Vector Tools for Computer Graphics

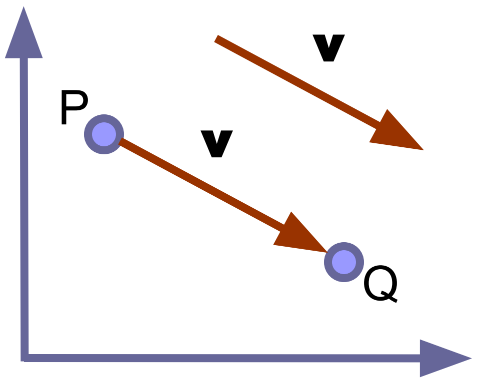
Computer Graphics

Basic Definitions

- Points specify location in space (or in the plane).
- Vectors have magnitude and direction (like velocity).

$$\boxed{\text{Points} \neq \text{Vectors}}$$

Basics of Vectors



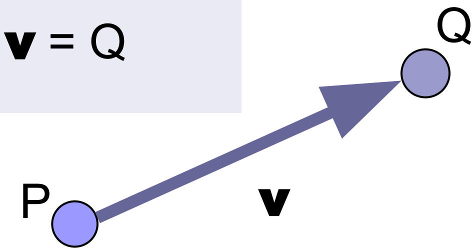
Vector as displacement:

v is a vector from point P to point Q.

The **difference** between two points is a vector: **v** = Q - P

Another way:

The **sum** of a point and a vector is a point : P + **v** = Q



Operations on Vectors

Two operations

Addition

$\mathbf{a} + \mathbf{b}$

$\mathbf{a} = (3,5,8)$, $\mathbf{b} = (-1,2,-4)$

$\mathbf{a} + \mathbf{b} = (2,7,4)$

Multiplication be scalars

$s\mathbf{a}$

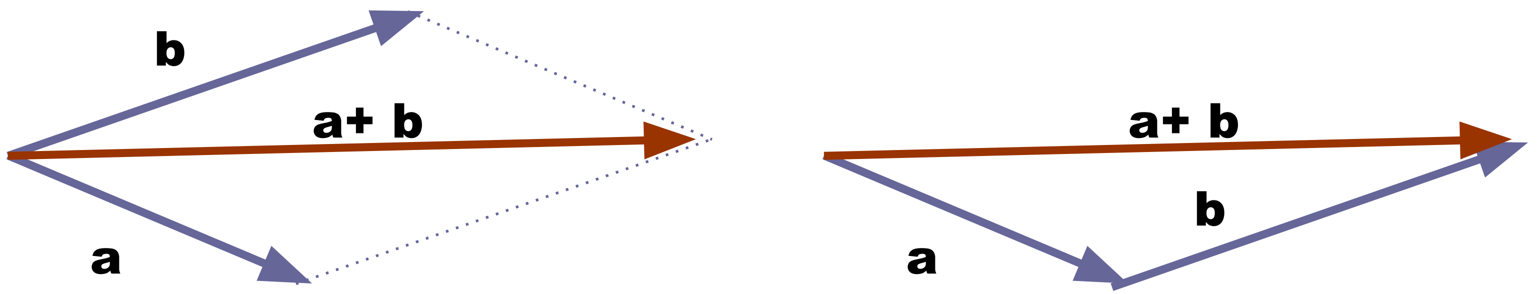
$\mathbf{a} = (3,-5,8)$, $s = 5$

$5\mathbf{a} = (15,-25,40)$

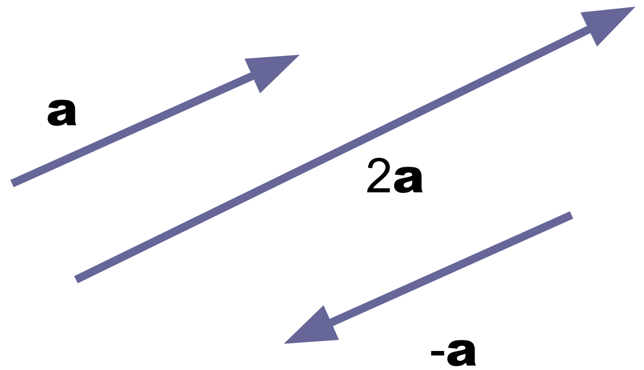
operations are done ***componentwise***

Operations on vectors

Addition

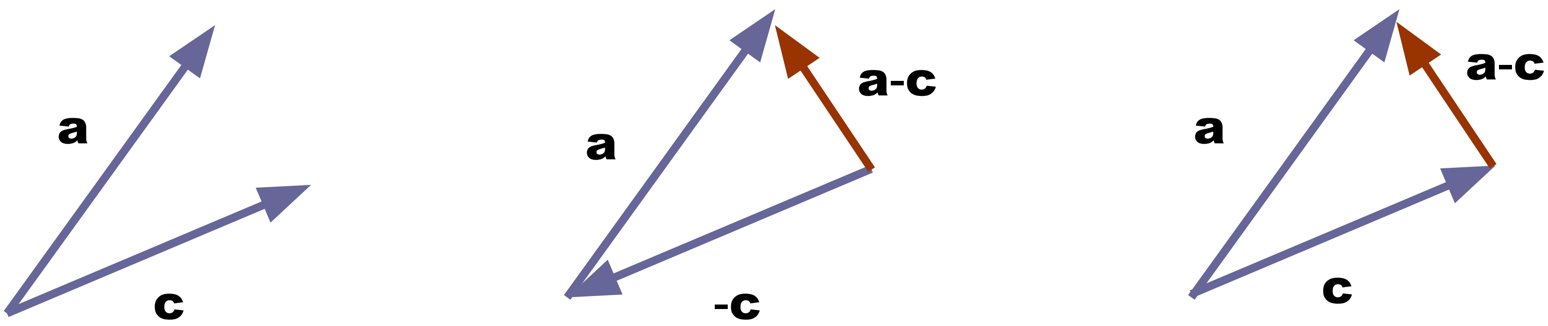


Multiplication by scalar



Operations on vectors

Subtraction



Properties of vectors

Length or size

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)$$

$$|\,\mathbf{w}\,| = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}$$

Unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

- The process is called **normalizing**
- Used to refer **direction**

The **standard unit vectors**: $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$

Dot Product

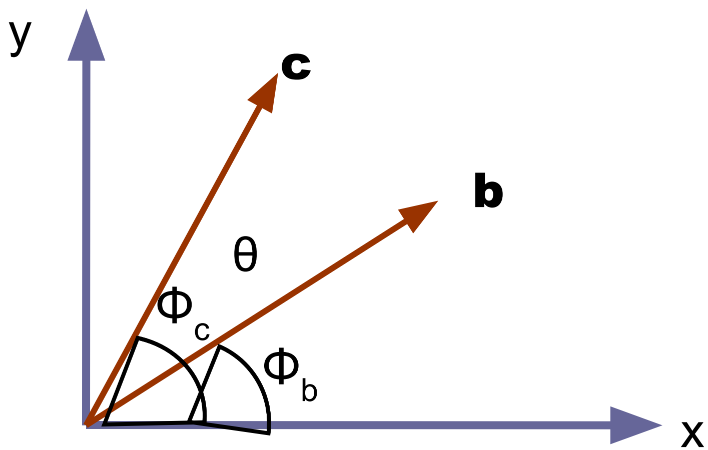
The dot product ***d*** of two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots w_n)$:

Properties

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a}+\mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$

Application of Dot Product

Angle between two unit vectors **b** and **c**



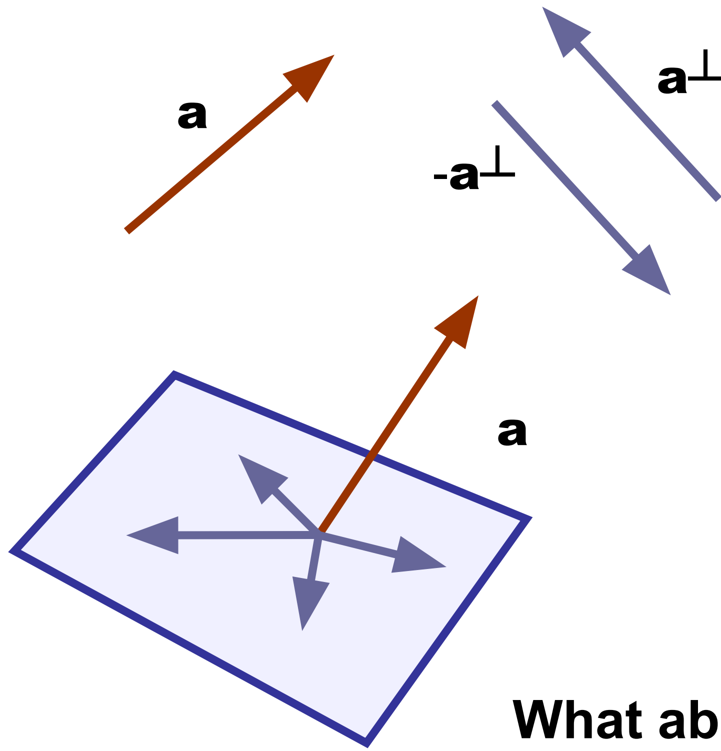
$$\cos(\theta) = \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}$$

Two vectors **b** and **c** are <u>**perpendicular**</u> (**orthogonal/normal**) if **b**·**c** = 0

2D perp Vector

Which vector is perpendicular to the 2D vector $\mathbf{a} = (a_x,a_y)$?

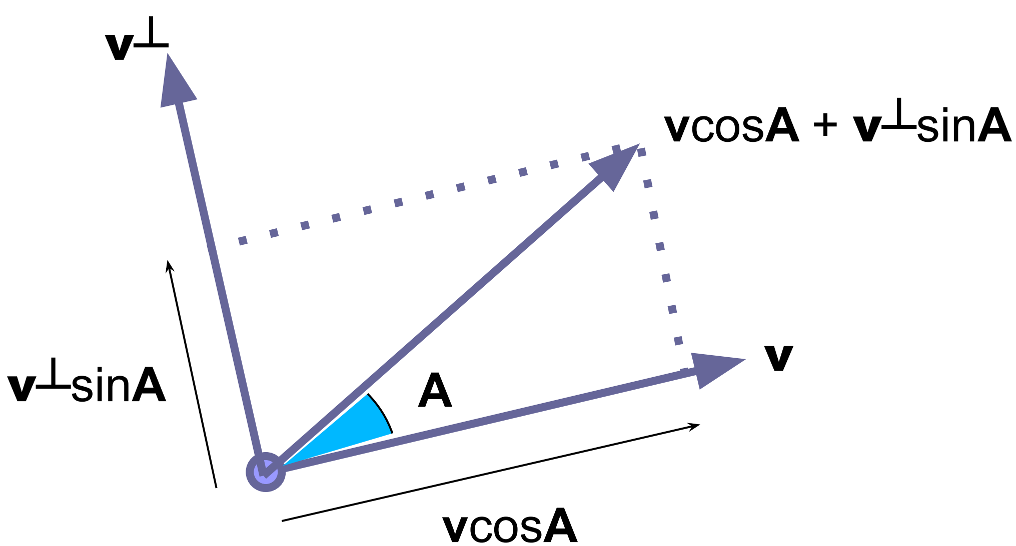
Let $\mathbf{a} = (a_x,a_y)$.
Then
$\mathbf{a}^{\perp} = (-a_y,a_x)$
is the
counterclockwise perpendicular to **a**.



What about 3D case?

