Convex hulls

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Computational geometry

Computational geometry is devoted to the study of algorithms which can be stated in terms of geometry

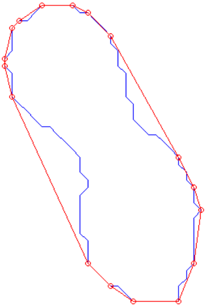

Figure 1: Convex hull 2D

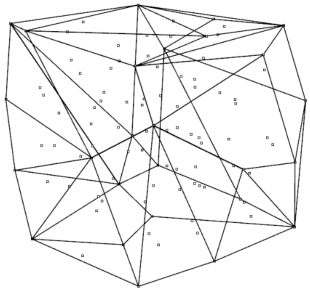

Figure 2: Convex hull 3D

Computational geometry

One of the oldest sections in computer science.
Divided in 2 types of problems: static and dynamic problems
Important applications include

- Computer graphics
- Computer vision
- Computer aided design (CAD)
- Mathematical visualization
- Robotics
- Integrated circuit design

Convex set

First we have to understand what a convex set is. A subset $S \subseteq P$ is convex, if $\forall x, y \in S$, $\overline{xy}$ is contained entirely in S too.

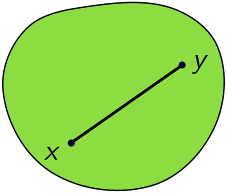

Figure 3: Convex set

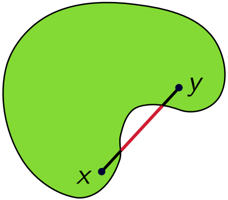

Figure 4: Non convex set

Convex hulls

A convex hull of a set P is therefore the smallest convex set CH that contains it.

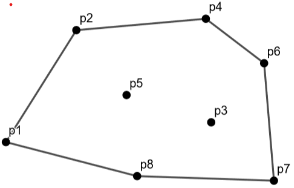

Figure 5: Convex hull with enumerations

$$P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$$
$$CH = (p_1, p_2, p_4, p_6, p_7, p_8)$$

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Naive approach

First of all: What do we expect from our algorithm?

Input: $P = \{p_1, p_2 \ldots p_n\}$
Output: ordered list of vertices representing convex hull of P in clockwise order, e.g. $CH = (p_1, p_2, p_4, \ldots, p_8)$

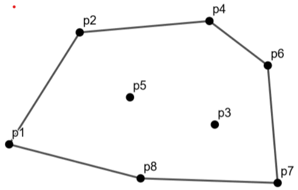


Figure 6: Convex hull

Argument 1

Let $p, q \in P$ and (p, q) edge of $CH(P)$. Then $\forall r \in P \setminus \{p, q\}$, r lies to the right of (p, q).

Idea of using this argument to compute $CH(P)$:

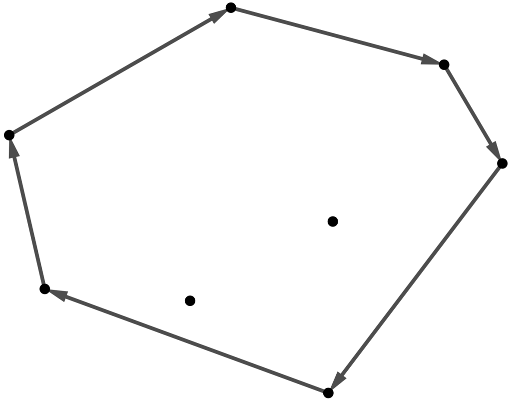
1. Loop through all pairs of points in P
2. Check if Argument 1 applies
3. Construct $CH(P)$ out of valid pairs

Let's construct an algorithm for the idea:

$E = \emptyset$
$Pairs = (p, q) \in P \times P$ with $p \neq q$

Loop trough all $(p, q) \in Pairs$ and check $\forall r \in P \setminus \{p, q\}$, that r lies on the right of (p, q).
If $\exists r$ that not lies on the right, (p, q) can't be part of $CH(P)$.
Else (p, q) is part of the convex hull and add (p, q) to E.

