Contents

1 Basic

1.1 Default Code

```
#include <bits/stdc++.h>
using namespace std;

#define iter(v) v.begin(),v.end()
#define SZ(v) int(v.size())
#define pb emplace_back
#define ff first
#define ss second

using ll = long long;
using pii = pair<int, int>;
using pll = pair<ll, ll>;

#ifdef zisk
void debug(){cerr << "\n";}
template<class T, class ... U>
void debug(T a, U ... b){cerr << a << " ", debug(b...)
    ;}
template<class T> void pary(T l, T r){
  while (l != r) cerr << *l << " ", l++;
  cerr << "\n";
}
#else
#define debug(...) void()
#define pary(...) void()
#endif

template<class A, class B>
ostream& operator<<(ostream& o, pair<A,B> p)
{ return o << '(' << p.ff << ',' << p.ss << ')'; }

int main(){
  ios_base::sync_with_stdio(0); cin.tie(0);
}
```

1.2 .vimrc

```
sy on
se nu rnu bs=2 sw=4 ts=4 hls ls=2 si acd bo=all mouse=a
map <F9> :w<bar>!g++ "%" -o %:r -std=c++17 -Wall -
    Wextra -Wshadow -O2 -Dzisk -g -fsanitize=undefined,
    address<CR>
map <F8> :!./%:r<CR>
inoremap {<CR> {<CR>}<ESC>ko
# -D_GLIBCXX_ASSERTIONS, -D_GLIBCXX_DEBUG
```

1.3 Fast IO

```
// from JAW
inline int my_getchar() {
  const int N = 1<<20;
  static char buf[N];
  static char *p = buf , *end = buf;
  if(p == end) {
    if((end = buf + fread(buf , 1 , N , stdin)) == buf)
      return EOF;
    p = buf;
  }
  return *p++;
}

inline int readint(int &x) {
  static char c , neg;
  while((c = my_getchar()) < '-') {
    if(c == EOF) return 0;
  }
  neg = (c == '-') ? -1 : 1;
  x = (neg == 1) ? c - '0' : 0;
  while((c = my_getchar()) >= '0') x = (x << 3) + (x <<
      1) + (c - '0');
  x *= neg;
  return 1;
}

const int kBufSize = 524288;
char inbuf[kBufSize];
char buf_[kBufSize]; size_t size_;
inline void Flush_() { write(1, buf_, size_); size_ =
    0; }
inline void CheckFlush_(size_t sz) { if (sz + size_ >
    kBufSize) Flush_(); }

inline void PutInt(int a) {
  static char tmp[22] = "01234567890123456789\n";
  CheckFlush_(10);
  if(a < 0){
    *(buf_ + size_) = '-';
    a = ~a + 1;
    size_++;
  }
  int tail = 20;
  if (!a) {
    tmp[--tail] = '0';
  } else {
    for (; a; a /= 10) tmp[--tail] = (a % 10) ^ '0';
  }
  memcpy(buf_ + size_, tmp + tail, 21 - tail);
  size_ += 21 - tail;
}

int main(){
  Flush_();
  return 0;
}
```

1.4 Random

```
mt19937 rng(chrono::system_clock::now().
    time_since_epoch().count());
```

1.5 Checker

```
#!/usr/bin/env bash
set -e
while :; do
    python3 gen.py > test.txt
    diff <(./a.exe < test.txt) <(./b.exe < test.txt)
done
```

1.6 PBDS Tree

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
using Tree = tree<int, null_type, less<>, rb_tree_tag,
    tree_order_node_statistics_update>;
// .find_by_order(x)
// .order_of_key(x)
```

1.7 Pragma

```
#pragma GCC optimize("Ofast,no-stack-protector")
#pragma GCC optimize("no-math-errno,unroll-loops")
#pragma GCC target("sse,sse2,sse3,ssse3,sse4")
#pragma GCC target("popcnt,abm,mmx,avx,arch=skylake")
__builtin_ia32_ldmxcsr(__builtin_ia32_stmxcsr()|0x8040)
```

1.8 SVG Writer

```
class SVG {
  void p(string_view s) { o << s; }
  void p(string_view s, auto v, auto... vs) {
    auto i = s.find('$');
    o << s.substr(0, i) << v, p(s.substr(i + 1), vs...)
        ;
  }
  ofstream o; string c = "red";
public: // SVG svg("test.svg", 0, 0, 100, 100)
  SVG(auto f, auto x1, auto y1, auto x2, auto y2) : o(f
      ) {
    p("<svg xmlns='http://www.w3.org/2000/svg' "
      "viewBox='$ $ $ $'>\n"
      "<style>*{stroke-width:0.5%;}</style>\n",
      x1, -y2, x2 - x1, y2 - y1); }
  ~SVG() { p("</svg>\n"); }
  void color(string nc) { c = nc; }
  void line(auto x1, auto y1, auto x2, auto y2) {
    p("<line x1='$' y1='$' x2='$' y2='$' stroke='$'/>\n
      ",
      x1, -y1, x2, -y2, c); }
  void circle(auto x, auto y, auto r) {
    p("<circle cx='$' cy='$' r='$' stroke='$' "
      "fill='none'/>\n", x, -y, r, c); }
  void text(auto x, auto y, string s, int w = 12) {
    p("<text x='$' y='$' font-size='$px'>$</text>\n",
      x, -y, w, s); }
};
```

2 Data Structure

2.1 Heavy-Light Decomposition

```
struct HLD{ // 1-based
  int n, ts = 0; // ord is 1-based
  vector<vector<int>> g;
  vector<int> par, top, down, ord, dpt, sub;
  explicit HLD(int _n): n(_n), g(n + 1),
  par(n + 1), top(n + 1), down(n + 1),
  ord(n + 1), dpt(n + 1), sub(n + 1) {}
  void add_edge(int u, int v){ g[u].pb(v); g[v].pb(u);
      }
  void dfs(int now, int p){
    par[now] = p; sub[now] = 1;
    for(int i : g[now]){
      if(i == p) continue;
      dpt[i] = dpt[now] + 1;
      dfs(i, now);
      sub[now] += sub[i];
      if(sub[i] > sub[down[now]]) down[now] = i;
    }
  }
  void cut(int now, int t){
    top[now] = t; ord[now] = ++ts;
    if(!down[now]) return;
    cut(down[now], t);
    for(int i : g[now]){
      if(i != par[now] && i != down[now])
        cut(i, i);
    }
  }
  void build(){ dfs(1, 1), cut(1, 1); }
  int query(int a, int b){
    int ta = top[a], tb = top[b];
    while(ta != tb){
      if(dpt[ta] > dpt[tb]) swap(ta, tb), swap(a, b);
      // ord[tb], ord[b]
      tb = top[b = par[tb]];
    }
    if(ord[a] > ord[b]) swap(a, b);
    // ord[a], ord[b]
    return a; // lca
  }
};
```

2.2 Link Cut Tree

```
// 1-based
template <typename Val, typename SVal> struct LCT {
  struct node {
    int pa, ch[2]; bool rev; int size;
    Val v, sum, rsum; SVal sv, sub, vir;
    node() : pa{0}, ch{0, 0}, rev{false}, size{1}, v{},
      sum{}, rsum{}, sv{}, sub{}, vir{} {}
  };
#define cur o[u]
#define lc cur.ch[0]
#define rc cur.ch[1]
  vector<node> o;
  bool is_root(int u) const {
    return o[cur.pa].ch[0]!=u && o[cur.pa].ch[1]!=u; }
  bool is_rch(int u) const {
    return o[cur.pa].ch[1] == u && !is_root(u); }
  void down(int u) {
    for (int c : {lc, rc}) if (c) {
      if (cur.rev) set_rev(c);
    }
    cur.rev = false;
  }
  void up(int u) {
    cur.sum = o[lc].sum + cur.v + o[rc].sum;
    cur.rsum = o[rc].rsum + cur.v + o[lc].rsum;
    cur.sub = cur.vir + o[lc].sub + o[rc].sub + cur.sv;
    cur.size = o[lc].size + o[rc].size + 1;
  }
  void set_rev(int u) {
    swap(lc, rc), swap(cur.sum, cur.rsum);
    cur.rev ^= 1;
  }
  /* --- */
  void rotate(int u) {
    int f = cur.pa, g = o[f].pa, l = is_rch(u);
    if (cur.ch[l ^ 1]) o[cur.ch[l ^ 1]].pa = f;
    if (not is_root(f)) o[g].ch[is_rch(f)] = u;
    o[f].ch[l] = cur.ch[l ^ 1], cur.ch[l ^ 1] = f;
    cur.pa = g, o[f].pa = u; up(f);
  }
  void splay(int u) {
    vector<int> stk = {u};
    while (not is_root(stk.back()))
      stk.push_back(o[stk.back()].pa);
    while (not stk.empty())
      down(stk.back()), stk.pop_back();
    for (int f = cur.pa; not is_root(u); f = cur.pa) {
      if (!is_root(f))
        rotate(is_rch(u) == is_rch(f) ? f : u);
      rotate(u);
    }
    up(u);
  }
  void access(int x) {
    for (int u = x, last = 0; u; u = cur.pa) {
      splay(u);
```

```
      cur.vir = cur.vir + o[rc].sub - o[last].sub;
      rc = last; up(last = u);
    }
    splay(x);
  }
  int find_root(int u) {
    int la = 0;
    for (access(u); u; u = lc) down(la = u);
    return la;
  }
  void split(int x, int y) { chroot(x); access(y); }
  void chroot(int u) { access(u); set_rev(u); }
  /* --- */
  LCT(int n = 0) : o(n + 1) { o[0].size = 0; }
  void set_val(int u, const Val &v) {
    splay(u); cur.v = v; up(u); }
  void set_sval(int u, const SVal &v) {
    access(u); cur.sv = v; up(u); }
  Val query(int x, int y) {
    split(x, y); return o[y].sum; }
  SVal subtree(int p, int u) {
    chroot(p); access(u); return cur.vir + cur.sv; }
  bool connected(int u, int v) {
    return find_root(u) == find_root(v); }
  void link(int x, int y) {
    chroot(x); access(y);
    o[y].vir = o[y].vir + o[x].sub;
    up(o[x].pa = y);
  }
  void cut(int x, int y) {
    split(x, y); o[y].ch[0] = o[x].pa = 0; up(y); }
#undef cur
#undef lc
#undef rc
};
```

2.3 Treap

```
mt19937 rng(880301);
struct node {
  ll data; int sz;
  node *l, *r;
  node(ll k = 0) : data(k), sz(1), l(0), r(0) {}
  void up() {
    sz = 1;
    if (l) sz += l->sz;
    if (r) sz += r->sz;
  }
  void down() {}
};
node pool[1000010]; int pool_cnt = 0;
node *newnode(ll k){ return &(pool[pool_cnt++] = node(k
    )); }
int sz(node *a) { return a ? a->sz : 0; }
node *merge(node *a, node *b) {
  if (!a || !b) return a ? a : b;
  if (int(rng() % (sz(a) + sz(b))) < sz(a))
    return a->down(), a->r = merge(a->r, b), a->up(),
          a;
  return b->down(), b->l = merge(a, b->l), b->up(), b;
}
// a: key <= k, b: key > k
void split(node *o, node *&a, node *&b, ll k) {
  if (!o) return a = b = 0, void();
  o->down();
  if (o->data <= k)
    a = o, split(o->r, a->r, b, k), a->up();
  else b = o, split(o->l, a, b->l, k), b->up();
}
// a: size k, b: size n - k
void split2(node *o, node *&a, node *&b, int k) {
  if (sz(o) <= k) return a = o, b = 0, void();
  o->down();
  if (sz(o->l) + 1 <= k)
    a = o, split2(o->r, a->r, b, k - sz(o->l) - 1);
  else b = o, split2(o->l, a, b->l, k);
  o->up();
}
node *kth(node *o, ll k) { // 1-based
  if (k <= sz(o->l)) return kth(o->l, k);
  if (k == sz(o->l) + 1) return o;
  return kth(o->r, k - sz(o->l) - 1);
}
int Rank(node *o, ll key) { // num of key < key
  if (!o) return 0;
  if (o->data < key)
    return sz(o->l) + 1 + Rank(o->r, key);
  else return Rank(o->l, key);
}
bool erase(node *&o, ll k) {
  if (!o) return 0;
  if (o->data == k) {
    node *t = o;
    o->down(), o = merge(o->l, o->r);
    return 1;
  }
  node *&t = k < o->data ? o->l : o->r;
  return erase(t, k) ? o->up(), 1 : 0;
}
void insert(node *&o, ll k) {
  node *a, *b;
  split(o, a, b, k),
    o = merge(a, merge(new node(k), b));
}
tuple<node*, node*, node*> interval(node *&o, int l,
    int r) { // 1-based
  node *a, *b, *c; // b: [l, r]
  split2(o, a, b, l - 1), split2(b, b, c, r - l + 1);
  return make_tuple(a, b, c);
}
```

2.4 KD Tree

```
namespace kdt {
  int root, lc[maxn], rc[maxn], xl[maxn], xr[maxn],
  yl[maxn], yr[maxn];
  point p[maxn];
  int build(int l, int r, int dep = 0) {
    if (l == r) return -1;
    function<bool(const point &, const point &)> f =
      [dep](const point &a, const point &b) {
        if (dep & 1) return a.x < b.x;
        else return a.y < b.y;
      };
    int m = (l + r) >> 1;
    nth_element(p + l, p + m, p + r, f);
    xl[m] = xr[m] = p[m].x;
    yl[m] = yr[m] = p[m].y;
    lc[m] = build(l, m, dep + 1);
    if (~lc[m]) {
      xl[m] = min(xl[m], xl[lc[m]]);
      xr[m] = max(xr[m], xr[lc[m]]);
      yl[m] = min(yl[m], yl[lc[m]]);
      yr[m] = max(yr[m], yr[lc[m]]);
    }
    rc[m] = build(m + 1, r, dep + 1);
    if (~rc[m]) {
      xl[m] = min(xl[m], xl[rc[m]]);
      xr[m] = max(xr[m], xr[rc[m]]);
      yl[m] = min(yl[m], yl[rc[m]]);
      yr[m] = max(yr[m], yr[rc[m]]);
    }
    return m;
  }
  bool bound(const point &q, int o, long long d) {
    double ds = sqrt(d + 1.0);
    if (q.x < xl[o] - ds || q.x > xr[o] + ds ||
        q.y < yl[o] - ds || q.y > yr[o] + ds)
      return false;
    return true;
  }
  long long dist(const point &a, const point &b) {
    return (a.x - b.x) * 1ll * (a.x - b.x) +
      (a.y - b.y) * 1ll * (a.y - b.y);
  }
  void dfs(
      const point &q, long long &d, int o, int dep = 0)
          {
    if (!bound(q, o, d)) return;
    long long cd = dist(p[o], q);
    if (cd != 0) d = min(d, cd);
    if ((dep & 1) && q.x < p[o].x ||
        !(dep & 1) && q.y < p[o].y) {
      if (~lc[o]) dfs(q, d, lc[o], dep + 1);
```

```
      if (~rc[o]) dfs(q, d, rc[o], dep + 1);
    } else {
      if (~rc[o]) dfs(q, d, rc[o], dep + 1);
      if (~lc[o]) dfs(q, d, lc[o], dep + 1);
    }
  }
  void init(const vector<point> &v) {
    for (int i = 0; i < v.size(); ++i) p[i] = v[i];
    root = build(0, v.size());
  }
  long long nearest(const point &q) {
    long long res = 1e18;
    dfs(q, res, root);
    return res;
  }
} // namespace kdt
```

2.5 Leftist Tree

```
struct node {
  ll v, data, sz, sum;
  node *l, *r;
  node(ll k)
    : v(0), data(k), sz(1), l(0), r(0), sum(k) {}
};
ll sz(node *p) { return p ? p->sz : 0; }
ll V(node *p) { return p ? p->v : -1; }
ll sum(node *p) { return p ? p->sum : 0; }
node *merge(node *a, node *b) {
  if (!a || !b) return a ? a : b;
  if (a->data < b->data) swap(a, b);
  a->r = merge(a->r, b);
  if (V(a->r) > V(a->l)) swap(a->r, a->l);
  a->v = V(a->r) + 1, a->sz = sz(a->l) + sz(a->r) + 1;
  a->sum = sum(a->l) + sum(a->r) + a->data;
  return a;
}
void pop(node *&o) {
  node *tmp = o;
  o = merge(o->l, o->r);
  delete tmp;
}
```

2.6 Convex 1D/1D

```
template<class T>
struct DynamicHull {
  struct seg { int x, l, r; };
  T f; int C; deque<seg> dq; // range: 1~C
  explicit DynamicHull(T _f, int _C): f(_f), C(_C) {}
  // max t s.t. f(x, t) >= f(y, t), x < y, maintain max
  int intersect(int x, int y) {
    int l = 0, r = C + 1;
    while (l + 1 < r) {
      int mid = (l + r) / 2;
      if (f(x, mid) >= f(y, mid)) l = mid;
      else r = mid;
    }
    return l;
  }
  void push_back(int x) {
    for (int i; !dq.empty() &&
        (i = dq.back().l, f(dq.back().x, i) < f(x, i));
            )
      dq.pop_back();
    if (dq.empty()) return dq.pb(seg({x, 1, C})), void
        ();
    dq.back().r = intersect(dq.back().x, x);
    dq.pb(seg({x, dq.back().l + 1, C}));
  }
  int query(int x) {
    while (dq.front().r < x) dq.pop_front();
    return dq.front().x;
  }
};
```

3 Flow & Matching

3.1 Dinic

