CMPE320 Spring 2022
Homework 06 Assigned 3/24/2017 Due 3/31/2017

Before starting:

From lecture, the magic integrating tricks!

1) $$\int_0^\infty x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}} \text{ for } n = 0,1,2,...$$

2) $$\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma((m+1)/2)}{2a^{(m+1)/2}}$$

3) $$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/4a}$$

Solutions:

1. The Rayleigh pdf is defined as $f_X(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, x \geq 0$

a) Show that the given pdf is, in fact, a pdf.

$$\int_0^\infty \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx \text{ Use the second integrating trick, } m = 1, a = \frac{1}{2\sigma^2}$$

$$\int_0^\infty \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{\frac{m+1}{2}}\sigma^2} = \frac{\Gamma(1)}{2a\sigma^2} = \frac{0!}{2\left(\frac{1}{2\sigma^2}\right)\sigma^2} = 1.$$

$$x \geq 0, e^{-x^2/2\sigma^2} > 0 \Rightarrow f_X(x) \geq 0. \text{ Therefore this is a pdf}$$

b) Compute the mean.

$$E[X] = \int_0^\infty x \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx \text{ Use the second integrating trick, } m = 2, a = \frac{1}{2\sigma^2}$$

$$\int_0^\infty \frac{x^2}{\sigma^2} e^{-x^2/2\sigma^2} dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{\frac{m+1}{2}}\sigma^2} = \frac{\Gamma\left(\frac{3}{2}\right)}{2a^{\frac{3}{2}}\sigma^2} = \frac{\frac{\sqrt{\pi}}{2}}{2\left(\frac{1}{2\sigma^2}\right)^{3/2}\sigma^2} = \frac{\frac{\sqrt{\pi}}{2}}{2\left(\frac{1}{2\sigma^2}\right)\left(\frac{1}{2\sigma^2}\right)^{1/2}\sigma^2}$$

$$= \frac{\sqrt{\pi}}{2} \times \sqrt{2\sigma^2} = \sqrt{\frac{\pi}{2}}\sigma$$

c) Compute the mean square value

$$E\left[X^2\right]=\int_0^\infty x^2\frac{x}{\sigma^2}e^{-x^2/2\sigma^2}\,dx \text{ Use the second integrating trick, } m=2, a=\frac{1}{2\sigma^2}$$

$$\int_0^\infty \frac{x^3}{\sigma^2}e^{-x^2/2\sigma^2}\,dx=\frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{\frac{m+1}{2}}\sigma^2}=\frac{\Gamma\left(\frac{4}{2}\right)}{2a^{\frac{3}{2}}\sigma^2}=\frac{\Gamma(2)}{2\left(\frac{1}{2\sigma^2}\right)^2\sigma^2}=\frac{1!}{\left(\frac{1}{2\sigma^2}\right)}=2\sigma^2$$

d) Compute the variance

$$\operatorname{var}\left[X\right]=E\left[X^2\right]-\left(E\left[X\right]\right)^2=2\sigma^2-\left(\sqrt{\frac{\pi}{2}}\sigma\right)^2=2\sigma^2-\frac{\pi}{2}\sigma^2=\frac{4-\pi}{2}\sigma^2$$

e) The *mode* is the numerical value $X=x_{Mode}$ for which $\Pr\left[x_{Mode}\le X<x_{Mode}+dx\right]$ is a maximum. What is the mode of this pdf?

$$\text{Maximum occurs at } x_{Mode}=\frac{df_X(x)}{dx}=0$$

$$\frac{d}{dx}\left[\frac{x}{\sigma^2}e^{-\frac{x}{2\sigma^2}}\right]=\frac{e^{-\frac{x}{2\sigma^2}}}{\sigma^2}+\frac{x}{\sigma^2}\left(-\frac{2x}{2\sigma^2}\right)e^{-\frac{x}{2\sigma^2}}=0$$

$$\frac{e^{-\frac{x}{2\sigma^2}}}{\sigma^2}\left(1-\frac{2x^2}{2\sigma^2}\right)=0$$

$$x_{Mode}=\sigma,\ \text{since } x\ge 0$$

f) The *median* is the numerical value $X=x_{Median}$ for which $\Pr\left[X\le x_{Median}\right]=\Pr\left[X>x_{Median}\right]=0.5$. Find the median value of this pdf.

