EECS 241A

HOMEWORK 4 SOLUTIONS

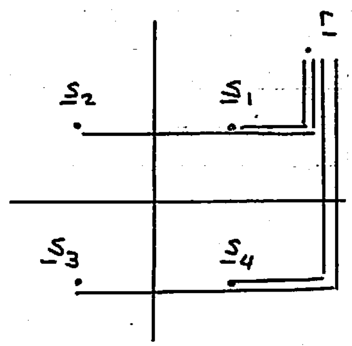
1. (a) $p(\underline{r} \mid \underline{s}_m) = \prod_{k=1}^{N} p_{N_k}(r_k - s_{mk}) = \left(\frac{a}{2}\right)^N e^{-a\sum_{k=1}^{N}|r_k - s_{mk}|}$ $\quad m = 1, 2, \ldots, M$

(b) $\log p(\underline{r} \mid \underline{s}_m) = \log\left(\frac{a}{2}\right)^N - a\sum_{k=1}^{N}|r_k - s_{mk}|$

(c) $D(\underline{r}, \underline{s}_m) = \sum_{k=1}^{N}|r_k - s_{mk}|$

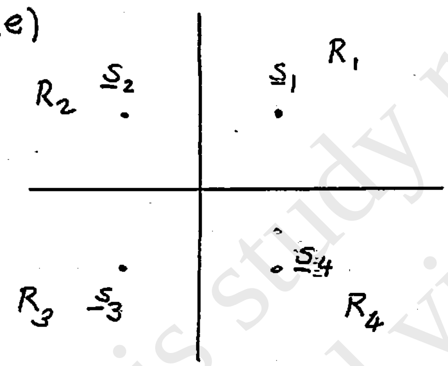
$m = 1, 2, \ldots, M$

(d)

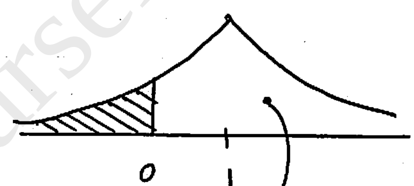


D is the sum of the absolute values along each dimension (Also known as L_1 metric or Manhattan metric)

(e)



(f)



$$P(C) = \left(\frac{1}{2} + \frac{a}{2}\int_0^1 e^{-ax}\,dx\right)^2$$

$$= \left(1 - \frac{e^{-a}}{2}\right)^2$$

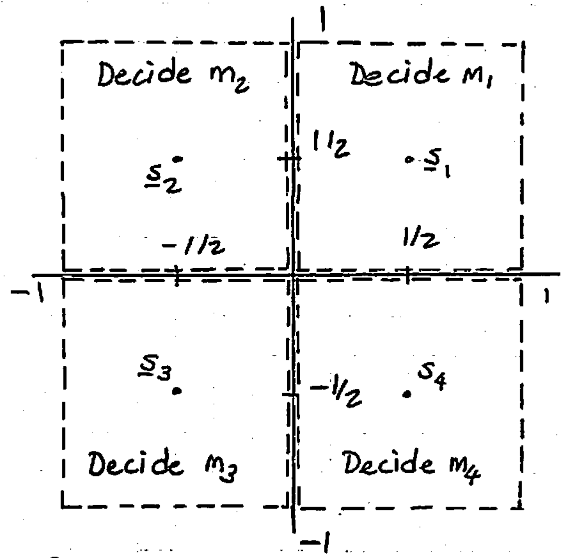
$$P(E) = 1 - \left(1 - \frac{e^{-a}}{2}\right)^2 = e^{-a} - \frac{e^{-2a}}{4}$$

(g) P(E) is zero since there is no probability of noise exceeding beyond decision regions

