1D Heat Conduction

FE formulation

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1 INTRODUCTION

2 DERIVATION

2.1 Governing partial differential equation and weak form

In the case of 1D heat conduction, the governing partial equation reads:

$$\rho c_p \frac{\partial T}{\partial z} - k_z \frac{\partial^2 T}{\partial z^2} - \dot{Q} = 0 \tag{1}$$

with T the temperature, ρc_p the volumetric heat capacity, k_z the thermal conductivity, and $\dot{Q}$ an internal source or sink, e.g. due to a phase transformation. The partial differential equation is subject to boundary conditions at the two ends of the domain, which can be of the Dirichlet type:

$$T(0,t) = T_{\text{left}}(t), \qquad T(L,t) = T_{\text{right}}(t),$$

or of the Neumann type:

$$-k_z \frac{\partial T}{\partial z}\bigg|_{z=0} = q_{\text{left}}(t), \qquad -k_z \frac{\partial T}{\partial z}\bigg|_{z=L} = q_{\text{right}}(t),$$

with T_{left} and T_{right} an imposed temperature and q_{left} and q_{right} an imposed heat flux density. The latter could be either directly imposed $\hat{q}$, for example as a result of laser heating, or be the result of convection:

$$q = h(T_\infty - T),$$

in which h and T_∞ are the heat transfer coefficient and the far field temperature, or the result of thermal radiation:

$$q = \epsilon\sigma(T_\infty^4 - T^4),$$

where ϵ is the surface emissivity and σ is Stefan's constant.

We can approximate the solution $T(z,t)$ of the partial differential equation using the weighted residual method:

$$\int_L w \left(\rho c_p \frac{\partial T}{\partial t} - k_z \frac{\partial^2 T}{\partial z^2} - \dot{Q} \right) dz = 0, \tag{2}$$

with w a weighting function.

2.2 Discretization in space

The domain in N is now split up in smaller elements, as is indicated in Figure 1 which shows an element (e) of length ℓ that is bounded by the nodes i and j. The weighted residual (Equation 2) can now be rewritten as:

$$\sum_{e=1}^{N} \int_{\ell} w \left(\rho c_p \frac{\partial T}{\partial t} - k_z \frac{\partial^2 T}{\partial z^2} - \dot{Q} \right) dz = 0. \tag{3}$$

We will now make sure that the weighted residual vanishes for each element. Further, we get rid of the second derivative with respect to z in the second term using integration by parts:

$$\int_{\ell} w \rho c_p \frac{\partial T}{\partial t} dz + \int_{\ell} \frac{dw}{dz} k_z \frac{\partial T}{\partial z} dz - w k_z \frac{\partial T}{\partial z} \bigg|_{z_i}^{z_j} - \int_{\ell} w \dot{Q} dz = 0,$$

with z_i and z_j the element end points. Realizing that the third term represents the flux ($q = -k\partial T/\partial z$), the equation can be rewritten as:

$$\int_{\ell} w \rho c_p \frac{\partial T}{\partial t} dz + \int_{\ell} \frac{dw}{dz} k_z \frac{\partial T}{\partial z} dz = \int_{\ell} w \dot{Q} dz - w q \bigg|_{z_i}^{z_j}. \tag{4}$$

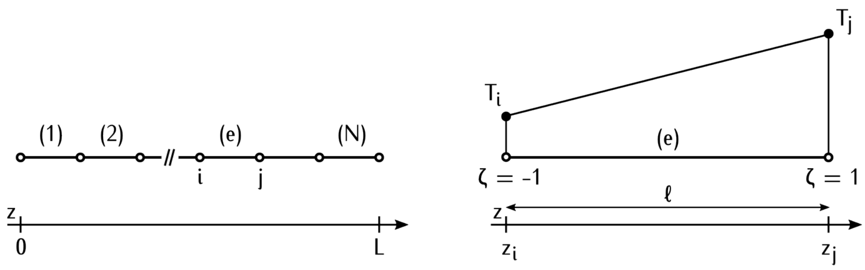

Figure 1: Left: Elements in 1D domain. Right: Definition of local coordinate system and linear interpolation of temperature.

