Wireless Communications and Optimization Notebook

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Category	**Narrowband**	**Wideband**
Key comparison	$B \ll B_c$	$B \gtrsim B_c$
Channel behavior	Frequency-flat	Frequency-selective
Multipath effect	Single effective tap	Multiple resolvable taps
ISI	No ISI	ISI present
Typical model	$y = hx + n$	$y(t) = \sum_\ell h_\ell x(t - \tau_\ell) + n(t)$
Typical systems	Single-carrier	OFDM, UWB

Table 1: Comparison between narrowband and wideband transmission

Category	**Slow Fading**	**Fast Fading**
Key comparison	$T_c \gg T_s$	$T_c \ll T_s$
Channel variation	Slow in time	Rapid in time
Cause	Low mobility	High mobility
Doppler spread	Small	Large
Channel model	Quasi-static	Time-varying
Performance metric	Outage probability	Ergodic capacity

Table 2: Comparison between slow fading and fast fading

1 The Wireless Channel

Large-scale fading represents the average signal power attenuation or path loss over large distances. It is generally slow-acting and influenced by the prominent terrain features between the transmitter and receiver.

1.1 Path Loss

Path loss is the deterministic reduction in power density as a wave propagates through space. A real-valued deterministic attenuation term modeled as $\Lambda_0 \propto R^\eta$

$$P_r = P_t K \left(\frac{R_0}{R}\right)^\eta$$

Important parameter: the path loss exponent η

- Determined empirically: $2 \leq \eta \leq 8$.

Path Loss: $\Lambda_0|_{\mathrm{dB}} = L_0|_{\mathrm{dB}} + 10\eta \log_{10}\left(\frac{R}{R_0}\right)$

- L_0 is the deterministic path-loss at a reference distance R_0

1.2 Shadowing

Shadowing occurs when large objects (buildings, hills) obstruct the main propagation path. It is modeled as a random variable with a **Log-Normal distribution**.

$$S|_{dB} \sim \mathcal{N}(0, \sigma_S^2) \tag{1}$$

is a zero-mean Gaussian random variable representing the random shadowing effect.

Small-scale fading refers to the rapid fluctuations of signal amplitude and phase over very short distances (wavelength scale), primarily caused by **multipath propagation**.

1.3 Multipath Effects

- **Rayleigh Fading:** Used when there is no dominant Line-of-Sight (LoS) path.
- **Rician Fading:** Used when a strong LoS component exists alongside multiple reflected paths.

For the Rayleigh Fading Channel, assuming that the signal reaches the receiver via a large number of paths of similar energy,

- Central Limit Theorem used to explain this model
- h is modeled such that its real and imaginary parts are i.i.d. zero mean Gaussian variables of variance σ^2 (circularly symmetric complex Gaussian variable).
- Recall $\mathcal{E}\{|h|^2\} = 2\sigma^2 = 1$.

2 Fading and Diversity

The system model can be written as,

$$y = \sqrt{E_s} hc + n \tag{2}$$

where the power of symbol $\mathbb{E}[\|c\|^2]$ equals 1 and the noise n follows $N(0, \sigma^2)$.

2.1 Maximum Likelihood Detection

- Decision rule: choose the hypothesis that maximizes the conditional density

$$\arg\max_x p(y|x) = \arg\max_x \log p(y|x)$$

- If real AWGN $y = x + n$ with $n \sim N(0, \sigma_n^2)$,

$$p(y|x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(y-x)^2}{2\sigma_n^2}\right)$$

  and

$$\arg\max_x p(y|x) = \arg\min_x (y-x)^2$$

- If $y = \sqrt{E_s} hc + n$, the ML decision rule becomes

$$\arg\min_c \left|y - \sqrt{E_s} hc\right|^2$$

2.2 array gain and diversity gain

- *Array Gain*: increase in average output SNR (i.e., at the input of the detector) relative to the single-branch average SNR ρ

$$g_a \triangleq \frac{\bar{\rho}_{out}}{\bar{\rho}} = \frac{\bar{\rho}_{out}}{\rho}$$

- *Diversity Gain*: increase in the error rate slope as a function of the SNR. Defined as the negative slope of the log-log plot of the average error probability $\bar{P}$ versus SNR

$$g_d^o(\rho) \triangleq -\frac{\log_2(\bar{P})}{\log_2(\rho)}.$$

The diversity gain is commonly taken as the asymptotic slope, i.e., for $\rho \to \infty$.

