CMPE320 Spring 2022
Homework #3 Assigned Feb 16, 2022 Due Feb 23, 2022.

Don't forget to start work on Project 1, due February 28, 2022!

1. T&T Problem 2.19 In the serial transmission of a byte of information (e.g. over a network) errors in each bit occur independently with probability q . For every 8-bit byte sent, a parity check bit is appended so that each byte transmitted actually consists of 9 bits. The parity bit is chosen so the group of 9 bits has "even parity", i.e., the number of 1's contained in the 9 bit group is even. Errors can be detected by checking the parity of the received 9 bit sequence.

   a) What is the probability that a single error occurs in a 9-bit sequence?
   The problem of a single error in any N bit sequence is a binomial distribution problem. If the probability of an error on any bit is q, and the bits are assumed to be independent,
   $$\Pr\left[1 \text{ error in 9 bits }\right]=\binom{9}{1}q^1(1-q)^8$$

   b) What is the probability that the errors occur but are not detected? Write your answer as an expression involving the error probability, q.
   This problem uses even parity across 9 bits. So the probability that an undetectable error occurs is the probability that an even number of errors occurs, because such an error will not affect the parity.
   $$\Pr\left[\text{undetectable error}\right]=\sum_{k=2,4,6,8}\binom{9}{k}q^k(1-q)^{9-k}$$
   $$=\binom{9}{2}q^2(1-q)^7+\binom{9}{4}q^4\left(1-q\right)^5+\binom{9}{6}q^6(1-q)^3+\binom{9}{8}q^8(1-q)$$

2. T&T Problem 2.33 In the process of transmitting binary data over a certain noisy communication channel, it is found that the errors occur in two-bit bursts. Given that a single bit error occurs, the probability that the *next* bit is in error is *twice* the probability of a single bit error. If it is known that two consecutive errors occur with probability 2×10^{-4}, what is the probability of a single bit error?

   Let b_k be the event that there is an error on the k-th bit
   $$\Pr\left[b_k\right]=p$$
   $$\Pr\left[b_{k+1}\,|\,b_k\right]=2p$$
   $$\Pr\left[b_{k+1}b_k\right]=\Pr\left[b_{k+1}\,|\,b_k\right]\Pr\left[b_k\right]=2\times10^{-4}$$
   $$=\left(2p\right)p=2p^2=2\times10^{-4}\Rightarrow p=0.01$$

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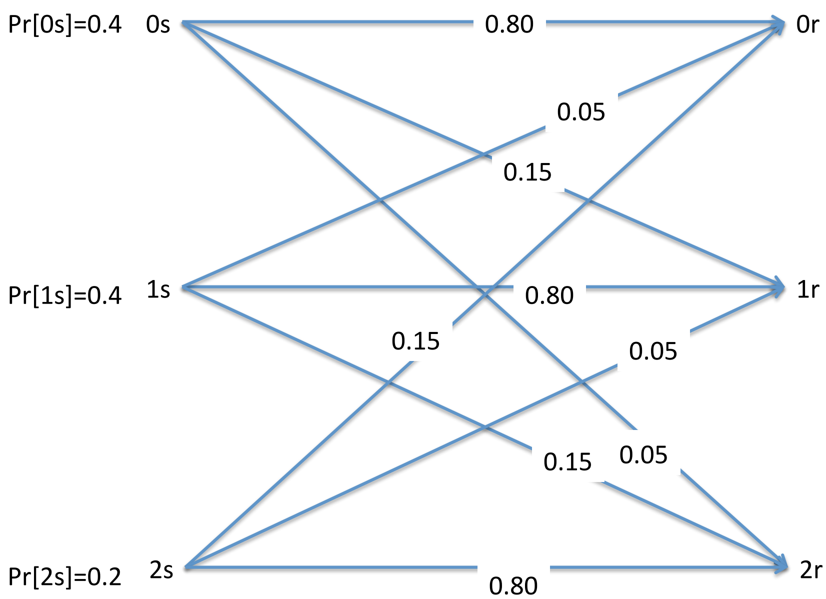
3. T&T Problem 2.46 Never-Ready Wireless has just invented a new signaling system known as 3PSK. The system involves transmitting one of three symbols $\{0,1,2\}$. The conditional probabilities for their network are given by the following transition table:

$\Pr[R\,|\,S]$	0_R	1_R	2_R
0_S	0.80	0.15	0.05
1_S	0.05	0.80	0.15
2_2	0.15	0.05	0.80

The prior probabilities of the symbols are known to be $\Pr[0_S]=0.4, \Pr[1_S]=0.4, \Pr[2_S]=0.2$.