2.2.1 *Linear shape functions*

The temperature inside an element is approximated as a linear interpolation between the bounding node temperatures as:

$$T(\zeta) = N_i(\zeta) T_i + N_j(\zeta) T_j, \tag{5}$$

with ζ a local coordinate, as is illustrated in Figure 1, defined as:

$$\zeta(z) = 2 \frac{z - (z_j - z_i)/2}{\ell},$$

while the two shape functions are:

$$N_i(\zeta) = \frac{1-\zeta}{2} \quad \text{and} \quad N_j(\zeta) = \frac{1+\zeta}{2}. \tag{6}$$

The spatial derivative of the temperature with respect to z can now be calculated as:

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial \zeta} \frac{\partial \zeta}{\partial z},$$

with:

$$\frac{\partial \zeta}{\partial z} = \frac{2}{\ell} \quad \text{and} \quad \frac{\partial T}{\partial \zeta} = \frac{T_2 - T_1}{2},$$

such that:

$$\frac{\partial T}{\partial z} = \frac{T_2 - T_1}{\ell}, \tag{7}$$

which also intuitively makes sense of course. Further, noting that the shape functions do not depend on time, we can rewrite the time derivative as:

$$\frac{\partial T}{\partial t} = N_i(\zeta)\frac{\partial T_i}{\partial t} + N_j(\zeta)\frac{\partial T_j}{\partial t}. \tag{8}$$

Following the Galerkin method, we choose our weighting function w to be our shape functions. The equation for the weighted residual for an element (Equation 4) can now be rewritten as:

$$\int_\ell N_k \rho c_p \frac{\partial T}{\partial t} dz + \int_\ell \frac{dN_k}{dz} k_z \frac{\partial T}{\partial z} dz = \int_\ell N_k \dot{Q} dz - N_k q \Big|_{z_i}^{z_j} \quad \text{for: } k = 1,2. \tag{9}$$

with N_i and N_j the two shape functions as defined in Equation 6.

Matrix-vector notation

Before evaluating the integrals, we first rewrite the expressions from the previous section into a matrix-vector form. Starting with Equation 5, which can be written as:

$$T(\zeta) = \mathbf{NT},$$

in which:

$$\mathbf{N} = [N_i(\zeta), N_j(\zeta)] \quad \text{and} \quad \mathbf{T} = \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

The spatial derivative of the temperature (Equation 7) yields:

$$\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}(\mathbf{NT}) = \mathbf{BT},$$

with:

$$\mathbf{B} = \frac{\partial \mathbf{N}}{\partial z} = \left[\frac{\partial N_i}{\partial z}, \frac{\partial N_j}{\partial z}\right] = \left[-\frac{1}{\ell}, \frac{1}{\ell}\right],$$

while the time-derivative of the temperature can be rewritten as:

$$\frac{\partial T}{\partial t} = \mathbf{N\dot{T}}.$$

Further, for convenience, we will write our weighting functions as:

$$w = \mathbf{N}^T = \begin{Bmatrix} N_i \\ N_j \end{Bmatrix}.$$

We can now evaluate the integrals, starting with the first term:

$$\int_\ell w \rho c_p \frac{\partial T}{\partial t} dz = \rho c_p \int_\ell \mathbf{N}^T \mathbf{N} dz\, \mathbf{\dot{T}}.$$

