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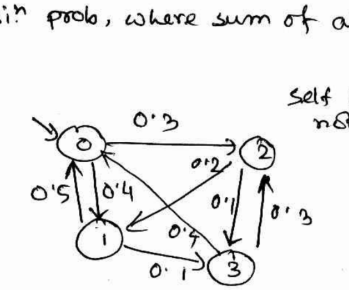
State-Space Modeling

- Views a system to be in one of several possible states.
  - Represents the system through a state transition dia
  - Quantify transition probabilities to determine the prob of the system being in each state.

Markov chains & models

- ~~is~~ Represented by a state dia with transi^n prob, where sum of all transition prob out of each state is 1

Transition matrix ; $M = \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0 \\ 0.5 & 0.4 & 0 & 0.1 \\ 0 & 0.2 & 0.7 & 0.1 \\ 0.4 & 0 & 0.3 & 0.3 \end{pmatrix}$



$S(t+1) = S(t) M$

$S(t+n) = S(t) M^h$

Example: $(s_0, s_1, s_2, s_3) = \underbrace{(0.5, 0.5, 0, 0)}_{\text{initial state}} M = (0.4, 0.4, 0.15, 0.05)$

Stochastic Sequential m/c

- Transition taken from state s under input j is not uniquely determined. Rather, a # of states may be entered with different probabilities
- There will be a separate transition (Markov) matrix for each input value.

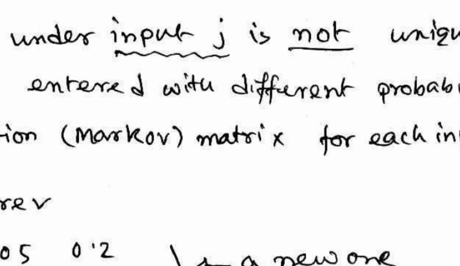
Transition, $j=0$, M = as prev

Transition, $j=1$, $M = \begin{pmatrix} 0.5 & 0.2 & \\ 0.1 & \cdots & \\ \vdots & & \end{pmatrix}$ ← a new one

- A markov chain can be viewed as a stochastic sequential m/c with no input

Sample app^n of Markov Modeling

- Programmer workflow



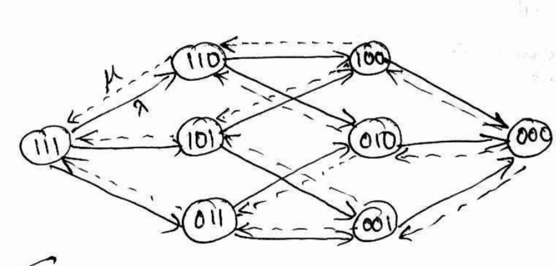
- Speech recognition problem

Markov models

	System state is fully observable	System state is partially observable
System is autonomous	Markov chain	Hidden Markov model (contains hidden states)
System is controlled	Markov decision process (outcome is partly under control of	Partially observable Markov decision process

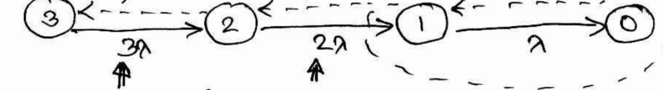
Merging states in a Markov model

- There are three identical units (1 = Unit is up; 0 = Unit is down)



All solid lines λ
Dashed lines μ

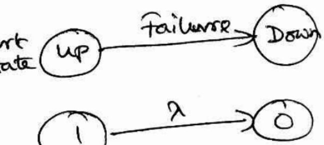
Simpler equivalent model for 3-unit fail-soft system



