$$|\mathbf{A}_2| = -3(-5Q_2 + 900 - 2Q_3 + 400) - (Q_1 - 150)(-5 - 4) + 1(Q_3 - 200 - 2Q_2 + 360)$$
$$= -5090 + 9Q_1 + 13Q_2 + 7Q_3$$
$$P_2 = \frac{-5090 + 9Q_1 + 13Q_2 + 7Q_3}{-36} = 141.39 - 0.25Q_1 - 0.36Q_2 - 0.19Q_3$$
$$|\mathbf{A}_3| = -3(-4Q_3 + 800 - Q_2 + 180) - 1(Q_3 - 200 - 2Q_2 + 360) + (Q_1 - 150)(1 + 8)$$
$$= -4450 + 9Q_1 + 5Q_2 + 11Q_3$$
$$P_3 = \frac{-4450 + 9Q_1 + 5Q_2 + 11Q_3}{-36} = 123.61 - 0.25Q_1 - 0.14Q_2 - 0.31Q_3$$

b)
$$\Pi = P_1Q_1 + P_2Q_2 + P_3Q_3 - \mathrm{TC}$$
$$= 138.33Q_1 + 141.39Q_2 + 123.61Q_3 - 1.42Q_1Q_2$$
$$-1.33Q_2Q_3 - 1.42Q_1Q_3 - 1.5Q_1^2 - 2.36Q_2^2 - 1.31Q_3^2$$
$$\Pi_1 = 138.33 - 1.42Q_2 - 1.42Q_3 - 3Q_1 = 0$$
$$\Pi_2 = 141.39 - 1.42Q_1 - 1.33Q_3 - 4.72Q_2 = 0$$
$$\Pi_3 = 123.61 - 1.33Q_2 - 1.42Q_1 - 2.62Q_3 = 0$$

Thus,
$$\begin{bmatrix} -3 & -1.42 & -1.42 \\ -1.42 & -4.72 & -1.33 \\ -1.42 & -1.33 & -2.62 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} -138.33 \\ -141.39 \\ -123.61 \end{bmatrix}$$
$$|\mathbf{A}| = -22.37$$
$$|\mathbf{A}_1| = -612.27 \qquad \bar{Q}_1 = \frac{-612.27}{-22.37} \approx 27.37$$
$$|\mathbf{A}_2| = -329.14 \qquad \bar{Q}_2 = \frac{-329.14}{-22.37} \approx 14.71$$
$$|\mathbf{A}_3| = -556.64 \qquad \bar{Q}_3 = \frac{-556.64}{-22.37} \approx 24.88$$

c)
$$|H_1| = \begin{vmatrix} -3 & -1.42 & -1.42 \\ -1.42 & -4.72 & -1.33 \\ -1.42 & -1.33 & -2.62 \end{vmatrix}$$

$|H_1| = -3$, $|H_2| = 12.14$, and $|H_3| = |\mathbf{A}| = -22.37$. And Π is maximized.

THE BORDERED HESSIAN IN CONSTRAINED OPTIMIZATION

12.19. Maximize utility $u = 2xy$ subject to a budget constraint equal to $3x + 4y = 90$ by (*a*) finding the critical values $\bar{x}$, $\bar{y}$, and $\bar{\lambda}$ and (*b*) using the bordered Hessian $|\bar{\mathbf{H}}|$ to test the second-order condition.

a) The Lagrangian function is $\qquad U = 2xy + \lambda(90 - 3x - 4y)$

The first-order conditions are

$$U_x = 2y - 3\lambda = 0 \qquad U_y = 2x - 4\lambda = 0 \qquad U_\lambda = 90 - 3x - 4y = 0$$

In matrix form,

$$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & -4 \\ -3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -90 \end{bmatrix} \qquad (12.12)$$

Solving by Cramer's rule, $|\mathbf{A}| = 48$, $|\mathbf{A}_1| = 720$, $|\mathbf{A}_2| = 540$, and $|\mathbf{A}_3| = 360$. Thus, $\bar{x} = 15$, $\bar{y} = 11.25$, and $\bar{\lambda} = 7.5$.

