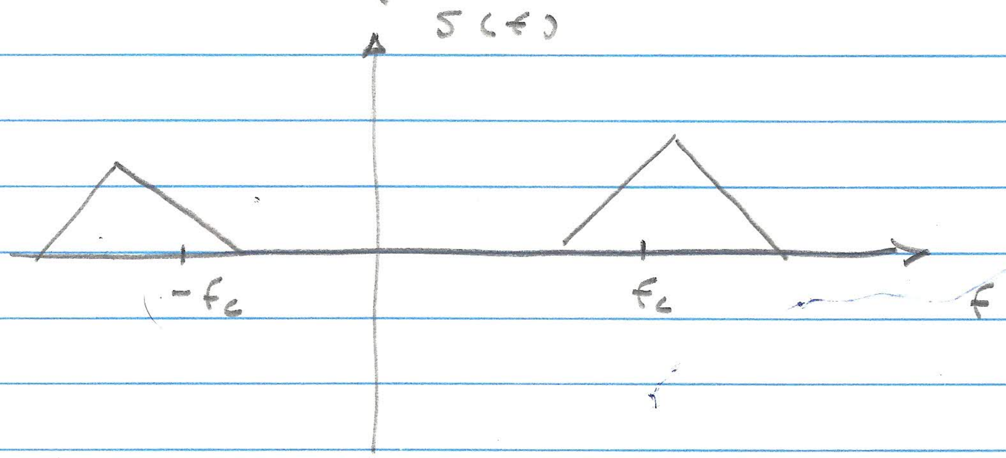
# #5

## Bandpass signals.

- There are 3 ways to characterize BP signals
- Most digitally modulated signals are BP.
- Let $u(\cdot)$ be the step function
- Look at a general situation:




$f_c$ does <u>not</u> need to be in the middle!

## Pre-envelope.

$$S_+(f) \stackrel{\Delta}{=} 2 \cdot u(f) \cdot S(f) \quad , \text{ then}$$

$$s_+(t) = s(t) + j \cdot \hat{s}(t) \quad (\text{why?})$$

$$\hat{s}(t) = \frac{1}{\pi t} * s(t) \quad (\text{Hilbert transform})$$

$$\hat{S}(f) = -j\,\mathrm{sgn}(f) \cdot S(f)$$

Then, $s(t) = \mathrm{Re}\{s_+(t)\}$,

however

$$s_+(t) = s_\ell(t) \cdot \exp(+j2\pi f_c t)$$

Thus,

$$s(t) = \mathrm{Re}\{s_\ell(t) \exp(+j2\pi f_c t)\}$$

Also, $s_\ell(t) = s_I(t) + j\, s_Q(t)$, thus

$$s(t) = s_I(t) \cdot \cos(2\pi f_c t) - s_Q(t) \cdot \sin(2\pi f_c t),$$

and

$$s(t) = r(t) \cos(2\pi f_c t + \theta(t)),$$

with

$$r(t) = \{s_I^2(t) + s_Q^2(t)\}^{1/2}$$

&

$$\theta(t) = \tan^{-1}\left(\frac{s_Q(t)}{s_I(t)}\right)$$

Note:

$$S(f) = \mathcal{F}\{s(t)\} =$$

$$= \int_{-\infty}^{+\infty} s(t) \cdot \exp(-j2\pi f t)\, dt =$$

$$= \int_{-\infty}^{+\infty} \mathrm{Re}\{ s_\ell(t) \exp(+j2\pi f_c t)\} \exp(-j2\pi f t)\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} s_\ell(t) \exp(-j2\pi (f-f_c)t)\, dt$$

$$+ \frac{1}{2} \int s_\ell^*(t) \exp\left(-j2\pi (f+f_c)t\right) dt =$$

$$= \frac{1}{2} \left[ S_\ell(f-f_c) + S_\ell^*(-f-f_c) \right]$$

Signal energy:

$$\mathcal{E}_s = \int_{-\infty}^{+\infty} |s(t)|^2\, dt = \int_{-\infty}^{+\infty} |S(f)|^2\, df =$$

$$= \frac{1}{4} \cdot \int_{-\infty}^{+\infty} |S_\ell(f-f_c)|^2 + \frac{1}{4} \int_{-\infty}^{+\infty} |S_\ell(-f-f_c)|^2\, df$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} |S_\ell(f)|^2\, df = \frac{1}{2} \mathcal{E}_{s\ell}$$

## LTI BP Systems

$$h(t) = 2 \cdot \mathrm{Re}\{ h_\ell(t) \exp(+j2\pi f_c t)\}$$

$$x(t) = \mathrm{Re}\{ x_\ell(t) \cdot \exp(+j2\pi f_c t)\}$$

$$y(t) = \int_{-\infty}^{+\infty} h(u)\, x(t-u)\, du =$$

$$= \int_{-\infty}^{+\infty} \cancel{2} \cdot \frac{1}{\cancel{2}} \{ h_\ell(u) \exp(+j2\pi f_c u) + h_\ell^*(u) \cdot \exp(-j2\pi f_c u)\} \cdot$$

$$\frac{1}{2} \{ x_\ell(t-u) \exp(+j2\pi f_c (t-u)) + x_\ell^*(t-u) \exp(-j2\pi f_c (t-u))\} \, du =$$

$$= \frac{1}{2} \cdot \left( \int_{-\infty}^{+\infty} h_\ell(u) \cdot x_\ell(t-u)\, du \right) \cdot \exp(+j2\pi f_c t)$$

$$+ \frac{1}{2} \cdot \left( \int_{-\infty}^{+\infty} h_\ell^*(u) \cdot x_\ell^*(t-u)\, du \right) \cdot \exp(-j2\pi f_c t) =$$

$$= \mathrm{Re}\{ \underbrace{\int_{-\infty}^{+\infty} h_\ell(u)\, x_\ell(t-u)\, du} \exp(+j2\pi f_c t)\}$$

