Dr. Bill Capehart
CEE 284, Fall 2015

NONLINEAR ERRORS
MATHCAD SHEET EXAMPLE

Index Convention starts at 0
Units Convention: SI

This is an example of a heavilly commented Mathcad sheet with my demonstrative comments written in ***purple****. My "baseline" text comments are blue but I reserve the option to use color to hightlight things.*

For a general rule-of-thumb outside of the Mathcad environment consider what you would expect in a well-written Lab report. My general guidance is to make it so that a person who is new to Mathcad can follow what you are doing. Keep it complete and professional and "pretty" (i.e., easy on the eyes so that it can be read and understood by a large audience of people with a reasonable technical background.

For this activity we are charged to explore how the average of a functinon, f(x), is not necessarily equal to the function of the average values of x.

Our null hypothesis (Ho) is that $f[avg(x)] = avg(f(x))$

Our alternative hypothesis (Ha) is $f[avg(x)] \neq avg(f(x))$

...where avg(x) is the average since I can't do an overbar in the Mathcad editor here. I could be clever and insert a picture from a screenshot from something I make in Microsoft Word,

$$f(\bar{x}) = \overline{f(x)}$$

but that's just too much for a beginner so don't worry for now if it means not getting your English Lit paper in. At this phase consider it to be "cake icing"

Consider the temperature dataset (shown below-left), *T*. It is entered into Matchad by hand. The temperatures are all approximate to one another with a signfiicant higher-value outlier.

$$T := \begin{bmatrix} 40 \\ 35 \\ 37 \\ 39 \\ 34 \\ 120 \end{bmatrix} {}^{\circ}F \qquad \text{so that} \qquad T = \begin{bmatrix} 277.594 \\ 274.817 \\ 275.928 \\ 277.039 \\ 274.261 \\ 322.039 \end{bmatrix} K$$

For our f(x) we will be using the Stefan-Boltzmann Law for the for outgoing radiated energy from a surface (*M*)
(you may have had this in University Physics II)

$$M(T) := \sigma \cdot T^4$$

Where σ is the Stefan-Boltzmann constant which is internally declared in Mathcad as $\sigma = \left(5.67 \cdot 10^{-8}\right) \dfrac{kg}{s^3 \cdot K^4}$

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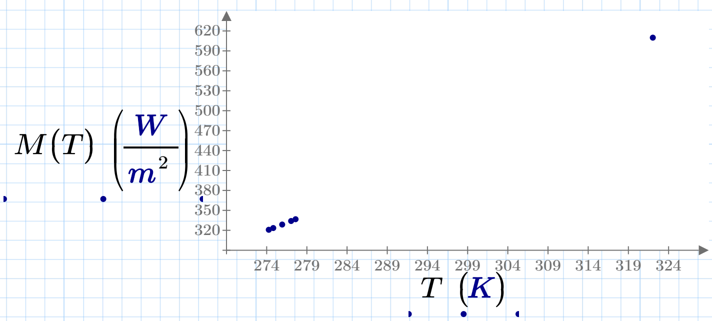
When we apply the formula to our temperatures, we get the following:

$$M(T) = \begin{bmatrix} 336.711 \\ 323.434 \\ 328.697 \\ 334.023 \\ 320.827 \\ 609.884 \end{bmatrix} \frac{W}{m^2}$$

We note here the magnitude of the outliers outgoing energy which is considerably more than its cohorts

[As a convention we like to use W/m^2 for M and you can manually change the units in Mathcad. It will "adjust" the units for you. You can try this for fun for units like Joules and other combined units.]

The graph to the right also demonstrates this. Also that is is not linear (as can be seen if you extrapolate a line from the first lower five values to higher temperatures.)



We now calculate the average value of T both with an intrinsic function in Mathcad and also with a "manual" calculation as a demonstration if Mathcad's symbolic math.

$$avgT1 := \frac{1}{\mathrm{rows}(T)} \sum T = 283.613\ K$$

$$avgT := \mathrm{mean}(T) = 283.613\ K$$

$$avgT - avgT1 = 0\ K$$

The two values are equal

Let us now apply our formula to the temperatures:

The value for M[avg(T)] is: $M(avgT) = 366.875\ \frac{W}{m^2}$

The average value of M(T) is: $\mathrm{mean}(M(T)) = 375.596\ \frac{W}{m^2}$

We presume that the average of M(T) is the correct value here. The base and relative errors, therefore are:

$$M(avgT) - \mathrm{mean}(M(T)) = -8.721\ \frac{W}{m^2}$$

$$PercError := \frac{(M(avgT) - \mathrm{mean}(M(T)))\ 100}{\mathrm{mean}(M(T))} = -2.322 \quad \%$$

(I'm using a text box for that percent value)

The two values do not *appear* equal. If the precision of the instrument or standard of accuracy is less than 8.7 W m-2 then we may presume that these values are effectively the same, otherwise we can reject the Null Hypothesis (In Stats you would also use more formal Hypothesis Testing)

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But let's not stop there: Let's look at how to associate error that is derived from uncertainty estimates when we put them to work for us in functions and equation.

This example makes a good case study for this.

Once again, let's look at the function.

$$M(T) := \sigma \cdot T^4$$

We have a constant times temperature raised to the fourth power.

With this you saw earlier, as you go to higher values of T, the infrared heat emitted by the object will increase more and more.