b) Taking the second partials of U with respect to x and y and the first partials of the constraint with respect to x and y to form the bordered Hessian,

$$U_{xx} = 0 \qquad U_{yy} = 0 \qquad U_{xy} = 2 = U_{yx} \qquad c_x = 3 \qquad c_y = 4$$

From Section 12.5,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{vmatrix} \qquad \text{or} \qquad |\bar{\mathbf{H}}| = \begin{vmatrix} 0 & 3 & 4 \\ 3 & 0 & 2 \\ 4 & 2 & 0 \end{vmatrix}$$

$$|\bar{H}_2| = |\bar{\mathbf{H}}| = -2(-12) + 3(8) = 48 > 0 \qquad |\bar{H}_2| = |\bar{\mathbf{H}}| = -3(-8) + 4(6) = 48 > 0$$

The bordered Hessian can be set up in either of the above forms without affecting the value of the principal minor. With $|\bar{\mathbf{H}}| = |\mathbf{A}| > 0$, from the rules of Section 12.5 $|\bar{\mathbf{H}}|$ is negative definite, and U is maximized.

12.20. Maximize utility $u = xy + x$ subject to the budget constraint $6x + 2y = 110$, by using the techniques of Problem 12.19.

a)
$$U = xy + x + \lambda(110 - 6x - 2y)$$
$$U_x = y + 1 - 6\lambda = 0 \qquad U_y = x - 2\lambda = 0 \qquad U_\lambda = 110 - 6x - 2y = 0$$

In matrix form,

$$\begin{bmatrix} 0 & 1 & -6 \\ 1 & 0 & -2 \\ -6 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -110 \end{bmatrix}$$

Solving by Cramer's rule, $\bar{x} = 9\frac{1}{3}$, $\bar{y} = 27$, and $\bar{\lambda} = 4\frac{2}{3}$.

b) Since $U_{xx} = 0$, $U_{yy} = 0$, $U_{xy} = 1 = U_{yx}$, $c_x = 6$, and $c_y = 2$,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 0 & 1 & 6 \\ 1 & 0 & 2 \\ 6 & 2 & 0 \end{vmatrix} \qquad |\bar{H}_2| = |\bar{\mathbf{H}}| = 24$$

With $|\bar{H}_2| > 0$, $|\bar{\mathbf{H}}|$ is negative definite, and U is maximized.

12.21. Minimize a firm's total costs $c = 45x^2 + 90xy + 90y^2$ when the firm has to meet a production quota g equal to $2x + 3y = 60$ by (*a*) finding the critical values and (*b*) using the bordered Hessian to test the second-order conditions.

a)
$$C = 45x^2 + 90xy + 90y^2 + \lambda(60 - 2x - 3y)$$
$$C_x = 90x + 90y - 2\lambda = 0 \qquad C_y = 90x + 180y - 3\lambda = 0$$
$$C_\lambda = 60 - 2x - 3y = 0$$

In matrix form,

$$\begin{bmatrix} 90 & 90 & -2 \\ 90 & 180 & -3 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -60 \end{bmatrix}$$

Solving by Cramer's rule, $\bar{x} = 12$, $\bar{y} = 12$, and $\bar{\lambda} = 1080$.

b) Since $C_{xx} = 90$, $C_{yy} = 180$, $C_{xy} = 90 = C_{yx}$, $g_x = 2$, and $g_y = 3$,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 90 & 90 & 2 \\ 90 & 180 & 3 \\ 2 & 3 & 0 \end{vmatrix}$$

$|\bar{H}_2| = -450$. With $|\bar{H}_2| < 0$, $|\bar{\mathbf{H}}|$ is positive definite and C is minimized.

