1.2 Periodicity

$\exists T>0,\ x(t) = x(t+T)\ \forall t \Rightarrow$ periodic

$\sin(At+B)\ \cos(At+B)$: $T_p = \frac{2\pi}{A}$

$\tan(At+B)$: $T_p = \frac{\pi}{A}$

e^{jt} : $T_p = 2\pi$

$f_p = \frac{1}{T_p}$

1.2.5 Complex Signal

$z = a + jb$, a: Re, b: Im ($\checkmark$ j)

Conjugate : $z \to z^*$ change j to $-j$

$z = re^{j\theta}$

$r = |z| \geq 0$ mag

$\theta = \angle z = \arg(z)$

$-\pi < \theta \leq \pi$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ $\xrightarrow{\text{with adjustment}}$

$(-\frac{\pi}{2}, \frac{\pi}{2})$ — so for $(-\pi, \pi]$

- $a>0$ (no need)
- $a<0,\ b>0$, $+\pi$
- $a<0,\ b<0$, $-\pi$

Energy & Power Signal

$x(t)$ is energy signal iff

$$0 < \int_{-\infty}^{\infty} |x(t)|^2 dt = E < \infty$$

$x(t)$ is power signal iff

$$0 < \boxed{\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt} = P < \infty$$

for periodic signal, $\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt$

Basic Signal

A. $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases} = \int_{-\infty}^{t} \delta(\tau)\, d\tau$

B. $\mathrm{sgn}(t) = \begin{cases} +1, & t \geq 0 \\ -1, & t<0 \end{cases}$

C. $\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1; & -\frac{\tau}{2} \leq t < \frac{\tau}{2} \\ 0; & \text{elsewhere} \end{cases}$ $\left(\frac{\text{duration}}{\tau}\right)$

D. $\mathrm{tri}\left(\frac{t}{\tau}\right) = \begin{cases} 1-\frac{|t|}{\tau}; & |t| \leq \tau \\ 0; & |t| > \tau \end{cases}$ $\left(\frac{\text{duration}}{2\tau}\right)$

Gradient of $A\,\mathrm{tri}\left(\frac{t}{\tau}\right)$ is $\frac{A}{\tau}$

2.

Time scaling, shifting

$x(t-\beta)$: $\beta>0$: delaying

$\beta<0$, advancing

2.1.5 Convolution

$\int_{-\infty}^{\infty} x(\alpha) y(t-\alpha)\, d\alpha = y * x$

- Commutative
- Associative
- Distributive

$F\{x * y\} = X \cdot Y$

2.2 δ

1) $\delta(t) = \delta(-t)$
2) $x(t)\,\delta(t-\lambda) = x(\lambda)\,\delta(t-\lambda)$
3) $x(t) * \delta(t-\lambda) = x(t-\lambda)$
4) $\int_{-\infty}^{\infty} x(t)\,\delta(t-\lambda) = x(\lambda)$

δ-comb: $\sum_{n=-\infty}^{\infty} \delta(t-nT)$

3. Discrete freq Spectrum

$$\tilde{x}(t) = Me^{j(2\pi f_0 t + \phi)} = \underbrace{\boxed{Me^{j\phi}}}_{\text{Coefficient spectrum}} \times e^{j2\pi f_0 t}$$

3.3 Fourier Series

bounded periodic signal

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t/T_p} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} = \boxed{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_p t}}$$

$$c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\, e^{-j2\pi k t/T_p}\, dt,\ k = 0, \pm1, \pm2$$

tips: use ω to when k is easier

3.3.2 Trigo FS

$a_k = \frac{c_{-k}+c_k}{2}$ $\quad b_k = \frac{c_{-k}-c_k}{j2}$

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{2\pi k t}{T_p}\right) + b_k \sin\left(\frac{2\pi k t}{T_p}\right)\right]$$

$$a_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t) \cos\left(\frac{2\pi k t}{T_p}\right) dt,\ k \geq 0$$

$$b_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t) \sin\left(\frac{2\pi k t}{T_p}\right) dt,\ k > 0$$

4. Continuous frequency Spectrum

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt \xrightarrow{\text{observe}} X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df \longrightarrow x(0) = \int_{-\infty}^{\infty} X(f)\, df$$

