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## 1 Basic

### 1.1 Default Code

```
//Challenge: Accepted
//#pragma GCC optimize("Ofast")
#include <bits/stdc++.h>
using namespace std;

#define io ios_base::sync_with_stdio(0);cin.tie(0);cerr.tie(0)
#define iter(v) v.begin(),v.end()
#define SZ(v) int(v.size())
#define pb emplace_back
#define ff first
#define ss second

using ll = long long;
using pii = pair<int, int>;
using pll = pair<ll, ll>;

#ifdef zisk
void debug(){cerr << "\n";}
template<class T, class ... U>
void debug(T a, U ... b){cerr << a << " ", debug(b...);}
template<class T> void pary(T l, T r){
    while (l != r) cerr << *l << " ", l++;
    cerr << "\n";
}
#else
#define debug(...) void()
#define pary(...) void()
#endif

template<class A, class B>
ostream& operator<<(ostream& o, pair<A,B> p)
{ return o << '(' << p.ff << ',' << p.ss << ')'; }

int main(){
    io;
}
```

## 1.2 .vimrc

```
sy on
se nu rnu bs=2 sw=4 ts=4 hls ls=2 si acd bo=all mouse=a
map <F9> :w<bar>!g++ "%" -o %:r -std=c++17 -Wall -Wextra -
    Wshadow -O2 -Dzisk -g -fsanitize=undefined,address<CR>
map <F8> :!./%:r<CR>
inoremap {<CR> {<CR>}<ESC>kO
# -D_GLIBCXX_ASSERTIONS, -D_GLIBCXX_DEBUG
```

## 1.3 Fast IO

```
// from JAW
inline int my_getchar() {
    const int N = 1<<20;
    static char buf[N];
    static char *p = buf, *end = buf;
    if(p == end) {
        if((end = buf + fread(buf, 1, N, stdin)) == buf)
            return EOF;
        p = buf;
    }
    return *p++;
}

inline int readint(int &x) {
    static char c, neg;
    while((c = my_getchar()) < '-') {
        if(c == EOF) return 0;
    }
    neg = (c == '-') ? -1 : 1;
    x = (neg == 1) ? c - '0' : 0;
    while((c = my_getchar()) >= '0') x = (x << 3) + (x << 1)
        + (c - '0');
    x *= neg;
    return 1;
}

const int kBufSize = 524288;
char inbuf[kBufSize];
char buf_[kBufSize]; size_t size_;
inline void Flush_() { write(1, buf_, size_); size_ = 0; }
inline void CheckFlush_(size_t sz) { if (sz + size_ >
    kBufSize) Flush_(); }

inline void PutInt(int a) {
    static char tmp[22] = "01234567890123456789\n";
    CheckFlush_(10);
    if(a < 0){
        *(buf_ + size_) = '-';
        a = ~a + 1;
        size_++;
    }
    int tail = 20;
    if (!a) {
        tmp[--tail] = '0';
    } else {
        for (; a; a /= 10) tmp[--tail] = (a % 10) ^ '0';
    }
    memcpy(buf_ + size_, tmp + tail, 21 - tail);
    size_ += 21 - tail;
}

int main(){
    Flush_();
    return 0;
}
```

## 1.4 Random

```
mt19937 rng(chrono::system_clock::now().time_since_epoch().
    count());
```

## 1.5 Checker

```
#!/usr/bin/env bash
set -e
while ;; do
    python3 gen.py > test.txt
    diff <(/a.exe < test.txt) <(/b.exe < test.txt)
done
```

## 1.6 PBDS Tree

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
using Tree = tree<int, null_type, less<>, rb_tree_tag,
    tree_order_node_statistics_update>;
// .find_by_order(x)
// .order_of_key(x)
```

## 2 Data Structure

### 2.1 Heavy-Light Decomposition

```
struct Heavy_light_Decomposition { // 1-base
    int n, up[maxn], dep[maxn], to[maxn], siz[maxn], pa[maxn];
    int C, ti[maxn], ord[maxn], wdown[maxn], edge[maxn], et = 0;
    vector<pii> G[maxn];
    void init(int _n) {
        n = _n, C = 0, et = 1;
        for (int i = 1; i <= n; i++)
            G[i].clear(), to[i] = 0;
    }
    void add_edge(int a, int b, int w) {
        G[a].push_back(pii(b, et)), G[b].push_back(pii(a, et));
        edge[et++] = w;
    }
    void dfs(int u, int f, int d) {
        siz[u] = 1, pa[u] = f, dep[u] = d;
        for (auto &v: G[u])
            if (v.ff != f) {
                dfs(v.ff, u, d+1), siz[u] += siz[v];
                if (siz[to[u]] < siz[v]) to[u] = v;
            }
    }
    void cut(int u, int link) {
        ti[u] = C;
        ord[C++] = u, up[u] = link;
        if (!to[u]) return;
        cut(to[u], link);
        for (auto v: G[u]) {
            if (v.ff != pa[u] && v.ff != to[u]) cut(v.ff, v.ff);
        }
    }
    void build() { dfs(1, 1, 1), cut(1, 1); }
    int query(int a, int b) {
        int ta = up[a], tb = up[b], re = 0;
        while (ta != tb)
            if (dep[ta] < dep[tb])
                /*query*/, tb = up[b = pa[tb]];
            else /*query*/, ta = up[a = pa[ta]];
        if (a == b) return re;
        if (ti[a] > ti[b]) swap(a, b);
        /*query*/
        return re;
    }
};
```

### 2.2 Link Cut Tree

```
struct Splay { // LCT + PATH add
    static Splay nil;
```

```

Splay *ch[2], *f;
int rev;
int sz;
ll val, sum, tag;
Splay() : rev(0), sz(1), val(1), sum(1), tag(0) {
    f = ch[0] = ch[1] = &nil;
}
bool isr() { return f->ch[0] != this && f->ch[1] != this;
}
int dir() { return f->ch[0] == this ? 0 : 1; }
void setCh(Splay *c, int d) {
    ch[d] = c;
    if (c != &nil) c->f = this;
    pull();
}
void push() {
    for(int i = 0; i < 2; i++){
        if(ch[i] == &nil) continue;
        if(rev) swap(ch[i]->ch[0], ch[i]->ch[1]), ch[i]->rev
            ^= 1;
        if(tag != 0){
            ch[i]->tag += tag;
            ch[i]->val += tag;
            ch[i]->sum += tag * ch[i]->sz;
        }
    }
    tag = 0;
    rev = 0;
}
void pull() {
    // take care of the nil!
    sz = 1;
    sum = val;
    for(int i = 0; i < 2; i++){
        if(ch[i] == &nil) continue;
        ch[i]->f = this;
        sz += ch[i]->sz;
        sum += ch[i]->sum;
    }
}
void rotate(){
    Splay *p = f;
    int d = dir();
    if (!p->isr()) p->f->setCh(this, p->dir());
    else f = p->f;
    p->setCh(ch[!d], d);
    setCh(p, !d);
    p->pull(), pull();
}
void update(){
    if(f != &nil) f->update();
    push();
}
void splay(){
    update();
    for(Splay* fa; fa = f, !isr(); rotate())
        if(!fa->isr()) (fa->dir() == dir() ? fa : this)->
            rotate();
}
Splay *access(Splay* q = &nil){
    splay();
    setCh(q, 1);
    pull();
    if (f != &nil) return f->access(this);
    else return q;
}
void root_path(){access(), splay();}
void chroot() {root_path(), swap(ch[0], ch[1]), rev = 1,
    push(), pull();}
void split(Splay* y){chroot(), y->root_path();}
void link(Splay* y){root_path(), y->chroot(), setCh(y, 1)
    ;}
void cut(Splay* y) {split(y), y->push(), y->ch[0] = y->ch
    [0]->f = &nil;}
Splay *get_root(){
    root_path();

```

```

    auto q = this;
    for(; q->ch[0] != &nil; q = q->ch[0]) q->push();
    return q;
}
Splay *lca(Splay* y){
    access(), y->root_path();
    return y->f == &nil ? &nil : y->f;
}
bool conn(Splay* y){return get_root() == y->get_root();}
} Splay::nil;

```

## 2.3 Treap

```

struct node {
    int data, sz;
    node *l, *r;
    node(int k) : data(k), sz(1), l(0), r(0) {}
    void up() {
        sz = 1;
        if (l) sz += l->sz;
        if (r) sz += r->sz;
    }
    void down() {}
};
int sz(node *a) { return a ? a->sz : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (rand() % (sz(a) + sz(b)) < sz(a))
        return a->down(), a->r = merge(a->r, b), a->up(),
            a;
    return b->down(), b->l = merge(a, b->l), b->up(), b;
}
void split(node *o, node *&a, node *&b, int k) {
    if (!o) return a = b = 0, void();
    o->down();
    if (o->data <= k)
        a = o, split(o->r, a->r, b, k), a->up();
    else b = o, split(o->l, a, b->l, k), b->up();
}
void split2(node *o, node *&a, node *&b, int k) {
    if (sz(o) <= k) return a = o, b = 0, void();
    o->down();
    if (sz(o->l) + 1 <= k)
        a = o, split2(o->r, a->r, b, k - sz(o->l) - 1);
    else b = o, split2(o->l, a, b->l, k);
    o->up();
}
node *kth(node *o, int k) {
    if (k <= sz(o->l)) return kth(o->l, k);
    if (k == sz(o->l) + 1) return o;
    return kth(o->r, k - sz(o->l) - 1);
}
int Rank(node *o, int key) {
    if (!o) return 0;
    if (o->data < key)
        return sz(o->l) + 1 + Rank(o->r, key);
    else return Rank(o->l, key);
}
bool erase(node *&o, int k) {
    if (!o) return 0;
    if (o->data == k) {
        node *t = o;
        o->down(), o = merge(o->l, o->r);
        delete t;
        return 1;
    }
    node *&t = k < o->data ? o->l : o->r;
    return erase(t, k) ? o->up(), 1 : 0;
}
void insert(node *&o, int k) {
    node *a, *b;
    split(o, a, b, k),
        o = merge(a, merge(new node(k), b));
}
void interval(node *&o, int l, int r) {
    node *a, *b, *c;

```

```

split2(o, a, b, l - 1), split2(b, b, c, r);
// operate
o = merge(a, merge(b, c));
}

```

## 2.4 KD Tree

```

namespace kdt {
    int root, lc[maxn], rc[maxn], xl[maxn], xr[maxn],
    yl[maxn], yr[maxn];
    point p[maxn];
    int build(int l, int r, int dep = 0) {
        if (l == r) return -1;
        function<bool(const point &, const point &)> f =
            [dep](const point &a, const point &b) {
                if (dep & 1) return a.x < b.x;
                else return a.y < b.y;
            };
        int m = (l + r) >> 1;
        nth_element(p + l, p + m, p + r, f);
        xl[m] = xr[m] = p[m].x;
        yl[m] = yr[m] = p[m].y;
        lc[m] = build(l, m, dep + 1);
        if (~lc[m]) {
            xl[m] = min(xl[m], xl[lc[m]]);
            xr[m] = max(xr[m], xr[lc[m]]);
            yl[m] = min(yl[m], yl[lc[m]]);
            yr[m] = max(yr[m], yr[lc[m]]);
        }
        rc[m] = build(m + 1, r, dep + 1);
        if (~rc[m]) {
            xl[m] = min(xl[m], xl[rc[m]]);
            xr[m] = max(xr[m], xr[rc[m]]);
            yl[m] = min(yl[m], yl[rc[m]]);
            yr[m] = max(yr[m], yr[rc[m]]);
        }
        return m;
    }
    bool bound(const point &q, int o, long long d) {
        double ds = sqrt(d + 1.0);
        if (q.x < xl[o] - ds || q.x > xr[o] + ds ||
            q.y < yl[o] - ds || q.y > yr[o] + ds)
            return false;
        return true;
    }
    long long dist(const point &a, const point &b) {
        return (a.x - b.x) * 1ll * (a.x - b.x) +
            (a.y - b.y) * 1ll * (a.y - b.y);
    }
    void dfs(
        const point &q, long long &d, int o, int dep = 0) {
        if (!bound(q, o, d)) return;
        long long cd = dist(p[o], q);
        if (cd != 0) d = min(d, cd);
        if ((dep & 1) && q.x < p[o].x ||
            !(dep & 1) && q.y < p[o].y) {
            if (~lc[o]) dfs(q, d, lc[o], dep + 1);
            if (~rc[o]) dfs(q, d, rc[o], dep + 1);
        } else {
            if (~rc[o]) dfs(q, d, rc[o], dep + 1);
            if (~lc[o]) dfs(q, d, lc[o], dep + 1);
        }
    }
    void init(const vector<point> &v) {
        for (int i = 0; i < v.size(); ++i) p[i] = v[i];
        root = build(0, v.size());
    }
    long long nearest(const point &q) {
        long long res = 1e18;
        dfs(q, res, root);
        return res;
    }
} // namespace kdt

```

## 2.5 Leftist Tree

```

struct node {
    ll v, data, sz, sum;
    node *l, *r;
    node(ll k)
        : v(0), data(k), sz(1), l(0), r(0), sum(k) {}
};
ll sz(node *p) { return p ? p->sz : 0; }
ll V(node *p) { return p ? p->v : -1; }
ll sum(node *p) { return p ? p->sum : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (a->data < b->data) swap(a, b);
    a->r = merge(a->r, b);
    if (V(a->r) > V(a->l)) swap(a->r, a->l);
    a->v = V(a->r) + 1, a->sz = sz(a->l) + sz(a->r) + 1;
    a->sum = sum(a->l) + sum(a->r) + a->data;
    return a;
}
void pop(node *o) {
    node *tmp = o;
    o = merge(o->l, o->r);
    delete tmp;
}

```

## 3 Flow & Matching

### 3.1 Dinic

```

struct Dinic { //  $\theta$ -based,  $O(V^2E)$ , unit flow:  $O(\min(V^{2/3}E, E^{3/2}))$ , bipartite matching:  $O(\sqrt{V}E)$ 
    struct edge {
        ll to, cap, flow, rev;
    };
    int n, s, t;
    vector<vector<edge>> g;
    vector<int> dis, ind;

    void init(int _n) {
        n = _n;
        g.assign(n, vector<edge>());
    }
    void reset() {
        for (int i = 0; i < n; ++i)
            for (auto &j : g[i]) j.flow = 0;
    }
    void add_edge(int u, int v, ll cap) {
        g[u].pb(edge{v, cap, 0, SZ(g[v])});
        g[v].pb(edge{u, 0, 0, SZ(g[u]) - 1});
        //change g[v] to cap for undirected graphs
    }
    bool bfs() {
        dis.assign(n, -1);
        queue<int> q;
        q.push(s), dis[s] = 0;
        while (!q.empty()) {
            int cur = q.front(); q.pop();
            for (auto &e : g[cur]) {
                if (dis[e.to] == -1 && e.flow != e.cap) {
                    q.push(e.to);
                    dis[e.to] = dis[cur] + 1;
                }
            }
        }
        return dis[t] != -1;
    }
    ll dfs(int u, ll cap) {
        if (u == t || !cap) return cap;
        for (int &i = ind[u]; i < SZ(g[u]); ++i) {
            edge &e = g[u][i];
            if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
                ll df = dfs(e.to, min(e.cap - e.flow, cap));
                if (df) {
                    e.flow += df;
                    g[e.to][e.rev].flow -= df;
                }
            }
        }
    }
}

```

```

        return df;
    }
}
dis[u] = -1;
return 0;
}
ll maxflow(int _s, int _t) {
    s = _s; t = _t;
    ll flow = 0, df;
    while (bfs()) {
        ind.assign(n, 0);
        while ((df = dfs(s, INF))) flow += df;
    }
    return flow;
}
};

```

### 3.2 Bounded Flow

