EE2023 Final cheatsheet by wjb75

1 Maths

1.1 Arctan Correction

$tan^{-1}x \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ \theta = \tan^{-1}(\frac{b}{a}) \in (-\pi, \pi)$

1. $a > 0$: no adjustment (1st or 4th q)
2. $a < 0, b > 0$: Add $180°(\pi)$ (2nd q)
3. $a < 0, b < 0$: Subtract $180°(-\pi)$ (3rd q)

1.2 Partial Fraction Decomposition

$$\tilde{\mathcal{F}}(s) = \frac{\tilde{\mathcal{C}}(s)}{\tilde{\mathcal{D}}(s)}$$

1. $M < N$, all roots of $\tilde{\mathcal{D}}(s)$ are distinct
   $\tilde{\mathcal{F}}(s) = \sum_{j=1}^{N} \frac{\alpha_j}{s+p_j}$
   $\alpha_j = (s+p_j) \times \tilde{\mathcal{F}}(s)\big|_{s=-p_j}$
2. $M < N$, repeated roots
   $\tilde{\mathcal{F}}(s) = \sum_{j=1}^{N-r} \frac{\alpha_j}{s+p_j} + \sum_{k=1}^{r} \frac{\gamma_k}{(s+p)^k}$
   $\gamma_k = \frac{1}{(r-k)!} \frac{d^{r-k}}{ds^{r-k}} [(s+p)^r \times \tilde{\mathcal{F}}(s)]\big|_{s=-p}$
3. $M \geq N$
   do long division first
4. Complex root
   solve the linear equation

1.3 Common Laplace Transform of Higher Order Derivatives

$\tilde{\mathcal{L}}[y'(t)] = s\tilde{\mathcal{Y}}(s) - y(0)$
$\tilde{\mathcal{L}}[y''(t)] = s^2\tilde{\mathcal{Y}}(s) - sy(0) - y'(0)$

2 Signals

2.1 Energy and Power Signals

2.1.1 Energy E and Energy Spectral Density $E_x(f)$

$E = \int_{-\infty}^{\infty} |x(t)^2|\, dt = \int_{-\infty}^{\infty} E_x(f)\, df = \int_{-\infty}^{\infty} |\mathbf{X(f)}|^2\, df$
if it is defined

2.1.2 Power and Power Spectral Density $P_x(f)$

$$P = \lim_{W\to\infty} \frac{1}{2W} \int_{-W}^{W} |x(t)^2|\, dt = \int_{-W}^{W} \lim_{W\to\infty} \frac{1}{2W} |X_W(f)|^2\, df$$

For periodic signals,
$P = \lim_{W\to\infty} \frac{1}{2W} \int_{-W}^{W} |x(t)^2|\, dt = \frac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p)$$

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

2.2 Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau$$

2.3 Dirac Delta

1. $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$
2. $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)\, dt = x(\lambda)$
3. $x(t) * \delta(t-\lambda) = x(t-\lambda)$
4. $x_p(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_p) = \sum_{n=-\infty}^{\infty} x(t - nT_p)$

3 Fourier Series

3.1 Complex Exp FS

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_p(t) e^{-j2\pi k f_p t}\, dt \qquad \forall k \in \mathbb{Z}$$

where $f_p = HCF\{sinusoidal\ frequencies\} = \frac{1}{T_p}$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}$$

DC-value $= c_0$

3.2 Trigonometric FS

$$a_k = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_p(t) cos(j2\pi k f_p t)\, dt \qquad k \geq 0$$
$$b_k = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_p(t) sin(j2\pi k f_p t)\, dt \qquad k > 0$$

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty} [a_k cos(2\pi kt/T_p) + b_k sin(2\pi kt/T_p)]$$

4 Fourier Tranform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$
$$x(t) \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$$

4.1 Spectral Properties of Real Signal

1. if x(t) is real, X(f) is conjugate symmetric
2. if x(t) is real and even, X(f) is real and even
3. if x(t) is real and odd, X(f) is imaginary and odd

