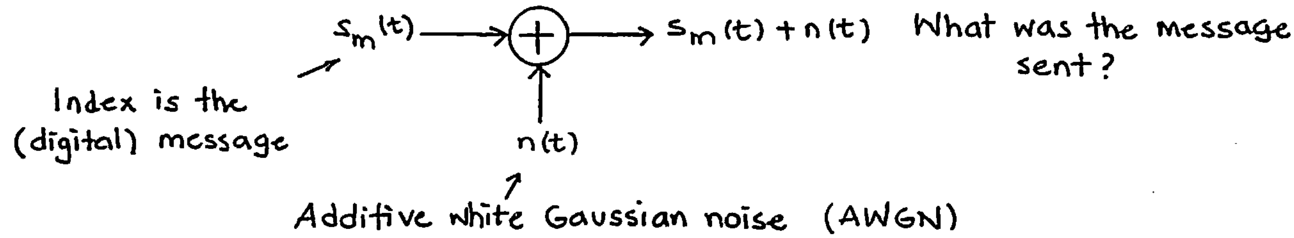
OPTIMUM RECEIVERS FOR SIGNALS IN ADDITIVE WHITE GAUSSIAN NOISE

CHANNEL MODEL AND RECEIVER



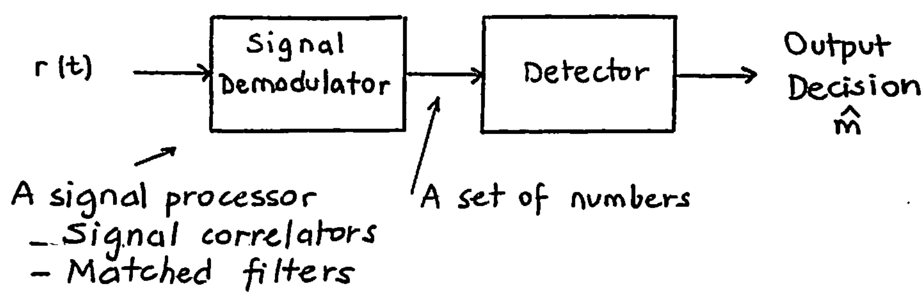
(Model for thermal receiver noise, also used in modem design to model quantization noise, also used in wireless)

$r(t) = s_m(t) + n(t) \qquad 0 \le t \le T$ Message transmitted in the period $[0,T]$

$S_N(f) = \frac{N_0}{2}$ Watts/Hz (if measurement is N_0, divide by 2 to include the effect of negative frequencies, a mathematical model)

Note $n(t)$ is zero mean from now on.

Receiver structure



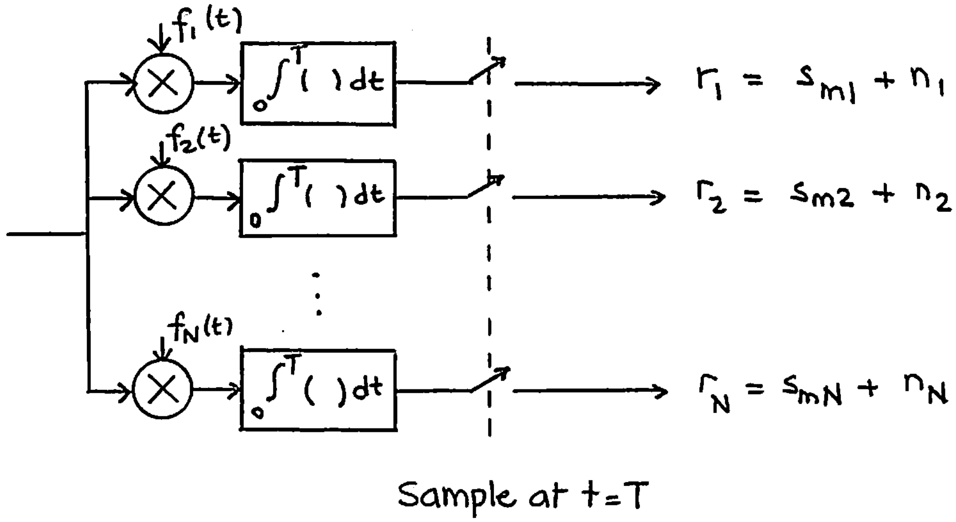
CORRELATION RECEIVER

Let $\{f_n(t), n=1,2,\dots,N\}$ be a set of orthonormal basis functions that span the signal space so that each message signal $s_m(t)$ can be represented as a linear combination of $\{f_n(t), n=1,2,\dots,N\}$.

$$\underbrace{\int_0^T r(t) f_k(t)\,dt}_{\substack{\text{Time correlation of} \\ \text{received signal with every basis function}}} = \int_0^T [s_m(t) + n(t)]\, f_k(t)\,dt = \underbrace{\int_0^T s_m(t) f_k(t)\,dt}_{s_{mk}} + \underbrace{\int_0^T n(t) f_k(t)\,dt}_{n_k}$$

$1 \le k \le N$

$$\underline{s}_m \triangleq (s_{m1}, s_{m2}, \dots, s_{mN})$$



$$r(t) = \sum_{k=1}^{N} s_{mk} f_k(t) + \sum_{k=1}^{N} n_k f_k(t) + n'(t)$$

Note this is the message index, not known at receiver

$$n'(t) \triangleq n(t) - \sum_{k=1}^{N} n_k f_k(t)$$

$$E[n'(t)] = \cancelto{0}{E[n(t)]} - \sum_{k=1}^{N} E[n_k] f_k(t) = 0$$

$$E[n_k] = \int_0^T \cancelto{0}{E[n(t)]} f_k(t) = 0$$

We will show that $n'(t)$ is not relevant to the decision as to which m was transmitted below.

