - p2);
    double s = a + b + c;
    return (a * p1 + b * p2 + c * p3) / s;
}
pdd masscenter(pdd p1, pdd p2, pdd p3)
{ return (p1 + p2 + p3) / 3; }
pdd orthcenter(pdd p1, pdd p2, pdd p3)
{ return masscenter(p1, p2, p3) * 3 - circenter(p1, p2,
    p3) * 2; }

```

#### 4.20 Rotating Sweep Line [f5f689]

```

struct Event {
    pll d; int u, v;
    bool operator<(const Event &b) const {
        int ret = cmp(d, b.d, false);
        return ret == -1 ? false : ret; } // no tie-break
};
void rotatingSweepLine(const vector<pll> &p) {
    const int n = SZ(p);
    vector<Event> e; e.reserve(n * (n - 1));
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++) // pos[i] < pos[j] when
            the event occurs
            if (i != j) e.pb(p[j] - p[i], i, j);
    sort(iter(e));
    vector<int> ord(n), pos(n);
    iota(iter(ord), 0);
    sort(iter(ord), [&](int i, int j) { // initial order
        return p[i].Y != p[j].Y ? p[i].Y < p[j].Y : p[i].
            X < p[j].X; });
    for (int i = 0; i < n; i++) pos[ord[i]] = i;
    // initialize
    for (int i = 0, j = 0; i < SZ(e); i = j) {
        // do something
        vector<pii> tmp;
        for (; j < SZ(e) && !(e[i] < e[j]); j++)
            tmp.pb(pii(e[j].u, e[j].v));
        sort(iter(tmp), [&](pii x, pii y) {
            return pii(pos[x.ff], pos[x.ss]) < pii(pos[y.ff
                ], pos[y.ss]); });
        for (auto [x, y] : tmp) // pos[x] + 1 == pos[y]
            tie(ord[pos[x]], ord[pos[y]], pos[x], pos[y]) =
                make_tuple(ord[pos[y]], ord[pos[x]], pos[y],
                    pos[x]);
    }
}

```

#### 4.21 Vector In Poly [c6d0fa]

```

// ori(a, b, c) >= 0, valid: "strict" angle from a-b to
// a-c
bool btwangle(pll a, pll b, pll c, pll p, int strict) {
    return ori(a, b, p) >= strict && ori(a, p, c) >=
        strict;
}
// whether vector{cur, p} in counter-clockwise order
// prv, cur, nxt
bool inside(pll prv, pll cur, pll nxt, pll p, int
    strict) {
    if (ori(cur, nxt, prv) >= 0)
        return btwangle(cur, nxt, prv, p, strict);
    return !btwangle(cur, prv, nxt, p, !strict);
}

```

## 4.22 Convex Hull DP [92fd4b]

```
sort(iter(pts), [&](p11 x, p11 y) {
    return x.Y != y.Y ? x.Y < y.Y : x.X < y.X;
});
auto getvec = [&](pii x) { return pts[x.ss] - pts[x.ff];
};
// DP for convex hull vertices (no points on edges)
auto solve = [&](int bottom) { // O(n^3)
    p11 o = pts[bottom];
    vector<pii> trans;
    for (int j = bottom + 1; j < n; j++)
        for (int k = bottom + 1; k < n; k++) {
            if (ori(o, pts[j], pts[k]) <= 0) continue;
            // check whether j->k is legal
            trans.pb(pii(j, k));
        }
    sort(iter(trans), [&](pii x, pii y) -> bool{
        int tmp = cmp(getvec(x), getvec(y), false);
        if (tmp != -1) return tmp;
        p11 v = getvec(x);
        return dot(v, pts[x.ff]) > dot(v, pts[y.ff]);
    });
    // vector<LL> dp(n);
    for (int j = bottom + 1; j < n; j++) {
        // check whether bottom -> j is legal
        // init trans -> j
    }
    for (auto [i, j] : trans) {
        // normal trans i -> j
    }
    for (int j = bottom + 1; j < n; j++) {
        // check whether j -> bottom is legal
        // end trans j ->
    }
};
for(int i = 0; i < n; i++) solve(i);
```

## 4.23 Calculate Points in Triangle [bf746f]

```
// all points are distinct
// cnt[i][j] = # of point k s.t. strictly above ij, and
//           i < k < j
// cnt2[i][j] = # of points k s.t. strictly in ij
// preprocess space: O(n^2), time: O(n^3), query time:
//           O(1)
vector cnt(n, vector<int>(n)), cnt2(n, vector<int>(n));
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++) {
        if (pts[i] >= pts[j]) continue;
        for (int k = 0; k < n; k++) {
            if (pts[i] < pts[k] && pts[k] < pts[j]) {
                int tmp = ori(pts[i], pts[j], pts[k]);
                if (tmp > 0) cnt[i][j]++; // only for i < j
                else if (tmp == 0) cnt2[i][j]++, cnt2[j][i]++;
            }
        }
    }
auto calc_tri = [&](array<int, 3> arr) { // strictly
    inside
    sort(iter(arr), [&](int x, int y){ return pts[x] <
        pts[y]; });
    auto [x, y, z] = arr;
    int tmp = ori(pts[x], pts[y], pts[z]);
    if (tmp == 0) return 0;
    else if (tmp < 0)
        return cnt[x][z] - cnt[x][y] - cnt[y][z] - cnt2[x][
            y] - cnt2[y][z] - 1;
    else return cnt[x][y] + cnt[y][z] - cnt[x][z] - cnt2[
        x][z];
};
```

# 5 Graph

## 5.1 BCC [d04ebe]

```
struct BCC{ // 0-based, allow multi edges but not allow
    loops
    int n, m, cnt = 0;
    // n:|V|, m:|E|, cnt:#bcc
    // bcc i : vertices bcc_v[i] and edges bcc_e[i]
    vector<vector<int>> bcc_v, bcc_e;
    vector<vector<pii>> g; // original graph
```

```
vector<pii> edges; // 0-based
BCC(int n, vector<pii> _edges):
    n(_n), m(SZ(_edges)), g(_n), edges(_edges){
    for(int i = 0; i < m; i++){
        auto [u, v] = edges[i];
        g[u].pb(pii(v, i)); g[v].pb(pii(u, i));
    }
}
void make_bcc(){ bcc_v.pb(); bcc_e.pb(); cnt++; }
// modify these if you need more information
void add_v(int v){ bcc_v.back().pb(v); }
void add_e(int e){ bcc_e.back().pb(e); }
void build(){
    vector<int> in(n, -1), low(n, -1), stk;
    vector<vector<int>> up(n);
    int ts = 0;
    auto _dfs = [&](auto dfs, int now, int par, int pe)
        -> void{
        if(pe != -1) up[now].pb(pe);
        in[now] = low[now] = ts++;
        stk.pb(now);
        for(auto [v, e] : g[now]){
            if(e == pe) continue;
            if(in[v] != -1){
                if(in[v] < in[now]) up[now].pb(e);
                low[now] = min(low[now], in[v]);
                continue;
            }
            dfs(dfs, v, now, e);
            low[now] = min(low[now], low[v]);
        }
        if((now != par && low[now] >= in[par]) || (now ==
            par && SZ(g[now]) == 0)){
            make_bcc();
            for(int v = stk.back(); v = stk.back()){
                stk.pop_back(), add_v(v);
                for(int e : up[v]) add_e(e);
                if(v == now) break;
            }
            if(now != par) add_v(par);
        }
    };
    for(int i = 0; i < n; i++)
        if(in[i] == -1) _dfs(_dfs, i, i, -1);
};
```

## 5.2 SCC [2c9a01]

```
struct SCC{ // 0-based, output reversed topo order
    int n, cnt = 0;
    vector<vector<int>> g;
    vector<int> sccid;
    explicit SCC(int _n): n(_n), g(n), sccid(n, -1) {}
    void add_edge(int u, int v){
        g[u].pb(v);
    }
    void build(){
        vector<int> in(n, -1), low(n), stk;
        vector<bool> instk(n);
        int ts = 0;
        auto dfs1 = [&](auto dfs, int now) -> void{
            stk.pb(now); instk[now] = true;
            in[now] = low[now] = ts++;
            for(int i : g[now]){
                if(in[i] == -1)
                    dfs(dfs, i), low[now] = min(low[now], low[i]);
                else if(instk[i] && in[i] < in[now])
                    low[now] = min(low[now], in[i]);
            }
            if(low[now] == in[now]){
                for(; stk.back() != now; stk.pop_back())
                    sccid[stk.back()] = cnt, instk[stk.back()] =
                    false;
                sccid[now] = cnt++, instk[now] = false, stk.
                    pop_back();
            }
        };
        for(int i = 0; i < n; i++)
            if(in[i] == -1) dfs1(dfs1, i);
    };
};
```

### 5.3 2-SAT [0686a5]

```
struct SAT { // 0-based
    int n;
    vector<bool> istrue;
    SCC scc;
    SAT(int _n): n(_n), istrue(n + n), scc(n + n) {}
    int neg(int a) {
        return a >= n ? a - n : a + n;
    }
    void add_clause(int a, int b) {
        scc.add_edge(neg(a), b), scc.add_edge(neg(b), a);
    }
    bool solve() {
        scc.build();
        for (int i = 0; i < n; ++i) {
            if (scc.sccid[i] == scc.sccid[i + n]) return false;
            istrue[i] = scc.sccid[i] < scc.sccid[i + n];
            istrue[i + n] = !istrue[i];
        }
        return true;
    }
};
```

### 5.4 Dominator Tree [2da9bb]

