CMPE320 Spring 2022
Homework #5 Solutions

Project 1 is due at 11:59 PM on March 1, 2021. This homework is due WEDNESDAY, March 3. HW05 will be due MONDAY, March 8 to get us back on schedule.

1) The noise in a communication system is known to be Additive White Gaussian Noise with mean zero and variance $\sigma^2 = 1\times 10^{-8}\ \mathrm{V}^2$. What is the form of the pdf of the noise power? *Hint: Noise voltage is Gaussian,* $P \propto V^2$.

$$V \sim N(0,\sigma^2)$$

$$P = aV^2, P \geq 0, a > 0$$

$$V = \pm\sqrt{\frac{P}{a}} \text{ (could be negative voltage)}$$

$$\frac{dP}{dV} = 2av$$

$$f_P(p) = \frac{1}{\left|2a\sqrt{\frac{p}{a}}\right|} f_N\left(\sqrt{\frac{p}{a}}\right) + \frac{1}{\left|2a\left(-\sqrt{\frac{p}{a}}\right)\right|} f_N\left(\sqrt{\frac{p}{a}}\right)$$

$$\frac{1}{\left|2a\sqrt{\frac{p}{a}}\right|} = \frac{1}{\left|2\sqrt{ap}\right|} = \frac{1}{2\sqrt{ap}} = \frac{1}{\left|2a\left(-\sqrt{\frac{p}{a}}\right)\right|}$$

$$f_P(p) = \frac{1}{2\sqrt{ap}} \times \frac{1}{\sqrt{2\pi\sigma^2}}\left[e^{-\left(\sqrt{\frac{p}{a}}-0\right)^2/2\sigma^2} + e^{-\left(-\sqrt{\frac{p}{a}}-0\right)^2/2\sigma^2}\right] = \frac{1}{2\sqrt{ap}} \times \frac{1}{\sqrt{2\pi\sigma^2}}\left[e^{-\left(\sqrt{\frac{p}{a}}\right)^2/2\sigma^2} + e^{-\left(-\sqrt{\frac{p}{a}}\right)^2/2\sigma^2}\right]$$

$$= \frac{1}{2\sqrt{ap}} \times \frac{1}{\sqrt{2\pi\sigma^2}}\left[e^{-p/(2a\sigma^2)} + e^{-p/(2a\sigma^2)}\right] = \frac{1}{\sqrt{ap}} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-p/(2a\sigma^2)}, p \geq 0, a > 0$$

Substituting for σ^2,

$$f_P(p) = \frac{1}{\sqrt{2\pi ap \times 10^{-8}}} e^{-p/(2a\times 1\times 10^{-8})}, p \geq 0, a > 0$$

The factor of $\frac{1}{\sqrt{p}}$ doesn't feel right, because it goes to infinite at $p = 0$. To check, let's integrate $f_P(p)$ to see if we get 1.

$$\int_0^\infty \frac{1}{\sqrt{2\pi a\sigma^2 p}} e^{-\frac{p}{2a\sigma^2}}\, dp$$

Let $u = \sqrt{p},\ u^2 = p$

$du = \frac{1}{2} \times \frac{1}{\sqrt{p}} dp$, or $2du = \frac{1}{\sqrt{p}} dp$

$p = 0 \rightarrow u = 0$

$p = \infty \rightarrow u = \infty$

Changing variables, $\int_0^\infty \frac{1}{\sqrt{2\pi a\sigma^2 p}} e^{-\frac{p}{2a\sigma^2}}\, dp = \frac{1}{\sqrt{2\pi a\sigma^2}} \int_0^\infty \frac{1}{\sqrt{p}} e^{-\frac{p}{2a\sigma^2}}\, dp$

$= \frac{2}{\sqrt{2\pi a\sigma^2}} \int_0^\infty e^{-\frac{u^2}{2a\sigma^2}}\, du = \frac{1}{\sqrt{2\pi a\sigma^2}} \int_{-\infty}^\infty e^{-\frac{u^2}{2a\sigma^2}}\, du$, which is the integral of a zero-mean Gaussian with variance $a\sigma^2$, which is equal to 1.

2) The probability density function of a random variable is given by

$$f_X(x) = \begin{cases} 0.5\sin x & 0 \le x \le c \\ 0.5\delta(x-1) & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the constant c and plot the pdf.

For any pdf, $\int_{-\infty}^{\infty} f_X(u)\, du = 1$. For this pdf, $\int_0^c 0.5\sin u\, du + \int_{1^-}^{1^+} 0.5\delta(u-1)\, du = 1$.

