CMPE320 Spring 2022
Homework #1 Assigned Jan 31, 2021, Due Feb 9, 2022 per detailed course schedule

In this homework, I want you to work with the algebra of events. *Events* are the results of experiments, and form a Borel Field. You don't need to worry about the formalism: just think of events as (proper) subsets of the *Sample Space,* $\mathcal{S}$, and that $\mathcal{S}$ is closed under both finite and infinite unions and intersections.

1. (T&T) Problem Draw Venn Diagrams to illustrate the following true statements in the algebra of events. $A,B,C \in \boldsymbol{\mathcal{S}}$ (Note Venn Diagrams are *illustrations*, not *proofs*!)

$$
\begin{aligned}
&(a) A(B+C) = AB + AC \\
&(b) A + A^C B = A + B \\
&(c) \left(A+B\right)\left(A+C\right) = A + BC \\
&(d) AB + B = B
\end{aligned}
$$

Solution: See hand-drawn diagrams on following pages.

2. (T&T) Starting with the expression for $\Pr\left[A_1 + A_2\right]$, prove that

$$
\begin{aligned}
\Pr\left[A_1 + A_2 + A_3\right] = \Pr\left[A_1\right] + \Pr\left[A_2\right] + \Pr\left[A_3\right] - \Pr\left[A_1A_2\right] - \Pr\left[A_1A_3\right] + \Pr\left[A_2A_3\right] \\
+\Pr\left[A_1A_2A_3\right]
\end{aligned}
$$

Solution :

$$
\begin{aligned}
&\text{Let } A_1, A_2 A_3 \in \boldsymbol{\mathcal{S}} && \\
&\Pr\left[A_1 + A_2\right] = \Pr\left[A_1\right] + \Pr\left[A_2\right] - \Pr\left[A_1A_2\right] && \text{proved in lecture} \\
&\qquad \text{Let } B = A_1 + A_2 . B \in \boldsymbol{\mathcal{S}} && \boldsymbol{\mathcal{S}} \text{ is closed under finite unions} \\
&\qquad \left(B + A_3\right) \in \boldsymbol{\mathcal{S}} && \boldsymbol{\mathcal{S}} \text{ is closed under finite unions} \\
&\Pr\left[B + A_3\right] = \Pr\left[B\right] + \Pr\left[A_3\right] - \Pr\left[BA_3\right] && \text{proved in lecture} \\
&= \left(\Pr\left[A_1\right] + \Pr\left[A_2\right] - \Pr\left[A_1A_2\right]\right) + \Pr\left[A_3\right] && \\
&\quad - \Pr\left[(A_1 + A_2)A_3\right] && \text{substitution} \\
&\Pr\left[\left(A_1 + A_2\right)A_3\right] = \Pr\left[A_1A_3 + A_2A_3\right] && \text{distributive law of algebra of events} \\
&= \Pr\left[A_1A_3\right] + \Pr\left[A_2A_3\right] - \Pr\left[A_1A_3A_2A_3\right] && \text{proved in lecture} \\
&= \Pr\left[A_1A_3\right] + \Pr\left[A_2A_3\right] - \Pr\left[A_1A_2A_3\right] && \text{algebra of events} \\
&\Pr\left[B + A_3\right] = \left(\Pr\left[A_1\right] + \Pr\left[A_2\right] - \Pr\left[A_1A_2\right]\right) && \\
&\quad + \Pr\left[A_3\right] - \left(\Pr\left[A_1A_3\right] + \Pr\left[A_2A_3\right] - \Pr\left[A_1A_2A_3\right]\right) && \text{substitution} \\
&\Pr\left[A_1 + A_2 + A_3\right] = \Pr\left[A_1\right] + \Pr\left[A_2\right] - \Pr\left[A_1A_2\right] + \Pr\left[A_3\right] && \\
&\quad - \Pr\left[A_1A_3\right] - \Pr\left[A_2A_3\right] + \Pr\left[A_1A_2A_3\right] && \text{substitution,simplification} \\
&= \Pr\left[A_1\right] + \Pr\left[A_2\right] + \Pr\left[A_3\right] - \Pr\left[A_1A_2\right] - \Pr\left[A_1A_3\right] && \\
&\quad - \Pr\left[A_2A_3\right] + \Pr\left[A_1A_2A_3\right] && \text{rearrange terms}
\end{aligned}
$$