$$\text{We want } \int_0^a \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\,dx=0.5$$

$$u=\frac{x^2}{2\sigma^2},\ du=\frac{2x}{2\sigma^2}=\frac{x}{\sigma^2}$$

$$x=0\Rightarrow u=0, x=a\Rightarrow u=\frac{a^2}{2\sigma^2}$$

$$\int_0^{\frac{a^2}{2\sigma^2}} e^{-u}\,du=1-e^{-\frac{a^2}{2\sigma^2}}=0.5\Rightarrow e^{-\frac{a^2}{2\sigma^2}}=0.5$$

$$-\log(2)=-\frac{a^2}{2\sigma^2}$$

$$a=\sigma\sqrt{2\log 2}=x_{Median}$$

2. The Erlang distribution is defined as $f_X(x;k,\lambda)=\dfrac{\lambda^k x^{k-1}e^{-\lambda x}}{(k-1)!}$, for $x,\lambda>0$, and $k\in\mathbb{N}$.

We view this as a function of the continuous variable $x > 0$ with parameters $\lambda > 0 \text{ and } k \in 1,2,3,\ldots$.

a) Show that the given pdf is, in fact, a pdf. *Hint: The magic integration formulas might help!*

$$f_X(x;k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \text{ for } x,\lambda > 0, \text{ and } k \in \mathbb{N}.$$

$$x,\lambda > 0, (k-1)! > 0, e^{-\lambda x} > 0 \Rightarrow f_X(x;k,\lambda) > 0$$

$$\int_0^\infty \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx = \frac{\lambda^k}{(k-1)!} \int_0^\infty x^{k-1} e^{-\lambda x} dx, \quad n = k-1, a = \lambda$$

From first integrating formula:

$$\frac{\lambda^k}{(k-1)!} \int_0^\infty x^{k-1} e^{-\lambda x} dx = \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(n+1)}{a^{n+1}}\right) = \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(k-1+1)}{\lambda^{k-1+1}}\right)$$

$$= \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(k)}{\lambda^k}\right) = \frac{\lambda^k (k-1)!}{(k-1)!\lambda^k} = 1, \text{ so } f_X(x;k,\lambda) \text{ satisfies both properties}$$

of a pdf.

b) Compute the mean.

$$E\left[X\right] = \int_0^\infty \frac{x\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx = \frac{\lambda^k}{(k-1)!} \int_0^\infty x^k e^{-\lambda x} dx, \quad n = k, a = \lambda$$

From first integrating formula:

$$\frac{\lambda^k}{(k-1)!} \int_0^\infty x^k e^{-\lambda x} dx = \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(n+1)}{a^{n+1}}\right) = \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(k+1)}{\lambda^{k+1}}\right)$$

$$= \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(k+1)}{\lambda^{k+1}}\right) = \frac{\lambda^k k!}{(k-1)!\lambda^{k+1}} = \frac{\lambda^k k(k-1)!}{(k-1)!\lambda^{k+1}} = \frac{k}{\lambda}$$

c) Compute the mean square value

$$E\left[X^2\right] = \int_0^\infty \frac{x^2\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} dx = \frac{\lambda^k}{(k-1)!} \int_0^\infty x^{k+1} e^{-\lambda x} dx, \quad n = k+1, a = \lambda$$