Rotation in 2d

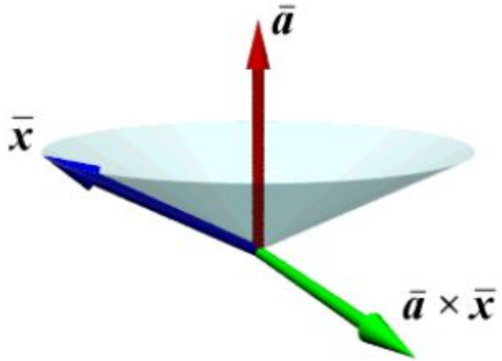
- We want to rotate a 2d vector **v** counterclockwise by an angle **A**
- First we determine perp(**v**), $\mathbf{v}^{\perp}$
- Then we scale **v** by cos**A** and scale $v^{\perp}$ by sin**A** and take their sum



Rotation in 3d: General Case

- Rodrigues Formula

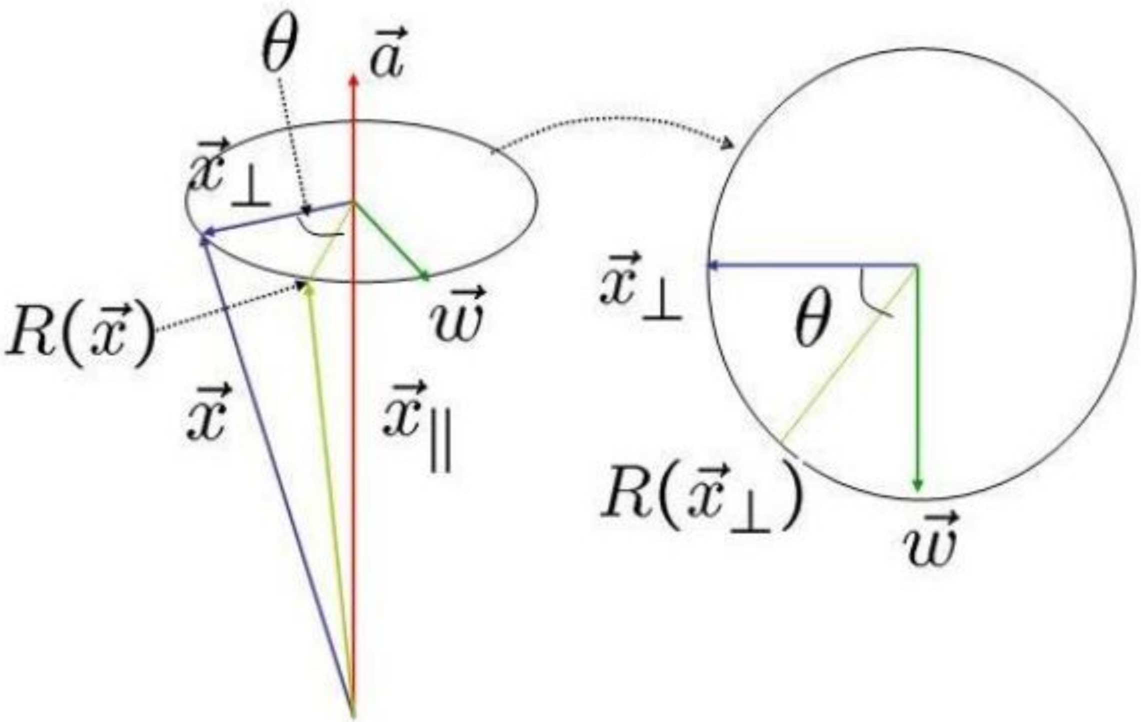
We can actually define a *natural* basis for rotation in terms of three defining vectors. These vectors are the rotation axis, a vector perpendicular to both the rotation axis and the vector being rotated, and the vector itself. These vectors correspond to the each respective term in the expression.

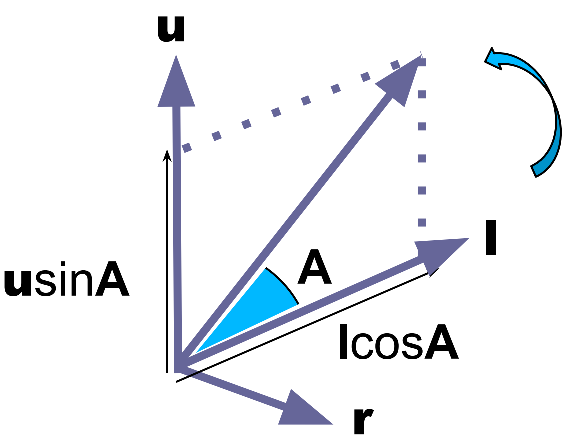


Let's look at this in greater detail

Rotation in 3d: A simple case

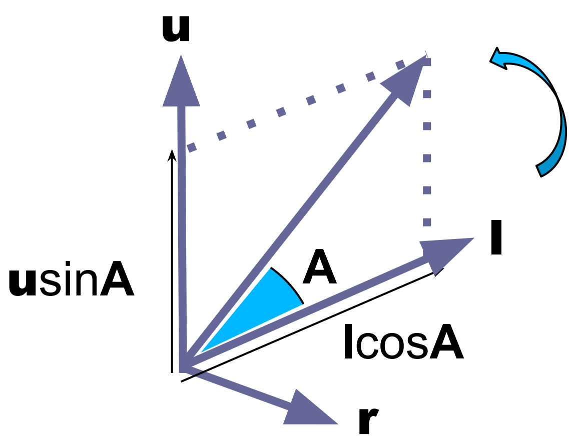
- Rodrigues Formula





Rotation in 3d

- We want to rotate a 3d vector **l** counterclockwise with respect to a 3d unit vector **r** by an angle **A,** where **l** and **r** are perpendicular to each other
- First we determine the vector **u**, that is perpendicular to both **l** and **r** and have a length equal to that of **l**
- So, **u** = **r** X **l**
- Then we scale **l** by cos**A** and scale **u** by sin**A** and take their sum



* note that, this method is applicable only in cases where the axis of rotation and the vector which is to be rotated are perpendicular to each other

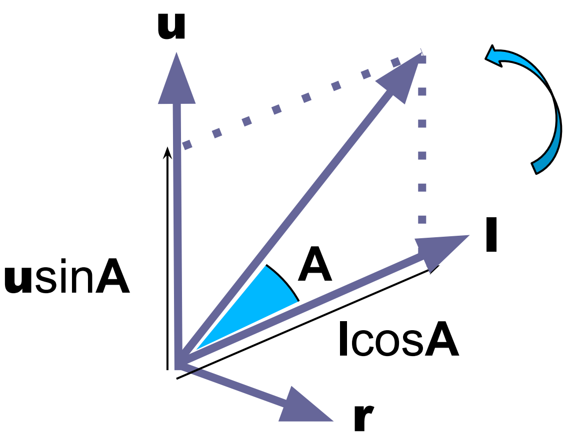
Rotation in 3d

- Example: Rotate (1, -1, 2) with respect to (-1, 3, 2), by 30°

- **l** = (1, -1, 2)
- **r** = (-1, 3, 2) / 12 = (-1/12, 1/4,1/6)

- **u** = **l x r** = ?

- Answer = **l** cos30° + **u** sin30° = ?



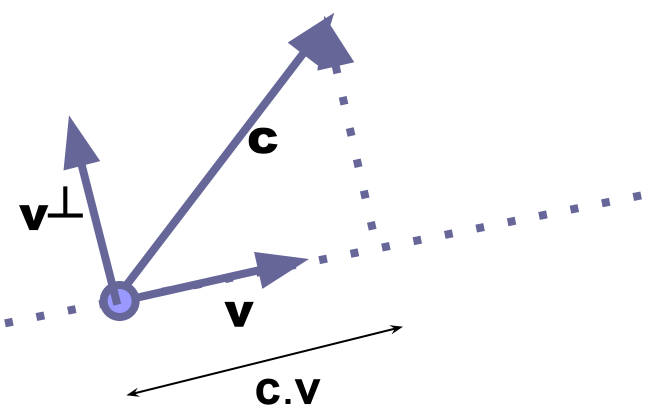
Why the formula works

- We simply get the other basis by cross multiplying l and r_unit.
- Why not rotation axis r instead of r_unit.
- |u| = |l|

Now like previous 2D rotation.

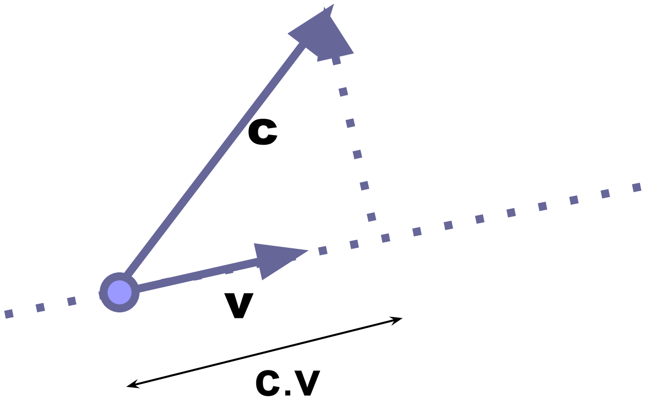
Orthogonal Projection

- We want to decompose the vector **c** into two vectors, one along the direction of a unit vector **v** and another along perp(**v**)
- The length of the orthogonal projection of **c** along **v** is **c.v**
  (as **v** is a unit vector)
- Thus the component (or orthogonal projection) of **c** along **v** is (**c.v**)**v**
- So the component of **c** along perp(**v**) is **c**-(**c.v**)**v**



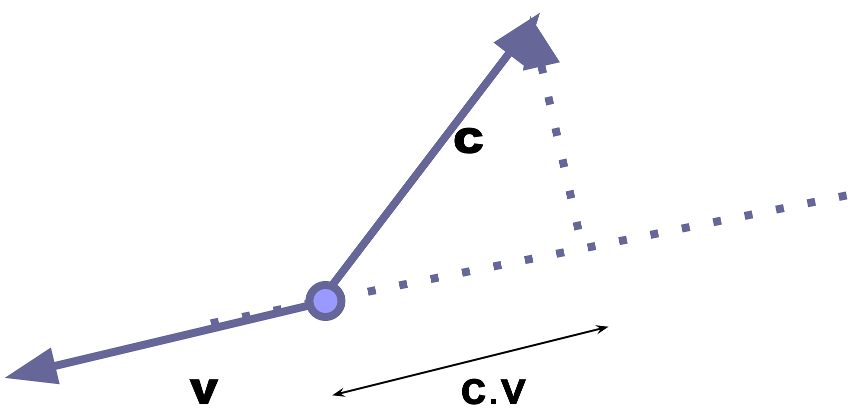
Projection and Component

Projection positive.
Component along **v**

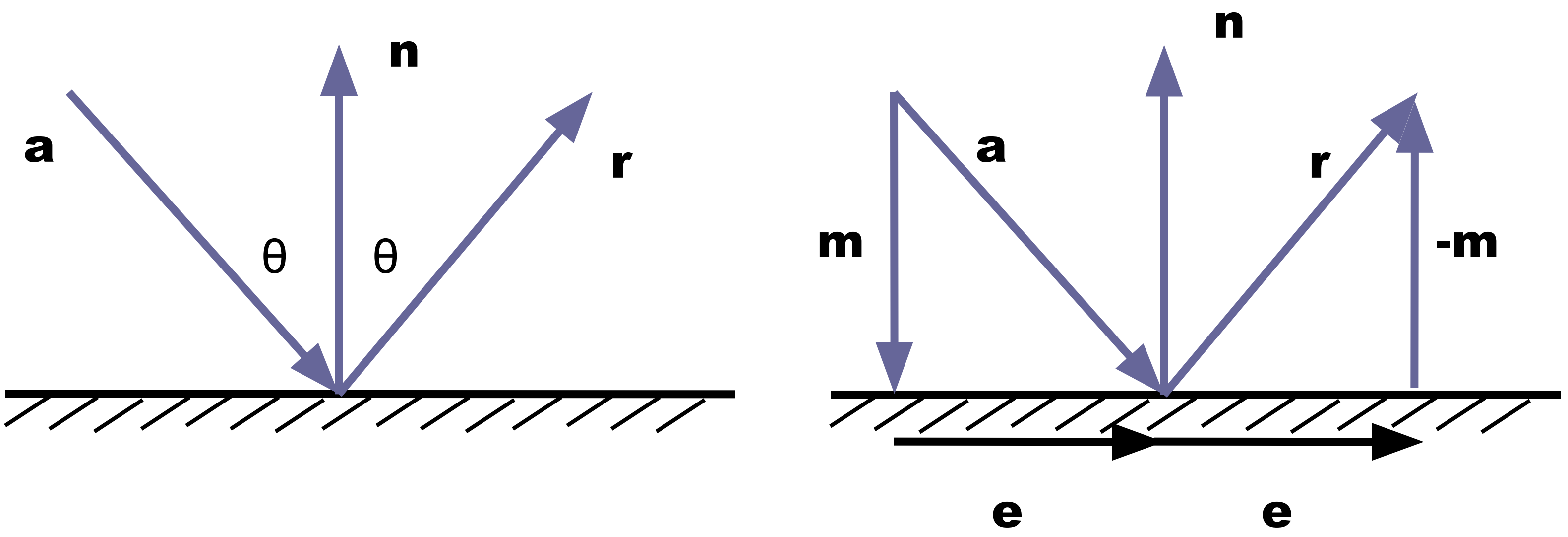


Projection and Component

Projection negative.
Component along **v**



Reflection [13]



