Computational Sensing and Imaging Notebook

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1 Computational Sensing & Imaging

Computational Imaging: Joint design of *optics capture* and *computational algorithms* to create novel systems.

1.1 Imaging Model

The imaging system can be described by the linear model:

$$Ax = b$$

where:

- x: scene
- A: system (e.g., CCDs and lens)
- b: image

To reconstruct the scene:

$$x = A^{-1}b$$

1.2 Wrapped Sample

Let the threshold be 64. For a raw sample of 70:

$$\text{wrapped sample} = 70 - 128 = -58$$

1.3 Perfect Recovery with Modulo Sampling

Our results allow for a perfect recovery of a bandlimited function whose amplitude exceeds the ADC threshold by an order of magnitude.

Theorem: Let $f(t)$ be a function with no frequencies higher than Ω (rad/s). Then a sufficient condition for recovery of $f(t)$ from its modulo samples:

$$y_k = \text{mod}_\lambda(f(t_k)), \quad t = kT, \quad k \in \mathbb{Z}$$

is:

$$T \leq \frac{1}{2\Omega}$$

In other words, it still satisfies the sampling theorem.

1.4 Quantization and Resolution

- N quantization levels $\Rightarrow$ resolution: N
- Modulo samples allow recovery with resolution: $2N$

1.5 Matrix Shapes in Reconstruction

- **Fat matrix:** More columns than rows
  A where knowns $\geq$ unknowns
- **Tall matrix:** More rows than columns
  A where unknowns $\geq$ knowns

2 1D Sampling, 2D Imaging, 3D Voxel Acquisition

2.1 Uniform Sampling

Given a continuous-time signal $f(t)$, we can analyze it via Fourier Transform to obtain $F(w)$. We sample the function every T_s seconds:

$$f[m] = f(mT_s)$$

The sampled signal in the frequency domain:

$$\hat{F}_{LS}(w) = \sum_{n\in\mathbb{Z}} F(w + n\Omega_s)$$

where the sample frequency is:

$$\Omega_s = \frac{2\pi}{T_s}$$

Definition (Bandlimitedness): A function is Ω-bandlimited if the largest frequency component in its Fourier transform is Ω, or

$$\hat{f}(w) = 0 \quad \text{for} \quad |w| > \Omega$$

To avoid aliasing:

$$\Omega_s \geq 2\Omega_0$$

2.2 Nyquist–Shannon Sampling Theorem

If a function $f(t)$ contains no frequency higher than Ω (rad/sec), then it can be reconstructed from its samples spaced no more than:

$$T_{\text{Nyq}} = \frac{\pi}{\Omega}$$

That is, it still satisfies the sampling theorem.

2.3 Examples of Sampling

- 1D sampling: Audio
- 2D sampling: Image
- 3D sampling: Video

2.4 Analog-to-Digital Converters (ADC)

The process can be described as:

$$f(t) \xrightarrow{\text{sample}} f[m] \xrightarrow{\text{Quantization}} \hat{f}[m]$$

Where:

- $f(t)$ is the analog signal
- $f[m]$ is the discrete signal
- $\hat{f}[m]$ is the digital signal

2.5 Sampling Methods

- **Point-wise sampling** (Theoretical)
- **Average sampling** (Practical)
- **One-bit sampling**
- **Modulo sampling** (New approach)

Average Sampling

$$f_\varphi(nT) = \frac{1}{\pi} \int_{(nT-\pi)}^{(nT+\pi)} 1 \cdot f(t)\, dt = (f * \varphi)(nT)$$

Where φ is the sampling kernel (e.g., $\varphi = 1$).

2.6 Denoising

2.7 One-bit Sampling

In one-bit sampling:

- Increasing sample rate and resolution increases power
- Bandlimitedness $\downarrow$ leads to lower dynamic range and sample rate

This method performs:

- Quantization in amplitude
- Quantization in time (sampling)

3 Optics Refreshers

Light: Waves & Particles

Light is regarded as both waves and particles.

