1. Rectangular Probability Density Function

if $s^2=\int_{-\infty}^{\infty}(x-\bar{x})^2 f(x)\,dx=\int_{-\infty}^{\infty}x^2 f(x)\,dx-\bar{x}^2$ and $f(x)=\begin{cases}\frac{1}{2a}, & (\bar{x}-a)<x<(\bar{x}+a)\\ 0, & \text{otherwise}\end{cases}$

then

(a)

$$S^2_{rectangle}=\int_{\bar{x}-a}^{\bar{x}+a} x^2\,\frac{1}{2a}\,dx-\bar{x}^2$$

$$(a^2+2ab+b^2)(a+b)=a^3+2a^2b+b^2a+a^2b+2ab^2+b^3$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$=\frac{(\bar{x}+a)^3}{6a}-\frac{(\bar{x}-a)^3}{6a}-\bar{x}^2$$

$$=\frac{\bar{x}^3+3\bar{x}^2a+3\bar{x}a^2+a^3}{6a}-\frac{\bar{x}^3-3\bar{x}^2a+3\bar{x}a^2-a^3}{6a}-\bar{x}^2$$

$$=\frac{\cancel{\bar{x}^3}+3\bar{x}^2a+\cancel{3\bar{x}a^2}+a^3-\cancel{\bar{x}^3}+3\bar{x}^2a-\cancel{3\bar{x}a^2}+a^3}{6a}-\bar{x}^2$$

$$=\frac{6\bar{x}^2a+2a^3}{6a}=\cancel{\bar{x}^2}+\frac{a^3}{3}-\cancel{\bar{x}^2}=\frac{a^2}{3}=S^2$$

So...

... then $S_{rectangle}=\frac{a}{\sqrt{3}}$

(b) $\frac{\text{total Area}}{\bar{x}\pm S_{rectangle}}=\frac{2a}{2\frac{a}{\sqrt{3}}}=\frac{1}{\sqrt{3}}\approx 0.5774\approx 58\%$

measunc

February 8, 2026

1 Measurement and Uncertainty

1-8 and 11-13

Table 1: Probabilites for two fair dice

Sum	Unique Combinations	Number	Probabilty
1	none	none	0:36
2	(1, 1)	1	1:36
3	(2, 1), (1, 2)	2	2:36
4	(3, 1), (2, 2), (1, 3)	3	3:36
5	(4, 1), (3, 2), (2, 3), (1, 4)	4	4:36
6	(5, 1), (4, 2), (3, 3), (2, 4), (1, 5)	5	5:36
7	(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)	6	6:36
8	(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)	5	5:36
9	(6, 3), (5, 4), (4, 5), (3, 6)	4	4:36
10	(6, 4), (5, 5), (4, 6)	3	3:36
11	(5, 6), (6, 5)	2	2:36
12	(6, 6)	1	1:36
13	none	0	0:36

Scatter Plots

[96]:
```
#Import libraries
import numpy as np
import matplotlib.pyplot as plt
```

[97]:
```
tt = np.arange(10, 26, 1)
vv = np.array([5, 6, 1, 3, 0, 0.1,0, -2, -2, -2, -2,-9,-15,-5,-7,-7])
sv = np.array([9,5, 10,2,3,0.2,4,4,7,13,17,6,1,21,22,25])
```

[98]:
```
plt.figure(figsize=(6,4))

plt.errorbar(
    tt,
    vv,
    yerr=sv,
```

```
    fmt='o',
    color='black',
    ecolor='black',
    elinewidth=1
)

plt.title("v versus t")
plt.xlabel("time [s]")
plt.ylabel("velocity [m/s]")
txt="Figure 1: Scatterplot of velocity versus time. Error bars indicate 1␣
↪standard uncertainty in the measurement."
plt.figtext(0.5, -0.1, txt, wrap=True, horizontalalignment='center',␣
↪fontsize=12)
plt.grid()
plt.show()
```

