Digital Signal Processing and Digital Filters Notebook

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1 Introduction

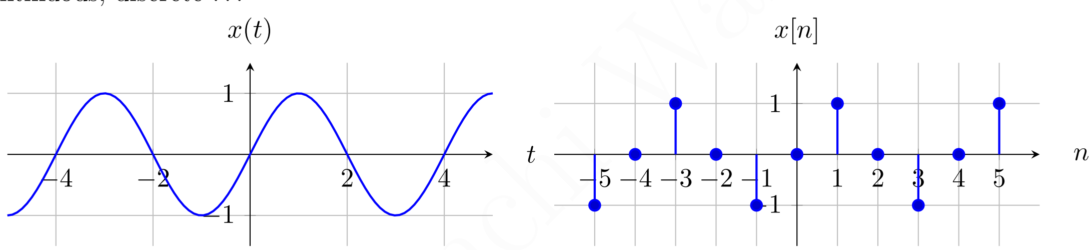
We live in an era where computing capacity is no longer a bottleneck.
We can utilize such computing capacity to do signal processing without thinking about its complexity.

1.1 Abstraction of a System

Input $\rightarrow$ $\boxed{\text{System}}$ $\rightarrow$ Output

1.2 Signals

Continuous, discrete ...



Sampling and reconstruction:

$$f(t) \xrightarrow{\text{sampling}} f(nT)$$

$$f(nT) \xrightarrow{\text{reconstruction}} f(t)$$

Reconstruction (Interpolation formula):

$$f(t) = \sum_n f(nT)\,\mathrm{sinc}\left(\frac{t}{T} - n\right)$$

1.3 Digital Transmission

$$x_{\mathrm{RX}}(t) = x_{\mathrm{TX}}(t) + \eta(t)$$

where $x_{\mathrm{TX}}(t)$ undergoes attenuation before transmission.

2 Lecture 02

2.1 Introduction

Special Sequences

- Unit step: $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

- Unit impulse: $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$
- Condition: $\delta[n] = 1$ only if condition is true, 0 otherwise.

Types of Signals

- Right-sided: $x[n] = 0$ for $n < n_{\min}$
- Left-sided: $x[n] = 0$ for $n > n_{\max}$
- Finite-length: $x[n] = 0$ for $n \notin [n_{\min}, n_{\max}]$
- Causal: $x[n] = 0$ for $n < 0$, Anticausal: $x[n] = 0$ for $n > 0$

Energy and Summability

- Finite energy: $\sum_n |x[n]|^2 < \infty$
- Absolutely summable: $\sum_n |x[n]| < \infty \quad \Rightarrow \quad$ finite energy

Proof

Assume $\sum_n |x[n]| = N_0$:

$$\sum_n |x[n]|^2 \leq \sum_n |x[n]| \max\{|x[n]|\} = N_0 \max\{|x[n]|\} < \infty$$

Note: Converse is not true!

$$\sum_n |x[n]|^2 \quad \not\Rightarrow \quad \sum_n |x[n]| < \infty$$

(i.e., $\sum_n |x[n]|$ can diverge even if $\sum_n |x[n]|^2$ converges.)

2.2 Z-transform

Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Convergence

A sequence of numbers $x[n]$ converges if there is some $x_0 \in \mathbb{C}$ such that:

$$\lim_{N \to \infty} x_0 - \sum_{n=1}^{N} x[n] = 0$$

Example: Power Series

If $x[n] = r^n$:

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

For $|r| < 1$:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

This defines the **region of convergence**.

Region of Convergence (ROC)

On the z-plane, the ROC is an annulus:

$$0 < r < |z| < R$$

(diagram: annulus in z-plane)

Why?

Let $z = re^{j\theta}$, then:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]z^{-n} &= \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-jn\theta} \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]|r^{-n}\end{aligned}$$

(Causal and Anticausal parts shown.)

Special Parts

- Causal part: $|r| > 1$
- Anticausal part: $|r| < 1$
- Convergence depends only on $r = |z|$ (amplitude of z)

2.3 Examples

- $u[n]$:

$$X(z) = \sum_{n=0}^{\infty} u[n]z^{-n} = \frac{1}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1$$

- $x[n] = \left(\frac{1}{2}\right)^n u[n]$:

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

2.4 Rational z-transforms

$$G(z) = g\prod_{i=1}^{M}(1 - p_i z^{-1}) \Big/ \prod_{k=1}^{K}(1 - r_k z^{-1})$$

Completely defined by the **poles**, **zeros**, and **gain**.
(Diagram of poles, zeros, and $G(z)$)

2.5 Another Example

$$2^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=0}^{\infty} (2)^n u[n] z^{-n} = \frac{1}{1 - (2z^{-1})}, \quad |z| > 2$$

$$X(z) = 2^n \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{-n} \quad \Rightarrow \quad X(z) = \frac{1}{1 - (2z^{-1})} \quad |z| > 2$$

$$-2^n u[-n-1] \quad \Rightarrow \quad X(z) = \frac{1}{1 - (2z^{-1})}, \quad |z| < 2$$

Note: Same z-transform, different ROCs.