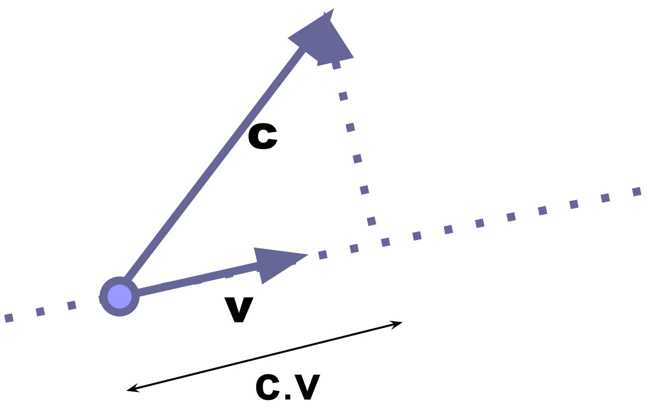
$$\bar{r} = \bar{a} - 2(\bar{a} \cdot \hat{n})\hat{n}$$

- Here **m** is the orthogonal projection of **a** along **n**
- **m** equals (**a.n**)**n** as **n** is a unit vector

Projection and Component

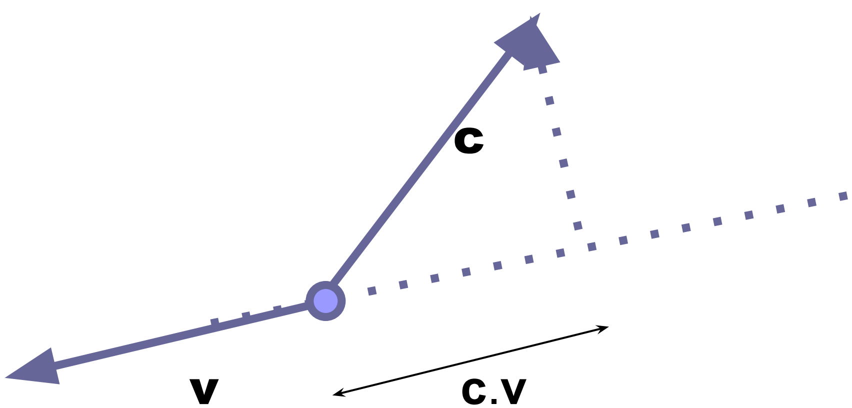
Projection positive.

Component along **v**



Projection and Component

Projection negative.
Component along **v**



Cross Product

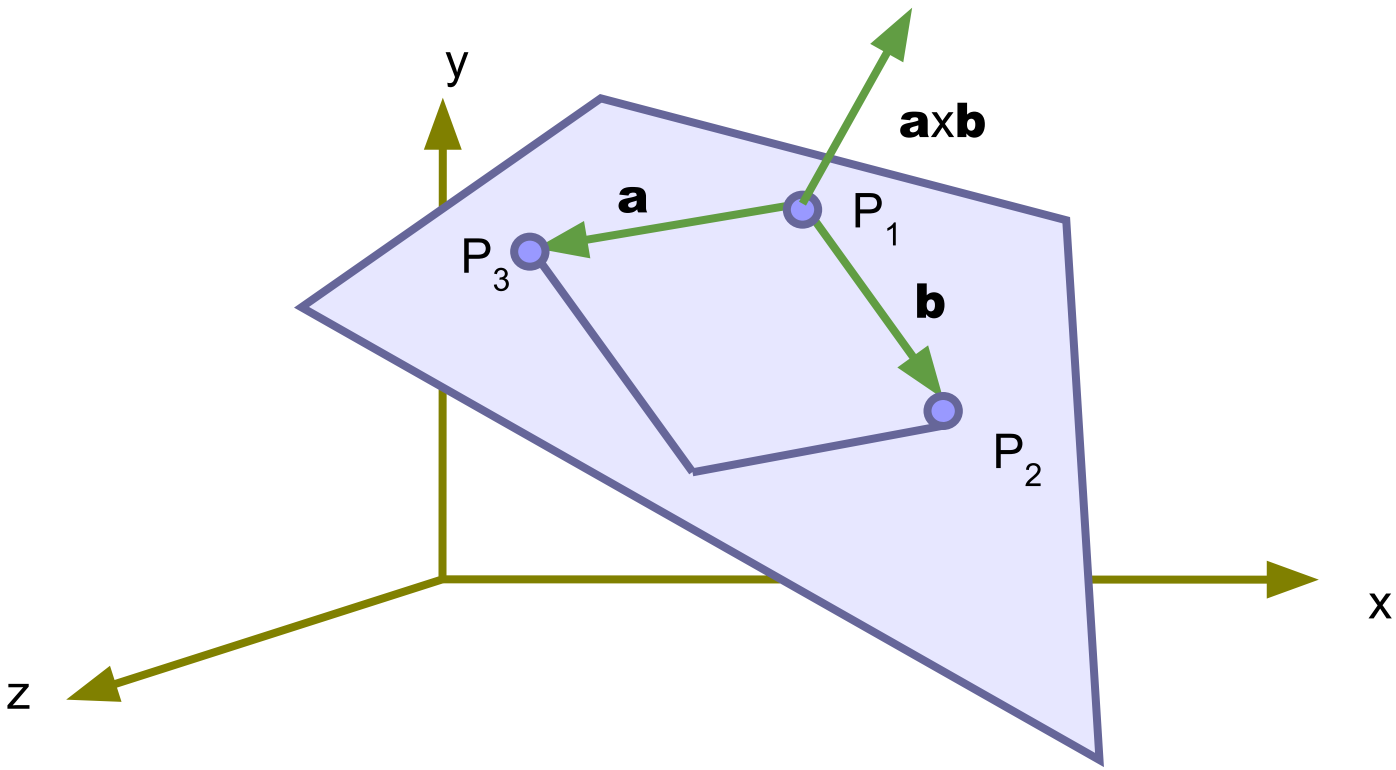
- Also called **vector product**.
- Defined for **3D** vectors only.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties

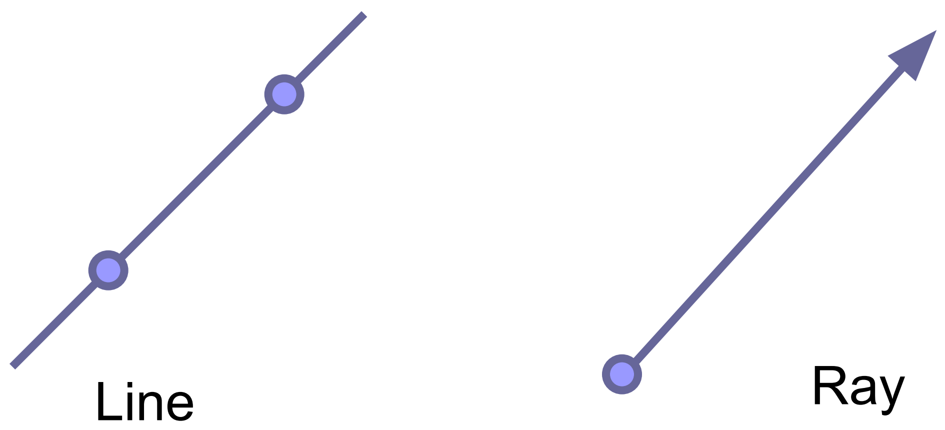
1. Antisymmetry: **a** X **b** = - **b** X **a**
2. Linearity: (**a** **+c**) X **b** = **a** X **b** + **c** X **b**
3. Homogeneity: (s**a**) X **b** = s(**a** X **b**)

Geometric Interpretation of Cross Product



1. **aXb** is perpendicular to both **a** and **b**
2. | **aXb |** = area of the parallelogram defined by **a** and **b**

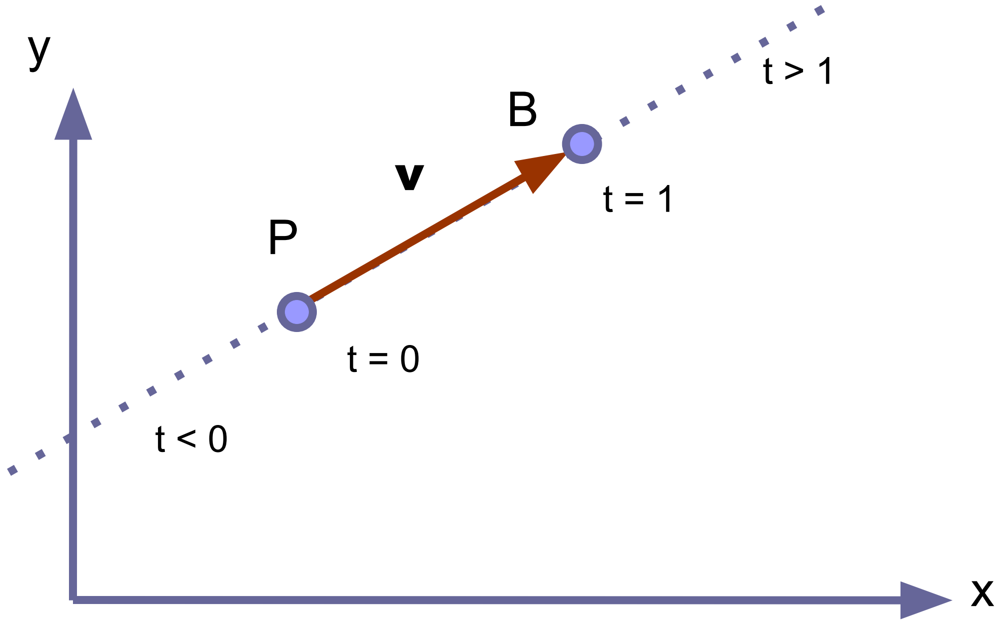
Representing Lines [11][12]



2 types of representations:

1. Two point form
2. Parametric representation

Parametric Representation of a Line

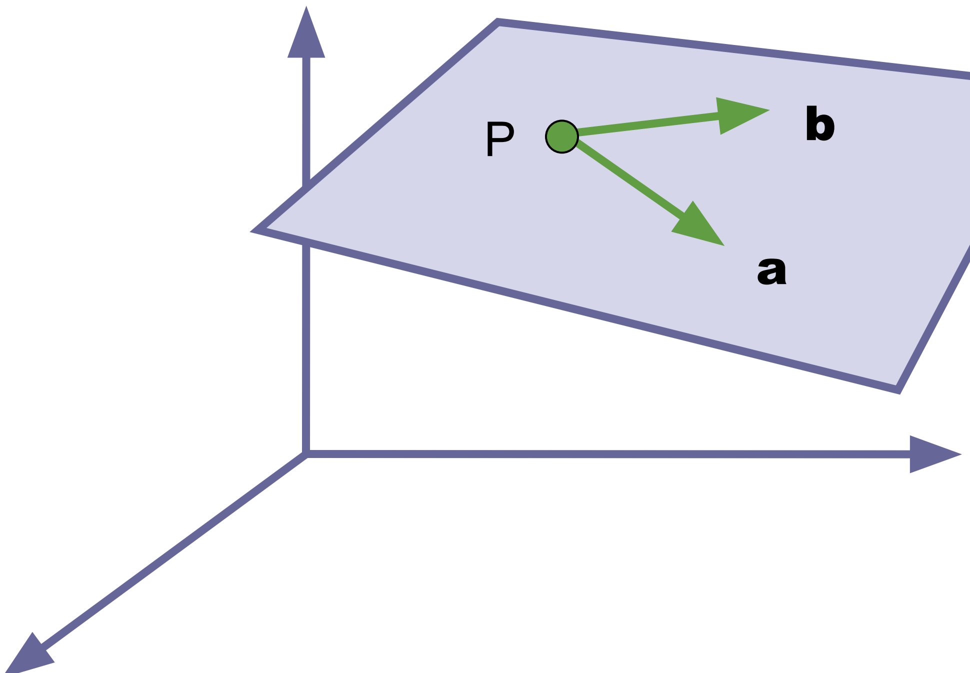


$$L(t) = P + t\boldsymbol{v}$$

Planes in 3D

- 4 fundamental forms
  - Three-point form
  - Parametric representation
  - Point normal form
  - Equation