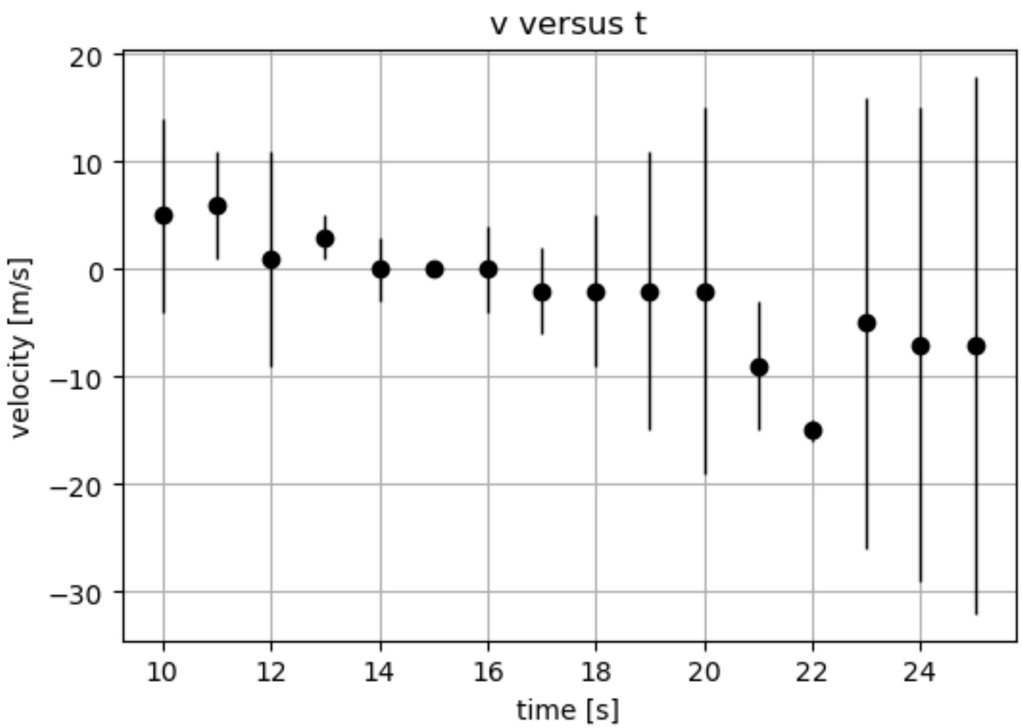


Figure 1: Scatterplot of velocity versus time. Error bars indicate 1 standard uncertainty in the measurement.

[99]:
```
sums = np.arange(1, 14, 1)
prob = np.array([0,1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36,0])
plt.plot(sums, prob, marker='o', linestyle='-')
```

```python
entries = []
for i in range(size):
    x = random.randrange(1, 7)
    y = random.randrange(1, 7)
    s = x + y
    entries.append(s)

relative_freq = [np.count_nonzero(entries == i)/size for i in range(1, 15)]
plt.hist(entries, bins=np.arange(0.5, 14.5, 1), density=True, edgecolor='black')

mean = np.mean(entries)
std = np.std(entries, ddof=0)   # population std


plt.title("Sum of Two Dice")
plt.xlabel("Sum")
plt.ylabel("Relative Frequency")
plt.legend()
info_text = (
    f"Entries   {size}\n"
    f"Mean      {mean:.2f}\n"
    f"Std Dev   {std:.3f}"
)

plt.text(0.95, 0.95, info_text,
         transform=plt.gca().transAxes,
         fontsize=10,
         verticalalignment='top',
         horizontalalignment='right',
         bbox=dict(facecolor='white', edgecolor='black'))


plt.show()
```

```
C:\Users\haru\AppData\Local\Temp\ipykernel_28240\1666612121.py:22: UserWarning:
No artists with labels found to put in legend.  Note that artists whose label
start with an underscore are ignored when legend() is called with no argument.
  plt.legend()
```

```
plt.axvline(x=7, ymin=0.05, ymax=0.95, color='black', linestyle='--',␣
 ↪label='a=6')
plt.annotate('a = 6',
             xy=(7, 0.0),
             xytext=(13, 0.0),
             ha='center',
             arrowprops=dict(arrowstyle='<->', linewidth=1.5))
txt="Figure 2: Probability distribution for two fair dice. Note the symmetry of␣
 ↪the distribution (an isosceles triangle). The connected dots define a␣
 ↪triangular PDF for the roll of two fair dice. The symmetry implies that the␣
 ↪center of the distribution is also the mean. Also shown is the half-width, a.
 ↪"
plt.figtext(0.5, -0.1, txt, wrap=True, horizontalalignment='center',␣
 ↪fontsize=12)
plt.show()
```