4.1.1 Periodic Signals

$$X_p(f) = \mathcal{F}\{x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}\} = \sum_{k=-\infty}^{\infty} c_k \delta(f - k/T_p)$$
$$= \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

refer to the power

5 Bandwidth

1. for bandpass signal, the bandwidths are two-sided.
2. careful about $E_x(f)$ or $|X(f)|$
3. Uncertainty : Band-limited signals are not time-limited, Time-limited signals are not band-limited

5.0.1 M% Energy Containment Bandwidth

Lowpass

$$\int_{-B}^{B} E_x(f)\, df = 2\int_0^B E_x(f)\, df = \frac{M}{100} \times E$$

Bandpass

$$\int_{f_c-0.5B}^{f_c+0.5B} E_x(f)\, df = \frac{M}{100} \times \frac{E}{2}$$

5.0.2 M% Power Containment Bandwidth

$$B = Kf_p$$

$$\sum_{k=-K}^{K} |c_k|^2 \geq \frac{M}{100} \times P(\text{careful about } c_0 \text{ in symmetry arguments})$$

6 LTI Systems

$h(t) = \mathbf{T}[\delta(t)]$
$y(t) = x(t) * h(t)$
$o(t) = \int_{-\infty}^{t} h(t)dt = \int_0^t h(t)dt = \mathcal{L}^{-1}\{\frac{1}{s}\tilde{H}(s)\}$

6.1 Frequency Response

$Y(f) = X(f)H(f)$

6.1.1 Sinusoidal Response Steady State

$H(f_0) = |H(f_0)| e^{j\angle H(f_0)}$
$H(j\omega) = |H(j\omega)| e^{j\angle \tilde{H}(j\omega)}$

$x(t) = Ae^{j(2\pi f_0 t + \phi)} \to H(f) \to y(t) = A|H(f_0)| e^{j(2\pi f_0 t + \phi + \angle H(f_0))}$

6.2 System Stability

6.2.1 BIBO Stable

$\lim_{t\to\infty} h(t) = 0$
all poles at the left half of the s-plane (Re{poles} < 0)

6.2.2 Marginally Stable

1. one or more distinct poles on Im axis
2. AND no poles on right half of s-plane

6.2.3 Unstable

1. at least 1 pole on right half of s-plane
2. OR repeated poles lying on Im axis

7 LTI Systems Models and Response

7.1 First Order System

Transfer function:

$$G(s) = \frac{K}{1+Ts} = \frac{K}{T}\frac{1}{s+1/T}$$

7.1.1 Unit Impulse Response

$$y_\delta(t) = \frac{K}{T} e^{-t/T} u(t)$$

7.1.2 Unit Step Response

$$y_{\text{step}}(t) = K\left(1 - e^{-t/T}\right) u(t)$$

Remarks: - Time constant: T - DC gain: K - Real pole at $s = -\frac{1}{T}$

7.2 Second Order System (Overdamped, $\zeta > 1$)

Transfer function:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2}$$

7.2.1 Unit Impulse Response

$$y_\delta(t) = [K_1 e^{-p_1 t} + K_2 e^{-p_2 t}]u(t)$$

7.2.2 Unit Step Response

$$y_{\text{step}}(t) = [K - \frac{K_1}{p_1}e^{-p_1 t} - \frac{K_2}{p_2}e^{-p_2 t}]u(t)$$

Remarks: - DC gain: K

$$p_{1,2} = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$K_1 = -K_2 = \frac{K\omega_n^2}{p_2 - p_1}$$