3 Lecture 03

3.1 Fourier Transform

Series:

$$f(x) = \sum_k \alpha_k B_k(x)$$

Classical Examples

1. **Maclaurin Series**:

$$f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \quad \text{Basis function: } x^k$$

$$C_k = \frac{1}{k!} f^{(k)}(0)$$

2. **Shannon Series**:

$$f(x) = C_0 \text{sinc}(x) + C_1 \text{sinc}(x-1) + C_2 \text{sinc}(x-2) + \cdots \quad \alpha_k = f(x)$$

Basis function: $\text{sinc}(x-k)$

3. **Fourier Series**:

$$f(x) = C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t} + \cdots$$

$$C_k = \frac{1}{T} \int_0^T f(x) e^{-j2\pi \frac{kx}{T}}\, dx \quad \Longleftrightarrow \quad \text{DFT} \quad \omega_0 = \frac{2\pi}{T}$$

Basis function: $e^{j\frac{2\pi}{T}kx}$

Orthogonal Basis

When the basis function is orthogonal, that is:

$$\langle B_m, B_n \rangle = \int B_m B_n^*\, dx = \delta_{m-n}$$

then the function can be expressed by the expansion:

$$f(x) = \sum_k \langle f, B_k \rangle B_k$$

(Geometric diagram showing orthogonal projection.)

3.2 Fourier Series Basis

$$B_k(t) = e^{j\frac{2\pi}{T}kt}$$

Orthogonality

$$\langle B_m, B_n \rangle = \int_0^T e^{j\frac{2\pi}{T}mt} e^{-j\frac{2\pi}{T}nt}\, dt = \int_0^T e^{j\frac{2\pi}{T}(m-n)t}\, dt = \delta_{m-n}$$

$$f(x) = \sum_k \langle B_k, f \rangle B_k(x)$$

where

$$\alpha_k = \langle f, B_k \rangle = \int_a^b f(x) B_k^*(x)\, dx = \int_a^b f(x) e^{-j\frac{2\pi}{T}kx}\, dx$$

Inner Product

$$\langle f(x), g(x) \rangle = \int_a^b f(x) g(x)^* \, dx$$

3.3 Fourier Transform

Three types of Fourier Transforms:

- **Continuous Time Fourier Transform (CTFT, FT)**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt$$

- **Discrete Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- **Discrete Fourier Transform (DFT)**

$$x[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

3.4 DTFT Properties

- DTFT is periodic in ω:

$$X(e^{j\omega}) = X(e^{j(\omega + 2\pi m)}) \quad \text{for integer } m$$

- DTFT is the z-transform evaluated at $z = e^{j\omega}$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{with} \quad z = e^{j\omega}$$

3.5 Parseval's Theorem

Fourier Transforms preserve "energy":

- **CTFT**:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

- **DTFT**:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

- **DFT**:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Note: When you integrate over frequency, you need a scaling factor.

3.6 Generalization

$$\sum_{n=0}^{N-1} x[n] \overline{y[n]} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \overline{Y[k]}$$

3.7 Convolution

$$x[n] = g[n] * h[n]$$

In the frequency domain:

$$X(e^{j\omega}) = G(e^{j\omega})H(e^{j\omega})$$

$$X[k] = G[k]H[k]$$

3.8 Uncertainty Principle

A signal cannot be concentrated in both **time** and **frequency**.

Example with a Gaussian function:

$$g_\delta(t) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{t^2}{2\delta^2}}$$

$$\hat{g}_\delta(\omega) = e^{-\frac{\delta^2\omega^2}{2}}$$

Thus:

$$g_\delta(t)\hat{g}_\delta(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{t^2}{2\delta^2} + \frac{\delta^2\omega^2}{2}\right)}$$

Conclusion: A nonzero signal cannot be localized in both time and frequency domains.

