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Averages of r.v.s :

$$\mathbb{E}(x) = m_x = \mu_x = \bar{X} = \int_{-\infty}^{+\infty} x \cdot P_x(x)\,dx$$

$$\mathbb{E}\{g(x)\} = \int_{-\infty}^{+\infty} g(x)\, P_x(x)\,dx$$

$$\sigma_x^2 = \mathrm{Var}(x) = \mathbb{E}(x - m_x)^2 = \mathbb{E}(x^2) - m_x^2 \geq 0$$

power

$$= \int_{-\infty}^{+\infty} (x - m_x)^2 \cdot P_x(x)\,dx$$

n th momement (non-central) :

$$\mathbb{E}(x^n) = \int_{-\infty}^{+\infty} x^n \cdot P_x(x)\,dx$$

nth central moment :

$$\mathbb{E}(x - m_x)^n = \int_{-\infty}^{+\infty} (x - m_x)^n \cdot P_x(x)\,dx$$

Standard deviation: $\sigma_X = \sqrt{Var(X)}$.

Joint moment of X_1, X_2:

$$\mathbb{E}\{X_1^j X_2^k\}$$

Joint central moment of X_1, X_2:

$$\mathbb{E}\{(X_1 - \bar{X}_1)^j \cdot (X_2 - \bar{X}_2)^k\}$$

Correlation between X_1, X_2:

$$\mathbb{E}(X_1 \cdot X_2)$$

Covariance between X_1, X_2:

$$Cov(X_1, X_2) = \mathbb{E}\{(X_1 - \bar{X}_1)(X_2 - \bar{X}_2)\} =$$

$$= \mathbb{E}(X_1 X_2) - \bar{X}_1 \cdot \bar{X}_2$$

X_1, X_2 are uncorrelated iff:

$$\mathbb{E}(X_1 \cdot X_2) = \mathbb{E}(X_1) \cdot \mathbb{E}(X_2) \iff$$

$$\iff Cov(X_1, X_2) = 0$$

- X_1, X_2 are $\perp$ iff

$$\mathbb{E}(X_1 X_2) = 0$$

- Covariance matrix of $X_1 \dots X_n$:

K is $n \times n$ matrix with

$$K[i,j] = \operatorname{Cov}(X_i, X_j)$$
$$1 \le i, j \le n$$

- Characteristic function of X:

① $$\Psi_X(jv) = \mathbb{E}\{e^{jvX}\} = \int_{-\infty}^{+\infty} e^{jvx} \cdot p_X(x)\, dx$$

, i.e., the IFT of $\underline{p_X(x)} \cdot 2\pi$

So, $$p_X(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \Psi_X(jv)\, e^{-jvx}\, dv$$

From ①,

$$\Psi_X(jv) = \sum_{n=0}^{+\infty} \frac{j^n \cdot v^n \cdot \mathbb{E}(x^n)}{n!} \rightsquigarrow$$

$$j^n\, \mathbb{E}(x^n) = \frac{\Psi_X^{(n)}(0)}{\cancel{\quad}} \rightsquigarrow$$

$$\mathbb{E}(x^n) = (-j)^n \cdot \frac{\Psi_X^{(n)}(0)}{\cancel{n}!}$$

Ex. Let $\{x_i\}_{i=1}^{n}$ be i.i.d. rv's.

Let $Y = \sum_{i=1}^{n} x_i$. What is $P_Y(y)$ in terms of $P_{x_i}(x_i)$?

$$\Psi_Y(jv) = \mathbb{E}\{e^{jvY}\} =$$

$$= \mathbb{E}\left\{e^{jv\sum_{i=1}^{n} x_i}\right\} = \prod_{i=1}^{n} \Psi_x(jv) =$$

$$= \Psi_x^n(jv). \text{ Thus,}$$

IFT $\quad P_{\cancel{Y}}(\cancel{y}) = \left(P_{x_1}(\cancel{x}) * \cdots * P_{x_n}(\cancel{y})\right)$

Generalization of $\Psi_x(jv)$:

$$\Psi_{x_1 \cdots x_n}(jv_1, \ldots, jv_n) =$$

$$= \mathbb{E}\left\{\exp\left(j\sum_{i=1}^{n} v_i x_i\right)\right\}$$

Gaussian r. v.