Construct $CH(P)$ out of E



Two steps are still not entirely clear

$E = \emptyset$
$Pairs = (p, q) \in P \times P$ with $p \neq q$

Loop trough all $(p, q) \in Pairs$ and check $\forall r \in P \setminus \{p, q\}$, that r lies on the right of (p, q).
If $\exists r$ that not lies on the right, (p, q) can't be part of $CH(P)$.
Else (p, q) is part of the convex hull and add (p, q) to E.

Construct $CH(P)$ out of E

How do we know if r lies to the right of (p, q)?

$p = (x_p, y_p), q = (x_q, y_q)$

$\overrightarrow{pq} = (x_q - x_p, y_q - y_p)$
$\overrightarrow{pr} = (x_r - x_p, y_r - y_p)$

Calculate the determinant of $(\overrightarrow{pq}, \overrightarrow{pr})$.

$$s_{pqr} = det \begin{pmatrix} x_q - x_p & x_r - x_p \\ y_q - y_p & y_r - y_p \end{pmatrix}$$

The sign of the determinant s_{pqr} tells us if r is to the right of (p, q)

- $s_{pqr} < 0$: r lies to the right
- $s_{pqr} > 0$: r lies to the left
- $s_{pqr} = 0$: r lies on the line of (p, q)

This can determine, where exactly the point is located. The time complexity of this check is $O(1)$, because it uses just a constant amount of arithmetical operations.

One step is still not entirely clear

$E = \emptyset$
$Pairs = (p, q) \in P \times P$ with $p \neq q$

Loop trough all $(p, q) \in Pairs$ and check $\forall r \in P \setminus \{p, q\}$, that r lies on the right of (p, q).
If $\exists r$ that not lies on the right, (p, q) can't be part of $CH(P)$.
Else (p, q) is part of the convex hull and add (p, q) to E.

Construct $CH(P)$ out of E

How do we construct $CH(P)$ out of E?

We know:

- $CH(P)$ should be a list of vertices, sorted in clockwise order
- E contains directed edges of the convex hull, not sorted

Directed edges means origin and destination of edge
All edges have all other points on their right
$CH(P)$ is list of origins and destination of edges in E

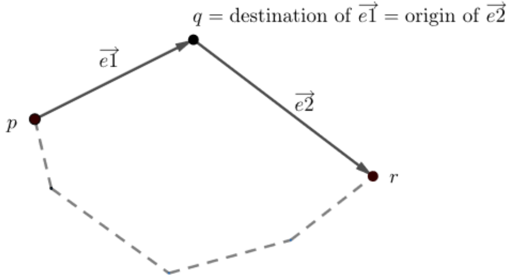


Figure 7: Directed edges in $CH(P)$

Let CH be an empty list
Remove any $(p, q) \in E$, append p to CH
Search in $(q, r) \in E$, remove (q, r) from E and append q to CH
Repeat the previous step until $r = p$

Problem: What do we do if more than 2 points lie on a straight line?

We check if point r lies on $\overline{pq}$.

If r lies on $\overline{pq}$, continue with next point.
Else $\overline{pq}$ cannot be part of $CH(P)$.

The time complexity of this naive convex hull computation is $O(n^3)$.