```
struct Dominator {
    int n;
    vector<vector<int>> g, r, rdom; int tk;
    vector<int> dfn, rev, fa, sdom, dom, val, rp;
    Dominator(int _n) : n(_n), g(n), r(n), rdom(n), tk(0) {
        dfn = rev = fa = sdom = dom =
        val = rp = vector<int>(n, -1); }
    void add_edge(int x, int y) { g[x].push_back(y); }
    void dfs(int x) {
        rev[dfn[x] = tk] = x;
        fa[tk] = sdom[tk] = val[tk] = tk; tk++;
        for (int u : g[x]) {
            if (dfn[u] == -1) dfs(u), rp[dfn[u]] = dfn[x];
            r[dfn[u]].push_back(dfn[x]);
        }
    }
    void merge(int x, int y) { fa[x] = y; }
    int find(int x, int c = 0) {
        if (fa[x] == x) return c ? -1 : x;
        if (int p = find(fa[x], 1); p != -1) {
            if (sdom[val[x]] > sdom[val[fa[x]]])
                val[x] = val[fa[x]];
            fa[x] = p;
            return c ? p : val[x];
        } else return c ? fa[x] : val[x];
    }
    vector<int> build(int s) {
        // return the father of each node in dominator tree
        dfs(s); // p[i] = -2 if i is unreachable, par[s] = -1
        for (int i = tk - 1; i >= 0; --i) {
            for (int u : r[i])
                sdom[i] = min(sdom[i], sdom[find(u)]);
            if (i) rdom[sdom[i]].push_back(i);
            for (int u : rdom[i]) {
                int p = find(u);
                dom[u] = (sdom[p] == i ? i : p);
            }
            if (i) merge(i, rp[i]);
        }
        vector<int> p(n, -2); p[s] = -1;
        for (int i = 1; i < tk; ++i)
            if (sdom[i] != dom[i]) dom[i] = dom[dom[i]];
        for (int i = 1; i < tk; ++i)
            p[rev[i]] = rev[dom[i]];
        return p;
    }
};
```

### 5.5 Virtual Tree [f977f8]

```
// copy from 8BQube
vector<int> vG[N];
int top, st[N];
int vrt = -1;
```

```
void insert(int u) {
    if (top == -1) return st[++top] = vrt = u, void();
    int p = LCA(st[top], u);
    if (dep[vrt] > dep[p]) vrt = p;
    if (p == st[top]) return st[++top] = u, void();
    while (top >= 1 && dep[st[top - 1]] >= dep[p])
        vG[st[top - 1]].pb(st[top]), --top;
    if (st[top] != p)
        vG[p].pb(st[top]), --top, st[++top] = p;
    st[++top] = u;
}

void reset(int u) {
    for (int i : vG[u]) reset(i);
    vG[u].clear();
}

void solve(vector<int> &v) {
    top = -1;
    sort(ALL(v),
        [&](int a, int b) { return dfn[a] < dfn[b]; });
    for (int i : v) insert(i);
    while (top > 0) vG[st[top - 1]].pb(st[top]), --top;
    // do something
    reset(vrt);
}
```

### 5.6 Fast DMST [7b274d]

```
struct E { int s, t; ll w; }; // 0-base
struct PQ {
    struct P {
        ll v; int i;
        bool operator>(const P &b) const { return v > b.v; }
    };
    priority_queue<P, vector<P>, greater<>> pq; ll tag;
    // min heap
    void push(P p) { p.v -= tag; pq.emplace(p); }
    P top() { P p = pq.top(); p.v += tag; return p; }
    void join(PQ &b) {
        if (pq.size() < b.pq.size())
            swap(pq, b.pq), swap(tag, b.tag);
        while (!b.pq.empty()) push(b.top()), b.pq.pop();
    }
}; // O(E log^2 V), use Leftist tree for O(E log V)
vector<int> dmst(const vector<E> &e, int n, int root) {
    vector<PQ> h(n * 2);
    for (int i = 0; i < int(e.size()); ++i)
        h[e[i].t].push({e[i].w, i});
    vector<int> a(n * 2); iota(iter(a), 0);
    vector<int> v(n * 2, -1), pa(n * 2, -1), r(n * 2);
    auto o = [&](auto Y, int x) -> int {
        return x == a[x] ? x : a[x] = Y(Y, a[x]); };
    auto S = [&](int i) { return o(o, e[i].s); };
    int pc = v[root] = n;
    for (int i = 0; i < n; ++i) if (v[i] == -1)
        for (int p = i; v[p] < 0 || v[p] == i; p = S(r[p])) {
            if (v[p] == i)
                for (int q = pc++; p != q; p = S(r[p])) {
                    h[p].tag -= h[p].top().v; h[q].join(h[p]);
                    pa[p] = a[p] = q;
                }
            while (S(h[p].top().i) == p) h[p].pq.pop();
            v[p] = i; r[p] = h[p].top().i;
        }
    vector<int> ans;
    for (int i = pc - 1; i >= 0; i--) if (v[i] != n) {
        for (int f = e[r[i]].t; f != -1 && v[f] != n; f = pa[f])
            v[f] = n;
        ans.push_back(r[i]);
    }
    return ans; // default minimize, returns edgeid array
}
```

### 5.7 Vizing [58a6ca]

```
// find D+1 edge coloring of a graph with max deg D, O(
nm)
struct Vizing { // returns maxdeg+1 edge coloring in
    adjacent matrix G
```



```

int n; // 1-based for vertices and colors, simple
graph
vector<vector<int>> C, G;
vector<int> X, vst;
Vizing(int _n): n(_n),
C(n + 1, vector<int>(n + 2)), G(n + 1, vector<int>(n
+ 1)),
X(n + 1, 1), vst(n + 1) {}
void solve(vector<pii> &E) {
    auto update = [&](int u)
    { for (X[u] = 1; C[u][X[u]]; ++X[u]); };
    auto color = [&](int u, int v, int c) {
        int p = G[u][v];
        G[u][v] = G[v][u] = c;
        C[u][c] = v, C[v][c] = u;
        C[u][p] = C[v][p] = 0;
        if (p) X[u] = X[v] = p;
        else update(u), update(v);
        return p;
    };
    auto flip = [&](int u, int c1, int c2) {
        int p = C[u][c1];
        swap(C[u][c1], C[u][c2]);
        if (p) G[u][p] = G[p][u] = c2;
        if (!C[u][c1]) X[u] = c1;
        if (!C[u][c2]) X[u] = c2;
        return p;
    };
    for (int t = 0; t < SZ(E); ++t) {
        int u = E[t].ff, v0 = E[t].ss, v = v0, c0 = X[u],
        c = c0, d;
        vector<pii> L;
        fill(iter(vst), 0);
        while (!G[u][v0]) {
            L.emplace_back(v, d = X[v]);
            if (!C[v][c]) for (int a = SZ(L) - 1; a >= 0;
                --a) c = color(u, L[a].ff, c);
            else if (!C[u][d]) for (int a = SZ(L) - 1; a >=
                0; --a) color(u, L[a].ff, L[a].ss);
            else if (vst[d]) break;
            else vst[d] = 1, v = C[u][d];
        }
        if (!G[u][v0]) {
            for (; v; v = flip(v, c, d), swap(c, d));
            if (int a; C[u][c0]) {
                for (a = SZ(L) - 2; a >= 0 && L[a].ss != c;
                    --a);
                for (; a >= 0; --a) color(u, L[a].ff, L[a].ss
                );
            }
            else --t;
        }
    }
}
};

```

## 5.8 Maximum Clique [d50aa9]

```

struct MaxClique { // fast when N <= 100
    bitset<N> G[N], cs[N];
    int ans, sol[N], q, cur[N], d[N], n;
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) G[i].reset();
    }
    void add_edge(int u, int v) {
        G[u][v] = G[v][u] = 1;
    }
    void pre_dfs(vector<int> &r, int l, bitset<N> mask) {
        if (l < 4) {
            for (int i : r) d[i] = (G[i] & mask).count();
            sort(ALL(r), [&](int x, int y) { return d[x] > d[
                y]; });
        }
        vector<int> c(SZ(r));
        int lft = max(ans - q + 1, 1), rgt = 1, tp = 0;
        cs[1].reset(), cs[2].reset();
        for (int p : r) {
            int k = 1;
            while ((cs[k] & G[p]).any()) ++k;
            if (k > rgt) cs[++rgt + 1].reset();
            cs[k][p] = 1;
            if (k < lft) r[tp++] = p;
        }
    }
};

```

```

}
for (int k = lft; k <= rgt; ++k)
    for (int p = cs[k]._Find_first(); p < N; p = cs[k]
        ]._Find_next(p))
        r[tp] = p, c[tp] = k, ++tp;
dfs(r, c, l + 1, mask);
}
void dfs(vector<int> &r, vector<int> &c, int l,
    bitset<N> mask) {
    while (!r.empty()) {
        int p = r.back();
        r.pop_back(), mask[p] = 0;
        if (q + c.back() <= ans) return;
        cur[q++] = p;
        vector<int> nr;
        for (int i : r) if (G[p][i]) nr.pb(i);
        if (!nr.empty()) pre_dfs(nr, l, mask & G[p]);
        else if (q > ans) ans = q, copy_n(cur, q, sol);
        c.pop_back(), --q;
    }
}
int solve() {
    vector<int> r(n);
    ans = q = 0, iota(ALL(r), 0);
    pre_dfs(r, 0, bitset<N>(string(n, '1')));
    return ans;
}
};

```

## 5.9 Number of Maximal Clique [11fa26]

```

struct BronKerbosch { // 1-base
    int n, a[N], g[N][N];
    int S, all[N][N], some[N][N], none[N][N];
    void init(int _n) {
        n = _n;
        for (int i = 1; i <= n; ++i)
            for (int j = 1; j <= n; ++j) g[i][j] = 0;
    }
    void add_edge(int u, int v) {
        g[u][v] = g[v][u] = 1;
    }
    void dfs(int d, int an, int sn, int nn) {
        if (S > 1000) return; // pruning
        if (sn == 0 && nn == 0) ++S;
        int u = some[d][0];
        for (int i = 0; i < sn; ++i) {
            int v = some[d][i];
            if (g[u][v]) continue;
            int tsu = 0, tnn = 0;
            copy_n(all[d], an, all[d + 1]);
            all[d + 1][an] = v;
            for (int j = 0; j < sn; ++j)
                if (g[v][some[d][j]])
                    some[d + 1][tsu++] = some[d][j];
            for (int j = 0; j < nn; ++j)
                if (g[v][none[d][j]])
                    none[d + 1][tnn++] = none[d][j];
            dfs(d + 1, an + 1, tsu, tnn);
            some[d][i] = 0, none[d][nn++] = v;
        }
    }
    int solve() {
        iota(some[0], some[0] + n, 1);
        S = 0, dfs(0, 0, n, 0);
        return S;
    }
};

```

## 5.10 Minimum Mean Cycle [3e5d2b]

