Graded Modalities as Linear Types

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We present an approach that models graded modal bindings using only linear ones. Inspired by Granule, we present an approach in Idris 2 to model limited grades using a construction, called "mu" (M). We present the construction of M, related ones, operations on them, and some useful properties

Todo list

Update everything

1 Introduction

One of the more interesting developments in Programming Language Theory is Quantitative Type Theory, or QTT. Based off Girard's linear logic, it forms the basis of the core syntax of Idris 2's core language, and also as a starting point for that of Linear Haskell [5, 3, 8, 1]. It has many theoretical applications, including creating a more concrete interpretation of the concept of a "real world", constraining memory usage, and allowing for safer foreign interfaces Need citation. Apart from just the theoretical insert, QTT has the potential to serve as an underlying logic for languages like Rust, where reasoning about resources takes the forefront.

However, new developments in Graded Modal Type Theory (GrTT), in particular with Granule, serve to create a finer grained[1] notion of usage than QTT. In GrTT, any natural number may serve as a usage, and in some cases even things that are first glance not natural numbers. GrTT is part of a larger trend of "types with algebras" being used to create inferable, simple, and intuitive systems for models of various things. Of these, some of the more notable ones are Koka, Effekt, and Flix, all of which seek to model effects with algebras [11, 4, 9].

[1]Hence the name

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```
data QNat : Type where
       Zero : QNat
       Succ : (1 k : QNat) -> QNat
```

Listing 1. The definition of `idris − mult9`'s natural numbers

2 Background

There are a couple constructions we must first define before we discuss the results posited here, however. Among these are the linear natural numbers, cloning and dropping, as well as existential types

2.1 Linear Natural Numbers

While linear values are quite useful in of themselves, they have one very large problem, the fact that reflection about them is limited. This is particarly problematic with linear natural numbers, which we define here as `QNat` in 1.

While addition is fine, once we get to multiplication we begin using one of the variables are variable amount of times. The *linear* library solves this by using inference to essientally clone the first value. Here, however, we opt for a different mechanism, that allows for easier reflection. Namely, we use the fact that erased functions are free to disregard multiplicities, and thence define a pair of multiplication functions, `lmul` and `lmul'`.

The first of these is the unrestricted "runtime" version of it, and its definition is *not exported*. The second of these is the erased "reflectable" version of it, who's definition follows the usual structure for natural numbers, and is much easier to infer about. We then define an assumption `mulRep` which is of the form `lmul` $=_?$ `lmul'`, and thereby can prove things about `lmul` through simpler proofs on `lmul'`

3 Conatural numbers

The conatural numbers are the natural numbers, $\mathbb{N}$, extended with a fixpoint, ∞ [**dependent_linear**].

Definition 3.1. The conatural numbers, $\mathbb{N}_\infty$, have two constructors, $\infty_{\mathbb{N}}\ 1 : \mathbb{N}_\infty$, and $\mathtt{Fin}_{\mathbb{N}} : \mathbb{N} \to_1 \mathbb{N}_\infty$ written in Idris as `Fix` and `Fin`, respectivly

Note that $\infty_{\mathbb{N}}\ 1$ is a simple constructor of $\mathbb{N}_\infty$, it is not special in any way, as opposed to the definition usually given in Rocq, which usually defines conatural numbers as merely a the natural numbers with some postulated "fixpoint" at some value. Granted, this makes conatural numbers in Rocq fairly easy to deal with, $\mathbb{N}_\infty$ tends to be rather difficult to work with, but we gain concreteness in exchange.

Often, we desire to talk about the "successors" of a given value. We define the successor, $S_\infty\ n$, as, for any finite value, its finite successor, and for $\infty_{\mathbb{N}}\ 1$, simply as $\infty_{\mathbb{N}}\ 1$. This unfourtantly means that successors of values are not neccaisarlly distinct from themselves. Perhaps even more troubling is the fact that this means that $\mathbb{N}_\infty$ *is not* well-founded.

However, fourtanetly, by defining `Finite`, we can create a set of

3.1 Existential Types

Existential types, in regards to dependent types, usually refers to dependent pair (Σ) types [14]. In this context, we see the first element of the pair as "evidence" for the type of the second element of the pair. In Idris, this is formalized (in Idris's *base*) by stating that the first arguement is runtime erased, in a similar way to which universal quantifiaction is a function that is erased.

