EE2012 Midterm cheatsheet by wjb75

1 Chap 1

Probability as a

1. subjective quantitative belief
2. Relative freq demended by law of large numbers $P(A) = lim_{n\to\infty} f_h(n) = lim_{n\to\infty} \frac{N_h}{n}$

1.1 Axioms of Probability

1. $0 \leq P(A_i) \leq 1 \; \forall A_i$
2. $P(S) = 1$
3. for any series of mutually exclusive events E_i,
$$P(\cup_{i=0}^{\infty} E_i) = \sum_{i=0}^{\infty} P(E_i)$$

note that individual outcomes are by default mutually exclusive events

1.2 properties of probability

1. $P(E^c) = 1 - P(E)$
2. $P(F) = P(EF) + P(E^cF)$
3. $E \subseteq F \implies P(E) \leq P(F)$
4. $P(E \cup F) = P(E) + P(F) - P(EF)$
5.
$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{1\leq i_1<i_2\leq n} P(E_{i_1} \cap E_{i_2})$$
$$+ \sum_{1\leq i_1<i_2<i_3\leq n} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) - \cdots$$
$$+(-1)^{r+1} \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots$$
$$+(-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n).$$

1.3 Combinatorics

Dont forget to use graphical methods

Die 1 \ Die 2	1	2	3	4	5	6
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)

1. n total Permutation $n!$
2. k-permutation from n :$nPk = \frac{n!}{(n-k)!}$
3. combination of n from k $\binom{n}{k} = \frac{n!}{(n-k)!(k!)}$
4. partition/ multinomial coefficient / not all distinct permutations : $\binom{n}{m1, m2, m3, ...} = \frac{n!}{m1!m2!m3!...}$
5. pascal identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
6. binomial theorem
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
so $\sum_{k=0}^{n} \binom{n}{k} = 2^n, \sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$
7. the equation $x_1 + x_2 + ... + x_r = n, x_i > 0 \forall i$ has $\binom{n-1}{r-1}$ positive integer solutions
8. the equation $x_1 + x_2 + ... + x_r = n, x_i \geq 0 \forall i$ has $\binom{n+r-1}{r-1}$ nonnegative integer solutions
9. vandermonde's identity
$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$
10. hockey stick identity
$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$
11. derangement
$$!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

2 Chap 2 conditional probability and bayes rule

1. $P(A|B) = \frac{P(AB)}{P(B)}$ if $P(B) \neq 0$
2. $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ if $P(B) \neq 0$
3. law of total probability
$$P(A) = \sum_i P(A|E_i)P(E_i)$$
where E_i forms a partition of the sample space (mutually exclusive and union is S)
4. Two events A and B are independent iff $P(AB) = P(A)P(B)$ In conditionnal probability, if $P(B|A) = P(A)$, $P(A) \neq 0$

2.1 Markov chain

1. probability to absorption states to reach a specific absorption state s, the probabilities from each state $a_1, a_2, .., a_m$ are unique solution to $a_s = 1, a_i = 0 \forall absorption\ state\ i$ and $a_i = \sum_{j=1}^{M} a_j p_{ij}$
2. expected time to absorption
$$t_i = 1 + \sum_{j=1}^{M} t_j p_{ij}$$
where the 1 is added for each step increment

3 Chap 3 discrete random variable

1. the PMF of a random variabe X is
$$P_X(k) = \begin{cases} P(X = k), & \text{if } k \in S_x, \\ 0, & otherwise \end{cases}$$
where S_x is the support of X
2. properties of PMF
   (a) $0 \leq P_X(i) \leq 1$
   (b) $\sum_{i=0}^{\infty} P_X(i) = 1$
3. CDF $F_X(x) = P(X \leq x)$ for $x \in R$
4. expectation
$$E[X] = \sum_{\text{all x}} x P_X(x)$$
Expectations are linear, regardless of independencies
5. Moments
$E[X^n] = \sum_{\text{all x}} x^n P_X(x)$ is called the n^{th} moment of X
$$E(X^n) \geq [E(X)]^n$$
(Jensen's inequality)
6. Variance and standard deviation
$$Var[X] = E((X - E[X])^2) = E[X^2] - E[X]^2$$
$$\sigma_X = \sqrt{Var[X]}$$
$$Var[aX + b] = a^2 Var[X]$$
$Var[X+Y] = Var[X] + Var[Y]$ iff X and Y are independent

4 Distribution.

Distribution	PMF	CDF	Mean	Variance
Bernoulli	$P(X=1)=p,\ P(X=0)=1-p$	$F(x)=\begin{cases}0 & x<0\\ 1-p & 0\leq x<1\\ 1 & x\geq 1\end{cases}$	p	$p(1-p)$
Binomial	$\binom{n}{k}p^k(1-p)^{n-k}$	$F(k)=\sum_{i=0}^{k}\binom{n}{i}p^i(1-p)^{n-i}$	np	$np(1-p)$
Uniform	$\frac{1}{b-a+1}$	$F(x)=\frac{x-a+1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Geometric	$(1-p)^k p$	$F(k)=1-(1-p)^{k+1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$\frac{e^{-\lambda}\lambda^k}{k!}$	$F(k)=e^{-\lambda}\sum_{i=0}^{k}\frac{\lambda^i}{i!}$	λ	λ

note that for geometric distribution $P(X>k)=(1-p)^k$

5 problem solving techniques

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