We can represent the change of M with respect to T with the following Calc-Fu: The derivative!

$$\boxed{\frac{d}{dT} M(T)} \rightarrow ?$$

Ooops. That T is still spoken for from our earlier scientific adventure violence. Luckily there is a function called "clear()" that will do what it says it will...

$$\mathbf{clear}\,(T)$$

$$\frac{d}{dT} M(T) \rightarrow 4 \cdot T^3 \cdot \sigma$$

That's better... So as we go to bigger values of T we get even bigger values of M

So what happens when we have an uncertainty in T?

Let's say that we swag that T is accurate to 0.1 K.

How does that manifest in an error for T?

Once again, this is a job for some Calc-Fu

$$\Delta M(T) := \Delta T \frac{d}{dT} M(T) \rightarrow 4 \cdot T^3 \cdot \sigma \cdot \Delta T$$

This is a simple formula to assess **The Propagation of Error** through an equation system. We symbolize it as Δ[whatever function we're assessing impact]
Thus, for our uncertainty of T of 0.1 K

$$\Delta T := 0.1\ K$$

$$\Delta T \frac{d}{dT} M(T) \rightarrow 0.4 \cdot K \cdot T^3 \cdot \sigma$$

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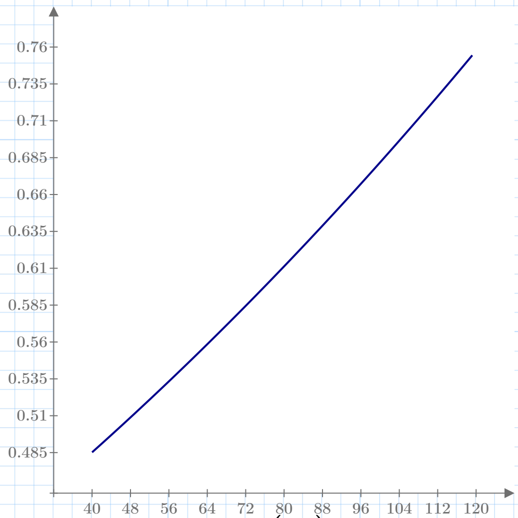
And when given a range of values of T such as in our early case...

$$T := 40\ {}^\circ F, 41.1\ {}^\circ F .. 120\ {}^\circ F = \begin{bmatrix} 277.594 \\ \vdots \end{bmatrix} K$$

$$\Delta M(T) := \Delta T \frac{d}{dT} M(T) \rightarrow 0.4 \cdot K \cdot T^3 \cdot \sigma$$

Darn: As you can see the units messed up our symbolic arrow. You can do this in your notebook then... OK?



$T\ ({}^\circ F)$

$\Delta M(T)\ \left(\frac{W}{m^2}\right)$

If you squint you can see that this isn't *completely* a straight line

(If you're playing along you may have to edit the units by "hand")

If you have a function with more than one uncertain dependant variable, the formula is expandable:

Let's try the function below.

$$f(x, y, z) := 4 \cdot x + 3 \cdot y^2 + 2 \cdot \ln(z)$$

To get the error propagation for f, just string along your deivatives and uncertainies, adding them along the way

$$\Delta f := \Delta x \frac{d}{dx} f(x, y, z) + \Delta y \frac{d}{dy} f(x, y, z) + \Delta z \frac{d}{dz} f(x, y, z) \rightarrow 4 \cdot \Delta x + \frac{2 \cdot \Delta z}{z} + 6 \cdot y \cdot \Delta y$$

Finally let's look at one more metric based on error propogation. $\mathbf{clear}(T, \Delta T)$
SENSITIVITY *Let's call it "S"*

Let's use our earlier formula

$$S_M := \frac{\Delta M(T)}{M(T)} \rightarrow \frac{0.4 \cdot K \cdot T^3}{T^4}$$

Darn again... ok let's just do it in pen and paper and come back...

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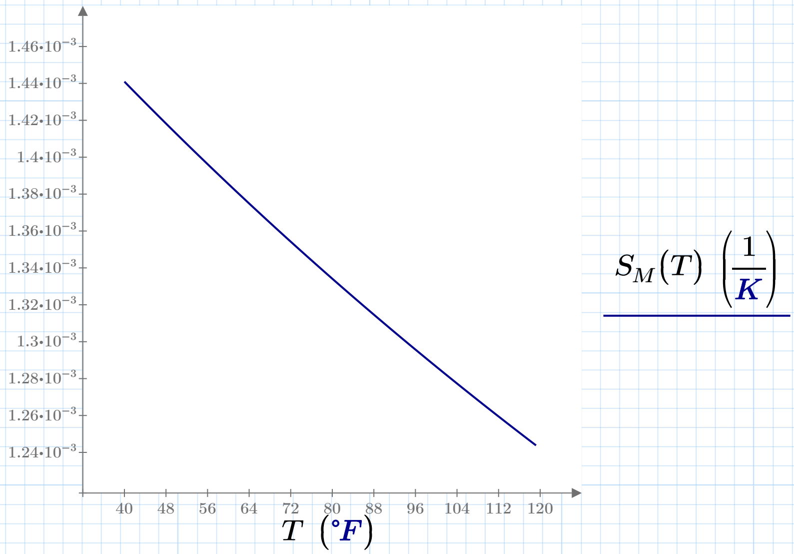
$$S_M(T) := \frac{0.4}{T}$$ Well that was easy...

$$T := 40\ °F, 41.1\ °F .. 120\ °F = \begin{bmatrix} 277.594 \\ 278.206 \\ 278.817 \\ \vdots \end{bmatrix} K$$



As you can see, the sensitivity increases with lower values of T in this case.