```
record Exists (t : Type) (p : (t -> Type)) where
        constructor Evidence
        0 fst' : t
        1 snd' : p fst'
```

Listing 2. Definition of linear existentials

```
^^I^^Idata Mu : (n : Nat) -> (t : Type) -> (w : t) -> Type where
^^I^^I^^IMZ :
^^I^^I^^I^^IMu 0 t w
^^I^^I^^IMS :
^^I^^I^^I^^I(1 w : t) ->
^^I^^I^^I^^I(1 xs : Inf (Mu n t w)) ->
^^I^^I^^I^^IMu (QSucc n) t w
```

Listing 3. The definition of M in Idris

However, for our usage here this is not suitable, as we are dealing with principally linear values, and `Exists` is unrestricted. Fourtanetly, however, the change from unrestricted to linear existentials is quite trivial, the construction of which may be found at 2

Not much is notable about this, we define the operations as is usual, except every time a function would be unrestricted over the second value it is instead linear. We also have a operator, #?, which serves as sugar for this, using Idris's `typebind`, mechanism.

4 Mu types

The core construction here is M, or, in Idris, `Mu`, type. This is made to model a "source" of a given value. It is indexed by a natural number, a type, and an erased value of that type. The definitions of this in Idris are given at 3.

Definition 4.1. M is a polymorphic type with signature $M : (m')n : \mathbb{N}) \to_{(S}(t : *) \to_0 (w : t) \to_0 *$

In addition, M has two constructors

Definition 4.2. There are two constructors of M, ⊠ `MZ`, and $\odot$, `MS`, Which have the signatures ⊠ $: M\, n\, t\, w$ and $\odot : (w : t) \to_1 M\, n\, t\, w \to_1 M\, (Sn)\, t\, w$

Firstly, it should be noted that the names were chosen due to the fact that the indices of them are related. That is, ⊠ or "mu zero" will always be indexed by 0, and $\odot$, "mu successor", is always indexed by the successor of whatever is given in.

Remark 4.3. $M\ 0\ t\ w$ can only be constructed by ⊠.

Intuitively, M represents n copies of t, all with the value w, very much inspired by the paper "How to Take the Inverse of a Type". For instance, if we want to construct $x : M\, 2$ `String` `"value"`, we can only construct this through $\odot$, an we know, bu nothing but the first argument of the value, that we must take as a initial argument `value`, and as a second value of the form $M\ 1\ \ String$`"value"`, which we then repeat one more time and get another `"value"` an finally we match $M\ 0$ `String` `"value"` and get that we must have ⊠.

We can then say that we know that the only constructors of $M\ 2$ `String` `"String"` is `"value"`$\odot$`"value"`$\odot$⊠. Note the similarity between the natural number and the constructors. Just as we get 2 by applying

Stwice to Z, we get M 2 `String` `"value"` by applying $\odot$ twice to $\boxtimes$. This relationship between M types and numbers is far more extensive, as we will cover later.

The windex or "witness" is the value being copied. Notably, if we remove the windex, we would have $\boxtimes : Mnt$, and $\odot : t \rightarrow_1 Mnt \rightarrow_1 M(Sn)t$, which is simply `LVect`. The reason that this is undesirable is that we don't want this as it allows for M to have heterogeneous elements. However, if we are talking about "copies" of something, we know that should all be the exact same.