$-0.5\cos u\Big|_0^c + 0.5 \times 1 = 1$

$0.5\cos u\Big|_c^0 = 0.5$

$1 - \cos c = 1$

$\cos c = 0$

$c = \frac{\pi}{2}.$

$$f_X(x) = \begin{cases} 0.5\sin x & 0 \le x \le \frac{\pi}{2} \\ 0.5\delta(x-1) & x = 1 \\ 0 & \textit{otherwise} \end{cases}$$

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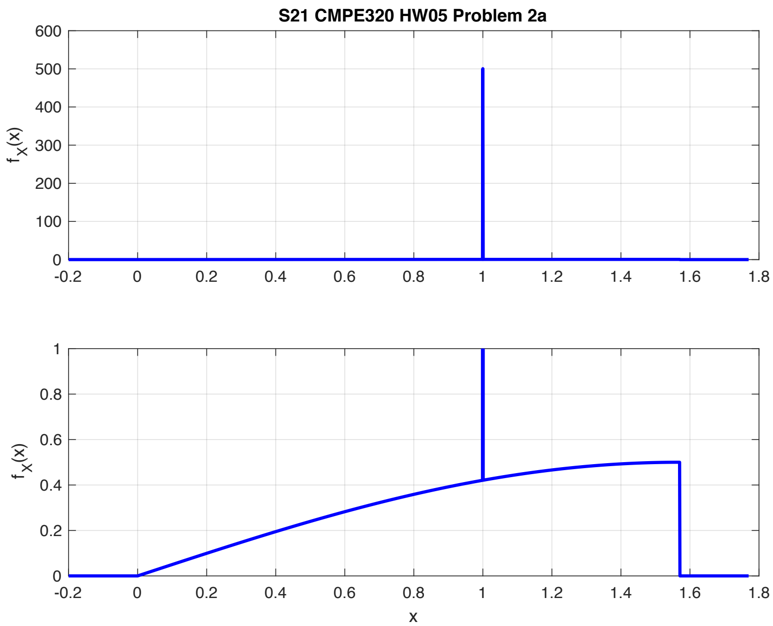


b) Compute and plot the CDF of x

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

For $0 \le x < 1$

$$\int_0^x 0.5\sin u\,du = -0.5\cos u\Big|_0^x = 0.5\left(1-\cos x\right)$$

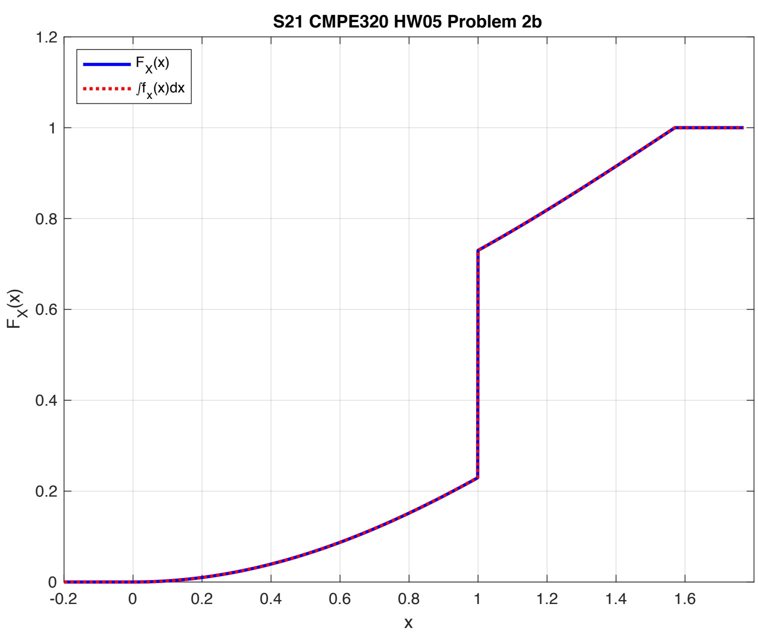
For $x = 1$,

$$\int_0^1 0.5\sin u\,du + \int_{1^-}^{1^+} 0.5\delta(u-1)\,du = 0.5\left(1-\cos 1\right) + 0.5 = 1 - 0.5\cos 1$$

for $1 < x \le \frac{\pi}{2}$

$$\int_0^x 0.5\sin u\,du + 0.5 = 0.5 + 0.5(1-\cos x) = 1 - 0.5\cos x$$

$$F_X(x) = \begin{cases} 0.5(1-\cos x) & 0 \le x < 1 \\ 1 - 0.5\cos 1 & x = 1 \\ 1 - 0.5\cos x & 1 < x \le \dfrac{\pi}{2} \\ 0 & otherwise \end{cases}$$