```
struct Dinic { // 0-based, O(V^2E), unit flow: O(min(V
    ^{2/3}E, E^{3/2})), bipartite matching: O(sqrt(V)E)
  struct edge {
    ll to, cap, flow, rev;
  };

  int n, s, t;
  vector<vector<edge>> g;
  vector<int> dis, ind;

  void init(int _n) {
    n = _n;
    g.assign(n, vector<edge>());
  }
  void reset() {
    for (int i = 0; i < n; ++i)
      for (auto &j : g[i]) j.flow = 0;
  }
  void add_edge(int u, int v, ll cap) {
    g[u].pb(edge{v, cap, 0, SZ(g[v])});
    g[v].pb(edge{u, 0, 0, SZ(g[u]) - 1});
    //change g[v] to cap for undirected graphs
  }
  bool bfs() {
    dis.assign(n, -1);
    queue<int> q;
    q.push(s), dis[s] = 0;
    while (!q.empty()) {
      int cur = q.front(); q.pop();
      for (auto &e : g[cur]) {
        if (dis[e.to] == -1 && e.flow != e.cap) {
          q.push(e.to);
          dis[e.to] = dis[cur] + 1;
        }
      }
    }
    return dis[t] != -1;
  }
  ll dfs(int u, ll cap) {
    if (u == t || !cap) return cap;
    for (int &i = ind[u]; i < SZ(g[u]); ++i) {
      edge &e = g[u][i];
      if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
        ll df = dfs(e.to, min(e.cap - e.flow, cap));
        if (df) {
          e.flow += df;
          g[e.to][e.rev].flow -= df;
          return df;
        }
      }
    }
    dis[u] = -1;
    return 0;
  }
  ll maxflow(int _s, int _t) {
    s = _s; t = _t;
    ll flow = 0, df;
    while (bfs()) {
      ind.assign(n, 0);
      while ((df = dfs(s, INF))) flow += df;
    }
    return flow;
  }
};
```

3.2 Bounded Flow

```
struct BoundedFlow : Dinic {
  vector<ll> tot;
  void init(int _n) {
    Dinic::init(_n + 2);
    tot.assign(n, 0);
  }
  void add_edge(int u, int v, ll lcap, ll rcap) {
    tot[u] -= lcap, tot[v] += lcap;
    g[u].pb(edge{v, rcap, lcap, SZ(g[v])});
    g[v].pb(edge{u, 0, 0, SZ(g[u]) - 1});
  }
  bool feasible() {
    ll sum = 0;
    int vs = n - 2, vt = n - 1;
    for(int i = 0; i < n - 2; ++i)
      if(tot[i] > 0)
        add_edge(vs, i, 0, tot[i]), sum += tot[i];
      else if(tot[i] < 0) add_edge(i, vt, 0, -tot[i]);
```

```
    if(sum != maxflow(vs, vt)) sum = -1;
    for(int i = 0; i < n - 2; i++)
      if(tot[i] > 0)
        g[vs].pop_back(), g[i].pop_back();
      else if(tot[i] < 0)
        g[i].pop_back(), g[vt].pop_back();
    return sum != -1;
  }
  ll boundedflow(int _s, int _t) {
    add_edge(_t, _s, 0, INF);
    if(!feasible()) return -1;
    ll x = g[_t].back().flow;
    g[_t].pop_back(), g[_s].pop_back();
    return x - maxflow(_t, _s); // min
    //return x + maxflow(_s, _t); // max
  }
};
```

3.3 MCMF

```
struct MCMF { // 0-based, O(SPFA * |f|)
  struct edge {
    ll from, to, cap, flow, cost, rev;
  };
  int n;
  int s, t; ll mx;
  //mx: maximum amount of flow
  vector<vector<edge>> g;
  vector<ll> dis, up;
  bool BellmanFord(ll &flow, ll &cost) {
    vector<edge*> past(n);
    vector<int> inq(n);
    dis.assign(n, INF); up.assign(n, 0);
    queue<int> q;
    q.push(s), inq[s] = 1;
    up[s] = mx - flow, past[s] = 0, dis[s] = 0;
    while (!q.empty()) {
      int u = q.front();
      q.pop(), inq[u] = 0;
      if (!up[u]) continue;
      for (auto &e : g[u])
        if (e.flow != e.cap &&
            dis[e.to] > dis[u] + e.cost) {
          dis[e.to] = dis[u] + e.cost, past[e.to] = &e;
          up[e.to] = min(up[u], e.cap - e.flow);
          if (!inq[e.to]) inq[e.to] = 1, q.push(e.to);
        }
    }
    if (dis[t] == INF) return 0;
    flow += up[t], cost += up[t] * dis[t];
    for (ll i = t; past[i]; i = past[i]->from) {
      auto &e = *past[i];
      e.flow += up[t], g[e.to][e.rev].flow -= up[t];
    }
    return 1;
  }
  pll MinCostMaxFlow(int _s, int _t) {
    s = _s, t = _t;
    ll flow = 0, cost = 0;
    while (BellmanFord(flow, cost));
    return pll(flow, cost);
  }
  void init(int _n, ll _mx) {
    n = _n, mx = _mx;
    g.assign(n, vector<edge>());
  }
  void add_edge(int a, int b, ll cap, ll cost) {
    g[a].pb(edge{a, b, cap, 0, cost, SZ(g[b])});
    g[b].pb(edge{b, a, 0, 0, -cost, SZ(g[a]) - 1});
  }
};
```

3.4 Min Cost Circulation

```
struct MinCostCirculation { // 0-based, O(VE * ElogC)
  struct edge {
    ll from, to, cap, fcap, flow, cost, rev;
  };
  int n;
  vector<edge*> past;
  vector<vector<edge>> g;
  vector<ll> dis;
  void BellmanFord(int s) {
    vector<int> inq(n);
    dis.assign(n, INF);
    queue<int> q;
    auto relax = [&](int u, ll d, edge *e) {
      if (dis[u] > d) {
        dis[u] = d, past[u] = e;
        if (!inq[u]) inq[u] = 1, q.push(u);
      }
    };
    relax(s, 0, 0);
    while (!q.empty()) {
      int u = q.front();
      q.pop(), inq[u] = 0;
      for (auto &e : g[u])
        if (e.cap > e.flow)
          relax(e.to, dis[u] + e.cost, &e);
    }
  }
  void try_edge(edge &cur) {
    if (cur.cap > cur.flow) return ++cur.cap, void();
    BellmanFord(cur.to);
    if (dis[cur.from] + cur.cost < 0) {
      ++cur.flow, --g[cur.to][cur.rev].flow;
      for (int i = cur.from; past[i]; i = past[i]->from
          ) {
        auto &e = *past[i];
        ++e.flow, --g[e.to][e.rev].flow;
      }
    }
    ++cur.cap;
  }
  void solve(int mxlg) { // mxlg >= log(max cap)
    for (int b = mxlg; b >= 0; --b) {
      for (int i = 0; i < n; ++i)
        for (auto &e : g[i])
          e.cap *= 2, e.flow *= 2;
      for (int i = 0; i < n; ++i)
        for (auto &e : g[i])
          if (e.fcap >> b & 1)
            try_edge(e);
    }
  }
  void init(int _n) {
    n = _n;
    past.assign(n, nullptr);
    g.assign(n, vector<edge>());
  }
  void add_edge(ll a, ll b, ll cap, ll cost) {
    g[a].pb(edge{a, b, 0, cap, 0, cost, SZ(g[b]) + (a
        == b)});
    g[b].pb(edge{b, a, 0, 0, 0, -cost, SZ(g[a]) - 1});
  }
};
```

3.5 Gomory Hu

```
void GomoryHu(Dinic &flow) { // 0-based
  int n = flow.n;
  vector<int> par(n);
  for (int i = 1; i < n; ++i) {
    flow.reset();
    add_edge(i, par[i], flow.maxflow(i, par[i]));
    for (int j = i + 1; j < n; ++j)
      if (par[j] == par[i] && ~flow.dis[j])
        par[j] = i;
  }
}
```

3.6 Stoer Wagner Algorithm

```
struct StoerWagner { // 0-based, O(V^3)
  int n;
  vector<int> vis, del;
  vector<ll> wei;
  vector<vector<ll>> edge;
  void init(int _n) {
    n = _n;
    del.assign(n, 0);
    edge.assign(n, vector<ll>(n));
  }
  void add_edge(int u, int v, ll w) {
    edge[u][v] += w, edge[v][u] += w;
  }
```

```
  void search(int &s, int &t) {
    vis.assign(n, 0); wei.assign(n, 0);
    s = t = -1;
    while (1) {
      ll mx = -1, cur = 0;
      for (int i = 0; i < n; ++i)
        if (!del[i] && !vis[i] && mx < wei[i])
          cur = i, mx = wei[i];
      if (mx == -1) break;
      vis[cur] = 1, s = t, t = cur;
      for (int i = 0; i < n; ++i)
        if (!vis[i] && !del[i]) wei[i] += edge[cur][i];
    }
  }
  ll solve() {
    ll ret = INF;
    for (int i = 0, x=0, y=0; i < n-1; ++i) {
      search(x, y), ret = min(ret, wei[y]), del[y] = 1;
      for (int j = 0; j < n; ++j)
        edge[x][j] = (edge[j][x] += edge[y][j]);
    }
    return ret;
  }
};
```

3.7 Bipartite Matching

```
//min vertex cover: take all unmatched vertices in L
    and find alternating tree,
//ans is not reached in L + reached in R
// O(VE)
int n; // 1-based, max matching
int mx[maxn], my[maxn];
bool adj[maxn][maxn], vis[maxn];
bool dfs(int u) {
  if (vis[u]) return 0;
  vis[u] = 1;
  for (int v = 1;v <= n;v++) {
    if (!adj[u][v]) continue;
    if (!my[v] || (my[v] && dfs(my[v]))) {
      mx[u] = v, my[v] = u;
      return 1;
    }
  }
  return 0;
}
// O(E sqrt(V)), O(E log V) for random sparse graphs
struct BipartiteMatching { // 0-based
  int nl, nr;
  vector<int> mx, my, dis, cur;
  vector<vector<int>> g;
  bool dfs(int u) {
    for (int &i = cur[u]; i < SZ(g[u]); ++i) {
      int e = g[u][i];
      if (!~my[e] || (dis[my[e]] == dis[u] + 1 && dfs(
          my[e])))
        return mx[my[e] = u] = e, 1;
    }
    dis[u] = -1;
    return 0;
  }
  bool bfs() {
    int ret = 0;
    queue<int> q;
    dis.assign(nl, -1);
    for (int i = 0; i < nl; ++i)
      if (!~mx[i]) q.push(i), dis[i] = 0;
    while (!q.empty()) {
      int u = q.front();
      q.pop();
      for (int e : g[u])
        if (!~my[e]) ret = 1;
        else if (!~dis[my[e]]) {
          q.push(my[e]);
          dis[my[e]] = dis[u] + 1;
        }
    }
    return ret;
  }
  int matching() {
    int ret = 0;
    mx.assign(nl, -1); my.assign(nr, -1);
    while (bfs()) {
      cur.assign(nl, 0);
      for (int i = 0; i < nl; ++i)
        if (!~mx[i] && dfs(i)) ++ret;
    }
    return ret;
  }
  void add_edge(int s, int t) { g[s].pb(t); }
  void init(int _nl, int _nr) {
    nl = _nl, nr = _nr;
    g.assign(nl, vector<int>());
  }
};
```

3.8 Kuhn Munkres Algorithm

```
struct KM { // 0-based, maximum matching, O(V^3)
  int n, ql, qr;
  vector<vector<ll>> w;
  vector<ll> hl, hr, slk;
  vector<int> fl, fr, pre, qu, vl, vr;
  void init(int _n) {
    n = _n;
    // -INF for perfect matching
    w.assign(n, vector<ll>(n, 0));
    pre.assign(n, 0);
    qu.assign(n, 0);
  }
  void add_edge(int a, int b, ll wei) {
    w[a][b] = wei;
  }
  bool check(int x) {
    if (vl[x] = 1, ~fl[x])
      return (vr[qu[qr++] = fl[x]] = 1);
    while (~x) swap(x, fr[fl[x] = pre[x]]);
    return 0;
  }
  void bfs(int s) {
    slk.assign(n, INF); vl.assign(n, 0); vr.assign(n,
        0);
    ql = qr = 0, qu[qr++] = s, vr[s] = 1;
    for (ll d;;) {
      while (ql < qr)
        for (int x = 0, y = qu[ql++]; x < n; ++x)
          if (!vl[x] && slk[x] >= (d = hl[x] + hr[y] -
              w[x][y])) {
            if (pre[x] = y, d) slk[x] = d;
            else if (!check(x)) return;
          }
      d = INF;
      for (int x = 0; x < n; ++x)
        if (!vl[x] && d > slk[x]) d = slk[x];
      for (int x = 0; x < n; ++x) {
        if (vl[x]) hl[x] += d;
        else slk[x] -= d;
        if (vr[x]) hr[x] -= d;
      }
      for (int x = 0; x < n; ++x)
        if (!vl[x] && !slk[x] && !check(x)) return;
    }
  }
  ll solve() {
    fl.assign(n, -1); fr.assign(n, -1); hl.assign(n, 0)
        ; hr.assign(n, 0);
    for (int i = 0; i < n; ++i)
      hl[i] = *max_element(iter(w[i]));
    for (int i = 0; i < n; ++i) bfs(i);
    ll res = 0;
    for (int i = 0; i < n; ++i) res += w[i][fl[i]];
    return res;
  }
};
```

3.9 Max Simple Graph Matching

```
struct Matching { // 0-based, O(V^3)
  queue<int> q; int n;
  vector<int> fa, s, vis, pre, match;
  vector<vector<int>> g;
  int Find(int u)
  { return u == fa[u] ? u : fa[u] = Find(fa[u]); }
  int LCA(int x, int y) {
    static int tk = 0; tk++; x = Find(x); y = Find(y);
```

```
    for (;; swap(x, y)) if (x != n) {
      if (vis[x] == tk) return x;
      vis[x] = tk;
      x = Find(pre[match[x]]);
    }
  }
  void Blossom(int x, int y, int l) {
    for (; Find(x) != l; x = pre[y]) {
      pre[x] = y, y = match[x];
      if (s[y] == 1) q.push(y), s[y] = 0;
      for (int z: {x, y}) if (fa[z] == z) fa[z] = l;
    }
  }
  bool Bfs(int r) {
    iota(iter(fa), 0); fill(iter(s), -1);
    q = queue<int>(); q.push(r); s[r] = 0;
    for (; !q.empty(); q.pop()) {
      for (int x = q.front(); int u : g[x])
        if (s[u] == -1) {
          if (pre[u] = x, s[u] = 1, match[u] == n) {
            for (int a = u, b = x, last;
                b != n; a = last, b = pre[a])
              last = match[b], match[b] = a, match[a] =
                   b;
            return true;
          }
          q.push(match[u]); s[match[u]] = 0;
        } else if (!s[u] && Find(u) != Find(x)) {
          int l = LCA(u, x);
          Blossom(x, u, l); Blossom(u, x, l);
        }
    }
    return false;
  }
  Matching(int _n) : n(_n), fa(n + 1), s(n + 1), vis(n
       + 1), pre(n + 1, n), match(n + 1, n), g(n) {}
  void add_edge(int u, int v)
  { g[u].pb(v), g[v].pb(u); }
  int solve() {
    int ans = 0;
    for (int x = 0; x < n; ++x)
      if (match[x] == n) ans += Bfs(x);
    return ans;
  } // match[x] == n means not matched
};
```

3.10 Stable Marriage

Initialize $m \in M$ and $w \in W$ to *free*
while $\exists$ *free* man m who has a woman w to propose to **do**
    $w \leftarrow$ first woman on m's list to whom m has not yet proposed
    if $\exists$ some pair (m', w) **then**
        if w prefers m to m' **then**
            $m' \leftarrow$ *free*
            $(m, w) \leftarrow$ *engaged*
        end if
    else
        $(m, w) \leftarrow$ *engaged*
    end if
end while

3.11 Flow Model

- Maximum/Minimum flow with lower bound / Circulation problem
  1. Construct super source S and sink T.
  2. For each edge (x, y, l, u), connect $x \to y$ with capacity $u - l$.
  3. For each vertex v, denote by $in(v)$ the difference between the sum of incoming lower bounds and the sum of outgoing lower bounds.
  4. If $in(v) > 0$, connect $S \to v$ with capacity $in(v)$, otherwise, connect $v \to T$ with capacity $-in(v)$.
     - To maximize, connect $t \to s$ with capacity ∞ (skip this in circulation problem), and let f be the maximum flow from S to T. If $f \neq \sum_{v \in V, in(v)>0} in(v)$, there's no solution. Otherwise, the maximum flow from s to t is the answer.
     - To minimize, let f be the maximum flow from S to T. Connect $t \to s$ with capacity ∞ and let the flow from S to T be f'. If $f + f' \neq \sum_{v \in V, in(v)>0} in(v)$, there's no solution. Otherwise, f' is the answer.
  5. The solution of each edge e is $l_e + f_e$, where f_e corresponds to the flow of edge e on the graph.
- Construct minimum vertex cover from maximum matching M on bipartite graph (X, Y)
  1. Redirect every edge: $y \to x$ if $(x, y) \in M$, $x \to y$ otherwise.
  2. DFS from unmatched vertices in X.
  3. $x \in X$ is chosen iff x is unvisited.
  4. $y \in Y$ is chosen iff y is visited.
- Minimum cost cyclic flow
  1. Consruct super source S and sink T
  2. For each edge (x, y, c), connect $x \to y$ with $(cost, cap) = (c, 1)$ if $c > 0$, otherwise connect $y \to x$ with $(cost, cap) = (-c, 1)$
  3. For each edge with $c < 0$, sum these cost as K, then increase $d(y)$ by 1, decrease $d(x)$ by 1
  4. For each vertex v with $d(v) > 0$, connect $S \to v$ with $(cost, cap) = (0, d(v))$
  5. For each vertex v with $d(v) < 0$, connect $v \to T$ with $(cost, cap) = (0, -d(v))$
  6. Flow from S to T, the answer is the cost of the flow $C + K$
- Maximum density induced subgraph
  1. Binary search on answer, suppose we're checking answer T
  2. Construct a max flow model, let K be the sum of all weights
  3. Connect source $s \to v, v \in G$ with capacity K
  4. For each edge (u, v, w) in G, connect $u \to v$ and $v \to u$ with capacity w
  5. For $v \in G$, connect it with sink $v \to t$ with capacity $K + 2T - (\sum_{e \in E(v)} w(e)) - 2w(v)$
  6. T is a valid answer if the maximum flow $f < K|V|$
- Minimum weight edge cover
  1. Let $w'(u, v) = w(u, v) - \mu(u) - \mu(v)$, where $\mu(v)$ is the cost of the cheapest edge incident to v.
  2. Find the minimum weight matching M with w'. The answer is $\sum \mu(v) + w'(M)$.
- Project selection problem
  1. If $p_v > 0$, create edge (s, v) with capacity p_v; otherwise, create edge (v, t) with capacity $-p_v$.
  2. Create edge (u, v) with capacity w with w being the cost of choosing u without choosing v.
  3. The mincut is equivalent to the maximum profit of a subset of projects.
- Dual of minimum cost maximum flow
  1. Capacity c_{uv}, Flow f_{uv}, Cost w_{uv}, Required Flow difference for vertex b_u.
  2. If all w_{uv} are integers, then optimal solution can happen when all p_u are integers.

$$\begin{aligned} \text{min} \sum_{uv} w_{uv} f_{uv} & \\ -f_{uv} \geq -c_{uv} & \Leftrightarrow \text{min} \sum_{u} b_u p_u + \sum_{uv} c_{uv} \, \text{max}(0, p_v - p_u - w_{uv}) \\ \sum_v f_{vu} - \sum_v f_{uv} = -b_u & \qquad p_u \geq 0 \end{aligned}$$

4 Geometry

4.1 Geometry Template

