Thus, $\pi = (520 - 0.1Q_1)Q_1 + (410 - 0.05Q_2)Q_2 - (0.1Q_1^2 + 0.1Q_1Q_2 + 0.2Q_2^2 + 325)$
$$= 520Q_1 - 0.2Q_1^2 + 410Q_2 - 0.25Q_2^2 - 0.1Q_1Q_2 - 325 \tag{6.50}$$

Maximizing (*6.50*),

$$\pi_1 = 520 - 0.4Q_1 - 0.1Q_2 = 0 \qquad \pi_2 = 410 - 0.5Q_2 - 0.1Q_1 = 0$$

Thus, $\bar{Q}_1 = 1152.63$ and $\bar{Q}_2 = 589.47$. Checking the second-order condition, $\pi_{11} = -0.4$, $\pi_{22} = -0.5$, and $\pi_{12} = -0.1 = \pi_{21}$. Since $\pi_{11}, \pi_{22} < 0$ and $\pi_{11}\pi_{22} > (\pi_{12})^2$, π is maximized at $\bar{Q}_1 = 1152.63$ and $\bar{Q}_2 = 589.47$.

b) Substituting in (*6.48*) and (*6.49*),

$$P_1 = 520 - 0.1(1152.63) = 404.74 \qquad P_2 = 410 - 0.05(589.47) = 380.53$$

c) $$\pi = 420{,}201.32$$

CONSTRAINED OPTIMIZATION IN ECONOMICS

6.28. (*a*) What combination of goods x and y should a firm produce to minimize costs when the joint cost function is $c = 6x^2 + 10y^2 - xy + 30$ and the firm has a production quota of $x + y = 34$? (*b*) Estimate the effect on costs if the production quota is reduced by 1 unit.

a) Form a new function by setting the constraint equal to zero, multiplying it by λ, and adding it to the original or objective function. Thus,

$$C = 6x^2 + 10y^2 - xy + 30 + \lambda(34 - x - y)$$
$$C_x = 12x - y - \lambda = 0$$
$$C_y = 20y - x - \lambda = 0$$
$$C_\lambda = 34 - x - y = 0$$

Solving simultaneously, $\bar{x} = 21$, $\bar{y} = 13$, and $\bar{\lambda} = 239$. Thus, $C = 4093$. Second-order conditions are discussed in Section 12.5.

b) With $\lambda = 239$, a decrease in the constant of the constraint (the production quota) will lead to a cost reduction of approximately 239.

6.29. (*a*) What output mix should a profit-maximizing firm produce when its total profit function is $\pi = 80x - 2x^2 - xy - 3y^2 + 100y$ and its maximum output capacity is $x + y = 12$? (*b*) Estimate the effect on profits if output capacity is expanded by 1 unit.

a)
$$\Pi = 80x - 2x^2 - xy - 3y^2 + 100y + \lambda(12 - x - y)$$
$$\Pi_x = 80 - 4x - y - \lambda = 0$$
$$\Pi_y = -x - 6y + 100 - \lambda = 0$$
$$\Pi_\lambda = 12 - x - y = 0$$

When solved simultaneously, $\bar{x} = 5$, $\bar{y} = 7$, and $\bar{\lambda} = 53$. Thus, $\pi = 868$.

b) With $\bar{\lambda} = 53$, an increase in output capacity should lead to increased profits of approximately 53.

6.30. A rancher faces the profit function

$$\pi = 110x - 3x^2 - 2xy - 2y^2 + 140y$$

where x = sides of beef and y = hides. Since there are two sides of beef for every hide, it follows that output must be in the proportion

$$\frac{x}{2} = y \qquad x = 2y$$

At what level of output will the rancher maximize profits?

$$\begin{aligned}\Pi &= 110x - 3x^2 - 2xy - 2y^2 + 140y + \lambda(x - 2y)\\ \Pi_x &= 110 - 6x - 2y + \lambda = 0\\ \Pi_y &= -2x - 4y + 140 - 2\lambda = 0\\ \Pi_\lambda &= x - 2y = 0\end{aligned}$$

Solving simultaneously, $\bar{x} = 20$, $\bar{y} = 10$, $\bar{\lambda} = 30$, and $\pi = 1800$.