```

struct BoundedFlow : Dinic {
    vector<ll> tot;
    void init(int _n) {
        Dinic::init(_n + 2);
        tot.assign(n, 0);
    }
    void add_edge(int u, int v, ll lcap, ll rcap) {
        tot[u] -= lcap, tot[v] += lcap;
        g[u].pb(edge{v, rcap, lcap, SZ(g[v]))});
        g[v].pb(edge{u, 0, 0, SZ(g[u]) - 1});
    }
    bool feasible() {
        ll sum = 0;
        int vs = n - 2, vt = n - 1;
        for(int i = 0; i < n - 2; ++i)
            if(tot[i] > 0)
                add_edge(vs, i, 0, tot[i]), sum += tot[i];
            else if(tot[i] < 0) add_edge(i, vt, 0, -tot[i]);
        if(sum != maxflow(vs, vt)) sum = -1;
        for(int i = 0; i < n - 2; ++i)
            if(tot[i] > 0)
                g[vs].pop_back(), g[i].pop_back();
            else if(tot[i] < 0)
                g[i].pop_back(), g[vt].pop_back();
        return sum != -1;
    }
    ll boundedflow(int _s, int _t) {
        add_edge(_t, _s, 0, INF);
        if(!feasible()) return -1;
        ll x = g[_t].back().flow;
        g[_t].pop_back(), g[_s].pop_back();
        return x - maxflow(_t, _s); // min
        //return x + maxflow(_s, _t); // max
    }
};

```

### 3.3 MCMF

```

struct MCMF { //  $\theta$ -based,  $O(SPFA * |f|)$ 
    struct edge {
        ll from, to, cap, flow, cost, rev;
    };
    int n;
    int s, t; ll mx;
    //mx: maximum amount of flow
    vector<vector<edge>> g;
    vector<ll> dis, up;
    bool BellmanFord(ll &flow, ll &cost) {
        vector<edge*> past(n);
        vector<int> inq(n);
        dis.assign(n, INF); up.assign(n, 0);
        queue<int> q;
        q.push(s), inq[s] = 1;
        up[s] = mx - flow, past[s] = 0, dis[s] = 0;
        while (!q.empty()) {

```

```

            int u = q.front();
            q.pop(), inq[u] = 0;
            if (!up[u]) continue;
            for (auto &e : g[u])
                if (e.flow != e.cap &&
                    dis[e.to] > dis[u] + e.cost) {
                    dis[e.to] = dis[u] + e.cost, past[e.to] = &e;
                    up[e.to] = min(up[u], e.cap - e.flow);
                    if (!inq[e.to]) inq[e.to] = 1, q.push(e.to);
                }
            }
            if (dis[t] == INF) return 0;
            flow += up[t], cost += up[t] * dis[t];
            for (ll i = t; past[i]; i = past[i]->from) {
                auto &e = *past[i];
                e.flow += up[t], g[e.to][e.rev].flow -= up[t];
            }
            return 1;
        }
    }
    pll MinCostMaxFlow(int _s, int _t) {
        s = _s, t = _t;
        ll flow = 0, cost = 0;
        while (BellmanFord(flow, cost));
        return pll(flow, cost);
    }
    void init(int _n, ll _mx) {
        n = _n, mx = _mx;
        g.assign(n, vector<edge>());
    }
    void add_edge(int a, int b, ll cap, ll cost) {
        g[a].pb(edge{a, b, cap, 0, cost, SZ(g[b]))});
        g[b].pb(edge{b, a, 0, 0, -cost, SZ(g[a]) - 1});
    }
};

```

### 3.4 Min Cost Circulation

```

struct MinCostCirculation { //  $\theta$ -based,  $O(VE * E \log C)$ 
    struct edge {
        ll from, to, cap, fcap, flow, cost, rev;
    };
    int n;
    vector<edge*> past;
    vector<vector<edge>> g;
    vector<ll> dis;
    void BellmanFord(int s) {
        vector<int> inq(n);
        dis.assign(n, INF);
        queue<int> q;
        auto relax = [&](int u, ll d, edge *e) {
            if (dis[u] > d) {
                dis[u] = d, past[u] = e;
                if (!inq[u]) inq[u] = 1, q.push(u);
            }
        };
        relax(s, 0, 0);
        while (!q.empty()) {
            int u = q.front();
            q.pop(), inq[u] = 0;
            for (auto &e : g[u])
                if (e.cap > e.flow)
                    relax(e.to, dis[u] + e.cost, &e);
        }
    }
    void try_edge(edge &cur) {
        if (cur.cap > cur.flow) return ++cur.cap, void();
        BellmanFord(cur.to);
        if (dis[cur.from] + cur.cost < 0) {
            ++cur.flow, --g[cur.to][cur.rev].flow;
            for (int i = cur.from; past[i]; i = past[i]->from) {
                auto &e = *past[i];
                ++e.flow, --g[e.to][e.rev].flow;
            }
        }
        ++cur.cap;
    }
};

```

```

void solve(int mxlg) { // mxlg >= log(max cap)
    for (int b = mxlg; b >= 0; --b) {
        for (int i = 0; i < n; ++i)
            for (auto &e : g[i])
                e.cap *= 2, e.flow *= 2;
        for (int i = 0; i < n; ++i)
            for (auto &e : g[i])
                if (e.fcap >> b & 1)
                    try_edge(e);
    }
}

void init(int _n) {
    n = _n;
    past.assign(n, nullptr);
    g.assign(n, vector<edge>());
}

void add_edge(ll a, ll b, ll cap, ll cost) {
    g[a].pb(edge{a, b, 0, cap, 0, cost, SZ(g[b]) + (a == b)});
    g[b].pb(edge{b, a, 0, 0, 0, -cost, SZ(g[a]) - 1});
}
};

```

### 3.5 Gomory Hu

```

void GomoryHu(Dinic &flow) { // 0-based
    int n = flow.n;
    vector<int> par(n);
    for (int i = 1; i < n; ++i) {
        flow.reset();
        add_edge(i, par[i], flow.maxflow(i, par[i]));
        for (int j = i + 1; j < n; ++j)
            if (par[j] == par[i] && ~flow.dis[j])
                par[j] = i;
    }
}

```

### 3.6 Stoer Wagner Algorithm

```

struct StoerWagner { // 0-based,  $O(V^3)$ 
    int n;
    vector<int> vis, del;
    vector<ll> wei;
    vector<vector<ll>> edge;
    void init(int _n) {
        n = _n;
        del.assign(n, 0);
        edge.assign(n, vector<ll>(n));
    }
    void add_edge(int u, int v, ll w) {
        edge[u][v] += w, edge[v][u] += w;
    }
    void search(int &s, int &t) {
        vis.assign(n, 0); wei.assign(n, 0);
        s = t = -1;
        while (1) {
            ll mx = -1, cur = 0;
            for (int i = 0; i < n; ++i)
                if (!del[i] && !vis[i] && mx < wei[i])
                    cur = i, mx = wei[i];
            if (mx == -1) break;
            vis[cur] = 1, s = t, t = cur;
            for (int i = 0; i < n; ++i)
                if (!vis[i] && !del[i]) wei[i] += edge[cur][i];
        }
    }
    ll solve() {
        ll ret = INF;
        for (int i = 0, x=0, y=0; i < n-1; ++i) {
            search(x, y), ret = min(ret, wei[y]), del[y] = 1;
            for (int j = 0; j < n; ++j)
                edge[x][j] = (edge[j][x] += edge[y][j]);
        }
        return ret;
    }
};

```

```
};
```

### 3.7 Bipartite Matching

```

//min vertex cover: take all unmatched vertices in L and
//find alternating tree,
//ans is not reached in L + reached in R
// O(VE)
int n; // 1-based, max matching
int mx[maxn], my[maxn];
bool adj[maxn][maxn], vis[maxn];
bool dfs(int u) {
    if (vis[u]) return 0;
    vis[u] = 1;
    for (int v = 1; v <= n; v++) {
        if (!adj[u][v]) continue;
        if (!my[v] || (my[v] && dfs(my[v]))) {
            mx[u] = v, my[v] = u;
            return 1;
        }
    }
    return 0;
}

// O(E sqrt(V)), O(E log V) for random sparse graphs
struct BipartiteMatching { // 0-based
    int nl, nr;
    vector<int> mx, my, dis, cur;
    vector<vector<int>> g;
    bool dfs(int u) {
        for (int &i = cur[u]; i < SZ(g[u]); ++i) {
            int e = g[u][i];
            if (!my[e] || (dis[my[e]] == dis[u] + 1 && dfs(my[e])))
                return mx[my[e] = u] = e, 1;
        }
        dis[u] = -1;
        return 0;
    }
    bool bfs() {
        int ret = 0;
        queue<int> q;
        dis.assign(nl, -1);
        for (int i = 0; i < nl; ++i)
            if (!mx[i]) q.push(i), dis[i] = 0;
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            for (int e : g[u])
                if (!my[e]) ret = 1;
                else if (!dis[my[e]]) {
                    q.push(my[e]);
                    dis[my[e]] = dis[u] + 1;
                }
        }
        return ret;
    }
    int matching() {
        int ret = 0;
        mx.assign(nl, -1); my.assign(nr, -1);
        while (bfs()) {
            cur.assign(nl, 0);
            for (int i = 0; i < nl; ++i)
                if (!mx[i] && dfs(i)) ++ret;
        }
        return ret;
    }
    void add_edge(int s, int t) { g[s].pb(t); }
    void init(int _nl, int _nr) {
        nl = _nl, nr = _nr;
        g.assign(nl, vector<int>());
    }
};

```

### 3.8 Kuhn Munkres Algorithm

```

struct KM { //  $\theta$ -based, maximum matching,  $O(V^3)$ 
    int n, ql, qr;
    vector<vector<ll>> w;
    vector<ll> hl, hr, slk;
    vector<int> fl, fr, pre, qu, vl, vr;
    void init(int _n) {
        n = _n;
        // -INF for perfect matching
        w.assign(n, vector<ll>(n, 0));
        pre.assign(n, 0);
        qu.assign(n, 0);
    }
    void add_edge(int a, int b, ll wei) {
        w[a][b] = wei;
    }
    bool check(int x) {
        if (vl[x] = 1, ~fl[x])
            return (vr[qu[qr++]] = fl[x]) = 1;
        while (~x) swap(x, fr[fl[x] = pre[x]]);
        return 0;
    }
    void bfs(int s) {
        slk.assign(n, INF); vl.assign(n, 0); vr.assign(n, 0);
        ql = qr = 0, qu[qr++] = s, vr[s] = 1;
        for (ll d;;) {
            while (ql < qr)
                for (int x = 0, y = qu[ql++]; x < n; ++x)
                    if (!vl[x] && slk[x] >= (d = hl[x] + hr[y] - w[x][y])) {
                        if (pre[x] = y, d) slk[x] = d;
                        else if (!check(x)) return;
                    }
            d = INF;
            for (int x = 0; x < n; ++x)
                if (!vl[x] && d > slk[x]) d = slk[x];
            for (int x = 0; x < n; ++x) {
                if (vl[x]) hl[x] += d;
                else slk[x] -= d;
                if (vr[x]) hr[x] -= d;
            }
            for (int x = 0; x < n; ++x)
                if (!vl[x] && !slk[x] && !check(x)) return;
        }
        ll solve() {
            fl.assign(n, -1); fr.assign(n, -1); hl.assign(n, 0); hr.assign(n, 0);
            for (int i = 0; i < n; ++i)
                hl[i] = *max_element(iter(w[i]));
            for (int i = 0; i < n; ++i) bfs(i);
            ll res = 0;
            for (int i = 0; i < n; ++i) res += w[i][fl[i]];
            return res;
        }
    }
};

```

### 3.9 Max Simple Graph Matching

```

struct Matching { //  $\theta$ -based,  $O(V^3)$ 
    queue<int> q; int n;
    vector<int> fa, s, vis, pre, match;
    vector<vector<int>> g;
    int Find(int u)
    { return u == fa[u] ? u : fa[u] = Find(fa[u]); }
    int LCA(int x, int y) {
        static int tk = 0; tk++; x = Find(x); y = Find(y);
        for (;;) swap(x, y) if (x != n) {
            if (vis[x] == tk) return x;
            vis[x] = tk;
            x = Find(pre[match[x]]);
        }
    }
    void Blossom(int x, int y, int l) {
        for (; Find(x) != l; x = pre[y]) {
            pre[x] = y, y = match[x];
            if (s[y] == 1) q.push(y), s[y] = 0;
        }
    }
};

```

```

        for (int z: {x, y}) if (fa[z] == z) fa[z] = l;
    }
}
bool Bfs(int r) {
    iota(iter(fa), 0); fill(iter(s), -1);
    q = queue<int>(); q.push(r); s[r] = 0;
    for (; !q.empty(); q.pop()) {
        for (int x = q.front(); int u : g[x])
            if (s[u] == -1) {
                if (pre[u] = x, s[u] = 1, match[u] == n) {
                    for (int a = u, b = x, last;
                        b != n; a = last, b = pre[a])
                        last = match[b], match[b] = a, match[a] = b;
                    return true;
                }
                q.push(match[u]); s[match[u]] = 0;
            }
        else if (!s[u] && Find(u) != Find(x)) {
            int l = LCA(u, x);
            Blossom(x, u, l); Blossom(u, x, l);
        }
    }
    return false;
}
Matching(int _n) : n(_n), fa(n + 1), s(n + 1), vis(n + 1),
    pre(n + 1, n), match(n + 1, n), g(n) {}
void add_edge(int u, int v)
{ g[u].pb(v), g[v].pb(u); }
int solve() {
    int ans = 0;
    for (int x = 0; x < n; ++x)
        if (match[x] == n) ans += Bfs(x);
    return ans;
} // match[x] == n means not matched
};

```

### 3.10 Stable Marriage

- 1: Initialize  $m \in M$  and  $w \in W$  to free
- 2: **while**  $\exists$  free man  $m$  who has a woman  $w$  to propose to **do**
- 3:    $w \leftarrow$  first woman on  $m$ 's list to whom  $m$  has not yet proposed
- 4:   **if**  $\exists$  some pair  $(m', w)$  **then**
- 5:     **if**  $w$  prefers  $m$  to  $m'$  **then**
- 6:        $m' \leftarrow$  free
- 7:        $(m, w) \leftarrow$  engaged
- 8:     **end if**
- 9:   **else**
- 10:      $(m, w) \leftarrow$  engaged
- 11:   **end if**
- 12: **end while**

### 3.11 Flow Model

- Maximum/Minimum flow with lower bound / Circulation problem
  1. Construct super source  $S$  and sink  $T$ .
  2. For each edge  $(x, y, l, u)$ , connect  $x \rightarrow y$  with capacity  $u - l$ .
  3. For each vertex  $v$ , denote by  $in(v)$  the difference between the sum of incoming lower bounds and the sum of outgoing lower bounds.
  4. If  $in(v) > 0$ , connect  $S \rightarrow v$  with capacity  $in(v)$ , otherwise, connect  $v \rightarrow T$  with capacity  $-in(v)$ .
    - To maximize, connect  $t \rightarrow s$  with capacity  $\infty$  (skip this in circulation problem), and let  $f$  be the maximum flow from  $S$  to  $T$ . If  $f \neq \sum_{v \in V, in(v) > 0} in(v)$ , there's no solution. Otherwise, the maximum flow from  $s$  to  $t$  is the answer.
    - To minimize, let  $f$  be the maximum flow from  $S$  to  $T$ . Connect  $t \rightarrow s$  with capacity  $\infty$  and let the flow from  $S$  to  $T$  be  $f'$ . If  $f + f' \neq \sum_{v \in V, in(v) > 0} in(v)$ , there's no solution. Otherwise,  $f'$  is the answer.
  5. The solution of each edge  $e$  is  $l_e + f_e$ , where  $f_e$  corresponds to the flow of edge  $e$  on the graph.
- Construct minimum vertex cover from maximum matching  $M$  on bipartite graph  $(X, Y)$ 
  1. Redirect every edge:  $y \rightarrow x$  if  $(x, y) \in M$ ,  $x \rightarrow y$  otherwise.
  2. DFS from unmatched vertices in  $X$ .
  3.  $x \in X$  is chosen iff  $x$  is unvisited.
  4.  $y \in Y$  is chosen iff  $y$  is visited.
- Minimum cost cyclic flow
  1. Construct super source  $S$  and sink  $T$

2. For each edge  $(x, y, c)$ , connect  $x \rightarrow y$  with  $(cost, cap) = (c, 1)$  if  $c > 0$ , otherwise connect  $y \rightarrow x$  with  $(cost, cap) = (-c, 1)$
3. For each edge with  $c < 0$ , sum these cost as  $K$ , then increase  $d(y)$  by 1, decrease  $d(x)$  by 1
4. For each vertex  $v$  with  $d(v) > 0$ , connect  $S \rightarrow v$  with  $(cost, cap) = (0, d(v))$
5. For each vertex  $v$  with  $d(v) < 0$ , connect  $v \rightarrow T$  with  $(cost, cap) = (0, -d(v))$
6. Flow from  $S$  to  $T$ , the answer is the cost of the flow  $C + K$
- Maximum density induced subgraph
  1. Binary search on answer, suppose we're checking answer  $T$
  2. Construct a max flow model, let  $K$  be the sum of all weights
  3. Connect source  $s \rightarrow v, v \in G$  with capacity  $K$
  4. For each edge  $(u, v, w)$  in  $G$ , connect  $u \rightarrow v$  and  $v \rightarrow u$  with capacity  $w$
  5. For  $v \in G$ , connect it with sink  $v \rightarrow t$  with capacity  $K + 2T - (\sum_{e \in E(v)} w(e))$
6.  $T$  is a valid answer if the maximum flow  $f < K|V|$
- Minimum weight edge cover
  1. For each  $v \in V$  create a copy  $v'$ , and connect  $u' \rightarrow v'$  with weight  $w(u, v)$ .
  2. Connect  $v \rightarrow v'$  with weight  $2\mu(v)$ , where  $\mu(v)$  is the cost of the cheapest edge incident to  $v$ .
  3. Find the minimum weight perfect matching on  $G'$ .
- Project selection problem
  1. If  $p_v > 0$ , create edge  $(s, v)$  with capacity  $p_v$ ; otherwise, create edge  $(v, t)$  with capacity  $-p_v$ .
  2. Create edge  $(u, v)$  with capacity  $w$  with  $w$  being the cost of choosing  $u$  without choosing  $v$ .
  3. The mincut is equivalent to the maximum profit of a subset of projects.
- Dual of minimum cost maximum flow
  1. Capacity  $c_{uv}$ , Flow  $f_{uv}$ , Cost  $w_{uv}$ , Required Flow difference for vertex  $b_u$ .
  2. If all  $w_{uv}$  are integers, then optimal solution can happen when all  $p_u$  are integers.

$$\min \sum_{uv} w_{uv} f_{uv} \quad \min \sum_u b_u p_u + \sum_{uv} c_{uv} \max(0, p_v - p_u - w_{uv})$$

$$-f_{uv} \geq -c_{uv} \Leftrightarrow \sum_v f_{vu} - \sum_v f_{uv} = -b_u \quad p_u \geq 0$$

## 4 Geometry

### 4.1 Geometry Template