Note $E[n_j n_k] = \int_0^T\int_0^T E[n(t)n(\tau)]\, f_j(t) f_k(\tau)\,dt\,d\tau$

$$= \int_0^T\int_0^T \frac{N_0}{2}\,\delta(t-\tau)\, f_j(t) f_k(\tau)\,dt\,d\tau$$

$$= \int_0^T \frac{N_0}{2} f_j(t) f_k(t)\,dt = \frac{N_0}{2}\delta_{jk}$$

$$E[n'(t)\,r_k] = \cancelto{0}{E[n'(t)]}\, s_{mk} + E[n'(t)\,n_k]$$

$$= E\left\{\left[n(t) - \sum_{j=1}^{N} n_j f_j(t)\right] n_k\right\}$$

$$= \int_0^T \underbrace{E[n(t)\,n(\tau)]}_{\frac{N_0}{2}\delta(t-\tau)} f_k(\tau)\,d\tau - \sum_{j=1}^{N} \underbrace{E[n_j n_k]}_{\frac{N_0}{2}\delta_{jk}} f_j(t)$$

$$E[n'(t) r_k] = \frac{N_0}{2} f_k(t) - \frac{N_0}{2} f_k(t) = 0$$

This means $n'(t)$ is uncorrelated with r_k. Note that

$$n_k = \int_0^T n(t) f_k(t)\, dt$$

is obtained from a Gaussian random process via linear filtering and is therefore a Gaussian r.v. . Also note that

$$n'(t) = n(t) - \sum_{k=1}^{N} n_k f_k(t)$$

is obtained from a Gaussian random process via linear filtering and is therefore Gaussian. Finally,

$$r_k = s_{mk} + n_k$$

Is Gaussian since s_{mk} is deterministic. Two Gaussian random variables (and processes) that are uncorrelated are statistically independent. Therefore $n'(t)$ is statistically independent of r_k, $1 \le k \le N$, and can be safely ignored as to the decision of m.

Note $\{n_k\}_{k=1}^{N}$ are uncorrelated Gaussian r.v.s and therefore they are statistically independent.

$$n_k \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$$

Notation (script N) for Gaussian (normal) with zero mean and $\sigma^2 = \frac{N_0}{2}$

$$\underline{n} \triangleq (n_1, n_2, \ldots, n_N) \sim \mathcal{N}\left(\underline{0}, \frac{N_0}{2} \underline{I}_{N \times N}\right)$$

Distributed as

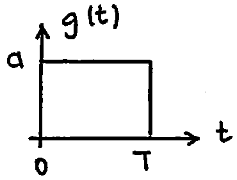
$$p(r_k \mid s_{mk}) = \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} \exp\left[-\frac{(r_k - s_{mk})^2}{2 N_0/2}\right]$$

$$p(\underline{r} \mid \underline{s}_m) = \prod_{k=1}^{N} p(r_k \mid s_{mk}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_0}\right]$$

$m = 1, 2, \ldots, M$

Ex M-ary baseband PAM with $g(t)$ the basic pulse shape.

The noise is additive, zero-mean, Gaussian.



$$\mathcal{E}_g = \int_0^T g^2(t)\, dt = \int_0^T a^2\, dt = a^2 T$$

There is only one basis function $f(t)$

$$f(t) = \frac{1}{\sqrt{\mathcal{E}_g}}\, g(t) = \begin{cases} 1/\sqrt{T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

The correlation receiver output is

$$r = \int_0^T r(t) f(t)\, dt = \int_0^T [s_m(t) + n(t)] \frac{1}{\sqrt{T}}\, dt$$

$$= \underbrace{\frac{1}{\sqrt{T}} \int_0^T s_m(t)\, dt}_{s_m} + \underbrace{\frac{1}{\sqrt{T}} \int_0^T n(t)\, dt}_{n}$$

$$r = s_m + n$$

$$E[n] = \frac{1}{\sqrt{T}} \int_0^T E[n(t)] = 0$$

$$E[n^2] = \sigma_n^2 = \frac{1}{T} \int_0^T \int_0^T E[n(t)\, n(\tau)]\, dt\, d\tau = \frac{1}{T} \int_0^T \int_0^T \frac{N_0}{2} \delta(t-\tau)\, dt\, d\tau$$

$$\sigma_n^2 = \frac{N_0}{2}$$

Note s_m is deterministic. $r \mid s_m \sim \mathcal{N}\left(s_m, \frac{N_0}{2}\right)$

$$p(r \mid s_m) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(r - s_m)^2\right]$$

MATCHED FILTER DEMODULATOR

Equivalent system, different implementation.

Let the outputs of N linear filters be sampled at time $t=T$, and let the kth filter have the impulse response

$$h_k(t) = f_k(T-t) \qquad 0 \le t \le T, \quad 1 \le k \le N$$

N basis functions

Note $h_k(t) = 0$ elsewhere.