a) Draw the trinary channel diagram for this system (*Hint: Just extend the BSC from lecture*)



b) What is the probability of an error, given that a 0 was sent?

$$\Pr[\text{error}|0\text{s}] = \Pr[1\text{r}|0\text{s} + 2\text{r}|0\text{s}]$$

$(1\text{r}|0\text{s}) \cap (2\text{r}|0\text{s}) = \varnothing$, so

$$\Pr[\text{error}|0\text{s}] = \Pr[1\text{r}|0\text{s}\ ] + \Pr[+2\text{r}|0\text{s}] = 0.15 + 0.05 = 0.2$$

Or

$$\Pr[\text{error}|0\text{s}] = 1 - \Pr[\text{no error }|0\text{s}] = 1 - \Pr[0\text{r}|0\text{s}] = 1 - 0.8 = 0.2$$

c) What is the (unconditional) probability of an error. *Hint: Consider all the terms*

The Principle of Total Probability says that if $\{A_k\}$ is a partition of $\boldsymbol{\mathcal{S}}$

$$\Pr[B] = \sum_k \Pr[A_k B] = \sum_k \Pr[B \mid A_k]\Pr[A_k]$$

In this problem, $A_k = \{k \text{ sent}\}, k = 0,1,2$ is a partition, and $B = \{\text{error}\}$,

so $\Pr[\text{error}] = \sum_{k=0}^{2} \Pr[\text{error}|\text{k sent}]\Pr[\text{k sent}]$

$$= \left(\Pr[\text{1r}|\text{0s}] + \Pr[\text{2r}|\text{0s}]\right)\Pr[\text{0s}] + \left(\Pr[0r \mid 1s] + \Pr[2r \mid 1s]\right)\Pr[1s]$$
$$+ \left(\Pr[0r \mid 2s] + \Pr[1r \mid 2s]\right)\Pr[2s]$$
$$= (0.15 + 0.05)0.4 + (0.15 + 0.05)0.4 + (0.15 + 0.05)0.2$$
$$= 0.2(0.4 + 0.4 + 0.2) = 0.2$$

d) Given that a 1 is received, what is the probability that a 1 was sent?

Hint: Use Bayes' Rule and the Principle of Total Probability

$$\Pr[1s \mid 1r] = \frac{\Pr[1s1r]}{\Pr[1r]} = \frac{\Pr[1r \mid 1s]\Pr[1s]}{\sum_{k=0,1,2} \Pr[1r \mid ks]\Pr[ks]}$$
$$= \frac{(0.8)(0.4)}{(0.15)(0.4) + (0.8)(0.4) + (0.05)(0.2)} = \frac{0.32}{0.06 + 0.32 + 0.01}$$
$$= \frac{0.32}{0.39} = 0.8205$$

e) Given that a 1 is received, what is the probability that this was an error?

Hint: Use Bayes' Rule and the Principle of Total Probability

$$\Pr[error \mid 1r] = \Pr[0s \mid 1r + 2s \mid 1r]$$
$$(0s \mid 1r) \cap (2s \mid 1r) = \varnothing,$$
$$\Pr[error \mid 1r] = \Pr[0s \mid 1r] + \Pr[2s \mid 1r]$$
$$= \frac{\Pr[0s1r]}{\sum_{k=0,1,2} \Pr[1r \mid ks]\Pr[ks]} + \frac{\Pr[2s1r]}{\sum_{k=0,1,2} \Pr[1r \mid ks]\Pr[ks]}$$
$$I = \frac{\Pr[1r \mid 0s]\Pr[0s] + \Pr[1r \mid 2s]\Pr[2s]}{\sum_{k=0,1,2} \Pr[1r \mid ks]\Pr[ks]}$$
$$= \frac{(0.15)(0.4) + (0.05)(0.2)}{(0.15)(0.4) + (0.8)(0.4) + (0.05)(0.2)} = \frac{0.06 + 0.01}{0.06 + 0.32 + 0.01} = \frac{0.07}{0.39} = 0.1795$$

Of course, we could just have said $\Pr[error \mid 1r] = 1 - \Pr[1s \mid 1r]$
$= 1 - 0.8205 = 0.1795,$ using the results of part d).