3.9 Real Frequency / Normalized Frequency

- Real Frequency: Ω
- Normalized Frequency: w

Relation:

$$\omega = \Omega \cdot T_s$$

Proof

$$\Omega = \frac{2\pi}{T}, \quad w = \frac{2\pi}{N}$$

Given $N = T \cdot f_s$, then:

$$w = \frac{2\pi}{N} = \frac{2\pi}{T \cdot f_s} = \frac{\Omega}{f_s} = \Omega \cdot T_s$$

where T_s is the sampling period.

Time Scaling

To make $f_s = 1$, we divide all "real" time by T_s and all "real" frequency by f_s:

$$x(t) = x(nT_s) \quad n = \frac{t}{T_s} \quad w = \frac{\Omega}{f_s}$$

4 Lecture 04: Discrete Cosine Transform

4.1 Introduction

For information compression, the DFT has several problems:

- DFT is periodic.
- DFT has frequency leakage.

DCT can overcome these problems.

4.2 Definition

Formal DCT:

$$X_c[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi(n+\frac{1}{2})k}{N}\right)$$

Inverse DCT:

$$x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k] \cos\left(\frac{\pi(n+\frac{1}{2})k}{N}\right)$$

4.3 Frame-Based Coding

- Divide continuous signal into frames.
- Apply DCT in each frame.
- Apply IDCT in each frame to recover it.

Problem: There exist discontinuities at frame edges.

4.4 Solution: Lapped Transform (Modified DCT)

To solve the discontinuity problem:

$$\begin{aligned}
&x[0:2N-1] \to X_c[0:N-1] \to y_1[0:2N-1] \\
&x[N:3N-1] \to X_c[N:2N-1] \to y_2[N:3N-1] \\
&x[2N:4N-1] \to X_c[2N:3N-1] \to y_3[2N:4N-1]
\end{aligned}$$

Final signal:

$$y = y_1 + y_2 + y_3$$

The discontinuities disappear.

4.5 Properties

- Energy Conservation
- Energy Compaction (important for Information Compression)

4.6 Modified DCT (MDCT)

MDCT:

$$X[k] = \sum_{n=0}^{2N-1} x[n] \cos\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{N}\right)$$

Inverse MDCT (IMDCT):

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cos\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{N}\right)$$

4.7 Overlapping

Overlapping: 50%

- Frame 1: $0 \sim 63$
- Frame 2: $32 \sim 95$
- Frame 3: $64 \sim 127$

This is a **slided window with 50% overlap**.

5 Lecture 05: Linear-Time Invariant System

5.1 Definition

A system is linear if:

$$\mathcal{H}(\alpha u[n] + \beta v[n]) = \alpha \mathcal{H}(u[n]) + \beta \mathcal{H}(v[n])$$

Time invariant if:

$$y[n] = \mathcal{H}(x[n]) \quad \Rightarrow \quad y[n+\tau] = \mathcal{H}(x[n+\tau])$$

5.2 Proof of Convolution Representation

Starting from:

$$x[n] = \sum_{r=-\infty}^{\infty} x[r]\delta[n-r]$$

Applying $\mathcal{H}$:

$$y[n] = \mathcal{H}(x[n]) = \mathcal{H}\left(\sum_{r=-\infty}^{\infty} x[r]\delta[n-r]\right)$$

$$= \sum_{r=-\infty}^{\infty} x[r]\mathcal{H}(\delta[n-r])$$

Define:

$$h[n] = \mathcal{H}(\delta[n])$$

Thus:

$$y[n] = x[n] * h[n]$$

where $h[n]$ is the impulse response.

5.3 Properties

Commutative

$$u[n] * v[n] = v[n] * u[n]$$

Associative

$$x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$$

Associativity is not always true for non-linear systems.
Example:

$$h[n] = \delta[n] - \delta[n-1], \quad g[n] = u[n], \quad f[n] = c_0$$

$$(h[n] * f[n]) * g[n] = 0$$

$$(h[n] * g[n]) * f[n] = c_0$$

thus:

$$(f[n] * h[n]) * g[n] \neq (f[n] * g[n]) * h[n]$$

Identity

$$x[n] * \delta[n] = x[n]$$

5.4 BIBO Stability

The following are equivalent:

- An LTI system is BIBO stable.
- $h[n]$ is absolutely summable: $\sum_n |h[n]| < \infty$.
- $H(z)$'s region of absolute convergence includes $|z| = 1$.

Proof Outline

Assume $|x[n]| \leq B$:

$$|y[n]| = |h[n] * x[n]| \leq B \sum_k |h[k]|$$

Thus, $\sum_k |h[k]| < \infty$ implies BIBO stability.