failed state if TMR

3λ for each of the 3 working units

Modeling Nonrepairable Systems

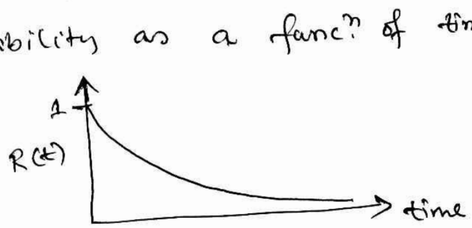


→ Rate of change for the probability of being at state 1 is $-\lambda$

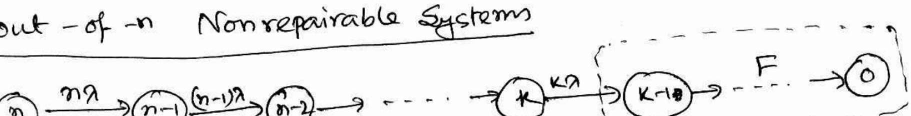
So, $P_1' = -\lambda P_1$
Also, $P_0 + P_1 = 1$
Initial condn: $P_1(0) = 1$

$\Rightarrow$ $P_0 = 1 - e^{-\lambda t}$
$P_1 = e^{-\lambda t}$
[derived earlier by solving $\frac{dP_1}{P_1} = -\lambda dt$]

⇒ Reliability as a funcn of time: $R(t) = P_1(t) = e^{-\lambda t}$
prob of being working, i.e., at state 1)



k-out-of-n Nonrepairable Systems



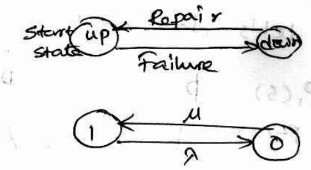
Initial condn: $P_n(0) = 1$

$$P_n' = -n\lambda P_n$$
$$P_{n-1}' = \underbrace{n\lambda P_n}_{\text{incoming}} \underbrace{-(n-1)\lambda P_{n-1}}_{\text{outgoint}}$$
$$\vdots$$
$$P_k' = (k+1)\lambda P_{k+1} - k\lambda P_k$$
And $P_n + P_{n-1} + \cdots + P_k + P_F = 1$

$\Rightarrow$
$$P_n = e^{-n\lambda t}$$
$$P_{n-1} = n e^{-(n-1)\lambda t}(1 - e^{-\lambda t})$$
$$\vdots$$
$$P_k = \binom{n}{k} e^{-(n-k)\lambda t}(1 - e^{-\lambda t})^k$$
$$P_F = 1 - \sum_{j=k}^{n} P_j$$

[Using Laplace Xformatn?]

Modeling Repairable Systems



$$\left.\begin{array}{l} -\lambda p_1 + \mu p_0 = 0 \\ p_0 + p_1 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} p_1 = \mu/(\lambda+\mu) \\ p_0 = \lambda/(\lambda+\mu) \end{array}$$

⇑

prob of each state @ steady-state or equilibrium [transitions into/out-of each state must "balance out" → $\lambda p_1 = \mu p_0$]

Q: What about time-variant balance?

or

What is the availability?

$$A(t) = p_1(t) = \mu/(\lambda+\mu) + \lambda/(\lambda+\mu) \cdot e^{-(\lambda+\mu)t}$$

Derivation: Using Laplace transformation

First, derive time variant equations to get transient ~~equations~~ solution.

$$p_1'(t) = -\lambda p_1(t) + \mu p_0(t) \quad [\text{(+)ve for incoming, (−)ve for outgoing}]$$
$$p_0'(t) = -\mu p_0(t) + \lambda p_1(t) \quad [1 \xleftarrow{\mu} 0 \;,\; 1 \xrightarrow{\lambda} 0]$$

Now, to solve linear differential equations with constant coefficient:

Step1: Convert to algebric equations using laplace transform

Step2: Solve the algebric equations

Step3: Use inverse Laplace transform to find original solⁿ

For step1:

The Laplace transform converts a time-domain function f(t) to its transform-domain counterpart F(s) by:

$$F(s) = \text{Laplace}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

[t → real argument ($t \geq 0$)
s → complex " ($s = \sigma + i\omega$)
L → Linear operator of f(t)]

For example, if $f(t) = k$, a constant

$$\text{then } F(s) = \int_0^\infty e^{-st} k\,dt = k \cdot \left(\frac{1}{-s}\right)\left[e^{-st}\right]_0^\infty = \frac{k}{s}$$