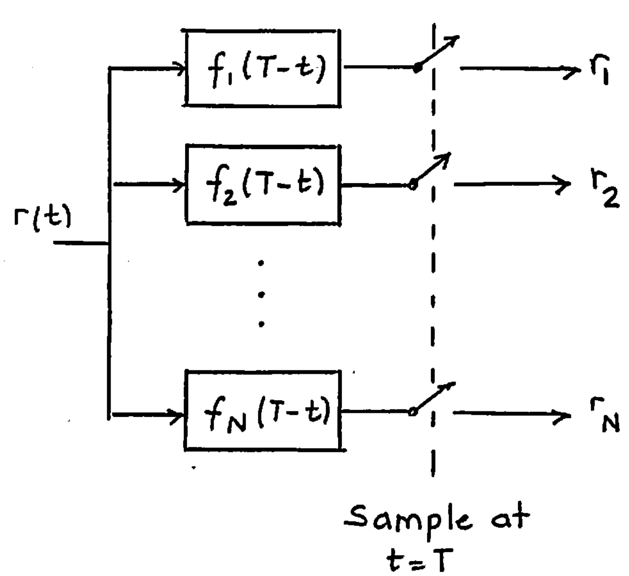
$$y_k(t) = \int_0^t r(\tau)\, h_k(t-\tau)\, dt \qquad k = 1, 2, \ldots, N$$

$$= \int_0^t r(\tau)\, f_k(T-t+\tau)\, dt$$

Sample the filter output at time $t=T$

$$y_k(T) = \int_0^T r(\tau)\, f_k(\tau)\, d\tau = r_k \qquad k = 1, 2, \ldots, N$$

A filter whose impulse response $h(t) = s(T-t)$ where $s(t)$ is confined to $0 \le t \le T$ is called the filter matched to the signal $s(t)$.



Fact: If a signal $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio (SNR).

Proof:

$$y(T) = \int_0^T r(\tau)\, h(T-\tau)\, d\tau$$

$$= \underbrace{\int_0^T s(\tau)\, h(T-\tau)\, d\tau}_{y_s(T)} + \underbrace{\int_0^T n(\tau)\, h(T-\tau)\, d\tau}_{y_n(T)}$$

$$= y_s(T) + y_n(T)$$

$$SNR_o \triangleq \frac{y_s^2(T)}{E[y_n^2(T)]}$$

$$E[y_n^2(T)] = \int_0^T\int_0^T E[n(t)n(\tau)]\, h(T-\tau)\, h(T-t)\, dt\, d\tau$$

$$= \frac{N_o}{2} \int_0^T\int_0^T \delta(t-\tau)\, h(T-\tau)\, h(t-\tau)\, dt\, d\tau$$

$$= \frac{N_o}{2} \int_0^T h^2(T-t)\, dt$$

$$SNR_o = \frac{\left[\int_0^T s(\tau)\, h(T-\tau)\, d\tau\right]^2}{\frac{N_o}{2}\int_0^T h^2(T-t)\, dt} \underset{\text{Cauchy-Schwarz inequality}}{\leq} \frac{\int_0^T s^2(\tau)\, d\tau \cancel{\int_0^T h^2(T-\tau)\, d\tau}}{\frac{N_o}{2}\cancel{\int_0^T h^2(T-t)\, dt}}$$

$$SNR_o \leq \frac{2}{N_o}\int_0^T s^2(\tau)\, d\tau$$

and the maximum is achieved if $h(T-t) = s(t) \quad 0 \leq t \leq T$.

The maximum SNR_o is $\frac{2}{N_o}\mathcal{E}$.

The matched filter has a frequency domain interpretation

$$h(t) = s(T-t) \qquad 0 \leq t \leq T$$

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} s(\underbrace{T-t}_{\tau})\, e^{-j2\pi ft}\, dt$$

$$= -\int_{\infty}^{-\infty} s(\tau)\, e^{-j2\pi fT}\, e^{+j2\pi f\tau}\, d\tau$$

$$= e^{-j2\pi fT}\left[\int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi f\tau}\, d\tau\right]^{*} \qquad \text{Note } s(t) \text{ is real}$$

$$= e^{-j2\pi fT}\; S^{*}(f)$$

Represents a delay of T

$|H(f)| = |S(f)|$ so that the magnitude response of the matched filter is identical to the signal spectrum while its phase is the negative of that of the signal

Note $y_s(t) = \mathcal{F}^{-1}\{H(f)S(f)\} = \mathcal{F}^{-1}\{|S(f)|^2 e^{-j2\pi fT}\}$

$$= \int_{-\infty}^{\infty} |S(f)|^2\, e^{-j2\pi fT}\, e^{j2\pi ft}\, df$$

$$y_s(T) = \int_{-\infty}^{\infty} |S(f)|^2\, df = \int_{-\infty}^{\infty} s^2(t)\, dt = \mathcal{E}$$

The output noise PSD is

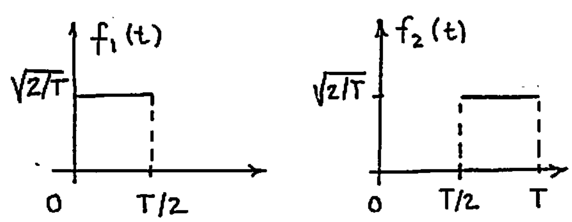
$$S_o(f) = |H(f)|^2 S_N(f) = \frac{1}{2}|H(f)|^2 N_o$$

$$P_n = \int_{-\infty}^{\infty} S_o(f)\, df = \frac{N_o}{2}\int_{-\infty}^{\infty} |S(f)|^2\, df = \frac{N_o}{2}\mathcal{E}$$

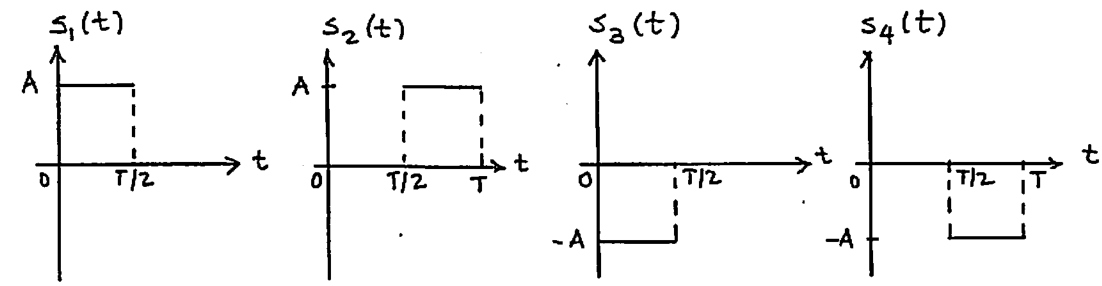
Noise power at the output

$$SNR_o = \frac{[y_s(T)]^2}{P_n} = \frac{\mathcal{E}^{\cancel{2}}}{\frac{N_o}{2}\cancel{\mathcal{E}}} = \frac{2\mathcal{E}}{N_o}$$