5.5 Frequency Response

For LTI system:

$$Y(\omega) = H(\omega)X(\omega)$$

where $H(\omega)$ is the frequency response.

5.6 Causality

A causal system cannot "see into the future", meaning output at time n depends only on inputs up to time n.

Formal definition: if $u[n] = x[n]$ for $n \leq n_0$, then $\mathcal{H}(u[n]) = \mathcal{H}(x[n])$ for $n \leq n_0$.

Equivalently:

- An LTI system is causal.
- $h[n] = 0$ for $n < 0$.
- $H(z)$ converges for $z = \infty$.

5.7 FFT and Convolution

Using FFT to realize fast convolution:

$$x[n] \xrightarrow{\text{FFT}} X(\omega) \quad h[n] \xrightarrow{\text{FFT}} H(\omega)$$

$$Y(\omega) = X(\omega)H(\omega) \quad y[n] = \text{IFFT}(Y(\omega))$$

5.8 Proofs for Causality and Stability

First Equivalence Proof

Assume system is causal, then:

$$y[n] = \sum_k h[k]x[n-k]$$

For $k > 0$, $x[n-k]$ is in the past.
For $k < 0$, $x[n-k]$ is in the future $\Rightarrow h[k] = 0$.

Thus:

$$h[k] = 0 \quad \text{for} \quad k < 0$$

Second Equivalence Proof

If $h[k] = 0$ for $k < 0$:

$$H(z) = \sum_k h[k]z^{-k} = \sum_{k=0}^{\infty} h[k]z^{-k}$$

For $z \to \infty$, $z^{-1} \to 0$:

$$H(z) = h[0] + \sum_{k=1}^{\infty} h[k]0^k = h[0]$$

which is finite.

Thus:

$$H(z) \quad \text{converges at} \quad z = \infty$$

6 Lecture 06: Filters

6.1 Introduction

Most useful LTI systems can be described by difference equations:

$$y[n] = \sum_{r=0}^{M} b[r]x[n-r] - \sum_{r=1}^{N} a[r]y[n-r]$$

which can be rewritten as:

$$\sum_{r=0}^{N} a[r]y[n-r] = \sum_{r=0}^{M} b[r]x[n-r]$$

Or equivalently:

$$a[n] * y[n] = b[n] * x[n]$$

Thus, in z-domain:

$$Y(z) = \frac{B(z)}{A(z)}X(z)$$

Assume $a[0] = 1$:

- Always causal.
- Order of system is $\max(M, N)$.
- System is BIBO stable if all roots of $A(z)$ lie inside the unit circle (poles).

6.2 FIR Filter

If $A(z) = 1$:

$$Y(z) = B(z)X(z)$$

$$y[n] = \sum_{r=0}^{M} b[r]x[n-r]$$

Rule of thumb: Fastest possible transition $\Delta\omega \gtrsim \frac{2\pi}{M}$.

6.3 IIR Filter

General form:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b[0]\prod_{m=1}^{M}(1 - q_i z^{-1})}{\prod_{n=1}^{N}(1 - p_i z^{-1})}$$

- Zeros at roots of $B(z)$.
- Poles at roots of $A(z)$.

Magnitude response:

$$|H(e^{j\omega})| = \frac{|B(1)|\,|\prod_{m=1}^{M}(1 - q_i e^{-j\omega})|}{|A(1)|\,|\prod_{n=1}^{N}(1 - p_i e^{-j\omega})|}$$

Frequency response is determined by poles and zeros.

6.4 First-Order Low Pass Filter

$$y[n] = (1 - p)x[n] + py[n - 1]$$

In z-domain:

$$H(z) = \frac{1 - p}{1 - pz^{-1}}$$

6.5 All Pass Filter

Let $b[n] = \alpha^n$ for $M = N$:

$$H(e^{j\omega}) = \frac{\sum_{n=0}^{N} b[n]e^{-j\omega n}}{\sum_{n=0}^{N} \alpha^n e^{-j\omega n}}$$

Simplified:

$$|H(e^{j\omega})| = 1 \quad \text{(All Pass)}$$

Note: $|H(e^{j\omega})|^2 = 1$, but phase is not constant.
Zeros and poles are reflected inside the unit circle.

6.6 Minimum Phase and Linear Phase

Minimum Phase

A filter with all zeros inside the unit circle is minimum phase:

$$\angle H(e^{j\omega}) \approx 0$$

Lowest possible group delay.