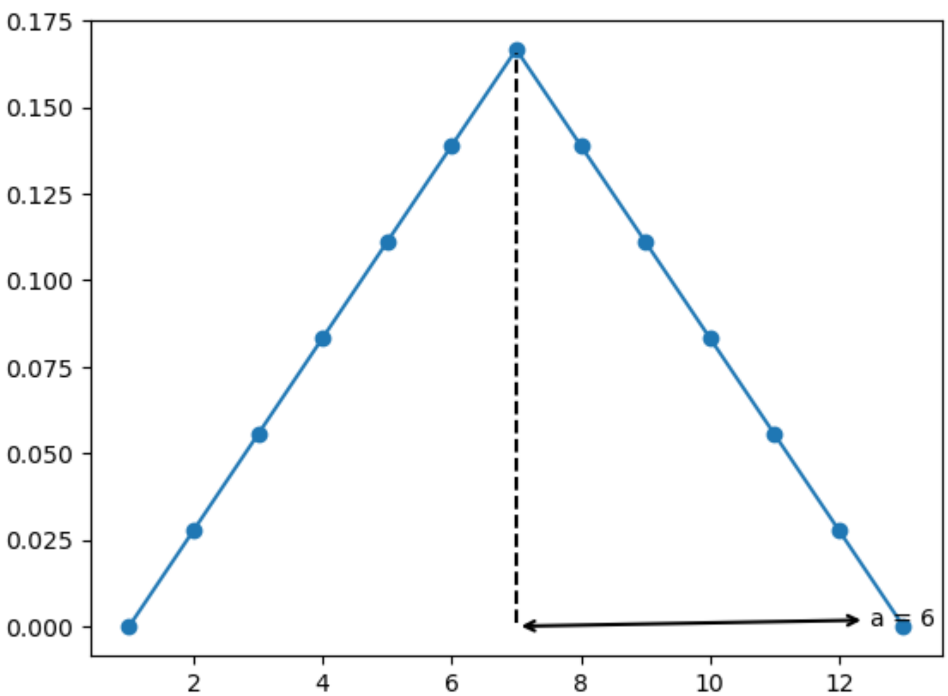


Figure 2: Probability distribution for two fair dice. Note the symmetry of the distribution (an isosceles triangle). The connected dots define a triangular PDF for the roll of two fair dice. The symmetry implies that the center of the distribution is also the mean. Also shown is the half-width, a.

[100]:
```
import random

sums = np.arange(1,15, 1)
size = 36
```

Triangular Probability Density Function

5.

Case 1: central value

it's 7

Case 2: Discrete Distribution

$\sum_i sum_i = 91$

$91/13 = 7$

Case 3: continous distribution

$$\bar{x} = \frac{1}{36}\left[\int_1^7 x(x-1)\,dx + \int_7^{13} x(13-x)\,dx\right]$$

$$= \frac{1}{36}\left[\int_1^7 x^2 - x\,dx + \int_7^{13} 13x - x^2\,dx\right]$$

$$= \frac{1}{36}\left(\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_1^7 + \left[\frac{13}{2}x^2 - \frac{1}{3}x^3\right]_7^{13}\right)$$

$$= \left(\frac{1}{3}(7^3) - \frac{1}{2}(7^2)\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$\left(\frac{13}{2}(13^2) - \frac{1}{3}13^3\right) - \left(\frac{13}{2}(7^2) - \frac{1}{3}(7^3)\right)$$

$$\frac{1}{36}(90 - 162)$$

$$\bar{x} = 7$$

6.