12.22. Minimize a firm's costs $c = 3x^2 + 5xy + 6y^2$ when the firm must meet a production quota of $5x + 7y = 732$, using the techniques of Problem 12.21.

a)
$$C = 3x^2 + 5xy + 6y^2 + \lambda(732 - 5x - 7y)$$
$$C_x = 6x + 5y - 5\lambda = 0 \qquad C_y = 5x + 12y - 7\lambda = 0$$
$$C_\lambda = 732 - 5x - 7y = 0$$

Solving simultaneously, $\bar{x} = 75 \qquad \bar{y} = 51 \qquad \bar{\lambda} = 141$

b) With $C_{xx} = 6$, $C_{yy} = 12$, $C_{xy} = 5 = C_{yx}$, $g_x = 5$, and $g_y = 7$,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 6 & 5 & 5 \\ 5 & 12 & 7 \\ 5 & 7 & 0 \end{vmatrix}$$

$|\bar{H}_2| = 5(35 - 60) - 7(42 - 25) = -244$. Thus, $|\bar{\mathbf{H}}|$ is positive definite, and C is minimized.

12.23. Redo Problem 12.21 by maximizing utility $u = x^{0.5}y^{0.3}$ subject to the budget constraint $10x + 3y = 140$.

a)
$$U = x^{0.5}y^{0.3} + \lambda(140 - 10x - 3y)$$
$$U_x = 0.5x^{-0.5}y^{0.3} - 10\lambda = 0 \qquad U_y = 0.3x^{0.5}y^{-0.7} - 3\lambda = 0$$
$$U_\lambda = 140 - 10x - 3y = 0$$

Solving simultaneously, as shown in Example 10 of Chapter 6,

$$\bar{x} = 8.75 \qquad \bar{y} = 17.5 \qquad \text{and} \qquad \bar{\lambda} = 0.04$$

b) With $U_{xx} = -0.25x^{-1.5}y^{0.3}$, $U_{yy} = -0.21x^{0.5}y^{-1.7}$, $U_{xy} = U_{yx} = 0.15x^{-0.5}y^{-0.7}$, $g_x = 10$, and $g_y = 3$,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} -0.25x^{-1.5}y^{0.3} & 0.15x^{-0.5}y^{-0.7} & 10 \\ 0.15x^{-0.5}y^{-0.7} & -0.21x^{0.5}y^{-1.7} & 3 \\ 10 & 3 & 0 \end{vmatrix}$$

Expanding along the third column,

$$|\bar{H}_2| = 10(0.45x^{-0.5}y^{-0.7} + 2.1x^{0.5}y^{-1.7}) - 3(-0.75x^{-1.5}y^{0.3} - 1.5x^{-0.5}y^{-0.7})$$
$$= 21x^{0.5}y^{-1.7} + 9x^{-0.5}y^{-0.7} + 2.25x^{-1.5}y^{0.3} > 0$$

since x and $y > 0$, and a positive number x raised to a negative power $-n$ equals $1/x^n$, which is also positive. With $|\bar{H}_2| > 0$, $|\bar{\mathbf{H}}|$ is negative definite, and U is maximized.

12.24. Maximize utility $u = x^{0.25}y^{0.4}$ subject to the budget constraint $2x + 8y = 104$, as in Problem 12.23.

a)
$$U = x^{0.25}y^{0.4} + \lambda(104 - 2x - 8y)$$
$$U_x = 0.25x^{-0.75}y^{0.4} - 2\lambda = 0 \qquad U_y = 0.4x^{0.25}y^{-0.6} - 8\lambda = 0$$
$$U_\lambda = 104 - 2x - 8y = 0$$

Solving simultaneously, $\bar{x} = 20$, $\bar{y} = 8$, and $\bar{\lambda} = 0.03$.

b)
$$|\bar{\mathbf{H}}| = \begin{vmatrix} -0.1875x^{-1.75}y^{0.4} & 0.1x^{-0.75}y^{-0.6} & 2 \\ 0.1x^{-0.75}y^{-0.6} & -0.24x^{0.25}y^{-1.6} & 8 \\ 2 & 8 & 0 \end{vmatrix}$$

Expanding along the third row,

$$|\bar{H}_2| = 2(0.8x^{-0.75}y^{-0.6} + 0.48x^{0.25}y^{-1.6}) - 8(-1.5x^{-1.75}y^{0.4} - 0.2x^{-0.75}y^{-0.6})$$
$$= 0.96x^{0.25}y^{-1.6} + 3.2x^{-0.75}y^{-0.6} + 12x^{-1.75}y^{0.4} > 0$$

Thus, $|\bar{\mathbf{H}}|$ is negative definite, and U is maximized.