```

// from 8BQube
ll road[N][N]; // input here
struct MinimumMeanCycle {
    ll dp[N + 5][N], n;
    pll solve() {
        ll a = -1, b = -1, L = n + 1;
        for (int i = 2; i <= L; ++i)
            for (int k = 0; k < n; ++k)
                for (int j = 0; j < n; ++j)
                    dp[i][j] =
                        min(dp[i - 1][k] + road[k][j], dp[i][j]);
        for (int i = 0; i < n; ++i) {

```

```

    if (dp[L][i] >= INF) continue;
    ll ta = 0, tb = 1;
    for (int j = 1; j < n; ++j)
        if (dp[j][i] < INF &&
            ta * (L - j) < (dp[L][i] - dp[j][i]) * tb)
            ta = dp[L][i] - dp[j][i], tb = L - j;
    if (ta == 0) continue;
    if (a == -1 || a * tb > ta * b) a = ta, b = tb;
}
if (a != -1) {
    ll g = __gcd(a, b);
    return pll(a / g, b / g);
}
return pll(-1LL, -1LL);
}
void init(int _n) {
    n = _n;
    for (int i = 0; i < n; ++i)
        for (int j = 0; j < n; ++j) dp[i + 2][j] = INF;
}
};

```

## 5.11 Minimum Steiner Tree [21acea]

```

// from 8BQube
// O(V 3^T + V^2 2^T)
struct SteinerTree { // 0-base
    static const int T = 10, N = 105, INF = 1e9;
    int n, dst[N][N], dp[1 << T][N], tdst[N];
    int vcost[N]; // the cost of vertices
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) {
            for (int j = 0; j < n; ++j) dst[i][j] = INF;
            dst[i][i] = vcost[i] = 0;
        }
    }
    void add_edge(int ui, int vi, int wi) {
        dst[ui][vi] = min(dst[ui][vi], wi);
    }
    void shortest_path() {
        for (int k = 0; k < n; ++k)
            for (int i = 0; i < n; ++i)
                for (int j = 0; j < n; ++j)
                    dst[i][j] =
                        min(dst[i][j], dst[i][k] + dst[k][j]);
    }
    int solve(const vector<int> &ter) {
        shortest_path();
        int t = SZ(ter);
        for (int i = 0; i < (1 << t); ++i)
            for (int j = 0; j < n; ++j) dp[i][j] = INF;
        for (int i = 0; i < n; ++i) dp[0][i] = vcost[i];
        for (int msk = 1; msk < (1 << t); ++msk) {
            if (!(msk & (msk - 1))) {
                int who = __lg(msk);
                for (int i = 0; i < n; ++i)
                    dp[msk][i] =
                        vcost[ter[who]] + dst[ter[who]][i];
            }
            for (int i = 0; i < n; ++i)
                for (int submsk = (msk - 1) & msk; submsk;
                    submsk = (submsk - 1) & msk)
                    dp[msk][i] = min(dp[msk][i],
                        dp[submsk][i] + dp[msk ^ submsk][i] -
                        vcost[i]);
            for (int i = 0; i < n; ++i) {
                tdst[i] = INF;
                for (int j = 0; j < n; ++j)
                    tdst[i] =
                        min(tdst[i], dp[msk][j] + dst[j][i]);
            }
            for (int i = 0; i < n; ++i) dp[msk][i] = tdst[i];
        }
        int ans = INF;
        for (int i = 0; i < n; ++i)
            ans = min(ans, dp[(1 << t) - 1][i]);
        return ans;
    }
};

```

## 5.12 Count Cycles [c7e8f2]

```

// ord = sort by deg decreasing, rk[ord[i]] = i
// D[i] = edge point from rk small to rk big
for (int x : ord) { // c3
    for (int y : D[x]) vis[y] = 1;
    for (int y : D[x]) for (int z : D[y]) c3 += vis[z];
    for (int y : D[x]) vis[y] = 0;
}
for (int x : ord) { // c4
    for (int y : D[x]) for (int z : adj[y])
        if (rk[z] > rk[x]) c4 += vis[z]++;
    for (int y : D[x]) for (int z : adj[y])
        if (rk[z] > rk[x]) --vis[z];
} // both are O(M*sqrt(M))

```

## 6 Math

### 6.1 Extended Euclidean Algorithm [c51ae9]

```

// ax+ny = 1, ax+ny == ax == 1 (mod n)
void extgcd(ll x, ll y, ll &g, ll &a, ll &b) {
    if (y == 0) g = x, a = 1, b = 0;
    else extgcd(y, x % y, g, b, a), b -= (x / y) * a;
}

```

### 6.2 Floor & Ceil [134881]

```

ll ifloor(ll a, ll b) {
    return a / b - (a % b && (a < 0) ^ (b < 0));
}
ll iceil(ll a, ll b) {
    return a / b + (a % b && (a < 0) ^ (b > 0));
}

```

### 6.3 Legendre [4e4b23]

```

// the Jacobi symbol is a generalization of the
// Legendre symbol,
// such that the bottom doesn't need to be prime.
// (n|p) -> same as Legendre
// (n|ab) = (n|a)(n|b)
// work with long long
int Jacobi(int a, int m) {
    int s = 1;
    for (; m > 1; ) {
        a %= m;
        if (a == 0) return 0;
        const int r = __builtin_ctz(a);
        if ((r & 1) && ((m + 2) & 4)) s = -s;
        a >>= r;
        if (a & m & 2) s = -s;
        swap(a, m);
    }
    return s;
}

// 0: a == 0
// -1: a isn't a quad res of p
// else: return X with X^2 % p == a
// doesn't work with long long
int QuadraticResidue(int a, int p) {
    if (p == 2) return a & 1;
    if (int jc = Jacobi(a, p); jc <= 0) return jc;
    int b, d;
    for (; ; ) {
        b = rand() % p;
        d = (1LL * b * b + p - a) % p;
        if (Jacobi(d, p) == -1) break;
    }
    int f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
    for (int e = (1LL + p) >> 1; e; e >>= 1) {
        if (e & 1) {
            tmp = (1LL * g0 * f0 + 1LL * d * (1LL * g1 * f1 %
                p)) % p;
            g1 = (1LL * g0 * f1 + 1LL * g1 * f0) % p;
            g0 = tmp;
        }
        tmp = (1LL * f0 * f0 + 1LL * d * (1LL * f1 * f1 % p
            )) % p;
        f1 = (2LL * f0 * f1) % p;
        f0 = tmp;
    }
    return g0;
}

```

## 6.4 Simplex [aa7741]

```
// maximize c^T x
// subject to Ax <= b, x >= 0
// and stores the solution;
typedef long double T; // Long double, Rational, double
+ mod<P>...
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-9, inf = 1/.0;
#define ltj(X) if(s == -1 || mp(X[j],N[j]) < mp(X[s],N[s])) s=j
#define rep(i, 1, n) for(int i = 1; i < n; i++)

struct LPSolver {
    int m, n;
    vector<int> N, B;
    vvd D;

    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(SZ(b)), n(SZ(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
        rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
        rep(i,0,m) { B[i] = n+1; D[i][n] = -1; D[i][n+1] = b[i]; }
        rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m+1][n] = 1;
    }

    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
        rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
            T *b = D[i].data(), inv2 = b[s] * inv;
            rep(j,0,n+2) b[j] -= a[j] * inv2;
            b[s] = a[s] * inv2;
        }
        rep(j,0,n+2) if (j != s) D[r][j] *= inv;
        rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]);
    }

    bool simplex(int phase) {
        int x = m + phase - 1;
        for (;;) {
            int s = -1;
            rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
            if (D[x][s] >= -eps) return true;
            int r = -1;
            rep(i,0,m) {
                if (D[i][s] <= eps) continue;
                if (r == -1 || mp(D[i][n+1] / D[i][s], B[i]) < mp(D[r][n+1] / D[r][s], B[r])) r = i;
            }
            if (r == -1) return false;
            pivot(r, s);
        }
    }

    T solve(vd &x) {
        int r = 0;
        rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
        if (D[r][n+1] < -eps) {
            pivot(r, n);
            if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
        }
        rep(i,0,m) if (B[i] == -1) {
            int s = 0;
            rep(j,1,n+1) ltj(D[i]);
            pivot(i, s);
        }
        bool ok = simplex(1); x = vd(n);
        rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
        return ok ? D[m][n+1] : inf;
    }
};
```

## 6.5 Simplex Construction

Standard form: maximize  $\sum_{1 \leq i \leq n} c_i x_i$  such that  $\sum_{1 \leq i \leq n} A_{ji} x_i \leq b_j$  for all  $1 \leq j \leq m$  and  $x_i \geq 0$  for all  $1 \leq i \leq n$ .

1. In case of minimization, let  $c'_i = -c_i$

- $\sum_{1 \leq i \leq n} A_{ji} x_i \geq b_j \rightarrow \sum_{1 \leq i \leq n} -A_{ji} x_i \leq -b_j$
- $\sum_{1 \leq i \leq n} A_{ji} x_i = b_j \rightarrow \text{add } \leq \text{ and } \geq$ .
- If  $x_i$  has no lower bound, replace  $x_i$  with  $x_i - x'_i$

## 6.6 DiscreteLog [da27bf]

```
int DiscreteLog(int s, int x, int y, int m) {
    constexpr int kStep = 32000;
    unordered_map<int, int> p;
    int b = 1;
    for (int i = 0; i < kStep; ++i) {
        p[y] = i;
        y = 1LL * y * x % m;
        b = 1LL * b * x % m;
    }
    for (int i = 0; i < m + 10; i += kStep) {
        s = 1LL * s * b % m;
        if (p.find(s) != p.end()) return i + kStep - p[s];
    }
    return -1;
}

int DiscreteLog(int x, int y, int m) {
    if (m == 1) return 0;
    int s = 1;
    for (int i = 0; i < 100; ++i) {
        if (s == y) return i;
        s = 1LL * s * x % m;
    }
    if (s == y) return 100;
    int p = 100 + DiscreteLog(s, x, y, m);
    if (fpow(x, p, m) != y) return -1;
    return p; // returns: x^p = y (mod m)
}
```

## 6.7 Miller Rabin & Pollard Rho [d3ecd2]