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3) The pdf of a random variable is given by

$$f_X(x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find and plot the CDF.

$$F_X(x) = \Pr\left[X \le x\right] = \int_{-\infty}^{x} f_X(u)du$$

$$= 0 \text{ for } x < 0$$

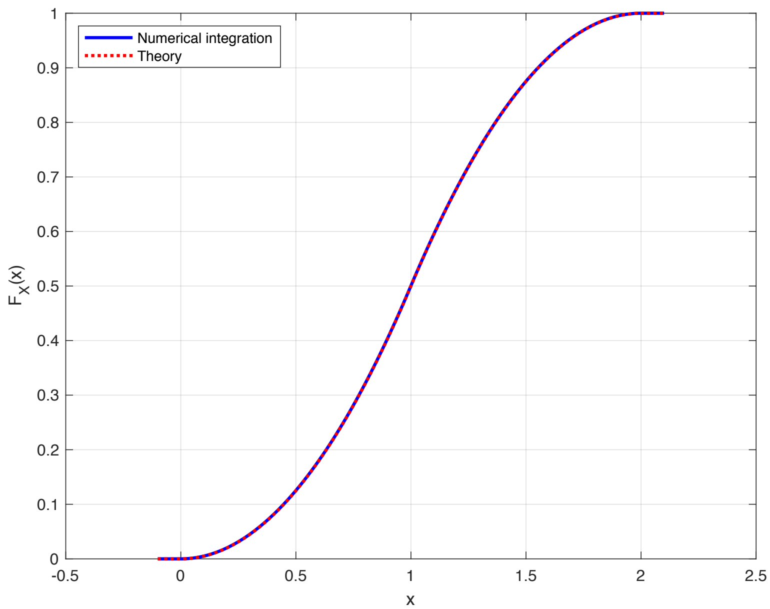
$$= \int_0^x u\,du = \left.\frac{u^2}{2}\right|_0^x = \frac{x^2}{2},\ 0 \le x < 1$$

$$= \int_0^1 u\,du + \int_1^x (2-u)\,du = \frac{1}{2} + \left(2u - \frac{u^2}{2}\right)_1^x = \frac{1}{2} + 2\left(x-1\right) - 0 - \left(\frac{x^2}{2} - \frac{1}{2}\right) = \frac{1}{2} + 2x - 2 - \frac{x^2}{2} + \frac{1}{2}$$

$$= \frac{1}{2} + (2x-2) - 0 - \frac{x^2}{2} + \frac{1}{2} = 2x - 1 - \frac{x^2}{2},\ 1 \le x < 2$$

$$= \int_0^1 u\,du + \int_1^2 (2-u)\,du = 2(2) - 1 + \frac{2^2}{2} = 4 - 1 - 2 = 1 \text{ for } x \ge 2$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{2} & 0 \le x < 1 \\ 2x - 1 - \dfrac{x^2}{2} & 1 \le x < 2 \\ & x \ge 2 \end{cases}$$



b) Find the following probabilities

$a)\ \Pr\left[0.5 \le X < 1.5\right]$

$= F_X(1.5) - F_X(0.5) = \int_{0.5}^{1.5} f_X(u)\,du = \int_{0.5}^{1} u\,du + \int_{1}^{1.5} (2-u)\,du$

$= \underbrace{\left(2(1.5) - 1 - \frac{1.5^2}{2}\right)}_{F_X(1.5)} - \underbrace{\left(\frac{0.5^2}{2}\right)}_{F_X(0.5)} = \left(3 - 1 - \frac{2.25}{2}\right) - \frac{0.25}{2} = 2 - 1.125 - 0.125 = 2 - 1.25 = 0.75$

$b)\ \Pr[1 \le X < 5]$

$= F_X(5) - F_X(1) = \int_{1}^{2} f_X(u)\,du + \int_{2}^{5} f_X(u) = \int_{1}^{2} f_X(u)\,du = \left(2(2) - \frac{2^2}{2}\right) - \left(2(1) - \frac{1}{2}\right) = 2 - (1.5) = 0.5$

$c)\ \Pr[0.75 \le X < 0.7501]$

Let $dx = 0.0001,\ f_X(0.75)dx = \Pr[0.75 \le X < 0.75 + 0.0001] = 0.75 \times 0.0001 = 0.0075$