$|Pairs|$ is $n^2 - n$. For each of this pair we check another $n - 2$ pairs if they lie on the right side. This results in $O(n^3)$

The step to check if a point lies on the right side takes like seen before $O(1)$.
The computation of $CH(P)$ in the last step takes $O(n^2)$.

$$O(n^3) + O(1) + O(n^2) = O(n^3)$$

Incremental algorithm

The naive approach is slow, not robust and handles things in weird ways.
We try a proper algorithm technique:
Incremental algorithms

Short explanation: Updating existing solution after adding an item of input

The idea:

1. Input: Sequence of points, sorted by x-coordinate
2. Add one by one point to a list that holds the convex line/hull until that point
3. Compute first upper hull, then lower hull
4. Connect the two hulls

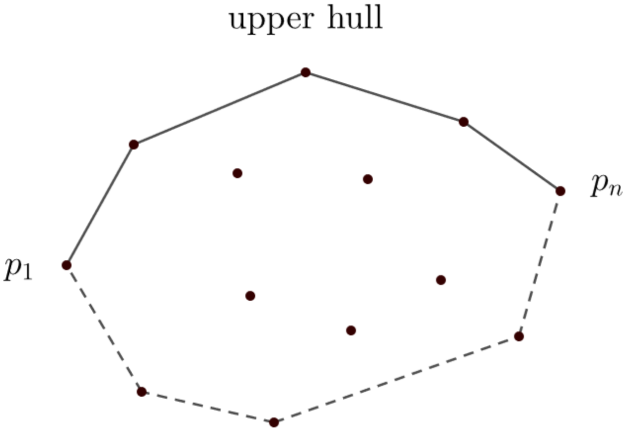


Figure 8: Representation of an upper hull

Computing upper hull

Sorting list of points by x-coordinate P, getting $S = (p_1, \ldots, p_n)$
Given a valid upper hull of points $p_1 \ldots p_{i-1}$, we compute the hull for $p_1 \ldots p_i$.

How do we compute/check the new hull after adding a point?

Argument 2

Walking a convex hull in clockwise order, there are only right turns at every vertex.

By knowing argument 2, we then can check if after adding p_i to the convex hull until p_{i-1}, the last 3 points of the hull are still making a right turn. If they do so, p_i is valid in the hull and we continue.

What if after adding p_i, the last 3 points do not make a right turn?
We remove the middle point and check again.

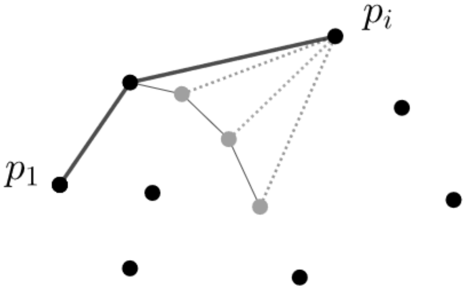


Figure 9: Removing middle point until right turn is made

Computing the upper hull

Let CH_{upper} be a list which contains the upper vertices in left-to-right order.
Appending p_i to CH_{upper} is valid, because p_i is the most right point.
Now check if the last three points, p_{i-2}, p_{i-1} and p_i are making a right turn.

If so, continue with adding p_{i+1}, else repeat removing the middle point until last 3 points make a right turn.

Computing the lower hull is exactly the same, except that CH_{lower} starts at p_n and works it way from right to left.

The algorithm

Sort points from P by x-coordinate, getting $(p_1, \ldots, p_n)$

$CH_{upper} \leftarrow [p_1, p_2]$

Loop i = 3 to n

Append p_i to CH_{upper}

while $|CH_{upper}| > 2$ **and** last 3 points not make a right turn

remove middle point

And again some issues to resolve:

1. 2 points with same x-coordinate
2. Collinear points should not appear in convex hull, only endpoints