We can rewrite this integral in terms of ζ, by making use of the derivative of ζ with respect to z:

$$\frac{d\zeta}{dz} = \frac{2}{\ell} \quad \to \quad dz = \frac{\ell}{2} d\zeta,$$

such that:

$$\rho c_p \int_\ell \mathbf{N}^T \mathbf{N} dz\, \mathbf{\dot{T}} = \frac{\ell \rho c_p}{2} \int_{-1}^{1} \mathbf{N}^T \mathbf{N} d\zeta\, \mathbf{\dot{T}} = \mathbf{C\dot{T}}, \tag{10}$$

with:

$$\mathbf{C} = \frac{\ell \rho c_p}{2} \int_{-1}^{1} \mathbf{N}^T \mathbf{N} d\zeta = \frac{\ell \rho c_p}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In the same manner, the second term yields:

$$\int_{\ell} \frac{dN_k}{dz} k_z \frac{\partial T}{\partial z} dz = \frac{\ell k_z}{2} \int_{-1}^{1} \mathbf{B}^T \mathbf{B} d\zeta\, \mathbf{T} = \mathbf{KT}, \tag{11}$$

with:

$$\mathbf{K} = \frac{\ell k_z}{2} \int_{-1}^{1} \mathbf{B}^T \mathbf{B} d\zeta = \frac{k_z}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The first term on the right hand side yields:

$$\int_{\ell} \mathbf{N}^T \dot{Q} dz = \frac{\ell}{2} \int_{-1}^{1} \mathbf{N} d\zeta\, \dot{Q} = \frac{\dot{Q}\ell}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

with $\dot{Q}$ the heat source term for the element between nodes i and j. The second term with the heat flux q on the boundary is first expanded to include both a direct heat flux $\hat{q}$ and a flux due to convection:

$$q = \hat{q} + h(T_\infty - T),$$

which yields:

$$N_k q \Big|_{z_i}^{z_j} = N_k \hat{q} \Big|_{z_i}^{z_j} + N_k h(T_\infty - T) \Big|_{z_i}^{z_j}.$$

The term with the direct heat flux $\hat{q}$ is evaluated as:

$$N_k \hat{q} \Big|_{z_i}^{z_j} = \begin{Bmatrix} N_i(z_j)q_j - N_i(z_i)\hat{q}_i \\ N_j(z_j)q_j - N_j(z_i)\hat{q}_i \end{Bmatrix} = \begin{Bmatrix} -\hat{q}_i \\ \hat{q}_j \end{Bmatrix},$$

with $\hat{q}_k$ the heat flux on the k-th node. The convective term can be accounted for using a stiffness matrix for convection:

$$N_k h T \Big|_{z_i}^{z_j} = \mathbf{HT} \quad \textit{with:} \quad \mathbf{H} = h \begin{bmatrix} N_i N_i & N_i N_j \\ N_j N_i & N_j N_j \end{bmatrix}, \tag{12}$$

and an additional term in the force vector:

$$N_k h T_\infty \Big|_{z_i}^{z_j} = h \begin{Bmatrix} -T_{\infty,i} \\ T_{\infty,j} \end{Bmatrix}.$$

As an example for the stiffness matrix $\mathbf{H}$, in case of a convective boundary condition at the j-th node, where $N_i = 0$, this term would evaluate as:

$$\mathbf{H} = \begin{bmatrix} N_i N_i & N_i N_j \\ N_j N_i & N_j N_j \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which intuitively makes sense. The force vector is now combined as:

$$\mathbf{f} = \int_{\ell} N_k \dot{Q} dz - N_k q \Big|_{z_i}^{z_j} - N_k h T_\infty \Big|_{z_i}^{z_j} = \frac{\dot{Q}\ell}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} \hat{q}_i \\ -\hat{q}_j \end{Bmatrix} + h \begin{Bmatrix} T_{\infty,i} \\ -T_{\infty,j} \end{Bmatrix}. \tag{13}$$

The final element equation can now be assembled from by subsituting Equations 10, 11, 12 and 13 in Equation 9:

$$\mathbf{C}\dot{\mathbf{T}} + (\mathbf{K} + \mathbf{H})\mathbf{T} = \mathbf{f}.$$

With the local damping and stiffness matrices determined for each element, we can assemble the global matrices using the element locations in the global system.

Table 1: Domain properties.