12.25. Minimize costs $c = 3x + 4y$ subject to the constraint $2xy = 337.5$, using the techniques of Problem 12.21(a) and (b). (c) Discuss the relationship between this solution and that for Problem 12.19.

a)
$$C = 3x + 4y + \lambda(337.5 - 2xy)$$
$$C_x = 3 - 2\lambda y = 0 \qquad \lambda = \frac{1.5}{y} \tag{12.13}$$
$$C_y = 4 - 2\lambda x = 0 \qquad \lambda = \frac{2}{x} \tag{12.14}$$
$$C_\lambda = 337.5 - 2xy = 0 \tag{12.15}$$

Equate λ's in (*12.13*) and (*12.14*).

$$\frac{1.5}{y} = \frac{2}{x} \qquad y = 0.75x$$

Substitute in (*12.15*).

$$337.5 = 2x(0.75x) = 1.5x^2$$
$$x^2 = 225 \qquad \bar{x} = 15$$

Thus, $\bar{y} = 11.25$ and $\bar{\lambda} = 0.133$.

b) With $C_{xx} = 0$, $C_{yy} = 0$, and $C_{xy} = C_{yx} = -2\lambda$ and from the constraint $2xy = 337.5$, $g_x = 2y$, and $g_y = 2x$,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 0 & -2\lambda & 2y \\ -2\lambda & 0 & 2x \\ 2y & 2x & 0 \end{vmatrix}$$

$|\bar{H}_2| = -(-2\lambda)(-4xy) + 2y(-4x\lambda) = -16\lambda xy$. With $\bar{\lambda}$, $\bar{x}$, $\bar{y} > 0$, $|\bar{H}_2| < 0$. Hence $|\bar{\mathbf{H}}|$ is positive definite, and C is minimized.

c) This problem and Problem 12.19 are the same, except that the objective functions and constraints are reversed. In Problem 12.19, the objective function $u = 2xy$ was maximized subject to the constraint $3x + 4y = 90$; in this problem the objective function $c = 3x + 4y$ was minimized subject to the constraint $2xy = 337.5$. Therefore, one may maximize utility subject to a budget constraint *or* minimize the cost of achieving a given level of utility.

12.26. Minimize the cost of 434 units of production for a firm when $Q = 10K^{0.7}L^{0.1}$ and $P_K = 28$, $P_L = 10$ by (a) finding the critical values and (b) using the bordered Hessian. (c) Check the answer with that of Problem 6.41(b).

a) The objective function is $c = 28K + 10L$, and the constraint is $10K^{0.7}L^{0.1} = 434$. Thus,

$$C = 28K + 10L + \lambda(434 - 10K^{0.7}L^{0.1})$$
$$C_K = 28 - 7\lambda K^{-0.3}L^{0.1} = 0 \tag{12.16}$$
$$C_L = 10 - \lambda K^{0.7}L^{-0.9} = 0 \tag{12.17}$$
$$C_\lambda = 434 - 10K^{0.7}L^{0.1} = 0 \tag{12.18}$$

Rearranging and dividing (*12.16*) by (*12.17*) to eliminate λ,

$$\frac{28}{10} = \frac{7\lambda K^{-0.3}L^{0.1}}{\lambda K^{0.7}L^{-0.9}}$$
$$2.8 = \frac{7L}{K} \qquad K = 2.5L$$

Substituting in (*12.18*) and using a calculator,

$$434 = 10(2.5)^{0.7}L^{0.7}L^{0.1} \qquad 434 = 19L^{0.8}$$
$$\bar{L} = (22.8)^{1/0.8} = (22.8)^{1.25} \approx 50$$

Thus, $\bar{K} = 125$ and $\bar{\lambda} = 11.5$.

b) With $C_{KK} = 2.1\lambda K^{-1.3}L^{0.1}$, $C_{LL} = 0.9\lambda K^{0.7}L^{-1.9}$, and $C_{KL} = -0.7\lambda K^{-0.3}L^{-0.9} = C_{LK}$ and from the constraint $g_K = 7K^{-0.3}L^{0.1}$ and $g_L = K^{0.7}L^{-0.9}$,

