CMPE320 Spring 2022
Homework 06 Assigned 3/09/2022 Due 3/16/2022

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1. (T&T 2nd 5.4) A joint PMF is defined by

$$f_{K_1K_2}(k_1,k_2) = \begin{cases} C & 0 \le k_1 \le 5,\ 0 \le k_2 \le k_1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant, C.

It's a pmf, so k_1 and k_2 are discrete.

$\sum_{k_1=0}^{5}\sum_{k_2=0}^{k_1} C = 1$. Note that the sums start at 0, and that k_2 sum doesn't include a k_2 value. Therefore, the inner sum is equal to $(k_1+1)C$, with the extra C because we're summing from zero and not 1.

$$\sum_{k_1=0}^{5}(k_1+1)C = \sum_{k_1=0}^{5}k_1C + \sum_{k_1=0}^{5}C = 0C + \sum_{k_1=1}^{5}k_1C + (5+1)C$$

$$= 6C + \sum_{k_1=1}^{5}k_1C = 6C + \frac{5(5+1)}{2}C = 6C + 15C = 21C = 1$$

Therefore $C = \frac{1}{21}$.

(b) Compute and sketch the marginal PMFs, $f_{K_1}(k_1), f_{K_2}(k_2)$. Are the random variables K_1, K_2 independent?

$$f_{K_1}(k_1) = \sum_{k_2=0}^{k_1} C = (k_1+1)C = \frac{k_1+1}{21}\ ,\ 0 \le k_1 \le 5$$

Just to check $\sum_{k_1=0}^{5} f_{K_1}(k_1) = \sum_{k_1=0}^{5}\frac{k_1+1}{21} = \frac{15}{21} + \frac{6}{21} = 1$

$$f_{K_2}(k_2) = \sum_{k_1=0}^{5-k_2} C = \frac{(5-k_2+1)}{21} = \frac{6-k_2}{21}, 0 \le k_2 \le 5$$

Just to check $\sum_{k_1=0}^{5} f_{K_2}(k_2) = \sum_{k_2=0}^{5}\frac{6-k_2}{21} = \frac{6\times 6}{21} - \frac{5(5+1)}{2\times 21} = \frac{36}{21} - \frac{15}{21} = 1$

$f_{K_1K_2}(k_1,k_2) \ne f_{K_1}(k_1)f_{K_2}(k_2)$, therefore K_1 and K_2 are not independent.

(c) Compute the conditional PMFs $f_{K_1|K_2}(k_1 \mid k_2), f_{K_2|K_1}(k_2 \mid k_1)$ and show that both satisfy the characteristics of a PMF.

$$f_{K_1|K_2}(k_1|k_2) = \frac{f_{K_1K_2}(k_1,k_2)}{f_{K_2}(k_2)} = \frac{(1/21)}{(6-k_2)/21} = \frac{1}{6-k_2},\ 0 \le k_1 \le 5-k_2$$

Checking: fix k_2, $\sum_{k_1=5-k_2}^{5} \frac{1}{6-k_2}$, let $m = 5-k_2$, then when $k_1 = 5, m = 0$, when $k_1 = 5-k_1, k_1 = m$. Also $6-k_2 = 1+(5-k_2) = m+1$.

Then $\sum_{k_1=5-k_2}^{m} \frac{1}{6-k_2} = \sum_{k_1=0}^{m} \frac{1}{m+1} = \frac{m+1}{m+1} = 1.$

$$f_{K_2|K_1}(k_2|k_1) = \frac{f_{K_1K_2}(k_1,k_2)}{f_{K_1}(k_1)} = \frac{1/21}{(k_1+1)/21} = \frac{1}{k_1+1}, 0 \le k_2 \le k_1$$

Checking: fix $k_1, \sum_{k_2=0}^{k_1} \frac{1}{k_1+1} = \frac{k_1+1}{k_1+1} = 1$

The key issues here are the limits on the sums. $0 \le k_2 \le k_1$ is clear from the conditions on the problems. Fixing a value of k_2, means that $5 \ge k_1 \ge 5-k_2$. Changing the indices for the sum via the use of the interim variable m is straightforward.