Dirichlet Condition

1) finite # of min/max
2) finite # of discontinuities
3) $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

4.2 Properties of Fourier Transform

1) Linearity
2) . . .
3) . . . see slides

4.3 Spectral Properties of real signal

If $x(t)$ is real, $x^*(t) = x(t)$ — $X^*(f) = X(-f)$ weak

$|X(f)| = |X(-f)|$, $\angle X(f) = -\angle X(-f)$

$|C_k| = |C_{-k}|$, $\angle C_k = -\angle C_{-k}$

$x(t)$ Real and even, $X(f)$ Real and even

$X^*(f) = X(f)$, $X(-f) = X(f)$

$C_k^* = C_k$, $C_k = C_{-k}$

$x(t)$ Real and odd, $X(f)$ imaginary and odd

$X^*(f) = -X(f)$, $X(-f) = -X(f)$

$C_k^* = -C_k$, $C_{-k} = -C_k$

Fourier Transform of periodic signal

$\{x_p(t);\ T = T_p,\ f_p = \frac{1}{T_p}\}$

$$X_p(f) = F\{x_p(t)\} = F\left\{\sum_{k=-\infty}^{\infty} C_k e^{j2\pi k t/T_p}\right\}$$

$$= \sum F(\ldots)$$

$$\boxed{X_p(f) = \sum_{k=-\infty}^{\infty} C_k\, \delta(f - k f_p)}$$

5. Spectral Density and Bandwidth

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad (J)$$

$$= \int_{-\infty}^{\infty} |X(f)|^2\, df$$

看哪个比较方便

$$E_x(f) = |X(f)|^2 \quad (J/Hz)$$

5.2 Power Spectral Density

$$P = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} \lim_{\tau\to\infty} \frac{1}{2\tau} |X_\tau(f)|^2\, df$$

$$P_x(f) = \lim_{\tau\to\infty} \frac{1}{2\tau} |X_\tau(f)|^2$$

For periodic signal

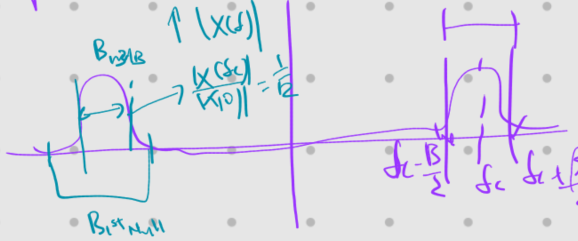
$$P_x(f) = \sum_{k=-\infty}^{\infty} |C_k|^2\, \delta(f - k f_p)$$

$$P = \int_{-\infty}^{\infty} P_x(f) = \sum_{k=-\infty}^{\infty} |C_k|^2$$

5.3 Bandlimited signal

Lowpass: $|X(f)| = 0$, $|f| > B$

Bandpass $|X(f)| = 0$; $|f| - f_c > B/2$

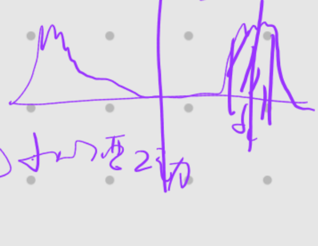


5.3.2 Bandwidth

1) M% Energy Containment Bandwidth

$$\int_{-B}^{B} E_x(f)\, df = \frac{M}{100} \times \int_{-\infty}^{\infty} E_x(f)\, df$$

E



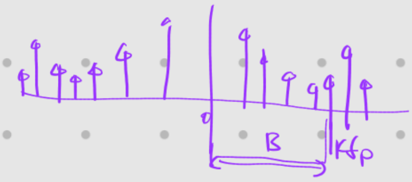
$$\int_{f_c - \frac{B}{2}}^{f_c + \frac{B}{2}} E_x(f)\, df = \frac{M}{100} \times \frac{E}{2}$$

! 如何是2边

2) M% Power Containment Bandwidth

$B = K f_p$ st K smallest that satisfy

$$\sum_{k=-K}^{K} |C_k|^2 \geq \frac{M}{100} \times \sum_{k=-\infty}^{\infty} |C_k|^2 \quad P$$



! 小心 ① 加 C_0 2边