- Real distinct poles: at

$$s = -p_1, s = -p_2$$

7.3 Second Order System (Critically Damped, $\zeta = 1$)

Transfer function:

$$G(s) = \frac{K\omega_n^2}{(s+\omega_n)^2}$$

7.3.1 Unit Impulse Response

$$y_\delta(t) = K\omega_n^2 t e^{-\omega_n t}u(t)$$

7.3.2 Unit Step Response

$$y_{\text{step}}(t) = K\left[(1 - (1+\omega_n t)e^{-\omega_n t})\right]u(t)$$

Remarks: - DC gain: K - Real repeated poles at $s = -\omega_n$

7.4 Second Order System (Underdamped, $0 < \zeta < 1$)

Transfer function:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K(\sigma^2 + \omega_d^2)}{(s+\sigma)^2 + \omega_d^2}$$

Let:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$\sigma = \zeta\omega_n$, $\omega_n^2 = \sigma^2 + \omega_d^2$, $tan(\phi) = \frac{\omega_d}{\sigma}$

7.4.1 Unit Impulse Response

$$y_\delta(t) = \frac{K\omega_n^2}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)u(t)$$

7.4.2 Unit Step Response

$$y_{\text{step}}(t) = K\left[1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\left(sin(\omega_d t + \phi)\right)\right]u(t)$$

Remarks: - Damped natural frequency: ω_d - Complex conjugate poles: at

$$s = -\zeta\omega_n \pm j\omega_d$$

7.5 Second Order System (Undamped, $\zeta = 0$)

Transfer function:

$$G(s) = \frac{K\omega_n^2}{s^2 + \omega_n^2}$$

7.5.1 Unit Impulse Response

$$y_\delta(t) = K\omega_n \sin(\omega_n t)u(t)$$

7.5.2 Unit Step Response

$$y_{\text{step}}(t) = K\left(1 - \cos(\omega_n t)\right)u(t)$$

Remarks: - Poles at $s = \pm j\omega_n$ - Pure oscillation at undamped natural frequency ω_n

7.6 Second Order Resonance Summary

Resonance occurs for $0 \le \zeta < \frac{1}{\sqrt{2}}$.

Resonant frequency:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

Resonance peak magnitude:

$$M_r = \left|G(j\omega_r)\right| = \frac{K}{2\zeta\sqrt{1-\zeta^2}}$$

8 Bode Plot

- Magnitude plot:

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(\left|\tilde{H}(j\omega)\right|) \ \ dB$$

- Phase plot in degrees: $\angle\tilde{H}(j\omega)$
- Cascade of frequency response: $\tilde{H}(j\omega) = \Pi_k \tilde{H}_k(j\omega) \rightarrow$ linear sum of magnitude and phase

8.1 DC Gain $\tilde{\mathbf{H}}(\mathbf{s}) = K_{dc}$

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(\left|\tilde{H}(K_{dc})\right|) \ \ dB$$

$\angle\tilde{H}(j\omega) = 0°$

8.2 Differentiator $\tilde{\mathbf{H}}(\mathbf{s}) = K_d s$

$$\tilde{H}(j\omega) = K_d j\omega = K_d\omega e^{j90°}$$

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(K_d) + 20log_{10}(\omega) \ \ dB$$

$20dB/decade$ slope at all freq, passing through $(1/K_d\,, 0), (1, 20log_{10}K_d)$

8.3 Integrator $\tilde{\mathbf{H}}(\mathbf{s}) = \frac{K_i}{s}$

$$\tilde{H}(j\omega) = \frac{K_i}{j\omega} = \frac{K_i}{\omega}e^{-j90°}$$

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(K_i) - 20log_{10}(\omega) \ \ dB$$

$-20dB/decade$ slope at all freq, passing through $(K_i\,, 0), (1, 20log_{10}K_i)$

8.4 Zero Factor $\tilde{\mathbf{H}}(\mathbf{s}) = \frac{s}{z_m} + 1; z_m > 0$

$$\tilde{H}(j\omega) = \frac{j\omega}{z_m} + 1$$

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(\frac{\omega}{z_m}) = 20log_{10}(\omega) - 20log_{10}(z_m) \text{ (after } \omega >> z_m \text{) } dB$$

$+20dB/decade$ slope change at $\omega = z_m$, while actual magnitude is $3dB$ at $\omega = z_m$