```
using ld = ll;
using pdd = pair<ld, ld>;
using Line = pair<pdd, pdd>;
#define X first
#define Y second
// ld eps = 1e-7;

pdd operator+(pdd a, pdd b)
{ return {a.X + b.X, a.Y + b.Y}; }
pdd operator-(pdd a, pdd b)
{ return {a.X - b.X, a.Y - b.Y}; }
pdd operator*(ld i, pdd v)
{ return {i * v.X, i * v.Y}; }
pdd operator*(pdd v, ld i)
{ return {i * v.X, i * v.Y}; }
pdd operator/(pdd v, ld i)
{ return {v.X / i, v.Y / i}; }
ld dot(pdd a, pdd b)
{ return a.X * b.X + a.Y * b.Y; }
ld cross(pdd a, pdd b)
{ return a.X * b.Y - a.Y * b.X; }
ld abs2(pdd v)
{ return v.X * v.X + v.Y * v.Y; }
ld abs(pdd v)
{ return sqrt(abs2(v)); }
int sgn(ld v)
{ return v > 0 ? 1 : (v < 0 ? -1 : 0); }
// int sgn(ld v){ return v > eps ? 1 : (v < -eps ? -1 : 0); }

int ori(pdd a, pdd b, pdd c)
{ return sgn(cross(b - a, c - a)); }
bool collinearity(pdd a, pdd b, pdd c)
```

```
{ return ori(a, b, c) == 0; }
bool btw(pdd p, pdd a, pdd b)
{ return collinearity(p, a, b) && sgn(dot(a - p, b - p)) <= 0; }

bool seg_intersect(Line a, Line b){
    pdd p1, p2, p3, p4;
    tie(p1, p2) = a; tie(p3, p4) = b;
    if(btw(p1, p3, p4) || btw(p2, p3, p4) || btw(p3, p1, p2) || btw(p4, p1, p2))
        return true;
    return ori(p1, p2, p3) * ori(p1, p2, p4) < 0 && ori(p3, p4, p1) * ori(p3, p4, p2) < 0;
}

pdd intersect(Line a, Line b){
    pdd p1, p2, p3, p4;
    tie(p1, p2) = a; tie(p3, p4) = b;
    ld a123 = cross(p2 - p1, p3 - p1);
    ld a124 = cross(p2 - p1, p4 - p1);
    return (p4 * a123 - p3 * a124) / (a123 - a124);
}

pdd perp(pdd p1)
{ return pdd(-p1.Y, p1.X); }
pdd projection(pdd p1, pdd p2, pdd p3)
{ return p1 + (p2 - p1) * dot(p3 - p1, p2 - p1) / abs2(p2 - p1); }
pdd reflection(pdd p1, pdd p2, pdd p3)
{ return p3 + perp(p2 - p1) * cross(p3 - p1, p2 - p1) / abs2(p2 - p1) * 2; }
pdd linearTransformation(pdd p0, pdd p1, pdd q0, pdd q1, pdd r) {
    pdd dp = p1 - p0, dq = q1 - q0, num(cross(dp, dq), dot(dp, dq));
    return q0 + pdd(cross(r - p0, num), dot(r - p0, num)) / abs2(dp);
} // from line p0--p1 to q0--q1, apply to r
```

### 4.2 Convex Hull

```
vector<int> getConvexHull(vector<pdd>& pts){
    vector<int> id(SZ(pts));
    iota(iter(id), 0);
    sort(iter(id), [&](int x, int y){ return pts[x] < pts[y]; });
    vector<int> hull;
    for(int tt = 0; tt < 2; tt++){
        int sz = SZ(hull);
        for(int j : id){
            pdd p = pts[j];
            while(SZ(hull) - sz >= 2 && cross(pts[hull.back()] - pts[hull[SZ(hull) - 2]], p - pts[hull[SZ(hull) - 2]]) <= 0)
                hull.pop_back();
            hull.pb(j);
        }
        hull.pop_back();
        reverse(iter(id));
    }
    return hull;
}
```

### 4.3 Minimum Enclosing Circle

```
using ld = long double;
pair<pdd, ld> circumcenter(pdd a, pdd b, pdd c);
pair<pdd, ld> MinimumEnclosingCircle(vector<pdd> &pts){
    random_shuffle(iter(pts));
    pdd c = pts[0];
    ld r = 0;
    for(int i = 1; i < SZ(pts); i++){
        if(abs(pts[i] - c) <= r) continue;
        c = pts[i]; r = 0;
        for(int j = 0; j < i; j++){
            if(abs(pts[j] - c) <= r) continue;
            c = (pts[i] + pts[j]) / 2;
        }
    }
}
```



```

    r = abs(pts[i] - c);
    for(int k = 0; k < j; k++){
        if(abs(pts[k] - c) > r)
            tie(c, r) = circumcenter(pts[i], pts[j], pts[k]);
    }
}
return {c, r};
}

```

#### 4.4 Minkowski Sum

```

void reorder_poly(vector<pdd>& pnts){
    int mn = 0;
    for(int i = 1; i < (int)pnts.size(); i++){
        if(pnts[i].Y < pnts[mn].Y || (pnts[i].Y == pnts[mn].Y
            && pnts[i].X < pnts[mn].X))
            mn = i;
    }
    rotate(pnts.begin(), pnts.begin() + mn, pnts.end());
}

vector<pdd> minkowski(vector<pdd> P, vector<pdd> Q){
    reorder_poly(P);
    reorder_poly(Q);
    int psz = P.size();
    int qsz = Q.size();
    P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
    vector<pdd> ans;
    int i = 0, j = 0;
    while(i < psz || j < qsz){
        ans.pb(P[i] + Q[j]);
        int t = sgn(cross(P[i + 1] - P[i], Q[j + 1] - Q[j]));
        if(t >= 0) i++;
        if(t <= 0) j++;
    }
    return ans;
}

```

#### 4.5 Polar Angle Comparator

```

// -1: a // b (if same), 0/1: a < b
int cmp(pll a, pll b, bool same = true){
#define is_neg(k) (sgn(k.Y) < 0 || (sgn(k.Y) == 0 && sgn(k.X) < 0))
    int A = is_neg(a), B = is_neg(b);
    if(A != B)
        return A < B;
    if(sgn(cross(a, b)) == 0)
        return same ? abs2(a) < abs2(b) : -1;
    return sgn(cross(a, b)) > 0;
}

```

#### 4.6 Half Plane Intersection

```

// from 8BQube
pll area_pair(Line a, Line b)
{ return pll(cross(a.Y - a.X, b.X - a.X), cross(a.Y - a.X,
    b.Y - a.X)); }
bool isin(Line l0, Line l1, Line l2) {
    // Check inter(l1, l2) strictly in l0
    auto [a02X, a02Y] = area_pair(l0, l2);
    auto [a12X, a12Y] = area_pair(l1, l2);
    if (a12X - a12Y < 0) a12X *= -1, a12Y *= -1;
    return ((__int128) a02Y * a12X - (__int128) a02X * a12Y >
        0; // C^4
}
/* Having solution, check size > 2 */
/* --^-- Line.X --^-- Line.Y --^-- */
vector<Line> halfPlaneInter(vector<Line> arr) {
    sort(iter(arr), [&](Line a, Line b) -> int {
        if (cmp(a.Y - a.X, b.Y - b.X, 0) != -1)
            return cmp(a.Y - a.X, b.Y - b.X, 0);
        return ori(a.X, a.Y, b.Y) < 0;
    });
}

```

```

});
deque<Line> dq(1, arr[0]);
for (auto p : arr) {
    if (cmp(dq.back().Y - dq.back().X, p.Y - p.X, 0) == -1)
        continue;
    while (SZ(dq) >= 2 && !isin(p, dq[SZ(dq) - 2], dq.back()))
        dq.pop_back();
    while (SZ(dq) >= 2 && !isin(p, dq[0], dq[1]))
        dq.pop_front();
    dq.pb(p);
}
while (SZ(dq) >= 3 && !isin(dq[0], dq[SZ(dq) - 2], dq.back()))
    dq.pop_back();
while (SZ(dq) >= 3 && !isin(dq.back(), dq[0], dq[1]))
    dq.pop_front();
return vector<Line>(iter(dq));
}

```

#### 4.7 Dynamic Convex Hull

```

struct Line{
    ll a, b, l = MIN, r = MAX;
    Line(ll a, ll b): a(a), b(b) {}
    ll operator()(ll x) const{
        return a * x + b;
    }
    bool operator<(Line b) const{
        return a < b.a;
    }
    bool operator<(ll b) const{
        return r < b;
    }
};

ll iceil(ll a, ll b){
    if(b < 0) a *= -1, b *= -1;
    if(a > 0) return (a + b - 1) / b;
    else return a / b;
}

ll intersect(Line a, Line b){
    return iceil(a.b - b.b, b.a - a.a);
}

struct DynamicConvexHull{
    multiset<Line, less<>> ch;

    void add(Line ln){
        auto it = ch.lower_bound(ln);
        while(it != ch.end()){
            Line tl = *it;
            if(tl(tl.r) <= ln(tl.r)){
                it = ch.erase(it);
            }
            else break;
        }
        auto it2 = ch.lower_bound(ln);
        while(it2 != ch.begin()){
            Line tl = *prev(it2);
            if(tl(tl.l) <= ln(tl.l)){
                it2 = ch.erase(prev(it2));
            }
            else break;
        }
        it = ch.lower_bound(ln);
        if(it != ch.end()){
            Line tl = *it;
            if(tl(tl.l) >= ln(tl.l)) ln.r = tl.l - 1;
            else{
                ll pos = intersect(ln, tl);
                tl.l = pos;
                ln.r = pos - 1;
                ch.erase(it);
                ch.insert(tl);
            }
        }
    }
}

```

```

    }
}
it2 = ch.lower_bound(ln);
if(it2 != ch.begin()){
    Line tl = *prev(it2);
    if(tl.tl.r >= ln.tl.r) ln.l = tl.r + 1;
    else{
        ll pos = intersect(tl, ln);
        tl.r = pos - 1;
        ln.l = pos;
        ch.erase(prev(it2));
        ch.insert(tl);
    }
}
if(ln.l <= ln.r) ch.insert(ln);
}

ll query(ll pos){
    auto it = ch.lower_bound(pos);
    if(it == ch.end()) return 0;
    return (*it)(pos);
}
};

```

## 4.8 3D Point

// Copy from 8BQube

```

struct Point {
    double x, y, z;
    Point(double _x = 0, double _y = 0, double _z = 0): x(_x)
        , y(_y), z(_z){}
    Point(pdd p) { x = p.X, y = p.Y, z = abs2(p); }
};

Point operator-(Point p1, Point p2)
{ return Point(p1.x - p2.x, p1.y - p2.y, p1.z - p2.z); }
Point operator+(Point p1, Point p2)
{ return Point(p1.x + p2.x, p1.y + p2.y, p1.z + p2.z); }
Point operator*(Point p1, double v)
{ return Point(p1.x * v, p1.y * v, p1.z * v); }
Point operator/(Point p1, double v)
{ return Point(p1.x / v, p1.y / v, p1.z / v); }
Point cross(Point p1, Point p2)
{ return Point(p1.y * p2.z - p1.z * p2.y, p1.z * p2.x - p1.x * p2.z, p1.x * p2.y - p1.y * p2.x); }
double dot(Point p1, Point p2)
{ return p1.x * p2.x + p1.y * p2.y + p1.z * p2.z; }
double abs(Point a)
{ return sqrt(dot(a, a)); }
Point cross3(Point a, Point b, Point c)
{ return cross(b - a, c - a); }
double area(Point a, Point b, Point c)
{ return abs(cross3(a, b, c)); }
double volume(Point a, Point b, Point c, Point d)
{ return dot(cross3(a, b, c), d - a); }
//Azimuthal angle (Longitude) to x-axis in interval [-pi, pi]
double phi(Point p) { return atan2(p.y, p.x); }
//Zenith angle (Latitude) to the z-axis in interval [0, pi]
double theta(Point p) { return atan2(sqrt(p.x * p.x + p.y * p.y), p.z); }
Point masscenter(Point a, Point b, Point c, Point d)
{ return (a + b + c + d) / 4; }
pdd proj(Point a, Point b, Point c, Point u) {
    // proj. u to the plane of a, b, and c
    Point e1 = b - a;
    Point e2 = c - a;
    e1 = e1 / abs(e1);
    e2 = e2 - e1 * dot(e2, e1);
    e2 = e2 / abs(e2);
    Point p = u - a;
    return pdd(dot(p, e1), dot(p, e2));
}
Point rotate_around(Point p, double angle, Point axis) {
    double s = sin(angle), c = cos(angle);
    Point u = axis / abs(axis);
    return u * dot(u, p) * (1 - c) + p * c + cross(u, p) * s;
}

```