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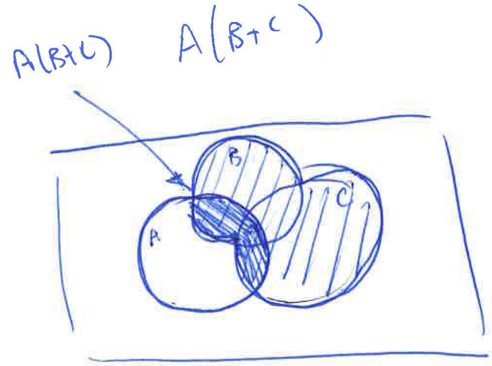
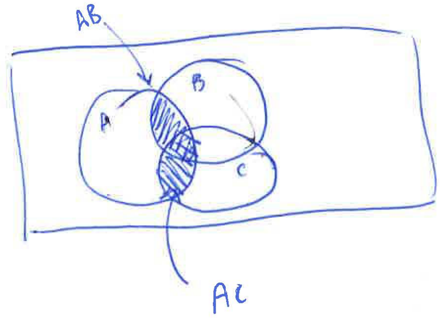
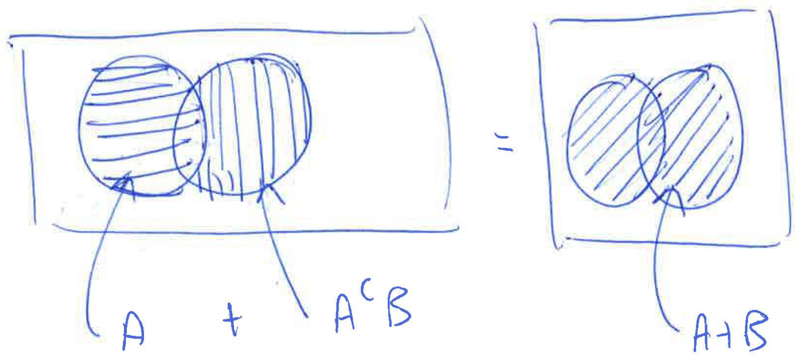
a)





b)

$A + A^c B = A + B$



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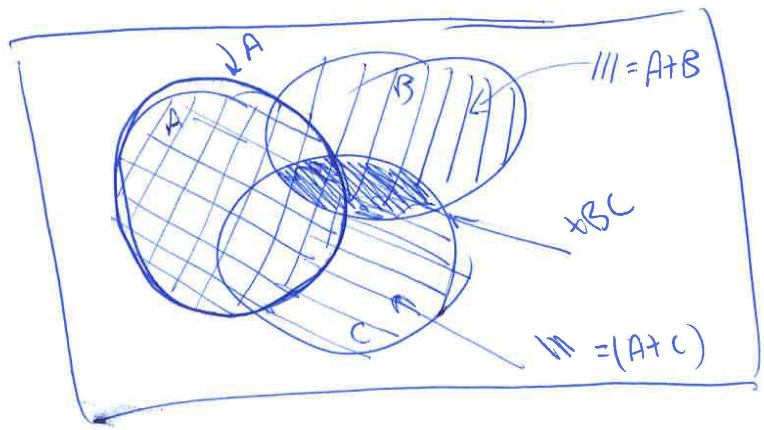
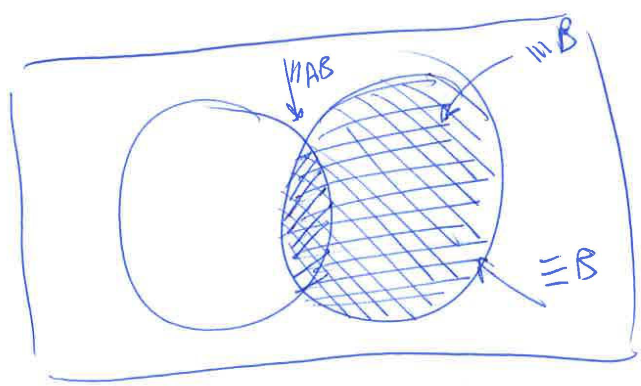
c $(A+B)(A+C) = A+BC$