Linear Phase

$$\angle H(e^{j\omega}) = \theta_0 - \omega n$$

$$\tau(\omega) = n \quad \text{(constant group delay)}$$

Linear phase filter must be symmetric or antisymmetric, i.e.

$$h[n] = h[M - n] \quad \text{or else} \quad h[n] = -h[M - n]$$

6.7 Group Delay

Group delay is defined as:

$$\tau(\omega) = -\frac{d}{d\omega}\angle H(e^{j\omega})$$

From:

$$\ln(H(e^{j\omega})) = \ln(|H(e^{j\omega})|) + j\angle H(e^{j\omega})$$

Thus:

$$-\frac{d}{d\omega}\angle H(e^{j\omega}) = \mathcal{I}\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right)$$

6.8 cutoff frequency

The cutoff frequency is the frequency that the filter has -3dB gain.

$$10\log_{10}\|H(e^{j\omega})\|^2 = -3\text{dB}$$

$$\|H(e^{j\omega})\|^2 = \frac{1}{2}$$

6.9 phase

$$H(e^{j\omega}) = \alpha e^{j\theta}$$

$$\angle H(e^{j\omega}) = \begin{cases} \theta & \alpha > 0 \\ \theta + \pi & \alpha \le 0 \end{cases} \qquad = \theta + (1 - \text{sign}(\alpha))\frac{\pi}{2}$$

6.10 roots of unity

The roots of unity are the complex solutions to the equation:

$$z^N = 1$$

for some integer $N \ge 1$. These roots are called the Nth roots of unity, and there are exactly N of them, given by:

$$z_k = e^{j\frac{2\pi k}{N}}, \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

6.11 Summary

- Stable if all poles $|p_i| < 1$.
- Reflecting a zero in the unit circle keeps $|H(e^{j\omega})|$ unchanged.
- Minimum phase if all zeros $|q_i| < 1$.

7 Lecture 07: Window Filter Design

7.1 Introduction

Given $H(e^{j\omega})$, the IDFT will give you $h[n]$:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Problem: $h[n]$ is infinite and non-causal.
Solution: Multiply $h[n]$ by a window.

7.2 Window Filter Design

- Multiply by a window to truncate $h[n]$.
- Delay by $M/2$ to make it causal.

7.3 Common Windows

- **Rectangular:** $w[n] = 1$
- **Hanning:** $w[n] = 0.5 + 0.5\cos\left(\frac{2\pi n}{M}\right)$
- **Hamming:** $w[n] = 0.54 + 0.46\cos\left(\frac{2\pi n}{M}\right)$

7.4 Dirichlet Kernel

When using a rectangular window:

$$W(e^{j\omega}) = \frac{\sin(0.5M\omega)}{\sin(0.5\omega)}$$

Thus:

$$H_{\text{win}}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) * W(e^{j\omega})$$

Convolve the frequency response with Dirichlet kernel.

7.5 Order Estimation

Order of filter must meet the requirement:

$$\epsilon \approx \sum \text{error}, \quad \text{transition bandwidth } \Delta\omega = \omega_2 - \omega_1$$

Order estimation formula:

$$M = \frac{-8 \cdot 20\log_{10}\epsilon}{22\Delta\omega}$$

Normalize the error.

7.6 Frequency Sampling Design

(Second design method)

Take $M + 1$ samples of $H(e^{j\omega})$, then take IDFT to get $h[n]$.

$$h[n] = \text{IDFT of sampled } H(e^{j\omega})$$

Advantage

Exact match at sample points.

Disadvantage

Poor intermediate approximation if spectrum is varying rapidly.

8 Lecture 08: Optimal FIR Filters

8.1 Even-Symmetric, Causal FIR Filter

We consider even-symmetric, causal FIR filter of length $M + 1$:

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

Due to symmetry:

$$H(e^{j\omega}) = h[0] + 2\sum_{n=1}^{M/2} h[n]\cos(\omega n)$$

To make it causal, delay by $M/2$:

$$H_{\text{del}}(e^{j\omega}) = H(e^{j\omega})e^{-j\omega M/2}$$

8.2 Chebyshev Polynomial

The Chebyshev polynomial $T_m(x)$ satisfies:

$$\cos(m\omega) = T_m(\cos\omega)$$

Examples:

$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$

Recursive relation:

$$T_0(x) = 1, \quad T_1(x) = x$$
$$T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x)$$