1. Sort points first by ascending x-coordinate, then by ascending y-coordinate.

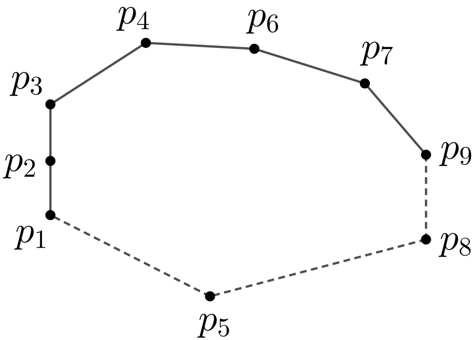


Figure 10: Sorted points with same x-coordinates

2. If 3 points are collinear $\Rightarrow$ no right turn is made (check fails and middle gets removed)

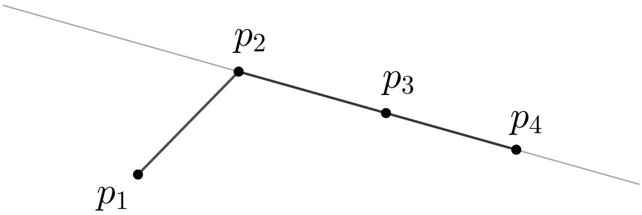


Figure 11: Collinear points

p_3 gets removed, no right turn is made

The time complexity of IncrementalConvexHull is $O(n \log n)$
Sorting points: $O(n \log n)$
for-loop bounded by n

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Introduction

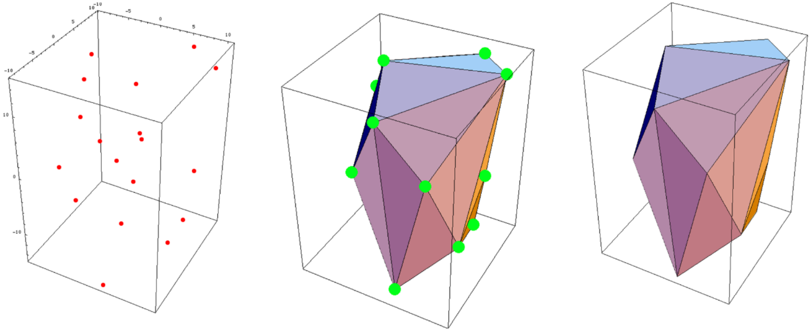


Figure 12: Convex hull in 3 dimensions

Convex hulls also exist in 3 dimensions.
Used for instance in collision detection in computer animations.

But why?
We want to know if 2 objects, Obj_1 and Obj_2 intersect. Most of the time, they do not $\Rightarrow$ faster to just approximate Obj_1 and Obj_2 and if convex hulls of them intersect, make more complex test with the exact object.

Computation

Again incremental algorithm.
Let $P = \{p_1, \ldots, p_n\}$

1. We choose 4 points in P.
   They should not lie on a common plane $\Rightarrow$ convex hull is tetrahedron
2. Random permutation for $p_5, \ldots p_n$. For $r >= 1$, $P_r := \{p_1, \ldots, p_r\}$.
3. For $r = 5$ to n
   $CH(P_r) =$ insert p_r into $CH(P_{r-1})$

To store $CH(P_r)$ we use a DCEL (doubly-connected edge list). The vertices are 3 dimensional.

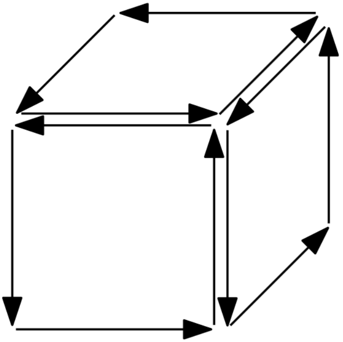


Figure 13: DCEL-representation

But how to insert p_r into $CH(P_{r-1})$?

We want to achieve:

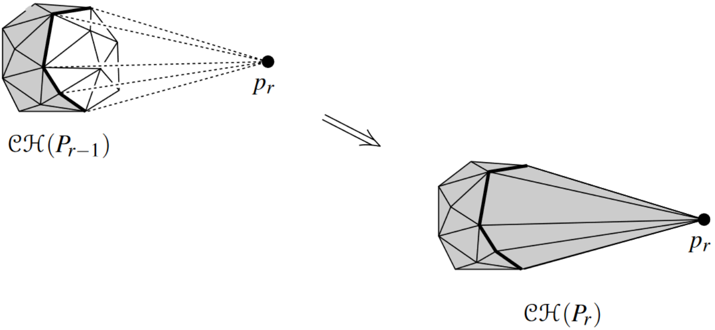


Figure 14: Adding point to $CH(P_{r-1})$

But how to insert p_r into $CH(P_{r-1})$?