Again if $f(t) = e^{-at}$, then, $F(s) = \int_0^\infty e^{-st} e^{-at} dt = \frac{1}{(-s-a)}\left[e^{-(s+a)t}\right]_0^\infty = \frac{1}{s+a}$

Similarly:

f(t): time domain function	F(s): Transform-domain funcⁿ $\left[L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt\right]$
k: constant	k/s
e^{-at}	$1/(s+a)$
$t^k/k!$; $k \geq 0$	$1/s^{k+1}$
$t^k e^{-at}/k!$	$1/(s+a)^{k+1}$
$k \cdot h(t)$; k constant	$k \cdot H(s)$
$h(kt)$; Constant $k > 0$	$H(s/k)/k$
$t\,h(t)$	$-dH(s)/ds$ ⟸ Wrong in book (in book it is mentioned as $dH(s)/ds$)
$g(t) + h(t)$	$G(s) + H(s)$
*** $h'(t)$: derivative of $h(t)$	$sH(s) - h(0)$; $h(0)$ is the initial value of h

Laplace transform of derivative:

$$H(s) = \int_0^\infty e^{-st} h(t)\,dt = \left[-\frac{e^{-st} h(t)}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} h'(t)\,dt$$

So, $s\,H(s) = h(0) + L(h'(t))$

~~So, H(s) = sH(s) − f(0)~~

So, $L(h'(t)) = s\,H(s) - h(0) = s\,L(h(t)) - h(0)$

Similarly, $L(h''(t)) = s^2 L(h(t)) - s\,h(0) - h'(0)$

⋮ So, $L(h^n(t)) = s^n L(h(t)) - \sum_{i=1}^{n} s^{i-1} h^{n-i}(0)$

order of derivative

for $f(t) = \int_0^t h(t)\,dt$

$$F(s) = L(f(t)) = \int_0^\infty e^{-st}\left(\int_0^t h(t)\,dt\right)dt$$
$$= \left[\frac{e^{-st} f(t)}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} h(t)\,dt = \frac{f(0)}{s} + \frac{H(s)}{s}$$
$$= H(s)/s$$

Now, let's do step1 for the two equations we got :

$$sP_1(s) - \cancel{p_1(0)}^{\,1} = -\lambda P_1(s) + \mu P_0(s)$$
$$sP_0(s) - \cancel{p_0(0)}^{\,0} = -\mu P_0(s) + \lambda P_1(s)$$

L.H.S. $\Rightarrow [h'(t) \rightarrow sH(s) - h(0)]$
R.H.S. $\Rightarrow [k\,h(t) \rightarrow k\,H(s)]$

Step 2:

$$P_1(s) = (s+\mu) / [s^2 + (\lambda+\mu)s]$$
$$P_0(s) = \lambda / [s^2 + (\lambda+\mu)s]$$

Step 3:

$$P_1(s) = \frac{s+\mu}{s^2+(\lambda+\mu)s} = \frac{1}{s+(\lambda+\mu)} + \frac{\mu}{s(s+(\lambda+\mu))}$$

Let, $\frac{1}{s(s+(\lambda+\mu))} = \frac{a}{s} + \frac{b}{s+(\lambda+\mu)}$

Transform each into sum of terms, each of which takes one of the forms of H(s)

So, $a[s+(\lambda+\mu)] + bs = 1$

So, $a+b=0$ [as coefficient of s is 0 in R.H.S.]