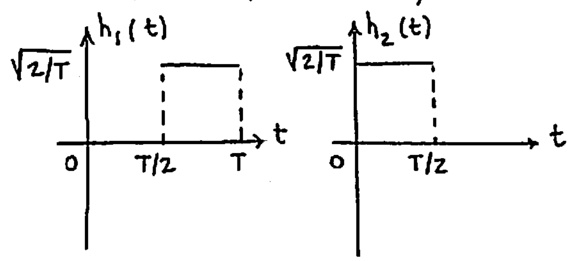
Ex Consider two orthogonal signals $f_1(t), f_2(t)$ $(N=2)$



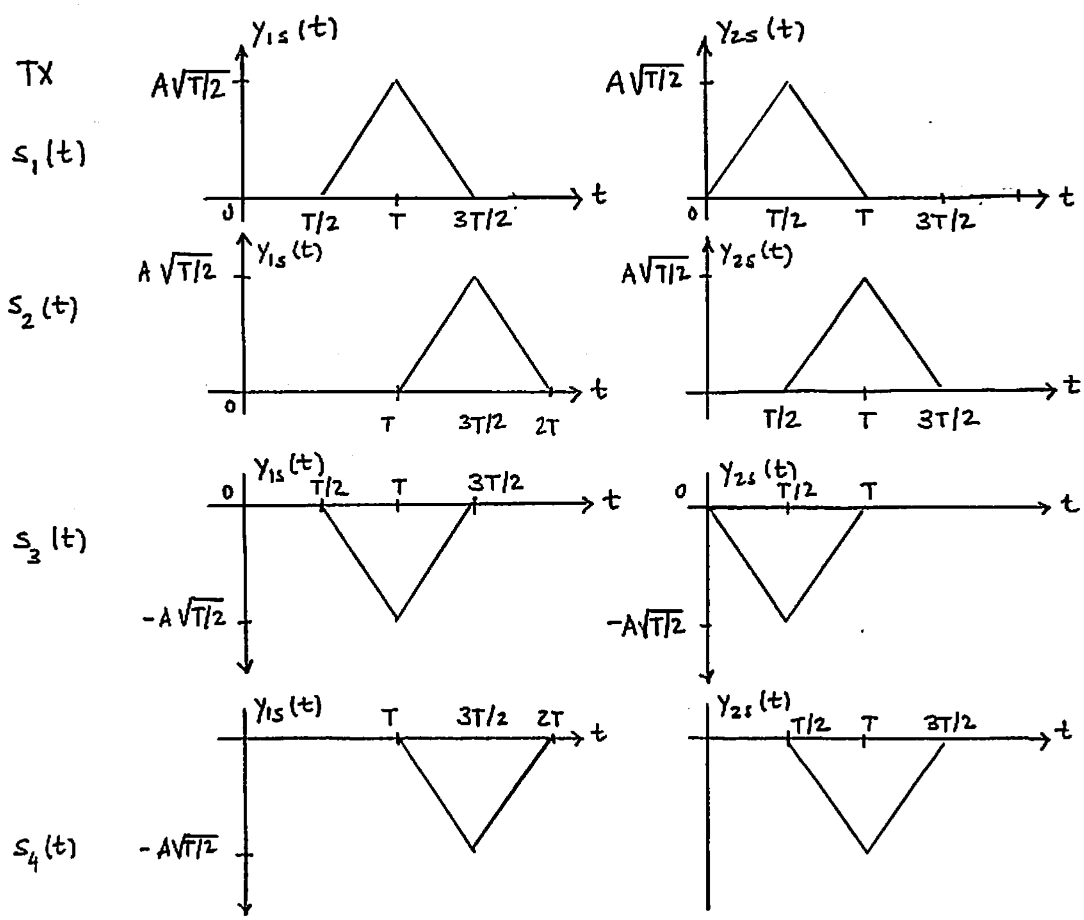
Construct $M=4$ biorthogonal signal from this set :



Let $h_i(t) = f_i(T-t)$, $i = 1, 2$



Consider filter responses in the absence of noise when $s_i(t)$ is sent $i=1,2,3,4$.



$$\mathcal{E} = \int_0^{T/2} A^2\,dt = A^2 T/2$$

$$\bar{r} = \begin{cases} (\sqrt{\varepsilon} + n_1 \, , \, n_2) & s_1 \text{ transmitted} \\ (n_1 \, , \, \sqrt{\varepsilon} + n_2) & s_2 \text{ transmitted} \\ (-\sqrt{\varepsilon} + n_1 \, , \, n_2) & s_3 \text{ transmitted} \\ (n_1 \, , \, -\sqrt{\varepsilon} + n_2) & s_4 \text{ transmitted} \end{cases}$$

$$\sigma_{n1}^2 = \sigma_{n2}^2 = E\left[\int_0^T n(\tau) f_i(\tau)\, d\tau \int_0^T n(t) f_i(t)\, dt\right]$$

$$= \int_0^T\int_0^T \frac{N_0}{2} \, \delta(t-\tau) \, f_i(t) \, f_i(\tau) \, dt \, d\tau$$

$$= \frac{N_0}{2}$$

$SNR_o = 2\varepsilon / N_o$ at the output of $\begin{cases} h_1(t) \text{ when } s_1 \text{ or } s_3 \text{ is sent} \\ h_2(t) \text{ when } s_2 \text{ or } s_4 \text{ is sent} \end{cases}$

THE OPTIMUM DETECTOR

A correlation receiver or (equivalently) a matched filter demodulator produces a vector which contains all the relevant information in a received signal waveform for an AWGN channel. What is the optimum detector, how to make optimum detection decisions?