Property	Value
Domain length, L	0.01 m
Conductivity, k	0.72 W/m K
Density, ρ	1560 kg/m^3
Specific heat, c_p	1450 J/kg K

2.2.2 *Quadratic shape functions*

bla

2.3 Temporal discretization

The final step is to integrate the equation with time. For this purpose, we will descretize the temporal variable will using the so-called Θ-method:

$$\mathbf{C}\frac{\mathbf{T}_{n+1}-\mathbf{T}_n}{\Delta t}+(1-\Theta)(\mathbf{K}+\mathbf{H})\mathbf{T}_n+\Theta(\mathbf{K}+\mathbf{H})\mathbf{T}_{n+1}=(1-\Theta)\mathbf{f}_n+\Theta\mathbf{f}_{n+1}, \tag{14}$$

where $\Theta \in [0, 1]$. Common values of Θ are:

$$\begin{aligned}\Theta = &\quad 0, &&\text{(Explit Euler)}\\ \Theta = &\quad 1/2, &&\text{(Crank Nicolson)}\\ \Theta = &\quad 1, &&\text{(Implicit Euler).}\end{aligned}$$

Equation 14 can be rearranged as:

$$\Big(\mathbf{C}+\Delta t\Theta(\mathbf{K}+\mathbf{H})\Big)\mathbf{T}_{n+1}=\Big(\mathbf{C}-\Delta t(1-\Theta)(\mathbf{K}+\mathbf{H})\Big)\mathbf{T}_n+\Delta t(1-\Theta)\mathbf{f}_n+\Delta t\Theta\mathbf{f}_{n+1}.$$

3 VALIDATION

3.1 Step temperature at boundary

Consider a domain of length L with a uniform initial temperature T_0. For $t > 0$ the temperature at one end is raised to a value of T_{end}, while the other end is kept at the initial temperature:

$$\begin{aligned}T(x,0) = &\quad T_0\\ T(0,t) = &\quad T_0\\ T(L,t) = &\quad T_{end}\end{aligned}$$

In case the initial temperature equals 0.0 °C , the analytical solution[1] yields:

$$T(x)=\frac{T_{end}x}{L}+\frac{2}{\pi}\sum_{N=1}^{\infty}\frac{T_{end}\cos N\pi}{N}\sin\left(\frac{N\pi x}{L}\right)\exp\left(-\alpha N^2\pi^2 t/L^2\right), \tag{15}$$

with $\alpha = k/\rho c_p$ the thermal diffusivity. The left graph in Figure 2 shows the temperature distribution at different times for a domain with properties as listed in Table 1. The right graphs shows the corresponding finite element solution for 10 linear elements of equal length. Good comparison is obtained between the numerical and analytical solution. The code for this comparison is available in the Python file `step_change.py`.

1 The Mathematics of Diffusion, Crank, 1975

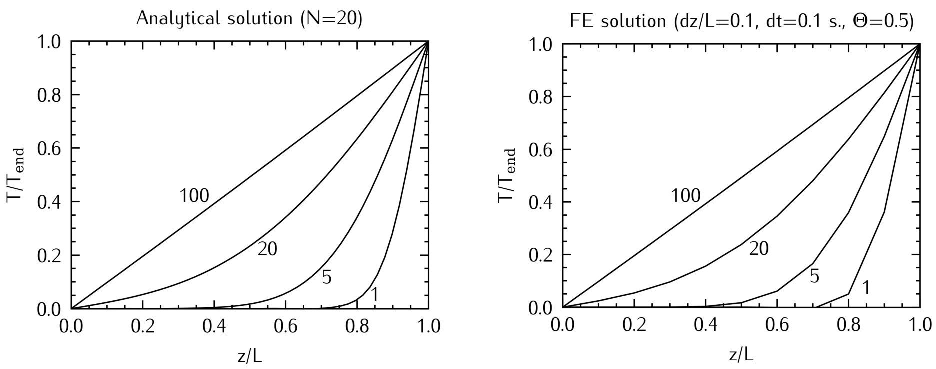

Figure 2: Comparison of the analytical and FE solution at different times. The numbers in the graphs indicate the time in seconds.

3.2 Constant heat flux at boundary of semi-infinite solid

4 USE CASES

4.1 Press forming

4.2 Laser assisted fiber placement