```
using ld = ll;
using pdd = pair<ld, ld>;
#define X first
#define Y second
// ld eps = 1e-7;

pdd operator+(pdd a, pdd b)
{ return {a.X + b.X, a.Y + b.Y}; }
pdd operator-(pdd a, pdd b)
{ return {a.X - b.X, a.Y - b.Y}; }
pdd operator*(ld i, pdd v)
{ return {i * v.X, i * v.Y}; }
pdd operator*(pdd v, ld i)
{ return {i * v.X, i * v.Y}; }
pdd operator/(pdd v, ld i)
{ return {v.X / i, v.Y / i}; }
ld dot(pdd a, pdd b)
{ return a.X * b.X + a.Y * b.Y; }
ld cross(pdd a, pdd b)
{ return a.X * b.Y - a.Y * b.X; }
ld abs2(pdd v)
{ return v.X * v.X + v.Y * v.Y; };
ld abs(pdd v)
{ return sqrt(abs2(v)); };
int sgn(ld v)
{ return v > 0 ? 1 : (v < 0 ? -1 : 0); }
// int sgn(ld v){ return v > eps ? 1 : ( v < -eps ? -1
    : 0); }
int ori(pdd a, pdd b, pdd c)
{ return sgn(cross(b - a, c - a)); }
bool collinearity(pdd a, pdd b, pdd c)
{ return ori(a, b, c) == 0; }
bool btw(pdd p, pdd a, pdd b)
{ return collinearity(p, a, b) && sgn(dot(a - p, b - p)
    ) <= 0; }
```

```cpp
bool seg_intersect(pdd p1, pdd p2, pdd p3, pdd p4){
  if(btw(p1, p3, p4) || btw(p2, p3, p4) || btw(p3, p1,
      p2) || btw(p4, p1, p2))
    return true;
  return ori(p1, p2, p3) * ori(p1, p2, p4) < 0 &&
    ori(p3, p4, p1) * ori(p3, p4, p2) < 0;
}
pdd intersect(pdd p1, pdd p2, pdd p3, pdd p4){
  ld a123 = cross(p2 - p1, p3 - p1);
  ld a124 = cross(p2 - p1, p4 - p1);
  return (p4 * a123 - p3 * a124) / (a123 - a124);
}
pdd perp(pdd p1)
{ return pdd(-p1.Y, p1.X); }
pdd projection(pdd p1, pdd p2, pdd p3)
{ return p1 + (p2 - p1) * dot(p3 - p1, p2 - p1) / abs2(
    p2 - p1); }
pdd reflection(pdd p1, pdd p2, pdd p3)
{ return p3 + perp(p2 - p1) * cross(p3 - p1, p2 - p1) /
     abs2(p2 - p1) * 2; }
pdd linearTransformation(pdd p0, pdd p1, pdd q0, pdd q1
    , pdd r) {
  pdd dp = p1 - p0, dq = q1 - q0, num(cross(dp, dq),
      dot(dp, dq));
  return q0 + pdd(cross(r - p0, num), dot(r - p0, num))
       / abs2(dp);
} // from line p0--p1 to q0--q1, apply to r
```

4.2 Polar Angle Comparator

```cpp
// -1: a // b (if same), 0/1: a < b
int cmp(pll a, pll b, bool same = true){
#define is_neg(k) (sgn(k.Y) < 0 || (sgn(k.Y) == 0 &&
    sgn(k.X) < 0))
  int A = is_neg(a), B = is_neg(b);
  if(A != B)
    return A < B;
  if(sgn(cross(a, b)) == 0)
    return same ? abs2(a) < abs2(b) : -1;
  return sgn(cross(a, b)) > 0;
}
```

4.3 Minkowski Sum

```cpp
void reorder_poly(vector<pdd>& pnts){
  int mn = 0;
  for(int i = 1; i < (int)pnts.size(); i++)
    if(pnts[i].Y < pnts[mn].Y || (pnts[i].Y == pnts[mn
        ].Y && pnts[i].X < pnts[mn].X))
      mn = i;
  rotate(pnts.begin(), pnts.begin() + mn, pnts.end());
}

vector<pdd> minkowski(vector<pdd> P, vector<pdd> Q){
  reorder_poly(P);
  reorder_poly(Q);
  int psz = P.size();
  int qsz = Q.size();
  P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
  vector<pdd> ans;
  int i = 0, j = 0;
  while(i < psz || j < qsz){
    ans.pb(P[i] + Q[j]);
    int t = sgn(cross(P[i + 1] - P[i], Q[j + 1] - Q[j])
        );
    if(t >= 0) i++;
    if(t <= 0) j++;
  }
  return ans;
}
```

4.4 Intersection of Circle and Convex Polygon

```cpp
double _area(pdd pa, pdd pb, double r){
  if(abs(pa)<abs(pb)) swap(pa, pb);
  if(abs(pb)<eps) return 0;
  double S, h, theta;
  double a=abs(pb),b=abs(pa),c=abs(pb-pa);
  double cosB = dot(pb,pb-pa) / a / c, B = acos(cosB);
  double cosC = dot(pa,pb) / a / b, C = acos(cosC);
  if(a > r){
    S = (C/2)*r*r;
    h = a*b*sin(C)/c;
    if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt
        (r*r-h*h));
  }
  else if(b > r){
    theta = PI - B - asin(sin(B)/r*a);
    S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
  }
  else S = .5*sin(C)*a*b;
  return S;
}
double areaPolyCircle(const vector<pdd> poly,const pdd
    &O,const double r){
  double S=0;
  for(int i=0;i<SZ(poly);++i)
    S+=_area(poly[i]-O,poly[(i+1)%SZ(poly)]-O,r)*ori(O,
        poly[i],poly[(i+1)%SZ(poly)]);
  return fabs(S);
}
```

4.5 Intersection of Circles

```cpp
bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
  pdd o1 = a.O, o2 = b.O;
  double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d =
      sqrt(d2);
  if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2)
      return 0;
  pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1
      * r1) / (2 * d2));
  double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 +
       r2 - d) * (-r1 + r2 + d));
  pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2)
      ;
  p1 = u + v, p2 = u - v;
  return 1;
}
```

4.6 Tangent Line of Circles

```cpp
vector<Line> CCtang( const Cir& c1 , const Cir& c2 ,
    int sign1 ){
  vector<Line> ret;
  double d_sq = abs2( c1.O - c2.O );
  if (sgn(d_sq) == 0) return ret;
  double d = sqrt(d_sq);
  pdd v = (c2.O - c1.O) / d;
  double c = (c1.R - sign1 * c2.R) / d; // cos t
  if (c * c > 1) return ret;
  double h = sqrt(max( 0.0, 1.0 - c * c)); // sin t
  for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
    pdd n = pdd(v.X * c - sign2 * h * v.Y,
        v.Y * c + sign2 * h * v.X);
    pdd p1 = c1.O + n * c1.R;
    pdd p2 = c2.O + n * (c2.R * sign1);
    if (sgn(p1.X - p2.X) == 0 and
        sgn(p1.Y - p2.Y) == 0)
      p2 = p1 + perp(c2.O - c1.O);
    ret.pb(Line(p1, p2));
  }
  return ret;
}
```

4.7 Intersection of Line and Convex Polygon

```cpp
int TangentDir(vector<pll> &C, pll dir) {
  return cyc_tsearch(SZ(C), [&](int a, int b) {
    return cross(dir, C[a]) > cross(dir, C[b]);
  });
}
#define cmpL(i) sign(cross(C[i] - a, b - a))
pii lineHull(pll a, pll b, vector<pll> &C) {
  int A = TangentDir(C, a - b);
  int B = TangentDir(C, b - a);
  int n = SZ(C);
  if (cmpL(A) < 0 || cmpL(B) > 0)
    return pii(-1, -1); // no collision
  auto gao = [&](int l, int r) {
    for (int t = l; (l + 1) % n != r; ) {
      int m = ((l + r + (l < r ? 0 : n)) / 2) % n;
      (cmpL(m) == cmpL(t) ? l : r) = m;
    }
    return (l + !cmpL(r)) % n;
```

```
  };
  pii res = pii(gao(B, A), gao(A, B)); // (i, j)
  if (res.X == res.Y) // touching the corner i
    return pii(res.X, -1);
  if (!cmpL(res.X) && !cmpL(res.Y)) // along side i, i
      +1
    switch ((res.X - res.Y + n + 1) % n) {
      case 0: return pii(res.X, res.X);
      case 2: return pii(res.Y, res.Y);
    }
  /* crossing sides (i, i+1) and (j, j+1)
  crossing corner i is treated as side (i, i+1)
  returned in the same order as the line hits the
      convex */
  return res;
} // convex cut: (r, l]
```

4.8 Intersection of Line and Circle

```
vector<pdd> circleLineIntersection(pdd c, double r, pdd
     a, pdd b) {
  pdd p = a + (b - a) * dot(c - a, b - a) / abs2(b - a)
      ;
  double s = cross(b - a, c - a), h2 = r * r - s * s /
      abs2(b - a);
  if (sgn(h2) < 0) return {};
  if (sgn(h2) == 0) return {p};
  pdd h = (b - a) / abs(b - a) * sqrt(h2);
  return {p - h, p + h};
}
```

4.9 Point in Circle

```
// return q's relation with circumcircle of tri(p[0],p
    [1],p[2])
bool in_cc(const array<pll, 3> &p, pll q) {
  __int128 det = 0;
  for (int i = 0; i < 3; ++i)
    det += __int128(abs2(p[i]) - abs2(q)) * cross(p[(i
        + 1) % 3] - q, p[(i + 2) % 3] - q);
  return det > 0; // in: >0, on: =0, out: <0
}
```

4.10 Point in Convex

```
bool PointInConvex(const vector<pll> &C, pll p, bool
    strict = true) {
  int a = 1, b = SZ(C) - 1, r = !strict;
  if (SZ(C) == 0) return false;
  if (SZ(C) < 3) return r && btw(C[0], C.back(), p);
  if (ori(C[0], C[a], C[b]) > 0) swap(a, b);
  if (ori(C[0], C[a], p) >= r || ori(C[0], C[b], p) <=
      -r)
    return false;
  while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (ori(C[0], C[c], p) > 0 ? b : a) = c;
  }
  return ori(C[a], C[b], p) < r;
}
```

4.11 Half Plane Intersection

```
// from 8BQube
pll area_pair(Line a, Line b)
{ return pll(cross(a.Y - a.X, b.X - a.X), cross(a.Y - a
    .X, b.Y - a.X)); }
bool isin(Line l0, Line l1, Line l2) {
  // Check inter(l1, l2) strictly in l0
  auto [a02X, a02Y] = area_pair(l0, l2);
  auto [a12X, a12Y] = area_pair(l1, l2);
  if (a12X - a12Y < 0) a12X *= -1, a12Y *= -1;
  return (__int128) a02Y * a12X - (__int128) a02X *
      a12Y > 0; // C^4
}
/* Having solution, check size > 2 */
/* --^-- Line.X --^-- Line.Y --^-- */
vector<Line> halfPlaneInter(vector<Line> arr) {
  sort(iter(arr), [&](Line a, Line b) -> int {
    if (cmp(a.Y - a.X, b.Y - b.X, 0) != -1)
      return cmp(a.Y - a.X, b.Y - b.X, 0);
    return ori(a.X, a.Y, b.Y) < 0;
  });
  deque<Line> dq(1, arr[0]);
  for (auto p : arr) {
    if (cmp(dq.back().Y - dq.back().X, p.Y - p.X, 0) ==
         -1)
      continue;
    while (SZ(dq) >= 2 && !isin(p, dq[SZ(dq) - 2], dq.
        back()))
      dq.pop_back();
    while (SZ(dq) >= 2 && !isin(p, dq[0], dq[1]))
      dq.pop_front();
    dq.pb(p);
  }
  while (SZ(dq) >= 3 && !isin(dq[0], dq[SZ(dq) - 2], dq
      .back()))
    dq.pop_back();
  while (SZ(dq) >= 3 && !isin(dq.back(), dq[0], dq[1]))
    dq.pop_front();
  return vector<Line>(iter(dq));
}
```

4.12 Minimum Enclosing Circle

```
using ld = long double;
pair<pdd, ld> circumcenter(pdd a, pdd b, pdd c);
pair<pdd, ld> MinimumEnclosingCircle(vector<pdd> &pts){
  random_shuffle(iter(pts));
  pdd c = pts[0];
  ld r = 0;
  for(int i = 1; i < SZ(pts); i++){
    if(abs(pts[i] - c) <= r) continue;
    c = pts[i]; r = 0;
    for(int j = 0; j < i; j++){
      if(abs(pts[j] - c) <= r) continue;
      c = (pts[i] + pts[j]) / 2;
      r = abs(pts[i] - c);
      for(int k = 0; k < j; k++){
        if(abs(pts[k] - c) > r)
          tie(c, r) = circumcenter(pts[i], pts[j], pts[
              k]);
      }
    }
  }
  return {c, r};
}
```

4.13 3D Point

```
// Copy from 8BQube
struct Point {
  double x, y, z;
  Point(double _x = 0, double _y = 0, double _z = 0): x
      (_x), y(_y), z(_z){}
  Point(pdd p) { x = p.X, y = p.Y, z = abs2(p); }
};
Point operator-(Point p1, Point p2)
{ return Point(p1.x - p2.x, p1.y - p2.y, p1.z - p2.z);
    }
Point operator+(Point p1, Point p2)
{ return Point(p1.x + p2.x, p1.y + p2.y, p1.z + p2.z);
    }
Point operator*(Point p1, double v)
{ return Point(p1.x * v, p1.y * v, p1.z * v); }
Point operator/(Point p1, double v)
{ return Point(p1.x / v, p1.y / v, p1.z / v); }
Point cross(Point p1, Point p2)
{ return Point(p1.y * p2.z - p1.z * p2.y, p1.z * p2.x -
     p1.x * p2.z, p1.x * p2.y - p1.y * p2.x); }
double dot(Point p1, Point p2)
{ return p1.x * p2.x + p1.y * p2.y + p1.z * p2.z; }
double abs(Point a)
{ return sqrt(dot(a, a)); }
Point cross3(Point a, Point b, Point c)
{ return cross(b - a, c - a); }
double area(Point a, Point b, Point c)
{ return abs(cross3(a, b, c)); }
double volume(Point a, Point b, Point c, Point d)
{ return dot(cross3(a, b, c), d - a); }
//Azimuthal angle (longitude) to x-axis in interval [-
    pi, pi]
double phi(Point p) { return atan2(p.y, p.x); }
//Zenith angle (latitude) to the z-axis in interval [0,
     pi]
double theta(Point p) { return atan2(sqrt(p.x * p.x + p
    .y * p.y), p.z); }
```

```
Point masscenter(Point a, Point b, Point c, Point d)
{ return (a + b + c + d) / 4; }
pdd proj(Point a, Point b, Point c, Point u) {
// proj. u to the plane of a, b, and c
  Point e1 = b - a;
  Point e2 = c - a;
  e1 = e1 / abs(e1);
  e2 = e2 - e1 * dot(e2, e1);
  e2 = e2 / abs(e2);
  Point p = u - a;
  return pdd(dot(p, e1), dot(p, e2));
}
Point rotate_around(Point p, double angle, Point axis)
    {
  double s = sin(angle), c = cos(angle);
  Point u = axis / abs(axis);
  return u * dot(u, p) * (1 - c) + p * c + cross(u, p)
      * s;
}
```

4.14 ConvexHull3D

```
struct convex_hull_3D {
struct Face {
  int a, b, c;
  Face(int ta, int tb, int tc): a(ta), b(tb), c(tc) {}
}; // return the faces with pt indexes
vector<Face> res;
vector<Point> P;
convex_hull_3D(const vector<Point> &_P): res(), P(_P) {
// all points coplanar case will WA, O(n^2)
  int n = SZ(P);
  if (n <= 2) return; // be careful about edge case
  // ensure first 4 points are not coplanar
  swap(P[1], *find_if(iter(P), [&](auto p) { return sgn
      (abs2(P[0] - p)) != 0; }));
  swap(P[2], *find_if(iter(P), [&](auto p) { return sgn
      (abs2(cross3(p, P[0], P[1]))) != 0; }));
  swap(P[3], *find_if(iter(P), [&](auto p) { return sgn
      (volume(P[0], P[1], P[2], p)) != 0; }));
  vector<vector<int>> flag(n, vector<int>(n));
  res.emplace_back(0, 1, 2); res.emplace_back(2, 1, 0);
  for (int i = 3; i < n; ++i) {
    vector<Face> next;
    for (auto f : res) {
      int d = sgn(volume(P[f.a], P[f.b], P[f.c], P[i]))
          ;
      if (d <= 0) next.pb(f);
      int ff = (d > 0) - (d < 0);
      flag[f.a][f.b] = flag[f.b][f.c] = flag[f.c][f.a]
          = ff;
    }
    for (auto f : res) {
      auto F = [&](int x, int y) {
        if (flag[x][y] > 0 && flag[y][x] <= 0)
          next.emplace_back(x, y, i);
      };
      F(f.a, f.b); F(f.b, f.c); F(f.c, f.a);
    }
    res = next;
  }
}
bool same(Face s, Face t) {
  if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.a])) != 0)
       return 0;
  if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.b])) != 0)
       return 0;
  if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.c])) != 0)
       return 0;
  return 1;
}
int polygon_face_num() {
  int ans = 0;
  for (int i = 0; i < SZ(res); ++i)
    ans += none_of(res.begin(), res.begin() + i, [&](
        Face g) { return same(res[i], g); });
  return ans;
}
double get_volume() {
  double ans = 0;
  for (auto f : res)
    ans += volume(Point(0, 0, 0), P[f.a], P[f.b], P[f.c
        ]);
  return fabs(ans / 6);
}
double get_dis(Point p, Face f) {
  Point p1 = P[f.a], p2 = P[f.b], p3 = P[f.c];
  double a = (p2.y - p1.y) * (p3.z - p1.z) - (p2.z - p1
      .z) * (p3.y - p1.y);
  double b = (p2.z - p1.z) * (p3.x - p1.x) - (p2.x - p1
      .x) * (p3.z - p1.z);
  double c = (p2.x - p1.x) * (p3.y - p1.y) - (p2.y - p1
      .y) * (p3.x - p1.x);
  double d = 0 - (a * p1.x + b * p1.y + c * p1.z);
  return fabs(a * p.x + b * p.y + c * p.z + d) / sqrt(a
       * a + b * b + c * c);
}
};
// n^2 delaunay: facets with negative z normal of
// convexhull of (x, y, x^2 + y^2), use a pseudo-point
// (0, 0, inf) to avoid degenerate case
```

4.15 Delaunay Triangulation

