

# Digital Signal Processing and Digital Filters Notebook

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## 1 Introduction

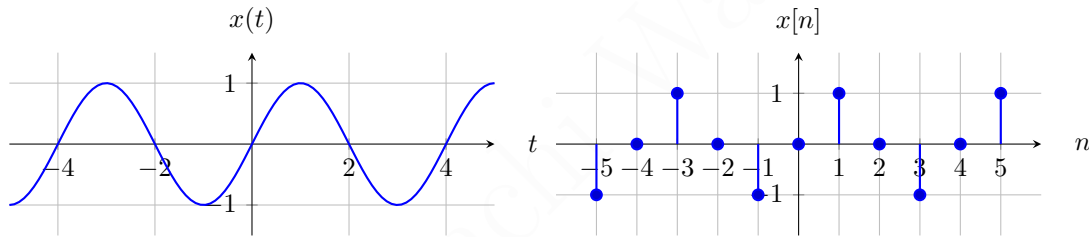
We live in an era where computing capacity is no longer a bottleneck.  
We can utilize such computing capacity to do signal processing without thinking about its complexity.

### 1.1 Abstraction of a System

Input  $\rightarrow$  System  $\rightarrow$  Output

### 1.2 Signals

Continuous, discrete ...



Sampling and reconstruction:

$$f(t) \xrightarrow{\text{sampling}} f(nT)$$
$$f(nT) \xrightarrow{\text{reconstruction}} f(t)$$

Reconstruction (Interpolation formula):

$$f(t) = \sum_n f(nT) \text{sinc}\left(\frac{t}{T} - n\right)$$

### 1.3 Digital Transmission

$$x_{\text{RX}}(t) = x_{\text{TX}}(t) + \eta(t)$$

where  $x_{\text{TX}}(t)$  undergoes attenuation before transmission.

## 2 Lecture 02

### 2.1 Introduction

Special Sequences

- Unit step:  $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

- Unit impulse:  $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$
- Condition:  $\delta[n] = 1$  only if condition is true, 0 otherwise.

### Types of Signals

- Right-sided:  $x[n] = 0$  for  $n < n_{\min}$
- Left-sided:  $x[n] = 0$  for  $n > n_{\max}$
- Finite-length:  $x[n] = 0$  for  $n \notin [n_{\min}, n_{\max}]$
- Causal:  $x[n] = 0$  for  $n < 0$ , Anticausal:  $x[n] = 0$  for  $n > 0$

### Energy and Summability

- Finite energy:  $\sum_n |x[n]|^2 < \infty$
- Absolutely summable:  $\sum_n |x[n]| < \infty \Rightarrow$  finite energy

### Proof

Assume  $\sum_n |x[n]| = N_0$ :

$$\sum_n |x[n]|^2 \leq \sum_n |x[n]| \max\{|x[n]|\} = N_0 \max\{|x[n]|\} < \infty$$

**Note:** Converse is not true!

$$\sum_n |x[n]|^2 \not\Rightarrow \sum_n |x[n]| < \infty$$

(i.e.,  $\sum_n |x[n]|$  can diverge even if  $\sum_n |x[n]|^2$  converges.)

## 2.2 Z-transform

### Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

### Convergence

A sequence of numbers  $x[n]$  converges if there is some  $x_0 \in \mathbb{C}$  such that:

$$\lim_{N \rightarrow \infty} x_0 - \sum_{n=1}^N x[n] = 0$$

### Example: Power Series

If  $x[n] = r^n$ :

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

For  $|r| < 1$ :

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

This defines the **region of convergence**.

## Region of Convergence (ROC)

On the  $z$ -plane, the ROC is an annulus:

$$0 < r < |z| < R$$

(diagram: annulus in  $z$ -plane)

## Why?

Let  $z = re^{j\theta}$ , then:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]z^{-n} &= \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-jn\theta} \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]|r^{-n}\end{aligned}$$

(Causal and Anticausal parts shown.)

## Special Parts

- Causal part:  $|r| > 1$
- Anticausal part:  $|r| < 1$
- Convergence depends only on  $r = |z|$  (amplitude of  $z$ )

## 2.3 Examples

- $u[n]$ :

$$X(z) = \sum_{n=0}^{\infty} u[n]z^{-n} = \frac{1}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1$$

- $x[n] = \left(\frac{1}{2}\right)^n u[n]$ :

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

## 2.4 Rational z-transforms

$$G(z) = g \prod_{i=1}^M (1 - p_i z^{-1}) / \prod_{k=1}^K (1 - r_k z^{-1})$$

Completely defined by the **poles**, **zeros**, and **gain**.