From first integrating formula:

$$\frac{\lambda^k}{(k-1)!} \int_0^\infty x^{k+1} e^{-\lambda x} dx = \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(n+1)}{a^{n+1}}\right) = \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(k+2)}{\lambda^{k+2}}\right)$$

$$= \left(\frac{\lambda^k}{(k-1)!}\right)\left(\frac{\Gamma(k+2)}{\lambda^{k+2}}\right) = \frac{\lambda^k (k+1)!}{(k-1)!\lambda^{k+2}} = \frac{\lambda^k (k+1)k(k-1)!}{(k-1)!\lambda^{k+3}} = \frac{(k+1)k}{\lambda^2}$$

d) Compute the variance

$$\text{var}\left[X\right] = E\left[X^2\right] - \left(E\left[X\right]\right)^2 = \frac{(k+1)k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k^2+k-k^2}{\lambda^2} = \frac{k}{\lambda^2}$$

e) The *mode* is the numerical value $X = x_{Mode}$ for which $\Pr\left[x_{Mode} \le X < x_{Mode} + dx\right]$ is a maximum. What is the mode of this pdf?

$$\frac{d}{dx}\left[\frac{\lambda^k x^{k-1}e^{-\lambda x}}{(k-1)!}\right]=\left(\frac{\lambda^k}{(k-1)!}\right)\left((k-1)x^{k-2}e^{-\lambda x}+x^{k-1}(-\lambda x)e^{-\lambda x}\right)=0$$
$$(kx^{k-2}e^{-\lambda x}-x^{k-2}e^{-\lambda x}-\lambda x^{k-1}e^{-\lambda x}=\left(kx^{k-2}-x^{k-2}-\lambda x^k\right)e^{-\lambda x}=0$$
$$x^{k-1}\left((k-1)x^{-1}-\lambda\right)=0,\ x>0,\ \text{so first term } \neq 0$$
$$(k-1)x^{-1}=\lambda,\ x=\frac{(k-1)}{\lambda}$$

3. A 12-bit analog to digital converter (ADC) is designed to operate between the values of $\pm 1.65\text{V}$.

a) Assuming that the quantization error is characterized by $U\left(-\frac{q}{2},\frac{q}{2}\right)$, where q is the voltage represented by the least significant bit, find the mean, mean square, and variance.

$$E:\ \text{quantization error} \sim U\left(-\frac{q}{2},\frac{q}{2}\right) \Rightarrow f_E(e)=\begin{cases}\frac{1}{q} & -\frac{q}{2}<e\leq\frac{q}{2}\\ 0 & \text{otherwise}\end{cases}$$
$$E[E]=\int_{-q/2}^{q/2}\frac{1}{q}u\,du=\frac{1}{q}\frac{u^2}{2}\bigg|_{u=-q/2}^{q/2}=\frac{1}{q}\left(\frac{q^2}{8}-\frac{q^2}{8}\right)=0$$

Of course, you could recognize this as the integral of an odd function, u, times a constant, $\frac{1}{q}$, integrated over symmetric limits, which is always zero.

$$E\left[E^2\right]=\int_{-q/2}^{q/2}\frac{1}{q}u^2\,du=\frac{1}{q}\frac{u^3}{3}\bigg|_{u=-q/2}^{q/2}=\frac{1}{q}\left(\frac{q^3}{24}-\frac{-q^3}{24}\right)=\frac{q^2}{12}$$
$$\sigma_E^2=E\left[e^2\right]-\left(E\left[E\right]\right)^2=\frac{q^2}{12}-0^2=\frac{q^2}{12}$$

b) Assuming that the ADC is designed to drive a $1\text{k}\Omega$ resistive load. Define the quantization noise power supplied to the load as the mean square voltage divided by the resistance and compute the noise power in the proper units.

The random variable E has units of volts.

$$P=\frac{E\left[E^2\right]}{R}=\frac{q^2}{12R}$$
$$q=\frac{(1.65-(-1.65)\text{Volts}}{2^{12}}=\frac{3.3\text{ V}}{4096}=0.8\text{ mV}$$
$$R=1000\Omega$$
$$P=\frac{(8\times10^{-4}\text{V})^2}{1.2\times10^4\Omega}=\frac{6.4\times10^{-7}\text{V}^2}{1.2\times10^4\text{V}\Omega}=5.4\times10^{-11}\text{W}=54\text{ pW}$$
$$=-103\text{ dBW}=-73\text{ dBm}$$