When described as a wave:

$$f = \frac{c}{\lambda}$$

where:

- f: frequency
- λ: wavelength
- c: speed of light

Photon Energy

The energy transferred by a single photon is:

$$E = h\frac{c}{\lambda} = hf$$

where:

- h: Planck's constant

Refraction and Snell's Law

Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where:

- n: index of refraction
- θ: angle of incidence/refraction

Thin Lens Equation

The thin lens formula:

$$\frac{1}{f} = \frac{1}{s_1} + \frac{1}{s_2}$$

where:

- f: focal length
- s_1: distance from sensor to lens
- s_2: distance from lens to scene

Pipeline of Digital Image Formation

Camera Irradiance (Physics)

- Optics $\rightarrow$ Aperture $\rightarrow$ Shutter

Sensor (RAW)

- CMOS/CCD Sensor $\rightarrow$ Gain (ISO) $\rightarrow$ ADC

Digital Signal Processing (JPEG)

- Demosaic $\rightarrow$ (Sharpen)
- White Balance $\rightarrow$ Gamma Correction $\rightarrow$ Compression

4 Modeling: Forward vs Inverse Problems

The general forward model:

$$y(t) = \mathcal{T}[f(t)] + n(t)$$

where:

- $f(t)$: scene (true signal)
- $y(t)$: observation
- $\mathcal{T}$: transformation/system
- $n(t)$: noise

Forward process: $f(t) \rightarrow y(t)$ (easy) **Inverse process:** $y(t) \rightarrow f(t)$ (difficult)

Linear Time-Invariant (LTI) System

In an LTI system:

$$y(t) = h(t) * f(t)$$

Discrete representation:

$$y(t) = \sum_i h(i) f(t-i) = h(t) * f(t)$$

Matrix form:

$$\mathbf{y} = \mathbf{H} \cdot \mathbf{f}$$

where $\mathbf{H}$ is a Toeplitz matrix (convolution matrix):

$$\mathbf{H} = \begin{bmatrix} h_1 & 0 & 0 & \cdots \\ h_2 & h_1 & 0 & \cdots \\ h_3 & h_2 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Examples of Inverse Problems ("Recovery")

- Deconvolution
- Denoising
- Superresolution
- Phase Retrieval
- Inpainting

Deconvolution

Model:

$$g(t) = f(t) * h(t)$$

Apply Fourier Transform:

$$G(\omega) = F(\omega) \cdot H(\omega) \quad \Rightarrow \quad F(\omega) = \frac{G(\omega)}{H(\omega)} \quad \text{(one possible solution)}$$

Since $H(\omega)$ may have low values in high frequencies, the reconstructed image $F(\omega)$ may amplify noise significantly.

Wiener Filter

To address this, use Wiener filtering:

$$H'(\omega) = \frac{S_{ff}(\omega) H^*(\omega)}{S_{ff}(\omega)|H(\omega)|^2 + S_{nn}(\omega)}$$

where:

- $S_{ff}(\omega)$: power spectral density of the signal
- $S_{nn}(\omega)$: power spectral density of the noise
- $H^*(\omega)$: complex conjugate of $H(\omega)$

$H'(\omega)$ is the **recovery transfer function**.

5 Signal Processing, Linear Algebra, and Optimization

Linear Systems

A system $\mathcal{L}$ is linear if:

$$\mathcal{L}[au + bv] = a\mathcal{L}[u] + b\mathcal{L}[v]$$

Time Invariant Systems

A system is time-invariant if:

$$g(t) = \mathcal{L}[f(t)] \Rightarrow g(t - \tau) = \mathcal{L}[f(t - \tau)]$$

LTI System Representation

$$\begin{aligned}
q(t) &= \mathcal{L}[f](t) \\
&= \mathcal{L}\left[\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} f(\tau)\mathcal{L}[\delta(t-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \\
&= f(t) * h(t) = h(t) * f(t)
\end{aligned}$$

Causality

The output depends only on current and past inputs, not future inputs.

BIBO Stability

$$\text{Bounded input } |x(t)| < \infty \Rightarrow \text{Bounded output } |y(t)| < \infty$$

Eigenfunctions of LTI Systems

If $e(t)$ is an eigenfunction:

$$\mathcal{L}[e(t)] = \lambda e(t)$$

Let $f(t) = e^{j\omega t}$:

$$\begin{aligned}
g(t) &= \int h(\tau)f(t-\tau)d\tau \\
&= \int h(\tau)e^{j\omega(t-\tau)}d\tau \\
&= e^{j\omega t}\int h(\tau)e^{-j\omega\tau}d\tau \\
&= \lambda e^{j\omega t}
\end{aligned}$$

Convolution Theorem

$$h(t) = (f * g)(t) \Rightarrow \hat{h}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$$

Norms and Matrix Properties

Euclidean Norm Functions

$$\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{1/p}, \quad \|f(t)\|_p = \left(\int |f(t)|^p dt\right)^{1/p}$$