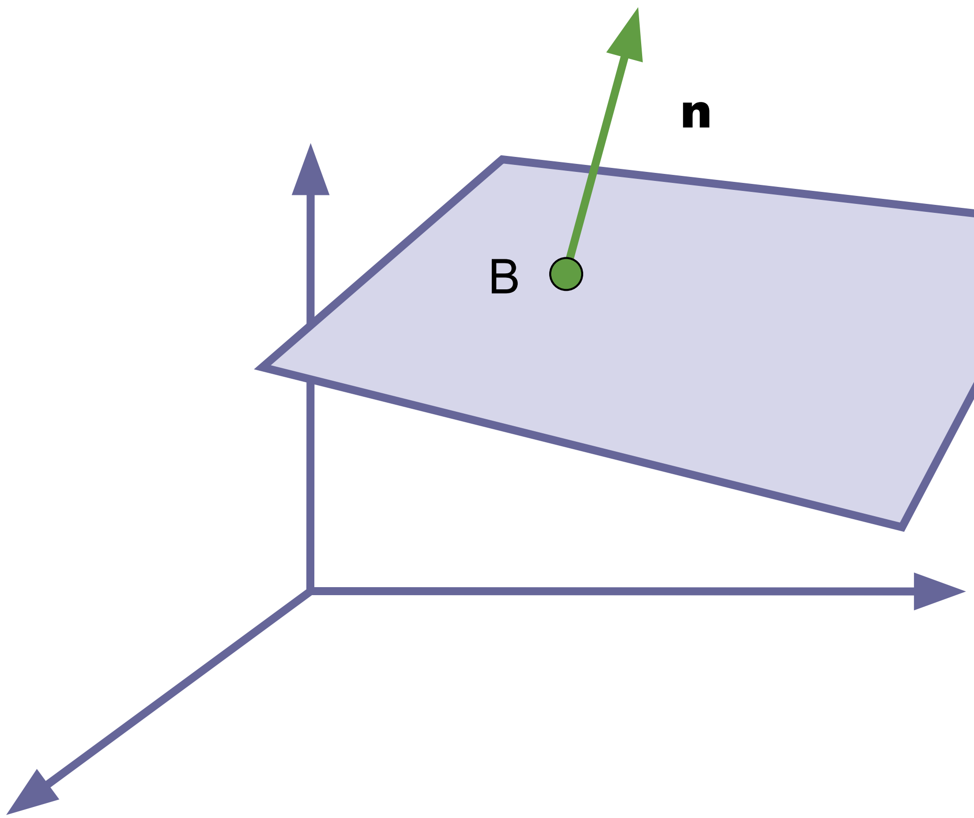
Parametric Representation of Plane



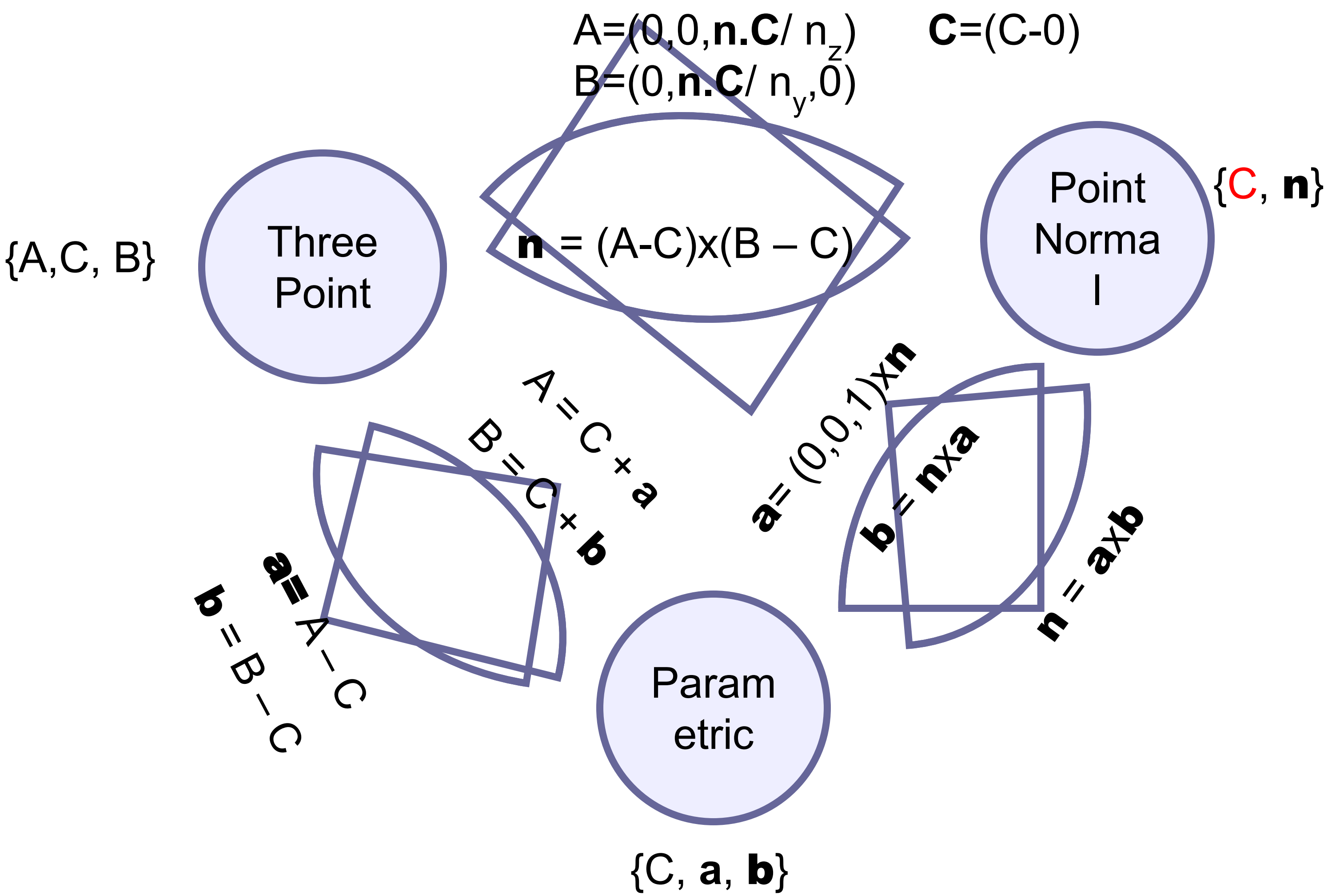
$$U(s,t) = P + s\boldsymbol{a} + t\boldsymbol{b}$$

a and **b** cannot be parallel

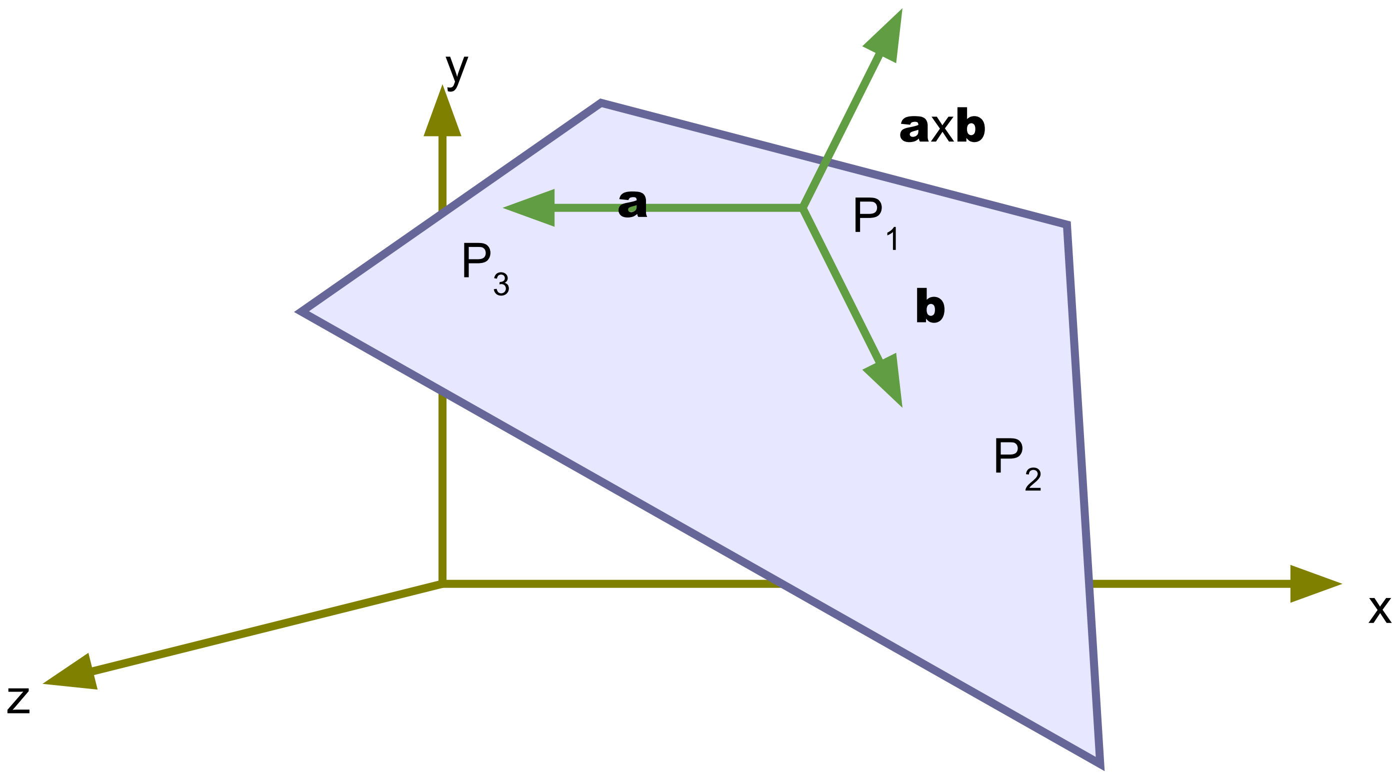
Point Normal Form of a Plane



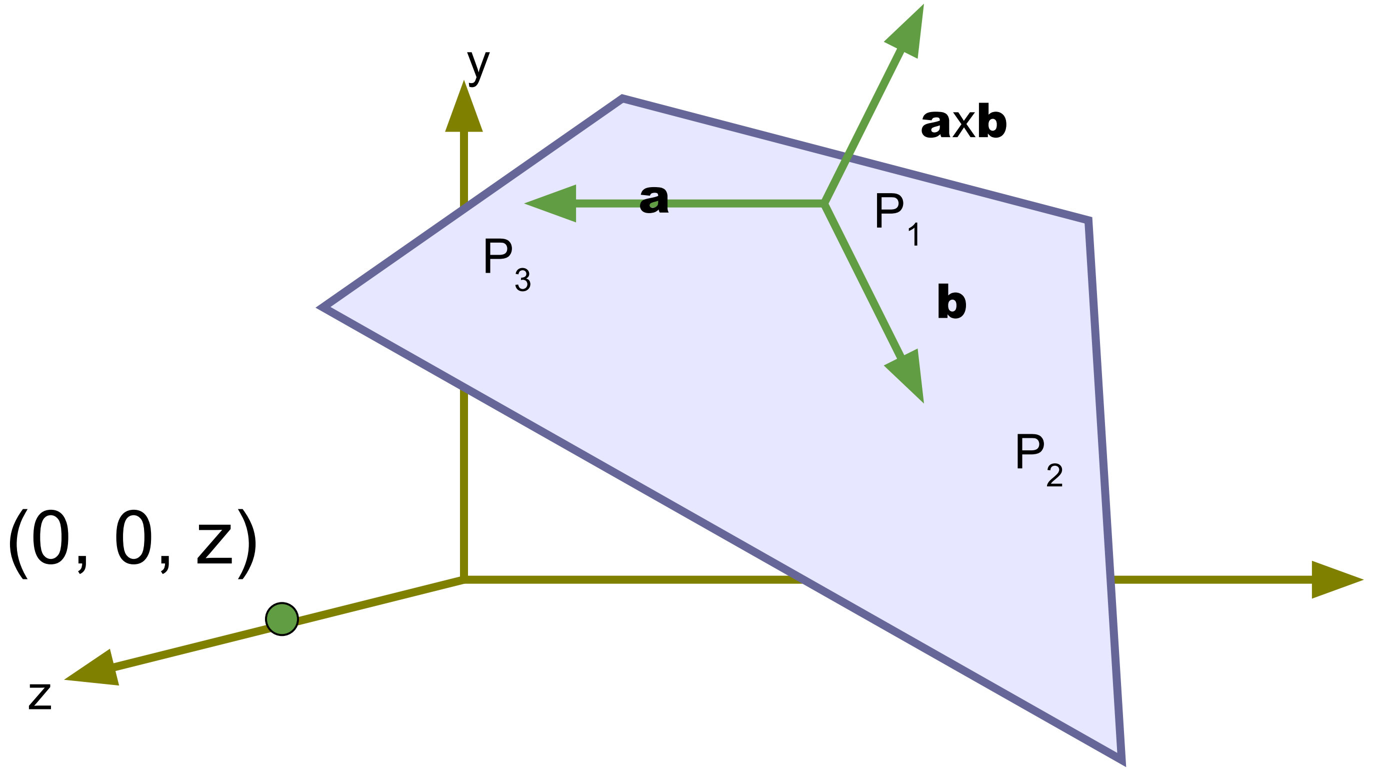
Representations of Plane [13]