So, $a = 1/(\lambda+\mu)$, $b = -1/(\lambda+\mu)$

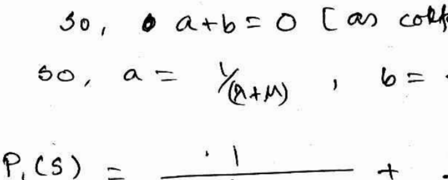
$$\text{So, } P_1(s) = \frac{1}{s+(\lambda+\mu)} + \frac{\mu}{(\lambda+\mu)s} - \frac{\mu}{(\lambda+\mu)(s+(\lambda+\mu))}$$

$$\text{So, } \cancel{P_1(t)} = \frac{\mu}{(\lambda+\mu)s} + \frac{\lambda}{(\lambda+\mu)(s+(\lambda+\mu))}$$

$$\text{So, } p_1(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t}$$

$H(s) = k/s$ gives $h(t) = k$

$kH(s)$ gives $k\,h(t)$ and $H(s) = \frac{1}{s+a}$ gives $h(t) = e^{-at}$



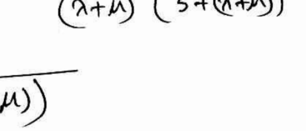
Steady-state availability $\Rightarrow$ for $t \to \infty$, $A(t) = \frac{\mu}{\lambda+\mu}$, a constant which we have got in our steady-state analysis $[p_1 = \frac{\mu}{\lambda+\mu}]$

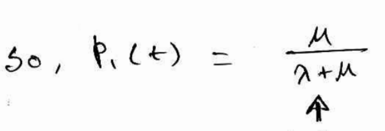
$\boxed{\text{So, we can get steady-state sol}^n \text{ by solving transient equ}^n \text{ and then putting } t\to\infty}$

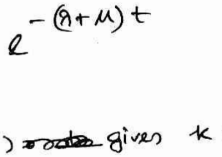
Multiple Failure States

Steady-state analysis:

$$\left.\begin{aligned} -\lambda p_2 + \mu p_1 + \mu p_0 &= 0 \\ -\mu p_1 + \lambda_1 p_2 &= 0 \\ p_2 + p_1 + p_0 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} p_2 &= \mu/(\lambda+\mu) \\ p_1 &= \lambda_1/(\lambda+\mu) \\ p_0 &= \lambda_0/(\lambda+\mu) \end{aligned}$$







$\lambda = \lambda_0 + \lambda_1$

Safety evaluation

Total risk of the system: $\sum_{i \in \text{failure states set}} C_i p_i$

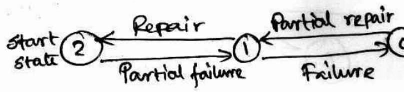
$C_i \Rightarrow$ cost of penalty associated with state i

Modeling Fail-soft Systems

$-\lambda_2 p_2 + \mu_2 p_1 = 0$
$\lambda_1 p_1 - \mu_1 p_0 = 0$
$p_2 + p_1 + p_0 = 1$

let $S = \dfrac{1}{1 + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2}}$

~~Given~~, we get, $p_2 = S$
$p_1 = S\lambda_2/\mu_2$
$p_0 = \dfrac{S\lambda_1\lambda_2}{\mu_1\mu_2}$



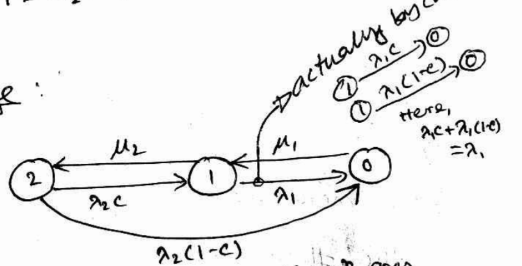
Operational state i has a benefit b_i associated with it

Performability evaluaⁿ:
$$\text{Performability} = \sum_{i \in \text{set of all } \underline{\text{operational}} \text{ states}} b_i p_i$$

Example: $\lambda_2 = 2\lambda$, $\lambda_1 = \lambda$, $\mu_1 = \mu_2 = \mu$
and, $b_2 = 2$, $b_1 = 1$, $b_0 = 0$
$P = 2p_2 + p_1 = 2S + 2S\lambda/\mu$

Fail-soft system with Imperfect Coverage:

$-\lambda_2 p_2 + \mu_2 p_1 = 0$
$\lambda_2(1-c)p_2 + \lambda_1 p_1 - \mu_1 p_0 = 0$
$p_2 + p_1 + p_0 = 1$