2.3 SIMO System

The SIMO system model is,

$$\mathbf{y} = \sqrt{E_s}\mathbf{h}c + \mathbf{n} \tag{3}$$

The gain combing can be described as $z = \mathbf{g}\mathbf{y}$

2.4 Maximal Ratio Combining

- *Maximal Ratio Combining*: weights are chosen as $g_n = h_n^*$.
  - MRC maximizes the output SNR with white noise

    $$\bar{\rho}_{out} = \mathcal{E}\left\{\frac{E_s|\sum_{n=1}^{n_r} h_n^* h_n|^2}{\|\mathbf{h}\|^2\sigma_n^2}\right\} = \frac{E_s}{\sigma_n^2}\mathcal{E}\left\{\frac{\|\mathbf{h}\|^4}{\|\mathbf{h}\|^2}\right\} = \rho\mathcal{E}\left\{\|\mathbf{h}\|^2\right\} = \rho n_r.$$

    The array gain g_a is thus always equal to n_r, or equivalently, the output SNR is the sum of the SNR levels of all branches (holds true irrespective of the correlation between the branches).
  - For BPSK transmission, the symbol error rate reads as

    $$\bar{P} = \int_0^\infty Q(\sqrt{2\rho u})p_u(u)\,du$$

    where $u = \|\mathbf{h}\|^2$ is χ^2 distribution with $2n_r$ degrees of freedom when the different channels are i.i.d. Rayleigh

    $$p_u(u) = \frac{1}{(n_r-1)!}u^{n_r-1}e^{-u}.$$

    At high SNR, $\bar{P}$ becomes

    $$\bar{P} = (4\rho)^{-n_r}\binom{2n_r-1}{n_r}.$$

    The diversity gain is again equal to n_r.

2.5 MMSE Combining

- So far noise was white Gaussian. When the noise (and interference) is colored, MRC is not optimal anymore.

- Let us denote the combined noise plus interference signal as $\mathbf{n}_i$ such that $\mathbf{y} = \sqrt{E_s}\mathbf{h}c + \mathbf{n}_i$.

- An optimal gain combining technique is the minimum mean square error (MMSE) combining, where the weights are chosen in order to minimize the mean square error between the transmitted symbol c and the combiner output z, i.e.,

$$\mathbf{g}^{\star} = \arg\min_{\mathbf{g}} \mathcal{E}_{\mathbf{n}_i,c}\left\{|\mathbf{g}\mathbf{y} - c|^2\right\}.$$

- Two popular solutions:

  - The optimal weight vector $\mathbf{g}^{\star}$ is given by

  $$\mathbf{g}^{\star} = \alpha\mathbf{h}^H\mathbf{R}_{\mathbf{yy}}^{-1},$$

  where $\mathbf{R}_{\mathbf{yy}} = \mathcal{E}\left\{\mathbf{y}\mathbf{y}^H\right\}$ is the covariance matrix of the received signal. α is a scalar.

  - The optimal weight vector $\mathbf{g}^{\star}$ is given by

  $$\mathbf{g}^{\star} = \beta\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1},$$

  where $\mathbf{R}_{\mathbf{n}_i} = \mathcal{E}\left\{\mathbf{n}_i\mathbf{n}_i^H\right\}$ is the covariance matrix of the combined noise plus interference signal $\mathbf{n}_i$. β is a scalar.

  Those two solutions have the same direction (more later).

- The Signal to Interference plus Noise Ratio (SINR) at the output of the MMSE combiner simply writes as $\rho_{out} = E_s\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1}\mathbf{h}$.

2.6 MISO system

$$y = \sqrt{E_s}\mathbf{h}\mathbf{x} + n = \sqrt{E_s}\mathbf{h}\mathbf{w}c + n \tag{4}$$

where $\mathbf{w}$ is the precoder. There are basically two different ways of achieving *direct transmit diversity*:

- when Tx has a *perfect channel knowledge*, beamforming can be performed to achieve both diversity and array gains,
- when Tx has a *partial or no channel knowledge of the channel*, space-time coding is used to achieve a diversity gain (but no array gain in the absence of any channel knowledge).

3 Channel Capacity

Definition

The capacity of an AWGN channel is

$$C = \log_2(1 + SNR) \quad [bits/s/H].$$

Definition

The capacity of a deterministic (time-invariant) SISO channel h is

$$C = \log_2(1 + \rho|h|^2) \quad [bits/s/H].$$

Instantaneous SNR is $\rho|h|^2$.