(Diagram of poles, zeros, and  $G(z)$ )

## 2.5 Another Example

$$\begin{aligned}2^n u[n] &\Rightarrow X(z) = \sum_{n=0}^{\infty} (2)^n u[n] z^{-n} = \frac{1}{1 - (2z^{-1})}, \quad |z| > 2 \\ X(z) = 2^n \left(\frac{1}{2}\right)^n &= \left(\frac{1}{2}\right)^{-n} \Rightarrow X(z) = \frac{1}{1 - (2z^{-1})} \quad |z| > 2 \\ -2^n u[-n-1] &\Rightarrow X(z) = \frac{1}{1 - (2z^{-1})}, \quad |z| < 2\end{aligned}$$

*Note: Same  $z$ -transform, different ROCs.*

## 3 Lecture 03

### 3.1 Fourier Transform

Series:

$$f(x) = \sum_k \alpha_k B_k(x)$$

#### Classical Examples

##### 1. Maclaurin Series:

$$f(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots \quad \text{Basis function: } x^k$$

$$C_k = \frac{1}{k!} f^{(k)}(0)$$

##### 2. Shannon Series:

$$f(x) = C_0 \text{sinc}(x) + C_1 \text{sinc}(x-1) + C_2 \text{sinc}(x-2) + \dots \quad \alpha_k = f(x)$$

Basis function:  $\text{sinc}(x-k)$

##### 3. Fourier Series:

$$f(x) = C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t} + \dots$$

$$C_k = \frac{1}{T} \int_0^T f(x) e^{-j2\pi \frac{kx}{T}} dx \quad \Longleftrightarrow \quad \text{DFT} \quad \omega_0 = \frac{2\pi}{T}$$

Basis function:  $e^{j\frac{2\pi}{T} kx}$

#### Orthogonal Basis

When the basis function is orthogonal, that is:

$$\langle B_m, B_n \rangle = \int B_m B_n^* dx = \delta_{m-n}$$

then the function can be expressed by the expansion:

$$f(x) = \sum_k \langle f, B_k \rangle B_k$$

(Geometric diagram showing orthogonal projection.)

### 3.2 Fourier Series Basis

$$B_k(t) = e^{j\frac{2\pi}{T} kt}$$

#### Orthogonality

$$\langle B_m, B_n \rangle = \int_0^T e^{j\frac{2\pi}{T} mt} e^{-j\frac{2\pi}{T} nt} dt = \int_0^T e^{j\frac{2\pi}{T} (m-n)t} dt = \delta_{m-n}$$

$$f(x) = \sum_k \langle B_k, f \rangle B_k(x)$$

where

$$\alpha_k = \langle f, B_k \rangle = \int_a^b f(x) B_k^*(x) dx = \int_a^b f(x) e^{-j\frac{2\pi}{T} kx} dx$$

## Inner Product

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)^* dx$$

### 3.3 Fourier Transform

Three types of Fourier Transforms:

- Continuous Time Fourier Transform (CTFT, FT)

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt$$

- Discrete Time Fourier Transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Discrete Fourier Transform (DFT)

$$x[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

### 3.4 DTFT Properties

- DTFT is periodic in  $\omega$ :

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi m)}) \quad \text{for integer } m$$

- DTFT is the  $z$ -transform evaluated at  $z = e^{j\omega}$ :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{with } z = e^{j\omega}$$

### 3.5 Parseval's Theorem

Fourier Transforms preserve "energy":

- CTFT:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- DTFT:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

- DFT:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

**Note:** When you integrate over frequency, you need a scaling factor.

### 3.6 Generalization

$$\sum_{n=0}^{N-1} x[n]\overline{y[n]} = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\overline{Y[k]}$$

### 3.7 Convolution

$$x[n] = g[n] * h[n]$$

In the frequency domain:

$$\begin{aligned} X(e^{j\omega}) &= G(e^{j\omega})H(e^{j\omega}) \\ X[k] &= G[k]H[k] \end{aligned}$$

### 3.8 Uncertainty Principle

A signal cannot be concentrated in both **time** and **frequency**.

Example with a Gaussian function:

$$\begin{aligned} g_\delta(t) &= \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{t^2}{2\delta^2}} \\ \hat{g}_\delta(\omega) &= e^{-\frac{\delta^2\omega^2}{2}} \end{aligned}$$

Thus:

$$g_\delta(t)\hat{g}_\delta(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{t^2}{2\delta^2} + \frac{\delta^2\omega^2}{2}\right)}$$

*Conclusion:* A nonzero signal cannot be localized in both time and frequency domains.