```
// n < 4,759,123,141      3 : 2, 7, 61
// n < 1,122,004,669,633  4 : 2, 13, 23, 1662803
// n < 3,474,749,660,383  6 : primes <= 13
// n < 2^64              7 :
// 2, 325, 9375, 28178, 450775, 9780504, 1795265022
ll mul(ll a, ll b, ll n){
    return (__int128)a * b % n;
}

bool Miller_Rabin(ll a, ll n) {
    if ((a = a % n) == 0) return 1;
    if (n % 2 == 0) return n == 2;
    ll tmp = (n - 1) / ((n - 1) & (1 - n));
    ll t = __lg(((n - 1) & (1 - n))), x = 1;
    for (; tmp; tmp >>= 1, a = mul(a, a, n))
        if (tmp & 1) x = mul(x, a, n);
    if (x == 1 || x == n - 1) return 1;
    while (--t)
        if ((x = mul(x, x, n)) == n - 1) return 1;
    return 0;
}

bool prime(ll n){
    vector<ll> tmp = {2, 325, 9375, 28178, 450775,
        9780504, 1795265022};
    for(ll i : tmp)
        if(!Miller_Rabin(i, n)) return false;
    return true;
}

map<ll, int> cnt;
void PollardRho(ll n) {
    if (n == 1) return;
    if (prime(n)) return ++cnt[n], void();
    if (n % 2 == 0) return PollardRho(n / 2), ++cnt[2], void();
    ll x = 2, y = 2, d = 1, p = 1;
    #define f(x, n, p) ((mul(x, x, n) + p) % n)
    while (true) {
        if (d != n && d != 1) {
            PollardRho(n / d);
            PollardRho(d);
            return;
        }
        if (d == n) ++p;
        x = f(x, n, p), y = f(f(y, n, p), n, p);
        d = gcd(abs(x - y), n);
    }
}
```

## 6.8 XOR Basis [006505]

```
const int digit = 60; // [0, 2^digit)
struct Basis{
    int total = 0, rank = 0;
    vector<ll> b;
    Basis(): b(digit) {}
    bool add(ll v){ // Gauss Jordan Elimination
        total++;
        for(int i = digit - 1; i >= 0; i--){
            if(!(1LL << i & v)) continue;
            if(b[i] != 0){
                v ^= b[i];
                continue;
            }
            for(int j = 0; j < i; j++){
                if(1LL << j & v) v ^= b[j];
            }
            for(int j = i + 1; j < digit; j++){
                if(1LL << i & b[j]) b[j] ^= v;
            }
            b[i] = v;
            rank++;
            return true;
        }
        return false;
    }
    ll getMax(ll x = 0){
        for(ll i : b) x = max(x, x ^ i);
        return x;
    }
    ll getMin(ll x = 0){
        for(ll i : b) x = min(x, x ^ i);
        return x;
    }
    bool can(ll x){
        return getMin(x) == 0;
    }
    ll kth(ll k){ // kth smallest, 0-indexed
        vector<ll> tmp;
        for(ll i : b) if(i) tmp.pb(i);
        ll ans = 0;
        for(int i = 0; i < SZ(tmp); i++){
            if(1LL << i & k) ans ^= tmp[i];
        }
        return ans;
    }
};
```

## 6.9 Linear Equation [ab487b]

```
vector<int> RREF(vector<vector<ll>> &mat){
    int N = mat.size(), M = mat[0].size();
    int rk = 0;
    vector<int> cols;
    for (int i = 0; i < M; i++) {
        int cnt = -1;
        for (int j = N-1; j >= rk; j--){
            if(mat[j][i] != 0) cnt = j;
        }
        if(cnt == -1) continue;
        swap(mat[rk], mat[cnt]);
        ll lead = mat[rk][i];
        for (int j = 0; j < M; j++) mat[rk][j] = mat[rk][j] *
            modinv(lead) % mod;
        for (int j = 0; j < N; j++) {
            if(j == rk) continue;
            ll tmp = mat[j][i];
            for (int k = 0; k < M; k++)
                mat[j][k] = (mat[j][k] - mat[rk][k] * tmp % mod
                    + mod) % mod;
        }
        cols.pb(i);
        rk++;
    }
    return cols;
}
struct LinearEquation{
    bool ok;
    vector<ll> par; //particular solution (Ax = b)
    vector<vector<ll>> homo; //homogenous (Ax = 0)
    vector<vector<ll>> rref;
    //first M columns are matrix A
    //last column of eq is vector b
    void solve(const vector<vector<ll>> &eq){
        int M = (int)eq[0].size() - 1;
        rref = eq;
```

```
        auto piv = RREF(rref);
        int rk = piv.size();
        if(piv.size() && piv.back() == M){
            ok = 0; return;
        }
        ok = 1;
        par.resize(M);
        vector<bool> ispiv(M);
        for (int i = 0; i < rk; i++) {
            par[piv[i]] = rref[i][M];
            ispiv[piv[i]] = 1;
        }
        for (int i = 0; i < M; i++) {
            if (ispiv[i]) continue;
            vector<ll> h(M);
            h[i] = 1;
            for (int j = 0; j < rk; j++) h[piv[j]] = rref[j][i]
                ? mod-rref[j][i] : 0;
            homo.pb(h);
        }
    }
};
```

## 6.10 Chinese Remainder Theorem [6ef4a3]

```
pll solve_crt(ll x1, ll m1, ll x2, ll m2){
    ll g = gcd(m1, m2);
    if ((x2 - x1) % g) return {0, 0}; // no sol
    m1 /= g; m2 /= g;
    ll _, p, q;
    extgcd(m1, m2, _, p, q); // p <= C
    ll lcm = m1 * m2 * g;
    ll res = ((int128)p * (x2 - x1) % lcm * m1 % lcm +
        x1) % lcm;
    // be careful with overflow, C^3
    return {(res + lcm) % lcm, lcm}; // (x, m)
}
```

## 6.11 Sqrt Decomposition [8d7bc0]

```
// for all i in [L, r], floor(n / i) = x
for(int l = 1, r; l <= n; l = r + 1){
    int x = ifloor(n, l);
    r = ifloor(n, x);
}
// for all i in [L, r], ceil(n / i) = x
for(int l = 1, r = n; r >= 1; r = l - 1){
    int x = iceil(n, r);
    l = iceil(n, x);
}
```

## 6.12 Floor Sum

- $m = \lfloor \frac{an+b}{c} \rfloor$
- Time complexity:  $O(\log n)$

$$f(a, b, c, n) = \sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$$

$$= \begin{cases} \lfloor \frac{a}{c} \rfloor \cdot \frac{n(n+1)}{2} + \lfloor \frac{b}{c} \rfloor \cdot (n+1) \\ + f(a \bmod c, b \bmod c, c, n), & a \geq c \vee b \geq c \\ 0, & n < 0 \vee a = 0 \\ nm - f(c, c-b-1, a, m-1), & \text{otherwise} \end{cases}$$

$$g(a, b, c, n) = \sum_{i=0}^n i \lfloor \frac{ai+b}{c} \rfloor$$

$$= \begin{cases} \lfloor \frac{a}{c} \rfloor \cdot \frac{n(n+1)(2n+1)}{6} + \lfloor \frac{b}{c} \rfloor \cdot \frac{n(n+1)}{2} \\ + g(a \bmod c, b \bmod c, c, n), & a \geq c \vee b \geq c \\ 0, & n < 0 \vee a = 0 \\ \frac{1}{2} \cdot (n(n+1)m - f(c, c-b-1, a, m-1) \\ - h(c, c-b-1, a, m-1)), & \text{otherwise} \end{cases}$$

$$h(a, b, c, n) = \sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor^2$$

$$= \begin{cases} \lfloor \frac{a}{c} \rfloor^2 \cdot \frac{n(n+1)(2n+1)}{6} + \lfloor \frac{b}{c} \rfloor^2 \cdot (n+1) \\ + \lfloor \frac{a}{c} \rfloor \cdot \lfloor \frac{b}{c} \rfloor \cdot n(n+1) \\ + h(a \bmod c, b \bmod c, c, n) \\ + 2 \lfloor \frac{a}{c} \rfloor \cdot g(a \bmod c, b \bmod c, c, n) \\ + 2 \lfloor \frac{b}{c} \rfloor \cdot f(a \bmod c, b \bmod c, c, n), & a \geq c \vee b \geq c \\ 0, & n < 0 \vee a = 0 \\ nm(m+1) - 2g(c, c-b-1, a, m-1) \\ - 2f(c, c-b-1, a, m-1) - f(a, b, c, n), & \text{otherwise} \end{cases}$$

## 7 Polynomial

### 7.1 FWHT [c9cdb6]

```

/* x: a[j], y: a[j + (L >> 1)]
or: (y += x * op), and: (x += y * op)
xor: (x, y = (x + y) * op, (x - y) * op)
invop: or, and, xor = -1, -1, 1/2 */
void fwt(int *a, int n, int op) { //or
    for (int L = 2; L <= n; L <= 1)
        for (int i = 0; i < n; i += L)
            for (int j = i; j < i + (L >> 1); ++j)
                a[j + (L >> 1)] += a[j] * op;
}
const int N = 21;
int f[N][1 << N], g[N][1 << N], h[N][1 << N], ct[1 << N];
void subset_convolution(int *a, int *b, int *c, int L)
{
    // c_k = \sum_{i+j=k, i&j=0} a_i * b_j
    int n = 1 << L;
    for (int i = 1; i < n; ++i)
        ct[i] = ct[i & (i - 1)] + 1;
    for (int i = 0; i < n; ++i)
        f[ct[i]][i] = a[i], g[ct[i]][i] = b[i];
    for (int i = 0; i <= L; ++i)
        fwt(f[i], n, 1), fwt(g[i], n, 1);
    for (int i = 0; i <= L; ++i)
        for (int j = 0; j <= i; ++j)
            for (int x = 0; x < n; ++x)
                h[i][x] += f[j][x] * g[i - j][x];
    for (int i = 0; i <= L; ++i) fwt(h[i], n, -1);
    for (int i = 0; i < n; ++i) c[i] = h[ct[i]][i];
}

```

### 7.2 FFT [13ec2f]

// Errichto: FFT for double works when the result < 1e15, and < 1e18 with long double