```
/* Delaunay Triangulation:
Given a sets of points on 2D plane, find a
triangulation such that no points will strictly
inside circumcircle of any triangle. */
struct Edge {
  int id; // oidx[id]
  list<Edge>::iterator twin;
  Edge(int _id = 0):id(_id) {}
};
struct Delaunay { // 0-base
  int n, oidx[N];
  list<Edge> head[N]; // result udir. graph
  pll p[N];
  void init(int _n, pll _p[]) {
    n = _n, iota(oidx, oidx + n, 0);
    for (int i = 0; i < n; ++i) head[i].clear();
    sort(oidx, oidx + n, [&](int a, int b)
    { return _p[a] < _p[b]; });
    for (int i = 0; i < n; ++i) p[i] = _p[oidx[i]];
    divide(0, n - 1);
  }
  void addEdge(int u, int v) {
    head[u].push_front(Edge(v));
    head[v].push_front(Edge(u));
    head[u].begin()->twin = head[v].begin();
    head[v].begin()->twin = head[u].begin();
  }
  void divide(int l, int r) {
    if (l == r) return;
    if (l + 1 == r) return addEdge(l, l + 1);
    int mid = (l + r) >> 1, nw[2] = {l, r};
    divide(l, mid), divide(mid + 1, r);
    auto gao = [&](int t) {
      pll pt[2] = {p[nw[0]], p[nw[1]]};
      for (auto it : head[nw[t]]) {
        int v = ori(pt[1], pt[0], p[it.id]);
        if (v > 0 || (v == 0 && abs2(pt[t ^ 1] - p[it.
            id]) < abs2(pt[1] - pt[0])))
          return nw[t] = it.id, true;
      }
      return false;
    };
    while (gao(0) || gao(1));
    addEdge(nw[0], nw[1]); // add tangent
    while (true) {
      pll pt[2] = {p[nw[0]], p[nw[1]]};
      int ch = -1, sd = 0;
      for (int t = 0; t < 2; ++t)
          for (auto it : head[nw[t]])
              if (ori(pt[0], pt[1], p[it.id]) > 0 && (
                  ch == -1 || in_cc({pt[0], pt[1], p[ch
                  ]}, p[it.id])))
                  ch = it.id, sd = t;
      if (ch == -1) break; // upper common tangent
      for (auto it = head[nw[sd]].begin(); it != head[
          nw[sd]].end(); )
        if (seg_strict_intersect(pt[sd], p[it->id], pt[
            sd ^ 1], p[ch]))
          head[it->id].erase(it->twin), head[nw[sd]].
              erase(it++);
        else ++it;
```

```
      nw[sd] = ch, addEdge(nw[0], nw[1]);
    }
  }
} tool;
```

4.16 Voronoi Diagram

```
// all coord. is even, you may want to call
    halfPlaneInter after then
vector<vector<Line>> vec;
void build_voronoi_line(int n, pll *arr) {
  tool.init(n, arr); // Delaunay
  vec.clear(), vec.resize(n);
  for (int i = 0; i < n; ++i)
    for (auto e : tool.head[i]) {
      int u = tool.oidx[i], v = tool.oidx[e.id];
      pll m = (arr[v] + arr[u]) / 2LL, d = perp(arr[v]
          - arr[u]);
      vec[u].pb(Line(m, m + d));
    }
}
```

4.17 Polygon Union

```
// from 8BQube
ld rat(pll a, pll b) {
  return sgn(b.X) ? (ld)a.X / b.X : (ld)a.Y / b.Y;
} // all poly. should be ccw
ld polyUnion(vector<vector<pll>> &poly) {
  ld res = 0;
  for (auto &p : poly)
    for (int a = 0; a < SZ(p); ++a) {
      pll A = p[a], B = p[(a + 1) % SZ(p)];
      vector<pair<ld, int>> segs = {{0, 0}, {1, 0}};
      for (auto &q : poly) {
        if (&p == &q) continue;
        for (int b = 0; b < SZ(q); ++b) {
          pll C = q[b], D = q[(b + 1) % SZ(q)];
          int sc = ori(A, B, C), sd = ori(A, B, D);
          if (sc != sd && min(sc, sd) < 0) {
            ld sa = cross(D - C, A - C), sb = cross(D -
                C, B - C);
            segs.pb(sa / (sa - sb), sgn(sc - sd));
          }
          if (!sc && !sd && &q < &p && sgn(dot(B - A, D
              - C)) > 0) {
            segs.pb(rat(C - A, B - A), 1);
            segs.pb(rat(D - A, B - A), -1);
          }
        }
      }
      sort(iter(segs));
      for (auto &s : segs) s.X = clamp(s.X, 0.0, 1.0);
      ld sum = 0;
      int cnt = segs[0].second;
      for (int j = 1; j < SZ(segs); ++j) {
        if (!cnt) sum += segs[j].X - segs[j - 1].X;
        cnt += segs[j].Y;
      }
      res += cross(A, B) * sum;
    }
  return res / 2;
}
```

4.18 Tangent Point to Convex Hull

```
// from 8BQube
/* The point should be strictly out of hull
  return arbitrary point on the tangent line */
pii get_tangent(vector<pll> &C, pll p) {
  auto gao = [&](int s) {
    return cyc_tsearch(SZ(C), [&](int x, int y)
    { return ori(p, C[x], C[y]) == s; });
  };
  return pii(gao(1), gao(-1));
} // return (a, b), ori(p, C[a], C[b]) >= 0
```

4.19 Heart

```
pdd circenter(pdd p0, pdd p1, pdd p2) { // radius = abs
    (center)
  p1 = p1 - p0, p2 = p2 - p0;
  double x1 = p1.X, y1 = p1.Y, x2 = p2.X, y2 = p2.Y;
  double m = 2. * (x1 * y2 - y1 * x2);
  pdd center;
  center.X = (x1 * x1 * y2 - x2 * x2 * y1 + y1 * y2 * (
      y1 - y2)) / m;
  center.Y = (x1 * x2 * (x2 - x1) - y1 * y1 * x2 + x1 *
       y2 * y2) / m;
  return center + p0;
}
pdd incenter(pdd p1, pdd p2, pdd p3) { // radius = area
     / s * 2
  double a = abs(p2 - p3), b = abs(p1 - p3), c = abs(p1
     - p2);
  double s = a + b + c;
  return (a * p1 + b * p2 + c * p3) / s;
}
pdd masscenter(pdd p1, pdd p2, pdd p3)
{ return (p1 + p2 + p3) / 3; }
pdd orthcenter(pdd p1, pdd p2, pdd p3)
{ return masscenter(p1, p2, p3) * 3 - circenter(p1, p2,
     p3) * 2; }
```

4.20 Rotating Sweep Line

```
struct Event {
  pll d; int u, v;
  bool operator<(const Event &b) const {
    int ret = cmp(d, b.d, false);
    return ret == -1 ? false : ret; } // no tie-break
};
void rotatingSweepLine(const vector<pll> &p) {
  const int n = SZ(p);
  vector<Event> e; e.reserve(n * (n - 1));
  for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++) // pos[i] < pos[j] when
         the event occurs
      if (i != j) e.pb(p[j] - p[i], i, j);
  sort(iter(e));
  vector<int> ord(n), pos(n);
  iota(iter(ord), 0);
  sort(iter(ord), [&](int i, int j) { // initial order
      return p[i].Y != p[j].Y ? p[i].Y < p[j].Y : p[i].
          X < p[j].X; });
  for (int i = 0; i < n; i++) pos[ord[i]] = i;
  // initialize
  for (int i = 0, j = 0; i < SZ(e); i = j) {
    // do something
    vector<pii> tmp;
    for (; j < SZ(e) && !(e[i] < e[j]); j++)
      tmp.pb(pii(e[j].u, e[j].v));
    sort(iter(tmp), [&](pii x, pii y){
        return pii(pos[x.ff], pos[x.ss]) < pii(pos[y.ff
            ], pos[y.ss]); });
    for (auto [x, y] : tmp) // pos[x] + 1 == pos[y]
      tie(ord[pos[x]], ord[pos[y]], pos[x], pos[y]) =
        make_tuple(ord[pos[y]], ord[pos[x]], pos[y],
            pos[x]);
  }
}
```

4.21 Vector In Poly

```
// ori(a, b, c) >= 0, valid: "strict" angle from a-b to
     a-c
bool btwangle(pll a, pll b, pll c, pll p, int strict) {
  return ori(a, b, p) >= strict && ori(a, p, c) >=
      strict;
}
// whether vector{cur, p} in counter-clockwise order
    prv, cur, nxt
bool inside(pll prv, pll cur, pll nxt, pll p, int
    strict) {
  if (ori(cur, nxt, prv) >= 0)
    return btwangle(cur, nxt, prv, p, strict);
  return !btwangle(cur, prv, nxt, p, !strict);
}
```

5 Graph

5.1 BCC

```
struct BCC{ // 0-based, allow multi edges but not allow
     loops
  int n, m, cnt = 0;
  // n:|V|, m:|E|, cnt:#bcc
```

```cpp
  // bcc i : vertices bcc_v[i] and edges bcc_e[i]
  vector<vector<int>> bcc_v, bcc_e;
  vector<vector<pii>> g; // original graph
  vector<pii> edges; // 0-based
  BCC(int _n, vector<pii> _edges):
    n(_n), m(SZ(_edges)), g(_n), edges(_edges){
      for(int i = 0; i < m; i++){
        auto [u, v] = edges[i];
        g[u].pb(pii(v, i)); g[v].pb(pii(u, i));
      }
    }
  void make_bcc(){ bcc_v.pb(); bcc_e.pb(); cnt++; }
  // modify these if you need more information
  void add_v(int v){ bcc_v.back().pb(v); }
  void add_e(int e){ bcc_e.back().pb(e); }
  void build(){
    vector<int> in(n, -1), low(n, -1), stk;
    vector<vector<int>> up(n);
    int ts = 0;
    auto _dfs = [&](auto dfs, int now, int par, int pe)
        -> void{
      if(pe != -1) up[now].pb(pe);
      in[now] = low[now] = ts++;
      stk.pb(now);
      for(auto [v, e] : g[now]){
        if(e == pe) continue;
        if(in[v] != -1){
          if(in[v] < in[now]) up[now].pb(e);
          low[now] = min(low[now], in[v]);
          continue;
        }
        dfs(dfs, v, now, e);
        low[now] = min(low[now], low[v]);
      }
      if((now != par && low[now] >= in[par]) || (now ==
          par && SZ(g[now]) == 0)){
        make_bcc();
        for(int v = stk.back();; v = stk.back()){
          stk.pop_back(), add_v(v);
          for(int e : up[v]) add_e(e);
          if(v == now) break;
        }
        if(now != par) add_v(par);
      }
    };
    for(int i = 0; i < n; i++)
      if(in[i] == -1) _dfs(_dfs, i, i, -1);
  }
};
```

5.2 SCC

```cpp
struct SCC{ // 0-based, output reversed topo order
  int n, cnt = 0;
  vector<vector<int>> g;
  vector<int> sccid;
  explicit SCC(int _n): n(_n), g(n), sccid(n, -1) {}
  void add_edge(int u, int v){
    g[u].pb(v);
  }
  void build(){
    vector<int> in(n, -1), low(n), stk;
    vector<bool> instk(n);
    int ts = 0;
    auto dfs1 = [&](auto dfs, int now) -> void{
      stk.pb(now); instk[now] = true;
      in[now] = low[now] = ts++;
      for(int i : g[now]){
        if(in[i] == -1)
          dfs(dfs, i), low[now] = min(low[now], low[i])
            ;
        else if(instk[i] && in[i] < in[now])
          low[now] = min(low[now], in[i]);
      }
      if(low[now] == in[now]){
        for(; stk.back() != now; stk.pop_back())
          sccid[stk.back()] = cnt, instk[stk.back()] =
            false;
        sccid[now] = cnt++, instk[now] = false, stk.
          pop_back();
      }
    };
    for(int i = 0; i < n; i++)
      if(in[i] == -1) dfs1(dfs1, i);
  }
};
```

5.3 2-SAT

```cpp
struct SAT { // 0-based
  int n;
  vector<bool> istrue;
  SCC scc;
  SAT(int _n): n(_n), istrue(n + n), scc(n + n) {}
  int neg(int a) {
    return a >= n ? a - n : a + n;
  }
  void add_clause(int a, int b) {
    scc.add_edge(neg(a), b), scc.add_edge(neg(b), a);
  }
  bool solve() {
    scc.build();
    for (int i = 0; i < n; ++i) {
      if (scc.sccid[i] == scc.sccid[i + n]) return
          false;
      istrue[i] = scc.sccid[i] < scc.sccid[i + n];
      istrue[i + n] = !istrue[i];
    }
    return true;
  }
};
```

5.4 Dominator Tree

```cpp
struct Dominator {
  int n;
  vector<vector<int>> g, r, rdom; int tk;
  vector<int> dfn, rev, fa, sdom, dom, val, rp;
  Dominator(int _n) : n(_n), g(n), r(n), rdom(n), tk(0)
      {
    dfn = rev = fa = sdom = dom =
      val = rp = vector<int>(n, -1); }
  void add_edge(int x, int y) { g[x].push_back(y); }
  void dfs(int x) {
    rev[dfn[x] = tk] = x;
    fa[tk] = sdom[tk] = val[tk] = tk; tk++;
    for (int u : g[x]) {
      if (dfn[u] == -1) dfs(u), rp[dfn[u]] = dfn[x];
      r[dfn[u]].push_back(dfn[x]);
    }
  }
  void merge(int x, int y) { fa[x] = y; }
  int find(int x, int c = 0) {
    if (fa[x] == x) return c ? -1 : x;
    if (int p = find(fa[x], 1); p != -1) {
      if (sdom[val[x]] > sdom[val[fa[x]]])
        val[x] = val[fa[x]];
      fa[x] = p;
      return c ? p : val[x];
    } else return c ? fa[x] : val[x];
  }
  vector<int> build(int s) {
    // return the father of each node in dominator tree
    dfs(s); // p[i] = -2 if i is unreachable, par[s] =
        -1
    for (int i = tk - 1; i >= 0; --i) {
      for (int u : r[i])
        sdom[i] = min(sdom[i], sdom[find(u)]);
      if (i) rdom[sdom[i]].push_back(i);
      for (int u : rdom[i]) {
        int p = find(u);
        dom[u] = (sdom[p] == i ? i : p);
      }
      if (i) merge(i, rp[i]);
    }
    vector<int> p(n, -2); p[s] = -1;
    for (int i = 1; i < tk; ++i)
      if (sdom[i] != dom[i]) dom[i] = dom[dom[i]];
    for (int i = 1; i < tk; ++i)
      p[rev[i]] = rev[dom[i]];
    return p;
  }
};
```

5.5 Virtual Tree

```
// copy from 8BQube
vector<int> vG[N];
int top, st[N];
int vrt = -1;

void insert(int u) {
  if (top == -1) return st[++top] = vrt = u, void();
  int p = LCA(st[top], u);
    if(dep[vrt] > dep[p]) vrt = p;
  if (p == st[top]) return st[++top] = u, void();
  while (top >= 1 && dep[st[top - 1]] >= dep[p])
    vG[st[top - 1]].pb(st[top]), --top;
  if (st[top] != p)
    vG[p].pb(st[top]), --top, st[++top] = p;
  st[++top] = u;
}

void reset(int u) {
  for (int i : vG[u]) reset(i);
  vG[u].clear();
}

void solve(vector<int> &v) {
  top = -1;
  sort(ALL(v),
      [&](int a, int b) { return dfn[a] < dfn[b]; });
  for (int i : v) insert(i);
  while (top > 0) vG[st[top - 1]].pb(st[top]), --top;
  // do something
  reset(vrt);
}
```

5.6 Fast DMST