### 3.9 Real Frequency / Normalized Frequency

- Real Frequency:  $\Omega$
- Normalized Frequency:  $w$

Relation:

$$W = \Omega \cdot T_s$$

**Proof**

$$\Omega = \frac{2\pi}{T}, \quad w = \frac{2\pi}{N}$$

Given  $N = T \cdot f_s$ , then:

$$w = \frac{2\pi}{N} = \frac{2\pi}{T \cdot f_s} = \frac{\Omega}{f_s} = \Omega \cdot T_s$$

where  $T_s$  is the sampling period.

#### Time Scaling

To make  $f_s = 1$ , we divide all "real" time by  $T_s$  and all "real" frequency by  $f_s$ :

$$x(t) = x(nT_s) \quad n = \frac{t}{T_s} \quad w = \frac{\Omega}{f_s}$$

## 4 Lecture 04: Discrete Cosine Transform

### 4.1 Introduction

For information compression, the DFT has several problems:

- DFT is periodic.
- DFT has frequency leakage.

DCT can overcome these problems.

## 4.2 Definition

Formal DCT:

$$X_c[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi(n + \frac{1}{2})k}{N}\right)$$

Inverse DCT:

$$x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos\left(\frac{\pi(n + \frac{1}{2})k}{N}\right)$$

## 4.3 Frame-Based Coding

- Divide continuous signal into frames.
- Apply DCT in each frame.
- Apply IDCT in each frame to recover it.

**Problem:** There exist discontinuities at frame edges.

## 4.4 Solution: Lapped Transform (Modified DCT)

To solve the discontinuity problem:

$$\begin{aligned} x[0 : 2N - 1] &\rightarrow X_c[0 : N - 1] \rightarrow y_1[0 : 2N - 1] \\ x[N : 3N - 1] &\rightarrow X_c[N : 2N - 1] \rightarrow y_2[N : 3N - 1] \\ x[2N : 4N - 1] &\rightarrow X_c[2N : 3N - 1] \rightarrow y_3[2N : 4N - 1] \end{aligned}$$

Final signal:

$$y = y_1 + y_2 + y_3$$

The discontinuities disappear.

## 4.5 Properties

- Energy Conservation
- Energy Compaction (important for Information Compression)

## 4.6 Modified DCT (MDCT)

MDCT:

$$X[k] = \sum_{n=0}^{2N-1} x[n] \cos\left(\frac{\pi(n + \frac{1}{2})(k + \frac{1}{2})}{N}\right)$$

Inverse MDCT (IMDCT):

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos\left(\frac{\pi(n + \frac{1}{2})(k + \frac{1}{2})}{N}\right)$$

## 4.7 Overlapping

Overlapping: 50%

- Frame 1: 0 ~ 63
- Frame 2: 32 ~ 95
- Frame 3: 64 ~ 127

This is a **slided window with 50% overlap**.

## 5 Lecture 05: Linear-Time Invariant System

### 5.1 Definition

A system is linear if:

$$\mathcal{H}(\alpha u[n] + \beta v[n]) = \alpha \mathcal{H}(u[n]) + \beta \mathcal{H}(v[n])$$

Time invariant if:

$$y[n] = \mathcal{H}(x[n]) \Rightarrow y[n + \tau] = \mathcal{H}(x[n + \tau])$$

### 5.2 Proof of Convolution Representation

Starting from:

$$x[n] = \sum_{r=-\infty}^{\infty} x[r] \delta[n - r]$$

Applying  $\mathcal{H}$ :

$$\begin{aligned} y[n] &= \mathcal{H}(x[n]) = \mathcal{H}\left(\sum_{r=-\infty}^{\infty} x[r] \delta[n - r]\right) \\ &= \sum_{r=-\infty}^{\infty} x[r] \mathcal{H}(\delta[n - r]) \end{aligned}$$

Define:

$$h[n] = \mathcal{H}(\delta[n])$$

Thus:

$$y[n] = x[n] * h[n]$$

where  $h[n]$  is the impulse response.