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 2.1\lambda K^{-1.3}L^{0.1} & -0.7\lambda K^{-0.3}L^{-0.9} & 7K^{-0.3}L^{0.1} \\ -0.7\lambda K^{-0.3}L^{-0.9} & 0.9\lambda K^{0.7}L^{-1.9} & K^{0.7}L^{-0.9} \\ 7K^{-0.3}L^{0.1} & K^{0.7}L^{-0.9} & 0 \end{vmatrix}$$

Expanding along the third row,

$$|\bar{H}_2| = 7K^{-0.3}L^{0.1}(-0.7\lambda K^{0.4}L^{-1.8} - 6.3\lambda K^{0.4}L^{-1.8}) - K^{0.7}L^{-0.9}(2.1\lambda K^{-0.6}L^{-0.8} + 4.9\lambda K^{-0.6}L^{-0.8})$$
$$= -49\lambda K^{0.1}L^{-1.7} - 7\lambda K^{0.1}L^{-1.7} = -56\lambda K^{0.1}L^{-1.7}$$

With $K, L, \lambda > 0$, $|\bar{H}_2| < 0$; $|\bar{\mathbf{H}}|$ is positive definite, and C is minimized.

c) The answers are identical with those in Problem 6.41(*b*), but note the difference in the work involved when the linear function is selected as the objective function and not the constraint. See also the bordered Hessian for Problem 6.41(*b*), which is calculated in Problem 12.27(*c*).

12.27. Use the bordered Hessian to check the second-order conditions for (*a*) Example 7 of Chapter 6, (*b*) Problem 6.41(*a*), and (*c*) Problem 6.41(*b*).

a)
$$|\bar{\mathbf{H}}| = \begin{vmatrix} 16 & -1 & 1 \\ -1 & 24 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$|\bar{H}_2| = 1(-1-24) - 1(16+1) = -42$. With $|\bar{H}_2| < 0$, $|\bar{\mathbf{H}}|$ is positive definite and C is minimized.

b)
$$|\bar{\mathbf{H}}| = \begin{vmatrix} -0.21K^{-1.7}L^{0.5} & 0.15K^{-0.7}L^{-0.5} & 6 \\ 0.15K^{-0.7}L^{-0.5} & -0.25K^{0.3}L^{-1.5} & 2 \\ 6 & 2 & 0 \end{vmatrix}$$

$$|\bar{H}_2| = 6(0.30K^{-0.7}L^{-0.5} + 1.5K^{0.3}L^{-1.5}) - 2(-0.42K^{-1.7}L^{0.5} - 0.9K^{-0.7}L^{-0.5})$$
$$= 9K^{0.3}L^{-1.5} + 3.6K^{-0.7}L^{-0.5} + 0.84K^{-1.7}L^{0.5} > 0$$

With $|\bar{H}_2| > 0$, $|\bar{\mathbf{H}}|$ is negative definite, and Q is maximized.

c)
$$|\bar{\mathbf{H}}| = \begin{vmatrix} -2.1K^{-1.3}L^{0.1} & 0.7K^{-0.3}L^{-0.9} & 28 \\ 0.7K^{-0.3}L^{-0.9} & -0.9K^{0.7}L^{-1.9} & 10 \\ 28 & 10 & 0 \end{vmatrix}$$

$$|\bar{H}_2| = 28(7K^{-0.3}L^{-0.9} + 25.2K^{0.7}L^{-1.9}) - 10(-21K^{-1.3}L^{0.1} - 19.6K^{-0.3}L^{-0.9})$$
$$= 705.6K^{0.7}L^{-1.9} + 392K^{-0.3}L^{-0.9} + 210K^{-1.3}L^{0.1} > 0$$

With $|\bar{H}_2| > 0$, $|\bar{\mathbf{H}}|$ is negative definite, and Q is maximized.

12.28. Use the bordered Hessian to check the second-order conditions in Problem 5.12(*c*), where $4xyz^2$ was optimized subject to the constraint $x + y + z = 56$; the first-order conditions were $F_x = 4yz^2 - \lambda = 0$, $F_y = 4xz^2 - \lambda = 0$, and $F_z = 8xyz - \lambda = 0$; and the critical values were $\bar{x} = 14$, $\bar{y} = 14$, and $\bar{z} = 28$.