There are two very basic functions that bear mention with respect to M. The first of these is `witness`, which is of the form `witness` $:_0 \forall(w : t) \rightarrow_0 M\ n\ t\ w \rightarrow t$. Note that this is an *erased* function. Its implementation is quite simple, just being `witness`$\{w\}\_ := w$, which we can create, as erased functions can return erased values Need citation. In addition, we also have `drop` $: M\ 0\ t\ w \rightarrow \top$, which allows us to drop a value. Its signature is simple `drop`$\boxtimes := ()$.

Finally, we have `once`, which takes a value of form $M1$ and extracts it into the value itself. It has a signature `once` $: M\ 1\ t\ w \rightarrow_1 t$, where we have `once`$(x \odot \boxtimes) := x$

4.1 Uniqueness

While `w`serves a great purpose in the interpretation of M it perhaps serves an even greater purpose in terms of values of M. We can, using w, prove that there exists exactly one inhabitant of the type $M\ n\ t\ w$, so long as that type is well formed.

Segway into proof

Lemma 4.4. $x : M\ n\ t\ w$ *only if the concrete value of* x *contains* n *applications of* $\odot$.

Proof. Induct on n, the first case, $M\ 0\ t\ w$, contains zero applications of $\odot$, as it is $\boxtimes$. The second one splits has $x : M(Sn')tw$, and we can destruct on the only possible constructor of M for S, $\odot$, and get $y :_1 t$ and $z :_1 Mntw$ where $x =_? (y \odot z)$. We know by the induction hypothesis that z contains exactly n' uses of $\odot$, and we therefore know that the constructor occurs one more times then that, or Sn' □

This codifies the relationship between M types and natural numbers. Next we prove that we can establish equality between any two elements of a given M instance. This is equivalent to the statement "there exists at most one M" Need citation.

Lemma 4.5. *If both* x *and* y *are of type* $M\ n\ t\ w$*, then* $x =_? y$.

Proof. Induct on n.

- The first case, where x and y are both $M\ 0\ t\ w$, is trivial because, as per 4.3 they both must be $\boxtimes$
- The inductive case, where, from the fact that for any a and b (both of $M\ n'\ tw$) we have $a =_? b$, we prove that we have, for x and y of $M\ (Sn')\ t\ w$ that $x =_? y$. We note that we can destruct both of these, with $x_1 : M\ n'\ tw$ and $y_1 : M\ n'\ tw$, into $x = x_0 \odot x_1$ and $y = y_0 \odot y_1$, where we note that both x_0 and y_0 must be equal to w, and then we just have the induction hypothesis, $x_1 =_? y_1$

We thereby can simply this to the fact that $M\ n'\ tw$ has the above property, which is the induction hypothesis. □

Finish proof

However, we can make a even more specific statement, given the fact we know that w is an inhabitant of t.

Theorem 4.6 (Uniqueness)**.** *If* $M\ n\ t\ w$ *is well formed, then it must have* exactly *one inhabitant.*

Proof. We know by 4.5 that there is *at most* one inhabitant of $M\ n\ t\ w$. We then induct on n to show that there must also exist *at least* one inhabitant of the type.