$\bar{x} = 7$ or the half width

$$s^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx$$

$$s^2 = \int_1^7 (x-\bar{x})^2 \frac{1}{36}(x-1)\,dx + \int_7^{13} (x-\bar{x})^2 \frac{1}{36}(13-x)\,dx$$

$$= \frac{1}{36}\left[\int_1^7 (x-7)^2 (x-1)\,dx + \int_7^{13} (x-\bar{x})^2 (13-x)\,dx\right.$$

$$= \frac{2}{36} \int_1^7 (x-7)^2 (x-1)\,dx$$

$(x^2 - 14x + 49)(x-1)$

$(x^3 - 14x^2 + 49x - x^2 + 14x - 49)$

$7(x^3 - 15x^2 + 63x - 49)$

$$= \int_1^7 x^3 - 15x^2 + 63x - 49\,dx$$

$$= \left[\frac{1}{4}x^4 - 5x^3 + \frac{63}{2}x^2 - 49x\right]_1^7$$

$$= \frac{2}{36}\left(\left[\frac{1}{4}(7)^4 - 5(7^3) + \frac{63}{2}(7^2) - 49(7)\right] - \left[\frac{1}{4} - 5 + \frac{63}{2} - 49\right]\right)$$

$$s^2 = 6$$

$$s = \sqrt{6} \quad \rightarrow \quad s_{triangle} = \frac{a}{\sqrt{6}}$$

7

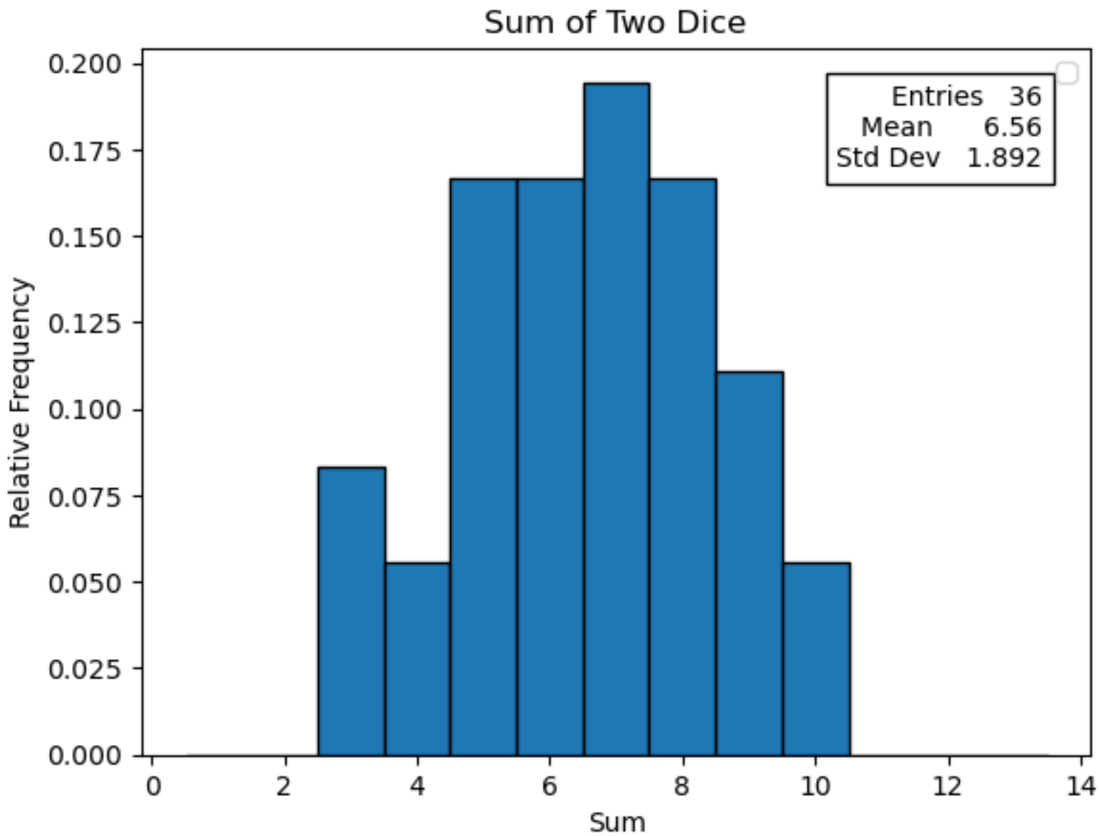
$$f(x) = \begin{cases} 1/a^2(x-(\bar{x}-a)), & \bar{x}-a \le x \le \bar{x} \\ 1/a^2((\bar{x}+a)-x), & \bar{x} \le x \le \bar{x}+a \\ 0 & \text{otherwise} \end{cases}$$