```

}

```

## 4.9 ConvexHull3D

```

struct convex_hull_3D {
    struct Face {
        int a, b, c;
        Face(int ta, int tb, int tc): a(ta), b(tb), c(tc) {}
    }; // return the faces with pt indexes
    vector<Face> res;
    vector<Point> P;
    convex_hull_3D(const vector<Point> &P): res(), P(P) {
        // all points coplanar case will WA, O(n^2)
        int n = SZ(P);
        if (n <= 2) return; // be careful about edge case
        // ensure first 4 points are not coplanar
        swap(P[1], *find_if(iter(P), [&](auto p) { return sgn(
            abs2(P[0] - p)) != 0; }));
        swap(P[2], *find_if(iter(P), [&](auto p) { return sgn(
            abs2(cross3(p, P[0], P[1]))) != 0; }));
        swap(P[3], *find_if(iter(P), [&](auto p) { return sgn(
            volume(P[0], P[1], P[2], p)) != 0; }));
        vector<vector<int>> flag(n, vector<int>(n));
        res.emplace_back(0, 1, 2); res.emplace_back(2, 1, 0);
        for (int i = 3; i < n; ++i) {
            vector<Face> next;
            for (auto f : res) {
                int d = sgn(volume(P[f.a], P[f.b], P[f.c], P[i]));
                if (d <= 0) next.pb(f);
                int ff = (d > 0) - (d < 0);
                flag[f.a][f.b] = flag[f.b][f.c] = flag[f.c][f.a] = ff;
            }
            for (auto f : res) {
                auto F = [&](int x, int y) {
                    if (flag[x][y] > 0 && flag[y][x] <= 0)
                        next.emplace_back(x, y, i);
                };
                F(f.a, f.b); F(f.b, f.c); F(f.c, f.a);
            }
            res = next;
        }
    }
    bool same(Face s, Face t) {
        if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.a])) != 0)
            return 0;
        if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.b])) != 0)
            return 0;
        if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.c])) != 0)
            return 0;
        return 1;
    }
    int polygon_face_num() {
        int ans = 0;
        for (int i = 0; i < SZ(res); ++i)
            ans += none_of(res.begin(), res.begin() + i, [&](Face g) {
                return same(res[i], g);
            });
        return ans;
    }
    double get_volume() {
        double ans = 0;
        for (auto f : res)
            ans += volume(Point(0, 0, 0), P[f.a], P[f.b], P[f.c]);
        return fabs(ans / 6);
    }
    double get_dis(Point p, Face f) {
        Point p1 = P[f.a], p2 = P[f.b], p3 = P[f.c];
        double a = (p2.y - p1.y) * (p3.z - p1.z) - (p2.z - p1.z) * (p3.y - p1.y);
        double b = (p2.z - p1.z) * (p3.x - p1.x) - (p2.x - p1.x) * (p3.z - p1.z);
        double c = (p2.x - p1.x) * (p3.y - p1.y) - (p2.y - p1.y) * (p3.x - p1.x);
        double d = 0 - (a * p1.x + b * p1.y + c * p1.z);
        return fabs(a * p.x + b * p.y + c * p.z + d) / sqrt(a * a + b * b + c * c);
    }
}

```

```

}
};
// n^2 delaunay: facets with negative z normal of
// convexhull of (x, y, x^2 + y^2), use a pseudo-point
// (0, 0, inf) to avoid degenerate case

```

## 4.10 Circle Operations

```

// from 8BQube
const double PI=acos(-1);
vector<pdd> circleLineIntersection(pdd c, double r, pdd a,
    pdd b) {
    pdd p = a + (b - a) * dot(c - a, b - a) / abs2(b - a);
    double s = cross(b - a, c - a), h2 = r * r - s * s / abs2
        (b - a);
    if (sgn(h2) < 0) return {};
    if (sgn(h2) == 0) return {p};
    pdd h = (b - a) / abs(b - a) * sqrt(h2);
    return {p - h, p + h};
}
double _area(pdd pa, pdd pb, double r){
    if(abs(pa)<abs(pb)) swap(pa, pb);
    if(abs(pb)<eps) return 0;
    double S, h, theta;
    double a=abs(pb),b=abs(pa),c=abs(pb-pa);
    double cosB = dot(pb,pb-pa) / a / c, B = acos(cosB);
    double cosC = dot(pa,pb) / a / b, C = acos(cosC);
    if(a > r){
        S = (C/2)*r*r;
        h = a*b*sin(C)/c;
        if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt(r*r
            -h*h));
    }
    else if(b > r){
        theta = PI - B - asin(sin(B)/r*a);
        S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
    }
    else S = .5*sin(C)*a*b;
    return S;
}
double areaPolyCircle(const vector<pdd> poly,const pdd &O,
    const double r){
    double S=0;
    for(int i=0;i<SZ(poly);++i)
        S+=_area(poly[i]-O,poly[(i+1)%SZ(poly)]-O,r)*ori(O,poly
            [i],poly[(i+1)%SZ(poly)]);
    return fabs(S);
}
bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
    pdd o1 = a.O, o2 = b.O;
    double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d = sqrt(
        d2);
    if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2) return
        0;
    pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1 * r1
        ) / (2 * d2));
    double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 + r2
        - d) * (-r1 + r2 + d));
    pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2);
    p1 = u + v, p2 = u - v;
    return 1;
}
vector<Line> CCTang( const Cir& c1 , const Cir& c2 , int
    sign1 ){
    vector<Line> ret;
    double d_sq = abs2( c1.O - c2.O );
    if (sgn(d_sq) == 0) return ret;
    double d = sqrt(d_sq);
    pdd v = (c2.O - c1.O) / d;
    double c = (c1.R - sign1 * c2.R) / d; // cos t
    if (c * c > 1) return ret;
    double h = sqrt(max( 0.0, 1.0 - c * c )); // sin t
    for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
        pdd n = pdd(v.X * c - sign2 * h * v.Y,
            v.Y * c + sign2 * h * v.X);
        pdd p1 = c1.O + n * c1.R;

```

```

        pdd p2 = c2.O + n * (c2.R * sign1);
        if (sgn(p1.X - p2.X) == 0 and
            sgn(p1.Y - p2.Y) == 0)
            p2 = p1 + perp(c2.O - c1.O);
        ret.pb(Line(p1, p2));
    }
    return ret;
}

```

## 4.11 Delaunay Triangulation

```

/* Delaunay Triangulation:
Given a sets of points on 2D plane, find a
triangulation such that no points will strictly
inside circumcircle of any triangle. */
struct Edge {
    int id; // oidx[id]
    list<Edge>::iterator twin;
    Edge(int _id = 0):id(_id) {}
};
struct Delaunay { // 0-base
    int n, oidx[N];
    list<Edge> head[N]; // result udir. graph
    pll p[N];
    void init(int _n, pll _p[]) {
        n = _n, iota(oidx, oidx + n, 0);
        for (int i = 0; i < n; ++i) head[i].clear();
        sort(oidx, oidx + n, [&](int a, int b)
            { return _p[a] < _p[b]; });
        for (int i = 0; i < n; ++i) p[i] = _p[oidx[i]];
        divide(0, n - 1);
    }
    void addEdge(int u, int v) {
        head[u].push_front(Edge(v));
        head[v].push_front(Edge(u));
        head[u].begin()->twin = head[v].begin();
        head[v].begin()->twin = head[u].begin();
    }
    void divide(int l, int r) {
        if (l == r) return;
        if (l + 1 == r) return addEdge(l, l + 1);
        int mid = (l + r) >> 1, nw[2] = {l, r};
        divide(l, mid), divide(mid + 1, r);
        auto gao = [&](int t) {
            pll pt[2] = {p[nw[0]], p[nw[1]]};
            for (auto it : head[nw[t]]) {
                int v = ori(pt[1], pt[0], p[it.id]);
                if (v > 0 || (v == 0 && abs2(pt[t ^ 1] - p[it.id])
                    < abs2(pt[1] - pt[0])))
                    return nw[t] = it.id, true;
            }
            return false;
        };
        while (gao(0) || gao(1));
        addEdge(nw[0], nw[1]); // add tangent
        while (true) {
            pll pt[2] = {p[nw[0]], p[nw[1]]};
            int ch = -1, sd = 0;
            for (int t = 0; t < 2; ++t)
                for (auto it : head[nw[t]])
                    if (ori(pt[0], pt[1], p[it.id]) > 0 && (ch ==
                        -1 || in_cc({pt[0], pt[1], p[ch]}, p[it.
                            id])))
                        ch = it.id, sd = t;
            if (ch == -1) break; // upper common tangent
            for (auto it = head[nw[sd]].begin(); it != head[nw[sd]
                ].end(); )
                if (seg_strict_intersect(pt[sd], p[it->id], pt[sd ^
                    1], p[ch]))
                    head[it->id].erase(it->twin), head[nw[sd]].erase(
                        it++);
            else ++it;
            nw[sd] = ch, addEdge(nw[0], nw[1]);
        }
    }
} tool;

```

## 4.12 Voronoi Diagram

```
// all coord. is even, you may want to call halfPlaneInter
// after then
vector<vector<Line>> vec;
void build_voronoi_line(int n, pll *arr) {
    tool.init(n, arr); // Delaunay
    vec.clear(), vec.resize(n);
    for (int i = 0; i < n; ++i)
        for (auto e : tool.head[i]) {
            int u = tool.oidx[i], v = tool.oidx[e.id];
            pll m = (arr[v] + arr[u]) / 2LL, d = perp(arr[v] -
                arr[u]);
            vec[u].pb(Line(m, m + d));
        }
}
```

## 4.13 Polygon Union

```
// from 8BQube
ld rat(pll a, pll b) {
    return sgn(b.X) ? (ld)a.X / b.X : (ld)a.Y / b.Y;
} // all poly. should be ccw
ld polyUnion(vector<vector<pll>> &poly) {
    ld res = 0;
    for (auto &p : poly)
        for (int a = 0; a < SZ(p); ++a) {
            pll A = p[a], B = p[(a + 1) % SZ(p)];
            vector<pair<ld, int>> segs = {{0, 0}, {1, 0}};
            for (auto &q : poly) {
                if (&p == &q) continue;
                for (int b = 0; b < SZ(q); ++b) {
                    pll C = q[b], D = q[(b + 1) % SZ(q)];
                    int sc = ori(A, B, C), sd = ori(A, B, D);
                    if (sc != sd && min(sc, sd) < 0) {
                        ld sa = cross(D - C, A - C), sb = cross(D - C,
                            B - C);
                        segs.pb(sa / (sa - sb), sgn(sc - sd));
                    }
                    if (!sc && !sd && &q < &p && sgn(dot(B - A, D - C
                        )) > 0) {
                        segs.pb(rat(C - A, B - A), 1);
                        segs.pb(rat(D - A, B - A), -1);
                    }
                }
            }
            sort(iter(segs));
            for (auto &s : segs) s.X = clamp(s.X, 0.0, 1.0);
            ld sum = 0;
            int cnt = segs[0].second;
            for (int j = 1; j < SZ(segs); ++j) {
                if (!cnt) sum += segs[j].X - segs[j - 1].X;
                cnt += segs[j].Y;
            }
            res += cross(A, B) * sum;
        }
    return res / 2;
}
```

## 4.14 Tangent Point to Convex Hull

```
// from 8BQube
/* The point should be strictly out of hull
   return arbitrary point on the tangent line */
pii get_tangent(vector<pll> &C, pll p) {
    auto gao = [&](int s) {
        return cyc_tsearch(SZ(C), [&](int x, int y)
            { return ori(p, C[x], C[y]) == s; });
    };
    return pii(gao(1), gao(-1));
} // return (a, b), ori(p, C[a], C[b]) >= 0
```

## 5 Graph

### 5.1 Block Cut Tree

```
struct BCC{
    vector<int> v, e, cut;
};
struct BlockCutTree{ // 0-based, allow multi edges but not
    // allow loops
    int n, m, cnt = 0;
    // n:|V|, m:|E|, cnt:|bcc|
    vector<BCC> bcc;
    vector<vector<pii>> g; // original graph
    vector<pii> edges; // 0-based
    vector<vector<int>> vbcc;
    // vbcc[i] = BCCs containing vertex i, vbcc[i].size()>1
    // iff i is an articulation
    vector<int> ebcc;
    // edge i is a bridge iff bcc[ebcc[i]].e.size() == 1
    // block cut tree:
    // adj[bcc i]: bcc[i].cut
    // adj[cut i]: vbcc[i]

    BlockCutTree(int _n, vector<pii> _edges):
        n(_n), m(SZ(_edges)), g(_n), edges(_edges), vbcc(_n),
        ebcc(SZ(_edges)){
        for(int i = 0; i < m; i++){
            auto [u, v] = edges[i];
            g[u].pb(pii(v, i)); g[v].pb(pii(u, i));
        }
    }

    void build(){
        vector<int> in(n, -1), low(n, -1);
        vector<vector<int>> up(n);
        vector<int> stk;
        int ts = 0;
        auto _dfs = [&](auto dfs, int now, int par, int pe) ->
            void{
                if(pe != -1) up[now].pb(pe);
                in[now] = low[now] = ts++;
                stk.pb(now);
                for(auto [v, e] : g[now]){
                    if(e == pe) continue;
                    if(in[v] != -1){
                        if(in[v] < in[now]) up[now].pb(e);
                        low[now] = min(low[now], in[v]);
                        continue;
                    }
                    dfs(dfs, v, now, e);
                    low[now] = min(low[now], low[v]);
                }
                if((now != par && low[now] >= in[par]) || (now == par
                    && SZ(g[now]) == 0)){
                    bcc.pb();
                    while(true){
                        int v = stk.back();
                        stk.pop_back();
                        vbcc[v].pb(cnt);
                        bcc[cnt].v.pb(v);
                        for(int e : up[v]){
                            ebcc[e] = cnt;
                            bcc[cnt].e.pb(e);
                        }
                        if(v == now) break;
                    }
                    if(now != par){
                        vbcc[par].pb(cnt);
                        bcc[cnt].v.pb(par);
                    }
                    cnt++;
                }
            };
        for(int i = 0; i < n; i++){
            if(in[i] == -1) _dfs(_dfs, i, i, -1);
        }
        for(int i = 0; i < cnt; i++)
```

```

    for(int j : bcc[i].v)
        if(SZ(vbcc[j]) > 1) bcc[i].cut.pb(j);
}
};

```

## 5.2 2-SAT

```

struct SAT{ // 0-based, [n, 2n) is neg of [0, n)
    int n;
    vector<vector<int>> g, rg;
    bool ok = true;
    vector<bool> ans;

    void init(int _n){
        n = _n;
        g.resize(2 * n);
        rg.resize(2 * n);
        ans.resize(n);
    }

    int neg(int v){
        return v < n ? v + n : v - n;
    }

    void addEdge(int u, int v){
        g[u].pb(v);
        rg[v].pb(u);
    }

    void addClause(int a, int b){
        addEdge(neg(a), b);
        addEdge(neg(b), a);
    }

    void build(){
        vector<bool> vst(2 * n, false);
        vector<int> tmp, scc(2 * n, -1);
        int cnt = 1;
        function<void(int)> dfs = [&](int now){
            vst[now] = true;
            for(int i : rg[now]){
                if(vst[i]) continue;
                dfs(i);
            }
            tmp.pb(now);
        };
        for(int i = 0; i < 2 * n; i++){
            if(!vst[i]) dfs(i);
        }
        reverse(all(tmp));
        function<void(int, int)> dfs2 = [&](int now, int id){
            scc[now] = id;
            for(int i : g[now]){
                if(scc[i] != -1) continue;
                dfs2(i, id);
            }
        };
        for(int i : tmp){
            if(scc[i] == -1) dfs2(i, cnt++);
        }
        debug(scc);
        for(int i = 0; i < n; i++){
            if(scc[i] == scc[neg(i)]){
                ok = false;
                return;
            }
            if(scc[i] < scc[neg(i)]) ans[i] = true;
            else ans[i] = false;
        }
    }
};

```

## 5.3 Dominator Tree

```

// copy from 8BQube
struct dominator_tree { // 1-base
    vector<int> G[N], rG[N];
    int n, pa[N], dfn[N], id[N], Time;

```