```
0 unique :
        {n : Nat} -> {t : Type} -> {w : t} ->
        {a : Mu n t w} -> {b : Mu n t w} ->
        (a === b)
unique {n=Z} {w=w} {a=MZ,b=MZ} = Refl
unique {n=(S n')} {w=w} {a=MS w xs, b=MS w ys} = rewrite__impl
        (\zs => MS w xs === MS w zs)
        (unique {a=ys} {b=xs})
        Refl
```

Listing 4. Proof of 4.5 in Idris

- The first case is that $M\ 0\ t\ w$ is always constructible, which is trivial, as this is just ⊠.
- The second case, that $M\ n\ '\ tw$ provides $M\ (Sn')\ t\ w$ being constructible is also trivial, namely, if we have the construction on $M\ n\ '\ tw$ as x, we *know* that the provided value must be w.

Given that we can prove that there must be at least and at most one inhabitant, we can prove that there is exactly one inhabitant. □

This is very important for proofs on M. It corresponds to the fact that there is only one way to copy something, to provide another value that is the exact same as the first.

4.2 Graded Modalities With Mu

A claim was made earlier that M types can be used to model graded modalities within QTT. The way it does this, however, is not by directly equating M types with [] types [13]. It instead does this in a similar way to how others have embedded QTT in Agda [7, 6].

Namely, rather than viewing $M\ n\ t\ w$ as the *type* $[t]_n$, we instead view it as the *judgment* $[w]\ :\ [t]_r$. Fortunately, due to the fact that this is Idris 2, we don't need a separate ⊩, as M is a type, which, like any other, can be bound linearly, so, the equivalent to the GrTT statement $\Gamma \vdash [x]\ :\ [a]_r$ is $\Gamma \vdash \phi\ :\ M\ r\ a\ x$, which can be manipulated like any other type.

This is incredibly powerful. Not only can we reason about graded modalities in Idris, we can reason about them in the language itself, rather than as part of the syntax, which allows us to employ regular proofs on them. This is very apparent in the way constructions are devised. For instance, while Granule requires separate rules for dereliction, we do not, and per as a matter of fact we just define it as `once`; a similar relationship exists between weakening and `drop`where the exact relationship is shown in **??**.

Remark 4.7. We assume that $Mntw$ is equivalent to $[w]\ :\ [t]_n$.

Unfortunately, there is no way to prove this in either language, as M can't be constructed in Granule, and graded modal types don't exist in Idris 2.

4.3 Operations on Mu

There are number of operations that are very important on M. The first of these that we will discuss is combination. We define this as $\otimes\ :\ M\ m\ t\ w \rightarrow_1 M\ n\ t\ w \rightarrow_1 M\ (m+n)\ t\ w$ Need code, and we define it inductively as $\odot \otimes x\ := x$, and $(a \odot b) \otimes x\ := a \odot (b \otimes x)$. Note the similarity between the natural number indices and the values, where $0 + x\ := x$ and $(Sn) + x\ := S(n + x)$

Fix the aligne- ment of these

$$\frac{\Gamma \vdash \alpha}{\Gamma, [w] : [t]_0 \vdash \alpha}\ \text{Weak}$$