```

using val_t = complex<double>;
template<int MAXN>
struct FFT {
    const double PI = acos(-1);
    val_t w[MAXN];
    FFT() {
        for (int i = 0; i < MAXN; ++i) {
            double arg = 2 * PI * i / MAXN;
            w[i] = val_t(cos(arg), sin(arg));
        }
    }
    void bitrev(vector<val_t> &a, int n) //same as NTT
    void trans(vector<val_t> &a, int n, bool inv = false)
    {
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
                for (int j = i, x = 0; j < i + dl; ++j, x += dx)
                {
                    val_t tmp = a[j + dl] * (inv ? conj(w[x]) : w[x]);
                    a[j + dl] = a[j] - tmp;
                    a[j] += tmp;
                }
            }
        }
        if (inv) {
            for (int i = 0; i < n; ++i) a[i] /= n;
        }
    }
    //multiplying two polynomials A * B:
    //fft.trans(A, siz, 0), fft.trans(B, siz, 0);
    //A[i] *= B[i], fft.trans(A, siz, 1);
};

```

### 7.3 NTT [bf683f]

```

//(2^16)+1, 65537, 3
//7*17*(2^23)+1, 998244353, 3
//1255*(2^20)+1, 1315962881, 3
//51*(2^25)+1, 1711276033, 29

```

```

// only works when sz(A) + sz(B) - 1 <= MAXN
template<int MAXN, ll P, ll RT> //MAXN must be 2^k
struct NTT {
    ll w[MAXN];
    ll mpow(ll a, ll n);
    ll minv(ll a) { return mpow(a, P - 2); }
    NTT() {
        ll dw = mpow(RT, (P - 1) / MAXN);
        w[0] = 1;
        for (int i = 1; i < MAXN; ++i) w[i] = w[i - 1] * dw % P;
    }
    void bitrev(vector<ll> &a, int n) {
        int i = 0;
        for (int j = 1; j < n - 1; ++j) {
            for (int k = n >> 1; (i ^ k) < k; k >>= 1);
            if (j < i) swap(a[i], a[j]);
        }
    }
    void operator()(vector<ll> &a, int n, bool inv = false) { //0 <= a[i] < P
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
                for (int j = i, x = 0; j < i + dl; ++j, x += dx)
                {
                    ll tmp = a[j + dl] * w[x] % P;
                    if ((a[j + dl] = a[j] - tmp) < 0) a[j + dl] += P;
                    if ((a[j] += tmp) >= P) a[j] -= P;
                }
            }
        }
        if (inv) {
            reverse(a.begin() + 1, a.begin() + n);
            ll invn = minv(n);
            for (int i = 0; i < n; ++i) a[i] = a[i] * invn % P;
        }
    }
};

```

### 7.4 Polynomial Operation [77a8a8]

```

// Copy from 8BQube
#define fi(s, n) for (int i = (int)(s); i < (int)(n); ++i)
#define neg(x) (x ? P - x : 0)
#define V (*this)
template<int MAXN, ll P, ll RT> // MAXN = 2^k
struct Poly : vector<ll> { // coefficients in [0, P)
    using vector<ll>::vector;
    static inline NTT<MAXN, P, RT> ntt;
    int n() const { return (int)size(); } // n() >= 1
    Poly(const Poly &p, int m) : vector<ll>(m) { copy_n(p.data(), min(p.n(), m), data()); }
    Poly &rev() { return reverse(data(), data() + n()), V; }
    Poly &isz(int m) { return resize(m), V; }
    static ll minv(ll x) { return ntt.minv(x); }
    Poly &iadd(const Poly &rhs) { // n() == rhs.n()
        fi(0, n()) if ((V[i] += rhs[i]) >= P) V[i] -= P;
        return V;
    }
    Poly &imul(ll k) {
        fi(0, n()) V[i] = V[i] * k % P;
        return V;
    }
    Poly Mul(const Poly &rhs) const {
        int m = 1;
        while (m < n() + rhs.n() - 1) m <= 1;
        assert(m <= MAXN);
        Poly X(V, m), Y(rhs, m);
        ntt(X, m), ntt(Y, m);
        fi(0, m) X[i] = X[i] * Y[i] % P;
        ntt(X, m, true);
        return X.isz(n() + rhs.n() - 1);
    }
    Poly Inv() const { // V[0] != 0, 2*sz<=MAXN
        if (n() == 1) return {minv(V[0])};
        int m = 1;

```



```

while (m < n() * 2) m <= 1;
assert(m <= MAXN);
Poly Xi = Poly(V, (n() + 1) / 2).Inv().isz(m);
Poly Y(V, m);
ntt(Xi, m), ntt(Y, m);
fi(0, m) {
    Xi[i] *= (2 - Xi[i] * Y[i]) % P;
    if ((Xi[i] % P) < 0) Xi[i] += P;
}
ntt(Xi, m, true);
return Xi.isz(n());
}
Poly &shift_inplace(const ll &c) { // 2 * sz <= MAXN
    int n = V.n();
    vector<ll> fc(n), ifc(n);
    fc[0] = ifc[0] = 1;
    for (int i = 1; i < n; i++) {
        fc[i] = fc[i - 1] * i % P;
        ifc[i] = minv(fc[i]);
    }
    for (int i = 0; i < n; i++) V[i] = V[i] * fc[i] % P;
    Poly g(n);
    ll cp = 1;
    for (int i = 0; i < n; i++) g[i] = cp * ifc[i] % P,
        cp = cp * c % P;
    V = V.irev().Mul(g).isz(n).irev();
    for (int i = 0; i < n; i++) V[i] = V[i] * ifc[i] % P;
    return V;
}
Poly shift(const ll &c) const { return Poly(V).
    shift_inplace(c); }
Poly _Sqrt() const { // Jacobi(V[0], P) = 1
    if (n() == 1) return {QuadraticResidue(V[0], P)};
    Poly X = Poly(V, (n() + 1) / 2)._Sqrt().isz(n());
    return X.iadd(Mul(X.Inv()).isz(n())).imul(P / 2 + 1);
}
Poly Sqrt() const { // 2 * sz <= MAXN
    Poly a;
    bool has = 0;
    for (int i = 0; i < n(); i++) {
        if (V[i]) has = 1;
        if (has) a.push_back(V[i]);
    }
    if (!has) return V;
    if ((n() + a.n()) % 2 || Jacobi(a[0], P) != 1) {
        return Poly();
    }
    a = a.isz((n() + a.n()) / 2)._Sqrt();
    int sz = a.n();
    a.isz(n());
    rotate(a.begin(), a.begin() + sz, a.end());
    return a;
}
pair<Poly, Poly> DivMod(const Poly &rhs) const { //
    (rhs).back() != 0
    if (n() < rhs.n()) return {{0}, V};
    const int m = n() - rhs.n() + 1;
    Poly X(rhs);
    X.irev().isz(m);
    Poly Y(V);
    Y.irev().isz(m);
    Poly Q = Y.Mul(X.Inv()).isz(m).irev();
    X = rhs.Mul(Q), Y = V;
    fi(0, n()) if ((Y[i] - X[i]) < 0) Y[i] += P;
    return {Q, Y.isz(max(1, rhs.n() - 1))};
}
Poly Dx() const {
    Poly ret(n() - 1);
    fi(0, ret.n()) ret[i] = (i + 1) * V[i + 1] % P;
    return ret.isz(max(1, ret.n()));
}
Poly Sx() const {
    Poly ret(n() + 1);
    fi(0, n()) ret[i + 1] = minv(i + 1) * V[i] % P;
    return ret;
}
Poly _tmul(int nn, const Poly &rhs) const {
    Poly Y = Mul(rhs).isz(n()) + nn - 1;
    return Poly(Y.data() + n() - 1, Y.data() + Y.n());
}

```

```

}
vector<ll> _eval(const vector<ll> &x, const vector<
    Poly> &up) const {
    const int m = (int)x.size();
    if (!m) return {};
    vector<Poly> down(m * 2);
    // down[1] = DivMod(up[1]).second;
    // fi(2, m * 2) down[i] = down[i / 2].DivMod(up[i])
    .second;
    down[1] = Poly(up[1]).irev().isz(n()).Inv().irev().
        _tmul(m, V);
    fi(2, m * 2) down[i] = up[i ^ 1]._tmul(up[i].n() - 1, down[i / 2]);
    vector<ll> y(m);
    fi(0, m) y[i] = down[m + i][0];
    return y;
}
static vector<Poly> _tree1(const vector<ll> &x) {
    const int m = (int)x.size();
    vector<Poly> up(m * 2);
    fi(0, m) up[m + i] = {neg(x[i]), 1};
    for (int i = m - 1; i > 0; --i) up[i] = up[i * 2].
        Mul(up[i * 2 + 1]);
    return up;
}
vector<ll> Eval(const vector<ll> &x) const { // 1e5,
    1s
    auto up = _tree1(x);
    return _eval(x, up);
}
static Poly Interpolate(const vector<ll> &x, const
    vector<ll> &y) { // 1e5, 1.4s
    const int m = (int)x.size();
    vector<Poly> up = _tree1(x), down(m * 2);
    vector<ll> z = up[1].Dx()._eval(x, up);
    fi(0, m) z[i] = y[i] * minv(z[i]) % P;
    fi(0, m) down[m + i] = {z[i]};
    for (int i = m - 1; i > 0; --i)
        down[i] = down[i * 2].Mul(up[i * 2 + 1]).iadd(
            down[i * 2 + 1].Mul(up[i * 2]));
    return down[1];
}
Poly Ln() const { // V[0] == 1, 2*sz<=MAXN
    return Dx().Mul(Inv()).Sx().isz(n());
}
Poly Exp() const { // V[0] == 0, 2*sz<=MAXN
    if (n() == 1) return {1};
    Poly X = Poly(V, (n() + 1) / 2).Exp().isz(n());
    Poly Y = X.Ln();
    Y[0] = P - 1;
    fi(0, n()) if ((Y[i] = V[i] - Y[i]) < 0) Y[i] += P;
    return X.Mul(Y).isz(n());
}
// M := P(P - 1). If k >= M, k := k % M + M.
Poly Pow(ll k) const { // 2*sz<=MAXN
    int nz = 0;
    while (nz < n() && !V[nz]) ++nz;
    if (nz * min(k, (ll)n()) >= n()) return Poly(n());
    if (!k) return Poly(Poly{1}, n());
    Poly X(data() + nz, data() + nz + n() - nz * k);
    const ll c = ntt.mpow(X[0], k % (P - 1));
    return X.Ln().imul(k % P).Exp().imul(c).irev().isz(
        n()).irev();
}
// sum_j w_j [x^j] f(x^i) for i in [0, m]
Poly power_projection(Poly wt, int m) { // 4*sz <=
    MAXN!
    assert(n() == wt.n());
    if (!n()) {
        return Poly(m + 1, 0);
    }
    if (V[0] != 0) {
        ll c = V[0];
        V[0] = 0;
        Poly A = V.power_projection(wt, m);
        fi(0, m + 1) A[i] = A[i] * fac[i] % P; //
        factorial
        Poly B(m + 1);
        ll pow = 1;
        fi(0, m + 1) B[i] = pow * ifac[i] % P, pow = pow
            * c % P; // inv. of fac
        A = A.Mul(B).isz(m + 1);
    }
}

```