Geometric Interpretation of Cross Product

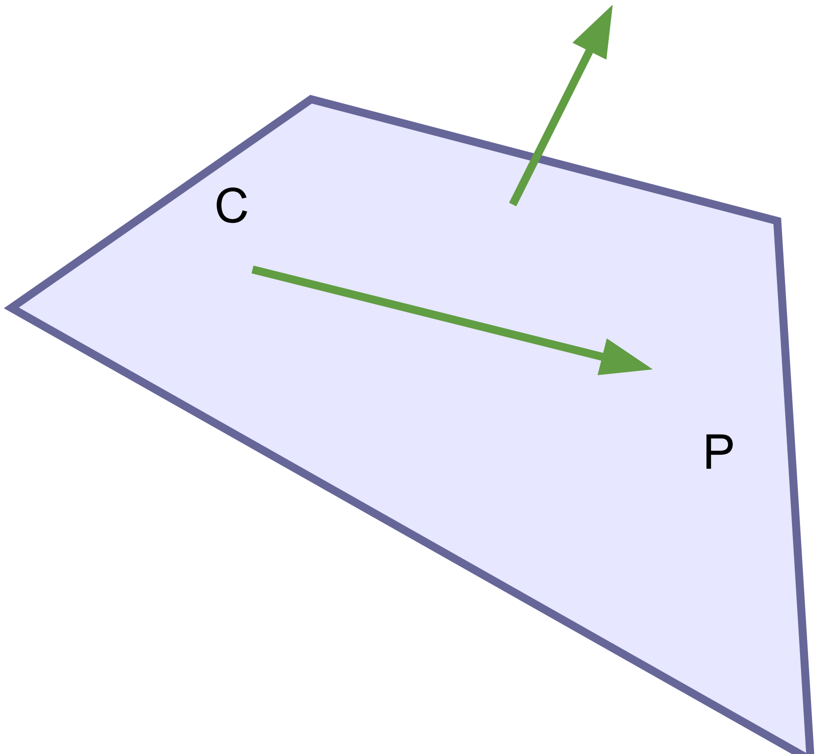


Point normal to 3 point form



Equation of a Plane from Point normal form

- Given {C, <u>n</u>}
- We take any point P(x, y, z)
- <u>n</u>*<u>PC</u> = 0



x

Equation of a Plane from Point normal form

- Given {C, <u>n</u>}
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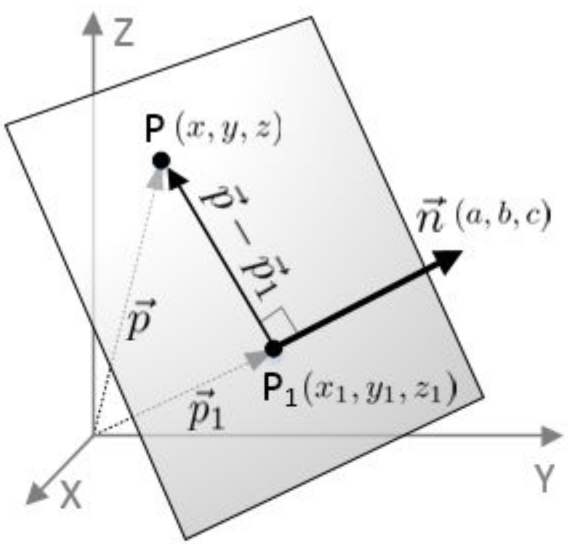
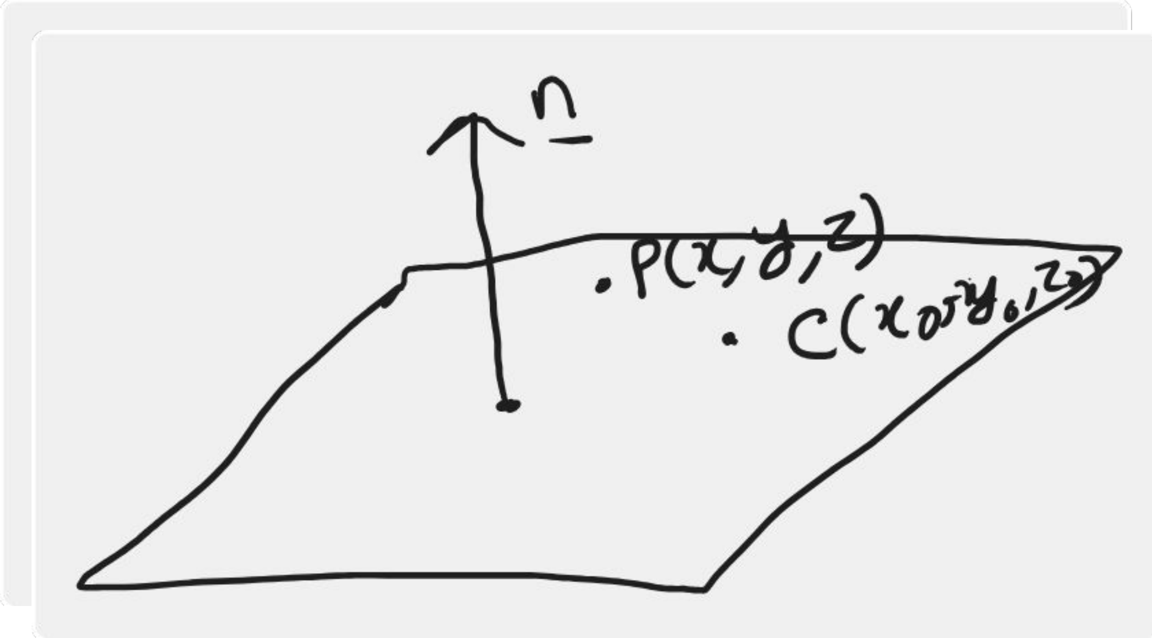
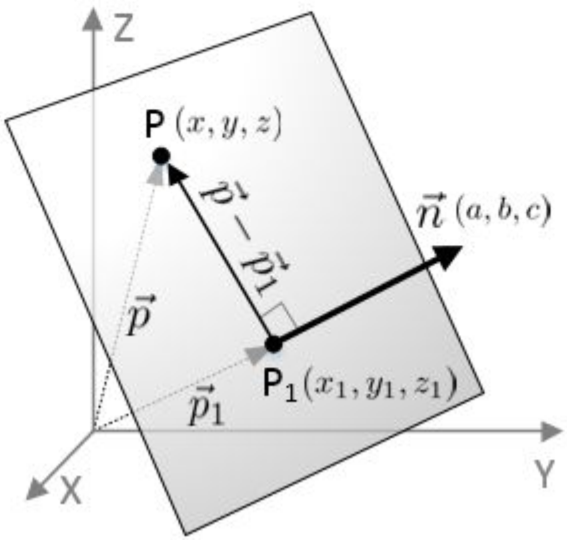




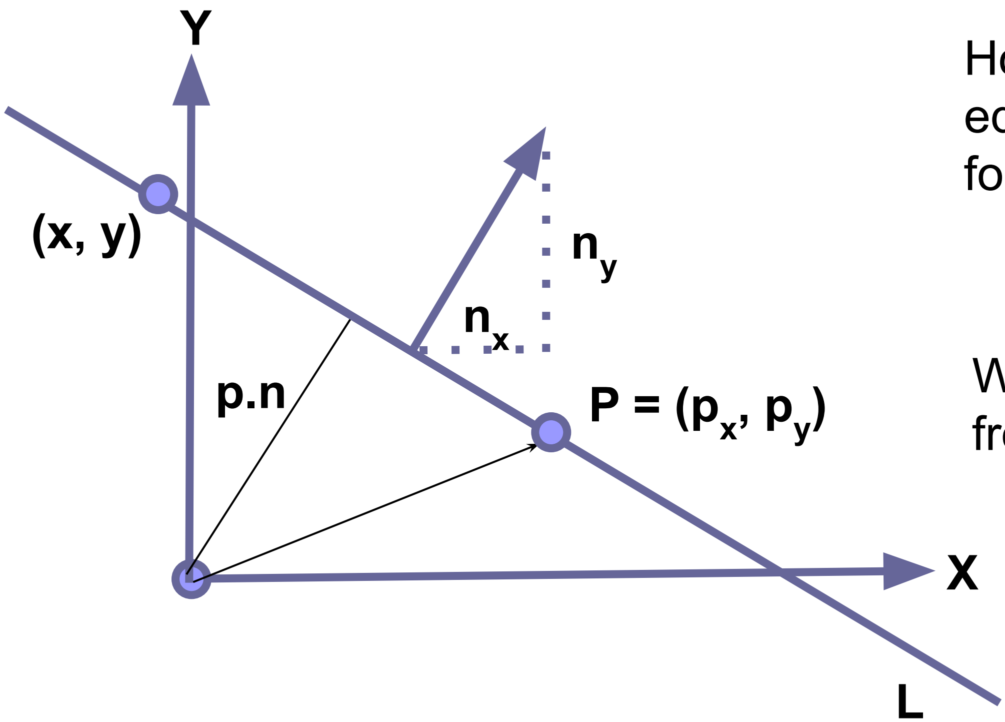
Equation of a Plane from Point normal form

- Given {C, n}
- We take any point P(x, y, z)
- n*PP_1 = 0



Equation of a Plane [1][2][3][4]

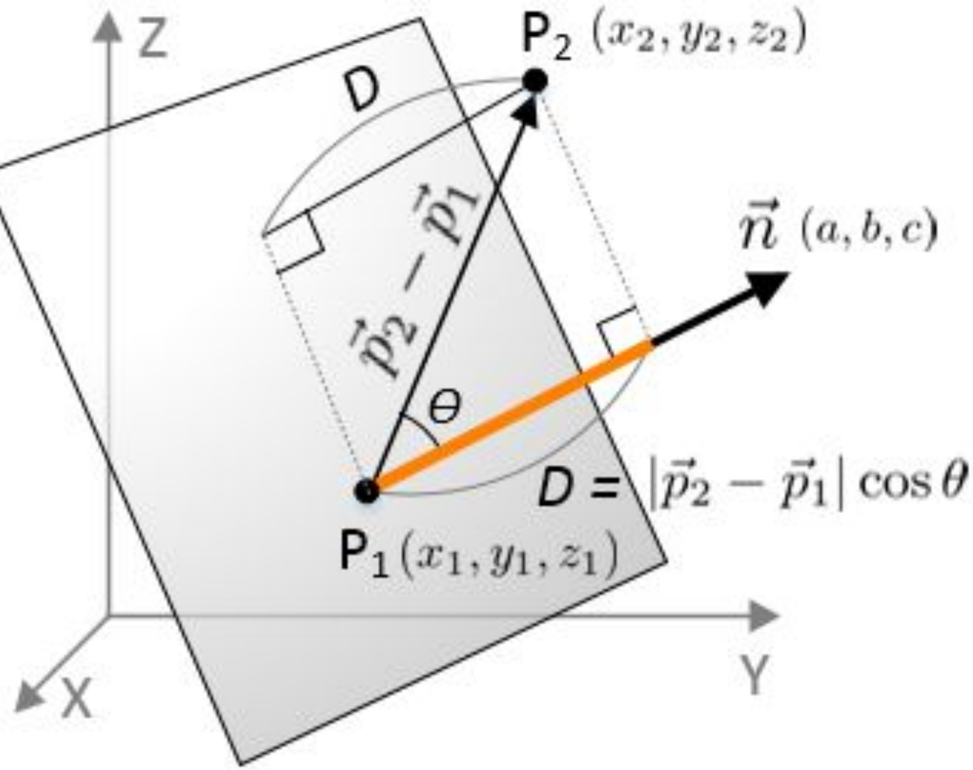
- $ax + by + cz + d = 0$ is the standard equation of a plane in 3d
- If sqrt(a^2 + b^2 + c^2) = 1, then it is called the normalized form
- In the normalized form, |d| equals the distance of the plane from the origin



How to convert between equation and the other forms?