```

    int semi[N], idom[N], best[N];
    vector<int> tree[N]; // dominator_tree
    void init(int _n) {
        n = _n;
        for (int i = 1; i <= n; ++i)
            G[i].clear(), rG[i].clear();
    }
    void add_edge(int u, int v) {
        G[u].pb(v), rG[v].pb(u);
    }
    void dfs(int u) {
        id[dfn[u] = ++Time] = u;
        for (auto v : G[u])
            if (!dfn[v]) dfs(v), pa[dfn[v]] = dfn[u];
    }
    int find(int y, int x) {
        if (y <= x) return y;
        int tmp = find(pa[y], x);
        if (semi[best[y]] > semi[best[pa[y]]])
            best[y] = best[pa[y]];
        return pa[y] = tmp;
    }
    void tarjan(int root) {
        Time = 0;
        for (int i = 1; i <= n; ++i) {
            dfn[i] = idom[i] = 0;
            tree[i].clear();
            best[i] = semi[i] = i;
        }
        dfs(root);
        for (int i = Time; i > 1; --i) {
            int u = id[i];
            for (auto v : rG[u])
                if (v = dfn[v]) {
                    find(v, i);
                    semi[i] = min(semi[i], semi[best[v]]);
                }
            tree[semi[i]].pb(i);
            for (auto v : tree[pa[i]]) {
                find(v, pa[i]);
                idom[v] =
                    semi[best[v]] == pa[i] ? pa[i] : best[v];
            }
            tree[pa[i]].clear();
        }
        for (int i = 2; i <= Time; ++i) {
            if (idom[i] != semi[i]) idom[i] = idom[idom[i]];
            tree[id[idom[i]]].pb(id[i]);
        }
    }
};

```

## 5.4 Virtual Tree

```

// copy from 8BQube
vector<int> vG[N];
int top, st[N];
int vrt = -1;

void insert(int u) {
    if (top == -1) return st[++top] = vrt = u, void();
    int p = LCA(st[top], u);
    if (dep[vrt] > dep[p]) vrt = p;
    if (p == st[top]) return st[++top] = u, void();
    while (top >= 1 && dep[st[top - 1]] >= dep[p])
        vG[st[top - 1]].pb(st[top]), --top;
    if (st[top] != p)
        vG[p].pb(st[top]), --top, st[++top] = p;
    st[++top] = u;
}

void reset(int u) {
    for (int i : vG[u]) reset(i);
    vG[u].clear();
}

```

```

void solve(vector<int> &v) {
    top = -1;
    sort(ALL(v),
        [&](int a, int b) { return dfn[a] < dfn[b]; });
    for (int i : v) insert(i);
    while (top > 0) vG[st[top - 1]].pb(st[top]), --top;
    // do something
    reset(vrt);
}

```

## 5.5 Directed Minimum Spanning Tree

```

const ll INF = LLONG_MAX;
struct edge{
    int u = -1, v = -1;
    ll w = INF;
    int id = -1;
};

// 0-based, E[i].id = i
bool DMST(int n, vector<edge> &E, int root, vector<edge> &
    sol){
    vector<int> id(n), vis(n);
    vector<edge> in(n);
    for(edge e : E)
        if(e.u != e.v && e.w < in[e.v].w && e.v != root)
            in[e.v] = e;
    for(int i = 0; i < n; i++)
        if(i != root && in[i].u == -1) return false; // no sol
    int cnt = 0;
    fill(iter(id), -1); fill(iter(vis), -1);
    for(int u = 0; u < n; u++){
        int v = u;
        while(vis[v] != u && id[v] == -1 && in[v].u != -1)
            vis[v] = u, v = in[v].u;
        if(v != root && id[v] == -1){
            for(int x = in[v].u; x != v; x = in[x].u)
                id[x] = cnt;
            id[v] = cnt++;
        }
    }
    if(!cnt) return sol = in, true; // no cycle
    for(int u = 0; u < n; u++)
        if(id[u] == -1) id[u] = cnt++;
    vector<edge> nE;
    for(int i = 0; i < SZ(E); i++){
        edge tmp = E[i];
        tmp.u = id[tmp.u], tmp.v = id[tmp.v];
        if(in[E[i].v].w != INF) tmp.w -= in[E[i].v].w;
        nE.pb(tmp);
    }
    vector<edge> tsol;
    if(!DMST(cnt, nE, id[root], tsol)) return false;
    sol.resize(n);
    for(int i = 0; i < cnt; i++){
        if(i == id[root]) continue;
        int t = tsol[i].id;
        sol[E[t].v] = E[t];
    }
    for(int i = 0; i < n; i++)
        if(sol[i].id == -1) sol[i] = in[i];
    return true;
}

```

## 5.6 Fast DMST

```

struct Edge { int a, b; ll w; };
struct Node { /// Lazy skew heap node
    Edge key;
    Node *l, *r;
    ll delta;
    void prop() {
        key.w += delta;
        if (l) l->delta += delta;
        if (r) r->delta += delta;
    }
};

```

```

    delta = 0;
}
Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
    if (!a || !b) return a ? b;
    a->prop(), b->prop();
    if (a->key.w > b->key.w) swap(a, b);
    swap(a->l, (a->r = merge(b, a->r)));
    return a;
}
void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }

pair<ll, vi> dmst(int n, int r, vector<Edge>& g) {
    RollbackUF uf(n); // need to implement this
    vector<Node*> heap(n);
    for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node{e});
    ll res = 0;
    vi seen(n, -1), path(n), par(n);
    seen[r] = r;
    vector<Edge> Q(n), in(n, {-1, -1}), comp;
    deque<tuple<int, int, vector<Edge>>> cycs;
    rep(s, 0, n) {
        int u = s, qi = 0, w;
        while (seen[u] < 0) {
            if (!heap[u]) return {-1, {}};
            Edge e = heap[u]->top();
            heap[u]->delta -= e.w, pop(heap[u]);
            Q[qi] = e, path[qi++] = u, seen[u] = s;
            res += e.w, u = uf.find(e.a);
            if (seen[u] == s) { /// found cycle, contract
                Node* cyc = 0;
                int end = qi, time = uf.time();
                do cyc = merge(cyc, heap[w = path[--qi]]);
                while (uf.join(u, w));
                u = uf.find(u), heap[u] = cyc, seen[u] = -1;
                cycs.push_front({u, time, {Q[qi], &Q[end]}});
            }
        }
        rep(i, 0, qi) in[uf.find(Q[i].b)] = Q[i];
    }

    for (auto& [u, t, comp] : cycs) { // restore sol (optional)
        uf.rollback(t);
        Edge inEdge = in[u];
        for (auto& e : comp) in[uf.find(e.b)] = e;
        in[uf.find(inEdge.b)] = inEdge;
    }
    rep(i, 0, n) par[i] = in[i].a;
    return {res, par};
}

```

## 5.7 Vizing

```

// find D+1 edge coloring of a graph with max deg D
struct vizing { // returns edge coloring in adjacent matrix
    G. 1 - based
    const int N = 105;
    int C[N][N], G[N][N], X[N], vst[N], n; // ans: G[i][j]
    void init(int _n) { n = _n; // n = |V|+1
        for (int i = 0; i <= n; ++i)
            for (int j = 0; j <= n; ++j)
                C[i][j] = G[i][j] = 0;
    }
    void solve(vector<pii> &E) {
        auto update = [&](int u)
            { for (X[u] = 1; C[u][X[u]]; ++X[u]); };
        auto color = [&](int u, int v, int c) {
            int p = G[u][v];
            G[u][v] = G[v][u] = c;
            C[u][c] = v, C[v][c] = u;
            C[u][p] = C[v][p] = 0;
            if (p) X[u] = X[v] = p;
            else update(u), update(v);
            return p;
        };
    }
};

```

```

};
auto flip = [&](int u, int c1, int c2) {
    int p = C[u][c1];
    swap(C[u][c1], C[u][c2]);
    if (p) G[u][p] = G[p][u] = c2;
    if (!C[u][c1]) X[u] = c1;
    if (!C[u][c2]) X[u] = c2;
    return p;
};
fill_n(X + 1, n, 1);
for (int t = 0; t < SZ(E); ++t) {
    int u = E[t].X, v0 = E[t].Y, v = v0, c0 = X[u], c =
        c0, d;
    vector<pii> L;
    fill_n(vst + 1, n, 0);
    while (!G[u][v0]) {
        L.emplace_back(v, d = X[v]);
        if (!C[v][c]) for (int a = SZ(L) - 1; a >= 0; --a)
            c = color(u, L[a].X, c);
        else if (!C[u][d]) for (int a = SZ(L) - 1; a >= 0;
            --a) color(u, L[a].X, L[a].Y);
        else if (vst[d]) break;
        else vst[d] = 1, v = C[u][d];
    }
    if (!G[u][v0]) {
        for (; v; v = flip(v, c, d), swap(c, d));
        if (int a; C[u][c0]) {
            for (a = SZ(L) - 2; a >= 0 && L[a].Y != c; --a);
            for (; a >= 0; --a) color(u, L[a].X, L[a].Y);
        }
        else --t;
    }
}
};

```

## 5.8 Maximum Clique

```

struct MaxClique { // fast when N <= 100
    bitset<N> G[N], cs[N];
    int ans, sol[N], q, cur[N], d[N], n;
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) G[i].reset();
    }
    void add_edge(int u, int v) {
        G[u][v] = G[v][u] = 1;
    }
    void pre_dfs(vector<int> &r, int l, bitset<N> mask) {
        if (l < 4) {
            for (int i : r) d[i] = (G[i] & mask).count();
            sort(ALL(r), [&](int x, int y) { return d[x] > d[y]; });
        }
        vector<int> c(SZ(r));
        int lft = max(ans - q + 1, 1), rgt = 1, tp = 0;
        cs[1].reset(), cs[2].reset();
        for (int p : r) {
            int k = 1;
            while ((cs[k] & G[p]).any()) ++k;
            if (k > rgt) cs[++rgt].reset();
            cs[k][p] = 1;
            if (k < lft) r[tp++] = p;
        }
        for (int k = lft; k <= rgt; ++k)
            for (int p = cs[k]._Find_first(); p < N; p = cs[k]._Find_next(p))
                r[tp] = p, c[tp] = k, ++tp;
        dfs(r, c, l + 1, mask);
    }
    void dfs(vector<int> &r, vector<int> &c, int l, bitset<N>
        mask) {
        while (!r.empty()) {
            int p = r.back();
            r.pop_back(), mask[p] = 0;
            if (q + c.back() <= ans) return;

```

```

            cur[q++] = p;
            vector<int> nr;
            for (int i : r) if (G[p][i]) nr.pb(i);
            if (!nr.empty()) pre_dfs(nr, l, mask & G[p]);
            else if (q > ans) ans = q, copy_n(cur, q, sol);
            c.pop_back(), --q;
        }
    }
    int solve() {
        vector<int> r(n);
        ans = q = 0, iota(ALL(r), 0);
        pre_dfs(r, 0, bitset<N>(string(n, '1')));
        return ans;
    }
};

```

## 5.9 Number of Maximal Clique

```

struct BronKerbosch { // 1-base
    int n, a[N], g[N][N];
    int S, all[N][N], some[N][N], none[N][N];
    void init(int _n) {
        n = _n;
        for (int i = 1; i <= n; ++i)
            for (int j = 1; j <= n; ++j) g[i][j] = 0;
    }
    void add_edge(int u, int v) {
        g[u][v] = g[v][u] = 1;
    }
    void dfs(int d, int an, int sn, int nn) {
        if (S > 1000) return; // pruning
        if (sn == 0 && nn == 0) ++S;
        int u = some[d][0];
        for (int i = 0; i < sn; ++i) {
            int v = some[d][i];
            if (g[u][v]) continue;
            int tsu = 0, tnn = 0;
            copy_n(all[d], an, all[d + 1]);
            all[d + 1][an] = v;
            for (int j = 0; j < sn; ++j)
                if (g[v][some[d][j]])
                    some[d + 1][tsu++] = some[d][j];
            for (int j = 0; j < nn; ++j)
                if (g[v][none[d][j]])
                    none[d + 1][tnn++] = none[d][j];
            dfs(d + 1, an + 1, tsu, tnn);
            some[d][i] = 0, none[d][nn++] = v;
        }
    }
    int solve() {
        iota(some[0], some[0] + n, 1);
        S = 0, dfs(0, 0, n, 0);
        return S;
    }
};

```

## 5.10 Minimum Mean Cycle

```

// from 8BQube
ll road[N][N]; // input here
struct MinimumMeanCycle {
    ll dp[N + 5][N], n;
    pll solve() {
        ll a = -1, b = -1, L = n + 1;
        for (int i = 2; i <= L; ++i)
            for (int k = 0; k < n; ++k)
                for (int j = 0; j < n; ++j)
                    dp[i][j] =
                        min(dp[i - 1][k] + road[k][j], dp[i][j]);
        for (int i = 0; i < n; ++i) {
            if (dp[L][i] >= INF) continue;
            ll ta = 0, tb = 1;
            for (int j = 1; j < n; ++j)
                if (dp[j][i] < INF &&
                    ta * (L - j) < (dp[L][i] - dp[j][i]) * tb)

```

```

    ta = dp[L][i] - dp[j][i], tb = L - j;
    if (ta == 0) continue;
    if (a == -1 || a * tb > ta * b) a = ta, b = tb;
}
if (a != -1) {
    ll g = __gcd(a, b);
    return pll(a / g, b / g);
}
return pll(-1LL, -1LL);
}
void init(int _n) {
    n = _n;
    for (int i = 0; i < n; ++i)
        for (int j = 0; j < n; ++j) dp[i + 2][j] = INF;
}
};

```

## 5.11 Minimum Steiner Tree

```

// from 8BQube
// O(V 3^T + V^2 2^T)
struct SteinerTree { // 0-base
    static const int T = 10, N = 105, INF = 1e9;
    int n, dst[N][N], dp[1 << T][N], tdst[N];
    int vcost[N]; // the cost of vertexs
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) {
            for (int j = 0; j < n; ++j) dst[i][j] = INF;
            dst[i][i] = vcost[i] = 0;
        }
    }
    void add_edge(int ui, int vi, int wi) {
        dst[ui][vi] = min(dst[ui][vi], wi);
    }
    void shortest_path() {
        for (int k = 0; k < n; ++k)
            for (int i = 0; i < n; ++i)
                for (int j = 0; j < n; ++j)
                    dst[i][j] =
                        min(dst[i][j], dst[i][k] + dst[k][j]);
    }
    int solve(const vector<int> &ter) {
        shortest_path();
        int t = SZ(ter);
        for (int i = 0; i < (1 << t); ++i)
            for (int j = 0; j < n; ++j) dp[i][j] = INF;
        for (int i = 0; i < n; ++i) dp[0][i] = vcost[i];
        for (int msk = 1; msk < (1 << t); ++msk) {
            if (!(msk & (msk - 1))) {
                int who = __lg(msk);
                for (int i = 0; i < n; ++i)
                    dp[msk][i] =
                        vcost[ter[who]] + dst[ter[who]][i];
            }
            for (int i = 0; i < n; ++i)
                for (int submsk = (msk - 1) & msk; submsk; submsk = (submsk - 1) & msk)
                    dp[msk][i] = min(dp[msk][i],
                        dp[submsk][i] + dp[msk ^ submsk][i] -
                        vcost[i]);
            for (int i = 0; i < n; ++i) {
                tdst[i] = INF;
                for (int j = 0; j < n; ++j)
                    tdst[i] =
                        min(tdst[i], dp[msk][j] + dst[j][i]);
            }
            for (int i = 0; i < n; ++i) dp[msk][i] = tdst[i];
        }
        int ans = INF;
        for (int i = 0; i < n; ++i)
            ans = min(ans, dp[(1 << t) - 1][i]);
        return ans;
    }
};

```

## 6 Math

### 6.1 Extended Euclidean Algorithm

```

// ax+ny = 1, ax+ny == ax == 1 (mod n)
void extgcd(ll x, ll y, ll &g, ll &a, ll &b) {
    if (y == 0) g = x, a = 1, b = 0;
    else extgcd(y, x % y, g, b, a), b -= (x / y) * a;
}

```

### 6.2 Floor & Ceil

```

int ifloor(int a, int b) {
    return a / b - (a % b && (a < 0) ^ (b < 0));
}
int iceil(int a, int b) {
    return a / b + (a % b && (a < 0) ^ (b > 0));
}

```

### 6.3 Legendre