$$P = \int_{\bar{x}-\frac{a}{\sqrt{6}}}^{\bar{x}+\frac{a}{\sqrt{6}}} f(x)\,dx = 2\int_{\bar{x}-a/\sqrt{6}}^{\bar{x}} f(x)\,dx$$

$$= 2\int_{\bar{x}-a/\sqrt{6}}^{\bar{x}} \frac{1}{a^2}(x-\bar{x}+a)\,dx \qquad \text{Let } u = x-\bar{x}$$

$$P = 2\int_{-a/\sqrt{6}}^{0} \frac{1}{a^2}(u+a)\,du = 2\left[\frac{1}{a^2}\left(\frac{u^2}{2}+au\right)\right]_{-a/\sqrt{6}}^{0}$$

$$= 2\left(\frac{1}{\sqrt{6}} - \frac{1}{12}\right) = \frac{\sqrt{6}}{3} - \frac{1}{6} \approx 0.6548 = 65\,\%$$



11. Confidence Levels

[101]:
```
from scipy.stats import norm

# (a) Confidence level for ±2
conf_2 = norm.cdf(2) - norm.cdf(-2)

# (b) Confidence level for ±3
conf_3 = norm.cdf(3) - norm.cdf(-3)

# (c) Coverage factor n for 90% confidence
# 90% symmetric → upper tail at 0.95
n_90 = norm.ppf(0.95)

print("Confidence for ±2 :", conf_2)
print("Confidence for ±3 :", conf_3)
print("Coverage factor for 90%:", n_90)
```

```
Confidence for ±2 : 0.9544997361036416
Confidence for ±3 : 0.9973002039367398
Coverage factor for 90%: 1.6448536269514722
```

Confidence Levels

11 $$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

a) Confidence levels being claimed by $x \pm 2s_{normal}$

Let $s_{normal} = \sigma$

$$\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $z = \frac{x-\mu}{\sigma} \rightarrow \frac{\cancel{\mu}-2\sigma-\cancel{\mu}}{\sigma} = -2$

$\sigma z = x-\mu \qquad \frac{\mu+2\sigma-\mu}{\sigma} = 2$

$x = \sigma z + \mu$

$dx = \sigma dz$

$$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

Integral is unsolvable through analytical means so scipy will be used

13 Show that

$$z = \frac{1}{y_1 y_2} \rightarrow S_z = \frac{1}{y_1 y_2}\sqrt{\left(\frac{S_1}{y_1}\right)^2 + \left(\frac{S_2}{y_2}\right)^2}$$

$$S_z^2 = \sum_{n=1}^{N} \left(\frac{\partial z}{\partial x_i}\right)^2 S_i^2$$

$$\frac{\partial z}{\partial y_1} = -\frac{1}{y_1^2 y_2}, \quad \frac{\partial z}{\partial y_2} = -\frac{1}{y_1 y_2^2} \Rightarrow S^2 = \frac{S_1^2}{y_1^4 y_2^2} + \frac{S_2^2}{y_1^2 y_2^4}$$

$$S^2 = \frac{1}{y_1^2 y_2^2}\left[\left(\frac{S_1}{y_1}\right)^2 + \left(\frac{S_2}{y_2}\right)^2\right]$$

$$S_z = \frac{1}{y_1 y_2}\sqrt{\left(\frac{S_1}{y_1}\right)^2 + \left(\frac{S_2}{y_2}\right)^2}$$