```
weak : (ctx -@ a) -@ ((LPair ctx (Mu 0 t w)) -@ a)
weak f (x # MZ) = f x
```

Fig. 1. The meta-logical drop rule and its Idris 2 equivalent

$$⊠ \otimes x := x \tag{1}$$
$$Z + x := x \tag{2}$$
$$(a \odot b) \otimes x := a \odot (b \otimes x) \tag{3}$$
$$(Sn) + x := S(n + x) \tag{4}$$

In addition, using the assumption of 4.7, we can liken the function ⊗ to context concatenation [13]. However, unlike Granule, we define this in the language itself. Per as a matter of fact, it isn't actually possible to construct ⊗in Granule, as it would require a way to reason about type level equality, which isn't possible.

Lemma 4.8. *⊗ is commutative*[2]*, that is,* $x \otimes y =_? y \otimes x$.

Proof. These types are the same, and by 4.6, they are equal. □

Given that we can "add" (Or, as we will see later, multiply) two M, it would seem natural that we could also subtract them. We can this function, which is the inverse of `combine`, `split`, which has the signature `split` $: (n : \mathbb{N}) \to_1 (- : M\ (m+n)\ t\ w) \to_1 M\ m\ t\ w \times^1 M\ n\ t\ w$. We actually use this a fair bit more than we use `combine`, as `split`doesn't impose any restrictions on the witness, which is very useful for when we discuss $^\wedge$types.

In a similar manner to how we proved (in 4.8) the commutativity of ⊗, we can prove that these are inverses.

Add something on meta-logical interpretation

Lemma 4.9. *Given that we have* $f :=$ `split`*, and* $g :=$ `uncurry`$^1(\otimes)$[3]*, then f and g are inverses.*

Proof. The type of f is $M\ (m+n)\ t\ w \to_1 M\ m\ t\ w \times^1 M\ n\ t\ w$, and that of g is $M\ m\ t\ w \times^1 M\ n\ t\ w \to_1 M\ (m+n)\ t\ w$, so there composition $f \circ g : (- : M\ (m+n)\ t\ w) \to_1 M\ (m+n)\ t\ w$ (the same for $g \circ f$). Any function from a unique object and that same unique object is an identity, thereby these are inverses Need citation □

Another relevant construction is multiplicity "joining" and its inverse, multiplicity "expanding".

Definition 4.10. We define `join` $: M\ m\ (M\ n\ t\ w)\ ?$[4]. Its definition is like that of natural number multiplication, with it defined as follows:

`join`⊠ := ⊠

`join`$(x_0 \odot x_1) := x_0 \otimes ($`join`$x_1)$

This is equivalent to flattening of values, and bears the same name as the equivalent monadic operation, `join`. Just like ⊗had the inverse `split`, `join` has the inverse `expand`

[2]You can also prove associativity and related properties, the proof is the same

[3]Given that we have $uncurry^1 : (a \to_1 b \to_1 c) \to (a \times^1 b) \to_1 c$

[4]For simplicity, we infer the second witness

