Advanced Communication Theory Notebook

Junchi Wang

May 2025

1 Advanced Communication Theory

1.1 Wireless System Components

A wireless system can be partitioned into three main parts:

- TX (the "source" that sends/transmits some information)
- Wireless channel (the physical propagation paths)
- RX (the "sink" that receives the transmitted waves)

The objective in general is to increase the communication speed without sacrificing the quality of service.

1.2 Basic Block Diagram

$$\text{Transmitter (TX)} \longrightarrow \text{Wireless Channel} \longrightarrow \text{Receiver (RX)}$$

1.3 Classification

- SISO (Single Input, Single Output)
- SIMO (Single Input, Multiple Output)
- MISO (Multiple Input, Single Output)
- MIMO (Multiple Input, Multiple Output)

1.4 Array Antenna

An array antenna consists of multiple antenna elements typically arranged in a grid pattern to enhance signal processing capabilities.

2 Revisiting the EM Fields

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp(j2\pi f_c t - j\frac{2\pi}{\lambda}\vec{u}^T\vec{r})$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 \exp(j2\pi f_c t - j\frac{2\pi}{\lambda}\vec{u}^T\vec{r})$$

where $\vec{r} = [x,y,z]^T$, $\|\vec{r}\| = R$, and $\vec{u} \in \mathbb{R}^3$ is a unit vector:

$$\vec{u} = \frac{\vec{r}}{\|\vec{r}\|}$$

2.1 Electric Field at RX-Array Reference Point

At the RX-array's reference point (Cartesian origin $[0, 0, 0]^T$):

$$\vec{E}(\vec{r}, t)\big|_{\vec{r}=0} = \vec{E}(0, t) = \vec{E}_0 \exp(j2\pi f_c t)$$

2.2 K-th Antenna Displacement

Let the k-th antenna of an array of N elements be displaced from the origin by:

$$\vec{r}_k = [x_k, y_k, z_k]^T, \quad \Delta t_k = \frac{\Delta \vec{r}_k}{c} = \frac{\vec{u}^T \vec{r}_k}{c}$$

2.3 Electrical Field at the K-th Antenna

$$\vec{E}(\vec{r}_k, t) = \vec{E}(0, t - \Delta t_k) = \vec{E}_0 \exp(j2\pi f_c t) \exp\left(-j\frac{2\pi f_c}{c}\vec{u}^T \vec{r}_k\right)$$

Assuming $\vec{E}(0, t) = 1$, we define:

$$\text{k-th antenna response} = \exp\left(-j\frac{2\pi}{\lambda}\vec{u}^T \vec{r}_k\right)$$

$$\text{Array response vector:} \quad \begin{bmatrix} \exp(-j\frac{2\pi}{\lambda}\vec{u}^T \vec{r}_1) \\ \exp(-j\frac{2\pi}{\lambda}\vec{u}^T \vec{r}_2) \\ \vdots \\ \exp(-j\frac{2\pi}{\lambda}\vec{u}^T \vec{r}_N) \end{bmatrix} = \exp\left(-j\boldsymbol{k}(\theta, \varphi)^T \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_N \end{bmatrix}\right)$$

The column vector $\vec{S}(\theta, \varphi) \in \mathbb{C}^{N \times 1}$ is called the **array manifold vector**.

3 Notations to Remember

- N: number of RX array elements
- θ: elevation angle
- φ: azimuth angle
- $\vec{u}$: unit vector, $\|\vec{u}\| = 1$
- c: speed of light
- f_c: carrier frequency
- λ: wavelength
- $k = \frac{2\pi}{\lambda}$: wavenumber

3.1 Array Aperture

$$\text{Array aperture} = \max_{i,j} \|\vec{r}_i - \vec{r}_j\|$$

3.2 Manifold Vector Definition

$$\vec{S}(\theta, \varphi) = \exp\left(-j\,[\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N]^T\, \boldsymbol{k}(\theta, \varphi)\right)$$

$$\boldsymbol{k}(\theta, \varphi) = \frac{2\pi}{\lambda}\vec{u}(\theta, \varphi), \quad \|\vec{u}(\theta, \varphi)\| = 1$$

4 Differential Geometry

In physics, Albert Einstein (Nobel Prize 1921) used differential geometry to express his general theory of relativity.

Differential geometry is closely related to SISO, MISO, and MIMO systems.