6.31. (*a*) Minimize costs for a firm with the cost function $c = 5x^2 + 2xy + 3y^2 + 800$ subject to the production quota $x + y = 39$. (*b*) Estimate additional costs if the production quota is increased to 40.

a)
$$\begin{aligned}C &= 5x^2 + 2xy + 3y^2 + 800 + \lambda(39 - x - y)\\ C_x &= 10x + 2y - \lambda = 0\\ C_y &= 2x + 6y - \lambda = 0\\ C_\lambda &= 39 - x - y = 0\end{aligned}$$

When solved simultaneously, $\bar{x} = 13$, $\bar{y} = 26$, $\bar{\lambda} = 182$, and $c = 4349$.

b) Since $\bar{\lambda} = 182$, an increased production quota will lead to additional costs of approximately 182.

6.32. A monopolistic firm has the following demand functions for each of its products x and y:

$$x = 72 - 0.5P_x \tag{6.51}$$

$$x = 120 - P_y \tag{6.52}$$

The combined cost function is $c = x^2 + xy + y^2 + 35$, and maximum joint production is 40. Thus $x + y = 40$. Find the profit-maximizing level of (*a*) output, (*b*) price, and (*c*) profit.

a) From (*6.51*) and (*6.52*),

$$P_x = 144 - 2x \tag{6.53}$$

$$P_y = 120 - y \tag{6.54}$$

Thus, $\pi = (144 - 2x)x + (120 - y)y - (x^2 + xy + y^2 + 35) = 144x - 3x^2 - xy - 2y^2 + 120y - 35$.

Incorporating the constraint,

$$\Pi = 144x - 3x^2 - xy - 2y^2 + 120y - 35 + \lambda(40 - x - y)$$

Thus,
$$\begin{aligned}\Pi_x &= 144 - 6x - y - \lambda = 0\\ \Pi_y &= -x - 4y + 120 - \lambda = 0\\ \Pi_\lambda &= 40 - x - y = 0\end{aligned}$$

and, $\bar{x} = 18$, $\bar{y} = 22$, and $\bar{\lambda} = 14$.

b) Substituting in (*6.53*) and (*6.54*),

$$P_x = 144 - 2(18) = 108 \qquad P_y = 120 - 22 = 98$$

c)
$$\pi = 2861$$

6.33. A manufacturer of parts for the tricycle industry sells three tires (x) for every frame (y). Thus,

$$\frac{x}{3} = y \qquad x = 3y$$

If the demand functions are

$$x = 63 - 0.25P_x \tag{6.55}$$

$$y = 60 - \tfrac{1}{3}P_y \tag{6.56}$$

and costs are

$$c = x^2 + xy + y^2 + 190$$

find the profit-maximizing level of (*a*) output, (*b*) price, and (*c*) profit.

a) From (*6.55*) and (*6.56*),

$$P_x = 252 - 4x \tag{6.57}$$

$$P_y = 180 - 3y \tag{6.58}$$

Thus, $\pi = (252 - 4x)x + (180 - 3y)y - (x^2 + xy + y^2 + 190) = 252x - 5x^2 - xy + 180y - 190 - 4y^2$

Forming a new, constrained function,

$$\Pi = 252x - 5x^2 - xy - 4y^2 + 180y - 190 + \lambda(x - 3y)$$

Hence, $\Pi_x = 252 - 10x - y + \lambda = 0 \qquad \Pi_y = -x - 8y + 180 - 3\lambda = 0 \qquad \Pi_\lambda = x - 3y = 0$

and $\bar{x} = 27$, $\bar{y} = 9$, and $\bar{\lambda} = 27$.

b) From (*6.57*) and (*6.58*), $P_x = 144$ and $P_y = 153$.

c) $$\pi = 4022$$

6.34. Problem 4.22 dealt with the profit-maximizing level of output for a firm producing a single product that is sold in two distinct markets when it does and does not discriminate. The functions given were

$$Q_1 = 21 - 0.1P_1 \tag{6.59}$$

$$Q_2 = 50 - 0.4P_2 \tag{6.60}$$

$$c = 2000 + 10Q \qquad \text{where } Q = Q_1 + Q_2 \tag{6.61}$$

Use multivariable calculus to check your solution to Problem 4.22.