If a units malfuncⁿ goes undetected, the system fails

For $\lambda_2 = 2\lambda$, $\lambda_1 = \lambda$, $\mu_2 = \mu_1 = \mu$,

$p_0 = 2[(1-c)\rho + 1] / [1 + (4-2c)\rho + 2\rho^2]$
$p_1 = 2\rho / [1 + (4-2c)\rho + 2\rho^2]$
$p_2 = \rho^2 / [1 + (4-2c)\rho + 2\rho^2]$

where $\rho = \mu/\lambda$

=> we can also consider coverage for the repair direcⁿ

Dependability Modeling in practice

Ex: (1) Birth & death [state j to state $j+1$] => in case of ~~the~~ of considering ~~the~~ # of operational units
(2) Arrival & departure [state $j+1$ to state j] => in case of considering the # of failed units

For arrival or birth of an operational unit [arrival of fault will give reverse direcⁿ]

(Similar to: $(2) \rightleftarrows (1) \rightleftarrows (0)$ of a fail-soft system
Here, 2 is the # of operational ~~systems~~ units.
Now, if we consider the # of failed units, then the same ~~graph~~ state dia becomes $(0) \rightleftarrows (1) \rightleftarrows (2)$

=> closed-form solution for state probabilities are difficult to obtain in general
=> Steady-state prob.s are easily obtained: $p_j = p_0 \lambda_0 \lambda_1 \cdots \lambda_{j-1} / (\mu_1 \mu_2 \cdots \mu_j)$
[Similar to the case of Fail-soft system (at the top of this page)]

Special case 1: Constant arrival & departure rates

$$0 \underset{\lambda}{\overset{\mu}{\rightleftarrows}} 1 \underset{\lambda}{\overset{\mu}{\rightleftarrows}} 2 \underset{\lambda}{\overset{\mu}{\rightleftarrows}} 3 \rightleftarrows \cdots$$

Let $\rho = \lambda/\mu$, ratio betn arrival & departure rates

$\Rightarrow$ General case steady-state prob: $p_j = p_0 \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} = p_0 \dfrac{\lambda^j}{\mu^j} = p_0 \rho^j$

$\Rightarrow$ $p_0 + p_1 + p_2 + \cdots = 1$ gives $p_0 (1 + \rho + \rho^2 + \cdots) = 1$

So, $p_0 \dfrac{1}{1-\rho} = 1$; So, ~~$\rho_0 = 1$~~ $p_0 = 1 - \rho$

and thus $p_j = p_0 \rho^j = (1-\rho)\rho^j$

$\Rightarrow$ If ~~the~~ n is the last state, then we can merge states having ~~state~~ indices $\geq n$ can be merged in one state. So,

$$P_n = p_n + p_{n+1} + \cdots = (1-\rho)(\rho^n + \rho^{n+1} + \cdots)$$
$$= (1-\rho)\rho^n (1 + \rho + \cdots) = (1-\rho)\rho^n \cdot \frac{1}{1-\rho} = \rho^n$$

[For more than service providers, repair rate can be $\mu, 2\mu, \ldots s\mu$, based on the # of available units in a state; Derivan in slide & book $\rightarrow$ not needed]

TMR System with Repair (only for the first failure)

$$3 \underset{3\lambda}{\overset{\mu}{\rightleftarrows}} 2 \xrightarrow{2\lambda} F$$

$-3\lambda p_3 + \mu p_2 = 0$

$-(\mu + 2\lambda) p_2 + 3\lambda p_3 = 0$

$p_3 + p_2 + p_F = 1$

steady-state analysis of no use:

$p_3 = p_2 = 0$, $p_F = 1$

[It happens due to having no repair from F, and all systems in the world must get to such failed state]

~~Assume~~ (1) Voter is perfect
(2) Upon first module ~~im~~ malfuncn, we switch to duplex operan for servicing
(3) System fails only if second consecutive failure occurs before servicing the first one