```

// the Jacobi symbol is a generalization of the Legendre
// symbol,
// such that the bottom doesn't need to be prime.
// (n/p) -> same as Legendre
// (n/ab) = (n/a)(n/b)
// work with long long
int Jacobi(int a, int m) {
    int s = 1;
    for (; m > 1; ) {
        a %= m;
        if (a == 0) return 0;
        const int r = __builtin_ctz(a);
        if ((r & 1) && ((m + 2) & 4)) s = -s;
        a >>= r;
        if (a & m & 2) s = -s;
        swap(a, m);
    }
    return s;
}

// 0: a == 0
// -1: a isn't a quad res of p
// else: return X with X^2 % p == a
// doesn't work with long long
int QuadraticResidue(int a, int p) {
    if (p == 2) return a & 1;
    const int jc = Jacobi(a, p);
    if (jc == 0) return 0;
    if (jc == -1) return -1;
    int b, d;
    for (; ; ) {
        b = rand() % p;
        d = (1LL * b * b + p - a) % p;
        if (Jacobi(d, p) == -1) break;
    }
    int f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
    for (int e = (1LL + p) >> 1; e; e >>= 1) {
        if (e & 1) {
            tmp = (1LL * g0 * f0 + 1LL * d * (1LL * g1 * f1 % p))
                % p;
            g1 = (1LL * g0 * f1 + 1LL * g1 * f0) % p;
            g0 = tmp;
        }
        tmp = (1LL * f0 * f0 + 1LL * d * (1LL * f1 * f1 % p)) %
            p;
        f1 = (2LL * f0 * f1) % p;
        f0 = tmp;
    }
    return g0;
}

```



## 6.4 Simplex

```
#pragma once

typedef double T; // Long double, Rational, double + mod<P>
>...
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(x) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s]))
    s=j

struct LPSolver {
    int m, n;
    vi N, B;
    vvd D;

    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
        rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
        rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
        rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m+1][n] = 1;
    }

    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
        rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
            T *b = D[i].data(), inv2 = b[s] * inv;
            rep(j,0,n+2) b[j] -= a[j] * inv2;
            b[s] = a[s] * inv2;
        }
        rep(j,0,n+2) if (j != s) D[r][j] *= inv;
        rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]);
    }

    bool simplex(int phase) {
        int x = m + phase - 1;
        for (;;) {
            int s = -1;
            rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
            if (D[x][s] >= -eps) return true;
            int r = -1;
            rep(i,0,m) {
                if (D[i][s] <= eps) continue;
                if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                    < MP(D[r][n+1] / D[r][s], B[r])) r = i;
            }
            if (r == -1) return false;
            pivot(r, s);
        }
    }

    T solve(vd &x) {
        int r = 0;
        rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
        if (D[r][n+1] < -eps) {
            pivot(r, n);
            if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
            rep(i,0,m) if (B[i] == -1) {
                int s = 0;
                rep(j,1,n+1) ltj(D[i]);
                pivot(i, s);
            }
        }
        bool ok = simplex(1); x = vd(n);
        rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
        return ok ? D[m][n+1] : inf;
    }
};
```

## 6.5 Floor Sum

```
// from 8BQube
ll floor_sum(ll n, ll m, ll a, ll b) {
    ll ans = 0;
    if (a >= m)
        ans += (n - 1) * n * (a / m) / 2, a %= m;
    if (b >= m)
        ans += n * (b / m), b %= m;
    ll y_max = (a * n + b) / m, x_max = (y_max * m - b);
    if (y_max == 0) return ans;
    ans += (n - (x_max + a - 1) / a) * y_max;
    ans += floor_sum(y_max, a, m, (a - x_max % a) % a);
    return ans;
} // sum^{n-1}_0 floor((a * i + b) / m) in log(n + m + a + b)
```

## 6.6 DiscreteLog

```
int DiscreteLog(int s, int x, int y, int m) {
    constexpr int kStep = 32000;
    unordered_map<int, int> p;
    int b = 1;
    for (int i = 0; i < kStep; ++i) {
        p[y] = i;
        y = 1LL * y * x % m;
        b = 1LL * b * x % m;
    }
    for (int i = 0; i < m + 10; i += kStep) {
        s = 1LL * s * b % m;
        if (p.find(s) != p.end()) return i + kStep - p[s];
    }
    return -1;
}

int DiscreteLog(int x, int y, int m) {
    if (m == 1) return 0;
    int s = 1;
    for (int i = 0; i < 100; ++i) {
        if (s == y) return i;
        s = 1LL * s * x % m;
    }
    if (s == y) return 100;
    int p = 100 + DiscreteLog(s, x, y, m);
    if (fpow(x, p, m) != y) return -1;
    return p; //returns: x^p = y (mod m)
}
```

## 6.7 Miller Rabin & Pollard Rho

```
// n < 4,759,123,141      3 : 2, 7, 61
// n < 1,122,004,669,633  4 : 2, 13, 23, 1662803
// n < 3,474,749,660,383  6 : primes <= 13
// n < 2^64              7 :
// 2, 325, 9375, 28178, 450775, 9780504, 1795265022
ll mul(ll a, ll b, ll n){
    return (__int128)a * b % n;
}

bool Miller_Rabin(ll a, ll n) {
    if ((a = a % n) == 0) return 1;
    if (n % 2 == 0) return n == 2;
    ll tmp = (n - 1) / ((n - 1) & (1 - n));
    ll t = __lg(((n - 1) & (1 - n))), x = 1;
    for (; tmp; tmp >>= 1, a = mul(a, a, n))
        if (tmp & 1) x = mul(x, a, n);
    if (x == 1 || x == n - 1) return 1;
    while (--t)
        if ((x = mul(x, x, n)) == n - 1) return 1;
    return 0;
}

bool prime(ll n){
    vector<ll> tmp = {2, 325, 9375, 28178, 450775, 9780504,
        1795265022};
    for(ll i : tmp)
        if(!Miller_Rabin(i, n)) return false;
}
```

```

    return true;
}
map<ll, int> cnt;
void PollardRho(ll n) {
    if (n == 1) return;
    if (prime(n)) return ++cnt[n], void();
    if (n % 2 == 0) return PollardRho(n / 2), ++cnt[2], void();
    ll x = 2, y = 2, d = 1, p = 1;
#define f(x, n, p) ((mul(x, x, n) + p) % n)
    while (true) {
        if (d != n && d != 1) {
            PollardRho(n / d);
            PollardRho(d);
            return;
        }
        if (d == n) ++p;
        x = f(x, n, p), y = f(f(y, n, p), n, p);
        d = gcd(abs(x - y), n);
    }
}

```

## 6.8 XOR Basis

```

const int digit = 60; // [0, 2^digit)
struct Basis {
    int total = 0, rank = 0;
    vector<ll> b;
    Basis(): b(digit) {}
    bool add(ll v) { // Gauss Jordan Elimination
        total++;
        for (int i = digit - 1; i >= 0; i--) {
            if (1LL << i & v) continue;
            if (b[i] != 0) {
                v ^= b[i];
                continue;
            }
            for (int j = 0; j < i; j++)
                if (1LL << j & v) v ^= b[j];
            for (int j = i + 1; j < digit; j++)
                if (1LL << i & b[j]) b[j] ^= v;
            b[i] = v;
            rank++;
            return true;
        }
        return false;
    }
    ll getMax(ll x = 0) {
        for (ll i : b) x = max(x, x ^ i);
        return x;
    }
    ll getMin(ll x = 0) {
        for (ll i : b) x = min(x, x ^ i);
        return x;
    }
    bool can(ll x) {
        return getMin(x) == 0;
    }
    ll kth(ll k) { // kth smallest, 0-indexed
        vector<ll> tmp;
        for (ll i : b) if (i) tmp.pb(i);
        ll ans = 0;
        for (int i = 0; i < SZ(tmp); i++)
            if (1LL << i & k) ans ^= tmp[i];
        return ans;
    }
};

```

## 6.9 Linear Equation

```

vector<int> RREF(vector<vector<ll>> &mat) {
    int N = mat.size(), M = mat[0].size();
    int rk = 0;
    vector<int> cols;
    for (int i = 0; i < M; i++) {

```

```

        int cnt = -1;
        for (int j = N - 1; j >= rk; j--)
            if (mat[j][i] != 0) cnt = j;
        if (cnt == -1) continue;
        swap(mat[rk], mat[cnt]);
        ll lead = mat[rk][i];
        for (int j = 0; j < M; j++) mat[rk][j] /= lead;
        for (int j = 0; j < N; j++) {
            if (j == rk) continue;
            ll tmp = mat[j][i];
            for (int k = 0; k < M; k++)
                mat[j][k] -= mat[rk][k] * tmp;
        }
        cols.pb(i);
        rk++;
    }
    return cols;
}
struct LinearEquation {
    bool ok;
    vector<ll> par; // particular solution (Ax = b)
    vector<vector<ll>> homo; // homogenous (Ax = 0)
    vector<vector<ll>> rref;
    // first M columns are matrix A
    // last column of eq is vector b
    void solve(const vector<vector<ll>> &eq) {
        int M = (int)eq[0].size() - 1;
        rref = eq;
        auto piv = RREF(rref);
        int rk = piv.size();
        if (piv.size() && piv.back() == M) {
            ok = 0; return;
        }
        ok = 1;
        par.resize(M);
        vector<bool> ispiv(M);
        for (int i = 0; i < rk; i++) {
            par[piv[i]] = rref[i][M];
            ispiv[piv[i]] = 1;
        }
        for (int i = 0; i < M; i++) {
            if (ispiv[i]) continue;
            vector<ll> h(M);
            h[i] = 1;
            for (int j = 0; j < rk; j++) h[piv[j]] = -rref[j][i];
            homo.pb(h);
        }
    }
};

```

## 6.10 Chinese Remainder Theorem

```

pll solve_crt(ll x1, ll m1, ll x2, ll m2) {
    ll g = gcd(m1, m2);
    if ((x2 - x1) % g) return {0, 0}; // no sol
    m1 /= g; m2 /= g;
    ll _, p, q;
    extgcd(m1, m2, _, p, q); // p <= C
    ll lcm = m1 * m2 * g;
    ll res = (((__int128)p * (x2 - x1) % lcm * m1 % lcm + x1) % lcm);
    // be careful with overflow, C^3
    return {(res + lcm) % lcm, lcm}; // (x, m)
}

```

## 6.11 Sqrt Decomposition

```

// for all i in [l, r], floor(n / i) = x
for (int l = 1, r; l <= n; l = r + 1) {
    int x = ifloor(n, l);
    r = ifloor(n, x);
}
// for all i in [l, r], ceil(n / i) = x
for (int l, r = n; r >= 1; r = l - 1) {
    int x = iceil(n, r);

```

```

    l = iceil(n, x);
}

```

## 7 Misc

### 7.1 Cyclic Ternary Search

```

/* bool pred(int a, int b);
f(0) ~ f(n - 1) is a cyclic-shift U-function
return idx s.t. pred(x, idx) is false forall x*/
int cyc_tsearch(int n, auto pred) {
    if (n == 1) return 0;
    int l = 0, r = n; bool rv = pred(1, 0);
    while (r - l > 1) {
        int m = (l + r) / 2;
        if (pred(0, m) ? rv: pred(m, (m + 1) % n)) r = m;
        else l = m;
    }
    return pred(l, r % n) ? l : r % n;
}

```

### 7.2 Matroid

我們稱一個二元組  $M = (E, \mathcal{I})$  為一個擬陣，其中  $\mathcal{I} \subseteq 2^E$  為  $E$  的子集所形成的非空集合，若：

- 若  $S \in \mathcal{I}$  以及  $S' \subseteq S$ ，則  $S' \in \mathcal{I}$
- 對於  $S_1, S_2 \in \mathcal{I}$  滿足  $|S_1| < |S_2|$ ，存在  $e \in S_2 \setminus S_1$  使得  $S_1 \cup \{e\} \in \mathcal{I}$

除此之外，我們有以下的定義：

- 位於  $\mathcal{I}$  中的集合我們稱之為獨立集 (independent set)，反之不在  $\mathcal{I}$  中的我們稱為相依集 (dependent set)
- 極大的獨立集為基底 (base)、極小的相依集為迴路 (circuit)
- 一個集合  $Y$  的秩 (rank)  $r(Y)$  為該集中最大的獨立子集，也就是  $r(Y) = \max\{|X| \mid X \subseteq Y \text{ 且 } X \in \mathcal{I}\}$

性質：

1.  $X \subseteq Y \wedge Y \in \mathcal{I} \implies X \in \mathcal{I}$
2.  $X \subseteq Y \wedge X \notin \mathcal{I} \implies Y \notin \mathcal{I}$
3. 若  $B$  與  $B'$  皆是基底且  $B \subseteq B'$ ，則  $B = B'$   
若  $C$  與  $C'$  皆是迴路且  $C \subseteq C'$ ，則  $C = C'$
4.  $e \in E \wedge X \subseteq E \implies r(X) \leq r(X \cup \{e\}) \leq r(X) + 1$  i.e. 加入一個元素後秩不會降底，最多增加 1
5.  $\forall Y \subseteq E, \exists X \subseteq Y, r(X) = |X| = r(Y)$

一些等價的性質：

1. 對於所有  $X \subseteq E$ ， $X$  的極大獨立子集都有相同的大小
2. 對於  $B_1, B_2 \in \mathcal{B} \wedge B_1 \neq B_2$ ，對於所有  $e_1 \in B_1 \setminus B_2$ ，存在  $e_2 \in B_2 \setminus B_1$  使得  $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$
3. 對於  $X, Y \in \mathcal{I}$  且  $|X| < |Y|$ ，存在  $e \in Y \setminus X$  使得  $X \cup \{e\} \in \mathcal{B}$
4. 如果  $r(X \cup \{e_1\}) = r(X \cup \{e_2\}) = r(X)$ ，則  $r(X \cup \{e_1, e_2\}) = r(X)$ 。如果  $r(X \cup \{e\}) = r(X)$  對於所有  $e \in E'$  都成立，則  $r(X \cup E') = r(X)$ 。

擬陣交

Data: 兩個擬陣  $M_1 = (E, \mathcal{I}_1)$  以及  $M_2 = (E, \mathcal{I}_2)$

Result:  $I$  為最大的位於  $\mathcal{I}_1 \cap \mathcal{I}_2$  中的獨立集

$I \leftarrow \emptyset$

$X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$

$X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$

**while**  $X_1 \neq \emptyset$  且  $X_2 \neq \emptyset$  **do**

**if**  $e \in X_1 \cap X_2$  **then**

$I \leftarrow I \cup \{e\}$

**else**

        構造交換圖  $\mathcal{D}_{M_1, M_2}(I)$

        在交換圖上找到一條  $X_1$  到  $X_2$  且沒有捷徑的路徑  $P$

$I \leftarrow I \Delta P$

**end if**

$X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$

$X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$

**end while**

## 8 Polynomial

### 8.1 FWHT

```

/* x: a[j], y: a[j + (L >> 1)]
or: (y += x * op), and: (x += y * op)
xor: (x, y = (x + y) * op, (x - y) * op)
invop: or, and, xor = -1, -1, 1/2 */
void fwt(int *a, int n, int op) { //or
    for (int L = 2; L <= n; L <= 1)
        for (int i = 0; i < n; i += L)
            for (int j = i; j < i + (L >> 1); ++j)
                a[j + (L >> 1)] += a[j] * op;
}
const int N = 21;
int f[N][1 << N], g[N][1 << N], h[N][1 << N], ct[1 << N];
void subset_convolution(int *a, int *b, int *c, int L) {
    // c_k = \sum_{i+j=k, i&j=0} a_i * b_j
    int n = 1 << L;
    for (int i = 1; i < n; ++i)
        ct[i] = ct[i & (i - 1)] + 1;
    for (int i = 0; i < n; ++i)
        f[ct[i]][i] = a[i], g[ct[i]][i] = b[i];
    for (int i = 0; i <= L; ++i)
        fwt(f[i], n, 1), fwt(g[i], n, 1);
    for (int i = 0; i <= L; ++i)
        for (int j = 0; j <= i; ++j)
            for (int x = 0; x < n; ++x)
                h[i][x] += f[j][x] * g[i - j][x];
    for (int i = 0; i <= L; ++i) fwt(h[i], n, -1);
    for (int i = 0; i < n; ++i) c[i] = h[ct[i]][i];
}

```

### 8.2 FFT

// Errichto: FFT for double works when the result < 1e15,  
and < 1e18 with long double