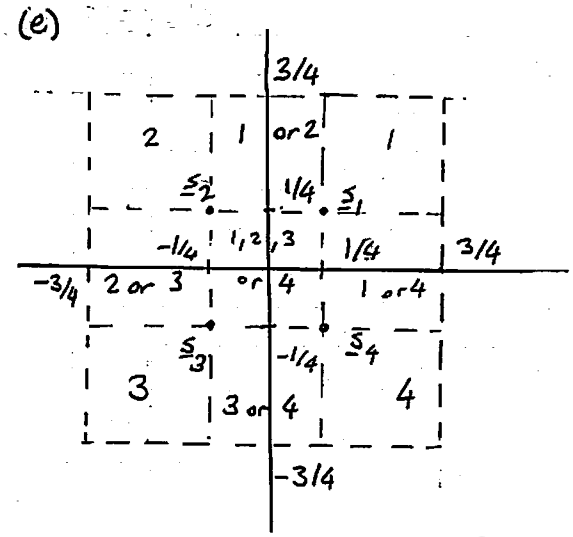
2. (a) $p(\underline{r} \mid \underline{s}_m) = p(\underline{r} - \underline{s}_m) = \prod_{k=1}^{N} \text{rect}(r_k - s_{mk})$

(b) $\log p(\underline{r} \mid \underline{s}_m) = \sum_{k=1}^{N} \log \text{rect}(r_k - s_{mk}) = \begin{cases} -\infty & \text{if } |r_k - s_{mk}| > \frac{1}{2} \text{ for any } k \\ 0 & \text{if } |r_k - s_{mk}| < \frac{1}{2} \text{ for all } k \end{cases}$

(c)



(e)



(d) $P(E) = 0$

3. (a) $p_{Y_1}(y_1 | x) = p_{N_1}(y_1 - x) = \frac{1}{2} e^{-|y_1 - x|} \qquad -\infty < y_1 < \infty$

$p_{Y_2}(y_2 | x) = p_{N_2}(y_2 - x) = \frac{1}{2} e^{-|y_2 - x|} \qquad -\infty < y_2 < \infty$

(b) $p_{\underline{Y}|x}(\underline{y} | x) = p_{Y_1, Y_2 | x} = p_{Y_1|x}(y_1 | x)\, p_{Y_2}(y_2 | x)$

$$= \frac{1}{4} e^{-|y_1 - x| - |y_2 - x|}$$

$$\log p_{\underline{Y}|x}(\underline{y} | x) = \log \frac{1}{4} - (|y_1 - x| + |y_2 - x|)$$

$$\max p_{\underline{Y}|x}(\underline{y} | x) \Rightarrow \min \left(|y_1 - x| + |y_2 - x| \right)$$

This rule becomes

$$|y_1 - x| + |y_2 - x| \underset{x=1}{\overset{x=-1}{\gtrless}} |y_1 + 1| + |y_2 + 1|$$

(c) You can conclude by plugging in numbers that

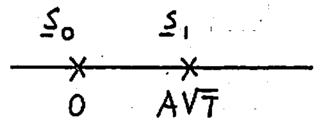
i) Region 1 : Decide $x = 1$

ii) Region 2 : Decide $x = -1$

iii) Region 3 : Both sides of the inequality have the same numeric value. Both $x = 1$ and $x = -1$ are optimum decisions

4\. (a) $f_1(t) = \begin{cases} 1/\sqrt{T} & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$ The orthonormal basis set $\{f_i(t)\}_{i=1}^{1}$ is unique.

(b) $s_0(t) = 0 \cdot f_1(t)$ $\quad s_1(t) = A\sqrt{T}\, f_1(t)$



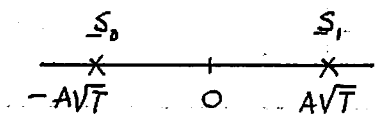
(c) $P_b = Q\left(\dfrac{A\sqrt{T}}{2\sqrt{N_o/2}}\right) = Q\left(A\sqrt{\dfrac{T}{2N_o}}\right)$

(d) $A\sqrt{\dfrac{T}{2N_o}} = 4.25 \quad (Q(4.25) = 10^{-5})$

$T = 72.25\ \mu s$

(e) $f_1(t) = \begin{cases} 1/\sqrt{T} & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$ The orthonormal basis set $\{f_i(t)\}_{i=1}^{1}$ is unique.