$$P_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot \exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)$$

$$F_X(x) = \int_{-\infty}^{x} P_X(u)\,du =$$

$$= \frac{1}{2} + \frac{1}{2}\,\mathrm{erfc}\left(\frac{x-m_X}{\sqrt{2}\,\sigma_X}\right) =$$

$$= 1 - Q\left(\frac{x-m_X}{\sigma_X}\right).$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\,dt$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{x}^{+\infty} e^{-t^2/2}\,dt$$

Tu (end)

Fact: $Y = \sum_{i=1}^{n} X_i$, X_i i.i.d.

Gaussian with mean m, variance

σ^2. Then Y is Gaussian.

$$\Psi_Y(jv) = \prod_{i=1}^{n} \Psi_X(jv) =$$

$$= \prod_{i=1}^{n} \exp\left(jvm_x - \frac{v^2 \sigma_x^2}{2} \right) =$$

$$= \exp\left(jv \cdot nm_x - \frac{v^2}{2} \cdot n\sigma_x^2 \right), \text{ i.e.,}$$

Gaussian w/ mean $n \cdot m_x$,

variance $= n \cdot \sigma_x^2$.

Multivariate Gaussian Random Vector:

$\underline{x} = [x_1 \; ... \; x_n]^T$ is a

Gaussian vector, iff

$$P_X(\underline{x}) = \frac{1}{(2\pi)^{n/2} \det^{1/2} K_X}$$

$$\exp\left(- \frac{(\underline{x} - \underline{m}_x)^T \, K_x^{-1} \, (\underline{x} - \underline{m}_x)}{2} \right)$$

Recall :

$$K_X[i,j] = \mathbb{E}\left[\left(X_i - m_i\right)\left(X_j - m_j\right)\right] =$$

$$= \rho_{ij}\,\sigma_i\,\sigma_j$$

$$m_i = \mathbb{E}(X_i)$$

For $n=2$, we get:

$$P_X(\underline{x}) = P_{X_1, X_2}(x_1, x_2) =$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1/2}{(1-\rho^2)}\right.$$

$$\left[\frac{(x_1-m_1)^2}{\sigma_1^2} - 2\rho\underbrace{\left(\frac{x_1-m_1}{\sigma_1}\right)}_{\tilde{x}_1}\underbrace{\left(\frac{x_2-m_2}{\sigma_2}\right)}_{\tilde{x}_2}\right.$$

$$\left.\left. + \frac{(x_2-m_2)^2}{\sigma_2^2}\right]\right)$$

* $|\rho| \le 1$, ρ is called correlation coefficient

Contour plots:

By rotating x_1, x_2 to y_1, y_2

(45° rotation):

New coordinates:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Thus, for constant value in the original,

$$\tilde{x}_1^2 - 2\rho \cdot \tilde{x}_1 \cdot \tilde{x}_2 + \tilde{x}_2^2 =$$

$$= y_1^2 \cdot (1-\rho) + y_2^2 \cdot (1+\rho) = \text{const}$$

$\Rightarrow$ ellipse

Theorem: If $\underline{x} \sim \mathcal{N}(\underline{x} \mid \underline{m}_x, \underline{K}_x)$,

then $\underline{y} = \underline{A} \cdot \underline{x}$ is

$$\mathcal{N}(\underline{y} \mid \underline{A} \cdot \underline{m}_x, \underline{A}\, \underline{K}_x\, \underline{A}^T)$$

$\longrightarrow$

C. L. T

$\{x_i\}_{i=1}^{n}$ i. i. d. r. v. s

$\Rightarrow Y = \sum_{i=1}^{n} x_i \longrightarrow \mathcal{N}(Y \mid n \cdot m, n\sigma^2)$

as $n \to +\infty$.

Generalize:

1. $\underline{x}$ Complex Gaussian vector

2. ~~Proper vect~~

$\underline{Y} = \underline{A} \cdot \underline{x} + \underline{n}$

$\underline{x}$, $\underline{n}$ independent,

then $\underline{x}$, $\underline{Y}$ jointly

Gaussian.