$d)\ \Pr[0.5 \le X < 1.5 \mid X < 1]$

$\Pr[0.5 \le X < 1.5 \mid X < 1] = \frac{\Pr[0.5 \le X < 1.5 \text{ and } X < 1]}{\Pr\left[X < 1\right]} = \frac{\Pr\left[0.5 \le X < 1\right]}{\Pr\left[X < 1\right]} = \frac{F_X(1) - F_X(0.5)}{F_X(1)}$

$= \frac{0.5 - 0.125}{0.5} = 0.75$

$e)\ \Pr[1 \le X < 5 \mid X > 1.5]$

$\Pr[1 \le X < 5 \mid X > 1.5] = \frac{\Pr[1 \le X < 5 \text{ and } X > 1.5]}{\Pr\left[X > 1.5\right]} = \frac{\Pr\left[1.5 \le X < 2\right]}{\Pr\left[X > 1.5\right]} = \frac{F_X(2) - F_X(1.5)}{1 - F_X(1.5)}$

$= \frac{1 - 0.875}{1 - 0.875} = 1.$ If we *know* $X > 1.5,$ then it is certain that $1 \le X < 5.$

4) Using the pdf of Problem 3, find the pdf of $Z = \log_2(X)$ *Hint: Remember that* $\log_a(x) = \frac{\log(x)}{\log(a)}$

Using the hint, $\log_2(x) = \dfrac{1}{\log 2}\log(x),$ where log is the natural log ("ln").

Using the General Method.

$Z = \log_2(X) \Rightarrow X = 2^Z$. This will be our $g^{-1}(z)$.

$$\frac{dz}{dx} = \frac{d}{dx}\left[\frac{1}{\log 2}\log(x)\right] = \frac{1}{\log 2}\frac{d\log(x)}{dx} = \frac{1}{x\log 2}$$

The General formula

$$f_Z(z) = \left[\frac{1}{|dz/dx|} f_X(x)\right]_{x=g^{-1}(z)}$$

When $0 < +x < 1,$ or $-\infty < z < 0,$

$$f_Z(z) = \left[\frac{1}{\left(\dfrac{1}{x\log 2}\right)} x\right]_{x=2^z} = x^2 \log 2\Big|_{x=2^z} = 2^{2z}\log 2$$

In this region, $0 < f_Z(z) < \log 2$.

When $1 \le x < 2,$ or $0 \le z < 1$

$$f_Z(z) = \left[\frac{1}{\left(\dfrac{1}{x\log 2}\right)}(2-x)\right]_{x=2^z} = \log 2\left[2x - x^2\right]_{x=2^z} = \log 2\left(2\times 2^z - 2^{2z}\right) = \log 2\left(2^{z+1} - 2^{2z}\right)$$

In this region, $\log 2 > f_Z(z) > 0$ Putting it together

$$f_Z(z) = \begin{cases} 2^{2z}\log 2 & -\infty < z < 0 \\ \log 2\left(2^{z+1} - 2^{2z}\right) & 0 \le z < 1 \\ 0 & \text{otherwise} \end{cases}$$

It's always a good idea to check the result:

$$f_Z(z) = \begin{cases} 2^{2z}\log 2 & -\infty < z < 0 \\ \log 2\left(2^{z+1} - 2^{2z}\right) & 0 \le z < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_Z(u)\,du = \int_{-\infty}^{0} 2^{2u}\log 2\,du + \int_{0}^{1} \log 2\left(2^{u+1} - 2^{2}\right)du$$

Remembering that $2 = e^{\log 2}$,

$$\int_{-\infty}^{0} e^{2\log 2u}\log 2\,du + \int_{0}^{1} \log 2\left(e^{(u+1)\log 2} - e^{2\log 2u}\right)du$$

$$= \frac{\log 2}{2\log 2}e^{2\log 2u}\Big|_{-\infty}^{0} + \frac{e^{\log 2}\log 2}{\log 2}e^{\log 2u}\Big|_{0}^{1} - \frac{\log 2}{2\log 2}e^{2\log 2u}\Big|_{0}^{1}$$

$$= \frac{1}{2}\left(2^0 - 0\right) + 2\left(2^1 - 2^0\right) - \frac{1}{2}\left(2^2 - 2^0\right)$$

$$= \frac{1}{2} + 4 - 2 - \frac{4}{2} + \frac{1}{2} = \frac{1}{2} + 4 - 2 - 2 + \frac{1}{2} = 1(!!!)$$