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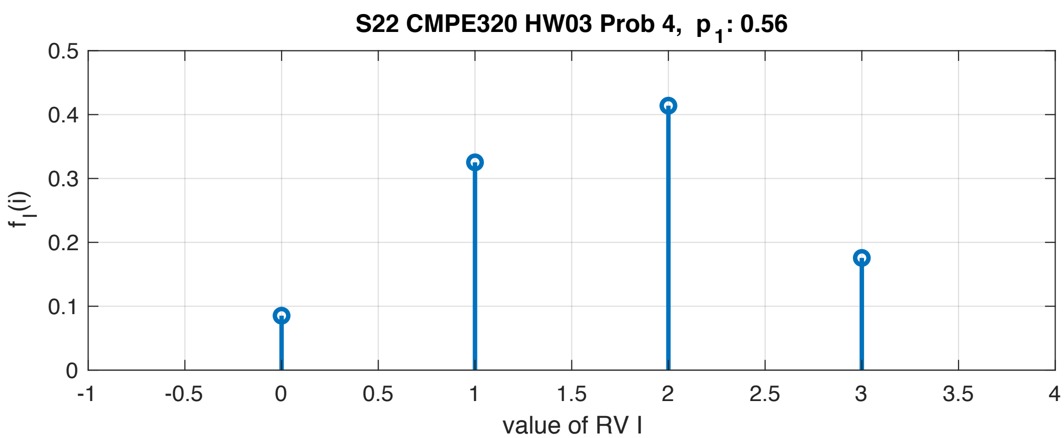
Hint: For problems 4 and 5, you might want to use MATLAB with the function stem(x,y) or bars(x,y) to plot the pmfs.

4. T&T Problem 3.5.
   Consider the transmission of packets with a fixed size of 3 bits. The probability of a binary 1 is 0.56, and the bits are independent within a packet. Let the number of binary 1s in a packet be the random variable I.
   a) Determine and plot the PMF of this random variable
      Random variable I is a count in a fixed number (3) of items, so this is Binomial PMF. We are given $p_1 = 0.56$, so $p_0 = (1-p_1) = 0.44$

$$f_I(i) = \begin{pmatrix} 3 \\ i \end{pmatrix} p_1^i p_0^{3-k},\ i = 0,1,2,3$$

      the plot is below.



   b) What is the probability that a given packet contains all binary 1's or all binary 0's?
      The $f_I(i)$ is the count of binary 1's. Therefore,

$$f_I(3) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} p_1^3 p_0^0 = 1 \times (0.56)^3 \times 1 = 0.1756 = \Pr\{\text{all 1's}\}$$

$$f_I(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} p_1^0 p_0^3 = 1 \times 1 \times (0.44)^3 = 0.0.0851 = \Pr\{\text{all 0's}\}$$

      Note that $\{\text{all 0's}\} \neq \{\text{all 1's}\}^C$

$$\{\text{all 1's}\}^C = \{\textit{any}\ \text{0's}\} = \{\text{not all 1's}\}$$

5. T&T Problem 3.10. For each of the following random variables, sketch the PMF, compute the probability that the random variable is greater than 5, and find the values of the parameter(s) such that $\Pr\left[K>5\right]\le 0.5$.

   a) geometric, $p=1/8$

   There are two versions of the geometric pmf (see text). Which one applies depends on the particular problem at hand.

   If we choose

   $f_K(k)=p(1-p)^{k-1}, k=1,2,3...,$ corresponding to $\left\{\text{first event occurs at } k\text{-th bit}\right\}$

   $$\Pr\left[K>5\right]=\sum_{k=6}^{\infty}p(1-p)^{k-1}=p\sum_{m=5}^{\infty}(1-p)^m=p(1-p)^5\sum_{n=0}^{\infty}(1-p)^n$$

   $$=p(1-p)^5\times\frac{1}{1-(1-p)}=\frac{p}{p}\times(1-p)^5=(1-p)^5$$

   For $p=0.125,\ (1-p)^5=0.5129$

   For $\Pr\left[K>5\right]\le 0.5,\ \left(1-p\right)^5\le 0.5,\ p\le 1-0.5^{0.2}=0.1294$

   If we choose

   $f_K(k)=p(1-p)^k, k=0,1,2...,$corresponding to $\left\{\text{first event occurs at } k+1\text{-st bit}\right\}$

   $$\Pr\left[k>5\right]=\sum_{k=6}^{\infty}p(1-p)^k=p(1-p)^6\sum_{n=0}^{\infty}(1-p)^n=(1-p)^6$$

   $(1-p)^6=0.4488$

   For $\Pr\left[K>5\right]\le 0.5,\ \left(1-p\right)^6\le 0.5,\ p\le 1-0.5^{0.1666}=0.1091$

   b) uniform, $m=0,n=7$ (that is, uniform discrete between 0 and 7, inclusive)

   $$f_K(k)=\begin{cases}0.125 & k=0,1,2,...,7\\ 0 & \text{otherwise}\end{cases}$$

   $$\Pr\left[K>5\right]=\sum_{k=6,7}f_K(k)=2\times 0.125=0.25$$

   For $\Pr\left[K>5\right]\le 0.5,$ we must have $\displaystyle\sum_{k=6}^{N}\frac{1}{N+1}=\frac{N-5}{N+1}\le 0.5$