```

    fi(0, m + 1) A[i] = A[i] * fac[i] % P;
    return A;
}

int n = 1;
while (n < V.n()) n *= 2;
isz(n), wt.isz(n).irev();
int k = 1;
Poly p(wt, 2 * n), q(V, 2 * n);
q.imul(P - 1);

while (n > 1) {
    Poly r(2 * n * k);
    fi(0, 2 * n * k) r[i] = (i % 2 == 0 ? q[i] : neg(
        q[i]));
    Poly pq = p.Mul(r).isz(4 * n * k);
    Poly qq = q.Mul(r).isz(4 * n * k);
    fi(0, 2 * n * k) {
        pq[2 * n * k + i] += p[i];
        qq[2 * n * k + i] += q[i] + r[i];
        pq[2 * n * k + i] %= P;
        qq[2 * n * k + i] %= P;
    }
    fill(p.begin(), p.end(), 0);
    fill(q.begin(), q.end(), 0);
    for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
        p[n * j + i] = pq[(2 * n) * j + (2 * i + 1)];
        q[n * j + i] = qq[(2 * n) * j + (2 * i + 0)];
    }
    n /= 2, k *= 2;
}
Poly ans(k);
fi(0, k) ans[i] = p[2 * i];
return ans.irev().isz(m + 1);
}

Poly FPSinv() {
    const int n = V.n() - 1;
    if (n == -1) return {};
    assert(V[0] == 0);
    if (n == 0) return V;
    assert(V[1] != 0);
    ll c = V[1], ic = minv(c);
    imul(ic);
    Poly wt(n + 1);
    wt[n] = 1;

    Poly A = V.power_projection(wt, n);
    Poly g(n);
    fi(1, n + 1) g[n - i] = n * A[i] % P * minv(i) % P;
    g = g.Pow(neg(minv(n)));
    g.insert(g.begin(), 0);

    ll pow = 1;
    fi(0, g.n()) g[i] = g[i] * pow % P, pow = pow * ic
        % P;
    return g;
}

Poly TMul(const Poly &rhs) const { // this[i] - rhs[j]
    j = k;
    return Poly(*this).irev().Mul(rhs).isz(n()).irev();
}

Poly FPScomp(Poly g) { // solves V(g(x))
    auto rec = [&](auto &rec, int n, int k, Poly Q) ->
        Poly {
            if (n == 1) {
                Poly p(2 * k);
                irev();
                fi(0, k) p[2 * i] = V[i];
                return p;
            }
            Poly R(2 * n * k);
            fi(0, 2 * n * k) R[i] = (i % 2 == 0 ? Q[i] : neg(
                Q[i]));
            Poly QQ = Q.Mul(R).isz(4 * n * k);
            fi(0, 2 * n * k) {
                QQ[2 * n * k + i] += Q[i] + R[i];
                QQ[2 * n * k + i] %= P;
            }
            Poly nxt_Q(2 * n * k);
            for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
                nxt_Q[n * j + i] = QQ[(2 * n) * j + (2 * i + 0)]
                    ;
            }

```

```

        }
        Poly nxt_p = rec(rec, n / 2, k * 2, nxt_Q);
        Poly pq(4 * n * k);
        for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
            pq[(2 * n) * j + (2 * i + 1)] += nxt_p[n * j +
                i];
            pq[(2 * n) * j + (2 * i + 1)] %= P;
        }
        Poly p(2 * n * k);
        fi(0, 2 * n * k) p[i] = (p[i] + pq[2 * n * k + i
            ]) % P;
        pq.pop_back();
        Poly x = pq.TMul(R);
        fi(0, 2 * n * k) p[i] = (p[i] + x[i]) % P;
        return p;
    };
    int sz = 1;
    while(sz < n() || sz < g.n()) sz <= 1;
    return isz(sz), rec(rec, sz, 1, g.imul(P-1).isz(2 *
        sz)).isz(sz).irev();
}
};
#undef fi
#undef V
#undef neg
using Poly_t = Poly<1 << 19, 998244353, 3>;

```

## 7.5 Generating Function

### Ordinary Generating Function

- $C(x) = A(rx)$ :  $c_n = r^n a_n$  的一般生成函數。
- $C(x) = A(x) + B(x)$ :  $c_n = a_n + b_n$  的一般生成函數。
- $C(x) = A(x)B(x)$ :  $c_n = \sum_{i=0}^n a_i b_{n-i}$  的一般生成函數。
- $C(x) = A(x)^k$ :  $c_n = \sum_{i_1+i_2+\dots+i_k=n} a_{i_1} a_{i_2} \dots a_{i_k}$  的一般生成函數。
- $C(x) = xA(x)'$ :  $c_n = na_n$  的一般生成函數。
- $C(x) = \frac{A(x)}{1-x}$ :  $c_n = \sum_{i=0}^n a_i$  的一般生成函數。
- $C(x) = A(1) + x \frac{A(1)-A(x)}{1-x}$ :  $c_n = \sum_{i=n}^{\infty} a_i$  的一般生成函數。

常用展開式

- $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$
- $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ ,  $\binom{a}{n} = \frac{a(a-1)(a-2)\dots(a-n+1)}{n!}$ .

常見生函

- 卡特蘭數:  $f(x) = \frac{1-\sqrt{1-4x}}{2x}$

### Exponential Generating Function

$a_0, a_1, \dots$  的指數生成函數:

$$\hat{A}(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \dots$$

- $\hat{C}(x) = \hat{A}(x) + \hat{B}(x)$ :  $c_n = a_n + b_n$  的指數生成函數
- $\hat{C}(x) = \hat{A}^{(k)}(x)$ :  $c_n = a_{n+k}$  的指數生成函數
- $\hat{C}(x) = x\hat{A}(x)$ :  $c_n = na_n$  的指數生成函數
- $\hat{C}(x) = \hat{A}(x)\hat{B}(x)$ :  $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$  的指數生成函數
- $\hat{C}(x) = \hat{A}(x)^k$ :  $\sum_{i_1+i_2+\dots+i_k=n} \binom{n}{i_1, i_2, \dots, i_k} a_{i_1} a_{i_2} \dots a_{i_k}$  的指數生成函數
- $\hat{C}(x) = \exp(A(x))$ : 假設  $A(x)$  是一個分量 (component) 的生成函數, 那麼  $\hat{C}(x)$  是將  $n$  個有編號的東西分成若干個分量的指數生成函數

Lagrange's Inversion Formula

如果  $F$  跟  $G$  互反, 則有  $F(0), G(0) = 0, F'(0), G'(0) \neq 0$ 。若  $H$  為任意 FPS, 則

$$n[x^n]G(x) = [x^{n-1}] \frac{1}{(F(x)/x)^n}$$

$$n[x^n]H(G(x)) = [x^{n-1}] H'(x) \frac{1}{(F(x)/x)^n}$$

## 7.6 Bostan Mori [41c3bc]

```

const ll mod = 998244353;
NTT<262144, mod, 3> ntt;
// Finds the k-th coefficient of P / Q in O(d Log d Log
    k)
// size of NTT has to > 2 * d
ll BostanMori(vector<ll> P, vector<ll> Q, long long k)
{
    int d = max((int)P.size(), (int)Q.size() - 1);
    vector M = {P, Q};

```

```

M[0].resize(d, 0);
M[1].resize(d + 1, 0);
int sz = (2 * d + 1 == 1 ? 2 : (1 << (lg(2 * d) + 1)));
vector<ll> Qn(sz);
vector N(2, vector<ll>(sz));
while(k) {
    fill(iter(Qn), 0);
    for(int i = 0; i < d + 1; i++){
        Qn[i] = M[1][i] * ((i & 1) ? -1 : 1);
        if(Qn[i] < 0) Qn[i] += mod;
    }
    ntt(Qn, sz, false);

    ll t[2] = {k & 1, 0};
    for(int i = 0; i < 2; i++){
        fill(iter(N[i]), 0);
        copy(iter(M[i]), N[i].begin());
        ntt(N[i], sz, false);
        for(int j = 0; j < sz; j++){
            N[i][j] = N[i][j] * Qn[j] % mod;
        }
        ntt(N[i], sz, true);
        for(int j = t[i]; j < 2 * sz(M[i]); j += 2){
            M[i][j >> 1] = N[i][j];
        }
    }
    k >>= 1;
}
return M[0][0] * ntt.minv(M[1][0]) % mod;
}

ll LinearRecursion(vector<ll> a, vector<ll> c, ll k) {
    // a_n = \sum_{j=1}^n c_j a_{n-j}
    int d = siz(a);
    int sz = (2 * d + 1 == 1 ? 2 : (1 << (lg(2 * d) + 1)));

    c[0] = mod - 1;
    for(ll &i : c) i = i % mod - i : 0;

    auto A = a; A.resize(sz);
    auto C = c; C.resize(sz);
    ntt(A, sz, false); ntt(C, sz, false);
    for(int i = 0; i < sz; i++) A[i] = A[i] * C[i] % mod;
    ntt(A, sz, true);
    A.resize(d);

    return BostanMori(A, c, k);
}

```

## 8 String

### 8.1 KMP Algorithm [c8b75f]