### 5.3 Properties

**Commutative**

$$u[n] * v[n] = v[n] * u[n]$$

**Associative**

$$x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$$

**Associativity is not always true** for non-linear systems.

Example:

$$h[n] = \delta[n] - \delta[n - 1], \quad g[n] = u[n], \quad f[n] = c_0$$

$$(h[n] * f[n]) * g[n] = 0$$

$$(h[n] * g[n]) * f[n] = c_0$$

thus:

$$(f[n] * h[n]) * g[n] \neq (f[n] * g[n]) * h[n]$$

**Identity**

$$x[n] * \delta[n] = x[n]$$



## 5.4 BIBO Stability

The following are equivalent:

- An LTI system is BIBO stable.
- $h[n]$  is absolutely summable:  $\sum_n |h[n]| < \infty$ .
- $H(z)$ 's region of absolute convergence includes  $|z| = 1$ .

### Proof Outline

Assume  $|x[n]| \leq B$ :

$$|y[n]| = |h[n] * x[n]| \leq B \sum_k |h[k]|$$

Thus,  $\sum_k |h[k]| < \infty$  implies BIBO stability.

## 5.5 Frequency Response

For LTI system:

$$Y(\omega) = H(\omega)X(\omega)$$

where  $H(\omega)$  is the frequency response.

## 5.6 Causality

A causal system cannot "see into the future", meaning output at time  $n$  depends only on inputs up to time  $n$ .

Formal definition: if  $u[n] = x[n]$  for  $n \leq n_0$ , then  $\mathcal{H}(u[n]) = \mathcal{H}(x[n])$  for  $n \leq n_0$ .

Equivalently:

- An LTI system is causal.
- $h[n] = 0$  for  $n < 0$ .
- $H(z)$  converges for  $z = \infty$ .

## 5.7 FFT and Convolution

Using FFT to realize fast convolution:

$$x[n] \xrightarrow{\text{FFT}} X(\omega) \quad h[n] \xrightarrow{\text{FFT}} H(\omega)$$

$$Y(\omega) = X(\omega)H(\omega) \quad y[n] = \text{IFFT}(Y(\omega))$$

## 5.8 Proofs for Causality and Stability

### First Equivalence Proof

Assume system is causal, then:

$$y[n] = \sum_k h[k]x[n-k]$$

For  $k > 0$ ,  $x[n-k]$  is in the past.

For  $k < 0$ ,  $x[n-k]$  is in the future  $\Rightarrow h[k] = 0$ .

Thus:

$$h[k] = 0 \quad \text{for } k < 0$$

## Second Equivalence Proof

If  $h[k] = 0$  for  $k < 0$ :

$$H(z) = \sum_k h[k]z^{-k} = \sum_{k=0}^{\infty} h[k]z^{-k}$$

For  $z \rightarrow \infty$ ,  $z^{-1} \rightarrow 0$ :

$$H(z) = h[0] + \sum_{k=1}^{\infty} h[k]0^k = h[0]$$

which is finite.

Thus:

$$H(z) \text{ converges at } z = \infty$$

## 6 Lecture 06: Filters

### 6.1 Introduction

Most useful LTI systems can be described by difference equations:

$$y[n] = \sum_{r=0}^M b[r]x[n-r] - \sum_{r=1}^N a[r]y[n-r]$$

which can be rewritten as:

$$\sum_{r=0}^N a[r]y[n-r] = \sum_{r=0}^M b[r]x[n-r]$$

Or equivalently:

$$a[n] * y[n] = b[n] * x[n]$$

Thus, in  $z$ -domain:

$$Y(z) = \frac{B(z)}{A(z)}X(z)$$

**Assume**  $a[0] = 1$ :

- Always causal.
- Order of system is  $\max(M, N)$ .
- System is BIBO stable if all roots of  $A(z)$  lie inside the unit circle (poles).

### 6.2 FIR Filter

If  $A(z) = 1$ :

$$Y(z) = B(z)X(z)$$

$$y[n] = \sum_{r=0}^M b[r]x[n-r]$$

**Rule of thumb:** Fastest possible transition  $\Delta\omega \gtrsim \frac{2\pi}{M}$ .

### 6.3 IIR Filter

General form:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b[0] \prod_{m=1}^M (1 - q_m z^{-1})}{\prod_{n=1}^N (1 - p_n z^{-1})}$$

- Zeros at roots of  $B(z)$ .
- Poles at roots of  $A(z)$ .