What about distance from an arbitrary point?

Distance from an any arbitrary point



Exercise:

the distance from a point (-1, -2, -3) to a plane x + 2y + 2z - 6 = 0 is

Line-Plane Intersection [8]

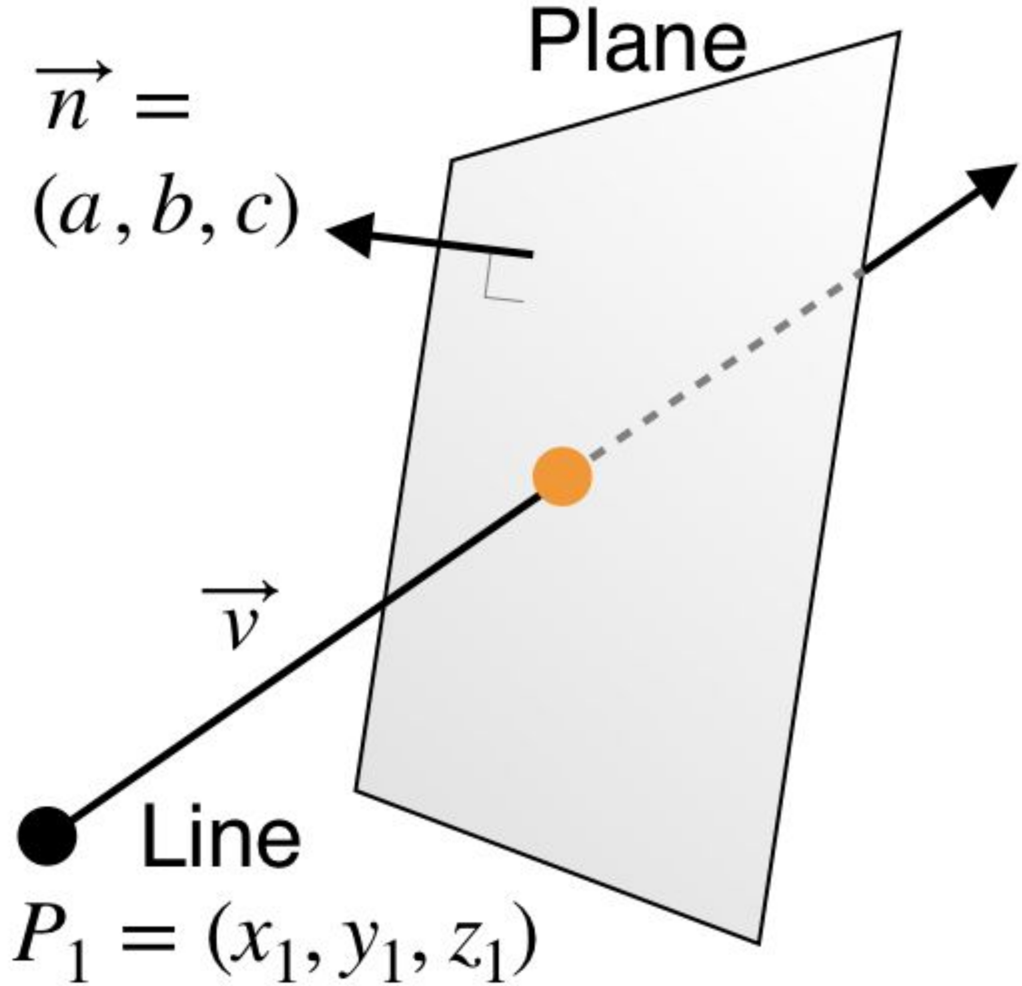
- Plane: ax + by + cz + d = 0
- Line: P + tV
- Determine the specific value of t (say t') for which the equation of the plane is satisfied, i.e., the point on the line lies on the plane

Exercise:

Find intersection of $x + 2y + 3z + 4 = 0$ and $(1, 2, 3) + t(3, 2, 1)$.

$$\vec{n} \cdot P_1 = ax_1 + by_1 + cz_1$$
$$\vec{n} \cdot \vec{v} = av_x + bv_y + cv_z$$

$$\therefore \; t = \frac{-(\vec{n} \cdot P_1 + d)}{\vec{n} \cdot \vec{v}}$$



$$
\begin{aligned}
ax + by + cz + d &= 0 \\
a(x_1 + tv_x) + b(y_1 + tv_y) + c(z_1 + tv_z) + d &= 0 \quad (\text{substitute } (x_1 + tv_x, y_1 + tv_y, z_1 + tv_z)) \\
ax_1 + by_1 + cz_1 + d + t(av_x + bv_y + cv_z) &= 0 \\
t(av_x + bv_y + cv_z) &= -(ax_1 + by_1 + cz_1 + d) \\
t &= \frac{-(ax_1 + by_1 + cz_1 + d)}{(av_x + bv_y + cv_z)}
\end{aligned}
$$

Line-Line Intersection [5][6][7]

- Four possible cases:
  - Coincident
  - Parallel
  - Not parallel and do not intersect
  - Not parallel and intersect

Line-Line Intersection

- L_1: $P_1 + tV_1$
- L_2: $P_2 + sV_2$
- Parallel if V_1 and V_2 are in the same or opposite direction (i.e., the angle between them are 0 degree or 180 degree)
- Coincident if they are parallel and have at least one point in common
- If they are not parallel, how to decide whether they intersect or not?

t (V1 X V2) = (L2 - L1) X V2

More on

Line-Line Intersection

- If they are not parallel, how to decide whether they intersect or not?
- One solution
  - Generate three equations for two unknowns
  - Solve the first two equations to find a solution
  - Check whether the solution satisfies the third equation
- Another solution
  - Check whether $(P_1 – P_2).(V_1 \text{ X } V_2) = 0$
  - If lines intersect this condition must hold

Line-Line Intersection

- If they do not intersect what is the minimum distance between them

Example: Given two lines (0,0,0) + s(1, 0, 0), (0, 0, 1) + t(1, 1, 0), find the minimum distance between them.

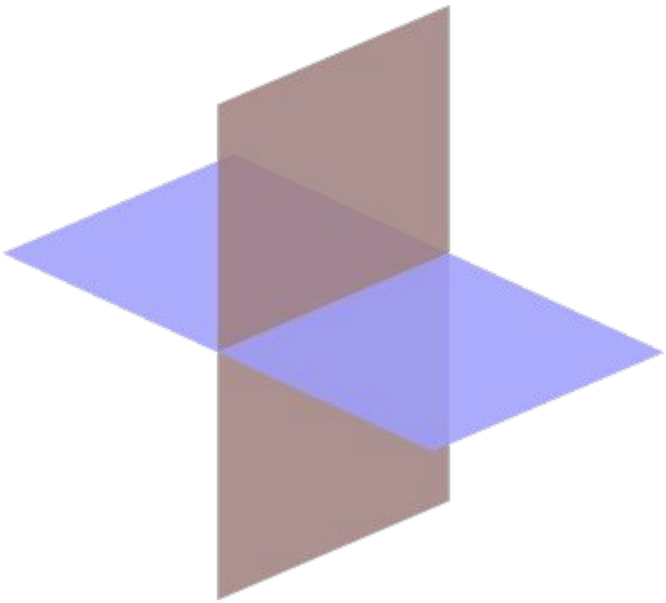
Line-Line Intersection

Exercise:

Find intersection between $(1,2,3) + t(1,1,1)$ and $(1,1,1) + s(1,2,3)$.

Plane-Plane Intersection [9][10]

- Intersection is a line
- So we need two points on the line, or one point and the direction
- How to get a point on the intersecting line?
- How to get the direction of the intersecting line?



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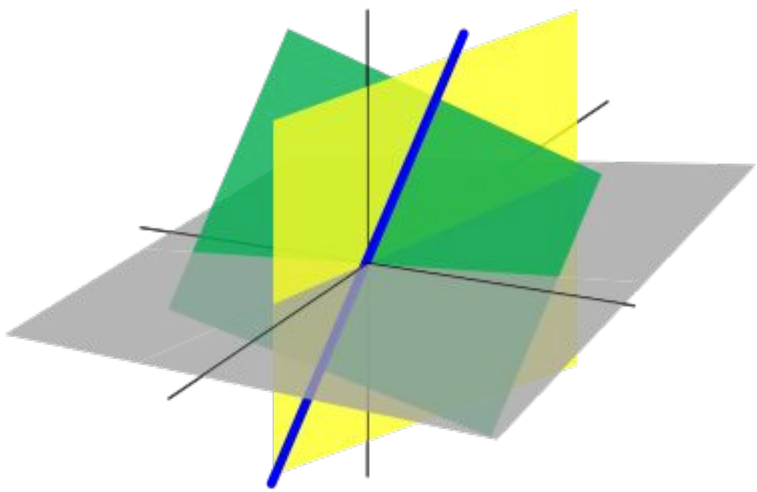
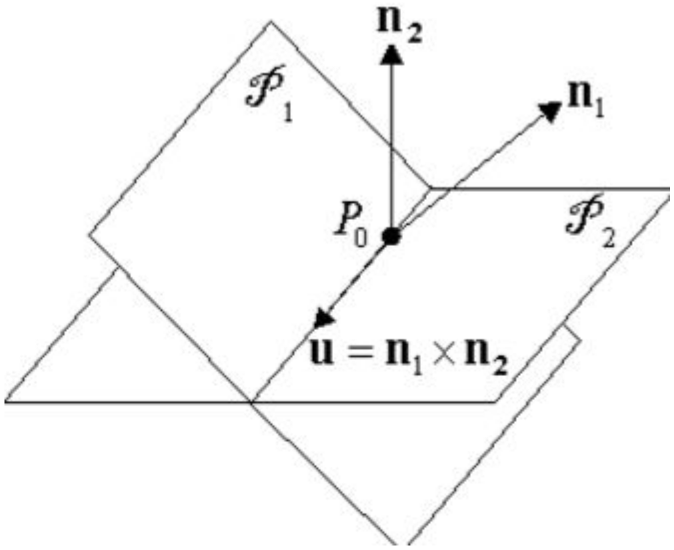


Plane-Plane Intersection

- How to get a point on the intersecting line?
  - Imagine another plane not parallel **to the intersecting line**, for example the plane z = 0 (the XY plane).
  - Now solve **three equations** to find their common intersection point

Plane-Plane Intersection

- How to get the direction of the intersecting line?
- Consider the planes in point-normal form
  - Plane 1: P_1, n_1
  - Plane 2: P_2, n_2
- n_1 and n_2 are both perpendicular to the intersecting line
- So the direction of the line of intersection is along n_1 X n_2

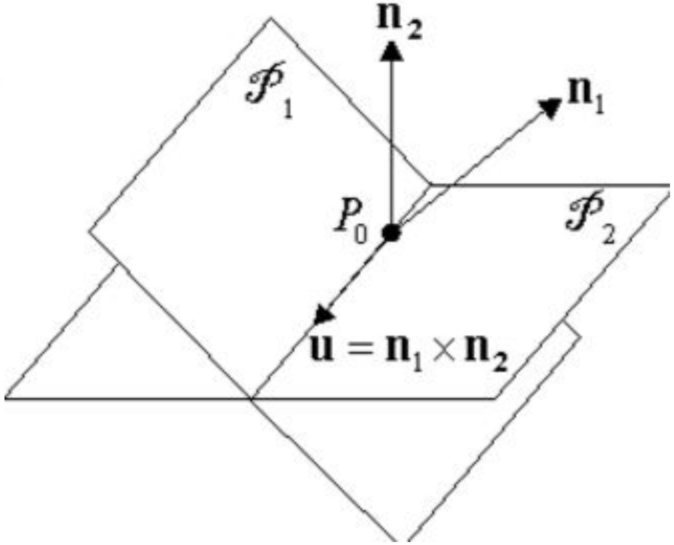


Plane-Plane Intersection

Exercise:

Find intersection of $x + 2y - z = 5$

and $x - 4y + z = 3$.