(f) $s_0(t) = -A\sqrt{T}\, f_1(t)$ $\quad s_1(t) = A\sqrt{T}\, f_1(t)$



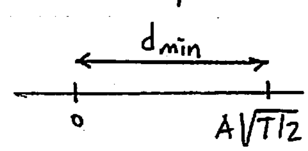
(g) $P_b = Q\left(\dfrac{2A\sqrt{T}}{2\sqrt{N_o/2}}\right) = Q\left(A\sqrt{\dfrac{2T}{N_o}}\right)$

(h) $A\sqrt{\dfrac{2T}{N_o}} = 4.25$ $\quad T = 18.06\ \mu s$

(i) Note the first system is on-off signaling while the second one is antipodal. The minimum distances between the two systems differ by 3 dB. As a result, for the same voltage level, the second system can work 4 times faster.

5\. (a) $\displaystyle\int_0^T s_1^2(t)\,dt = \int_0^T A^2 \sin^2\frac{\pi t}{T}\,dt = A^2\left[\frac{T}{2} - \frac{1}{2}\cancel{\int_0^T \cos\frac{2\pi t}{T}\,dt}^{\,0}\right] = \frac{A^2 T}{2}$

$\sigma^2 = N_o/2$



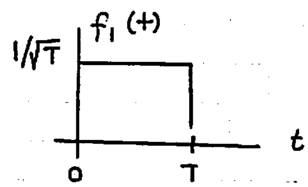
$$P(E) = Q\left(\frac{d_{min}}{2\sigma}\right) = Q\left(\frac{1}{2} A\sqrt{\frac{T}{2}}\sqrt{\frac{2}{N_o}}\right) = Q\left(\frac{A}{2}\sqrt{\frac{T}{N_o}}\right)$$

$$= Q\left(\frac{0.2\times10^{-3}}{2}\sqrt{\frac{2\times10^{-6}}{2\times10^{-15}}}\right) = Q(\sqrt{10}) = 7.83\times10^{-4}$$

(b) $Q(4.25) = 10^{-5}$

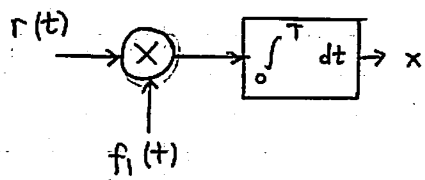
$$4.25 = \frac{A}{2\times 10^{-4}}\sqrt{10} \qquad A = \frac{8.5}{\sqrt{10}}\times 10^{-4} = 2.7\times 10^{-4} = 10 \text{ mV}$$

6. (a) $$f_1(t) = \frac{s_1(t)}{\sqrt{\int_0^T s_1^2(t)\,dt}}$$



$s_1(t) = A\sqrt{T}\, f_1(t)$

$s_2(t) = 0$



(b) MAP criterion : Maximize $p(\underline{r}\mid \underline{s}_m)\Pr(\underline{s}_m)$

$$p(\underline{r}\mid\underline{s}_1)\Pr(\underline{s}_1) \underset{2}{\overset{1}{\gtrless}} p(\underline{r}\mid\underline{s}_2)\Pr(\underline{s}_2)$$

$$\frac{p(\underline{r}\mid\underline{s}_1)}{p(\underline{r}\mid\underline{s}_2)} \underset{2}{\overset{1}{\gtrless}} \frac{P_2}{P_1}$$

$$-\frac{1}{N_0}\left[(r - A\sqrt{T})^2 - r^2\right] \underset{2}{\overset{1}{\gtrless}} \ln\frac{P_2}{P_1}$$

$$2A\sqrt{T}\,r - A^2T \underset{2}{\overset{1}{\gtrless}} N_0 \ln\frac{P_2}{P_1}$$

$$r \underset{2}{\overset{1}{\gtrless}} \underbrace{\frac{A\sqrt{T}}{2} - \frac{N_0}{2A\sqrt{T}}\ln\frac{P_1}{P_2}}_{\eta}$$