Magnitude response:

$$|H(e^{j\omega})| = \frac{|B(1)| \prod_{m=1}^M (1 - q_m e^{-j\omega})|}{|A(1)| \prod_{n=1}^N (1 - p_n e^{-j\omega})|}$$

*Frequency response is determined by poles and zeros.*

### 6.4 First-Order Low Pass Filter

$$y[n] = (1 - p)x[n] + py[n - 1]$$

In  $z$ -domain:

$$H(z) = \frac{1 - p}{1 - pz^{-1}}$$

### 6.5 All Pass Filter

Let  $b[n] = \alpha^n$  for  $M = N$ :

$$H(e^{j\omega}) = \frac{\sum_{n=0}^N b[n] e^{-j\omega n}}{\sum_{n=0}^N \alpha^n e^{-j\omega n}}$$

Simplified:

$$|H(e^{j\omega})| = 1 \quad (\text{All Pass})$$

**Note:**  $|H(e^{j\omega})|^2 = 1$ , but phase is not constant.

*Zeros and poles are reflected inside the unit circle.*

### 6.6 Minimum Phase and Linear Phase

#### Minimum Phase

A filter with all zeros inside the unit circle is minimum phase:

$$\angle H(e^{j\omega}) \approx 0$$

Lowest possible group delay.

#### Linear Phase

$$\angle H(e^{j\omega}) = \theta_0 - \omega n$$

$$\tau(\omega) = 2 \quad (\text{constant group delay})$$

### 6.7 Group Delay

Group delay is defined as:

$$\tau(\omega) = -\frac{d}{d\omega} \angle H(e^{j\omega})$$

From:

$$\ln(H(e^{j\omega})) = \ln(|H(e^{j\omega})|) + j \angle H(e^{j\omega})$$

Thus:

$$-\frac{d}{d\omega} \angle H(e^{j\omega}) = \mathcal{I} \left( \frac{-1}{H(e^{j\omega})} \frac{dH(e^{j\omega})}{d\omega} \right)$$

## 6.8 Summary

- Stable if all poles  $|p_i| < 1$ .
- Reflecting a zero in the unit circle keeps  $|H(e^{j\omega})|$  unchanged.
- Minimum phase if all zeros  $|q_i| < 1$ .

## 7 Lecture 07: Window Filter Design

### 7.1 Introduction

Given  $H(e^{j\omega})$ , the IDFT will give you  $h[n]$ :

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

Problem:  $h[n]$  is infinite and non-causal.

Solution: Multiply  $h[n]$  by a window.

### 7.2 Window Filter Design

- Multiply by a window to truncate  $h[n]$ .
- Delay by  $M/2$  to make it causal.

### 7.3 Common Windows

- **Rectangular:**  $w[n] = 1$
- **Hanning:**  $w[n] = 0.5 + 0.5 \cos\left(\frac{2\pi n}{M}\right)$
- **Hamming:**  $w[n] = 0.54 + 0.46 \cos\left(\frac{2\pi n}{M}\right)$

### 7.4 Dirichlet Kernel

When using a rectangular window:

$$W(e^{j\omega}) = \frac{\sin(0.5M\omega)}{\sin(0.5\omega)}$$

Thus:

$$H_{\text{win}}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) * W(e^{j\omega})$$

*Convolve the frequency response with Dirichlet kernel.*

### 7.5 Order Estimation

Order of filter must meet the requirement:

$$\epsilon \approx \sum \text{error}, \quad \text{transition bandwidth } \Delta\omega = \omega_2 - \omega_1$$

Order estimation formula:

$$M = \frac{-8 \cdot 20 \log_{10} \epsilon}{22 \Delta\omega}$$

*Normalize the error.*

## 7.6 Frequency Sampling Design

(Second design method)

Take  $M + 1$  samples of  $H(e^{j\omega})$ , then take IDFT to get  $h[n]$ .

$$h[n] = \text{IDFT of sampled } H(e^{j\omega})$$

### Advantage

Exact match at sample points.

### Disadvantage

Poor intermediate approximation if spectrum is varying rapidly.

## 8 Lecture 08: Optimal FIR Filters

### 8.1 Even-Symmetric, Causal FIR Filter

We consider even-symmetric, causal FIR filter of length  $M + 1$ :

$$H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n}$$

Due to symmetry:

$$H(e^{j\omega}) = h[0] + 2 \sum_{n=1}^{M/2} h[n] \cos(\omega n)$$

To make it causal, delay by  $M/2$ :

$$H_{\text{del}}(e^{j\omega}) = H(e^{j\omega})e^{-j\omega M/2}$$

### 8.2 Chebyshev Polynomial

The Chebyshev polynomial  $T_m(x)$  satisfies:

$$\cos(m\omega) = T_m(\cos \omega)$$

Examples:

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

Recursive relation:

$$T_0(x) = 1, \quad T_1(x) = x$$

$$T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x)$$

### Proof

Starting from:

$$\begin{aligned} \cos((n+1)\omega) &= \cos(n\omega)\cos\omega - \sin(n\omega)\sin\omega \\ &= 2\cos\omega\cos(n\omega) - \cos((n-1)\omega) \end{aligned}$$

Thus confirming the recursive relation.