5 Projection Matrix

Let $\mathbb{R}^N \to \mathbb{R}^M$, with $N > M$. Let:

$$B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_M], \quad \vec{a} = \sum \lambda_i \vec{b}_i$$

$$\langle \vec{a} - \vec{B}\vec{\lambda}, \vec{b}_i \rangle = 0, \quad \Rightarrow (\vec{a} - B\vec{\lambda})^T B = 0$$

Solving:

$$B^T B \vec{\lambda} = B^T \vec{a} \Rightarrow \vec{\lambda} = (B^T B)^{-1} B^T \vec{a}$$

Then the projection matrix is:

$$P = B(B^T B)^{-1} B^T$$

6 Diversity Theory

6.1 Diversity Combining Concept

Diversity is a technique that utilizes two or more copies of a signal with varying degrees of noise/interference to achieve higher message recovery performance than any single copy alone.

6.2 Diversity System Model

Let each branch be:

$$x_k = \beta_k s_d + n_k$$

$$s_{div} = w_1^H x_1 + w_2^H x_2 + \cdots + w_N^H x_N$$

6.3 Input and Output Expressions

Input:

$$x = \beta s_d + n$$

Output (recovery):

$$s_{div} = w^H x = w^H(\beta s_d + n) = w^H \beta s_d + w^H n$$

Desired and noise components:

$$\text{desired} = w^H \beta s_d, \quad \text{noise} = w^H n$$

6.4 Power of Desired Signal and Noise

$$\begin{aligned} P_{desired} &= \mathbb{E}[(w^H \beta s_d)(w^H \beta s_d)^H] \\ &= w^H \beta \mathbb{E}[s_d s_d^H] \beta^H w \\ &= P_d w^H R_{\beta\beta} w \end{aligned}$$

$$P_{noise} = \mathbb{E}[(w^H n)(w^H n)^H] = w^H R_{nn} w$$

$$\text{SNR} = \frac{P_{desired}}{P_{noise}} = \frac{P_d w^H R_{\beta\beta} w}{w^H R_{nn} w}$$

6.5 Covariance Matrices

$$R_{\beta\beta} = \text{cov}(\beta), \quad R_{nn} = \text{cov}(n)$$

6.6 Diversity Combining Strategies

- **Max Ratio Combining (MRC):**

$$w_{MRC} = \arg\max_{w} \text{SNR}_{out,div}$$

- **Selection Combining (SC):**

$$w_k = \begin{cases} 1, & \text{if } \text{SNR}_k > \text{SNR}_i \quad \forall i \\ 0, & \text{otherwise} \end{cases}$$

- **Equal Gain Combining (EGC):** All weights are equal:

$$w_1 = w_2 = \cdots = w_N$$

- **Scanning Combining (SCC):**

$$\text{if } \text{SNR}_k > \text{threshold} \Rightarrow w_k = 1, \quad w_j = 0 \quad \forall j \neq k$$

$$\text{else: } k \leftarrow k + 1 \quad \text{(repeat)}$$

6.7 Classification of Diversity

- **Multi-path diversity**
- **Time diversity**
- **Frequency diversity**
- **Space diversity:**
  - TX diversity
  - RX diversity
  - TX-RX diversity
- **Polarization diversity**

7 Multipath Diversity

7.1 Impulse Response

$$h(t) = \sum_i \beta_i \delta(t - \tau_i)$$

7.2 Delay Spread

The delay spread is a measure of multipath richness in a wireless channel.

Modern systems aim to:

- Resolve multipaths
- Estimate them
- Utilize them

7.3 Pulse Duration

$$\text{pulse duration} = \frac{1}{\text{Bandwidth}}, \quad \text{Number of resolvable paths} = \left\lfloor \frac{\text{Delay spread}}{\text{Pulse duration}} \right\rfloor + 1$$

8 Wireless System Design Considerations

8.1 Historical Perspective

Previously, multipath was seen as unwanted (self-interference). Now it's utilized for performance gain.

8.2 Refresher on Data Rates

$$C = B \log_2(1 + \text{SNR})$$

To increase data rates:

- Increase bandwidth
- Increase transmit power

Fading leads to unreliability; diversity helps mitigate fading.

8.3 Important Channel Parameters

- C: Channel capacity
- B: Channel bandwidth (Hz)
- B_{coh}: Coherence bandwidth (Hz)
- B_{dop}: Doppler spread (Hz)

8.4 Timing Parameters

- T_{cs}: Channel symbol duration
- T_{spread}: Delay spread
- T_{coh}: Coherence time

$$B = \frac{1}{T_{cs}}, \quad B_{coh} = \frac{1}{T_{spread}}, \quad B_{dop} = \frac{1}{T_{coh}}$$

8.5 Doppler Effect

$$f' = f \times \frac{v \pm v_o}{v \pm v_s}$$

Interpretations:

- $B_{dop} \uparrow \Rightarrow$ fast movement
- $B_{dop} \downarrow \Rightarrow$ slow movement
- $T_{spread} \uparrow \Rightarrow$ more multipath, frequency-selective fading
- $T_{spread} \downarrow \Rightarrow$ flat fading

9 Modeling of SIMO, MISO & MIMO Antenna Array

9.1 Notation

- $\vec{a}, \vec{A}$: Column vector
- A or (A): Matrix
- I_N: $N \times N$ identity matrix
- $\vec{1}_N$: Vector of N ones
- $\vec{0}_N$: Vector of N zeros
- $0_{N,M}$: $N \times M$ zero matrix
- $(\cdot)^T$: Transpose
- $(\cdot)^H$: Hermitian transpose
- $\odot, \oslash$: Hadamard product and division (element-wise)
- $\otimes$: Kronecker product
- $\exp(A), \exp(|A|)$: Element-wise exponential
- $\mathcal{L}\{A\}$: Linear space (subspace) spanned by columns of A
- $\mathcal{L}\{A\}^{\perp}$: Orthogonal complement of space $\mathcal{L}\{A\}$
- $P_{\mathcal{L}\{A\}}$: Projection operator onto subspace $\mathcal{L}\{A\}$
- $P_{\mathcal{L}\{A\}^{\perp}}$: Projection operator onto complement subspace

9.2 Vector Representation

Let

$$A = [\vec{b}_1 \; \vec{b}_2 \; \dots \; \vec{b}_M] \in \mathbb{R}^{N \times M}, \quad N > M$$

Any vector $\vec{x} \in \mathcal{L}\{A\}$ can be expressed as:

$$\vec{x} = \vec{b}_1 \lambda_1 + \vec{b}_2 \lambda_2 + \cdots + \vec{b}_M \lambda_M = A\vec{\lambda}$$

9.3 Projection Operator

The projection onto $\mathcal{L}\{A\}$ is:

$$P_{\mathcal{L}\{A\}} = A(A^H A)^{-1} A^H$$

$$\vec{a} = P_{\mathcal{L}\{A\}} \vec{b}$$

Complement subspace:

$$P_{\mathcal{L}\{A\}^\perp} = I_N - P_{\mathcal{L}\{A\}}$$

10 Fading Concepts

10.1 Types of Fading

- **Small-scale fading**: Due to multipath interference (constructive/destructive) over small displacements.
- **Large-scale fading**: Due to path loss over large distances.

10.2 Space-Selective Fading

- $T_{spread} \Rightarrow$ frequency-selective fading
- $B_{dop} \Rightarrow$ fast/slow fading
- $B_{spread} \Rightarrow$ space-selective fading

Space-selective: variation across spatial domain.

Spatial-coherence: transfer function remains constant within a coherence distance D_{coh}.

11 Scattering and Spectrum Concepts

11.1 Scattering Function

$$H(t, f, \vec{r}) \Rightarrow \text{Transfer function}$$

$$\phi_{HH}(\Delta t, \Delta f, \Delta \vec{r}) \Rightarrow \text{Autocorrelation} \Rightarrow \mathcal{F} \Rightarrow S(f, t, \vec{k}) \text{ (Scattering function)}$$

11.2 Angle Spectrum

$$S_H(\vec{k}) = \left(\frac{2\pi}{\lambda}\right)^3 \delta(\|\vec{k}\| - \frac{2\pi}{\lambda}) \cdot p(\theta, \varphi)$$

where $p(\theta, \varphi)$ is the angular power spectrum with:

- θ: Azimuth angle
- φ: Elevation angle

12 Local Area and Fading Behavior

12.1 Slow Fading

$$T_{cs} \leq T_{coh}, \quad \text{Local area: } d = T_{coh} \cdot c = \frac{c}{B}$$

$$B \uparrow \Rightarrow \text{Data rate} \uparrow, \quad d \downarrow \Rightarrow \text{Reliability} \downarrow$$

12.2 Homogeneous Plane Waves

In the local area, EM waves can be considered homogeneous plane waves:

$$H(\vec{r}) \approx \frac{1}{\|\vec{r}\|}$$

12.3 Dependencies and Transforms

- f: Doppler frequency (shift) from time
- τ: Delay
- $\vec{k}$: Wavevector

Transform domains:

$$\Delta t \leftrightarrow \Delta f, \quad \Delta\tau \leftrightarrow \Delta f, \quad \Delta\vec{r} \leftrightarrow \vec{k}$$

13 Wireless Channel Models

13.1 Wireless SIMO Channel

An array system is a collection of $N > 1$ sensors (transducing elements, receivers, antennas, etc.) distributed in the 3-D Cartesian space with a common reference point.