2. (T&T 2nd 5.8) A joint PDF is given to be

$$f_{X_1X_2}(x_1,x_2) = \begin{cases} \frac{1}{24} & -3 \le x_1 \le 3, -2 \le x_2 \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine $\Pr\left[-2 \le X_1 \le 1, X_2 \ge 0.5\right]$

This is a pdf, thus a continuous (or piecewise continuous) function.

$\Pr\left[-2 \le X_1 \le 1, -2 \le X_2 \le 2\right] = \int_{-2}^{1}\int_{-2}^{2} f_{X_1X_2}(x_1,x_2)\,dx_2\,dx_1$. The limits of integration are within the limits in the pdf definition, so we just proceed.

$$\Pr\left[-2 \le X_1 \le 1, -2 \le X_2 \le 2\right] = \int_{-2}^{1}\int_{-2}^{2} \frac{1}{24}\,dx_2\,dx_1 = \int_{-2}^{1} \frac{4}{24}\,dx_1 = \frac{4\times(1-(-2))}{24} = \frac{12}{24} = \frac{1}{2}$$

(b) Determine $\Pr\left[X_1 < 2X_2\right]$

$$\Pr\left[X_1 \le 2X_2\right] = \int_{-3}^{3}\int_{-2}^{2x_2} \frac{1}{24}\,dx_1\,dx_2 = \int_{-3}^{3} \frac{2x_2-(-2)}{24}\,dx_1 = \left.\frac{2x_2^2}{2\times 24}\right|_{-3}^{3} + \left.\frac{2x_2}{24}\right|_{-3}^{3}$$

$$= \frac{9-9}{24} + \frac{2(3-(-3))}{24} = 0 + \frac{12}{24} = \frac{1}{2}$$

Limits are the total range of x_2, but only the range from the minimum of x_1 to $2x_2$.
The x_1 integral must be done first, as the limit depends on x_2.

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3. (Extended T&T 2nd 5.9) The joint PDF for two random variables is

$$f_{X_1X_2}(x_1,x_2)=\begin{cases} C(4-x_1x_2) & 0\le x_1\le 4, 0\le x_2\le 1 \\ 0 & \text{otherwise}\end{cases}$$

(a) Find C to make this a valid PMF

$$1=\int_0^4\int_0^1 C\left(4-x_1x_2\right)dx_2\,dx_1=C\int_0^4\left(4-\frac{x_1}{2}\right)dx_1=$$

$$C\left(4x_1-\frac{x_1^2}{4}\right)_0^4=C(16-4)=12C;\ C=\frac{1}{12}$$

(b) Find the marginal density functions for X_1 and X_2. Show that they meet the requirements for a PDF.

$$f_{X_1}(x_1)=\frac{1}{12}\int_0^1(4-x_1x_2)\,dx_2=\frac{1}{12}\left(4x_2-\frac{x_1x_2^2}{2}\right)_0^1=\frac{1}{12}\left(4-\frac{x_1}{2}\right)$$

$$\int_0^4 f_{X_1}(x_1)dx_1=\int_0^4\frac{1}{12}\left(4-\frac{x_1}{2}\right)dx_1=\frac{1}{12}\left(4x_1-\frac{x_1^2}{4}\right)_0^4=\frac{1}{12}\left(16-4\right)=1$$

$$f_{X_2}(x_2)=\frac{1}{12}\int_0^4(4-x_1x_2)\,dx_1=\frac{1}{12}\left(4x_1-\frac{x_2x_1^2}{2}\right)_0^1=\frac{1}{12}\left(16-8x_2\right)$$

$$\int_0^1 f_{X_2}(x_2)dx_2=\int_0^1\frac{1}{12}\left(16-8x_2\right)dx_2=\frac{1}{12}\left(16x_2-8\frac{x_2^2}{2}\right)_0^1=\frac{1}{12}\left(16-4\right)=1$$

(c) Are the random variables independent.

$$\text{No}, f_{X_1X_2}(x_1,x_2)=\frac{1}{144}\left(4-\frac{x_1}{2}\right)\left(16-8x_2\right)$$

$$=\frac{1}{144}\left(64-8x_1-32x_2+4x_1x_2\right)\neq\frac{1}{12}(4-x_1x_2)$$

(d) Find the conditional expected values of $E\left[X_1\,|\,X_2\right]$ and $E\left[X_2\,|\,X_1\right]$.