```
struct E { int s, t; ll w; }; // 0-base
struct PQ {
  struct P {
    ll v; int i;
    bool operator>(const P &b) const { return v > b.v;
        }
  };
  priority_queue<P, vector<P>, greater<>> pq; ll tag;
      // min heap
  void push(P p) { p.v -= tag; pq.emplace(p); }
  P top() { P p = pq.top(); p.v += tag; return p; }
  void join(PQ &b) {
    if (pq.size() < b.pq.size())
      swap(pq, b.pq), swap(tag, b.tag);
    while (!b.pq.empty()) push(b.top()), b.pq.pop();
  }
}; // O(E log^2 V), use leftist tree for O(E log V)
vector<int> dmst(const vector<E> &e, int n, int root) {
  vector<PQ> h(n * 2);
  for (int i = 0; i < int(e.size()); ++i)
    h[e[i].t].push({e[i].w, i});
  vector<int> a(n * 2); iota(iter(a), 0);
  vector<int> v(n * 2, -1), pa(n * 2, -1), r(n * 2);
  auto o = [&](auto Y, int x) -> int {
    return x==a[x] ? x : a[x] = Y(Y, a[x]); };
  auto S = [&](int i) { return o(o, e[i].s); };
  int pc = v[root] = n;
  for (int i = 0; i < n; ++i) if (v[i] == -1)
    for (int p = i; v[p]<0 || v[p]==i; p = S(r[p])) {
      if (v[p] == i)
        for (int q = pc++; p != q; p = S(r[p])) {
          h[p].tag -= h[p].top().v; h[q].join(h[p]);
          pa[p] = a[p] = q;
        }
      while (S(h[p].top().i) == p) h[p].pq.pop();
      v[p] = i; r[p] = h[p].top().i;
    }
  vector<int> ans;
  for (int i = pc - 1; i >= 0; i--) if (v[i] != n) {
    for (int f = e[r[i]].t; f!=-1 && v[f]!=n; f = pa[f
        ])
      v[f] = n;
    ans.push_back(r[i]);
  }
  return ans; // default minimize, returns edgeid array
}
```

5.7 Vizing

```
// find D+1 edge coloring of a graph with max deg D
struct vizing { // returns edge coloring in adjacent
    matrix G. 1 - based
  const int N = 105;
  int C[N][N], G[N][N], X[N], vst[N], n; // ans: G[i][j
    ]
  void init(int _n) { n = _n; // n = |V|+1
    for (int i = 0; i <= n; ++i)
      for (int j = 0; j <= n; ++j)
        C[i][j] = G[i][j] = 0;
  }
  void solve(vector<pii> &E) {
    auto update = [&](int u)
    { for (X[u] = 1; C[u][X[u]]; ++X[u]); };
    auto color = [&](int u, int v, int c) {
      int p = G[u][v];
      G[u][v] = G[v][u] = c;
      C[u][c] = v, C[v][c] = u;
      C[u][p] = C[v][p] = 0;
      if (p) X[u] = X[v] = p;
      else update(u), update(v);
      return p;
    };
    auto flip = [&](int u, int c1, int c2) {
      int p = C[u][c1];
      swap(C[u][c1], C[u][c2]);
      if (p) G[u][p] = G[p][u] = c2;
      if (!C[u][c1]) X[u] = c1;
      if (!C[u][c2]) X[u] = c2;
      return p;
    };
    fill_n(X + 1, n, 1);
    for (int t = 0; t < SZ(E); ++t) {
      int u = E[t].X, v0 = E[t].Y, v = v0, c0 = X[u], c
          = c0, d;
      vector<pii> L;
      fill_n(vst + 1, n, 0);
      while (!G[u][v0]) {
        L.emplace_back(v, d = X[v]);
        if (!C[v][c]) for (int a = SZ(L) - 1; a >= 0;
            --a) c = color(u, L[a].X, c);
        else if (!C[u][d]) for (int a = SZ(L) - 1; a >=
             0; --a) color(u, L[a].X, L[a].Y);
        else if (vst[d]) break;
        else vst[d] = 1, v = C[u][d];
      }
      if (!G[u][v0]) {
        for (; v; v = flip(v, c, d), swap(c, d));
        if (int a; C[u][c0]) {
          for (a = SZ(L) - 2; a >= 0 && L[a].Y != c; --
              a);
          for (; a >= 0; --a) color(u, L[a].X, L[a].Y);
        }
        else --t;
      }
    }
  }
};
```

5.8 Maximum Clique

```
struct MaxClique { // fast when N <= 100
  bitset<N> G[N], cs[N];
  int ans, sol[N], q, cur[N], d[N], n;
  void init(int _n) {
    n = _n;
    for (int i = 0; i < n; ++i) G[i].reset();
  }
  void add_edge(int u, int v) {
    G[u][v] = G[v][u] = 1;
  }
  void pre_dfs(vector<int> &r, int l, bitset<N> mask) {
    if (l < 4) {
      for (int i : r) d[i] = (G[i] & mask).count();
      sort(ALL(r), [&](int x, int y) { return d[x] > d[
          y]; });
    }
    vector<int> c(SZ(r));
    int lft = max(ans - q + 1, 1), rgt = 1, tp = 0;
    cs[1].reset(), cs[2].reset();
    for (int p : r) {
```

```
      int k = 1;
      while ((cs[k] & G[p]).any()) ++k;
      if (k > rgt) cs[++rgt + 1].reset();
      cs[k][p] = 1;
      if (k < lft) r[tp++] = p;
    }
    for (int k = lft; k <= rgt; ++k)
      for (int p = cs[k]._Find_first(); p < N; p = cs[k
          ]._Find_next(p))
        r[tp] = p, c[tp] = k, ++tp;
    dfs(r, c, l + 1, mask);
  }
  void dfs(vector<int> &r, vector<int> &c, int l,
      bitset<N> mask) {
    while (!r.empty()) {
      int p = r.back();
      r.pop_back(), mask[p] = 0;
      if (q + c.back() <= ans) return;
      cur[q++] = p;
      vector<int> nr;
      for (int i : r) if (G[p][i]) nr.pb(i);
      if (!nr.empty()) pre_dfs(nr, l, mask & G[p]);
      else if (q > ans) ans = q, copy_n(cur, q, sol);
      c.pop_back(), --q;
    }
  }
  int solve() {
    vector<int> r(n);
    ans = q = 0, iota(ALL(r), 0);
    pre_dfs(r, 0, bitset<N>(string(n, '1')));
    return ans;
  }
};
```

5.9 Number of Maximal Clique

```
struct BronKerbosch { // 1-base
  int n, a[N], g[N][N];
  int S, all[N][N], some[N][N], none[N][N];
  void init(int _n) {
    n = _n;
    for (int i = 1; i <= n; ++i)
      for (int j = 1; j <= n; ++j) g[i][j] = 0;
  }
  void add_edge(int u, int v) {
    g[u][v] = g[v][u] = 1;
  }
  void dfs(int d, int an, int sn, int nn) {
    if (S > 1000) return; // pruning
    if (sn == 0 && nn == 0) ++S;
    int u = some[d][0];
    for (int i = 0; i < sn; ++i) {
      int v = some[d][i];
      if (g[u][v]) continue;
      int tsn = 0, tnn = 0;
      copy_n(all[d], an, all[d + 1]);
      all[d + 1][an] = v;
      for (int j = 0; j < sn; ++j)
        if (g[v][some[d][j]])
          some[d + 1][tsn++] = some[d][j];
      for (int j = 0; j < nn; ++j)
        if (g[v][none[d][j]])
          none[d + 1][tnn++] = none[d][j];
      dfs(d + 1, an + 1, tsn, tnn);
      some[d][i] = 0, none[d][nn++] = v;
    }
  }
  int solve() {
    iota(some[0], some[0] + n, 1);
    S = 0, dfs(0, 0, n, 0);
    return S;
  }
};
```

5.10 Minimum Mean Cycle

```
// from 8BQube
ll road[N][N]; // input here
struct MinimumMeanCycle {
  ll dp[N + 5][N], n;
  pll solve() {
    ll a = -1, b = -1, L = n + 1;
    for (int i = 2; i <= L; ++i)
      for (int k = 0; k < n; ++k)
        for (int j = 0; j < n; ++j)
          dp[i][j] =
            min(dp[i - 1][k] + road[k][j], dp[i][j]);
    for (int i = 0; i < n; ++i) {
      if (dp[L][i] >= INF) continue;
      ll ta = 0, tb = 1;
      for (int j = 1; j < n; ++j)
        if (dp[j][i] < INF &&
          ta * (L - j) < (dp[L][i] - dp[j][i]) * tb)
          ta = dp[L][i] - dp[j][i], tb = L - j;
      if (ta == 0) continue;
      if (a == -1 || a * tb > ta * b) a = ta, b = tb;
    }
    if (a != -1) {
      ll g = __gcd(a, b);
      return pll(a / g, b / g);
    }
    return pll(-1LL, -1LL);
  }
  void init(int _n) {
    n = _n;
    for (int i = 0; i < n; ++i)
      for (int j = 0; j < n; ++j) dp[i + 2][j] = INF;
  }
};
```

5.11 Minimum Steiner Tree

```
// from 8BQube
// O(V 3^T + V^2 2^T)
struct SteinerTree { // 0-base
  static const int T = 10, N = 105, INF = 1e9;
  int n, dst[N][N], dp[1 << T][N], tdst[N];
  int vcost[N]; // the cost of vertexs
  void init(int _n) {
    n = _n;
    for (int i = 0; i < n; ++i) {
      for (int j = 0; j < n; ++j) dst[i][j] = INF;
      dst[i][i] = vcost[i] = 0;
    }
  }
  void add_edge(int ui, int vi, int wi) {
    dst[ui][vi] = min(dst[ui][vi], wi);
  }
  void shortest_path() {
    for (int k = 0; k < n; ++k)
      for (int i = 0; i < n; ++i)
        for (int j = 0; j < n; ++j)
          dst[i][j] =
            min(dst[i][j], dst[i][k] + dst[k][j]);
  }
  int solve(const vector<int> &ter) {
    shortest_path();
    int t = SZ(ter);
    for (int i = 0; i < (1 << t); ++i)
      for (int j = 0; j < n; ++j) dp[i][j] = INF;
    for (int i = 0; i < n; ++i) dp[0][i] = vcost[i];
    for (int msk = 1; msk < (1 << t); ++msk) {
      if (!(msk & (msk - 1))) {
        int who = __lg(msk);
        for (int i = 0; i < n; ++i)
          dp[msk][i] =
            vcost[ter[who]] + dst[ter[who]][i];
      }
      for (int i = 0; i < n; ++i)
        for (int submsk = (msk - 1) & msk; submsk;
             submsk = (submsk - 1) & msk)
          dp[msk][i] = min(dp[msk][i],
            dp[submsk][i] + dp[msk ^ submsk][i] -
              vcost[i]);
      for (int i = 0; i < n; ++i) {
        tdst[i] = INF;
        for (int j = 0; j < n; ++j)
          tdst[i] =
            min(tdst[i], dp[msk][j] + dst[j][i]);
      }
      for (int i = 0; i < n; ++i) dp[msk][i] = tdst[i];
    }
    int ans = INF;
    for (int i = 0; i < n; ++i)
      ans = min(ans, dp[(1 << t) - 1][i]);
```

```
    return ans;
  }
};
```

6 Math

6.1 Extended Euclidean Algorithm

```
// ax+ny = 1, ax+ny == ax == 1 (mod n)
void extgcd(ll x,ll y,ll &g,ll &a,ll &b) {
  if (y == 0) g=x,a=1,b=0;
  else extgcd(y,x%y,g,b,a),b-=(x/y)*a;
}
```

6.2 Floor & Ceil

```
ll ifloor(ll a,ll b){
  return a / b - (a % b && (a < 0) ^ (b < 0));
}
ll iceil(ll a,ll b){
  return a / b + (a % b && (a < 0) ^ (b > 0));
}
```

6.3 Legendre

```
// the Jacobi symbol is a generalization of the
    Legendre symbol,
// such that the bottom doesn't need to be prime.
// (n|p) -> same as legendre
// (n|ab) = (n|a)(n|b)
// work with long long
int Jacobi(int a, int m) {
  int s = 1;
  for (; m > 1; ) {
    a %= m;
    if (a == 0) return 0;
    const int r = __builtin_ctz(a);
    if ((r & 1) && ((m + 2) & 4)) s = -s;
    a >>= r;
    if (a & m & 2) s = -s;
    swap(a, m);
  }
  return s;
}

// 0: a == 0
// -1: a isn't a quad res of p
// else: return X with X^2 % p == a
// doesn't work with long long
int QuadraticResidue(int a, int p) {
  if (p == 2) return a & 1;
  if(int jc = Jacobi(a, p); jc <= 0) return jc;
  int b, d;
  for (; ; ) {
    b = rand() % p;
    d = (1LL * b * b + p - a) % p;
    if (Jacobi(d, p) == -1) break;
  }
  int f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
  for (int e = (1LL + p) >> 1; e; e >>= 1) {
    if (e & 1) {
      tmp = (1LL * g0 * f0 + 1LL * d * (1LL * g1 * f1 %
          p)) % p;
      g1 = (1LL * g0 * f1 + 1LL * g1 * f0) % p;
      g0 = tmp;
    }
    tmp = (1LL * f0 * f0 + 1LL * d * (1LL * f1 * f1 % p
        )) % p;
    f1 = (2LL * f0 * f1) % p;
    f0 = tmp;
  }
  return g0;
}
```

6.4 Simplex

```
// maximize c^T x
// subject to Ax <= b, x >= 0
// and stores the solution;
typedef long double T; // long double, Rational, double
     + mod<P>...
typedef vector<T> vd;
typedef vector<vd> vvd;
const T eps = 1e-9, inf = 1/.0;
#define ltj(X) if(s == -1 || mp(X[j],N[j]) < mp(X[s],N[
    s])) s=j
#define rep(i, l, n) for(int i = l; i < n; i++)

struct LPSolver {
  int m, n;
  vector<int> N, B;
  vvd D;

  LPSolver(const vvd& A, const vd& b, const vd& c) :
    m(SZ(b)), n(SZ(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
      rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
      rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1]
          = b[i];}
      rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
      N[n] = -1; D[m+1][n] = 1;
    }

  void pivot(int r, int s) {
    T *a = D[r].data(), inv = 1 / a[s];
    rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
      T *b = D[i].data(), inv2 = b[s] * inv;
      rep(j,0,n+2) b[j] -= a[j] * inv2;
      b[s] = a[s] * inv2;
    }
    rep(j,0,n+2) if (j != s) D[r][j] *= inv;
    rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
    D[r][s] = inv;
    swap(B[r], N[s]);
  }

  bool simplex(int phase) {
    int x = m + phase - 1;
    for (;;) {
      int s = -1;
      rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
      if (D[x][s] >= -eps) return true;
      int r = -1;
      rep(i,0,m) {
        if (D[i][s] <= eps) continue;
        if (r == -1 || mp(D[i][n+1] / D[i][s], B[i])
            < mp(D[r][n+1] / D[r][s], B[r])) r = i;
      }
      if (r == -1) return false;
      pivot(r, s);
    }
  }

  T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
      pivot(r, n);
      if (!simplex(2) || D[m+1][n+1] < -eps) return -
          inf;
      rep(i,0,m) if (B[i] == -1) {
        int s = 0;
        rep(j,1,n+1) ltj(D[i]);
        pivot(i, s);
      }
    }
    bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
  }
};
```

6.5 Floor Sum

```
// from 8BQube
ll floor_sum(ll n, ll m, ll a, ll b) {
  assert(m);
  if(m < 0) return -floor_sum(n, -m, a, b-m-1);
  ll ans = 0;
  if (a >= m)
    ans += (n - 1) * n * (a / m) / 2, a %= m;
  if (b >= m)
    ans += n * (b / m), b %= m;
  ll y_max = (a * n + b) / m, x_max = (y_max * m - b);
  if (y_max == 0) return ans;
```

```
  ans += (n - (x_max + a - 1) / a) * y_max;
  ans += floor_sum(y_max, a, m, (a - x_max % a) % a);
  return ans;
}// sum^{n-1}_0 floor((a * i + b) / m) in log(n + m + a
     + b)
```

6.6 DiscreteLog

```
int DiscreteLog(int s, int x, int y, int m) {
  constexpr int kStep = 32000;
  unordered_map<int, int> p;
  int b = 1;
  for (int i = 0; i < kStep; ++i) {
    p[y] = i;
    y = 1LL * y * x % m;
    b = 1LL * b * x % m;
  }
  for (int i = 0; i < m + 10; i += kStep) {
    s = 1LL * s * b % m;
    if (p.find(s) != p.end()) return i + kStep - p[s];
  }
  return -1;
}
int DiscreteLog(int x, int y, int m) {
  if (m == 1) return 0;
  int s = 1;
  for (int i = 0; i < 100; ++i) {
    if (s == y) return i;
    s = 1LL * s * x % m;
  }
  if (s == y) return 100;
  int p = 100 + DiscreteLog(s, x, y, m);
  if (fpow(x, p, m) != y) return -1;
  return p; //returns: x^p = y (mod m)
}
```

6.7 Miller Rabin & Pollard Rho

```
// n < 4,759,123,141        3 : 2, 7, 61
// n < 1,122,004,669,633  4 : 2, 13, 23, 1662803
// n < 3,474,749,660,383  6 : primes <= 13
// n < 2^64                 7 :
// 2, 325, 9375, 28178, 450775, 9780504, 1795265022
ll mul(ll a, ll b, ll n){
  return (__int128)a * b % n;
}
bool Miller_Rabin(ll a, ll n) {
  if ((a = a % n) == 0) return 1;
  if (n % 2 == 0) return n == 2;
  ll tmp = (n - 1) / ((n - 1) & (1 - n));
  ll t = __lg(((n - 1) & (1 - n))), x = 1;
  for (; tmp; tmp >>= 1, a = mul(a, a, n))
    if (tmp & 1) x = mul(x, a, n);
  if (x == 1 || x == n - 1) return 1;
  while (--t)
    if ((x = mul(x, x, n)) == n - 1) return 1;
  return 0;
}
bool prime(ll n){
  vector<ll> tmp = {2, 325, 9375, 28178, 450775,
      9780504, 1795265022};
  for(ll i : tmp)
    if(!Miller_Rabin(i, n)) return false;
  return true;
}
map<ll, int> cnt;
void PollardRho(ll n) {
  if (n == 1) return;
  if (prime(n)) return ++cnt[n], void();
  if (n % 2 == 0) return PollardRho(n / 2), ++cnt[2],
      void();
  ll x = 2, y = 2, d = 1, p = 1;
#define f(x, n, p) ((mul(x, x, n) + p) % n)
  while (true) {
    if (d != n && d != 1) {
      PollardRho(n / d);
      PollardRho(d);
      return;
    }
    if (d == n) ++p;
    x = f(x, n, p), y = f(f(y, n, p), n, p);
    d = gcd(abs(x - y), n);
  }
}
```

6.8 XOR Basis