In terms of Chebyshev polynomials:

$$H(e^{j\omega}) = h[0] + 2 \sum_{n=1}^{M/2} h[n]T_n(\cos \omega)$$

### 8.3 Alternative Theorem

Let  $P(x)$  be a polynomial of degree  $r$ :

$$P(x) = \sum_{m=0}^r p_m x^m$$

Use  $P(x)$  to approximate  $D_p(x)$ . Define the error:

$$E(x) = W(x)(P(x) - D_p(x))$$

$P(x)$  is optimal if  $E(x)$  has at least  $r + 2$  sign alternations:

$$E(x_1) = -E(x_2) = E(x_3) = \cdots = \pm \|E_{\max}\|$$

### 8.4 Optimal Filters

Choose  $S(\omega)$  to control the error variation:

$$S(\omega) = \begin{cases} \delta^{-1}, & 0 < \omega < \omega_1 \\ -\delta^{-1}, & \omega_2 < \omega < \pi \end{cases}$$

Normalize all possible errors to 1.

### 8.5 Lagrange Interpolation

Lagrange interpolation formula:

$$P(x) = \sum_{i=1}^n y_i L_i(x)$$

where

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

This interpolates given points exactly.

## 9 Lecture 09: IIR Filter Transformation

### 9.1 Butterworth Filter

$$G^2(\Omega) = |H(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$$

Expanding:

$$G(\Omega) = 1 - \frac{1}{2}\Omega^2 + \frac{3}{8}\Omega^4 + \cdots$$

(Taylor series expansion at  $\Omega = 0$ .)

### 9.2 Chebyshev Filter

$$G^2(\Omega) = \frac{1}{1 + C^2 T_N^2(\Omega)}$$

where  $T_N(x)$  is the Chebyshev polynomial, satisfying:

$$\cos(N\omega) = T_N(\cos(\omega))$$

### 9.3 Bilinear Mapping

Transformation between analog and digital domain:

Mapping:

$$z = \frac{2+s}{2-s}, \quad s = 2\frac{z-1}{z+1}$$

Important correspondence:

$$\text{I axis } (s) \leftrightarrow \text{Unit circle } (z)$$

#### Proof

Given  $z = e^{j\omega}$ :

$$s = 2\frac{z-1}{z+1} = 2\frac{e^{j\omega}-1}{e^{j\omega}+1}$$

Simplifying gives:

$$s = 2j \tan(\omega/2)$$

thus, the imaginary axis in  $s$  maps to the unit circle in  $z$ .

#### Frequency Mapping

Frequency mapping between analog and digital domains:

$$\omega = \tan^{-1}\left(\frac{\Omega}{2}\right) \quad \text{or} \quad \Omega = 2 \tan\left(\frac{\omega}{2}\right)$$

Choosing  $\Omega_c$  ensures correct frequency mapping.

### 9.4 Spectral Transformation

Start with a low-pass prototype filter  $H(z)$  with cutoff  $\omega_0$ .

Substitutions for transformation:

- Low-pass  $\rightarrow$  Low-pass
- Low-pass  $\rightarrow$  High-pass
- Low-pass  $\rightarrow$  Band-pass
- Low-pass  $\rightarrow$  Band-stop

Substitution formula:

$$z^{-1} \rightarrow \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}}$$

where:

$$\lambda = \frac{\sin\left(\frac{\omega_1 + \omega_2}{2}\right)}{\sin\left(\frac{\omega_2 - \omega_1}{2}\right)}$$

### 9.5 Impulse Invariance

(Second design method)

Process:

$$H(s) \xrightarrow{\mathcal{L}^{-1}} h(t) \xrightarrow{\text{sampling}} h[n] \xrightarrow{\mathcal{Z}} H(z)$$

Impulse invariant method samples the analog impulse response.

## 10 Lecture 10: Optimal IIR

### 10.1 Minimizing Error

Want to minimize the error  $E_s(\omega)$ :

$$E_s(\omega) = W_s(\omega) (B(e^{j\omega}) - A(e^{j\omega})D(\omega))$$

where:

$$H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})}$$

After equalization:

$$E_s(\omega) = W_s(\omega) (B(e^{j\omega}) - A(e^{j\omega})D(\omega))$$

We minimize:

$$\int_{-\pi}^{\pi} |E_s(\omega)|^p d\omega$$

where  $p = 2$  or  $\infty$ .