$$E\left[X_1 \mid X_2\right] = \int x_1 f_{X_1|X_2}(x_1 \mid x_2)\,dx_1 = \int x_1 \frac{f_{X_1X_2}(x_1,x_2)}{f_{X_2}(x_2)}\,dx_1 = \int x_1 \frac{12(4-x_1x_2)}{16-8x_2}\,dx_1$$

$$= \frac{12}{16-8x_2}\int_0^4 4x_1 - x_1^2 x_2\,dx_1 = \frac{12}{16-8x_2}\left(\frac{4(4^2)}{2} - \frac{4^3 x_2}{3}\right) = \frac{12}{16-8x_2}\left(\frac{64}{2} - \frac{64x_2}{3}\right)$$

$$= \frac{12\times 64}{16-8x_2}\left(\frac{1}{2} - \frac{x_2}{3}\right) = \frac{768}{16-8x_2}\left(\frac{1}{2} - \frac{x_2}{3}\right)$$

$$E\left[X_2 \mid X_1 =\right] = \int x_2 f_{X_2|X_1}(x_2 \mid x_1)\,dx_2 = \int x_2 \frac{f_{X_1X_2}(x_1,x_2)}{f_{X_1}(x_1)}\,dx_1 = \int x_2 \frac{12(4-x_1x_2)}{4-\frac{x_1}{2}}\,dx_2$$

$$\frac{12}{4-\frac{x_1}{2}}\int_0^1 x_2(4-x_1x_2)\,dx_2 = \frac{12}{4-\frac{x_1}{2}}\left(\frac{4x_2^2}{2} - \frac{x_1x_2^3}{3}\right)_0^1 = \frac{12}{4-\frac{x_1}{2}}\left(\frac{4}{2} - \frac{x_1}{3}\right) = \frac{12\left(2-\frac{x_1}{3}\right)}{4-\frac{x_1}{2}}$$

$$== \frac{48-8x_1}{8-x_1} = \frac{8(6-x_1)}{(8-x_1)}$$

Note that both of the conditional expectations are functions of the conditioning variable, because the whole idea of conditioning is that we *know* the value of that variable.

4 . Compute the entropy of the following sources

$$\text{Source } X, \text{ with } f_X(x) = \begin{cases} 1/3 & x = 0 \\ 1/3 & x = 1 \\ 1/3 & x = 2 \end{cases}$$

$$H(X) = -E\left[\log_2(f_X(s))\right] = -\sum_x \log_2(f_X(x)) f_X(x)$$

$$= -\left(\log_2(1/3)\frac{1}{3} + \log_2(1/3)\frac{1}{3} + \log_2(1/3)\frac{1}{3}\right)$$

$$= -\log_2(1/3) = \log_2(3) = 1.585 \text{ bits.}$$

$$\text{Source } W \text{ with } f_W(w) = \begin{cases} 1/3 & w = -1 \\ 1/3 & w = -5 \\ 1/3 & w = -10 \end{cases}$$

The entropy, $H(W)$, depends on the PMF, not the actual values of the random variable. Therefore this answer is the same as the previous answer, or $H(W) = 1.585$ bits.

Source Y with $f_Y(y) = \begin{cases} 0.25 & y = 0 \\ 0.5 & y = 1 \\ 0.25 & y = 2 \end{cases}$

$$H(Y) = -E\left[\log_2(f_Y(y))\right] = -\sum_y \log_2(f_Y(y)) f_Y(y)$$
$$= -\left(\log_2\left(0.25\right) \times 0.25 + \log_2(0.5) \times 0.5 + \log_2(0.25) \times 0.25\right)$$
$$= -(-2 \times 0.25 + -1 \times 0.5 + -2 \times 0.25) = -\left(-0.5 - 0.5 - 0.5\right) = 1.5 \text{ bits.}$$

Source Z with $f_Z(z) = \begin{cases} 0.25 & z = -10 \\ 0.25 & z = +10 \\ 0.5 & z = +20 \end{cases}$

As with $H(W)$ and $H(X)$, the entropy depends only on the PMF.
Since the values of the PMF are the same for RV Z and RV Y, they have the same entropy and $H(Z) = 1.5$ bits.

Comparing Source X with Source W, and source Y with Source Z, what conclusions can you draw about the dependence of the entropy of a discrete RV with the *values* of that random variable?