Consider the posterior probabilities

$P(\underline{s}_m | \underline{r}) = Pr\,(\text{signal } \underline{s}_m \text{ was transmitted} \mid \underline{r}) \quad m = 1, 2, \ldots, M$

Choosing m based on maximizing $P(\underline{s}_m | \underline{r})$ is known as Maximum A Posteriori (MAP) decision criterion.

$$P(\underline{s}_m | \underline{r}) = \frac{p(\underline{r}|\underline{s}_m) P(\underline{s}_m)}{p(\underline{r})} = \frac{p(\underline{r}|\underline{s}_m) P(\underline{s}_m)}{\sum_{m=1}^{M} p(\underline{r}|\underline{s}_m) P(\underline{s}_m)}$$

Bayes' rule

Note independent of m, can be dropped

$p(\underline{r}|\underline{s}_m)$: Likelihood function

If $\underline{s}_m$ are equally probable, $P(\underline{s}_m)$ is a constant and maximizing $P(\underline{s}_m|\underline{r})$ is equivalent to maximizing $p(\underline{r}|\underline{s}_m)$.

Maximization of $p(\underline{r}|\underline{s}_m)$ is known as the <u>Maximum Likelihood (ML) detection criterion</u>.

$$p(\underline{r}|\underline{s}_m) = \frac{1}{(\pi N_o)^{N/2}} \exp\left[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_o}\right] \quad m=1,2,\ldots,M$$

$$\ln p(\underline{r}|\underline{s}_m) = -\frac{N}{2} \ln(\pi N_o) - \frac{1}{N_o} \sum_{k=1}^{N} (r_k - s_{mk})^2 \quad m=1,2,\ldots,M$$

Maximization of $p(\underline{r}|\underline{s}_m)$ is equivalent to to minimization of $\sum_{k=1}^{N} (r_k - s_{mk})^2 = \|\underline{r} - \underline{s}_m\|^2$, the squared distance between the received vector and $\underline{s}_m$ for all $m=1,2,\ldots,M$.

$$D(\underline{r}, \underline{s}_m) = \|\underline{r} - \underline{s}_m\|^2 = \sum_{k=1}^{N} (r_k - s_{mk})^2$$

Distance metrics

For the AWGN channel, the ML criterion results in minimizing the distance metrics, this is known as the minimum distance detection criterion.

$$D(\underline{r}, \underline{s}_m) = \|\underline{r} - \underline{s}_m\|^2 = \langle \underline{r} - \underline{s}_m, \underline{r} - \underline{s}_m \rangle$$
$$= \|\underline{r}\|^2 - 2\langle \underline{r}, \underline{s}_m \rangle + \|\underline{s}_m\|^2$$

Independent of m

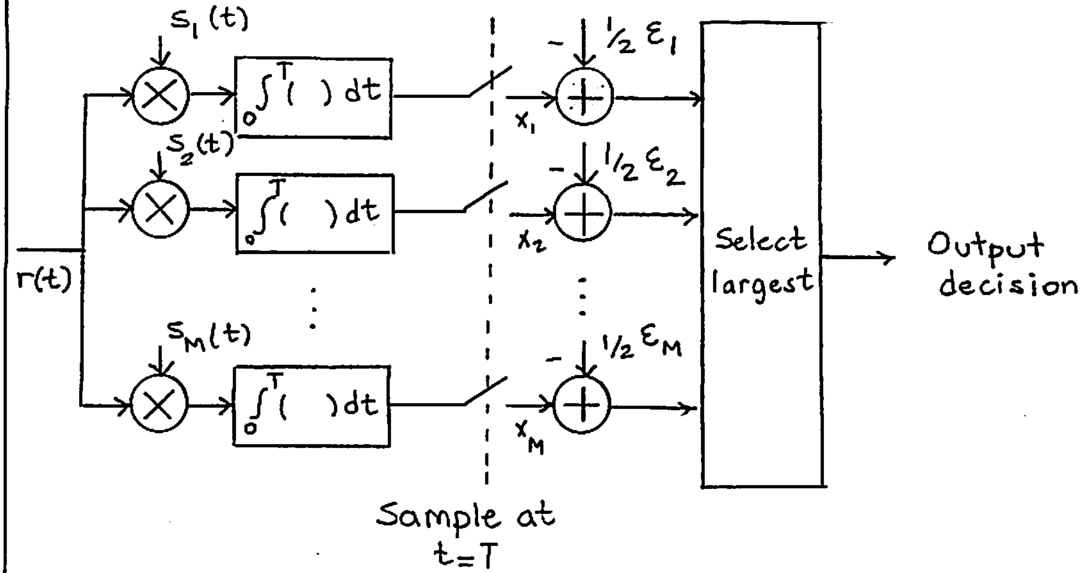
Minimizing $D(\underline{r}, \underline{s}_m)$ is therefore equivalent to minimizing

$$D'(\underline{r}, \underline{s}_m) = -2 \langle \underline{r}, \underline{s}_m \rangle + \| \underline{s}_m \|^2$$

or maximizing

$$C(\underline{r}, \underline{s}_m) = 2 \langle \underline{r}, \underline{s}_m \rangle - \| \underline{s}_m \|^2$$