Definition

The outage probability $P_{out}(R)$ of a wireless channel with a target rate R is given by

$$P_{out}(R) = P\left(\log_2(1 + \rho|h|^2) < R\right).$$

4 Basics of MIMO

$$\mathbf{y} = \sqrt{E_s}\mathbf{H}\mathbf{x} + \mathbf{n} \tag{5}$$

where $\mathbf{y} \in \mathbb{C}^{n_r \times 1}$, $\mathbf{H} \in C^{n_r \times n_t}$ and $\mathbf{x} \in C^{n_t \times 1}$

- when Tx has a *perfect channel knowledge*: (dominant and multiple) eigenmode transmission
- when Tx has *no knowledge of the channel*: space-time coding (with $\mathbf{x}_k = \mathbf{c}_k$)

4.1 Dominant Eigenmodes

MIMO with Perfect Transmit Channel Knowledge

Extension of Matched Beamforming to MIMO:

$$\mathbf{y} = \sqrt{E_s}\mathbf{H}\mathbf{x} + \mathbf{n} = \sqrt{E_s}\mathbf{H}\mathbf{w}\mathbf{c} + \mathbf{n},$$

$$z = \mathbf{g}\mathbf{y} = \sqrt{E_s}\mathbf{g}\mathbf{H}\mathbf{w}\mathbf{c} + \mathbf{g}\mathbf{n}.$$

Decompose:

$$\mathbf{H} = \mathbf{U_H}\mathbf{\Sigma_H}\mathbf{V_H^H},$$

$$\mathbf{\Sigma_H} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}.$$

Received SNR is maximized by matched filtering, leading to:

- $\mathbf{w} = \mathbf{v}_{max}$
- $\mathbf{g} = \mathbf{u}_{max}^H$

where $\mathbf{v}_{max}$ and $\mathbf{u}_{max}$ are respectively the right and left singular vectors corresponding to the maximum singular value of $\mathbf{H}$, $\sigma_{max} = \max\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}$. Equivalent channel: $z = \sqrt{E_s}\sigma_{max}c + \tilde{n}$ where $\tilde{n} = \mathbf{g}\mathbf{n}$ has a variance equal to σ_n^2.

4.2 Multiple Eigenmodes

Assume $n_r \geq n_t$ and that $r(\mathbf{H}) = n_t$, i.e., n_t singular values in $\mathbf{H}$. Spreading symbols over all non-zero eigenmodes of the channel:

- Tx side: multiply the input vector $\mathbf{c}$ ($n_t \times 1$) using $\mathbf{V_H}$, i.e. $\mathbf{c}' = \mathbf{V_H}\mathbf{c}$.

- Rx side: multiply the received vector $\mathbf{y}$ by $\mathbf{G} = \mathbf{U}_{\mathbf{H}}^{\mathbf{H}}$.

- Overall: $\mathbf{z} = \sqrt{E_s}\mathbf{GHx} + \mathbf{Gn} = \sqrt{E_s}\mathbf{U}_{\mathbf{H}}^{\mathbf{H}}\mathbf{HV}_{\mathbf{H}}\mathbf{c} + \mathbf{U}_{\mathbf{H}}^{\mathbf{H}}\mathbf{n} = \sqrt{E_s}\mathbf{\Sigma}_{\mathbf{H}}\mathbf{c} + \tilde{\mathbf{n}}$.

Channel decomposed into n_t parallel SISO channels given by $\{\sigma_1, \ldots, \sigma_{n_t}\}$.

The rate achievable in the MIMO channel is the sum of the SISO channel capacities:

$$R = \sum_{k=1}^{n_t} \log_2(1 + \rho s_k \sigma_k^2),$$

where $\{s_1, \ldots, s_{n_t}\}$ is the power allocation on each of the channel eigenmodes.

- The rate scales linearly in n_t.

- In general, the rate scales linearly with the rank of $\mathbf{H}$.

4.3 Water-Filling Algorithm

- $\mathbf{\Sigma_H} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}$ and $\lambda_k \triangleq \sigma_k^2$

The power allocation strategy $\{s_1, \ldots, s_n\} = \{s_1^\star, \ldots, s_n^\star\}$ that maximizes $\sum_{k=1}^n \log_2(1 + \rho\lambda_k s_k)$ under the power constraint $\sum_{k=1}^n s_k = 1$, is given by the water-filling solution:

$$s_k^\star = \left(\mu - \frac{1}{\rho\lambda_k}\right)^+, \quad k = 1, \ldots, n$$

where μ is chosen so as to satisfy the power constraint $\sum_{k=1}^n s_k^\star = 1$.