   $N-5\le 0.5N+0.5$

   $0.5N\le 5.5, N=11,$ so $f_K(k)=\dfrac{1}{12}.$

   $$\Pr\left[K>5\right]=\sum_{k=6}^{11}\frac{1}{12}=\frac{6}{12}=0.5$$

   c) Poisson, $\alpha=2$ (remember that $\alpha=\lambda t$)

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$$a = 2, f_K(k) = \frac{2^k}{k!}e^{-2},\ k = 0,1,2,...$$

$$\Pr\left[K > 5\right] = \sum_{k=6}^{\infty}\frac{2^k}{k!}e^{-2}, \text{ but there's no closed form,}$$

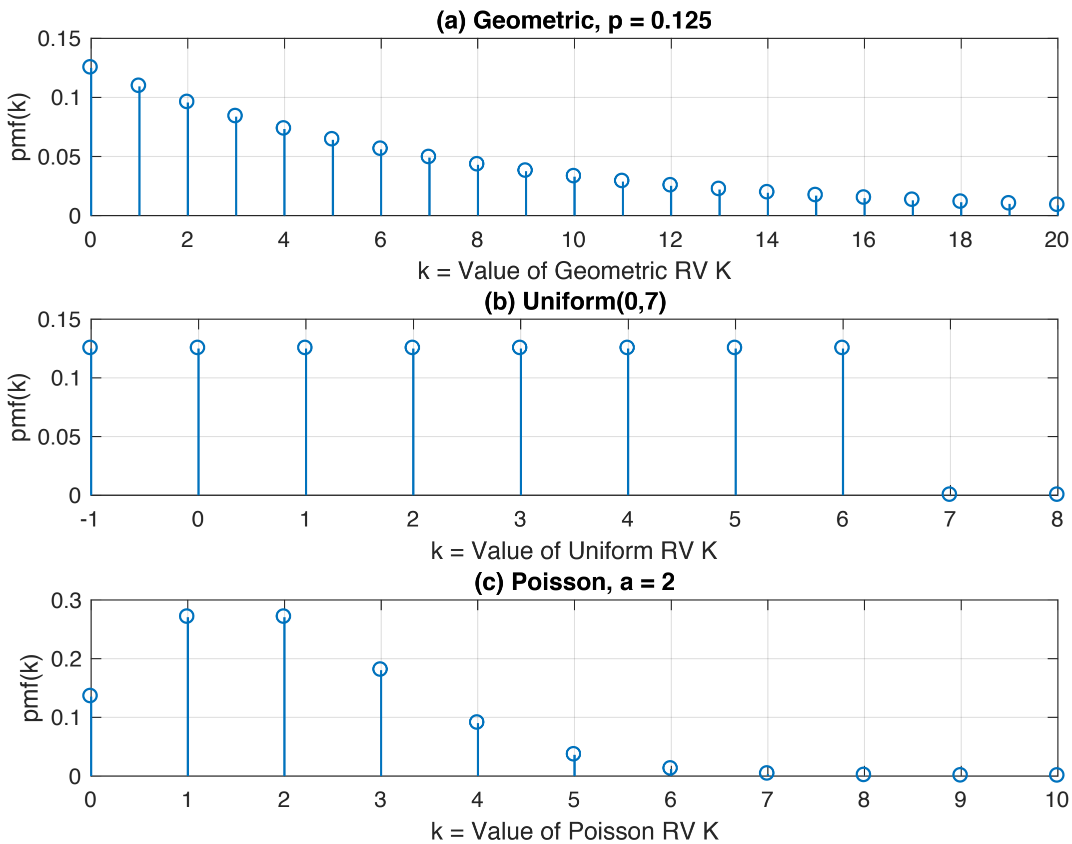
$$\text{MATLAB with 100 terms } \Pr\left[K > 5\right] = 0.0166$$

$$Or,$$

$$\Pr\left[K > 5\right] = 1 - \Pr\left[K \le 5\right] = 1 - \sum_{k=0}^{5}\frac{2^k}{k!}e^{-2}$$

$$= 1 - \left(0.1353\ +\ 0.2707\ +\ 0.2707\ +\ 0.1804\ + 0.0902\ + 0.0361\right)$$

$$= 1 - 0.9834 = 0.0166$$



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