```

using val_t = complex<double>;
template<int MAXN>
struct FFT {
    const double PI = acos(-1);
    val_t w[MAXN];
    FFT() {
        for (int i = 0; i < MAXN; ++i) {
            double arg = 2 * PI * i / MAXN;
            w[i] = val_t(cos(arg), sin(arg));
        }
    }
    void bitrev(vector<val_t> &a, int n) //same as NTT
    void trans(vector<val_t> &a, int n, bool inv = false) {
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
                for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
                    val_t tmp = a[j + dl] * (inv ? conj(w[x]) : w[x]);
                    a[j + dl] = a[j] - tmp;
                    a[j] += tmp;
                }
            }
        }
        if (inv) {
            for (int i = 0; i < n; ++i) a[i] /= n;
        }
    }
    //multiplying two polynomials A * B:
    //fft.trans(A, siz, 0), fft.trans(B, siz, 0):
    //A[i] *= B[i], fft.trans(A, siz, 1);
};

```

### 8.3 NTT

```

//(2^16)+1, 65537, 3
//7*17*(2^23)+1, 998244353, 3
//1255*(2^20)+1, 1315962881, 3
//51*(2^25)+1, 1711276033, 29
// only works when sz(A) + sz(B) - 1 <= MAXN

```

```

template<int MAXN, ll P, ll RT> //MAXN must be 2^k
struct NTT {
    ll w[MAXN];
    ll mpow(ll a, ll n);
    ll minv(ll a) { return mpow(a, P - 2); }
    NTT() {
        ll dw = mpow(RT, (P - 1) / MAXN);
        w[0] = 1;
        for (int i = 1; i < MAXN; ++i) w[i] = w[i - 1] * dw % P;
    }
    void bitrev(vector<ll> &a, int n) {
        int i = 0;
        for (int j = 1; j < n - 1; ++j) {
            for (int k = n >> 1; (i ^ k) < k; k >>= 1);
            if (j < i) swap(a[i], a[j]);
        }
    }
    void operator()(vector<ll> &a, int n, bool inv = false) {
        //0 <= a[i] < P
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
                for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
                    ll tmp = a[j + dl] * w[x] % P;
                    if ((a[j + dl] = a[j] - tmp) < 0) a[j + dl] += P;
                    if ((a[j] += tmp) >= P) a[j] -= P;
                }
            }
        }
        if (inv) {
            reverse(a.begin() + 1, a.begin() + n);
            ll invn = minv(n);
            for (int i = 0; i < n; ++i) a[i] = a[i] * invn % P;
        }
    }
};

```

## 8.4 Polynomial Operation

```

// Copy from 8BQube
#define fi(s, n) for (int i = (int)(s); i < (int)(n); ++i)
template<int MAXN, ll P, ll RT> // MAXN = 2^k
struct Poly : vector<ll> { // coefficients in [0, P)
    using vector<ll>::vector;
    static inline NTT<MAXN, P, RT> ntt;
    int n() const { return (int)size(); } // n() >= 1
    Poly(const Poly &p, int m) : vector<ll>(m) {
        copy_n(p.data(), min(p.n(), m), data());
    }
    Poly& irev() { return reverse(data(), data() + n()), *this; }
    Poly& isz(int m) { return resize(m), *this; }
    Poly& iadd(const Poly &rhs) { // n() == rhs.n()
        fi(0, n()) if (((*this)[i] += rhs[i]) >= P) (*this)[i] -= P;
        return *this;
    }
    Poly& imul(ll k) {
        fi(0, n()) (*this)[i] = (*this)[i] * k % P;
        return *this;
    }
    Poly Mul(const Poly &rhs) const {
        int m = 1;
        while (m < n() + rhs.n() - 1) m <= 1;
        assert(m <= MAXN);
        Poly X(*this, m), Y(rhs, m);
        ntt(X, m), ntt(Y, m);
        fi(0, m) X[i] = X[i] * Y[i] % P;
        ntt(X, m, true);
        return X.isz(n() + rhs.n() - 1);
    }
    Poly Inv() const { // (*this)[0] != 0, 1e5/95ms, 2*sz<=MAXN
        if (n() == 1) return {ntt.minv((*this)[0])};
    }
};

```

```

int m = 1;
while (m < n() * 2) m <= 1;
assert(m <= MAXN);
Poly Xi = Poly(*this, (n() + 1) / 2).Inv().isz(m);
Poly Y(*this, m);
ntt(Xi, m), ntt(Y, m);
fi(0, m) {
    Xi[i] *= (2 - Xi[i] * Y[i]) % P;
    if ((Xi[i] %= P) < 0) Xi[i] += P;
}
ntt(Xi, m, true);
return Xi.isz(n());
}
Poly& shift_inplace(const ll &c) { // 2 * sz <= MAXN
    int n = this->n();
    vector<ll> fc(n), ifc(n);
    fc[0] = ifc[0] = 1;
    for (int i = 1; i < n; ++i) {
        fc[i] = fc[i - 1] * i % P;
        ifc[i] = ntt.minv(fc[i]);
    }
    for (int i = 0; i < n; ++i) (*this)[i] = (*this)[i] * fc[i] % P;
    Poly g(n);
    ll cp = 1;
    for (int i = 0; i < n; ++i) g[i] = cp * ifc[i] % P, cp = cp * c % P;
    *this = (*this).irev().Mul(g).isz(n).irev();
    for (int i = 0; i < n; ++i) (*this)[i] = (*this)[i] * ifc[i] % P;
    return *this;
}
Poly shift(const ll &c) const { return Poly(*this).shift_inplace(c); }
Poly _Sqrt() const { // Jacobi((*this)[0], P) = 1
    if (n() == 1) return {QuadraticResidue((*this)[0], P)};
    Poly X = Poly(*this, (n() + 1) / 2)._Sqrt().isz(n());
    return X.iadd(Mul(X.Inv()).isz(n())).imul(P / 2 + 1);
}
Poly Sqrt() const { // 2 * sz <= MAXN
    Poly a;
    bool has = 0;
    for (int i = 0; i < n(); ++i) {
        if ((*this)[i]) has = 1;
        if (has) a.push_back((*this)[i]);
    }
    if (!has) return *this;
    if ((n() + a.n()) % 2 || Jacobi(a[0], P) != 1) {
        return Poly();
    }
    a = a.isz((n() + a.n()) / 2)._Sqrt();
    int sz = a.n();
    a.isz(n());
    rotate(a.begin(), a.begin() + sz, a.end());
    return a;
}
pair<Poly, Poly> DivMod(const Poly &rhs) const { // (rhs)back() != 0
    if (n() < rhs.n()) return {{0}, *this};
    const int m = n() - rhs.n() + 1;
    Poly X(rhs); X.irev().isz(m);
    Poly Y(*this); Y.irev().isz(m);
    Poly Q = Y.Mul(X.Inv()).isz(m).irev();
    X = rhs.Mul(Q), Y = *this;
    fi(0, n()) if ((Y[i] -= X[i]) < 0) Y[i] += P;
    return {Q, Y.isz(max(1, rhs.n() - 1))};
}
Poly Dx() const {
    Poly ret(n() - 1);
    fi(0, ret.n()) ret[i] = (i + 1) * (*this)[i + 1] % P;
    return ret.isz(max(1, ret.n()));
}
Poly Sx() const {
    Poly ret(n() + 1);
    fi(0, n()) ret[i + 1] = ntt.minv(i + 1) * (*this)[i] % P;
    return ret;
}

```

```

}
Poly _tmul(int nn, const Poly &rhs) const {
    Poly Y = Mul(rhs).isz(n() + nn - 1);
    return Poly(Y.data() + n() - 1, Y.data() + Y.n());
}
vector<ll> _eval(const vector<ll> &x, const vector<Poly>
    &up) const {
    const int m = (int)x.size();
    if (!m) return {};
    vector<Poly> down(m * 2);
    // down[1] = DivMod(up[1]).second;
    // fi(2, m * 2) down[i] = down[i / 2].DivMod(up[i]).
        second;
    down[1] = Poly(up[1]).irev().isz(n()).Inv().irev().
        _tmul(m, *this);
    fi(2, m * 2) down[i] = up[i ^ 1]._tmul(up[i].n() - 1,
        down[i / 2]);
    vector<ll> y(m);
    fi(0, m) y[i] = down[m + i][0];
    return y;
}
static vector<Poly> _tree1(const vector<ll> &x) {
    const int m = (int)x.size();
    vector<Poly> up(m * 2);
    fi(0, m) up[m + i] = {(x[i] ? P - x[i] : 0), 1};
    for (int i = m - 1; i > 0; --i) up[i] = up[i * 2].Mul(
        up[i * 2 + 1]);
    return up;
}
vector<ll> Eval(const vector<ll> &x) const { // 1e5, 1s
    auto up = _tree1(x); return _eval(x, up);
}
static Poly Interpolate(const vector<ll> &x, const vector
    <ll> &y) { // 1e5, 1.4s
    const int m = (int)x.size();
    vector<Poly> up = _tree1(x), down(m * 2);
    vector<ll> z = up[1].Dx()._eval(x, up);
    fi(0, m) z[i] = y[i] * ntt.minv(z[i]) % P;
    fi(0, m) down[m + i] = {z[i]};
    for (int i = m - 1; i > 0; --i) down[i] = down[i * 2].
        Mul(up[i * 2 + 1]).iadd(down[i * 2 + 1].Mul(up[i *
        2]));
    return down[1];
}
Poly Ln() const { // (*this)[0] == 1, 2*sz<=MAXN
    return Dx().Mul(Inv()).Sx().isz(n());
}
Poly Exp() const { // (*this)[0] == 0, 2*sz<=MAXN
    if (n() == 1) return {1};
    Poly X = Poly(*this, (n() + 1) / 2).Exp().isz(n());
    Poly Y = X.Ln(); Y[0] = P - 1;
    fi(0, n()) if ((Y[i] = (*this)[i] - Y[i]) < 0) Y[i] +=
        P;
    return X.Mul(Y).isz(n());
}
// M := P(P - 1). If k >= M, k := k % M + M.
Poly Pow(ll k) const { // 2*sz<=MAXN
    int nz = 0;
    while (nz < n() && !(*this)[nz]) ++nz;
    if (nz * min(k, (ll)n()) >= n()) return Poly(n());
    if (!k) return Poly(Poly {1}, n());
    Poly X(data() + nz, data() + nz + n() - nz * k);
    const ll c = ntt.mpow(X[0], k % (P - 1));
    return X.Ln().imul(k % P).Exp().imul(c).irev().isz(n())
        .irev();
}
static ll LinearRecursion(const vector<ll> &a, const
    vector<ll> &coef, ll n) { // a_n = \sum c_j a_{(n-j)}
    const int k = (int)a.size();
    assert((int)coef.size() == k + 1);
    Poly C(k + 1), W(Poly {1}, k), M = {0, 1};
    fi(1, k + 1) C[k - i] = coef[i] ? P - coef[i] : 0;
    C[k] = 1;
    while (n) {
        if (n % 2) W = W.Mul(M).DivMod(C).second;
        n /= 2, M = M.Mul(M).DivMod(C).second;
    }
}

```

```

    ll ret = 0;
    fi(0, k) ret = (ret + W[i] * a[i]) % P;
    return ret;
}
};
#undef fi
using Poly_t = Poly<1 << 20, 998244353, 3>;
// template<> decltype(Poly_t::ntt) Poly_t::ntt = {};

```

## 8.5 Generating Function

### 8.5.1 Ordinary Generating Function

- $C(x) = A(rx)$ :  $c_n = r^n a_n$  的一般生成函數。
- $C(x) = A(x) + B(x)$ :  $c_n = a_n + b_n$  的一般生成函數。
- $C(x) = A(x)B(x)$ :  $c_n = \sum_{i=0}^n a_i b_{n-i}$  的一般生成函數。
- $C(x) = A(x)^k$ :  $c_n = \sum_{i_1+i_2+\dots+i_k=n} a_{i_1} a_{i_2} \dots a_{i_k}$  的一般生成函數。
- $C(x) = xA(x)'$ :  $c_n = na_n$  的一般生成函數。
- $C(x) = \frac{A(x)}{1-x}$ :  $c_n = \sum_{i=0}^n a_i$  的一般生成函數。
- $C(x) = A(1) + x \frac{A(1)-A(x)}{1-x}$ :  $c_n = \sum_{i=n}^{\infty} a_i$  的一般生成函數。

常用展開式

- $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$
- $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ ,  $\binom{a}{n} = \frac{a(a-1)(a-2)\dots(a-n+1)}{n!}$ .

常見生函

- 卡特蘭數:  $f(x) = \frac{1-\sqrt{1-4x}}{2x}$

### 8.5.2 Exponential Generating Function

$a_0, a_1, \dots$  的指數生成函數:

$$\hat{A}(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \dots$$

- $\hat{C}(x) = \hat{A}(x) + \hat{B}(x)$ :  $c_n = a_n + b_n$  的指數生成函數
- $\hat{C}(x) = \hat{A}^{(k)}(x)$ :  $c_n = a_{n+k}$  的指數生成函數
- $\hat{C}(x) = x\hat{A}(x)$ :  $c_n = na_n$  的指數生成函數
- $\hat{C}(x) = \hat{A}(x)\hat{B}(x)$ :  $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$  的指數生成函數
- $\hat{C}(x) = \hat{A}(x)^k$ :  $\sum_{i_1+i_2+\dots+i_k=n} \binom{n}{i_1, i_2, \dots, i_k} a_{i_1} a_{i_2} \dots a_{i_k}$  的指數生成函數
- $\hat{C}(x) = \exp(A(x))$ : 假設  $A(x)$  是一個分量 (component) 的生成函數, 那  $\hat{C}(x)$  是將  $n$  個有編號的東西分成若干個分量的指數生成函數

## 8.6 Bostan Mori

```

NTT<262144, 998244353, 3> ntt;
// Finds the k-th coefficient of P / Q in O(d Log d Log k)
// size of NTT has to > 2 * d
ll BostanMori(vector<ll> P, vector<ll> Q, long long k) {
    int d = max((int)P.size(), (int)Q.size() - 1);
    P.resize(d, 0);
    Q.resize(d + 1, 0);
    int sz = (2 * d + 1 == 1 ? 2 : (1 << (__lg(2 * d) + 1)));
    while(k) {
        vector<ll> Qneg(sz);
        for(int i = 0; i < (int)Q.size(); i++){
            Qneg[i] = Q[i] * ((i & 1) ? -1 : 1);
            if(Qneg[i] < 0) Qneg[i] += mod;
        }
        ntt(Qneg, sz, false);

        vector<ll> U(sz), V(sz);
        for(int i = 0; i < (int)P.size(); i++)
            U[i] = P[i];
        for(int i = 0; i < (int)Q.size(); i++)
            V[i] = Q[i];

        ntt(U, sz, false);
        ntt(V, sz, false);
        for(int i = 0; i < sz; i++)
            U[i] = U[i] * Qneg[i] % mod;
    }
}

```

```

for(int i = 0; i < sz; i++)
    V[i] = V[i] * Qneg[i] % mod;
ntt(U, sz, true);
ntt(V, sz, true);

for(int i = k & 1; i <= 2 * d - 1; i += 2)
    P[i >> 1] = U[i];
for(int i = 0; i <= 2 * d; i += 2)
    Q[i >> 1] = V[i];
k >>= 1;
}
return P[0] * ntt.minv(Q[0]) % mod;
}

```

## 9 String

### 9.1 KMP Algorithm

```

// 0-based
// fail[i] = max k<i s.t. s[0..k] = s[i-k..i]
vector<int> kmp_build_fail(const string &s){
    int n = SZ(s);
    vector<int> fail(n, -1);
    int cur = -1;
    for(int i = 1; i < n; i++){
        while(cur != -1 && s[cur + 1] != s[i])
            cur = fail[cur];
        if(s[cur + 1] == s[i])
            cur++;
        fail[i] = cur;
    }
    return fail;
}

void kmp_match(const string &s, const vector<int> &fail,
    const string &t){
    int cur = -1;
    int n = SZ(s), m = SZ(t);
    for(int i = 0; i < m; i++){
        while(cur != -1 && (cur + 1 == n || s[cur + 1] != t[i]))
            cur = fail[cur];
        if(cur + 1 < n && s[cur + 1] == t[i])
            cur++;
        // cur = max k s.t. s[0..k] = t[i-k..i]
    }
}

```

### 9.2 Manacher Algorithm