Alternative optimum AWGN receiver



Recall ML is optimum if transmitted signals are equally likely. Otherwise MAP is the optimum decision criterion and is based on maximizing the metrics

$$PM(\underline{r}, \underline{s}_m) \triangleq p(\underline{r} \mid \underline{s}_m) \Pr(\underline{s}_m)$$

<u>Ex</u> Binary PAM signals $s_1 = -s_2 = \sqrt{\mathcal{E}_b}$ $\quad \mathcal{E}_b$: Bit energy $\Pr(s_1) = p$, $\Pr(s_2) = 1-p$. What is the optimum detector for an AWGN channel with two-sided noise PSD $\frac{N_0}{2}$.

$$r = \pm\sqrt{\mathcal{E}_b} + y_n(T) \qquad y_n(T) \sim \mathcal{N}(0, \underbrace{\tfrac{N_0}{2}}_{\sigma_n^2})$$

$$p(r \mid s_1) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left[-\frac{1}{2\sigma_n^2} (r - \sqrt{\mathcal{E}_b})^2 \right]$$

$$p(r \mid s_2) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left[-\frac{1}{2\sigma_n^2} (r + \sqrt{\mathcal{E}_b})^2 \right]$$

$$PM(\underline{r}, \underline{s}_1) = \frac{p}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{1}{2\sigma_n^2}(r-\sqrt{\mathcal{E}_b})^2}$$

$$PM(\underline{r}, \underline{s}_2) = \frac{(1-p)}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{1}{2\sigma_n^2}(r+\sqrt{\mathcal{E}_b})^2}$$

We choose m that maximizes $PM(\underline{r}, \underline{s}_m)$ (MAP criterion)

$$PM(\underline{r}, \underline{s}_1) \underset{s_2}{\overset{s_1}{\gtrless}} PM(\underline{r}, \underline{s}_2)$$

$$\frac{p}{\cancel{\sqrt{2\pi}\,\sigma_n}} e^{-\frac{1}{2\sigma_n^2}(r-\sqrt{\mathcal{E}_b})^2} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{(1-p)}{\cancel{\sqrt{2\pi}\,\sigma_n}} e^{-\frac{1}{2\sigma_n^2}(r+\sqrt{\mathcal{E}_b})^2}$$

$$\frac{p}{(1-p)} \underset{s_2}{\overset{s_1}{\gtrless}} e^{\frac{1}{2\sigma_n^2}\left[(r-\sqrt{\mathcal{E}_b})^2-(r+\sqrt{\mathcal{E}_b})^2\right]}$$

$$\ln\frac{p}{1-p} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{1}{2\sigma_n^2}\left[(r-\sqrt{\mathcal{E}_b})^2-(r+\sqrt{\mathcal{E}_b})^2\right]$$

$$2\sigma_n^2 \ln\frac{1-p}{p} \underset{s_1}{\overset{s_2}{\gtrless}} (r+\sqrt{\mathcal{E}_b})^2-(r-\sqrt{\mathcal{E}_b})^2$$

$$\sqrt{\mathcal{E}_b}\, r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\sigma_n^2}{2} \ln\frac{(1-p)}{p}$$

$$r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_o}{4\sqrt{\mathcal{E}_b}} \ln\frac{(1-p)}{p}$$

Note need not know N_o or $\sqrt{\mathcal{E}_b}$ if $p = 1/2$.

Fact: ML decision rule minimizes $Pr(\text{error})$ when the M signals are equally likely.

Proof: Define R_m as the region in N-dimensional space for which the ML criterion leads one to decide $\underline{s}_m$ was transmitted.

$$Pr(c \mid \underline{s}_m) = \int_{R_m} p(\underline{r} \mid \underline{s}_m)\, d\underline{r}$$

Prob (correct decision given message m is tx)

$$Pr(c) = \sum_{m=1}^{M} \frac{1}{M} Pr(c \mid \underline{s}_m)$$

$$= \sum_{m=1}^{M} \frac{1}{M} \int_{R_m} p(\underline{r} \mid \underline{s}_m)\, d\underline{r}$$

To maximize, R_m should be such that $p(\underline{r} \mid \underline{s}_m)$ is maximized (among all $p(\underline{r} \mid \underline{s}_k)$ $k=1,2,\ldots,M$) within R_m.

Fact: MAP decision rule minimizes Pr(error) even when M signals are not equally likely.

Proof: $$Pr(c) = \sum_{m=1}^{M} Pr(c \mid \underline{s}_m)\, Pr(\underline{s}_m)$$

$$= \sum_{m=1}^{M} \int_{R_m} p(\underline{r} \mid \underline{s}_m)\, Pr(\underline{s}_m)\, d\underline{r}$$

$$= \sum_{m=1}^{M} \int_{R_m} Pr(\underline{s}_m \mid \underline{r})\, p(\underline{r})\, d\underline{r}$$