From (*6.59*), (*6.60*), and (*6.61*),

$$P_1 = 210 - 10Q_1 \tag{6.62}$$

$$P_2 = 125 - 2.5Q_2 \tag{6.63}$$

$$c = 2000 + 10Q_1 + 10Q_2$$

With discrimination $P_1 \neq P_2$ since different prices are charged in different markets, and therefore

$$\begin{aligned}\pi &= (210 - 10Q_1)Q_1 + (125 - 2.5Q_2)Q_2 - (2000 + 10Q_1 + 10Q_2)\\ &= 200Q_1 - 10Q_1^2 + 115Q_2 - 2.5Q_2^2 - 2000\end{aligned}$$

Taking the first partials,

$$\pi_1 = 200 - 20Q_1 = 0 \qquad \pi_2 = 115 - 5Q_2 = 0$$

Thus, $\bar{Q}_1 = 10$ and $\bar{Q}_2 = 23$. Substituting in (*6.62*) and (*6.63*), $\bar{P}_1 = 110$ and $\bar{P}_2 = 67.5$.

If there is no discrimination, the same price must be charged in both markets. Hence $P_1 = P_2$. Substituting from (*6.62*) and (*6.63*),

$$\begin{aligned}210 - 10Q_1 &= 125 - 2.5Q_2\\ 2.5Q_2 - 10Q_1 &= -85\end{aligned}$$

Rearranging this as a constraint and forming a new function,

$$\Pi = 200Q_1 - 10Q_1^2 + 115Q_2 - 2.5Q_2^2 - 2000 + \lambda(85 - 10Q_1 + 2.5Q_2)$$

Thus, $\Pi_1 = 200 - 20Q_1 - 10\lambda = 0 \qquad \Pi_2 = 115 - 5Q_2 + 2.5\lambda = 0 \qquad \Pi_\lambda = 85 - 10Q_1 + 2.5Q_2 = 0$

and $\bar{Q}_1 = 13.4$, $\bar{Q}_2 = 19.6$, and $\bar{\lambda} = -6.8$. Substituting in (*6.62*) and (*6.63*),

$$P_1 = 210 - 10(13.4) = 76$$
$$P_2 = 125 - 2.5(19.6) = 76$$
$$Q = 13.4 + 19.6 = 33$$

6.35. Check your answers to Problem 4.23, given

$$Q_1 = 24 - 0.2P_1 \qquad Q_2 = 10 - 0.05P_2$$
$$c = 35 + 40Q \qquad \text{where } Q = Q_1 + Q_2$$

From the information given,

$$P_1 = 120 - 5Q_1 \tag{6.64}$$
$$P_2 = 200 - 20Q_2 \tag{6.65}$$
$$c = 35 + 40Q_1 + 40Q_2$$

With price discrimination,

$$\pi = (120 - 5Q_1)Q_1 + (200 - 20Q_2)Q_2 - (35 + 40Q_1 + 40Q_2) = 80Q_1 - 5Q_1^2 + 160Q_2 - 20Q_2^2 - 35$$

Thus, $\pi_1 = 80 - 10Q_1 = 0 \qquad \pi_2 = 160 - 40Q_2 = 0$

and $\bar{Q}_1 = 8$, $\bar{Q}_2 = 4$, $P_1 = 80$, and $P_2 = 120$.

If there is no price discrimination, $P_1 = P_2$. Substituting from (*6.64*) and (*6.65*),

$$120 - 5Q_1 = 200 - 20Q_2$$
$$20Q_2 - 5Q_1 = 80 \tag{6.66}$$

Forming a new function with (*6.66*) as a constraint,

$$\Pi = 80Q_1 - 5Q_1^2 + 160Q_2 - 20Q_2^2 - 35 + \lambda(80 + 5Q_1 - 20Q_2)$$

Thus, $\Pi_1 = 80 - 10Q_1 + 5\lambda = 0 \qquad \Pi_2 = 160 - 40Q_2 - 20\lambda = 0 \qquad \Pi_\lambda = 80 + 5Q_1 - 20Q_2 = 0$

and $\bar{Q}_1 = 6.4$, $\bar{Q}_2 = 5.6$, and $\bar{\lambda} = -3.2$. Substituting in (*6.64*) and (*6.65*),

$$P_1 = 120 - 5(6.4) = 88$$
$$P_2 = 200 - 20(5.6) = 88$$
$$Q = 6.4 + 5.6 = 12$$

6.36. (*a*) Maximize utility $u = Q_1Q_2$ when $P_1 = 1$, $P_2 = 4$, and one's budget $B = 120$. (*b*) Estimate the effect of a 1-unit increase in the budget.

a) The budget constraint is $Q_1 + 4Q_2 = 120$. Forming a new function to incorporate the constraint,

$$U = Q_1Q_2 + \lambda(120 - Q_1 - 4Q_2)$$

Thus, $U_1 = Q_2 - \lambda = 0 \qquad U_2 = Q_1 - 4\lambda = 0 \qquad U_\lambda = 120 - Q_1 - 4Q_2 = 0$

and $\bar{Q}_1 = 60$, $\bar{Q}_2 = 15$, and $\bar{\lambda} = 15$.

b) With $\bar{\lambda} = 15$, a \$1 increase in the budget will lead to an increase in the utility function of approximately 15. Thus, the marginal utility of money (or income) at $\bar{Q}_1 = 60$ and $\bar{Q}_2 = 15$ is approximately 15.