```
app : Mu n (t -@ u) wf -> Mu n t wx -@
↪  Mu n u (wf wx)
app MZ MZ = MZ
app (MS f fs) (MS x xs) = MS (f x) (app
↪  fs xs)
```

```
map : (f : t -@ u) -> Mu n t w -@ Mu n
↪  u (f w)
map f MZ = MZ
map f (MS x xs) = MS (f x) (map {w=x} f
↪  xs)
```

Listing 5. Definition of map and app

4.4 Applications over Mu

There is still one crucial operation that we have not yet mentioned, and that is application over M. That is, we want a way to be able to lift a linear function into M. We can do this, and we call it map, due to its similarity to functorial lifting and its definition may be found 5.

However, there is something unsatisfying about map. Namely, the first argument is unrestricted. However, we *know* exactly how many of the function we need. It will simply be the same as the number of arguments.

So, we define another function, app, which has the type app $: (f : M\ n\ (t \to_1 u)\ w_f) \to_1 (x : M\ n\ t\ w_x) \to_1$ and a definition given at 5

Notably, if we define a function genMu $: (x : !_* t) \to_1 \forall (n : \mathbb{N}) \to_1 M\ n\ t\ x$.unrestricted, which is simply defined as genMu(MkBang_)0 $:=$ ⊠ and genMu(MkBang$x$)(Sn) $:= (x \odot$(genMun(MkBang$x$))), we can define mapas map$\{n\} f x :=$ app(genMu $f n$)x .

This isn't exactly correct

4.5 Applicative Mu

We can use app to derive equivalents of the push and pull methods of Granule, in a similar way to how we can use $< * >$ to define mappings over pairs. In Idris, we define push $: M\ n\ (t \times_1 u)\ (w_0 *_1 w_1) \to_1 M\ n\ t$ and pull $: M\ n\ t\ w_0 \times_1 M\ n\ u\ w_1 \to_1 M\ n\ (t \times_1 u)\ (w_0 *_1 w_1)$. Their construction is simple, and may be found in idris $-$ mult9.

4.6 Infinite Mu

We noted before that M i sdefined not by the linear natural numbers but by the linear conatural numbers. This is as in Need citation, where the conatural numbers are used to model infinite "sources" of a specific value.

This gives M a very special distinction, it is either a inductive datatype or coinductive datatype, but in either case uses the constructor $\odot$. This is because of the dual nature of conatural numbers, they consist of a finite portion and a infinite fixpoint, and because we build M on top of them, we then havea duality in how we construct M.

Fourtanetly, we can include a Inf annotation on the inductive portion of M and the compiler will allow it even when M is actually acting as a regular datatype. While at first it seems like $M \infty_{\mathbb{N}}\ 1$ would not be that useful, it is actually quite useful for one very particular reason: we can use it, combined with the Ω construction, to create a model for Girard's "of course" construct.

It therein makes sense that we should be able to define a pair of functions, one to $M\ \infty_{\mathbb{N}}\ 1\ t\ w$ from $!_* t$ and another going in the opposite direction. Unfourtantly, Idris does not in general have a mechanism for combining infinitely many of the same value into one ⊠ binding, so only the second of these, gen $: 1 \to !_* t M\ \infty_{\mathbb{N}}\ 1\ t\ w$, exists, and we have already essientally seen it, so we do not write it out explicitly here

5 Resource Algebras as Types

In the languages of Flix, Koka, and others, the type system is enriched with an effect algebra [11, 9]. In a similar vein, Granule (and Idris to a much more limited extent) use a resource algebra to enrich the type system. In a language like Haskell however, there has been a question as to whether this can be embed into the language itself. This is the goal of the libraries such as $\texttt{fused}-\texttt{effect}$

Check name

Need citation

, which use the constraint and type polymorphism to form a limited algebra.

In Idris, we can do similar constructions. However, it remains a question as to whether we can construct a type of resource algebras to enrich Idris types with multiplicity polymorphism

Here, we propose a "simple" system of effect formulas that model functions on natural numbers. We view each possible multiplicity as a "member of" solution set. So, for instance, to get all possible multiplicities of two, we would use the model of $\lambda x.2 * x$.

The problem with this, however, is that solving functions themselves are not decidable.

5.1 Formula Types

So, we create a "restricted" effect formula type, called `Form` which represents a function from natural numbers to natural numbers. We formalize this notion of "modeling a given function" by defining a function $\texttt{Eval} :_0 \texttt{Form} \to \mathbb{N} \to \mathbb{N}$. This is the core means through which we interpret a given formula.

In addition, we have two further procedures, $\texttt{Solve}\phi x$. Its definition is $\texttt{Solve} f y :=_0 (\texttt{Exists}\mathbb{N}(\lambda x.(\texttt{Eval} f x$ $y)))$. We further define $\texttt{Unify}\phi\psi$, which takes in two formulas and returns a type, and is defined as $\forall(y : \mathbb{N}) \to y \in \phi \to y \in \psi$