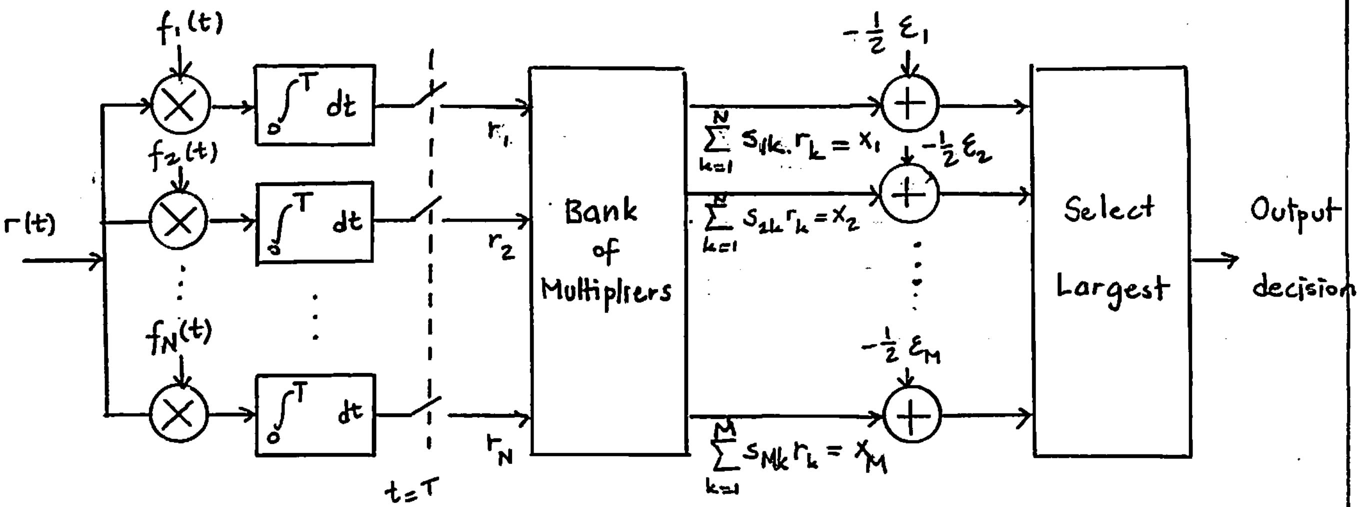
$\Rightarrow$ Maximize $Pr(\underline{s}_m \mid \underline{r})$ (MAP) maximizes $P(c)$.

ALTERNATIVE IMPLEMENTATION OF THE OPTIMUM RECEIVER

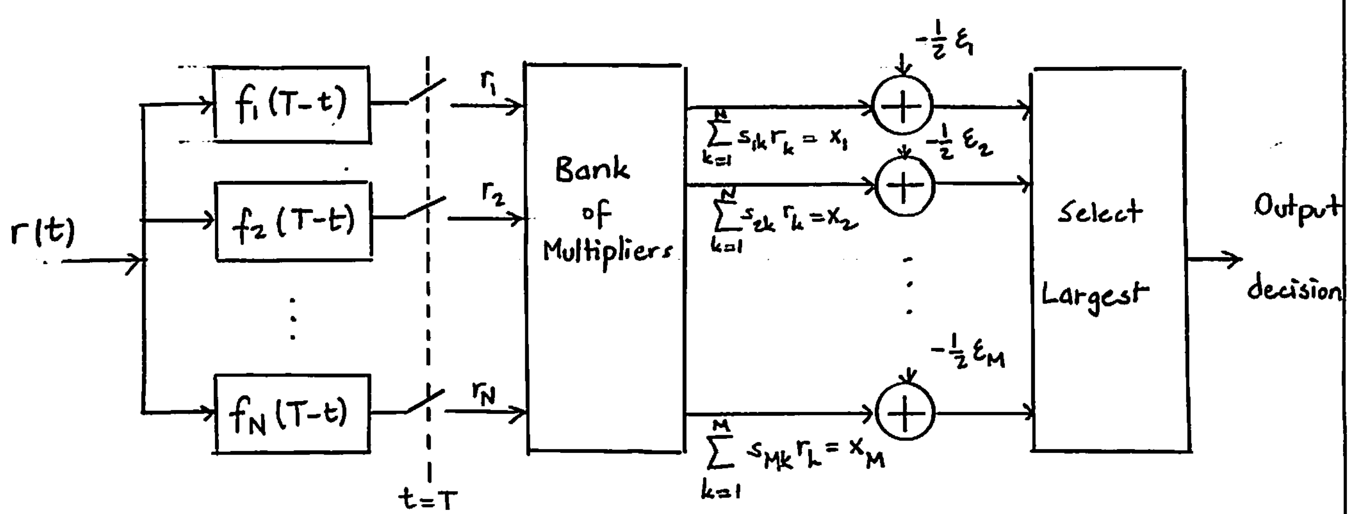
Recall that

$$s_m(t) = \sum_{k=1}^{N} s_{mk} f_k(t)$$

This motivates an alternative implementation of the correlation receiver as shown below



Similarly, the matched filter receiver can be alternatively implemented as shown below



Note the adders prior to the "Select Largest" block are not needed if $\mathcal{E}_i = \mathcal{E}_j \quad 1 \le i, j \le M$.

ALTERNATIVE DETECTION RULE: BAYES CRITERION

An alternative to MAP and ML decision rules can be obtained by casting the detection (or decision) problem as a hypothesis testing problem. This approach formulates the problem in a different way but also results in a likelihood ratio test, although potentially with a different threshold. This approach is described in the textbook "Detection, Estimation, and Modulation Theory, Part I" by H. L. Van Trees.

Consider a decision problem in which each of two source outputs corresponds to a hypothesis (H_0 and H_1, in a communications problem correspond to bit 0 is transmitted and bit 1 is transmitted). The observation space is assumed to be N-dimensional and the observation can be thought as a point in N-dimensional space

$$\underline{r} = (r_1, r_2, \ldots, r_N)^T.$$

Observations are generated with two known conditional densities $p_{\underline{R}|H_1}(\underline{r}|H_1)$ and $p_{\underline{R}|H_0}(\underline{r}|H_0)$. (to be abbreviated as $p(\underline{r}|H_1)$ and $p(\underline{r}|H_0)$.