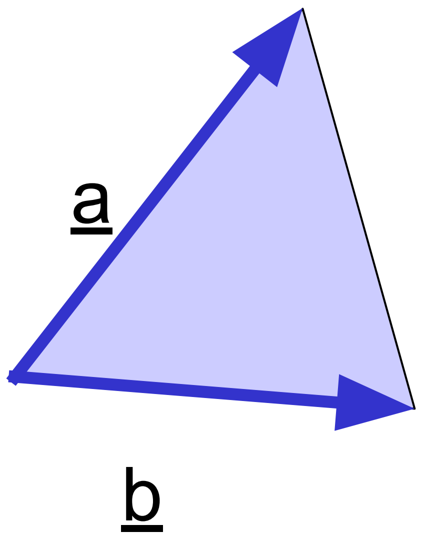


Linear combination of vector

- $\underline{r} = c_0\,\underline{a} + c_1\,\underline{b}$
- Affine if,

  $c_0 + c_1 = 1$

- Convex if,
  - Affine and
  - $0 \le c_0$ and $0 \le c_1$



Affine combination of vector

- $\underline{r} = c_0\,\underline{a} + c_1\,\underline{b}$
- Affine if,

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- Convex if,
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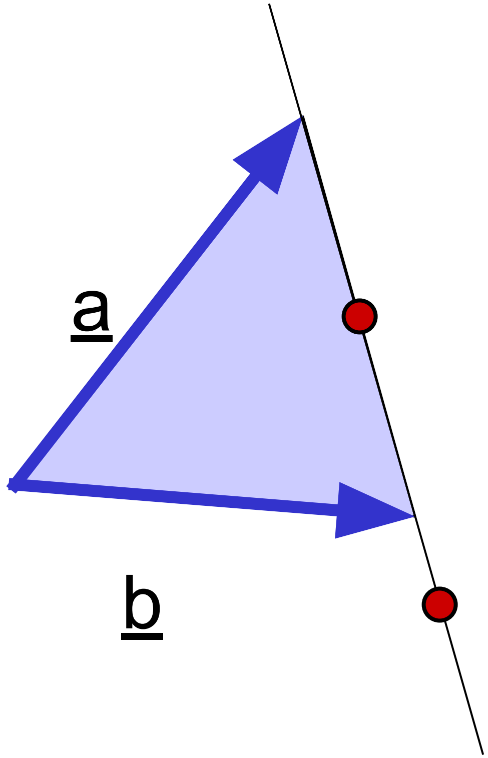
  - $0 \le c_0$ and $0 \le c_1$

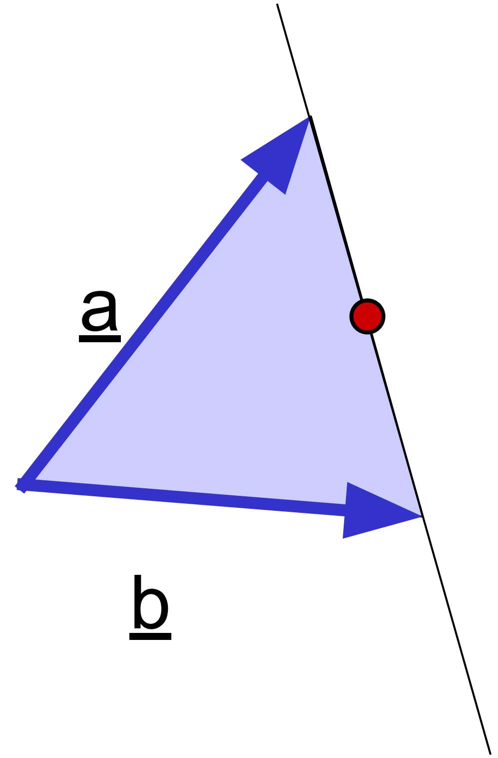


Convex combination of vector

- $\underline{r} = c_0\, \underline{a} + c_1\, \underline{b}$
- Affine if,

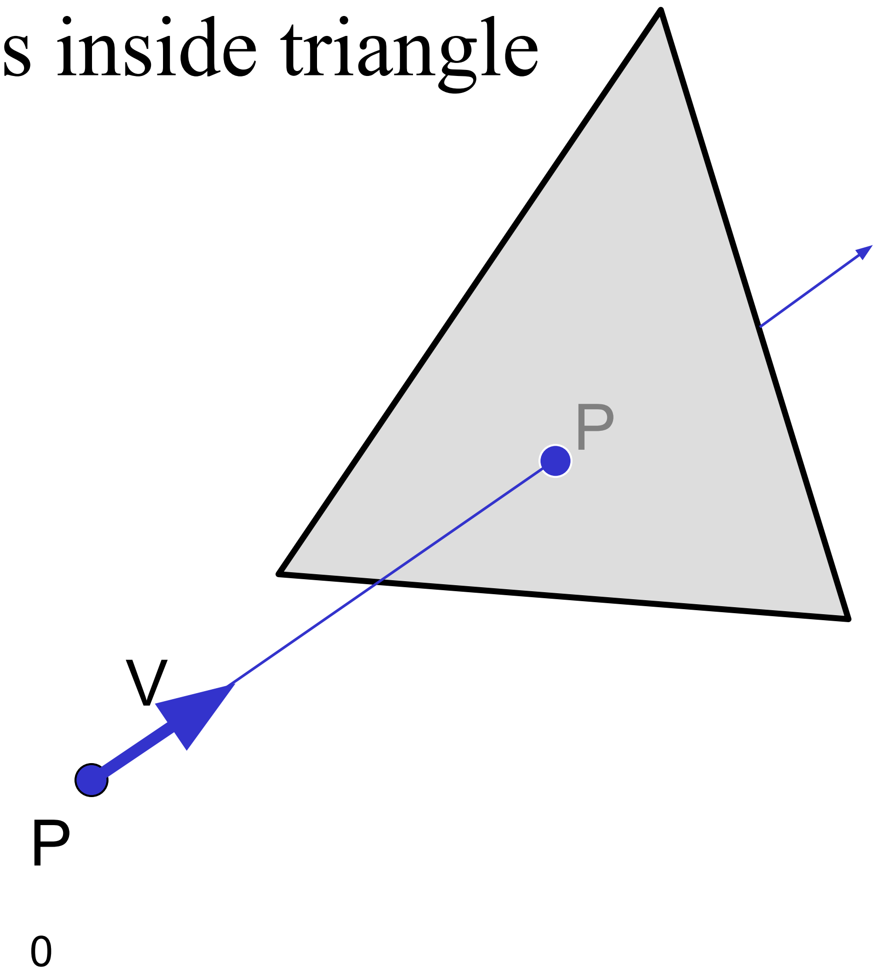
  $c_0 + c_1 = 1$

- Convex if,
  - Affine and
  - $0 \le c_0$ and $0 \le c_1$



Ray-Triangle Intersection

- First, intersect ray with plane
- Then, check if point is inside triangle



Ray-Triangle Intersection

- Check if point is inside triangle parametrically

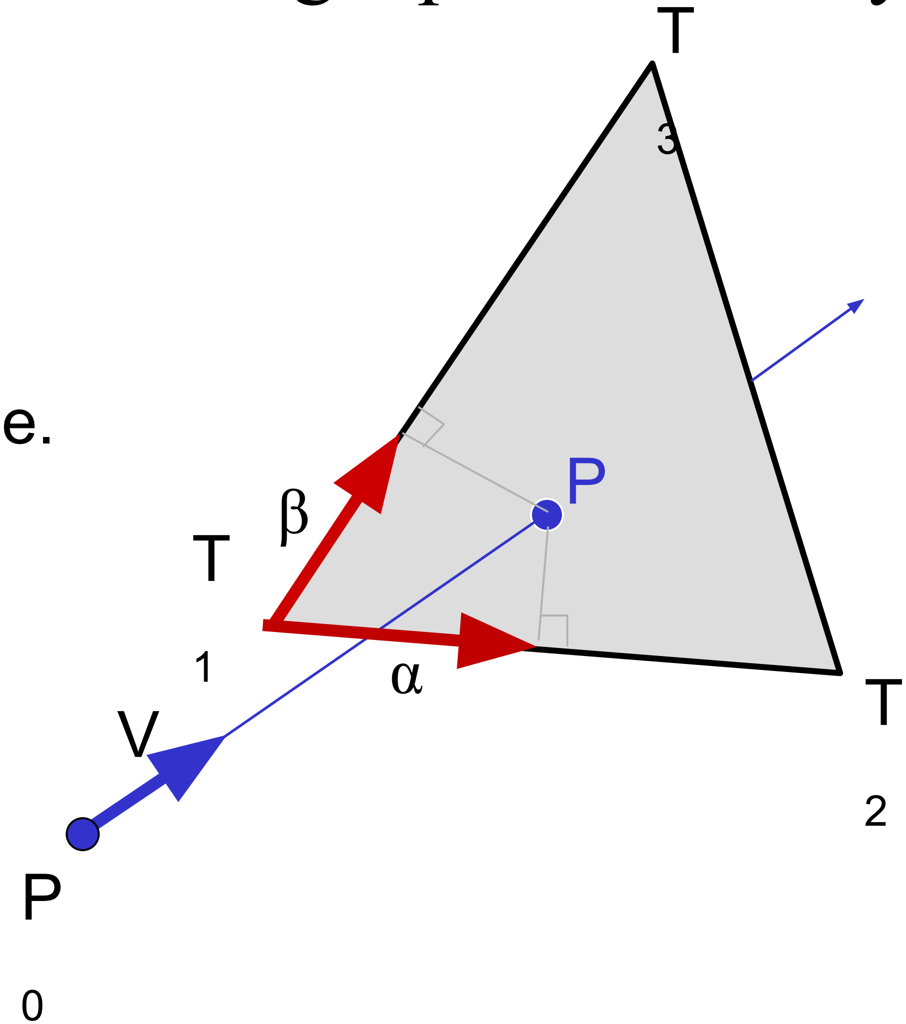
Compute α, β:

$$P = \alpha (T_2 - T_1) + \beta (T_3 - T_1)$$

Check if point inside triangle.

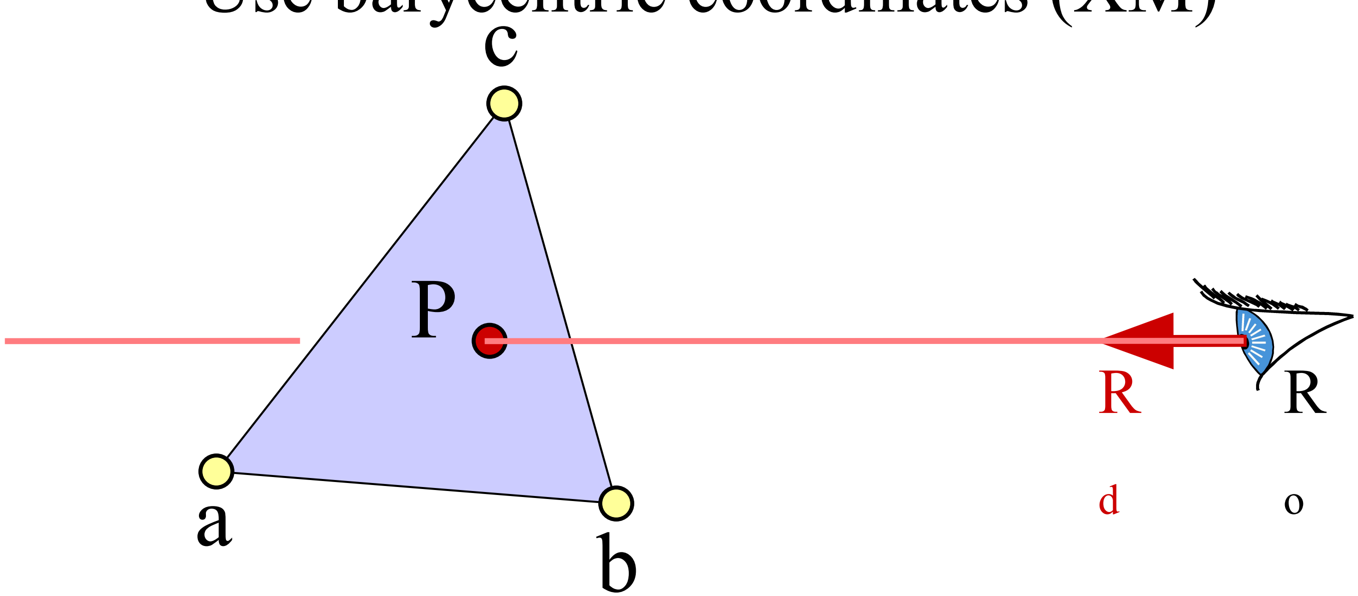
$$0 \leq \alpha \leq 1 \text{ and } 0 \leq \beta \leq 1$$

$$\alpha + \beta \leq 1$$

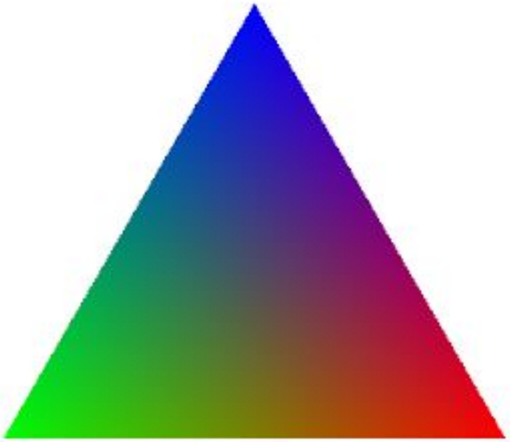


Ray-Triangle Intersection

- Use general ray-polygon
- Or try to be smarter
  - Use barycentric coordinates (XM)