2 cases:

- p_r lies already in $CH(P_{r-1})$. Then $CH(P_r) = CH(P_{r-1})$
- p_r lies outside of $CH(P_{r-1})$.

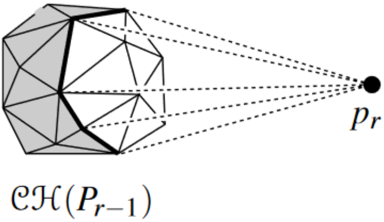
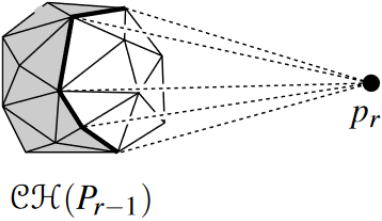


Figure 15: Horizon of visible and non visible points

We have visible regions (white) and invisible regions (gray) looking from p_r onto $CH(P_{r-1})$.



The 2 regions are seperated by edges from $CH(P_{r-1})$, we call this curve *horizon* of p_r on $CH(P_{r-1})$.

The horizon sets the boundary for faces we keep and faces that have to be removed in order to add p_r.

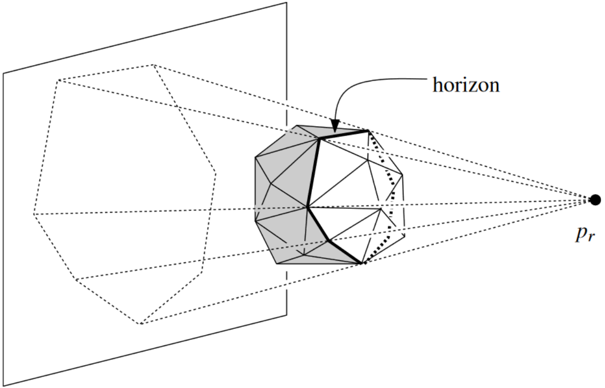


Figure 16: Projected horizon

Idea to insert p_r into $CH(p_{r-1})$:

- Keep invisible faces
- Remove visible faces
- Add new faces

Would be easy if we know all visible faces from $CH(P_r) \Rightarrow$ remove visible faces from the DCEL and compute new ones connecting p_r to *horizon*.
How do we find visible faces?

Naive:
Test every face for every point $\Rightarrow O(n^2)$

Conflict graph:
We maintain some additional information for each face $f \in CH(P_{r-1})$ and p_t with $t > r$

- $P_{conflict}(f) \subseteq \{p_r, \ldots, p_n\}$: all points that can see face f
  $\Rightarrow$ are in conflict because not both p and f can be part of the convex hull at the same time
- $F_{conflict}(p_t) \subseteq$ planes in $CH(P_r)$: contains all faces f that p_t can see

Bipartite conflict graph to store $P_{conflict}(f)$ and $F_{conflict}(p_t)$. One set is P_r (one node for each point not inserted yet) and the other is all faces f of the current convex hull.

A node $p_t \in P$ is connected to a face f of $CH(P_r)$ if $r < t$ and p_t can see f.

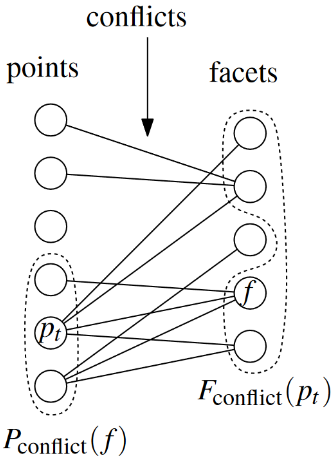


Figure 17: Conflict graph

Insertion of p_r into $CH(P_{r-1})$

We look up in $F_{conflict}(p_r)$, get the faces visible from p_r, and replace them by new faces connecting p_r to the horizon.