We know that either H_0 or H_1 is true. Each time the experiment is conducted one of four things can happen

1. H_0 true, choose H_0
2. H_0 true, choose H_1
3. H_1 true, choose H_0
4. H_1 true, choose H_1

The first and fourth are correct choices. The purpose of a decision criterion is to attach some relative importance to four possible outcomes. We do this by assigning costs to these outcomes $C_{00}, C_{01}, C_{10}, C_{11}$ where

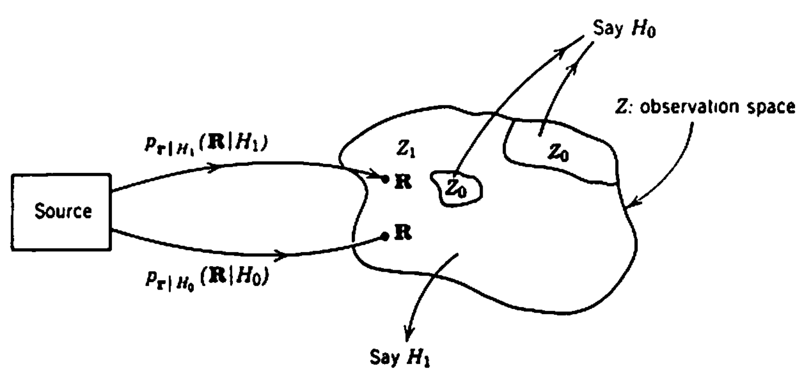
$$C_{ij}$$

hypothesis that was chosen (i)   hypothesis that was true (j)

The Bayes criterion tries to minimize the risk, defined as the expected value of the cost

$$\begin{aligned}\mathcal{R} = & \; C_{00} P_0 \Pr(\text{say } H_0 \mid H_0 \text{ true}) \\ & + C_{10} P_0 \Pr(\text{say } H_1 \mid H_0 \text{ true}) \\ & + C_{01} P_1 \Pr(\text{say } H_0 \mid H_1 \text{ true}) \\ & + C_{11} P_1 \Pr(\text{say } H_1 \mid H_1 \text{ true})\end{aligned}$$

where P_0 and P_1 are the a priori probabilities of H_0 and H_1 occurring, which we assume we know. Since after each observation, we will make a decision, the observation space is divided into Z_0 and Z_1: when $\underline{r} \in Z_0$ we choose H_0 and when $\underline{r} \in Z_1$ we choose H_1.



Then,

$$\mathcal{R} = C_{00} P_0 \int_{Z_0} p(\underline{r} \mid H_0)\, d\underline{r}$$

$$+ C_{10} P_0 \int_{Z_1} p(\underline{r} \mid H_0)\, d\underline{r}$$

$$+ C_{01} P_1 \int_{Z_0} p(\underline{r} \mid H_1)\, d\underline{r}$$

$$+ C_{11} P_1 \int_{Z_1} p(\underline{r} \mid H_1)\, d\underline{r}$$

We can assume that the cost of a wrong decision is higher than the cost of a correct decision

$$C_{10} > C_{00},$$

$$C_{01} > C_{11}.$$

Recall that the observation space is divided into Z_0 and Z_1:

$$Z = Z_0 \cup Z_1 = Z_0 + Z_1$$

$$\mathcal{R} = P_0 C_{00} \int_{Z_0} p(\underline{r} \mid H_0)\, d\underline{r} + P_0 C_{10} \int_{Z - Z_0} p(\underline{r} \mid H_0)\, d\underline{r}$$

$$+ P_1 C_{01} \int_{Z_0} p(\underline{r} \mid H_1)\, d\underline{r} + P_1 C_{11} \int_{Z - Z_0} p(\underline{r} \mid H_1)\, d\underline{r}$$

Note

$$\int_{Z} p(\underline{r} \mid H_i)\, d\underline{r} = 1$$

and

$$\int_{Z - Z_0} p(\underline{r} \mid H_i)\, d\underline{r} = 1 - \int_{Z_0} p(\underline{r} \mid H_i)\, d\underline{r}$$

for $i = 0, 1$.

$$\mathcal{R} = P_0 C_{10} + P_1 C_{11}$$

$$+ \int_{Z_0} \{ [P_1 (C_{01} - C_{11}) \; p(\underline{r} | H_1)] - [P_0 (C_{10} - C_{00}) \, p(\underline{r} | H_0)] \} \, d\underline{r}$$

Note that due to the assumptions on costs the two terms inside the brackets are positive. Observe that all values of $\underline{r}$ for which the second term is larger should be assigned to Z_0 because then they contribute a negative amount to the integral. Likewise, all values of $\underline{r}$ for which the first term is larger should be assigned to Z_1 so that they are not part of the integral because they would contribute positive to the integral. This specifies the decision rule

$$P_1 (C_{01} - C_{11}) \; p(\underline{r} | H_1) \underset{H_0}{\overset{H_1}{\gtrless}} P_0 (C_{10} - C_{00}) \; p(\underline{r} | H_0)$$

$$\frac{p(\underline{r} | H_1)}{p(\underline{r} | H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0 \; (C_{10} - C_{00})}{P_1 \; (C_{01} - C_{11})} \qquad (6.1)$$

Note the similarity of this result to the binary MAP example studied previously.

In addition to digital communication, this detection technique is used in (for example) radar problems. When the hypothesis is the presence or absence of an intruder, one can see how the costs associated with wrong decisions can be asymmetric. The cost of a false alarm is typically much less than the cost of a miss.

Note if $C_{10} = C_{01} = 1$ and $C_{00} = C_{11} = 0$ $\mathcal{R}$ becomes $Pr(E)$ ($Pr(E) = 1 - Pr(C)$) and the Bayes criterion for the binary hypothesis problem becomes equivalent to the MAP criterion for the same problem. If, furthermore, $P_0 = P_1$ then the Bayes criterion for this problem becomes the ML criterion for the same problem. Also note that the left hand side of equation (6.1) remains the same for using one of these criteria, only the right hand side changes.