UCI EECS
EECS241A
Fall 2017

HOMEWORK 1
(due 10/12/2017)

1. Let $Y = |X|$. Let the PDF of X be given as

$$p_X(x) = \text{rect}(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

i.e., X is uniformly distributed over $[-1/2, 1/2]$. Calculate and plot the PDF of Y, $p_Y(y)$.

2. Let $p_{X,Y}(x,y) = e^{-(x+y)}$ over the region $x > 0$ and $y > 0$. Let $U = X/(X+Y)$ be a transformation whose probability density function we wish to calculate. Define $V = X$ as an auxiliary transformation.
辅助的

   (a) What are $p_X(x)$ and $p_Y(y)$?

   (b) Express X and Y in terms of U and V, i.e., $X = g(U,V)$ and $Y = h(U,V)$.

   (c) What is the region of definition of the random variables U and V?

   (d) Calculate the Jacobian of the transformation

$$J = \det \begin{bmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial h}{\partial u} \\ \dfrac{\partial g}{\partial v} & \dfrac{\partial h}{\partial v} \end{bmatrix}.$$

   (e) Calculate $p_{U,V}(u,v)$.

   (f) Calculate $p_U(u)$. Can you recognize this density?

3. Let a random variable X be distributed as

$$p_X(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & x < 0. \end{cases}$$

Let $Y = X_1 + X_2 + \cdots + X_n$ where each X_i is distributed independently and identically as X, i.e., $p_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \Pi_{i=1}^n p_X(x_i)$.

   (a) Calculate the characteristic function $\psi_X(\nu) = E[e^{j\nu x}]$.

   (b) Calculate $E[X]$.

   (c) Calculate $E[X^2]$.

   (d) Generalize to $E[X^n]$.

   (e) Calculate $E[Y]$.

   (f) Calculate $E[Y^2]$.

(g) Generalize to $E[Y^m]$.

4. Let X_i be independent Gaussian random variables with mean m_{X_i} and variance $\sigma^2_{X_i}$ for $i = 1, 2, \ldots, N$. Let
$$Y_i = a_i X_i + b_i$$
for real scalars a_i and b_i, $i = 1, 2, \ldots, N$.

   (a) What is the probability density function of Y_i?

   (b) What is the probability density function of $Y = \sum_{i=1}^{N} Y_i$?

5. In this course we will need an identity for the sum of odd integers up to a given number. In this question, you will develop this identity. Consider
$$\begin{aligned} 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 &= \sum_{i=1}^{n} (2i-1)^2 \\ &= 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1. \end{aligned}$$

   (a) What is $\sum_{i=1}^{n} 1$?

   (b) Calculate $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$ for n even and n odd. Note you can group the sum as $(1+n) + (2+n-1) + (3+n-2) + \cdots$.

   (c) To calculate $\sum_{i=1}^{n} i^2$, first show by direct expansion as a telescoping sum that
$$\sum_{i=1}^{n} (1+i)^3 - i^3) = (1+n)^3 - 1.$$
   Then, expand $(1+i)^3$ to show
$$\sum_{i=1}^{n} (1+i)^3 - i^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
   and then calculate $\sum_{i=1}^{n} i^2$.

   (d) Now, show that
$$\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$