Thus, $y_\ell(t) = \int_{-\infty}^{+\infty} h_\ell(u) \cdot x_\ell(t-u)\, du$

$\rightarrow$ Note this way: $\mathcal{E}_{y_\ell} = 2\, \mathcal{E}_y$ ! $\leftarrow$

Also,

$$Y(f) = \frac{1}{2}\left\{ Y_\ell(f-f_0) + Y_\ell^*(-f-f_0) \right\} =$$

$$= \frac{1}{2}\left\{ H_\ell(f-f_0)\cdot X_\ell(f-f_0) + \right.$$

$$\left. + H_\ell^*(-f-f_0)\, X_\ell^*(-f-f_0) \right\}.$$

<u>Band-pass stationary stochastic process:</u>

$$n(t) = \mathrm{Re}\left\{ n_\ell(t) \cdot \exp(+j\,2\pi f_c t) \right\}$$

$\Rightarrow$ For $n(t)$ to be WSS, we <u>need</u>

$n_\ell(t) = n_I(t) + j\,n_Q(t)$, to have:

a) $n_I(\cdot)$, $n_Q(\cdot)$ are WSS, <u>jointly</u>

b) $R_{n_I}(\tau) = R_{n_Q}(\tau)$

c) $R_{n_I n_Q}(\tau) = -R_{n_Q n_I}(\tau)$

Then,

$$R_n(\tau) = R_{n_I}(\tau)\cos(2\pi f_c \tau) - R_{n_{QI}}(\tau)\sin(\cdot)$$

Thus,

$$R_n(\tau) = \mathrm{Re}\{R_{n_e}(\tau)\cdot \exp(+j2\pi f_c \tau)\}$$

with $$R_{n_e}(\tau) = R_{n_I}(\tau) + j\cdot R_{n_{QI}}(\tau)$$

Now, consider:

$$z(t) = n_I(t) + j n_Q(t)$$

$$R_z(\tau) = 2\cdot\left(R_{n_I}(\tau) + j R_{n_{QI}}(\tau)\right)$$

More properties of BPS R.PR.

1. $$S_n(f) = \mathcal{F}\{R_z(\tau)\, \exp(+j2\pi f_c\tau)\} =$$

$$= \frac{1}{2}\{S_z(f-f_c) + S_z^*(-f-f_c)\}$$

2. $$R_z(\tau) = \mathbb{E}\{z(t+\tau)\cdot z^*(t)\} =$$

$$= R_z^*(-\tau) \implies \underline{R_z^*(\tau) = R_z(-\tau)}$$

$$\implies S_z^*(-f) = S_z(-f) \iff S_z \text{ is } \underline{\text{real}}.$$

Thus, $$S_n(f) = \frac{1}{2}\{S_z(f-f_c) + S_z(-f-f_c)\}$$

3. $$R_{n_{QI}}(\tau) = -R_{n_{IQ}}(\tau) \implies$$

$\implies$ $n_I(t)$, $n_Q(t)$ are $\underline{\text{uncorrelated}}$

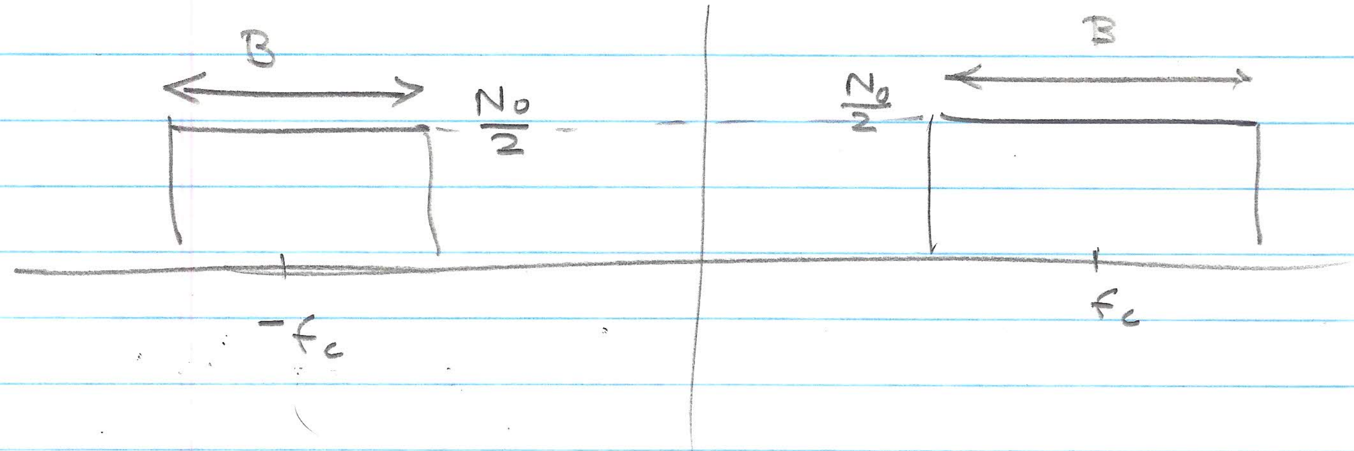
- White noise

Idealized concept

$$S_N(f) = \frac{N_0}{2} \quad W \cdot Hz^{-1}$$

for all $f$.

BP white noise




$$R_Z(\tau) = \begin{cases} 2N_0 & |f| < \frac{B}{2} \\ 0 & \text{else} \end{cases}$$

(no normalization by 1/2)

Thus, $R_{n_Q I}(\tau) = 0$ for all $\tau$.