```

// 0-based
// fail[i] = max k<i s.t. s[0..k] = s[i-k..i]
vector<int> kmp_build_fail(const string &s){
    int n = SZ(s);
    vector<int> fail(n, -1);
    int cur = -1;
    for(int i = 1; i < n; i++){
        while(cur != -1 && s[cur + 1] != s[i])
            cur = fail[cur];
        if(s[cur + 1] == s[i])
            cur++;
        fail[i] = cur;
    }
    return fail;
}

void kmp_match(const string &s, const vector<int> &fail, const string &t){
    int cur = -1;
    int n = SZ(s), m = SZ(t);
    for(int i = 0; i < m; i++){
        while(cur != -1 && (cur + 1 == n || s[cur + 1] != t[i]))
            cur = fail[cur];
        cur = fail[cur];
        if(cur + 1 < n && s[cur + 1] == t[i])
            cur++;
        // cur = max k s.t. s[0..k] = t[i-k..i]
    }
}

```

### 8.2 Manacher Algorithm [caf0f4]

```

/* center i: radius z[i * 2 + 1] / 2
   center i, i + 1: radius z[i * 2 + 2] / 2
   both aba, abba have radius 2 */
vector<int> manacher(const string &tmp){ // 0-based
    string s = "%";
    int l = 0, r = 0;
    for(char c : tmp) s += c, s += '%';
    vector<int> z(SZ(s));
    for(int i = 0; i < SZ(s); i++){
        z[i] = r > i ? min(z[2 * l - i], r - i) : 1;
        while(i - z[i] >= 0 && i + z[i] < SZ(s)
            && s[i + z[i]] == s[i - z[i]])
            ++z[i];
        if(z[i] + i > r) r = z[i] + i, l = i;
    }
    return z;
}

```

### 8.3 Lyndon Factorization [7c612b]

```

// partition s = w[0] + w[1] + ... + w[k-1],
// w[0] >= w[1] >= ... >= w[k-1]
// each w[i] strictly smaller than all its suffix
void duval(const string &s, vector<pii> &w) {
    for (int n = (int)s.size(), i = 0, j, k; i < n; ) {
        for (j = i + 1, k = i; j < n && s[k] <= s[j]; j++)
            k = (s[k] < s[j] ? i : k + 1);
        // if (i < n / 2 && j >= n / 2) {
        // for min cyclic shift, call duval(s + s)
        // then here s.substr(i, n / 2) is min cyclic shift
        // }
        for (; i <= k; i += j - k)
            w.pb(pii(i, j - k)); // s.substr(l, len)
    }
}

```

### 8.4 Suffix Array [cd67ea]

```

struct SuffixArray {
    vector<int> sa, lcp, rank; // lcp[i] is lcp of sa[i]
                                // and sa[i-1]
                                // sa[0] = s.size()
                                // character should be 1-
                                // based
    SuffixArray(string& s, int lim=256) { // or
        basic_string<int>
        int n = s.size() + 1, k = 0, a, b;
        vector<int> x(n, 0), y(n), ws(max(n, lim));
        rank.assign(n, 0);
        for (int i = 0; i < n - 1; i++) x[i] = s[i];
        sa = lcp = y, iota(sa.begin(), sa.end(), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2),
            lim = p) {
            p = j, iota(y.begin(), y.end(), n - j);
            for (int i = 0; i < n; i++)
                if (sa[i] >= j) y[p++] = sa[i] - j;
            for (int &i : ws) i = 0;
            for (int i = 0; i < n; i++) ws[x[i]]++;
            for (int i = 1; i < lim; i++) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            for(int i = 1; i < n; i++){
                a = sa[i - 1], b = sa[i];
                x[b] = (y[a] == y[b] && y[a + j] == y[b + j]) ?
                    p - 1 : p++;
            }
        }
        for (int i = 1; i < n; i++) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};

```

### 8.5 Suffix Automaton [016373]

```

struct exSAM {
    const int CNUM = 26;
    // len: maxlength, link: fail link
    // lenSorted: topo order, cnt: occur
    vector<int> len, link, lenSorted, cnt;
    vector<vector<int>> next;
};

```

```

int total = 0;
int newnode() {
    return total++;
}
void init(int n) { // total number of characters
    len.assign(2 * n, 0); link.assign(2 * n, 0);
    lenSorted.assign(2 * n, 0); cnt.assign(2 * n, 0);
    next.assign(2 * n, vector<int>(CNUM));
    newnode(), link[0] = -1;
}
int insertSAM(int last, int c) {
    // not exSAM: cur = newnode(), p = last
    int cur = next[last][c];
    len[cur] = len[last] + 1;
    int p = link[last];
    while (p != -1 && !next[p][c])
        next[p][c] = cur, p = link[p];
    if (p == -1) return link[cur] = 0, cur;
    int q = next[p][c];
    if (len[p] + 1 == len[q]) return link[cur] = q, cur;
    ;
    int clone = newnode();
    for (int i = 0; i < CNUM; ++i)
        next[clone][i] = len[next[q][i]] ? next[q][i] : 0;
    len[clone] = len[p] + 1;
    while (p != -1 && next[p][c] == q)
        next[p][c] = clone, p = link[p];
    link[link[cur]] = clone; link[q] = clone;
    link[q] = clone;
    return cur;
}
void insert(const string &s) {
    int cur = 0;
    for (auto ch : s) {
        int &nxt = next[cur][int(ch - 'a')];
        if (!nxt) nxt = newnode();
        cnt[cur = nxt] += 1;
    }
}
void build() {
    queue<int> q;
    q.push(0);
    while (!q.empty()) {
        int cur = q.front();
        q.pop();
        for (int i = 0; i < CNUM; ++i)
            if (next[cur][i])
                q.push(insertSAM(cur, i));
    }
    vector<int> lc(total);
    for (int i = 1; i < total; ++i) ++lc[len[i]];
    partial_sum(iter(lc), lc.begin());
    for (int i = 1; i < total; ++i) lenSorted[--lc[len[i]]] = i;
}
void solve() {
    for (int i = total - 2; i >= 0; --i)
        cnt[link[lenSorted[i]]] += cnt[lenSorted[i]];
}
};

```

## 8.6 Z-value Algorithm [488d87]

```

// z[i] = max k s.t. s[0..k-1] = s[i..i+k-1]
// i.e. Length of longest common prefix
// z[0] = 0
vector<int> z_function(const string &s) {
    int n = s.size();
    vector<int> z(n);
    for (int i = 1, l = 0, r = 0; i < n; i++) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
            z[i]++;
        if (i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
    return z;
}

```

## 8.7 Main Lorentz [fcfb8f]

```

struct Rep { int minl, maxl, len; };

```

```

vector<Rep> rep; // 0-base
// p \in [minl, maxl] => s[p, p + i) = s[p + i, p + 2i)
void main_lorentz(const string &s, int sft = 0) {
    const int n = s.size();
    if (n == 1) return;
    const int nu = n / 2, nv = n - nu;
    const string u = s.substr(0, nu), v = s.substr(nu,
        ru(u.rbegin(), u.rend()), rv(v.rbegin(), v.rend()
        ));
    main_lorentz(u, sft), main_lorentz(v, sft + nu);
    const auto z1 = z_function(ru), z2 = z_function(v + '#' + u),
        z3 = z_function(ru + '#' + rv), z4 =
        z_function(v);
    auto get_z = [](const vector<int> &z, int i) {
        return (0 <= i and i < (int)z.size()) ? z[i] : 0;
    };
    auto add_rep = [&](bool left, int c, int l, int k1,
        int k2) {
        const int L = max(1, l - k2), R = min(l - left, k1);
        ;
        if (L > R) return;
        if (left) rep.emplace_back(Rep({sft + c - R, sft +
            c - L, l}));
        else rep.emplace_back(Rep({sft + c - R - l + 1, sft
            + c - L - l + 1, l}));
    };
    for (int cntr = 0; cntr < n; cntr++) {
        int l, k1, k2;
        if (cntr < nu) {
            l = nu - cntr;
            k1 = get_z(z1, nu - cntr);
            k2 = get_z(z2, nv + 1 + cntr);
        } else {
            l = cntr - nu + 1;
            k1 = get_z(z3, nu + 1 + nv - 1 - (cntr - nu));
            k2 = get_z(z4, (cntr - nu) + 1);
        }
        if (k1 + k2 >= 1)
            add_rep(cntr < nu, cntr, l, k1, k2);
    }
}

```

## 8.8 AC Automaton [f529e6]

```

const int SIGMA = 26;
struct AC_Automaton {
    // child: trie, next: automaton
    vector<vector<int>> child, next;
    vector<int> fail, cnt, ord;
    int total = 0;
    int newnode() {
        return total++;
    }
    void init(int len) { // len >= 1 + total len
        child.assign(len, vector<int>(26, -1));
        next.assign(len, vector<int>(26, -1));
        fail.assign(len, -1); cnt.assign(len, 0);
        ord.clear();
        newnode();
    }
    int input(string &s) {
        int cur = 0;
        for (char c : s) {
            if (child[cur][c - 'A'] == -1)
                child[cur][c - 'A'] = newnode();
            cur = child[cur][c - 'A'];
        }
        return cur; // return the end node of string
    }
    void make_fl() {
        queue<int> q;
        q.push(0), fail[0] = -1;
        while (!q.empty()) {
            int R = q.front();
            q.pop(); ord.pb(R);
            for (int i = 0; i < SIGMA; ++i)
                if (child[R][i] != -1) {
                    int X = next[R][i] = child[R][i], Z = fail[R];
                    ;
                    while (Z != -1 && child[Z][i] == -1)
                        Z = fail[Z];
                    fail[X] = Z != -1 ? child[Z][i] : 0;
                }
        }
    }
}

```