Notation 5.1. We write $\texttt{Solve}\phi x$ as $x \in \phi$ and $\texttt{Unify}\phi\psi$ as $\phi \sqsubseteq \psi$

Intuitively, $y \in \phi$ is "ϕ can output x, and $\phi \sqsubseteq \psi$ is that if y is a solution of ϕ it is also a solution of ψ [13].

Note that $\sqsubseteq$ forms a pre-order on `Form`, that is, it is transitive and reflexive.

Lemma 5.2. *$\sqsubseteq$ is reflexive, that is,* $\forall(\phi : \texttt{Form}) \to \phi \sqsubseteq \phi$

Proof. Upon expansion, the result type becomes $\forall(y : \mathbb{N}) \to \texttt{Eval}\phi y \to \texttt{Eval}\phi y$, which is just identity □

Lemma 5.3. *$\sqsubseteq$ is transitive, that is,* $\forall(\phi_0 : \texttt{Form}) \to \forall(\phi_1 : \texttt{Form}) \to \forall(\phi_2 : \texttt{Form}) \to \phi_0 \sqsubseteq \phi_1 \to \phi_1 \sqsubseteq \phi_2$

Proof

Proof. □

5.2 The Simple Forms

The most basic constructor of the `Form` type is that of `FVar`, which is a nullary constructor of `Form`. It models the "argument" in a given formula, and its clause of `Eval` is $\texttt{EvalFVar}x := x$.

Note that `FVar` is more general than any other formula, as every single natural number is mapped to, so we have $\phi \sqsubseteq \texttt{FVar}$ for each formula ϕ

Add proof of this

Next up we have nearly as simple of a constructor, `FLit`. It is a unary constructor taking a natural number, which is essentially the "constant" formula. Its branch of `Eval` is $\texttt{Eval}(\texttt{FLit}k)x := k$, that is, it ignores its argument.

5.3 Binary Operations

We introduce four constructors of `Form`, all of which are quite similar, all taking in two subformulas as agreements to the constructors, and all model binary operations.

They are FAdd (addition), FMul (multiplication), FMax (joins), FMin (meets). The first two of these are sufficient to define Form as an instance of the Idris Num class, with the conversion being FLit, addition as FAdd and multiplication as FMul.

We then define the evaluation of all of these. Using FAdd as an example, we have $\texttt{Eval}(\texttt{FAdd} f\, g)x := (\texttt{Eval} f x) + (\texttt{Eval} g x)$, and we omit here the rest of the operators because they are quite similar.

The only notable fact about such of these is that they both use the same variable for both of the formulas, so FAddFVarFVar is a formula which the same variable is used twice. every instance of FVar in ϕ with ϕ

5.4 The Extensions

While so far we have defined Form to be a rig (ring sans negation), the rest of these forms all do something fairly unique. First among these is FApp, which is a binary constructor of a Form. Very roughly, $\texttt{FApp}\phi\psi$ represents the "composition" of the formulas. While this sounds complex, it is in practiced quite simple, with us defining the Eval branch as $\texttt{Eval}(\texttt{FApp}\phi\psi)x := \texttt{Eval}\phi(\texttt{Eval}\psi x)$.

Apart from being viewed as a composition, we can also view $\texttt{FApp}\phi\psi$ as the result of substituting

Lemma 5.4. FVar *is the identity formula with respect to* FApp

Proof. □ Proof

We also note compositions always are subformulas of their second formula

Theorem 5.5. *For any formulas ϕ ψ,* $\texttt{FApp}\phi\psi$ *is at most general as* ϕ

Proof. □ Proof

5.5 Completeness of Natural Numbers

There are two much more out of place constructors, FLeft and FRight, both of which are unary constructors on Form. These constructors have a seemingly random definition, but have a very important for allowing polyvariadic formulas.

Firstly, we define a function, pairing, which has the form $\mathbb{N} \to_1 \mathbb{N} \times \mathbb{N}$ it is defined as follows:

- pairing 0 is $(0, 0)$
- If n is not of the prime factorization form $2^x 3^y$, then it is equal to $(0, 0)$
- If, for some $x, y : \mathbb{N}$, it is of the form $2^x 3^y$, then it is equal to (x, y)

Corollary 5.6. *For all $x, y : \mathbb{N}$,* $\texttt{pairing}(2^x 3^y) =_? (x, y)$

Proof. □ Finish

6 Omega Types

Finish this section

The principle use of Form here is as part of the Ω construction, which uses it to constraint over mutliplicity, thereby giving us an equivalent of Granuleeffect formulas,

6.1 Poly-multiplicative Judgments

Definition 6.1. Ω is a type indexed by a formula, ϕ, a type t, and a witness of that type, such that $\Omega\ \phi\ t\ w :=_0 (n : \mathbb{N}) \to_1 M\ (\texttt{Eval}\ \phi\ n)\ t\ w$

That is, an Ω designates a function from a natural number to a certain number of t. However, this is often unedsierable, so we instead efine a helper function, reify, of the type $\Omega\ \phi\ t\ w \to_1 \forall(n : \mathbb{N}) \to_1 \forall(n \in \phi$ which, allows for us to instead consider if the number in is in the "solution set" as opposed to a value projected to.