### 10.2 Linear Least Squares

In a linear system  $Ax = b$ , the solution minimizing the error is:

$$x = (A^T A)^{-1} A^T b$$

where  $A^+$  denotes the Moore-Penrose inverse.

### 10.3 Iterative Solution

1. Select  $K$  frequencies  $\omega_k$  (uniformly spaced).
2. Initialize  $W(\omega) = W_s(\omega)$ .
3. Solve least squares system to minimize:

$$W(\omega) (B(e^{j\omega}) - A(e^{j\omega})D(\omega)) = 0$$

4. Force  $A(e^{j\omega})$  to be stable by replacing unstable poles.
5. Update weight:

$$W(\omega) = \frac{W_s(\omega)}{|A(e^{j\omega})|^m}$$

6. Repeat from step 3 until convergence.

### 10.4 Hilbert Relations

For any sequence:

$$h_0[n] = h_e[n] + jh_o[n]$$

where:

$$h_e[n] = \frac{h[n] + h[-n]}{2} \quad h_o[n] = \frac{h[n] - h[-n]}{2}$$

In Fourier domain:

$$\text{Re}\{H(e^{j\omega})\} = H_e(e^{j\omega}) \quad \text{Im}\{H(e^{j\omega})\} = -H_o(e^{j\omega})$$

Define:

$$t[n] = u[n-1] - u[n+1]$$



Circular convolution:

$$h_0[n] = h_e[n] * t[n]$$

Thus:

$$H_0(e^{j\omega}) = H_e(e^{j\omega}) \cdot T(e^{j\omega})$$

where:

$$T(e^{j\omega}) = -j \cot\left(\frac{\omega}{2}\right)$$

Thus:

$$\text{Im}\{H(e^{j\omega})\} = \text{Re}\{H(e^{j\omega})\} \cdot (-\cot(\omega/2))$$

This shows the relation between real and imaginary parts of the spectrum.

## 10.5 Relation Between Amplitude and Phase

Given:

$$\bar{X}(e^{j\omega}) = \log(X(e^{j\omega})) = \log|X(e^{j\omega})| + j\angle X(e^{j\omega})$$

Thus:

$$\angle X(e^{j\omega}) = -\log|X(e^{j\omega})| * \cot\left(\frac{\omega}{2}\right)$$

This shows the relation between amplitude and phase of a spectrum.

## 11 Lecture 11: Digital Filter Structure

### 11.1 Direct Form

The filter can be expressed as:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{r=0}^M b[r]z^{-r}}{1 + \sum_{r=1}^N a[r]z^{-r}}$$

Or the difference equation:

$$y[n] = \sum_{r=0}^M b[r]x[n-r] - \sum_{r=1}^N a[r]y[n-r]$$

#### Direct Form I

First compute  $B(z)$ , then divide by  $A(z)$ . (Refer to block diagram.)

#### Direct Form II

First compute  $1/A(z)$ , then multiply by  $B(z)$ . (More efficient in memory.)

### 11.2 Cascaded Biquad

Decompose  $H(z)$  as a product of second-order sections:

$$H(z) = g \prod_k \frac{(1 + b_{k1}z^{-1} + b_{k2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})}$$

Each block is a biquad (quadratic).

### 11.3 Linear Phase

Linear phase filters are always FIR filters with symmetric coefficients:

$$h[n] = h[M - n]$$

Example when  $M = 6$ : (Refer to block diagram.)

**Efficiency:** 4 multipliers and 6 storage units.

### 11.4 Transposition

Equivalent transposed form but with different internal structure:

- Reverse direction of each interconnection.
- Reverse direction of each multiplier.
- Swap junctions with adders and vice-versa.

## 12 Lecture 12: Multirate Systems

### 12.1 Introduction

A multirate system includes different sampling rates.

Example:

$$f_s = 32, 44.1, 48, 96 \text{ kHz}$$

(Audio systems are multirate systems.)

### 12.2 Building Blocks

- **Downsampling** by  $K$ :

$$y[n] = x[nK]$$

(Resampling operation.)

- **Upsampling** by  $K$ :

$$v[n] = \begin{cases} u\left[\frac{n}{K}\right], & n = 0 \pmod K \\ 0, & \text{otherwise} \end{cases}$$

### 12.3 Resampling Cascades

Resampling can be interchanged if  $P$  and  $Q$  are coprime.