Now,

$$\left.\begin{aligned} p_3' &= -3\lambda p_3 + \mu p_2 \\ p_2' &= 3\lambda p_3 - (2\lambda + \mu) p_2 \end{aligned}\right\} \Rightarrow \text{Solution (see ex: 4.11 in book)}$$

So, $s P_3(s) - \overset{1}{p_3(0)} = -3\lambda P_3(s) + \mu P_2(s)$

ধরি

$s P_2(s) - \overset{0}{p_2(0)} = 3\lambda P_3(s) - (2\lambda + \mu) P_2(s)$

Here, $p_3(0) = 1$
$p_2(0) = 0$

So, $s P_3(s) - 1 = -3\lambda P_3(s) + \mu P_2(s)$

$s P_2(s) = 3\lambda P_3(s) - (2\lambda + \mu) P_2(s) \rightarrow P_3(s) = \dfrac{(s + 2\lambda + \mu) P_2(s)}{3\lambda}$

So, $(s + 3\lambda) \cdot \dfrac{(s + 2\lambda + \mu) P_2(s)}{3\lambda} - \mu P_2(s) = 1$

So, $P_2(s) = \dfrac{3\lambda}{(s + 3\lambda)(s + 2\lambda + \mu) - 3\mu\lambda}$; So, $P_3(s) = \dfrac{(s + 2\lambda + \mu)}{(s + 3\lambda)(s + 2\lambda + \mu) - 3\mu\lambda}$

Now, $P_2(s) = \dfrac{3\lambda}{s^2+2\lambda s+\mu s+3\lambda s+6\lambda^2+3\lambda\mu-3\lambda\mu}$

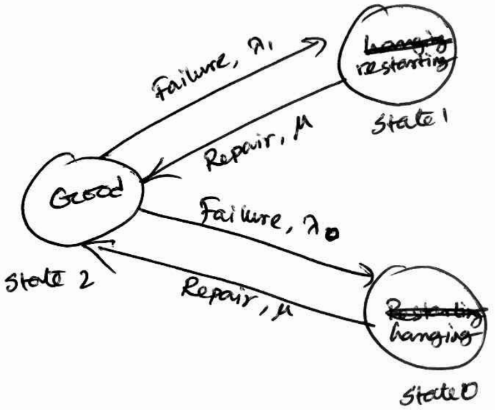
$$= \frac{3\lambda}{s^2+5\lambda s+\mu s+6\lambda^2}$$

$$\vdots$$

Ex: An industrial plant utilites a robot for manufacturing highly-precise aircrafts. Two parts of the robot have in-built defects. Defect of the first part causes a hanging state of the robot, whereas defect of the second part causes a continuous restarting state of the robot. Expected ~~times~~ intervals elapsed before experiencing these two states are 100 days and 50 days respectively. In case of any failure occurring, a service man takes 10 days for repairing the system.

Draw a state diagram rendering operation of the robot in different states. Perform a steady-state analysis based on your state diagram.

Ans.



Steady-state analysis

$\lambda = \lambda_1 + \lambda_2$

$$\left.\begin{aligned} -\lambda P_2 + \mu P_1 + \mu P_0 &= 0 \\ \lambda_1 P_2 - \mu P_1 &= 0 \\ P_0 + P_1 + P_2 &= 1 \end{aligned}\right\} \quad P_0 = \frac{\lambda_0}{\lambda+\mu}, \quad P_1 = \frac{\lambda_1}{\lambda+\mu}, \quad P_2 = \frac{\mu}{\lambda+\mu}$$

Now, $\lambda_0 = \frac{1}{100}\ \text{day}^{-1} = .01\ \text{day}^{-1}$

$\lambda_1 = \frac{1}{50}\ \text{day}^{-1} = .02\ \text{day}^{-1}$

$\mu = \frac{1}{10}\ \text{day}^{-1} = .1\ \text{day}^{-1}$

$P_0 = .08$

$P_1 = .15$

$P_2 = .77$