$AB+B = B$



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3. (Modified T&T 2.9) The following events and their probabilities are listed below

Event	A	B	C	D
Probability	1/2	1/3	¼	¼

Compute

$a)\Pr\left[A+B\right]$ assuming $AB=\varnothing$.
$b)\Pr\left[A+B\right]$ assuming A and B are independent
$c)$ Is it possible that A,B,C,D are mutually exclusive? Explain why or why not.

Solution:

$$\begin{array}{c|cccc} \Pr & A & B & C & D \\ \hline p & 1/4 & 1/3 & 1/4 & 1/4 \\ & & & & \end{array}$$

$\Pr\left[A+B\right]$ given that A,B are mutually exclusive.
$\Pr\left[A+B\right]=\Pr\left[A\right]+\Pr\left[B\right]-\Pr\left[AB\right]=\Pr\left[A\right]+\Pr\left[B\right]-0$
$=\left(1/4\right)+\left(1/3\right)-0=7/12$

$\Pr\left[A+B\right]$ given that A and B are independent
$\Pr\left[A+B\right]=\Pr\left[A\right]+\Pr\left[B\right]-\Pr\left[AB\right]=\Pr\left[A\right]+\Pr\left[B\right]-\Pr\left[A\right]\Pr\left[B\right]$
$=1/4+1/3-(1/3)(1/4)=7/12-1/12=1/2.$

Is it possible that A,B,C,D are mutually exclusive?
No! Assume A,B,C,D are mutually exclusive, then
$\Pr\left[A+B+C+D\right]=\Pr\left[A\right]+\Pr\left[B\right]+\Pr\left[C\right]+\Pr\left[D\right]=13/12>1$ (!!).
So they can't be mutually exclusive.

This is an example of a "proof by contradiction".
We proved that A,B,C,D were not
mutually exclusive by assuming that they *were* mutually exclusive and then showing
that a contradition $\left(\Pr\left[A+B+C+D\right]>1\right)$ occurred.

4. Consider the following propositions and prove (like in CMSC203!) the appropriate conclusions:

If A and B are independent events, prove that A^C and B^C are also independent.
In formal terms: A, B independent $\rightarrow A^C, B^C$ independent?
Is the *converse* true, i.e., A^C, B^C independent $\rightarrow A, B$ independent? If so, prove it, if not, provide a counterexample.

Solution:

If A and B are independent events, prove that A^C and B^C are also independent.
In formal terms: A, B independent $\rightarrow A^C, B^C$ independent?
Is the *converse* true, i.e., A^C, B^C independent $\rightarrow A, B$ independent?

Assume A, B are independent.
$A^C B^C = (A+B)^C$, by DeMorgan's Law
$\Pr[A^C B^C] = 1 - \Pr\left(A+B\right)$, by defn of probability
$= 1 - \left(\Pr[A] + \Pr[B] - \Pr[AB]\right) = 1 - \Pr[A] - \Pr[B] + \Pr[A]\Pr[B]$, by algebra and independence
$= \left(1 - \Pr[A]\right)\left(1 - \Pr[B]\right)$ by algebra for real numbers
$= \Pr[A^C]\Pr[B^C]$, by defn of probability.
Therefore, $\Pr[A^C B^C] = \Pr[A^C]\Pr[B^C]$, and A^C is independent of B^C.
QED.

In the other direction, assume A^C, B^C are independent.
Let $C = A^C, D = B^C$, so that C^C=A, $D^C = B$, and $C, D, C^C, D^C \in \mathcal{S}$ by algebra of events.
Then C, D independent $\rightarrow C^C, D^C$ independent by our previous proof. But $C^C = \left(A^C\right)^C = A$, and similarly $D^C = B$. So A^C, B^C independent $\rightarrow A, B$ independent. This proves the converse.
Thus, A, B independent $\leftrightarrow A^C, B^C$ are independent.
QED.