```

/* center i: radius z[i * 2 + 1] / 2
   center i, i + 1: radius z[i * 2 + 2] / 2
   both aba, abba have radius 2 */
vector<int> manacher(const string &tmp){ // 0-based
    string s = "%";
    int l = 0, r = 0;
    for(char c : tmp) s += c, s += '%';
    vector<int> z(SZ(s));
    for(int i = 0; i < SZ(s); i++){
        z[i] = r > i ? min(z[2 * l - i], r - i) : 1;
        while(i - z[i] >= 0 && i + z[i] < SZ(s)
            && s[i + z[i]] == s[i - z[i]])
            ++z[i];
        if(z[i] + i > r) r = z[i] + i, l = i;
    }
    return z;
}

```

### 9.3 Lyndon Factorization

```

// partition s = w[0] + w[1] + ... + w[k-1],
// w[0] >= w[1] >= ... >= w[k-1]

```

```

// each w[i] strictly smaller than all its suffix
void duval(const string &s, vector<pii> &w) {
    for (int n = (int)s.size(), i = 0, j, k; i < n; ) {
        for (j = i + 1, k = i; j < n && s[k] <= s[j]; j++)
            k = (s[k] < s[j] ? i : k + 1);
        // if (i < n / 2 && j >= n / 2) {
        // for min cyclic shift, call duval(s + s)
        // then here s.substr(i, n / 2) is min cyclic shift
        // }
        for (; i <= k; i += j - k)
            w.pb(pii(i, j - k)); // s.substr(l, len)
    }
}

```

### 9.4 Suffix Array

```

struct SuffixArray {
    vector<int> sa, lcp, rank; // lcp[i] is lcp of sa[i] and
                               // sa[i-1]
                               // sa[0] = s.size()
                               // character should be 1-based
    SuffixArray(string& s, int lim=256) { // or basic_string<
        int>
        int n = s.size() + 1, k = 0, a, b;
        vector<int> x(n, 0), y(n), ws(max(n, lim));
        rank.assign(n, 0);
        for (int i = 0; i < n - 1; i++) x[i] = s[i];
        sa = lcp = y, iota(sa.begin(), sa.end(), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim =
            p) {
            p = j, iota(y.begin(), y.end(), n - j);
            for (int i = 0; i < n; i++)
                if (sa[i] >= j) y[p++] = sa[i] - j;
            for (int &i : ws) i = 0;
            for (int i = 0; i < n; i++) ws[x[i]]++;
            for (int i = 1; i < lim; i++) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            for(int i = 1; i < n; i++){
                a = sa[i - 1], b = sa[i];
                x[b] = (y[a] == y[b] && y[a + j] == y[b + j]) ? p -
                    1 : p++;
            }
        }
        for (int i = 1; i < n; i++) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i+1]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};

```

### 9.5 Suffix Automaton

```

struct exSAM {
    const int CNUM = 26;
    // len: maxlength, link: fail link
    // lenSorted: topo order, cnt: occur
    vector<int> len, link, lenSorted, cnt;
    vector<vector<int>> next;
    int total = 0;
    int newnode() {
        return total++;
    }
    void init(int n) { // total number of characters
        len.assign(2 * n, 0); link.assign(2 * n, 0);
        lenSorted.assign(2 * n, 0); cnt.assign(2 * n, 0);
        next.assign(2 * n, vector<int>(CNUM));
        newnode(), link[0] = -1;
    }
    int insertSAM(int last, int c) {
        // not exSAM: cur = newnode(), p = last
        int cur = next[last][c];
        len[cur] = len[last] + 1;
        int p = link[last];
        while (p != -1 && !next[p][c])

```

```

    next[p][c] = cur, p = link[p];
    if (p == -1) return link[cur] = 0, cur;
    int q = next[p][c];
    if (len[p] + 1 == len[q]) return link[cur] = q, cur;
    int clone = newnode();
    for (int i = 0; i < CNUM; ++i)
        next[clone][i] = len[next[q][i]] ? next[q][i] : 0;
    len[clone] = len[p] + 1;
    while (p != -1 && next[p][c] == q)
        next[p][c] = clone, p = link[p];
    link[link[cur] = clone] = link[q];
    link[q] = clone;
    return cur;
}
void insert(const string &s) {
    int cur = 0;
    for (auto ch : s) {
        int &nxt = next[cur][int(ch - 'a')];
        if (!nxt) nxt = newnode();
        cnt[cur = nxt] += 1;
    }
}
void build() {
    queue<int> q;
    q.push(0);
    while (!q.empty()) {
        int cur = q.front();
        q.pop();
        for (int i = 0; i < CNUM; ++i)
            if (next[cur][i])
                q.push(insertSAM(cur, i));
    }
    vector<int> lc(total);
    for (int i = 1; i < total; ++i) ++lc[len[i]];
    partial_sum(iter(lc), lc.begin());
    for (int i = 1; i < total; ++i) lenSorted[--lc[len[i]]]
        = i;
}
void solve() {
    for (int i = total - 2; i >= 0; --i)
        cnt[link[lenSorted[i]]] += cnt[lenSorted[i]];
}
};

```

## 9.6 Z-value Algorithm

```

// z[i] = max k s.t. s[0..k-1] = s[i..i+k-1]
// i.e. Length of Longest common prefix
// z[0] = 0
vector<int> z_function(const string &s){
    int n = s.size();
    vector<int> z(n);
    for(int i = 1, l = 0, r = 0; i < n; i++){
        if(i <= r) z[i] = min(r - i + 1, z[i - l]);
        while(i + z[i] < n && s[z[i]] == s[i + z[i]])
            z[i]++;
        if(i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
    return z;
}

```

## 9.7 Main Lorentz

```

struct Rep{ int minl, maxl, len; };
vector<Rep> rep; // 0-base
// p \in [minl, maxl] => s[p, p + i) = s[p + i, p + 2i)
void main_lorentz(const string &s, int sft = 0) {
    const int n = s.size();
    if (n == 1) return;
    const int nu = n / 2, nv = n - nu;
    const string u = s.substr(0, nu), v = s.substr(nu,
        ru(u.rbegin(), u.rend()), rv(v.rbegin(), v.rend()));
    main_lorentz(u, sft), main_lorentz(v, sft + nu);
}

```

```

const auto z1 = z_function(ru), z2 = z_function(v + '#' +
    u),
    z3 = z_function(ru + '#' + rv), z4 =
        z_function(v);
auto get_z = [](const vector<int> &z, int i) {
    return (0 <= i and i < (int)z.size()) ? z[i] : 0; };
auto add_rep = [&](bool left, int c, int l, int k1, int
    k2) {
    const int L = max(1, l - k2), R = min(l - left, k1);
    if (L > R) return;
    if (left) rep.emplace_back(Rep({sft + c - R, sft + c -
        L, l}));
    else rep.emplace_back(Rep({sft + c - R - l + 1, sft + c
        - L - l + 1, l}));
};
for (int cntr = 0; cntr < n; cntr++) {
    int l, k1, k2;
    if (cntr < nu) {
        l = nu - cntr;
        k1 = get_z(z1, nu - cntr);
        k2 = get_z(z2, nv + 1 + cntr);
    } else {
        l = cntr - nu + 1;
        k1 = get_z(z3, nu + 1 + nv - 1 - (cntr - nu));
        k2 = get_z(z4, (cntr - nu) + 1);
    }
    if (k1 + k2 >= 1)
        add_rep(cntr < nu, cntr, l, k1, k2);
}
}

```

## 9.8 AC Automaton

```

const int SIGMA = 26;
struct AC_Automaton {
    // child: trie, next: automaton
    vector<vector<int>> child, next;
    vector<int> fail, cnt, ord;
    int total = 0;
    int newnode() {
        return total++;
    }
    void init(int len) { // len >= 1 + total len
        child.assign(len, vector<int>(26, -1));
        next.assign(len, vector<int>(26, -1));
        fail.assign(len, -1); cnt.assign(len, 0);
        ord.clear();
        newnode();
    }
    int input(string &s) {
        int cur = 0;
        for (char c : s) {
            if (child[cur][c - 'A'] == -1)
                child[cur][c - 'A'] = newnode();
            cur = child[cur][c - 'A'];
        }
        return cur; // return the end node of string
    }
    void make_fl() {
        queue<int> q;
        q.push(0), fail[0] = -1;
        while(!q.empty()) {
            int R = q.front();
            q.pop(); ord.pb(R);
            for (int i = 0; i < SIGMA; i++)
                if (child[R][i] != -1) {
                    int X = next[R][i] = child[R][i], Z = fail[R];
                    while (Z != -1 && child[Z][i] == -1)
                        Z = fail[Z];
                    fail[X] = Z != -1 ? child[Z][i] : 0;
                    q.push(X);
                }
            else next[R][i] = R ? next[fail[R]][i] : 0;
        }
    }
    void solve() {

```

```

    for (int i : ord | views::reverse)
        cnt[fail[i]] += cnt[i];
}
};

```

## 9.9 Palindrome Automaton

```

struct PalindromicTree {
    struct node {
        int nxt[26], fail, len; // num = depth of fail link
        int cnt, num; // cnt = occur, num = #pal_suffix of this node
        node(int l = 0) : nxt{}, fail(0), len(1), cnt(0), num(0) {}
    };
    vector<node> st; vector<int> s; int last, n;
    void init() {
        st.clear(); s.clear(); last = 1; n = 0;
        st.pb(0); st.pb(-1);
        st[0].fail = 1; s.pb(-1);
    }
    int getFail(int x) {
        while (s[n - st[x].len - 1] != s[n]) x = st[x].fail;
        return x;
    }
    void add(int c) {
        s.pb(c == 'a'); ++n;
        int cur = getFail(last);
        if (!st[cur].nxt[c]) {
            int now = SZ(st);
            st.pb(st[cur].len + 2);
            st[now].fail = st[getFail(st[cur].fail)].nxt[c];
            st[cur].nxt[c] = now;
            st[now].num = st[st[now].fail].num + 1;
        }
        last = st[cur].nxt[c]; ++st[last].cnt;
    }
    void dpCnt() {
        for(int i = SZ(st) - 1; i >= 0; i--){
            auto nd = st[i];
            st[nd.fail].cnt += nd.cnt;
        }
    }
    int size() { return (int)st.size() - 2; }
};

```

## 10 Formula

### 10.1 Recurrences

If  $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$ , and  $r_1, \dots, r_k$  are distinct roots of  $x^k + c_1 x^{k-1} + \dots + c_k$ , there are  $d_1, \dots, d_k$  s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots  $r$  become polynomial factors, e.g.  $a_n = (d_1 n + d_2) r^n$ .

### 10.2 Geometry

#### 10.2.1 Rotation Matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- rotate  $90^\circ$ :  $(x, y) \rightarrow (-y, x)$
- rotate  $-90^\circ$ :  $(x, y) \rightarrow (y, -x)$

#### 10.2.2 Triangles

Side lengths:  $a, b, c$

Semiperimeter:  $p = \frac{a+b+c}{2}$

Area:  $A = \sqrt{p(p-a)(p-b)(p-c)}$

Circumradius:  $R = \frac{abc}{4A}$

Inradius:  $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles):  $m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two):  $s_a = \sqrt{bc \left( 1 - \left( \frac{a}{b+c} \right)^2 \right)}$

Law of sines:  $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$

Law of cosines:  $a^2 = b^2 + c^2 - 2bc \cos \alpha$

Law of tangents:  $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

Incenter:

$$P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$$

$$s_1 = P_2 P_3, s_2 = P_1 P_3, s_3 = P_1 P_2$$

$$\frac{s_1 P_1 + s_2 P_2 + s_3 P_3}{s_1 + s_2 + s_3}$$

Circumcenter:

$$P_0 = (0, 0), P_1 = (x_1, y_1), P_2 = (x_2, y_2)$$

$$x_c = \frac{1}{2} \times \frac{y_2(x_1^2 + y_1^2) - y_1(x_2^2 + y_2^2)}{-x_2 y_1 + x_1 y_2}$$

$$y_c = \frac{1}{2} \times \frac{x_2(x_1^2 + y_1^2) - x_1(x_2^2 + y_2^2)}{-x_1 y_2 + x_2 y_1}$$

Check if  $(x_0, y_0)$  is in the circumcircle:

$$\begin{vmatrix} x_1 - x_0 & y_1 - y_0 & (x_1^2 + y_1^2) - (x_0^2 + y_0^2) \\ x_2 - x_0 & y_2 - y_0 & (x_2^2 + y_2^2) - (x_0^2 + y_0^2) \\ x_3 - x_0 & y_3 - y_0 & (x_3^2 + y_3^2) - (x_0^2 + y_0^2) \end{vmatrix}$$

0: on edge, > 0: inside, < 0: outside

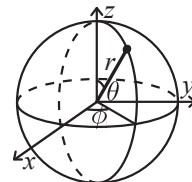
#### 10.2.3 Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ .

#### 10.2.4 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

#### 10.2.5 Green's Theorem

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{L^+} (P dx + Q dy)$$

$$\text{Area} = \frac{1}{2} \oint_L x dy - y dx$$

Circular sector:

$$x = x_0 + r \cos \theta$$

$$y = y_0 + r \sin \theta$$

$$\begin{aligned} A &= r \int_{\alpha}^{\beta} (x_0 + \cos \theta) \cos \theta + (y_0 + \sin \theta) \sin \theta d\theta \\ &= r(r\theta + x_0 \sin \theta - y_0 \cos \theta) \Big|_{\alpha}^{\beta} \end{aligned}$$

#### 10.2.6 Point-Line Duality

$$p = (a, b) \leftrightarrow p^* : y = ax - b$$

- $p \in l \iff l^* \in p^*$
- $p_1, p_2, p_3$  are collinear  $\iff p_1^*, p_2^*, p_3^*$  intersect at a point
- $p$  lies above  $l \iff l^*$  lies above  $p^*$
- lower convex hull  $\leftrightarrow$  upper envelope



## 10.3 Trigonometry

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sin n\pi = 0 \quad \cos n\pi = (-1)^n$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(2\alpha) = 2 \cos \alpha \sin \alpha$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$= 2 \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$(V + W) \tan(\alpha - \beta)/2 = (V - W) \tan(\alpha + \beta)/2$$

where  $V, W$  are lengths of sides opposite angles  $\alpha, \beta$ .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where  $r = \sqrt{a^2 + b^2}, \phi = \text{atan2}(b, a)$ .

## 10.4 Derivatives/Integrals

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln |\cos ax|}{a}$$

$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \text{erf}(x)$$

$$\int x e^{ax} = \frac{e^{ax}}{a^2} (ax - 1)$$

$$\int \sin^2(x) = \frac{x}{2} - \frac{1}{4} \sin 2x$$

$$\int \sin^3 x = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x$$

$$\int \cos^2(x) = \frac{x}{2} + \frac{1}{4} \sin 2x$$

$$\int \cos^3 x = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x$$

$$\int x \sin x = \sin x - x \cos x$$

$$\int x \cos x = \cos x + x \sin x$$

$$\int x e^x = e^x (x - 1)$$

$$\int x^2 e^x = e^x (x^2 - 2x + 2)$$

$$\int x^2 \sin x = 2x \sin x - (x^2 - 2) \cos x$$

$$\int x^2 \cos x = 2x \cos x + (x^2 - 2) \sin x$$

$$\int e^x \sin x = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^x \cos x = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int x e^x \sin x = \frac{1}{2} e^x (x \sin x - x \cos x + \cos x)$$

$$\int x e^x \cos x = \frac{1}{2} e^x (x \sin x + x \cos x - \sin x)$$

## 10.5 Sums

$$c^a + c^{a+1} + \cdots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

## 10.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \cdots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, (-\infty < x < \infty)$$