4.4 Partial Transmit Channel Knowledge

Proposition: In i.i.d. Rayleigh fading channels, the ergodic capacity with CDIT is achieved under an equal power allocation scheme $Q = \frac{I_{nt}}{n_t}$, i.e.,

$$\bar{C}_{CDIT} = \bar{I}_e = \mathbb{E}\left\{\log_2 \det\left[I_{nr} + \frac{\rho}{n_t} H_w H_w^H\right]\right\} = \mathbb{E}\left\{\sum_{k=1}^{n} \log_2\left[1 + \frac{\rho}{n_t}\lambda_k\right]\right\}.$$

Encoding requires a fixed-rate code (whose rate is given by the ergodic capacity) with encoding spanning many channel realizations.

5 Transmission and Reception Strategies

Spatial Multiplexing / V-BLAST / D-BLAST

- No transmit channel knowledge but we know it is i.i.d. Rayleigh fading.
- Spatial Multiplexing (SM), also called V-BLAST, is a full rate code ($r_s = n_t$) that consists in transmitting independent data streams on each transmit antenna.
- In uncoded transmissions, we assume one symbol duration ($T = 1$) and codeword C is a symbol vector of size $n_t \times 1$.

Example:

$$C = \frac{1}{\sqrt{n_t}} \begin{bmatrix} c_1 & \cdots & c_{n_t} \end{bmatrix}^T.$$

Each element c_q is a symbol chosen from a given constellation.

5.1 ML Decoding

- With instantaneous channel realizations perfectly known at the receive side, the ML decoder computes an estimate of the transmitted codeword according to

$$\hat{C} = \arg\min_{C \in \mathcal{C}} \left\| y - \sqrt{E_s} H C \right\|^2.$$

  where the minimization is performed over all possible codeword vectors $C \in \mathcal{C}$.

- What is C for SM?

$$C = \frac{1}{\sqrt{n_t}} \begin{bmatrix} c_1 & \cdots & c_{n_t} \end{bmatrix}^T.$$

- Assume all streams c_q use the same M-ary constellation of points $\mathcal{X}$, e.g., QPSK, 64-QAM, etc. Hence a search over M^{n_t} possible transmitted vectors is needed.
- Complexity gets prohibitive if M or the number of streams increases.

5.2 Zero-Forcing (ZF) Linear Receiver

- MIMO ZF receiver acts similarly to a ZF equalizer in frequency selective channels.
- ZF filtering effectively decouples the channel into n_t parallel channels.
  - Interference from other transmitted symbols is suppressed.

– Scalar decoding may be performed on each of these channels.

- The complexity of ZF decoding is similar to SISO ML decoding, but the inversion step is responsible for the noise enhancement (especially at low SNR).

- Assuming that a symbol vector $C = \frac{1}{\sqrt{n_t}} \begin{bmatrix} c_1 & \cdots & c_{n_t} \end{bmatrix}^T$ is transmitted, the output of the ZF filter G_{ZF} is given by

$$z = G_{ZF} y = \begin{bmatrix} c_1 & \cdots & c_{n_t} \end{bmatrix}^T + G_{ZF} n.$$

where G_{ZF} inverts the channel,

$$G_{ZF} = \sqrt{\frac{n_t}{E_s}} H^\dagger.$$

with $H^\dagger = (H^H H)^{-1} H^H$ denoting the Moore-Penrose pseudo inverse.

- Then each stream can be decoded $\hat{c}_q = \arg\min_{c_q \in \mathcal{X}} |z_q - c_q|, \forall q.$

5.3 Minimum Mean Squared Error (MMSE) Linear Receiver

- Recall the other MMSE solution based on

$$R_{yy} = \sigma_n^2 I_{nr} + \sum_p \frac{E_s}{n_t} h_p h_p^H$$

and the MMSE combiner for stream q is given by

$$\begin{aligned} g_{MMSE,q} &= \sqrt{\frac{E_s}{n_t}} h_q^H \left(\sigma_n^2 I_{nr} + \sum_p \frac{E_s}{n_t} h_p h_p^H \right)^{-1}. \\ &= \sqrt{\frac{E_s}{n_t}} h_q^H \left(\sigma_n^2 I_{nr} + \frac{E_s}{n_t} H H^H \right)^{-1} \\ &= \sqrt{\frac{n_t}{E_s}} h_q^H \left(\frac{n_t}{\rho} I_{nr} + H H^H \right)^{-1}. \end{aligned}$$

- We can stack up combiners of all streams into one matrix:

$$G_{MMSE} = \sqrt{\frac{n_t}{E_s}} \left(H^H H + \frac{n_t}{\rho} I_{nt} \right)^{-1} H^H$$

which is a popular representation of the MMSE filter. Alternatively, it can be written as,

$$G_{MMSE} = \sqrt{\frac{n_t}{E_s}} H^H \left(H H^H + \frac{n_t}{\rho} I_{nr} \right)^{-1}$$

- Bridge between matched filtering at low SNR and ZF at high SNR.