Proof

Starting from:

$$\begin{aligned}\cos((n+1)\omega) &= \cos(n\omega)\cos\omega - \sin(n\omega)\sin\omega \\ &= 2\cos\omega\cos(n\omega) - \cos((n-1)\omega)\end{aligned}$$

Thus confirming the recursive relation.
In terms of Chebyshev polynomials:

$$H(e^{j\omega}) = h[0] + 2\sum_{n=1}^{M/2} h[n]T_n(\cos\omega)$$

8.3 Alternative Theorem

Let $P(x)$ be a polynomial of degree r:

$$P(x) = \sum_{m=0}^{r} p_m x^m$$

Use $P(x)$ to approximate $D_p(x)$. Define the error:

$$E(x) = W(x)(P(x) - D_p(x))$$

$P(x)$ is optimal if $E(x)$ has at least $r + 2$ sign alternations:

$$E(x_1) = -E(x_2) = E(x_3) = \cdots = \pm\|E_{\max}\|$$

8.4 Optimal Filters

Choose $S(\omega)$ to control the error variation:

$$S(\omega) = \begin{cases} \delta^{-1}, & 0 < \omega < \omega_1 \\ -\delta^{-1}, & \omega_2 < \omega < \pi \end{cases}$$

Normalize all possible errors to 1.

8.5 Lagrange Interpolation

Lagrange interpolation formula:

$$P(x) = \sum_{i=1}^{n} y_i L_i(x)$$

where

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

This interpolates given points exactly.

9 Lecture 09: IIR Filter Transformation

9.1 Butterworth Filter

$$G^2(\Omega) = |H(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$$

Expanding:

$$G(\Omega) = 1 - \frac{1}{2}\Omega^2 + \frac{3}{8}\Omega^4 + \cdots$$

(Taylor series expansion at $\Omega = 0$.)

9.2 Chebyshev Filter

$$G^2(\Omega) = \frac{1}{1 + C^2 T_N^2(\Omega)}$$

where $T_N(x)$ is the Chebyshev polynomial, satisfying:

$$\cos(N\omega) = T_N(\cos(\omega))$$

9.3 Bilinear Mapping

Transformation between analog and digital domain:

Mapping:

$$z = \frac{\alpha + s}{\alpha - s}, \quad s = \alpha \frac{z - 1}{z + 1}$$

Important correspondence:

$$\text{I axis } (s) \quad \leftrightarrow \quad \text{Unit circle } (z)$$

Proof

Given $z = e^{j\omega}$:

$$s = \alpha \frac{z - 1}{z + 1} = \alpha \frac{e^{j\omega} - 1}{e^{j\omega} + 1}$$

Simplifying gives:

$$s = \alpha j \tan(\omega/2)$$

thus, the imaginary axis in s maps to the unit circle in z.

Frequency Mapping

Frequency mapping between analog and digital domains:

$$\omega = 2\tan^{-1}\left(\frac{\Omega}{\alpha}\right) \quad \text{or} \quad \Omega = \alpha \tan\left(\frac{\omega}{2}\right)$$

Choosing Ω_c ensures correct frequency mapping.

9.4 Spectral Transformation

Start with a low-pass prototype filter $H(z)$ with cutoff ω_0.

Substitutions for transformation:

- Low-pass $\to$ Low-pass
- Low-pass $\to$ High-pass
- Low-pass $\to$ Band-pass
- Low-pass $\to$ Band-stop

Substitution formula:

$$z^{-1} \to \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}}$$

where:

$$\lambda = \frac{\sin\left(\frac{\omega_1+\omega_2}{2}\right)}{\sin\left(\frac{\omega_2-\omega_1}{2}\right)}$$

9.5 Impulse Invariance

(Second design method)

Process:

$$H(s) \quad \xrightarrow{\mathcal{L}^{-1}} \quad h(t) \quad \xrightarrow{\text{sampling}} \quad h[n] \quad \xrightarrow{\mathcal{Z}} \quad H(z)$$

Impulse invariant method samples the analog impulse response.

9.6 Laplace Transform

$$X(s) = \int_0^{\infty} x(t)e^{-st}\,dt$$

$$e^{-at} \to \frac{1}{s+a}$$

10 Lecture 10: Optimal IIR

10.1 Minimizing Error

Want to minimize the error $E_s(\omega)$:

$$E_s(\omega) = W_s(\omega)\left(B(e^{j\omega}) - A(e^{j\omega})D(\omega)\right)$$

where:

$$H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})}$$

After equalization:

$$E_s(\omega) = W_s(\omega)\left(B(e^{j\omega}) - A(e^{j\omega})D(\omega)\right)$$

We minimize:

$$\int_{-\pi}^{\pi} |E_s(\omega)|^p\,d\omega$$

where $p = 2$ or ∞.