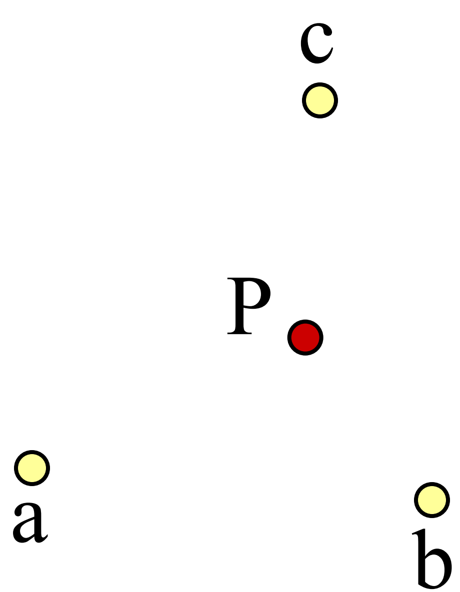
Barycentric triangle: Motivation



Barycentric Definition of a Plane

[Möbius, 1827]

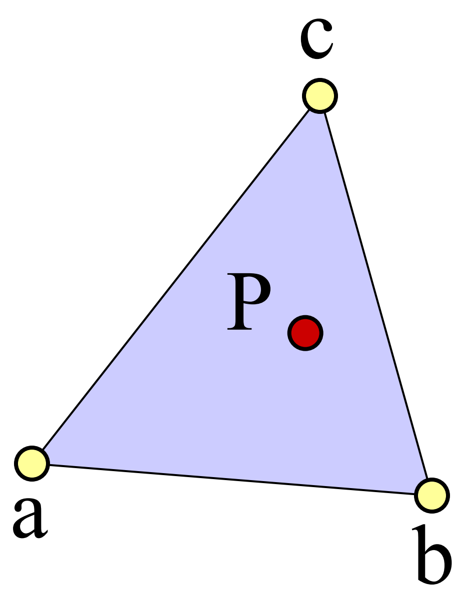
- $P(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$
  with $\alpha + \beta + \gamma = 1$



P is the *barycenter*:
the single point upon which the plane would balance if weights of size α, β, & γ are placed on points a, b, & c.

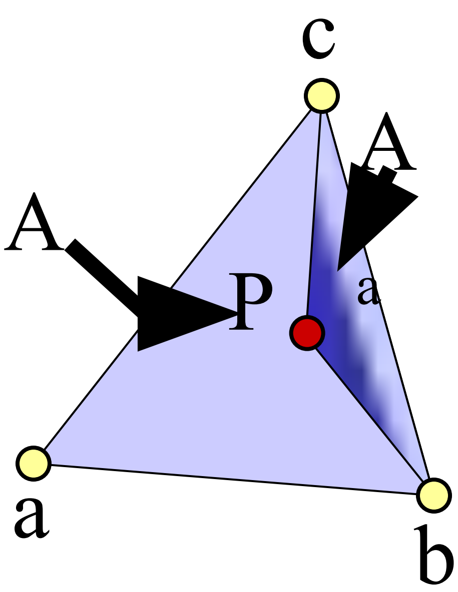
Barycentric Definition of a Triangle

- $P(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$
  with $\alpha + \beta + \gamma = 1$
- AND $0 < \alpha < 1$ & $0 < \beta < 1$ & $0 < \gamma < 1$



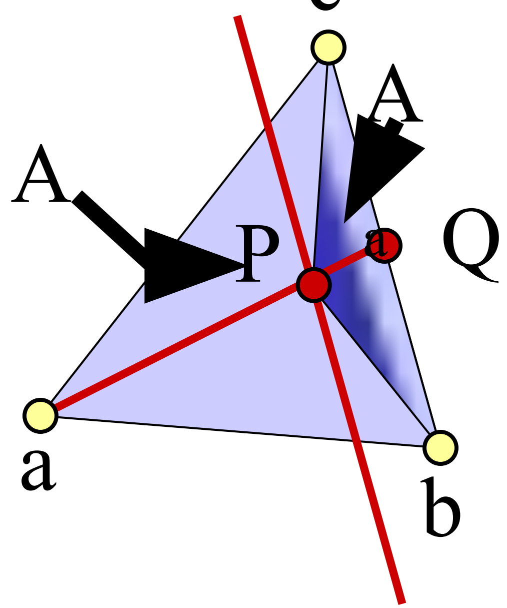
How Do We Compute α, β, γ ?

- Ratio of opposite sub-triangle area to total area
  - $\alpha = A_a/A \qquad \beta = A_b/A \qquad \gamma = A_c/A$
- Use signed areas for points outside the triangle



Intuition Behind Area Formula

- P is barycenter of a and Q
- A_a is the interpolation coefficient on aQ
- All points on lines parallel to bc have the same α (All such triangles have same height/area)



Simplify

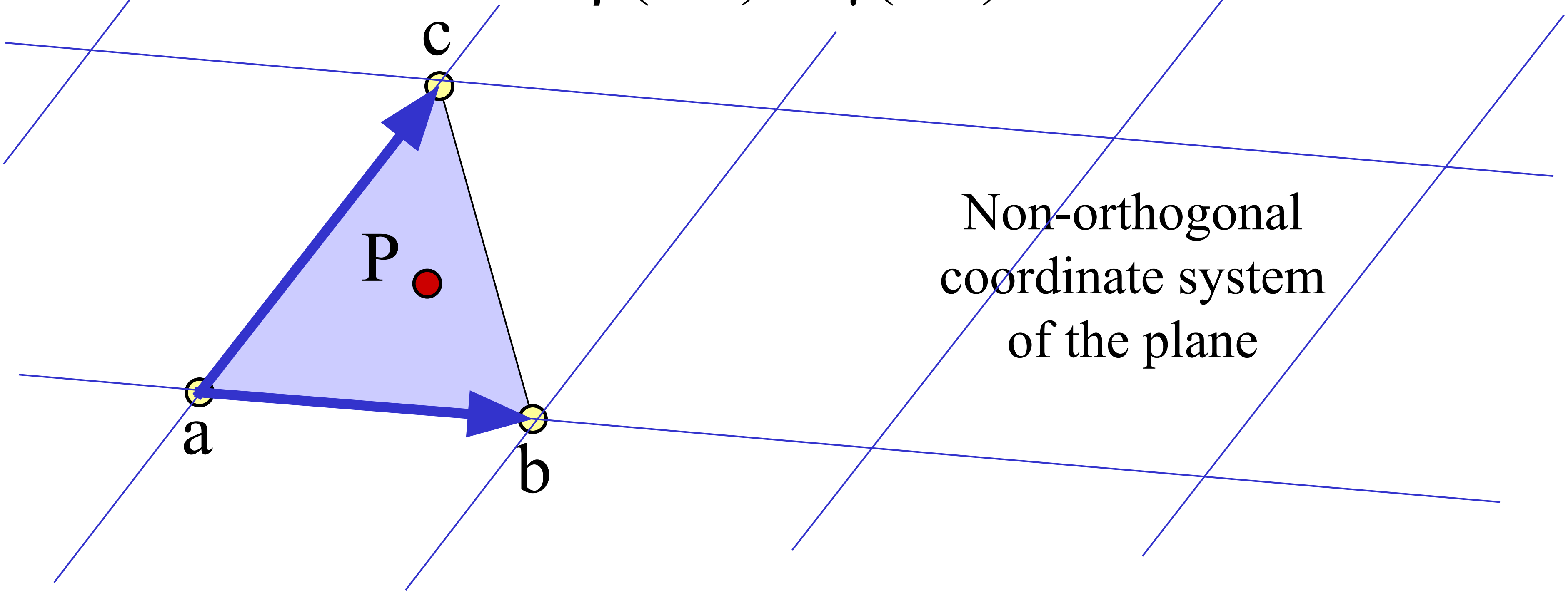
- Since $\alpha + \beta + \gamma = 1$, we can write $\alpha = 1 - \beta - \gamma$

$$P(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$$

rewrite

$$P(\beta, \gamma) = (1-\beta-\gamma)a + \beta b + \gamma c$$

$$= a + \beta(b-a) + \gamma(c-a)$$



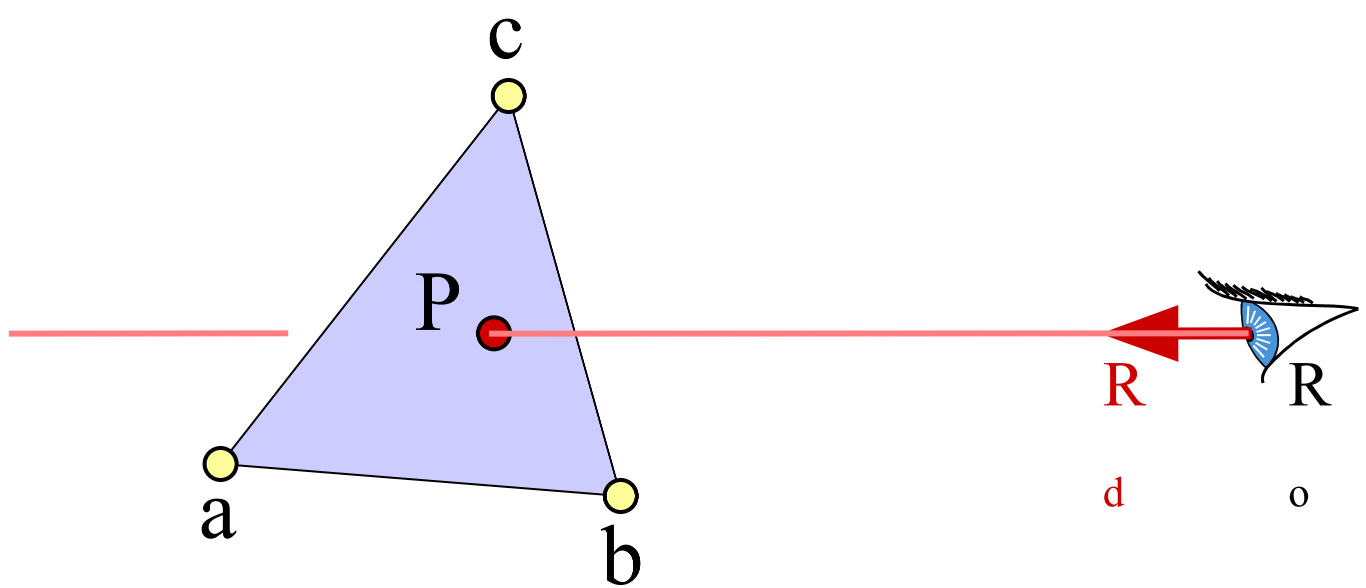
Intersection with Barycentric Triangle

- Set ray equation equal to barycentric equation

$$P(t) = P(\beta, \gamma)$$

$$P_o + t * \underline{v} = a + \beta(b-a) + \gamma(c-a)$$

- Intersection if $\beta + \gamma < 1$ & $\beta > 0$ & $\gamma > 0$



Links

[1] http://www.songho.ca/math/plane/plane.html

[2] http://mathinsight.org/distance_point_plane

[3] https://www.youtube.com/watch?v=gw-4wltP5tY

[4] https://www.youtube.com/watch?v=7rIFO8hct9g

[5] https://www.youtube.com/watch?v=nKVCvY-FW5Q

[6] https://www.youtube.com/watch?v=bJ56Xr9081k

[7] https://www.youtube.com/watch?v=r5DwyBFxD7Q

[8] https://www.youtube.com/watch?v=Td9CZGkqrSg

[9] https://www.youtube.com/watch?v=SoSTdgqknvY

[10] https://www.youtube.com/watch?v=LpardiBTAvU

Links

[11] https://www.youtube.com/watch?v=FILbI7DB0SM

[12] https://www.youtube.com/watch?v=nZ2mS5M4fcQ

[13] (textbook) Chapter 4, Computer Graphics using OpenGL (2nd edition) by Francis S Hill, Jr.