6.37. (*a*) Maximize utility $u = Q_1 Q_2$, subject to $P_1 = 10$, $P_2 = 2$, and $B = 240$. (*b*) What is the marginal utility of money?

a) Form the Lagrangian function $U = Q_1 Q_2 + \lambda(240 - 10Q_1 - 2Q_2)$.

$$U_1 = Q_2 - 10\lambda = 0 \qquad U_2 = Q_1 - 2\lambda = 0 \qquad U_\lambda = 240 - 10Q_1 - 2Q_2 = 0$$

Thus, $\bar{Q}_1 = 12$, $\bar{Q}_2 = 60$, and $\bar{\lambda} = 6$.

b) The marginal utility of money at $\bar{Q}_1 = 12$ and $\bar{Q}_2 = 60$ is approximately 6.

6.38. Maximize utility $u = Q_1 Q_2 + Q_1 + 2Q_2$, subject to $P_1 = 2$, $P_2 = 5$, and $B = 51$.

Form the Lagrangian function $U = Q_1Q_2 + Q_1 + 2Q_2 + \lambda(51 - 2Q_1 - 5Q_2)$.

$$U_1 = Q_2 + 1 - 2\lambda = 0 \qquad U_2 = Q_1 + 2 - 5\lambda = 0 \qquad U_\lambda = 51 - 2Q_1 - 5Q_2 = 0$$

Thus, $\bar{Q}_1 = 13$, $\bar{Q}_2 = 5$, and $\bar{\lambda} = 3$.

6.39. Maximize utility $u = xy + 3x + y$ subject to $P_x = 8$, $P_y = 12$, and $B = 212$.

The Lagrangian function is $U = xy + 3x + y + \lambda(212 - 8x - 12y)$.

$$U_x = y + 3 - 8\lambda = 0 \qquad U_y = x + 1 - 12\lambda = 0 \qquad U_\lambda = 212 - 8x - 12y = 0$$

Thus, $\bar{x} = 15$, $\bar{y} = 7\frac{2}{3}$, and $\bar{\lambda} = 1\frac{1}{3}$.

HOMOGENEITY AND RETURNS TO SCALE

6.40. Determine the level of homogeneity and returns to scale for each of the following production functions:

a) $Q = x^2 + 6xy + 7y^2$

here Q is homogeneous of degree 2, and returns to scale are increasing because

$$f(kx, ky) = (kx)^2 + 6(kx)(ky) + 7(ky)^2 = k^2(x^2 + 6xy + 7y^2)$$

b) $Q = x^3 - xy^2 + 3y^3 + x^2 y$

here Q is homogeneous of degree 3, and returns to scale are increasing because

$$f(kx, ky) = (kx)^3 - (kx)(ky)^2 + 3(ky)^3 + (kx)^2(ky) = k^3(x^3 - xy^2 + 3y^3 + x^2 y)$$

c) $Q = \dfrac{3x^2}{5y^2}$

here Q is homogeneous of degree 0, and returns to scale are decreasing because

$$f(kx, ky) = \frac{3(kx)^2}{5(ky)^2} = \frac{3x^2}{5y^2} \qquad \text{and} \qquad k^0 = 1$$

d) $Q = 0.9K^{0.2} L^{0.6}$

here Q is homogeneous of degree 0.8 and returns to scale are decreasing because

$$\begin{aligned} Q(kK, kL) &= 0.9(kK)^{0.2}(kL)^{0.6} = Ak^{0.2}K^{0.2}k^{0.6}L^{0.6} \\ &= k^{0.2+0.6}(0.9K^{0.2}L^{0.6}) = k^{0.8}(0.9K^{0.2}L^{0.6}) \end{aligned}$$

Note that the returns to scale of a Cobb-Douglas function will always equal the sum of the exponents $\alpha + \beta$, as is illustrated in part 6 of Example 8.