```
data Form : Type where
FVar : Form
FVal : (1 n : QNat) -> Form
FApp : (1 g : Form) -> (1 f : Form) -> Form
FAdd : (1 x : Form) -> (1 y : Form) -> Form
FMul : (1 x : Form) -> (1 y : Form) -> Form
FMin : (1 x : Form) -> (1 y : Form) -> Form
FMax : (1 x : Form) -> (1 y : Form) -> Form
FLeft : (1 f : Form) -> Form
FRight : (1 f : Form) -> Form
0 Eval : (1 f : Form) -> (1 x : QNat) -> QNat
```

```
Eval FVar x = x
Eval (FVal n) x = n
Eval (FApp g f) x = Eval g (Eval f x)
Eval (FAdd f g) x = ladd (Eval f x) (Eval g x)
Eval (FMul f g) x = lmul (Eval f x) (Eval g x)
Eval (FMin f g) x = lmin (Eval f x) (Eval g x)
Eval (FMax f g) x = lmax (Eval f x) (Eval g x)
Eval (FLeft f) x = let
        (y # z) = pairing x
        in Eval f y
Eval (FRight f) x = let
        (y # z) = pairing x
        in Eval f z
```

Listing 6. The definition of formulas and evaluation in `idris` – `mult9`

The most simple form of Ω is that where ϕ is `FVar` and thus is simply, $(n : \mathbb{N}) \rightarrow_1 M$ (`Eval FVar` n) $t\ w$, or simply $(n : \mathbb{N}) \rightarrow_1 M\ n\ t\ w$, which we specifically call ωtw, and $\omega tw :=_0 (n : \mathbb{N}) \rightarrow_1 M\ n\ t\ w$

6.2 Sources and Factories

7 Exponential and Existential Types

While the M and Ω types bear much insert theoretically, they, by themselves, have little practical use. This is because as demonstrated in 4.2, these types model the *judgments* about graded modalities, not the graded modalities themselves.

Fortunately, however, we can define a simple abstraction over them that allows them to behave more like true graded modalities at the term level. To do this, we must first introduce a linear existential type. A regular existential type is a dependent pair type that "doesn't care" about its first argument. In Idris 2, we can make the that fact part of the programming by giving the first argument a multiplicity of zero.

Idris 2 actually defines an existential type, however, it has the second argument have a multiplicity of ω, while we want it to have a multiplicity one. Fortunately, the modification of `Exists` to our type, `LExists`, is quite trivial[5], and we define it in 7

We can also define mapping like how Idris defines the mapping, with the signature described in 8 [2].

7.1 Existential Crisis (Solution)

This principle issue with using M and Ω in practice is that

[5]Idris, however, seems to have trouble correctly linearlizing constructor accessors, so we define the actual `fst` and `snd` accessors separately

```
record LExists {ty : Type} (f : (ty -> Type)) where
        constructor LEvidence
        0 fst' : ty
        1 snd' : f fst'
```

Listing 7. The definition of `LExists`

```
map :
        {0 p : a -> Type} ->
        {0 q : b -> Type} ->
        {0 m : (a -> b)} ->
        (1 f : forall x. p x -@ q (m x)) ->
        (LExists p -@ LExists q)
map f (LEvidence x y) = LEvidence (m x) (f y)
```

Listing 8. Definition of `map` for `LExists`

8 Using Mu and Friends

9 Conclusion

Related Work

GrTT and QTT. While QTT, in particular as described here, is quite useful, it of course has its limits. In particular, the work of languages like Granuleto create a generalized notion of this in a way that can easily be infered and checked is important. However, in terms of QTT (or even systems outside of it) this is far from complete.

The Syntax. For instance, even with the $^\wedge$types, the syntax for this, and more generally linearity in general, tends to be a bit hard to use. A question of how to integrate into the source syntax would be quite interesting. Also, in general, one of the advantages of making an algebra part of the core language itself (as opposed to a construction on top of it) is that it makes it easier to create an inference engine for that language..

Bump Allocation. QTT has been discussed as a potential theoretical model for ownership systems. One of the more useful constructs in such a system is bump allocation. With particular use seen in compilers, bump arena allocation, where memory is pre-allocated per phase, helps both seperate and simplify memory usage. It is possible that a usage of M types (given the fact that we know exactly how many times we need a value) as a form of modeling of arena allocation might be useful.

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Artifacts

All Idris code mentioned here is either directly from or derived from the code in the Idris library `idris-mult`, which may be found at its repository[6]

[6]Some listings may be modified for readability

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