```
const int digit = 60; // [0, 2^digit)
struct Basis{
  int total = 0, rank = 0;
  vector<ll> b;
  Basis(): b(digit) {}
  bool add(ll v){ // Gauss Jordan Elimination
    total++;
    for(int i = digit - 1; i >= 0; i--){
      if(!(1LL << i & v)) continue;
      if(b[i] != 0){
        v ^= b[i];
        continue;
      }
      for(int j = 0; j < i; j++)
        if(1LL << j & v) v ^= b[j];
      for(int j = i + 1; j < digit; j++)
        if(1LL << i & b[j]) b[j] ^= v;
      b[i] = v;
      rank++;
      return true;
    }
    return false;
  }
  ll getmax(ll x = 0){
    for(ll i : b) x = max(x, x ^ i);
    return x;
  }
  ll getmin(ll x = 0){
    for(ll i : b) x = min(x, x ^ i);
    return x;
  }
  bool can(ll x){
    return getmin(x) == 0;
  }
  ll kth(ll k){ // kth smallest, 0-indexed
    vector<ll> tmp;
    for(ll i : b) if(i) tmp.pb(i);
    ll ans = 0;
    for(int i = 0; i < SZ(tmp); i++)
      if(1LL << i & k) ans ^= tmp[i];
    return ans;
  }
};
```

6.9 Linear Equation

```
vector<int> RREF(vector<vector<ll>> &mat){
  int N = mat.size(), M = mat[0].size();
  int rk = 0;
  vector<int> cols;
  for (int i = 0;i < M;i++) {
    int cnt = -1;
    for (int j = N-1;j >= rk;j--)
      if(mat[j][i] != 0) cnt = j;
    if(cnt == -1) continue;
    swap(mat[rk], mat[cnt]);
    ll lead = mat[rk][i];
    for (int j = 0;j < M;j++) mat[rk][j] = mat[rk][j] *
        modinv(lead) % mod;
    for (int j = 0;j < N;j++) {
      if(j == rk) continue;
      ll tmp = mat[j][i];
      for (int k = 0;k < M;k++)
        mat[j][k] = (mat[j][k] - mat[rk][k] * tmp % mod
            + mod) % mod;
    }
    cols.pb(i);
    rk++;
  }
  return cols;
}
struct LinearEquation{
  bool ok;
  vector<ll> par; //particular solution (Ax = b)
  vector<vector<ll>> homo; //homogenous (Ax = 0)
  vector<vector<ll>> rref;
  //first M columns are matrix A
  //last column of eq is vector b
  void solve(const vector<vector<ll>> &eq){
    int M = (int)eq[0].size() - 1;
    rref = eq;
```

```
    auto piv = RREF(rref);
    int rk = piv.size();
    if(piv.size() && piv.back() == M){
      ok = 0;return;
    }
    ok = 1;
    par.resize(M);
    vector<bool> ispiv(M);
    for (int i = 0;i < rk;i++) {
      par[piv[i]] = rref[i][M];
      ispiv[piv[i]] = 1;
    }
    for (int i = 0;i < M;i++) {
      if (ispiv[i]) continue;
      vector<ll> h(M);
      h[i] = 1;
      for (int j = 0;j < rk;j++) h[piv[j]] = rref[j][i]
           ? mod-rref[j][i] : 0;
      homo.pb(h);
    }
  }
};
```

6.10 Chinese Remainder Theorem

```
pll solve_crt(ll x1, ll m1, ll x2, ll m2){
  ll g = gcd(m1, m2);
  if ((x2 - x1) % g) return {0, 0}; // no sol
  m1 /= g; m2 /= g;
  ll _, p, q;
  extgcd(m1, m2, _, p, q); // p <= C
  ll lcm = m1 * m2 * g;
  ll res = ((__int128)p * (x2 - x1) % lcm * m1 % lcm +
      x1) % lcm;
  // be careful with overflow, C^3
  return {(res + lcm) % lcm, lcm}; // (x, m)
}
```

6.11 Sqrt Decomposition

```
// for all i in [l, r], floor(n / i) = x
for(int l = 1, r; l <= n; l = r + 1){
  int x = ifloor(n, l);
  r = ifloor(n, x);
}
// for all i in [l, r], ceil(n / i) = x
for(int l, r = n; r >= 1; r = l - 1){
  int x = iceil(n, r);
  l = iceil(n, x);
}
```

7 Misc

7.1 Cyclic Ternary Search

```
/* bool pred(int a, int b);
f(0) ~ f(n - 1) is a cyclic-shift U-function
return idx s.t. pred(x, idx) is false forall x*/
int cyc_tsearch(int n, auto pred) {
  if (n == 1) return 0;
  int l = 0, r = n; bool rv = pred(1, 0);
  while (r - l > 1) {
    int m = (l + r) / 2;
    if (pred(0, m) ? rv: pred(m, (m + 1) % n)) r = m;
    else l = m;
  }
  return pred(l, r % n) ? l : r % n;
}
```

7.2 Matroid

我們稱一個二元組 $M=(E,\mathcal{I})$ 為一個擬陣，其中 $\mathcal{I}\subseteq 2^E$ 為 E 的子集所形成的**非空**集合，若：

- 若 $S\in\mathcal{I}$ 以及 $S'\subsetneq S$，則 $S'\in\mathcal{I}$
- 對於 $S_1,S_2\in\mathcal{I}$ 滿足 $|S_1|<|S_2|$，存在 $e\in S_2\setminus S_1$ 使得 $S_1\cup\{e\}\in\mathcal{I}$

除此之外，我們有以下的定義：

- 位於 $\mathcal{I}$ 中的集合我們稱之為獨立集（independent set），反之不在 $\mathcal{I}$ 中的我們稱為相依集（dependent set）
- 極大的獨立集為基底（base）、極小的相依集為迴路（circuit）
- 一個集合 Y 的秩 (rank) $r(Y)$ 為該集合中最大的獨立子集，也就是 $r(Y)=\max\{|X|\mid X\subseteq Y$ 且 $X\in\mathcal{I}\}$

性質：

1. $X\subseteq Y\wedge Y\in\mathcal{I}\implies X\in\mathcal{I}$
2. $X\subseteq Y\wedge X\notin\mathcal{I}\implies Y\notin\mathcal{I}$
3. 若 B 與 B' 皆是基底且 $B\subseteq B'$，則 $B=B'$
   若 C 與 C' 皆是迴路且 $C\subseteq C'$，則 $C=C'$
4. $e\in E\wedge X\subseteq E\implies r(X)\le r(X\cup\{e\})\le r(X)+1$ i.e. 加入一個元素後秩不會降底，最多增加 1
5. $\forall Y\subseteq E,\exists X\subseteq Y, r(X)=|X|=r(Y)$

一些等價的性質：

1. 對於所有 $X\subseteq E$，X 的極大獨立子集都有相同的大小
2. 對於 $B_1,B_2\in\mathcal{B}\wedge B_1\neq B_2$，對於所有 $e_1\in B_1\setminus B_2$，存在 $e_2\in B_2\setminus B_1$ 使得 $(B_1\setminus\{e_1\})\cup\{e_2\}\in\mathcal{B}$
3. 對於 $X,Y\in\mathcal{I}$ 且 $|X|<|Y|$，存在 $e\in Y\setminus X$ 使得 $X\cup\{e\}\in\mathcal{B}$
4. 如果 $r(X\cup\{e_1\})=r(X\cup\{e_2\})=r(X)$，則 $r(X\cup\{e_1,e_2\})=r(X)$。
   如果 $r(X\cup\{e\})=r(X)$ 對於所有 $e\in E'$ 都成立，則 $r(X\cup E')=r(X)$。

擬陣交

Data: 兩個擬陣 $M_1=(E,\mathcal{I}_1)$ 以及 $M_2=(E,\mathcal{I}_2)$
Result: I 為最大的位於 $\mathcal{I}_1\cap\mathcal{I}_2$ 中的獨立集
$I\leftarrow\emptyset$
$X_1\leftarrow\{e\in E\setminus I\mid I\cup\{e\}\in\mathcal{I}_1\}$
$X_2\leftarrow\{e\in E\setminus I\mid I\cup\{e\}\in\mathcal{I}_2\}$
while $X_1\neq\emptyset$ 且 $X_2\neq\emptyset$ **do**
  if $e\in X_1\cap X_2$ **then**
    $I\leftarrow I\cup\{e\}$
  else
    構造交換圖 $\mathcal{D}_{M_1,M_2}(I)$
    在交換圖上找到一條 X_1 到 X_2 且沒有捷徑的路徑 P
    $I\leftarrow I\triangle P$
  end if
  $X_1\leftarrow\{e\in E\setminus I\mid I\cup\{e\}\in\mathcal{I}_1\}$
  $X_2\leftarrow\{e\in E\setminus I\mid I\cup\{e\}\in\mathcal{I}_2\}$
end while

8 Polynomial

8.1 FWHT

```
/* x: a[j], y: a[j + (L >> 1)]
or: (y += x * op), and: (x += y * op)
xor: (x, y = (x + y) * op, (x - y) * op)
invop: or, and, xor = -1, -1, 1/2 */
void fwt(int *a, int n, int op) { //or
  for (int L = 2; L <= n; L <<= 1)
    for (int i = 0; i < n; i += L)
      for (int j = i; j < i + (L >> 1); ++j)
        a[j + (L >> 1)] += a[j] * op;
}
const int N = 21;
int f[N][1 << N], g[N][1 << N], h[N][1 << N], ct[1 << N
    ];
void subset_convolution(int *a, int *b, int *c, int L)
    {
  // c_k = \sum_{i | j = k, i & j = 0} a_i * b_j
  int n = 1 << L;
  for (int i = 1; i < n; ++i)
    ct[i] = ct[i & (i - 1)] + 1;
  for (int i = 0; i < n; ++i)
    f[ct[i]][i] = a[i], g[ct[i]][i] = b[i];
  for (int i = 0; i <= L; ++i)
    fwt(f[i], n, 1), fwt(g[i], n, 1);
  for (int i = 0; i <= L; ++i)
    for (int j = 0; j <= i; ++j)
      for (int x = 0; x < n; ++x)
        h[i][x] += f[j][x] * g[i - j][x];
  for (int i = 0; i <= L; ++i) fwt(h[i], n, -1);
  for (int i = 0; i < n; ++i) c[i] = h[ct[i]][i];
}
```

8.2 FFT

```
// Errichto: FFT for double works when the result < 1
    e15, and < 1e18 with long double

using val_t = complex<double>;
template<int MAXN>
struct FFT {
  const double PI = acos(-1);
  val_t w[MAXN];
  FFT() {
    for (int i = 0; i < MAXN; ++i) {
      double arg = 2 * PI * i / MAXN;
      w[i] = val_t(cos(arg), sin(arg));
    }
  }
  void bitrev(vector<val_t> &a, int n) //same as NTT
```

```
  void trans(vector<val_t> &a, int n, bool inv = false)
       {
    bitrev(a, n);
    for (int L = 2; L <= n; L <<= 1) {
      int dx = MAXN / L, dl = L >> 1;
      for (int i = 0; i < n; i += L) {
        for (int j = i, x = 0; j < i + dl; ++j, x += dx
            ) {
          val_t tmp = a[j + dl] * (inv ? conj(w[x]) : w
              [x]);
          a[j + dl] = a[j] - tmp;
          a[j] += tmp;
        }
      }
    }
    if (inv) {
      for (int i = 0; i < n; ++i) a[i] /= n;
    }
  }
  //multiplying two polynomials A * B:
  //fft.trans(A, siz, 0), fft.trans(B, siz, 0):
  //A[i] *= B[i], fft.trans(A, siz, 1);
};
```

8.3 NTT

```
//(2^16)+1, 65537, 3
//7*17*(2^23)+1, 998244353, 3
//1255*(2^20)+1, 1315962881, 3
//51*(2^25)+1, 1711276033, 29
// only works when sz(A) + sz(B) - 1 <= MAXN
template<int MAXN, ll P, ll RT> //MAXN must be 2^k
struct NTT {
  ll w[MAXN];
  ll mpow(ll a, ll n);
  ll minv(ll a) { return mpow(a, P - 2); }
  NTT() {
    ll dw = mpow(RT, (P - 1) / MAXN);
    w[0] = 1;
    for (int i = 1; i < MAXN; ++i) w[i] = w[i - 1] * dw
         % P;
  }
  void bitrev(vector<ll> &a, int n) {
    int i = 0;
    for (int j = 1; j < n - 1; ++j) {
      for (int k = n >> 1; (i ^= k) < k; k >>= 1);
      if (j < i) swap(a[i], a[j]);
    }
  }
  void operator()(vector<ll> &a, int n, bool inv =
      false) { //0 <= a[i] < P
    bitrev(a, n);
    for (int L = 2; L <= n; L <<= 1) {
      int dx = MAXN / L, dl = L >> 1;
      for (int i = 0; i < n; i += L) {
        for (int j = i, x = 0; j < i + dl; ++j, x += dx
            ) {
          ll tmp = a[j + dl] * w[x] % P;
          if ((a[j + dl] = a[j] - tmp) < 0) a[j + dl]
              += P;
          if ((a[j] += tmp) >= P) a[j] -= P;
        }
      }
    }
    if (inv) {
      reverse(a.begin()+1, a.begin()+n);
      ll invn = minv(n);
      for (int i = 0; i < n; ++i) a[i] = a[i] * invn %
          P;
    }
  }
};
```

8.4 Polynomial Operation

```
// Copy from 8BQube
#define fi(s, n) for (int i = (int)(s); i < (int)(n);
    ++i)
#define neg(x) (x ? P - x : 0)
#define V (*this)
template <int MAXN, ll P, ll RT>  // MAXN = 2^k
struct Poly : vector<ll> {       // coefficients in [0,
    P)
  using vector<ll>::vector;
  static inline NTT<MAXN, P, RT> ntt;
  int n() const { return (int)size(); }  // n() >= 1
  Poly(const Poly &p, int m) : vector<ll>(m) { copy_n(p
      .data(), min(p.n(), m), data()); }
  Poly &irev() { return reverse(data(), data() + n()),
      V; }
  Poly &isz(int m) { return resize(m), V; }
  static ll minv(ll x) { return ntt.minv(x); }
  Poly &iadd(const Poly &rhs) {  // n() == rhs.n()
    fi(0, n()) if ((V[i] += rhs[i]) >= P) V[i] -= P;
    return V;
  }
  Poly &imul(ll k) {
    fi(0, n()) V[i] = V[i] * k % P;
    return V;
  }
  Poly Mul(const Poly &rhs) const {
    int m = 1;
    while (m < n() + rhs.n() - 1) m <<= 1;
    assert(m <= MAXN);
    Poly X(V, m), Y(rhs, m);
    ntt(X, m), ntt(Y, m);
    fi(0, m) X[i] = X[i] * Y[i] % P;
    ntt(X, m, true);
    return X.isz(n() + rhs.n() - 1);
  }
  Poly Inv() const {  // V[0] != 0, 2*sz<=MAXN
    if (n() == 1) return {minv(V[0])};
    int m = 1;
    while (m < n() * 2) m <<= 1;
    assert(m <= MAXN);
    Poly Xi = Poly(V, (n() + 1) / 2).Inv().isz(m);
    Poly Y(V, m);
    ntt(Xi, m), ntt(Y, m);
    fi(0, m) {
      Xi[i] *= (2 - Xi[i] * Y[i]) % P;
      if ((Xi[i] %= P) < 0) Xi[i] += P;
    }
    ntt(Xi, m, true);
    return Xi.isz(n());
  }
  Poly &shift_inplace(const ll &c) {  // 2 * sz <= MAXN
    int n = V.n();
    vector<ll> fc(n), ifc(n);
    fc[0] = ifc[0] = 1;
    for (int i = 1; i < n; i++) {
      fc[i] = fc[i - 1] * i % P;
      ifc[i] = minv(fc[i]);
    }
    for (int i = 0; i < n; i++) V[i] = V[i] * fc[i] % P
        ;
    Poly g(n);
    ll cp = 1;
    for (int i = 0; i < n; i++) g[i] = cp * ifc[i] % P,
         cp = cp * c % P;
    V = V.irev().Mul(g).isz(n).irev();
    for (int i = 0; i < n; i++) V[i] = V[i] * ifc[i] %
        P;
    return V;
  }
  Poly shift(const ll &c) const { return Poly(V).
      shift_inplace(c); }
  Poly _Sqrt() const {  // Jacobi(V[0], P) = 1
    if (n() == 1) return {QuadraticResidue(V[0], P)};
    Poly X = Poly(V, (n() + 1) / 2)._Sqrt().isz(n());
    return X.iadd(Mul(X.Inv()).isz(n())).imul(P / 2 +
        1);
  }
  Poly Sqrt() const {  // 2 * sz <= MAXN
    Poly a;
    bool has = 0;
    for (int i = 0; i < n(); i++) {
      if (V[i]) has = 1;
      if (has) a.push_back(V[i]);
    }
    if (!has) return V;
    if ((n() + a.n()) % 2 || Jacobi(a[0], P) != 1) {
      return Poly();
    }
    a = a.isz((n() + a.n()) / 2)._Sqrt();
    int sz = a.n();
```