```

    q.push(X);
}
else next[R][i] = R ? next[fail[R]][i] : 0;
}
}
void solve() {
    for (int i : ord | views::reverse)
        if (i) cnt[fail[i]] += cnt[i];
}
};

```

## 8.9 Palindrome Automaton [8a071b]

```

struct PalindromicTree {
    struct node {
        int nxt[26], fail, len; // num = depth of fail link
        int cnt, num; // cnt = occur, num = #pal_suffix of this node
        node(int l = 0) : nxt{}, fail(0), len(l), cnt(0), num(0) {}
    };
    vector<node> st; vector<int> s; int last, n;
    void init() {
        st.clear(); s.clear(); last = 1; n = 0;
        st.pb(0); st.pb(-1);
        st[0].fail = 1; s.pb(-1);
    }
    int getFail(int x) {
        while (s[n - st[x].len - 1] != s[n]) x = st[x].fail;
        return x;
    }
    void add(int c) {
        s.pb(c -= 'a'); ++n;
        int cur = getFail(last);
        if (!st[cur].nxt[c]) {
            int now = SZ(st);
            st.pb(st[cur].len + 2);
            st[now].fail = st[getFail(st[cur].fail)].nxt[c];
            st[cur].nxt[c] = now;
            st[now].num = st[st[now].fail].num + 1;
        }
        last = st[cur].nxt[c]; ++st[last].cnt;
    }
    void dpcnt() {
        for (int i = SZ(st) - 1; i >= 0; i--) {
            auto nd = st[i];
            st[nd.fail].cnt += nd.cnt;
        }
    }
    int size() { return (int)st.size() - 2; }
};

```

## 9 Misc

### 9.1 Cyclic Ternary Search [9017cc]

```

/* bool pred(int a, int b);
f(0) ~ f(n - 1) is a cyclic-shift U-function
return idx s.t. pred(x, idx) is false forall x*/
int cyc_tsearch(int n, auto pred) {
    if (n == 1) return 0;
    int l = 0, r = n; bool rv = pred(1, 0);
    while (r - l > 1) {
        int m = (l + r) / 2;
        if (pred(0, m) ? rv : pred(m, (m + 1) % n)) r = m;
        else l = m;
    }
    return pred(1, r % n) ? 1 : r % n;
}

```

### 9.2 Matroid

$M = (E, \mathcal{I})$ , where  $\mathcal{I} \subseteq 2^E$  is nonempty, is a matroid if:

- If  $S \in \mathcal{I}$  and  $S' \subseteq S$ , then  $S' \in \mathcal{I}$ .
- For  $S_1, S_2 \in \mathcal{I}$  s.t.  $|S_1| < |S_2|$ , there exists  $e \in S_2 \setminus S_1$  s.t.  $S_1 \cup \{e\} \in \mathcal{I}$ .

Matroid intersection:

Start from  $S = \emptyset$ . In each iteration, let

- $Y_1 = \{x \notin S \mid S \cup \{x\} \in \mathcal{I}_1\}$
- $Y_2 = \{x \notin S \mid S \cup \{x\} \in \mathcal{I}_2\}$

If there exists  $x \in Y_1 \cap Y_2$ , insert  $x$  into  $S$ . Otherwise for each  $x \in S, y \notin S$ , create edges

- $x \rightarrow y$  if  $S - \{x\} \cup \{y\} \in \mathcal{I}_1$ .
- $y \rightarrow x$  if  $S - \{x\} \cup \{y\} \in \mathcal{I}_2$ .

Find a *shortest* path (with BFS) starting from a vertex in  $Y_1$  and ending at a vertex in  $Y_2$  which doesn't pass through any other vertices in  $Y_2$ , and alternate the path. The size of  $S$  will be incremented by 1 in each iteration. For the weighted case, assign weight  $w(x)$  to vertex  $x$  if  $x \in S$  and  $-w(x)$  if  $x \notin S$ . Find the path with the minimum number of edges among all minimum length paths and alternate it.

### 9.3 Simulate Annealing [ff826c]

```

ld anneal() {
    mt19937 rnd_engine(seed);
    uniform_real_distribution<ld> rnd(0, 1);
    const ld dT = 0.001;
    // Argument p
    ld S_cur = calc(p), S_best = S_cur;
    for (ld T = 2000; T > eps; T -= dT) {
        // Modify p to p_prime
        const ld S_prime = calc(p_prime);
        const ld delta_c = S_prime - S_cur;
        ld prob = min((ld)1, exp(-delta_c / T));
        if (rnd(rnd_engine) <= prob)
            S_cur = S_prime, p = p_prime;
        if (S_prime < S_best) // find min
            S_best = S_prime, p_best = p_prime;
    }
    return S_best;
}

```

### 9.4 Binary Search On Fraction [f6b9ec]

```

struct Q {
    ll p, q;
    Q go(Q b, ll d) { return {p + b.p * d, q + b.q * d};
    };
};
// returns smallest p/q in [lo, hi] such that
// pred(p/q) is true, and 0 <= p, q <= N
Q frac_bs(ll N, auto &&pred) {
    Q lo{0, 1}, hi{1, 0};
    if (pred(lo)) return lo;
    assert(pred(hi));
    bool dir = 1, L = 1, H = 1;
    for (; L || H; dir = !dir) {
        ll len = 0, step = 1;
        for (int t = 0; t < 2 && (t ? step /= 2 : step *= 2);)
            if (Q mid = hi.go(lo, len + step);
                mid.p > N || mid.q > N || dir ^ pred(mid))
                t++;
            else len += step;
        swap(lo, hi = hi.go(lo, len));
        (dir ? L : H) = !len;
    }
    return dir ? hi : lo;
}

```

### 9.5 Min Plus Convolution [09b5c3]

```

// a is convex a[i+1]-a[i] <= a[i+2]-a[i+1]
vector<int> min_plus_convolution(vector<int> &a, vector<int> &b) {
    int n = SZ(a), m = SZ(b);
    vector<int> c(n + m - 1, INF);
    auto dc = [&](auto Y, int l, int r, int jl, int jr) {
        if (l > r) return;
        int mid = (l + r) / 2, from = -1, &best = c[mid];
        for (int j = jl; j <= jr; ++j)
            if (int i = mid - j; i >= 0 && i < n)
                if (best > a[i] + b[j])
                    best = a[i] + b[j], from = j;
        Y(Y, l, mid - 1, jl, from), Y(Y, mid + 1, r, from, jr);
    };
    return dc(dc, 0, n - 1 + m - 1, 0, m - 1), c;
}

```

## 10 Notes

### 10.1 Geometry

#### Rotation Matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- rotate  $90^\circ$ :  $(x, y) \rightarrow (-y, x)$
- rotate  $-90^\circ$ :  $(x, y) \rightarrow (y, -x)$



## Triangles

Side lengths:  $a, b, c$

$$\text{Semiperimeter: } p = \frac{a+b+c}{2}$$

$$\text{Area: } A = \sqrt{p(p-a)(p-b)(p-c)}$$

$$\text{Circumradius: } R = \frac{abc}{4A}$$

$$\text{Inradius: } r = \frac{A}{p}$$

Length of median (divides triangle into two equal-area triangles):  $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$$\text{Length of bisector (divides angles in two): } s_a = \sqrt{bc \left( 1 - \left( \frac{a}{b+c} \right)^2 \right)}$$

$$\text{Law of sines: } \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$$

$$\text{Law of cosines: } a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$\text{Law of tangents: } \frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$$

## Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ .

## Spherical coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

## Green's Theorem

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{L^+} (P dx + Q dy)$$

$$\text{Area} = \frac{1}{2} \oint_L x dy - y dx$$

Circular sector:

$$x = x_0 + r \cos \theta$$

$$y = y_0 + r \sin \theta$$

$$\begin{aligned} A &= r \int_{\alpha}^{\beta} (x_0 + \cos \theta) \cos \theta + (y_0 + \sin \theta) \sin \theta d\theta \\ &= r(r\theta + x_0 \sin \theta - y_0 \cos \theta)|_{\alpha}^{\beta} \end{aligned}$$

## Point-Line Duality

$$p = (a, b) \leftrightarrow p^* : y = ax - b$$

- $p \in l \iff l^* \in p^*$
- $p_1, p_2, p_3$  are collinear  $\iff p_1^*, p_2^*, p_3^*$  intersect at a point
- $p$  lies above  $l \iff l^*$  lies above  $p^*$
- lower convex hull  $\leftrightarrow$  upper envelope

## 10.2 Trigonometry

$$\begin{aligned} \sinh x &= \frac{1}{2}(e^x - e^{-x}) & \cosh x &= \frac{1}{2}(e^x + e^{-x}) \\ \sin n\pi &= 0 & \cos n\pi &= (-1)^n \end{aligned}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(2\alpha) = 2 \cos \alpha \sin \alpha$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$= 2 \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$(V + W) \tan(\alpha - \beta)/2 = (V - W) \tan(\alpha + \beta)/2$$

where  $V, W$  are lengths of sides opposite angles  $\alpha, \beta$ .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where  $r = \sqrt{a^2 + b^2}$ ,  $\phi = \operatorname{atan2}(b, a)$ .

## 10.3 Calculus

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln |\cos ax|}{a}$$

$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$$

$$\int x e^{ax} = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int \sin^2(x) = \frac{x}{2} - \frac{1}{4} \sin 2x$$

$$\int \sin^3 x = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x$$

$$\int \cos^2(x) = \frac{x}{2} + \frac{1}{4} \sin 2x$$

$$\int \cos^3 x = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x$$

$$\int x \sin x = \sin x - x \cos x$$

$$\int x \cos x = \cos x + x \sin x$$

$$\int x e^x = e^x(x - 1)$$

$$\int x^2 e^x = e^x(x^2 - 2x + 2)$$

$$\int x^2 \sin x = 2x \sin x - (x^2 - 2) \cos x$$

$$\int x^2 \cos x = 2x \cos x + (x^2 - 2) \sin x$$

$$\int e^x \sin x = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^x \cos x = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int x e^x \sin x = \frac{1}{2} e^x (x \sin x - x \cos x + \cos x)$$

$$\int x e^x \cos x = \frac{1}{2} e^x (x \sin x + x \cos x - \sin x)$$

## 10.4 Sum & Series

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