Special cases:

- $p = 1 \Rightarrow$ Manhattan norm
- $p = 2 \Rightarrow$ Euclidean norm

Matrix Rank The rank is the number of linearly independent rows/columns.
Matrix Inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Optimization

Overdetermined vs Underdetermined Systems

- Overdetermined: more equations than unknowns
- Underdetermined: fewer equations than unknowns

Least Squares Solution

$$J(x) = \|y - Hx\|^2 \Rightarrow x = (H^T H)^{-1} H^T y$$

Lagrange Multipliers

$$\mathcal{L}(x, \lambda, \mu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_j \mu_j h_j(x)$$

Constrained Least Squares Example

$$\min \|y - Hx\|^2 \text{ s.t. } Cx = b$$

Lagrangian:

$$\mathcal{L}(x, \mu) = \|y - Hx\|^2 + \mu^T (Cx - b)$$

Solution:

$$x = (H^T H)^{-1}(H^T y - C^T (CC^T)^{-1}(CH^T H^{-1} H^T y - b))$$

Sparse Recovery

ℓ_1-Norm Sparsity

$$\min \|x\|_1 \text{ s.t. } y = Ax$$

Assuming $A = I$, solve:

$$\min \|y - x\|^2 + \lambda \|x\|_1$$

Derivative:

$$\frac{\partial}{\partial x} = 2(x - y) + \lambda \text{sgn}(x) = 0 \Rightarrow x = \text{soft}_\lambda(y)$$

Soft-thresholding:

$$\text{soft}_\lambda(y) = \begin{cases} y - \lambda & y \geq \lambda \\ y + \lambda & y \leq -\lambda \\ 0 & \text{otherwise} \end{cases}$$

Minimum Norm Solution for Underdetermined Systems

$$\min \|x\|_2^2 \text{ s.t. } Ax = y \Rightarrow x = A^T(AA^T)^{-1}y$$

6 Algorithmic Toolkit

Assume a sparse vector $x = [x_0, x_1, \ldots, x_{N-1}]^T$ such that

$$\begin{cases} x_i \neq 0, & i \in I \\ x_i = 0, & \text{else} \end{cases}$$

where $|I| = k \ll N$.

The observed measurements can be described by the linear process $A^{M \times N}$,

$$y = Ax \quad \text{with } M \ll N$$

Keywords: sparse approximation, compressed sensing

Classical and Modern Methods

- Tikhonov and followers: $\|Ax - b\|_2^2 + \lambda \|x\|_2^2$
- **Recent:**
  - LP: $\min \|x\|_1$ s.t. $Ax = b$
  - LASSO: $\min \|Ax - b\|_2^2 + \lambda \|x\|_1$

Orthogonal Matching Pursuit (OMP)

Given measurements y, recover a k-sparse vector x by solving:

$$x^* = \arg\min_x \|y - Ax\|_2^2 \quad \text{s.t. } \|x\|_0 \leq k$$

Key idea: identify non-zero entries based on their *importance* (contribution) to observed measurements. Often viewed as a "Subspace Technique."

OMP Algorithm Steps

1. Find the basis with the largest *importance*:

   $$i_1 = \arg\max_i |\langle y, a_i \rangle|$$

   Let $J \leftarrow J \cup \{i_1\}, H \leftarrow [H\ a_{i_1}]$

2. Compute residual:

   $$x^{[1]} = (H^T H)^{-1} H^T y, \quad y^{[2]} = y - Hx^{[1]}$$

3. Repeat to find next most important a_i:

   $$i_t = \arg\max_i |\langle a_i, y^{[t-1]} \rangle|$$

   Update: $J \leftarrow J \cup \{i_t\}, H \leftarrow [H\ a_{i_t}]$

   $$x^{[t]} = (H^T H)^{-1} H^T y, \quad y^{[t]} = y - Hx^{[t]}$$

4. Stop when k entries are selected.

Relation to SIC

OMP is similar to Successive Interference Cancellation (SIC):

1. Find strongest channel
2. Cancel its contribution
3. Repeat until all are decoded

Basis Pursuit

$$\min \|x\|_1 \quad \text{s.t. } Ax = b$$

Introduce auxiliary variable $t = [|x_0|, |x_1|, \ldots, |x_{N-1}|]^T$.
This becomes a linear program:

$$\min \mathbf{1}^T t \quad \text{s.t. } Ax = b \text{ and } -t \leq x \leq t$$

High-resolution Frequency Estimation

Prony's Method: Assumes $z_k = e^{j\omega_k}$, signal as a sum of exponentials:

$$y_n = \sum_{k=0}^{K-1} \alpha_k z_k^n$$

Define *annihilating filter* h_n, such that:

$$\sum_{m=0}^{K} h_m y_{n+m} = 0$$

Yields a system $T(y)h = 0$ for estimating h.