```
    a.isz(n());
    rotate(a.begin(), a.begin() + sz, a.end());
    return a;
  }
  pair<Poly, Poly> DivMod(const Poly &rhs) const {  //
      (rhs.)back() != 0
    if (n() < rhs.n()) return {{0}, V};
    const int m = n() - rhs.n() + 1;
    Poly X(rhs);
    X.irev().isz(m);
    Poly Y(V);
    Y.irev().isz(m);
    Poly Q = Y.Mul(X.Inv()).isz(m).irev();
    X = rhs.Mul(Q), Y = V;
    fi(0, n()) if ((Y[i] -= X[i]) < 0) Y[i] += P;
    return {Q, Y.isz(max(1, rhs.n() - 1))};
  }
  Poly Dx() const {
    Poly ret(n() - 1);
    fi(0, ret.n()) ret[i] = (i + 1) * V[i + 1] % P;
    return ret.isz(max(1, ret.n()));
  }
  Poly Sx() const {
    Poly ret(n() + 1);
    fi(0, n()) ret[i + 1] = minv(i + 1) * V[i] % P;
    return ret;
  }
  Poly _tmul(int nn, const Poly &rhs) const {
    Poly Y = Mul(rhs).isz(n() + nn - 1);
    return Poly(Y.data() + n() - 1, Y.data() + Y.n());
  }
  vector<ll> _eval(const vector<ll> &x, const vector<
      Poly> &up) const {
    const int m = (int)x.size();
    if (!m) return {};
    vector<Poly> down(m * 2);
    // down[1] = DivMod(up[1]).second;
    // fi(2, m * 2) down[i] = down[i / 2].DivMod(up[i])
        .second;
    down[1] = Poly(up[1]).irev().isz(n()).Inv().irev().
        _tmul(m, V);
    fi(2, m * 2) down[i] = up[i ^ 1]._tmul(up[i].n() -
        1, down[i / 2]);
    vector<ll> y(m);
    fi(0, m) y[i] = down[m + i][0];
    return y;
  }
  static vector<Poly> _tree1(const vector<ll> &x) {
    const int m = (int)x.size();
    vector<Poly> up(m * 2);
    fi(0, m) up[m + i] = {neg(x[i]), 1};
    for (int i = m - 1; i > 0; --i) up[i] = up[i * 2].
        Mul(up[i * 2 + 1]);
    return up;
  }
  vector<ll> Eval(const vector<ll> &x) const {  // 1e5,
       1s
    auto up = _tree1(x);
    return _eval(x, up);
  }
  static Poly Interpolate(const vector<ll> &x, const
      vector<ll> &y) {  // 1e5, 1.4s
    const int m = (int)x.size();
    vector<Poly> up = _tree1(x), down(m * 2);
    vector<ll> z = up[1].Dx()._eval(x, up);
    fi(0, m) z[i] = y[i] * minv(z[i]) % P;
    fi(0, m) down[m + i] = {z[i]};
    for (int i = m - 1; i > 0; --i)
      down[i] = down[i * 2].Mul(up[i * 2 + 1]).iadd(
          down[i * 2 + 1].Mul(up[i * 2]));
    return down[1];
  }
  Poly Ln() const {  // V[0] == 1, 2*sz<=MAXN
    return Dx().Mul(Inv()).Sx().isz(n());
  }
  Poly Exp() const {  // V[0] == 0,2*sz<=MAXN
    if (n() == 1) return {1};
    Poly X = Poly(V, (n() + 1) / 2).Exp().isz(n());
    Poly Y = X.Ln();
    Y[0] = P - 1;
    fi(0, n()) if ((Y[i] = V[i] - Y[i]) < 0) Y[i] += P;
    return X.Mul(Y).isz(n());
```

```
  }
  // M := P(P - 1). If k >= M, k := k % M + M.
  Poly Pow(ll k) const {  // 2*sz<=MAXN
    int nz = 0;
    while (nz < n() && !V[nz]) ++nz;
    if (nz * min(k, (ll)n()) >= n()) return Poly(n());
    if (!k) return Poly(Poly{1}, n());
    Poly X(data() + nz, data() + nz + n() - nz * k);
    const ll c = ntt.mpow(X[0], k % (P - 1));
    return X.Ln().imul(k % P).Exp().imul(c).irev().isz(
        n()).irev();
  }
  // sum_j w_j [x^j] f(x^i) for i \in [0, m]
  Poly power_projection(Poly wt, int m) {  // 4*sz <=
      MAXN!
    assert(n() == wt.n());
    if (!n()) {
      return Poly(m + 1, 0);
    }
    if (V[0] != 0) {
      ll c = V[0];
      V[0] = 0;
      Poly A = V.power_projection(wt, m);
      fi(0, m + 1) A[i] = A[i] * fac[i] % P;  //
          factorial
      Poly B(m + 1);
      ll pow = 1;
      fi(0, m + 1) B[i] = pow * ifac[i] % P, pow = pow
          * c % P;  // inv. of fac
      A = A.Mul(B).isz(m + 1);
      fi(0, m + 1) A[i] = A[i] * fac[i] % P;
      return A;
    }

    int n = 1;
    while (n < V.n()) n *= 2;
    isz(n), wt.isz(n).irev();
    int k = 1;
    Poly p(wt, 2 * n), q(V, 2 * n);
    q.imul(P - 1);

    while (n > 1) {
      Poly r(2 * n * k);
      fi(0, 2 * n * k) r[i] = (i % 2 == 0 ? q[i] : neg(
          q[i]));
      Poly pq = p.Mul(r).isz(4 * n * k);
      Poly qq = q.Mul(r).isz(4 * n * k);
      fi(0, 2 * n * k) {
        pq[2 * n * k + i] += p[i];
        qq[2 * n * k + i] += q[i] + r[i];
        pq[2 * n * k + i] %= P;
        qq[2 * n * k + i] %= P;
      }
      fill(p.begin(), p.end(), 0);
      fill(q.begin(), q.end(), 0);
      for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
        p[n * j + i] = pq[(2 * n) * j + (2 * i + 1)];
        q[n * j + i] = qq[(2 * n) * j + (2 * i + 0)];
      }
      n /= 2, k *= 2;
    }
    Poly ans(k);
    fi(0, k) ans[i] = p[2 * i];
    return ans.irev().isz(m + 1);
  }
  Poly FPSinv() {
    const int n = V.n() - 1;
    if (n == -1) return {};
    assert(V[0] == 0);
    if (n == 0) return V;
    assert(V[1] != 0);
    ll c = V[1], ic = minv(c);
    imul(ic);
    Poly wt(n + 1);
    wt[n] = 1;

    Poly A = V.power_projection(wt, n);
    Poly g(n);
    fi(1, n + 1) g[n - i] = n * A[i] % P * minv(i) % P;
    g = g.Pow(neg(minv(n)));
    g.insert(g.begin(), 0);
```

```
    ll pow = 1;
    fi(0, g.n()) g[i] = g[i] * pow % P, pow = pow * ic
        % P;
    return g;
  }
  Poly TMul(const Poly &rhs) const { // this[i] - rhs[j
      ] = k;
    return Poly(*this).irev().Mul(rhs).isz(n()).irev();
  }
  Poly FPScomp(Poly g) { // solves V(g(x))
    auto rec = [&](auto &rec, int n, int k, Poly Q) ->
        Poly {
      if (n == 1) {
        Poly p(2 * k);
        irev();
        fi(0, k) p[2 * i] = V[i];
        return p;
      }
      Poly R(2 * n * k);
      fi(0, 2 * n * k) R[i] = (i % 2 == 0 ? Q[i] : neg(
          Q[i]));
      Poly QQ = Q.Mul(R).isz(4 * n * k);
      fi(0, 2 * n * k) {
        QQ[2 * n * k + i] += Q[i] + R[i];
        QQ[2 * n * k + i] %= P;
      }
      Poly nxt_Q(2 * n * k);
      for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
        nxt_Q[n * j + i] = QQ[(2 * n) * j + (2 * i + 0)
            ];
      }
      Poly nxt_p = rec(rec, n / 2, k * 2, nxt_Q);
      Poly pq(4 * n * k);
      for(int j = 0; j < 2 * k; j++) fi(0, n / 2) {
        pq[(2 * n) * j + (2 * i + 1)] += nxt_p[n * j +
            i];
        pq[(2 * n) * j + (2 * i + 1)] %= P;
      }
      Poly p(2 * n * k);
      fi(0, 2 * n * k) p[i] = (p[i] + pq[2 * n * k + i
          ]) % P;
      pq.pop_back();
      Poly x = pq.TMul(R);
      fi(0, 2 * n * k) p[i] = (p[i] + x[i]) % P;
      return p;
    };
    int sz = 1;
    while(sz < n() || sz < g.n()) sz <<= 1;
    return isz(sz), rec(rec, sz, 1, g.imul(P-1).isz(2 *
        sz)).isz(sz).irev();
  }
};
#undef fi
#undef V
#undef neg
using Poly_t = Poly<1 << 19, 998244353, 3>;
```

8.5 Generating Function

Ordinary Generating Function

- $C(x) = A(rx)$: $c_n = r^n a_n$ 的一般生成函數。
- $C(x) = A(x) + B(x)$: $c_n = a_n + b_n$ 的一般生成函數。
- $C(x) = A(x)B(x)$: $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ 的一般生成函數。
- $C(x) = A(x)^k$: $c_n = \sum_{i_1+i_2+\ldots+i_k=n} a_{i_1} a_{i_2} \ldots a_{i_k}$ 的一般生成函數。
- $C(x) = xA(x)'$: $c_n = na_n$ 的一般生成函數。
- $C(x) = \frac{A(x)}{1-x}$: $c_n = \sum_{i=0}^{n} a_i$ 的一般生成函數。
- $C(x) = A(1) + x\frac{A(1)-A(x)}{1-x}$: $c_n = \sum_{i=n}^{\infty} a_i$ 的一般生成函數。

常用展開式

- $\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$
- $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$, $\binom{a}{n} = \frac{a(a-1)(a-2)\ldots(a-n+1)}{n!}$.

常見生函

- 卡特蘭數：$f(x) = \frac{1-\sqrt{1-4x}}{2x}$

Exponential Generating Function

$a_0, a_1, \ldots$ 的指數生成函數：

$$\hat{A}(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} = a_0 + a_1 x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \ldots$$

- $\hat{C}(x) = \hat{A}(x) + \hat{B}(x)$: $c_n = a_n + b_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}^{(k)}(x)$: $c_n = a_{n+k}$ 的指數生成函數
- $\hat{C}(x) = x\hat{A}(x)$: $c_n = na_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)\hat{B}(x)$: $c_n = \sum_{k=0}^{n} \binom{n}{i} a_k b_{n-k}$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)^k$: $\sum_{i_1+i_2+\cdots+i_k=n} \binom{n}{i_1, i_2, \ldots, i_k} a_i a_{i_2} \ldots a_{i_k}$ 的指數生成函數
- $\hat{C}(x) = \exp(A(x))$: 假設 $A(x)$ 是一個分量 (component) 的生成函數，那 $\hat{C}(x)$ 是將 n 個有編號的東西分成若干個分量的指數生成函數

Lagrange's Inversion Formula

如果 F 跟 G 互反，則有 $F(0), G(0) = 0$, $F'(0), G'(0) \neq 0$。若 H 為任意 FPS，則

$$n[x^n]G(x) = [x^{n-1}]\frac{1}{(F(x)/x)^n}$$

$$n[x^n]H(G(x)) = [x^{n-1}]H'(x)\frac{1}{(F(x)/x)^n}$$

8.6 Bostan Mori

```
const ll mod = 998244353;
NTT<262144, mod, 3> ntt;
// Finds the k-th coefficient of P / Q in O(d log d log
    k)
// size of NTT has to > 2 * d
ll BostanMori(vector<ll> P, vector<ll> Q, long long k)
    {
  int d = max((int)P.size(), (int)Q.size() - 1);
  vector M = {P, Q};
  M[0].resize(d, 0);
  M[1].resize(d + 1, 0);
  int sz = (2 * d + 1 == 1 ? 2 : (1 << (__lg(2 * d) +
      1)));
  vector<ll> Qn(sz);
  vector N(2, vector<ll>(sz));
  while(k) {
    fill(iter(Qn), 0);
    for(int i = 0; i < d + 1; i++){
      Qn[i] = M[1][i] * ((i & 1) ? -1 : 1);
      if(Qn[i] < 0) Qn[i] += mod;
    }
    ntt(Qn, sz, false);

    ll t[2] = {k & 1, 0};
    for(int i = 0; i < 2; i++){
      fill(iter(N[i]), 0);
      copy(iter(M[i]), N[i].begin());
      ntt(N[i], sz, false);
      for(int j = 0; j < sz; j++)
        N[i][j] = N[i][j] * Qn[j] % mod;
      ntt(N[i], sz, true);
      for(int j = t[i]; j < 2 * siz(M[i]); j += 2){
        M[i][j >> 1] = N[i][j];
      }
    }
    k >>= 1;
  }
  return M[0][0] * ntt.minv(M[1][0]) % mod;
}
ll LinearRecursion(vector<ll> a, vector<ll> c, ll k) {
    // a_n = \sum_{j=1}^{d} c_j a_(n-j)
  int d = siz(a);
  int sz = (2 * d + 1 == 1 ? 2 : (1 << (__lg(2 * d) +
      1)));

  c[0] = mod - 1;
  for(ll &i : c) i = i ? mod - i : 0;

  auto A = a; A.resize(sz);
  auto C = c; C.resize(sz);
  ntt(A, sz, false), ntt(C, sz, false);
  for(int i = 0; i < sz; i++) A[i] = A[i] * C[i] % mod;
  ntt(A, sz, true);
  A.resize(d);

  return BostanMori(A, c, k);
}
```

9 String

9.1 KMP Algorithm

```
// 0-based
```

```cpp
// fail[i] = max k<i s.t. s[0..k] = s[i-k..i]
vector<int> kmp_build_fail(const string &s){
  int n = SZ(s);
  vector<int> fail(n, -1);
  int cur = -1;
  for(int i = 1; i < n; i++){
    while(cur != -1 && s[cur + 1] != s[i])
      cur = fail[cur];
    if(s[cur + 1] == s[i])
      cur++;
    fail[i] = cur;
  }
  return fail;
}

void kmp_match(const string &s, const vector<int> &fail
    , const string &t){
  int cur = -1;
  int n = SZ(s), m = SZ(t);
  for(int i = 0; i < m; i++){
    while(cur != -1 && (cur + 1 == n || s[cur + 1] != t
        [i]))
      cur = fail[cur];
    if(cur + 1 < n && s[cur + 1] == t[i])
      cur++;
    // cur = max k s.t. s[0..k] = t[i-k..i]
  }
}
```

9.2 Manacher Algorithm

```cpp
/* center i: radius z[i * 2 + 1] / 2
   center i, i + 1: radius z[i * 2 + 2] / 2
   both aba, abba have radius 2 */
vector<int> manacher(const string &tmp){ // 0-based
  string s = "%";
  int l = 0, r = 0;
  for(char c : tmp) s += c, s += '%';
  vector<int> z(SZ(s));
  for(int i = 0; i < SZ(s); i++){
    z[i] = r > i ? min(z[2 * l - i], r - i) : 1;
    while(i - z[i] >= 0 && i + z[i] < SZ(s)
        && s[i + z[i]] == s[i - z[i]])
      ++z[i];
    if(z[i] + i > r) r = z[i] + i, l = i;
  }
  return z;
}
```

9.3 Lyndon Factorization

```cpp
// partition s = w[0] + w[1] + ... + w[k-1],
// w[0] >= w[1] >= ... >= w[k-1]
// each w[i] strictly smaller than all its suffix
void duval(const string &s, vector<pii> &w) {
  for (int n = (int)s.size(), i = 0, j, k; i < n; ) {
    for (j = i + 1, k = i; j < n && s[k] <= s[j]; j++)
      k = (s[k] < s[j] ? i : k + 1);
    // if (i < n / 2 && j >= n / 2) {
    // for min cyclic shift, call duval(s + s)
    // then here s.substr(i, n / 2) is min cyclic shift
    // }
    for (; i <= k; i += j - k)
      w.pb(pii(i, j - k)); // s.substr(l, len)
  }
}
```

9.4 Suffix Array

```cpp
struct SuffixArray {
  vector<int> sa, lcp, rank; // lcp[i] is lcp of sa[i]
      and sa[i-1]
                             // sa[0] = s.size()
                             // character should be 1-
                                 based
  SuffixArray(string& s, int lim=256) { // or
      basic_string<int>
    int n = s.size() + 1, k = 0, a, b;
    vector<int> x(n, 0), y(n), ws(max(n, lim));
    rank.assign(n, 0);
    for (int i = 0; i < n - 1; i++) x[i] = s[i];
    sa = lcp = y, iota(sa.begin(), sa.end(), 0);
    for (int j = 0, p = 0; p < n; j = max(1, j * 2),
        lim = p) {
      p = j, iota(y.begin(), y.end(), n - j);
      for (int i = 0; i < n; i++)
        if (sa[i] >= j) y[p++] = sa[i] - j;
      for (int &i : ws) i = 0;
      for (int i = 0; i < n; i++) ws[x[i]]++;
      for (int i = 1; i < lim; i++) ws[i] += ws[i - 1];
      for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
      swap(x, y), p = 1, x[sa[0]] = 0;
      for(int i = 1; i < n; i++){
        a = sa[i - 1], b = sa[i];
        x[b] = (y[a] == y[b] && y[a + j] == y[b + j]) ?
            p - 1 : p++;
      }
    }
    for (int i = 1; i < n; i++) rank[sa[i]] = i;
    for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
      for (k && k--, j = sa[rank[i] - 1];
          s[i + k] == s[j + k]; k++);
  }
};
```

9.5 Suffix Automaton

```cpp
struct exSAM {
  const int CNUM = 26;
  // len: maxlength, link: fail link
  // lenSorted: topo order, cnt: occur
  vector<int> len, link, lenSorted, cnt;
  vector<vector<int>> next;
  int total = 0;
  int newnode() {
    return total++;
  }
  void init(int n) { // total number of characters
    len.assign(2 * n, 0); link.assign(2 * n, 0);
    lenSorted.assign(2 * n, 0); cnt.assign(2 * n, 0);
    next.assign(2 * n, vector<int>(CNUM));
    newnode(), link[0] = -1;
  }
  int insertSAM(int last, int c) {
    // not exSAM: cur = newnode(), p = last
    int cur = next[last][c];
    len[cur] = len[last] + 1;
    int p = link[last];
    while (p != -1 && !next[p][c])
      next[p][c] = cur, p = link[p];
    if (p == -1) return link[cur] = 0, cur;
    int q = next[p][c];
    if (len[p] + 1 == len[q]) return link[cur] = q, cur
        ;
    int clone = newnode();
    for (int i = 0; i < CNUM; ++i)
      next[clone][i] = len[next[q][i]] ? next[q][i] :
          0;
    len[clone] = len[p] + 1;
    while (p != -1 && next[p][c] == q)
      next[p][c] = clone, p = link[p];
    link[link[cur] = clone] = link[q];
    link[q] = clone;
    return cur;
  }
  void insert(const string &s) {
    int cur = 0;
    for (auto ch : s) {
      int &nxt = next[cur][int(ch - 'a')];
      if (!nxt) nxt = newnode();
      cnt[cur = nxt] += 1;
    }
  }
  void build() {
    queue<int> q;
    q.push(0);
    while (!q.empty()) {
      int cur = q.front();
      q.pop();
      for (int i = 0; i < CNUM; ++i)
        if (next[cur][i])
          q.push(insertSAM(cur, i));
    }
    vector<int> lc(total);
    for (int i = 1; i < total; ++i) ++lc[len[i]];
    partial_sum(iter(lc), lc.begin());
```