Consider an antenna array $\mathbf{R}_k$ with locations given by the matrix

$$\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N]^T \in \mathbb{R}^{3\times N},$$

where $\mathbf{r}_k$ is a 3×1 vector denoting the location of the k-th sensor, $k = 1, 2, \ldots, N$.

The region over which the sensors are distributed is called the **aperture** of the array. The array aperture is defined as:

$$\text{array aperture} = \max_{i,j} \|\mathbf{r}_i - \mathbf{r}_j\|.$$

The array manifold vector Σ is defined as:

$$\Sigma(\theta, \varphi) = \exp\left(-j\frac{2\pi}{\lambda}[\mathbf{r}_1, \ldots, \mathbf{r}_N]^T \mathbf{k}(\theta, \varphi)\right),$$

where

$$\mathbf{k}(\theta, \varphi) = \frac{2\pi}{\lambda}\mathbf{u}(\theta, \varphi), \quad \mathbf{u}(\theta, \varphi) = \begin{bmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{bmatrix}.$$

13.2 Plane Wave Model

If the EM wave propagates in the x-y plane ($\varphi = 0$):

$$\mathbf{u}(\theta, 0) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad \Sigma(\theta, 0) = \exp\left(-j\frac{2\pi}{\lambda}(x_k\cos\theta + y_k\sin\theta)\right).$$

If the array is aligned along the x-axis ($y_k = z_k = 0$), then:

$$\Sigma(\theta, 0) = \exp\left(-j\frac{2\pi}{\lambda}x_k\cos\theta\right).$$

13.3 Single Path Propagation

The delay τ for a single path:

$$\tau = \frac{d + \Delta \mathbf{r}_k^T \mathbf{u}(\theta, \varphi)}{c}.$$

Assume TX transmits $s(t)$, then the received signal at RX becomes:

$$r(t) = \left(\frac{k}{d}\right)^2 \exp(j\varphi) \exp(j2\pi f_c(t-\tau)) s(t-\tau).$$

13.4 Channel Impulse Response

The impulse response is:

$$h_k(t) = \beta \cdot \Sigma(k) \cdot \delta\left(t - \frac{d}{c}\right), \quad \text{or generally, } \mathbf{h}(t) = \beta \cdot \Sigma(k) \cdot \delta\left(t - \frac{d}{c}\right).$$

13.5 Multipath SIMO

Assume transmitted signal arrives at RX via L paths. For the l-th path:

$$\mathbf{h}(t) = \sum_{l=1}^{L} \beta_l \Sigma(\theta_l, \varphi_l) \delta(t - \tau_l).$$

The received vector signal is:

$$\mathbf{x}(t) = \mathbf{h}(t) * \mathbf{m}(t) + \mathbf{n}(t) = \sum_{l=1}^{L} \beta_l \Sigma(\theta_l, \varphi_l) \mathbf{m}(t - \tau_l) + \mathbf{n}(t).$$

13.6 Wireless MISO Channel

Using the reciprocity theorem, we model a MISO channel by defining TX parameters with the same radiation pattern in both directions. The impulse response for a single path is:

$$h(t) = \beta \Sigma^H(\theta, \varphi) \delta\left(t - \frac{d}{c}\right).$$

Multipath MISO response:

$$h(t) = \sum_{l=1}^{L} \beta_l \Sigma^H(\theta_l, \varphi_l) \delta(t - \tau_l).$$

13.7 Wireless MIMO Channel

The received signal is modeled using:

$$h(t) = \beta \Sigma(\theta, \varphi) \Sigma^H(\bar{\theta}, \bar{\varphi}) \delta\left(t - \frac{d}{c}\right).$$

Multipath case:

$$H(t) = \sum_{l=1}^{L} \beta_l \Sigma(\theta_l, \varphi_l) \Sigma^H(\bar{\theta}_l, \bar{\varphi}_l) \delta(t - \tau_l).$$

13.8 Transmit Diversity

- With feedback: Receiver estimates CSI and transmits it back.
- Without feedback: Transmitter uses space-time codes (STBC).

Weights (example):

$$\mathbf{w} = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}.$$

13.9 Space-Time Block Coding (STBC)

Assume m_1, m_2 are two transmitted symbols:

$$\text{STBC Block: } \begin{bmatrix} m_1 & m_2 \\ -m_2^* & m_1^* \end{bmatrix}.$$

Received signals:

$$x_1 = m_1 h_1 + m_2 h_2 + n_1, \quad x_2 = -m_2^* h_1 + m_1^* h_2 + n_2.$$

The receiver computes:

$$\mathbf{g} = \mathbf{H}^H \mathbf{x},$$

to recover transmitted symbols.