MUSIC Algorithm: Formulate minimization:

$$\min \|T(y)h\|^2 \quad \text{s.t. } \|h\| = 1$$

Use eigenvector with smallest eigenvalue of $T(y)^T T(y)$. z_k are roots of resulting polynomial.

Sparse Deconvolution

Signal model:

$$y(t) = \sum_{k=0}^{K} \alpha_k \delta(t - \tau_k) * \varphi(t)$$

Taking Fourier transform:

$$Y(\omega) = \sum_k \alpha_k \Phi(\omega) e^{-j\omega\tau_k}$$

For better accuracy, require wide bandwidth $\Phi(\omega)$.

Refined Estimation: Minimize residual:

$$\min_{\alpha_k, \tau_k} \|Y_L(\omega_m) - C_L \Phi(\omega_m) e^{-j\omega_m \tau_k}\|^2$$

Align model with measurement in DFT domain.

One-bit Sampling

Delta-sigma modulation:

$$q[n] = \mathrm{sgn}(u[n]) \quad u[n] = u[n-1] + f[n] - q[n]$$

$q[n] \in \{-1, +1\}$

In frequency domain:

$$\hat{Q}(\omega) = \hat{F}(\omega) - \hat{U}(\omega) \cdot \hat{V}(\omega)$$

$\hat{V}(\omega) = [1, -1]$: high-pass filter. Reconstruction possible via low-pass filter.

7 Time-of-Flight Imaging

The signal model:

$$m(t) = \sum_{k=0}^{K} \Gamma_k \varphi(t - t_k)$$

This can be expressed as:

$$m(t) = \sum_{k=0}^{K} \sum_{m,k,n} \Phi_{mn} e^{jm\omega t} = VDY_m$$

which leads to a Vandermonde matrix form and diagonal matrix formulation. This can be processed using Prony's method.

Cross-Correlation

Given:

$$\begin{aligned} p(t) &= 1 + p_0 \cos \omega t \\ r(t) &= \Gamma_0 (1 + p_0 \cos \omega (t - t_0)) \end{aligned}$$

Cross-correlation:

$$R_{pr}(\tau) = \lim_{T \to \infty} \frac{\Gamma_0}{2T} \int_{-T}^{T} (1 + p_0 \cos \omega t)(1 + p_0 \cos \omega (t + \tau - t_0))\, dt$$

Evaluate to:

$$\Gamma_0 \left(1 + \frac{p_0^2}{2} \cos \omega (\tau + t_0) \right)$$

For $\tau = 0$, we define:

$$t_0 = \frac{2d}{c}$$

Lock-in Sensor (Four Bucket Method)

Measure input signal with four different phase shifts:

$$\begin{aligned} m_0 &= \frac{\Gamma_0}{2} (2 + p_0^2 \cos \omega t_0) \\ m_1 &= \frac{\Gamma_0}{2} (2 - p_0^2 \sin \omega t_0) \\ m_2 &= \frac{\Gamma_0}{2} (2 - p_0^2 \cos \omega t_0) \\ m_3 &= \frac{\Gamma_0}{2} (2 + p_0^2 \sin \omega t_0) \end{aligned}$$

Construct:

$$Z_\omega = (m_0 - m_2) + j(m_3 - m_1) = \Gamma_0 p_0^2 e^{j\omega t_0}$$

Solve:

$$\hat{t}_0 = \frac{|Z_\omega|}{\omega}, \quad \hat{d} = \frac{c|Z_\omega|}{2\omega}, \quad \hat{\Gamma}_0 = \frac{|Z_\omega|}{p_0^2}$$

System Model

$$r(t) = \chi(t) * h(t), \quad m(t) = p(t) \otimes r(t) = p(t) \otimes (\chi(t) * h(t))$$
$$m(t) = h(t) * p(t) * p(-t) \Rightarrow \hat{m}(\omega) = |\hat{p}(\omega)|^2 \hat{h}(\omega)$$