5.4 Successive Interference Canceler

- Successively decode one symbol (or more generally one layer/stream) and cancel the effect of this symbol from the received signal.
- Decoding order based on the SINR of each symbol/layer: the symbol/layer with the highest SINR is decoded first at each iteration.
- SM with (ordered) SIC is generally known as V-BLAST, and ZF and MMSE V-BLAST refer to SM with respectively ZF-SIC and MMSE-SIC receivers.
- The diversity order experienced by the decoded layer is increased by one at each iteration. Therefore, the symbol/layer detected at iteration i will achieve a diversity of $n_r - n_t + i$.
- Major issue: error propagation
  - The error performance is mostly dominated by the weakest stream.
  - Non-ordered SIC: diversity order approximately $n_r - n_t + 1$.
  - Ordered SIC: performance improved by reducing the error propagation caused by the first decoded stream. The diversity order remains lower than n_r.

- Broadcast Channel (BC) - downlink
- Multiple Access Channel (MAC) - uplink
- The capacity region $\mathcal{C}$ formulates this trade-off by expressing the set of all user rates $(R_1, \ldots, R_K)$ that are simultaneously achievable.

Definition

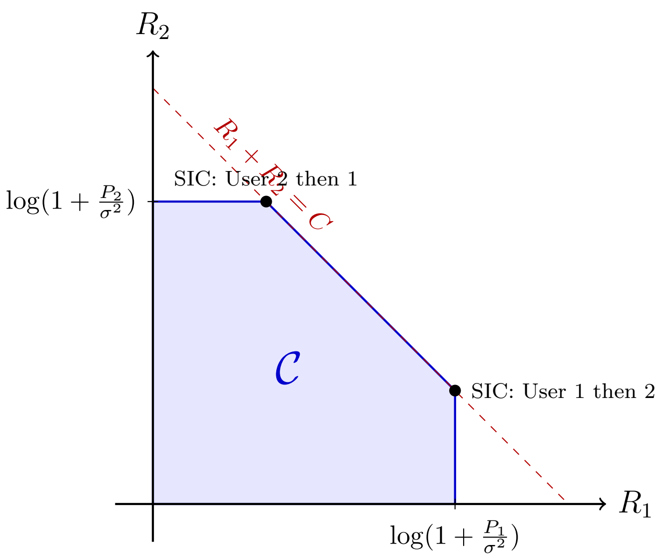
The capacity region $\mathcal{C}$ of a channel $\mathbf{H}_{ul}$ is the set of all rate vectors $(R_1, \ldots, R_K)$ such that simultaneously user 1 to user K can reliably communicate at rate R_1 to rate R_K, respectively.

Any rate vector not in the capacity region is not achievable (i.e. transmission at those rates will lead to errors).

Definition

The sum-rate capacity C of a capacity region $\mathcal{C}$ is the maximum achievable sum of rates

$$C = \max_{(R_1,\ldots,R_K)\in\mathcal{C}} \sum_{q=1}^{K} R_q.$$



5.5 Multi-user SISO

Consider the downlink case, the signal received by user q is,

$$y_q = \sqrt{E_s} h_q c + n_q \tag{6}$$

where,

$$c = \sum_q s_q c_q \tag{7}$$

Consider two-user case and the power allocations$s_1 > s_2$,

$$y_1 = h_1 s_1 c_1 + h_1 s_2 c_2 + n \tag{8}$$

$$y_2 = h_2 s_1 c_1 + h_2 s_2 c_2 + n \tag{9}$$

- Successive Interference Cancellation
- For user 1, it takes c_2 as noise and decodes c_1
- For user 2, it cancels c_1 and then decodes c_2

5.6 Multi-user MIMO

Consider the downlink case, the receive signal of user q can be written as,

$$y_q = \Lambda_q^{-1/2}\mathbf{h_q}c + n_q = \Lambda_q^{-1/2}\mathbf{h_q}\mathbf{w_q}c_q + \sum_{p\neq q}\Lambda_q^{-1/2}\mathbf{h_q}\mathbf{w_p}c_p + n_q \tag{10}$$

- includes intra-cell interference
- $h_q w_p = 0$ and $h_q w_q \neq 0$
- **ZFBF**: $HW = I$, then $W = H^\dagger$

- A *convex optimization* is one in which the objective and constraint functions are all convex[2], i.e.,

$$f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y}), \forall i = 0, \ldots, m \tag{11}$$

$-\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \forall \alpha, \beta \in \mathbb{R}, \alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$