Take the second partial derivatives of F and the first partials of the constraint, and set up the bordered Hessian, as follows:

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 0 & g_x & g_y & g_z \\ g_x & F_{xx} & F_{xy} & F_{xz} \\ g_y & F_{yx} & F_{yy} & F_{yz} \\ g_z & F_{zx} & F_{zy} & F_{zz} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 4z^2 & 8yz \\ 1 & 4z^2 & 0 & 8xz \\ 1 & 8yz & 8xz & 8xy \end{vmatrix}$$

Start with $|\bar{H}_2|$, the 3×3 submatrix in the upper left-hand corner.

$$|\bar{H}_2| = 0 - 1(-4z^2) + 1(4z^2) = 8z^2 > 0$$

Next evaluate $|\bar{H}_3|$, which here equals $|\bar{\mathbf{H}}|$. Expanding along the first row,

$$|\bar{H}_3| = 0 - 1\begin{vmatrix} 1 & 4z^2 & 8yz \\ 1 & 0 & 8xz \\ 1 & 8xz & 8xy \end{vmatrix} + 1\begin{vmatrix} 1 & 0 & 8yz \\ 1 & 4z^2 & 8xz \\ 1 & 8yz & 8xy \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 4z^2 \\ 1 & 4z^2 & 0 \\ 1 & 8yz & 8xz \end{vmatrix}$$

$$\begin{aligned} |\bar{H}_3| &= -1[1(0 - 8xz \cdot 8xz) - 4z^2(8xy - 8xz) + 8yz(8yz - 0)] \\ &\quad + 1[1(4z^2 \cdot 8xy - 8yz \cdot 8xz) - 0 + 8yz(8yz - 4z^2)] \\ &\quad -1[1(4z^2 \cdot 8xz - 0) - 0 + 4z^2(8yz - 4z^2)] \\ |\bar{H}_3| &= -1(-64x^2z^2 - 32xyz^2 + 32xz^3 + 64xyz^2) + 1(32xyz^2 - 64xyz^2 + 64y^2z^2 - 32yz^3) \\ &\quad -1(32xz^3 + 32yz^3 - 16z^4) \\ |\bar{H}_3| &= 16z^4 - 64xz^3 - 64yz^3 - 64xyz^2 + 64x^2z^2 + 64y^2z^2 \end{aligned}$$

Evaluated at $\bar{x} = 14$, $\bar{y} = 14$, $\bar{z} = 28$,

$$|\bar{H}_3| = -19{,}668{,}992 < 0$$

With $|\bar{H}_2| > 0$ and $|\bar{H}_3| < 0$, $|\bar{\mathbf{H}}|$ is negative definite and the function is maximized.

INPUT-OUTPUT ANALYSIS

12.29. Determine the total demand for industries 1, 2, and 3, given the matrix of technical coefficients **A** and the final demand vector **B** below.

$$\begin{array}{c} \text{Output industry} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \end{array}$$

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \ \text{Input industry} \qquad \mathbf{B} = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$$

From (*12.3*), the total demand vector is $\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where

$$\mathbf{I} - \mathbf{A} = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.3 \\ -0.3 & -0.5 & 0.8 \end{bmatrix}$$

Taking the inverse of $\mathbf{I} - \mathbf{A}$,

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{0.239}\begin{bmatrix} 0.57 & 0.34 & 0.27 \\ 0.41 & 0.58 & 0.32 \\ 0.47 & 0.49 & 0.60 \end{bmatrix}$$

Substituting in $\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$,

$$\mathbf{X} = \frac{1}{0.239}\begin{bmatrix} 0.57 & 0.34 & 0.27 \\ 0.41 & 0.58 & 0.32 \\ 0.47 & 0.49 & 0.60 \end{bmatrix}\begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix} = \frac{1}{0.239}\begin{bmatrix} 210.2 \\ 244.7 \\ 294.5 \end{bmatrix} = \begin{bmatrix} 879.50 \\ 1023.85 \\ 1232.22 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$