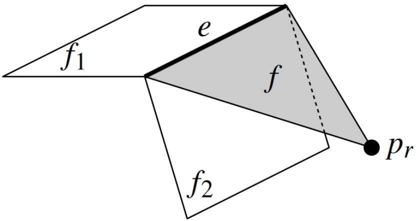


Figure 18: New faces to connect p_r

How to maintain correctness of conflict graph

Initialize conflict graph for $CH(P_4) \Rightarrow$ loop through points and determine it.

Update conflict graph after addition of p_r:

- Discard neighbors of p_r (visible faces from p_r)
- Discard p_r
- Insert nodes for new faces, which connect p_r to the *horizon*
- Find conflicts for all newly created faces

Find conflicts for all newly created faces

- p_t must have seen f_1 and f_2 in $CH(P_{r-1})$
- Therefore check all points in conflict lists of f_1 and f_2
- Add p_t if it sees face f and edge e

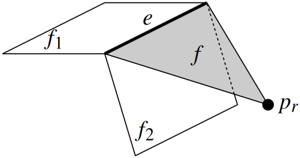


Figure 19: Adding face f

What we know now:
Rough idea of the algorithm
How to insert p_r into $CH(p_{r-1})$

Let's see the full algorithm to compute the convex hull

Input. A set P of n points in three-space.
Output. The convex hull $\mathcal{CH}(P)$ of P.

Find four points p_1, p_2, p_3, p_4 in P that form a tetrahedron.
$\mathcal{C} \leftarrow \mathcal{CH}(\{p_1, p_2, p_3, p_4\})$
Compute a random permutation $p_5, p_6, \ldots, p_n$ of the remaining points.
Initialize the conflict graph $\mathcal{G}$ with all visible pairs (p_t, f), where f is a facet of $\mathcal{C}$ and $t > 4$.
for $r \leftarrow 5$ **to** n
  do (∗ Insert p_r into $\mathcal{C}$: ∗)
    if $F_{\text{conflict}}(p_r)$ is not empty (∗ that is, p_r lies outside $\mathcal{C}$ ∗)
      then Delete all facets in $F_{\text{conflict}}(p_r)$ from $\mathcal{C}$.
        Walk along the boundary of the visible region of p_r (which consists exactly of the facets in $F_{\text{conflict}}(p_r)$) and create a list $\mathcal{L}$ of horizon edges in order.

for all $e \in \mathcal{L}$

  do Connect e to p_r by creating a triangular facet f.

    if f is coplanar with its neighbor facet f' along e

      then Merge f and f' into one facet, whose conflict list is the same as that of f'.

      else (∗ Determine conflicts for f: ∗)

        Create a node for f in $\mathcal{G}$.

        Let f_1 and f_2 be the facets incident to e in the old convex hull.

        $P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$

        for all points $p \in P(e)$

          do If f is visible from p, add (p, f) to $\mathcal{G}$.

Delete the node corresponding to p_r and the nodes corresponding to the facets in $F_{\text{conflict}}(p_r)$ from $\mathcal{G}$, together with their incident arcs.

return $\mathcal{C}$

Conclusion

What we saw in the last hour:

- Computational geometry
- Convex hulls in 2D:
  Naive approach, incremental algorithm
- Convex hulls in 3D

Any questions?

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Figure 2 https://www.genysis.cloud/documentation/convex-hull
Figure 3 https://en.wikipedia.org/wiki/Convex_set
Figure 4 https://en.wikipedia.org/wiki/Convex_set
Figure 5 Created by Samuel Kob
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Figure 11 Created by Samuel Kob
Figure 12 Wenk, Carola. *3D Convex Hulls*
Figure 13 Berg, Mark de et al. *Computational Geometry*
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