10.2 Linear Least Squares

In a linear system $Ax = b$, the solution minimizing the error is:

$$x = (A^T A)^{-1} A^T b$$

where $(A^T A)^{-1} A^T$ denotes the Moore-Penrose inverse.

10.3 Iterative Solution

1. Select K frequencies ω_k (uniformly spaced).
2. Initialize $W(\omega) = W_s(\omega)$.
3. Solve least squares system to minimize:

$$W(\omega)\left(B(e^{j\omega}) - A(e^{j\omega})D(\omega)\right) = 0$$

4. Force $A(e^{j\omega})$ to be stable by replacing unstable poles.
5. Update weight:

$$W(\omega) = \frac{W_s(\omega)}{|A(e^{j\omega})|^m}$$

6. Repeat from step 3 until convergence.

10.4 Hilbert Relations

For any sequence:

$$h_0[n] = h_e[n] + jh_o[n]$$

where:

$$h_e[n] = \frac{h[n] + h[-n]}{2} \quad h_o[n] = \frac{h[n] - h[-n]}{2}$$

In Fourier domain:

$$\mathrm{Re}\{H(e^{j\omega})\} = H_e(e^{j\omega}) \quad \mathrm{Im}\{H(e^{j\omega})\} = -H_o(e^{j\omega})$$

Define:

$$t[n] = u[n-1] - u[-n-1]$$

$$h_0[n] = h_e[n]t[n]$$

Thus:

$$H_0(e^{j\omega}) = H_e(e^{j\omega}) \circledast T(e^{j\omega})$$

where:

$$T(e^{j\omega}) = -j\cot\left(\frac{\omega}{2}\right)$$

Thus:

$$\mathrm{Im}\{H(e^{j\omega})\} = \mathrm{Re}\{H(e^{j\omega})\} \circledast (-\cot(\omega/2))$$

This shows the relation between real and imaginary parts of the spectrum.

10.5 Relation Between Amplitude and Phase

Given:

$$\bar{X}(e^{j\omega}) = \log(X(e^{j\omega})) = \log|X(e^{j\omega})| + j\angle X(e^{j\omega})$$

Thus:

$$\angle X(e^{j\omega}) = -\log|X(e^{j\omega})| \circledast \cot\left(\frac{\omega}{2}\right)$$

This shows the relation between amplitude and phase of a spectrum.

11 Lecture 11: Digital Filter Structure

11.1 Direct Form

The filter can be expressed as:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{r=0}^{M} b[r]z^{-r}}{1 + \sum_{r=1}^{N} a[r]z^{-r}}$$

Or the difference equation:

$$y[n] = \sum_{r=0}^{M} b[r]x[n-r] - \sum_{r=1}^{N} a[r]y[n-r]$$

Direct Form I

First compute $B(z)$, then divide by $A(z)$. (Refer to block diagram.)

Direct Form II

First compute $1/A(z)$, then multiply by $B(z)$. (More efficient in memory.)

11.2 Cascaded Biquad

Decompose $H(z)$ as a product of second-order sections:

$$H(z) = g \prod_{k} \frac{(1 + b_{k1}z^{-1} + b_{k2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})}$$

Each block is a biquad (quadratic).

11.3 Linear Phase

Linear phase filters are always FIR filters with symmetric coefficients:

$$h[n] = h[M-n]$$

Example when $M = 6$: (Refer to block diagram.)
Efficiency: 4 multipliers and 6 storage units.

11.4 Transposition

Equivalent transposed form but with different internal structure:

- Reverse direction of each interconnection.
- Reverse direction of each multiplier.
- Swap junctions with adders and vice-versa.

12 Lecture 12: Multirate Systems

12.1 Introduction

A multirate system includes different sampling rates.
Example:

$$f_s = 32, 44.1, 48, 96\,\text{kHz}$$

(Audio systems are multirate systems.)

12.2 Building Blocks

- **Downsampling** by K:

$$y[n] = x[nK]$$

  (Resampling operation.)

- **Upsampling** by K:

$$v[n] = \begin{cases} u\left[\frac{n}{K}\right], & n|K \\ 0, & \text{otherwise} \end{cases}$$

12.3 Resampling Cascades

Resampling can be interchanged if P and Q are coprime.