```
    for (int i = 1; i < total; ++i) lenSorted[--lc[len[
        i]]] = i;
  }
  void solve() {
    for (int i = total - 2; i >= 0; --i)
      cnt[link[lenSorted[i]]] += cnt[lenSorted[i]];
  }
};
```

9.6 Z-value Algorithm

```
// z[i] = max k s.t. s[0..k-1] = s[i..i+k-1]
// i.e. length of longest common prefix
// z[0] = 0
vector<int> z_function(const string &s){
  int n = s.size();
  vector<int> z(n);
  for(int i = 1, l = 0, r = 0; i < n; i++){
    if(i <= r) z[i] = min(r - i + 1, z[i - l]);
    while(i + z[i] < n && s[z[i]] == s[i + z[i]])
      z[i]++;
    if(i + z[i] - 1 > r)
      l = i, r = i + z[i] - 1;
  }
  return z;
}
```

9.7 Main Lorentz

```
struct Rep{ int minl, maxl, len; };
vector<Rep> rep; // 0-base
// p \in [minl, maxl] => s[p, p + i) = s[p + i, p + 2i)
void main_lorentz(const string &s, int sft = 0) {
  const int n = s.size();
  if (n == 1) return;
  const int nu = n / 2, nv = n - nu;
  const string u = s.substr(0, nu), v = s.substr(nu),
        ru(u.rbegin(), u.rend()), rv(v.rbegin(), v.rend
            ());
  main_lorentz(u, sft), main_lorentz(v, sft + nu);
  const auto z1 = z_function(ru), z2 = z_function(v + '
      #' + u),
             z3 = z_function(ru + '#' + rv), z4 =
                 z_function(v);
  auto get_z = [](const vector<int> &z, int i) {
    return (0 <= i and i < (int)z.size()) ? z[i] : 0;
        };
  auto add_rep = [&](bool left, int c, int l, int k1,
      int k2) {
    const int L = max(1, l - k2), R = min(l - left, k1)
        ;
    if (L > R) return;
    if (left) rep.emplace_back(Rep({sft + c - R, sft +
        c - L, l}));
    else rep.emplace_back(Rep({sft + c - R - l + 1, sft
         + c - L - l + 1, l}));
  };
  for (int cntr = 0; cntr < n; cntr++) {
    int l, k1, k2;
    if (cntr < nu) {
      l = nu - cntr;
      k1 = get_z(z1, nu - cntr);
      k2 = get_z(z2, nv + 1 + cntr);
    } else {
      l = cntr - nu + 1;
      k1 = get_z(z3, nu + 1 + nv - 1 - (cntr - nu));
      k2 = get_z(z4, (cntr - nu) + 1);
    }
    if (k1 + k2 >= l)
      add_rep(cntr < nu, cntr, l, k1, k2);
  }
}
```

9.8 AC Automaton

```
const int SIGMA = 26;
struct AC_Automaton {
  // child: trie, next: automaton
  vector<vector<int>> child, next;
  vector<int> fail, cnt, ord;
  int total = 0;
  int newnode() {
    return total++;
  }
  void init(int len) {  // len >= 1 + total len
    child.assign(len, vector<int>(26, -1));
    next.assign(len, vector<int>(26, -1));
    fail.assign(len, -1); cnt.assign(len, 0);
    ord.clear();
    newnode();
  }
  int input(string &s) {
    int cur = 0;
    for (char c : s) {
      if (child[cur][c - 'A'] == -1)
        child[cur][c - 'A'] = newnode();
      cur = child[cur][c - 'A'];
    }
    return cur; // return the end node of string
  }
  void make_fl() {
    queue<int> q;
    q.push(0), fail[0] = -1;
    while(!q.empty()) {
      int R = q.front();
      q.pop(); ord.pb(R);
      for (int i = 0; i < SIGMA; i++)
        if (child[R][i] != -1) {
          int X = next[R][i] = child[R][i], Z = fail[R
              ];
          while (Z != -1 && child[Z][i] == -1)
            Z = fail[Z];
          fail[X] = Z != -1 ? child[Z][i] : 0;
          q.push(X);
        }
        else next[R][i] = R ? next[fail[R]][i] : 0;
    }
  }
  void solve() {
    for (int i : ord | views::reverse)
      cnt[fail[i]] += cnt[i];
  }
};
```

9.9 Palindrome Automaton

```
struct PalindromicTree {
  struct node {
    int nxt[26], fail, len; // num = depth of fail link
    int cnt, num; // cnt = occur, num = #pal_suffix of
        this node
    node(int l = 0) : nxt{},fail(0),len(l),cnt(0),num
        (0) {}
  };
  vector<node> st; vector<int> s; int last, n;
  void init() {
    st.clear(); s.clear(); last = 1; n = 0;
    st.pb(0); st.pb(-1);
    st[0].fail = 1; s.pb(-1);
  }
  int getFail(int x) {
    while (s[n - st[x].len - 1] != s[n]) x = st[x].fail
        ;
    return x;
  }
  void add(int c) {
    s.pb(c -= 'a'); ++n;
    int cur = getFail(last);
    if (!st[cur].nxt[c]) {
      int now = SZ(st);
      st.pb(st[cur].len + 2);
      st[now].fail = st[getFail(st[cur].fail)].nxt[c];
      st[cur].nxt[c] = now;
      st[now].num = st[st[now].fail].num + 1;
    }
    last = st[cur].nxt[c]; ++st[last].cnt;
  }
  void dpcnt() {
    for(int i = SZ(st) - 1; i >= 0; i--){
      auto nd = st[i];
      st[nd.fail].cnt += nd.cnt;
    }
  }
  int size() { return (int)st.size() - 2; }
};
```

10 Notes

10.1 Geometry

Rotation Matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- rotate $90°$: $(x,y) \to (-y,x)$
- rotate $-90°$: $(x,y) \to (y,-x)$

Triangles

Side lengths: a,b,c

Semiperimeter: $p = \dfrac{a+b+c}{2}$

Area: $A = \sqrt{p(p-a)(p-b)(p-c)}$

Circumradius: $R = \dfrac{abc}{4A}$

Inradius: $r = \dfrac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of bisector (divides angles in two): $s_a = \sqrt{bc\left(1-\left(\dfrac{a}{b+c}\right)^2\right)}$

Law of sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = \dfrac{1}{2R}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Law of tangents: $\dfrac{a+b}{a-b} = \dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$

Quadrilaterals

With side lengths a,b,c,d, diagonals e,f, diagonals angle θ, area A and magic flux $F = b^2+d^2-a^2-c^2$:

$$4A = 2ef\cdot\sin\theta = F\tan\theta = \sqrt{4e^2f^2-F^2}$$

For cyclic quadrilaterals the sum of opposite angles is $180°$, $ef = ac+bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

Spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\sin\phi & \theta &= \operatorname{acos}(z/\sqrt{x^2+y^2+z^2}) \\ z &= r\cos\theta & \phi &= \operatorname{atan2}(y,x) \end{aligned}$$

Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{L^+} (Pdx + Qdy)$$

$$\text{Area} = \frac{1}{2}\oint_L x\,dy - y\,dx$$

Circular sector:

$$\begin{aligned} x &= x_0 + r\cos\theta \\ y &= y_0 + r\sin\theta \\ A &= r\int_\alpha^\beta (x_0+\cos\theta)\cos\theta + (y_0+\sin\theta)\sin\theta\,d\theta \\ &= r(r\theta + x_0\sin\theta - y_0\cos\theta)|_\alpha^\beta \end{aligned}$$

Point-Line Duality

$$p = (a,b) \leftrightarrow p^* : y = ax - b$$

- $p \in l \iff l^* \in p^*$
- p_1, p_2, p_3 are collinear $\iff p_1^*, p_2^*, p_3^*$ intersect at a point
- p lies above $l \iff l^*$ lies above p^*
- lower convex hull $\leftrightarrow$ upper envelope

10.2 Trigonometry

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sin n\pi = 0 \qquad \cos n\pi = (-1)^n$$

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(2\alpha) &= 2\cos\alpha\sin\alpha \\ \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \end{aligned}$$

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\ \sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) \\ \sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)) \\ \cos\alpha\sin\beta &= \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta)) \\ \cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)) \end{aligned}$$

$$(V+W)\tan(\alpha-\beta)/2 = (V-W)\tan(\alpha+\beta)/2$$

where V, W are lengths of sides opposite angles α, β.

$$\begin{aligned} a\cos x + b\sin x &= r\cos(x-\phi) \\ a\sin x + b\cos x &= r\sin(x+\phi) \end{aligned}$$

where $r = \sqrt{a^2+b^2}, \phi = \operatorname{atan2}(b,a)$.

10.3 Calculus

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

$$\begin{aligned} \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\arccos x &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\tan x &= 1+\tan^2 x & \frac{d}{dx}\arctan x &= \frac{1}{1+x^2} \\ \int\tan ax &= -\frac{\ln|\cos ax|}{a} & \int x\sin ax &= \frac{\sin ax - ax\cos ax}{a^2} \\ \int e^{-x^2} &= \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) & \int xe^{ax} &= \frac{e^{ax}}{a^2}(ax-1) \\ \int\sin^2(x) &= \frac{x}{2} - \frac{1}{4}\sin 2x & \int\sin^3 x &= \frac{1}{12}\cos 3x - \frac{3}{4}\cos x \\ \int\cos^2(x) &= \frac{x}{2} + \frac{1}{4}\sin 2x & \int\cos^3 x &= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x \\ \int x\sin x &= \sin x - x\cos x & \int x\cos x &= \cos x + x\sin x \\ \int xe^x &= e^x(x-1) & \int x^2e^x &= e^x(x^2-2x+2) \end{aligned}$$

$$\begin{aligned} \int x^2\sin x &= 2x\sin x - (x^2-2)\cos x \\ \int x^2\cos x &= 2x\cos x + (x^2-2)\sin x \\ \int e^x\sin x &= \frac{1}{2}e^x(\sin x - \cos x) \\ \int e^x\cos x &= \frac{1}{2}e^x(\sin x + \cos x) \\ \int xe^x\sin x &= \frac{1}{2}e^x(x\sin x - x\cos x + \cos x) \\ \int xe^x\cos x &= \frac{1}{2}e^x(x\sin x + x\cos x - \sin x) \end{aligned}$$

10.4 Sum & Series

$$c^a + c^{a+1} + \cdots + c^b = \frac{c^{b+1}-c^a}{c-1}, c \neq 1$$

$$\begin{aligned} 1+2+3+\cdots+n &= \frac{n(n+1)}{2} \\ 1^2+2^2+3^2+\cdots+n^2 &= \frac{n(2n+1)(n+1)}{6} \\ 1^3+2^3+3^3+\cdots+n^3 &= \frac{n^2(n+1)^2}{4} \\ 1^4+2^4+3^4+\cdots+n^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \ldots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, (-\infty < x < \infty)$$

10.5 Misc

- Cramer's rule

$$\begin{matrix} ax+by=e \\ cx+dy=f \end{matrix} \Rightarrow \begin{matrix} x = \dfrac{ed-bf}{ad-bc} \\ y = \dfrac{af-ec}{ad-bc} \end{matrix}$$

- Vandermonde's Identity

$$C(n+m,k) = \sum_{i=0}^{k} C(n,i)C(m,k-i)$$

- Kirchhoff's Theorem
  Denote L be a $n \times n$ matrix as the Laplacian matrix of graph G, where $L_{ii} = d(i)$, $L_{ij} = -c$ where c is the number of edge (i,j) in G.
  - The number of undirected spanning in G is $|\det(\tilde{L}_{11})|$.
  - The number of directed spanning tree rooted at r in G is $|\det(\tilde{L}_{rr})|$.
- Tutte's Matrix
  Let D be a $n \times n$ matrix, where $d_{ij} = x_{ij}$ (x_{ij} is chosen uniformly at random) if $i < j$ and $(i,j) \in E$, otherwise $d_{ij} = -d_{ji}$. $\frac{rank(D)}{2}$ is the maximum matching on G.
- Cayley's Formula
  - Given a degree sequence $d_1, d_2, \ldots, d_n$ for each *labeled* vertices, there are $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$ spanning trees.
  - Let $T_{n,k}$ be the number of *labeled* forests on n vertices with k components, such that vertex $1, 2, \ldots, k$ belong to different components. Then $T_{n,k} = kn^{n-k-1}$.
- Erdős–Gallai theorem
  A sequence of nonnegative integers $d_1 \geq \cdots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + \cdots + d_n$ is even and $\displaystyle\sum_{i-1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ holds for every $1 \leq k \leq n$.
- Gale–Ryser theorem
  A pair of sequences of nonnegative integers $a_1 \geq \cdots \geq a_n$ and $b_1, \ldots, b_n$ is bigraphic if and only if $\displaystyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and $\displaystyle\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{n} \min(b_i, k)$ holds for every $1 \leq k \leq n$.
- Fulkerson–Chen–Anstee theorem
  A sequence $(a_1, b_1), \ldots, (a_n, b_n)$ of nonnegative integer pairs with $a_1 \geq \cdots \geq a_n$ is digraphic if and only if $\displaystyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and $\displaystyle\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \min(b_i, k-1) + \sum_{i=k+1}^{n} \min(b_i, k)$ holds for every $1 \leq k \leq n$.
- Pick's theorem
  For simple polygon, when points are all integer, we have $A = \#\{\text{lattice points in the interior}\} + \frac{\#\{\text{lattice points on the boundary}\}}{2} - 1$.
- Möbius inversion formula
  - $f(n) = \sum_{d|n} g(d) \Leftrightarrow g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$
  - $f(n) = \sum_{n|d} g(d) \Leftrightarrow g(n) = \sum_{n|d} \mu(\frac{d}{n}) f(d)$
- Spherical cap
  - A portion of a sphere cut off by a plane.
  - r: sphere radius, a: radius of the base of the cap, h: height of the cap, θ: $\arcsin(a/r)$.
  - Volume $= \pi h^2(3r-h)/3 = \pi h(3a^2+h^2)/6 = \pi r^3(2+\cos\theta)(1-\cos\theta)^2/3$.
  - Area $= 2\pi rh = \pi(a^2+h^2) = 2\pi r^2(1-\cos\theta)$.
- Lagrange multiplier
  - Optimize $f(x_1, \ldots, x_n)$ when k constraints $g_i(x_1, \ldots, x_n) = 0$.
  - Lagrangian function $\mathcal{L}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k) = f(x_1, \ldots, x_n) - \sum_{i=1}^{k} \lambda_i g_i(x_1, \ldots, x_n)$.
  - The solution corresponding to the original constrained optimization is always a saddle point of the Lagrangian function.
- Nearest points of two skew lines
  - Line 1 : $\boldsymbol{v}_1 = \boldsymbol{p}_1 + t_1 \boldsymbol{d}_1$
  - Line 2 : $\boldsymbol{v}_2 = \boldsymbol{p}_2 + t_2 \boldsymbol{d}_2$
  - $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2$
  - $\boldsymbol{n}_1 = \boldsymbol{d}_1 \times \boldsymbol{n}$
  - $\boldsymbol{n}_2 = \boldsymbol{d}_2 \times \boldsymbol{n}$
  - $\boldsymbol{c}_1 = \boldsymbol{p}_1 + \frac{(\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{n}_2}{\boldsymbol{d}_1 \cdot \boldsymbol{n}_2} \boldsymbol{d}_1$
  - $\boldsymbol{c}_2 = \boldsymbol{p}_2 + \frac{(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot \boldsymbol{n}_1}{\boldsymbol{d}_2 \cdot \boldsymbol{n}_1} \boldsymbol{d}_2$
- Bernoulli numbers
  $B_0 - 1, B_1^{\pm} = \pm\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0$
  $\displaystyle\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0$, EGF is $B(x) = \frac{x}{e^x - 1} = \displaystyle\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$.
  $\displaystyle S_m(n) = \sum_{k=1}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k^+ n^{m+1-k}$
- Stirling numbers of the second kind Partitions of n distinct elements into exactly k groups.
  $S(n,k) = S(n-1,k-1) + kS(n-1,k), S(n,1) = S(n,n) = 1$
  $S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$
  $x^n = \sum_{i=0}^{n} S(n,i)(x)_i$
- Pentagonal number theorem
  $\displaystyle\prod_{n=1}^{\infty} (1-x^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right)$
- Catalan numbers
  $\displaystyle C_n^{(k)} = \frac{1}{(k-1)n+1} \binom{kn}{n}$
  $C^{(k)}(x) = 1 + x[C^{(k)}(x)]^k$
- Eulerian numbers
  Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.
  $E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$
  $E(n,0) = E(n,n-1) = 1$
  $E(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$

10.6 Number

- Some prime numbers:

12721, 13331, 14341, 75577, 123457, 222557, 556679, 999983, 1097774749, 1076767633, 100102021, 999997771, 1001010013, 1000512343, 987654361, 999991231, 999888733, 98789101, 987777733, 999991921, 1010101333, 1010102101, 1000000000039, 1000000000000037, 2305843009213693951, 4611686018427387847, 9223372036854775783, 18446744073709551557

- Number of paritions of n:

n	2	3	4	5	6	7	8	9	20	30	40	50	100
$p(n)$	2	3	5	7	11	15	22	30	627	5604	4e4	2e5	2e8

- Maximum number of divisors:

n	100	1e3	1e6	1e9	1e12	1e15	1e18
$d(i)$	12	32	240	1344	6720	26880	103680

-

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\binom{2n}{n}$	2	6	20	70	252	924	3432	12870	48620	184756	7e5	2e6	1e7	4e7	1.5e8

- Number of ways to partition a set of n labeled elements:

n	2	3	4	5	6	7	8	9	10	11	12	13
B_n	2	5	15	52	203	877	4140	21147	115975	7e5	4e6	3e7

- Fibonacci numbers:

n	1	2	3	4	5	31	45	88
F_n	1	1	3	5	8	1346269	1e9	1e18