*Note:*  $a|b$  means  $a$  divides  $b$  exactly (no remainder).

$$\frac{a}{b} = \text{No remainder} \quad \text{or} \quad \text{mod}(a, b) = 0$$

### 12.4 Noble Identities

$$\text{Downsampling: } H(z) \Rightarrow H(z^Q)$$

$$\text{Upsampling: } H(z) \Rightarrow H(z^{1/P})$$

Relationship:

$$h(n) = h_a(nQ)$$

## 12.5 Proofs

### 1. Downsampling Noble Identity

Starting with:

$$u[n] = x[nQ] \Rightarrow y[n] = u[n] * h[n]$$

Expand:

$$y[n] = \sum_r h[r]u[n-r] = \sum_r h[r]x[(n-r)Q]$$

Redefining:

$$= h_a[n] * x[n] = v[n] \Rightarrow y[n] = v[n]$$

Thus, proven.

### 2. Upsampling Noble Identity

Starting with:

$$u[n] = x[n] + \text{zeros inserted} \Rightarrow y[n] = u[n] * h[n]$$

Expand based on  $n \bmod Q$ :

$$y[n] = \begin{cases} \sum_r h_a[rQ]x[n-rQ], & n \equiv 0 \bmod Q \\ 0, & \text{otherwise} \end{cases}$$

Finally:

$$y[n] = h_a[n] * v[n] \Rightarrow y[n] = y_b[n]$$

Thus, proven.

## 12.6 Spectrum Effects in Multirate Systems

- Upsampling by  $K$ :

$$X_{\text{upsampled}}(e^{j\omega}) = X(z^K)$$

The spectrum compresses by a factor of  $K$ .

- Downsampling by  $K$ :

$$X_{\text{downsampled}}(e^{j\omega}) = \frac{1}{K} \sum_{n=0}^{K-1} X\left(e^{j\left(\frac{\omega+2\pi n}{K}\right)}\right)$$

Aliasing happens during downsampling because copies of the spectrum overlap.

## 13 Lecture 13: Subband Processing

### 13.1 Introduction

Subband processing divides  $x[n]$  into different subbands.

Using  $H_m(z)$  (analysis filters),  $G_m(z)$  (synthesis filters)

### 13.2 Goals

- (a) Good frequency selectivity in  $H_m(z)$ .
- (b) Perfect reconstruction:

$$\hat{y}[n] = x[n-d]$$

### 13.3 2-Band Filterbank Example

Input  $x(z)$  is split into two subbands:

Analysis:

$$N_m(z) = H_m(z)X(z)$$

$$U_m(z) = \frac{1}{K} \sum_{k=0}^{K-1} V_m \left( z^{1/K} e^{j \frac{2\pi k}{K}} \right)$$

For  $K = 2$ :

$$V_m(z) = V_m(z^2) + V_m(-z^2)$$

### 13.4 Output Reconstruction

The output  $Y(z)$  is reconstructed as:

$$Y(z) = W_0(z) + W_1(z)$$

where:

$$W_0(z) = G_0(z)U_0(z) \quad W_1(z) = G_1(z)U_1(z)$$

Substituting:

$$Y(z) = \frac{1}{2} ([H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) + [H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z))$$

**Perfect reconstruction condition:**

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-d}$$

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

This guarantees no aliasing and exact recovery (with delay  $d$ ).

### 13.5 Condition for Perfect Reconstruction

For perfect reconstruction without aliasing, the analysis filter bank  $H_0, H_1$  and synthesis filters  $G_0, G_1$  must satisfy

$$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 2z^{-d} \\ 0 \end{bmatrix}. \quad (1)$$

### 13.6 Solving for the Synthesis Filters

Equation (1) implies

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}^{-1} \begin{bmatrix} 2z^{-d} \\ 0 \end{bmatrix} \quad (2)$$

$$= \frac{2z^{-d}}{H_0(z)H_1(-z) - H_0(-z)H_1(z)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}. \quad (3)$$

### 13.7 Ensuring Finite-Impulse-Response (FIR) Filters

To keep all four filters FIR we require the denominator to reduce to a monomial

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = cz^{-k}, \quad (4)$$

with constants  $c \neq 0$  and integer  $k$ . Under this condition

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{c} z^{k-d} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}. \quad (5)$$

### 13.8 Remarks

- Equation (5) is the *canonical FIR* solution; other perfect-reconstruction solutions exist when the FIR or delay constraints are relaxed.
- The constant  $c$  merely rescales the analysis filters by  $c^{1/2}$  and the synthesis filters by  $c^{-1/2}$ .

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