Note: $a|b$ means a divides b exactly (no remainder).

$$\frac{a}{b} = \text{No remainder} \quad \text{or} \quad \text{mod}(a, b) = 0$$

12.4 Noble Identities

$$H(z^M) \downarrow M = \downarrow M\, H(z)$$

$$H(z) \uparrow M = \uparrow M\, H(z^M)$$

Relationship:

$$h(n) = h_a(nQ)$$

12.5 Proofs

1. Downsampling Noble Identity

Starting with:

$$u[n] = x[nQ] \quad \Rightarrow \quad y[n] = u[n] * h[n]$$

Expand:

$$y[n] = \sum_r h[r]u[n-r] = \sum_r h[r]x[(n-r)Q]$$

Redefining:

$$= h_a[n] * x[n] = v[n] \quad \Rightarrow \quad y[n] = v[n]$$

Thus, proven.

2. Upsampling Noble Identity

Starting with:

$$u[n] = x[n] + \text{zeros inserted} \quad \Rightarrow \quad y[n] = u[n] * h[n]$$

Expand based on $n \mod Q$:

$$y[n] = \begin{cases} \sum_r h_a[rQ]x[n-rQ], & n \equiv 0 \mod Q \\ 0, & \text{otherwise} \end{cases}$$

Finally:

$$y[n] = h_a[n] * v[n] \quad \Rightarrow \quad y[n] = y_b[n]$$

Thus, proven.

12.6 Spectrum Effects in Multirate Systems

- **Upsampling by K:**
$$X_{\text{upsampled}}(e^{j\omega}) = X(z^K)$$
The spectrum compresses by a factor of K.

- **Downsampling by K:**
$$X_{\text{downsampled}}(e^{j\omega}) = \frac{1}{K}\sum_{n=0}^{K-1} X\left(e^{j\left(\frac{\omega+2\pi n}{K}\right)}\right)$$
Aliasing happens during downsampling because copies of the spectrum overlap.

13 Lecture 13: Subband Processing

13.1 Introduction

Subband processing divides $x[n]$ into different subbands.

$$\text{Using } H_m(z) \quad \text{(analysis filters)}, \quad G_m(z) \quad \text{(synthesis filters)}$$

13.2 Goals

- (a) Good frequency selectivity in $H_m(z)$.
- (b) Perfect reconstruction:
$$\hat{y}[n] = x[n-d]$$

13.3 2-Band Filterbank Example

Input $x(z)$ is split into two subbands:

$$\text{Analysis:}$$
$$N_m(z) = H_m(z)X(z)$$
$$U_m(z) = \frac{1}{K}\sum_{k=0}^{K-1} V_m\left(z^{1/K} e^{j\frac{2\pi k}{K}}\right)$$

For $K = 2$:
$$V_m(z) = V_m(z^2) + V_m(-z^2)$$

13.4 Output Reconstruction

The output $Y(z)$ is reconstructed as:

$$Y(z) = W_0(z) + W_1(z)$$

where:
$$W_0(z) = G_0(z)U_0(z) \quad W_1(z) = G_1(z)U_1(z)$$

Substituting:
$$Y(z) = \frac{1}{2}\left([H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) + [H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z)\right)$$

Perfect reconstruction condition:
$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-d}$$
$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

This guarantees no aliasing and exact recovery (with delay d).

13.5 Condition for Perfect Reconstruction

For perfect reconstruction without aliasing, the analysis filter bank H_0, H_1 and synthesis filters G_0, G_1 must satisfy

$$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} z^{-d} \\ 0 \end{bmatrix}. \tag{1}$$

13.6 Solving for the Synthesis Filters

Equation (1) implies

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \left[\begin{smallmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{smallmatrix} \right]^{-1} \begin{bmatrix} 2z^{-d} \\ 0 \end{bmatrix} \tag{2}$$

$$= \frac{2z^{-d}}{H_0(z)H_1(-z) - H_0(-z)H_1(z)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}. \tag{3}$$

13.7 Ensuring Finite-Impulse-Response (FIR) Filters

To keep all four filters FIR we require the denominator to reduce to a monomial

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = c\, z^{-k}, \tag{4}$$

with constants $c \neq 0$ and integer k. Under this condition

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{c}\, z^{k-d} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}. \tag{5}$$

13.8 Remarks

- Equation (5) is the *canonical FIR* solution; other perfect-reconstruction solutions exist when the FIR or delay constraints are relaxed.
- The constant c merely rescales the analysis filters by $c^{1/2}$ and the synthesis filters by $c^{-1/2}$.