$\angle\tilde{H}(j\omega) = 0°$ at $z_m/10$, $\angle\tilde{H}(j\omega) = 45°$ at z_m, $\angle\tilde{H}(j\omega) = 90°$ at $10z_m$

8.5 Pole Factor $\tilde{\mathbf{H}}(\mathbf{s}) = \frac{1}{s/p_n + 1}; p_n > 0$

$$\tilde{H}(j\omega) = \frac{1}{j\omega/p_n + 1}$$

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(\frac{p_n}{\omega}) = 20log_{10}(p_n) - 20log_{10}(\omega) \text{ (after } \omega >> p_n \text{)}$$

$-20dB/decade$ slope change at $\omega = p_n$, while actual magnitude is $-3dB$ at $\omega = p_n$

$\angle\tilde{H}(j\omega) = 0°$ at $p_n/10$, $\angle\tilde{H}(j\omega) = -45°$ at p_n, $\angle\tilde{H}(j\omega) = -90°$ at $10p_n$

8.6 2nd Order Factor $\tilde{\mathbf{H}}(\mathbf{s}) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \omega_n > 0, 0 \le \zeta \le 1$

$$\left|\tilde{H}(j\omega)\right|_{dB} = 20log_{10}(\frac{\omega_n^2}{\omega^2}) = 40log_{10}(\omega_n) - 40log_{10}(\omega) \text{ (after } \omega >> \omega_n \text{)}$$

$-40dB/decade$ slope change at $\omega = \omega_n$, while actual magnitude might have resonance at ω_r depending on ζ not shown in Bode plot

$\angle\tilde{H}(j\omega) = 0°$ at $\omega_n/10$, $\angle\tilde{H}(j\omega) = -90°$ at ω_n, $\angle\tilde{H}(j\omega) = -180°$ at $10\omega_n$

8.7 Asymptotic Slope and Phase $\angle\tilde{H}(j\omega)$

let N be number of poles, M be number of zeroes, $N - M$ be pole-zero excess, D be number of differentiators I be the number of integrators

8.7.1 High Frequency

phase:

$$\lim_{\omega\to\infty}\angle\tilde{H}(j\omega) = (N - M) \times (-90°)$$

slope:

$$\lim_{\omega\to\infty}\left[\text{slope of } \left|\tilde{H}(j\omega)\right|_{dB}\right] = (N - M) \times (-20dB/decade)$$

8.7.2 Low Frequency

phase:

$$\lim_{\omega\to 0}\angle\tilde{H}(j\omega) = (I - D) \times (-90°)$$

slope:

$$\lim_{\omega\to 0}\left[\text{slope of } \left|\tilde{H}(j\omega)\right|_{dB}\right] = (I - D) \times (-20dB/decade)$$

9 Sampling and Reconstruction of Signals

$x_s(t) = x(t)\delta(t - nT_s)$ $X_s(f) = f_s \times [\sum_n X(f - nf_s)]$

Nyquist: $f_s \ge 2f_m$ for band-limited signals

9.1 Sampling Band-limited Bandpass Signal Below Nyquist Rate

9.1.1 Overlapping spectral images ($f_c > 0.5B$)

$k \le \lfloor\frac{2f_c}{B}\rfloor \in \mathbb{Z}^+$, $f_s = \frac{2f_c}{k}$

Reconstruction filter: $\frac{1}{2f_s}\left[rect(\frac{f-f_c}{B}) + rect(\frac{f+f_c}{B})\right]$

9.1.2 Non-Overlapping (Unaliased) spectral images ($f_c > 1.5B$)

$k \le \lfloor\frac{2f_c - B}{2B}\rfloor \in \mathbb{Z}^+$, $\frac{2f_c + B}{k+1} \le f_s \le \frac{2f_c - B}{k}$

Reconstruction filter: $\frac{1}{f_s}\left[rect(\frac{f-f_c}{B}) + rect(\frac{f+f_c}{B})\right]$