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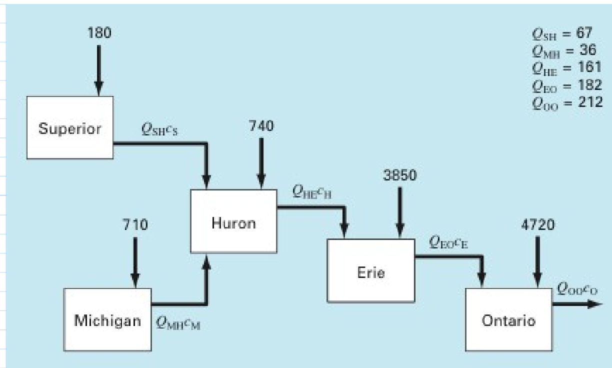
Making Linear Systems of Equations

Taking Our Great Lakes Transport Problem and putting it in Mathcad....



Our Transport Rates

$$Q_{SH} := 67 \qquad Q_{MH} := 36 \qquad Q_{HE} := 161$$
$$Q_{EO} := 182 \qquad Q_{OO} := 212$$

Our Local Chloride Contributions

$$q_S := 180 \qquad q_M := 710 \qquad q_H := 740$$
$$q_E := 3850 \qquad q_O := 4720$$

$$Q := \begin{bmatrix} -Q_{SH} & 0 & 0 & 0 & 0 \\ 0 & -Q_{MH} & 0 & 0 & 0 \\ Q_{SH} & Q_{MH} & -Q_{HE} & 0 & 0 \\ 0 & 0 & Q_{HE} & -Q_{EO} & 0 \\ 0 & 0 & 0 & Q_{EO} & -Q_{OO} \end{bmatrix} \quad c = \begin{bmatrix} c_S \\ c_M \\ c_H \\ c_E \\ c_O \end{bmatrix} \quad q := -\begin{bmatrix} q_S \\ q_M \\ q_H \\ q_E \\ q_O \end{bmatrix}$$

Our Steady-State Solution Matricies (c is just for show)

$$c_{matrix} := Q^{-1}\ q = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

And our steady state solution

Now let's set things up a Solve Block or Solve Arrow Approach...

I start by listing my equations as part of my documentation. I am using the "bold" comparative equal sign here to improve the look and feel of the sheet

$$\frac{d}{dt} c_S = q_S - Q_{SH} \cdot c_S$$

$$\frac{d}{dt} c_M = q_M - Q_{MH} \cdot c_M$$

$$\frac{d}{dt} c_H = q_H + Q_{SH} \cdot c_S + Q_{MH} \cdot c_M - Q_{HE} \cdot c_H$$

$$\frac{d}{dt} c_E = q_E + Q_{HE} \cdot c_H - Q_{EO} \cdot c_E$$

$$\frac{d}{dt} c_O = q_O + Q_{EO} \cdot c_E - Q_{OO} \cdot c_O$$

And here is our steady state solution using a solve block... "root-style"

Guess Values

$$c_S := 0 \qquad c_M := 0 \qquad c_H := 0$$
$$c_E := 0 \qquad c_O := 0$$

Constraints

$$0 = q_S - Q_{SH} \cdot c_S$$
$$0 = q_M - Q_{MH} \cdot c_M$$
$$0 = q_H + Q_{SH} \cdot c_S + Q_{MH} \cdot c_M - Q_{HE} \cdot c_H$$
$$0 = q_E + Q_{HE} \cdot c_H - Q_{EO} \cdot c_E$$
$$0 = q_O + Q_{EO} \cdot c_E - Q_{OO} \cdot c_O$$

Solver

$$c_{block} := \mathbf{find}\left(c_S, c_M, c_H, c_E, c_O\right) = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

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And our steady state solution using a solve arrow... Also "root-style"

$$dCdt(c_S, c_M, c_H, c_E, c_O) := \begin{bmatrix} q_S - Q_{SH} \cdot c_S \\ q_M - Q_{MH} \cdot c_M \\ q_H + Q_{SH} \cdot c_S + Q_{MH} \cdot c_M - Q_{HE} \cdot c_H \\ q_E + Q_{HE} \cdot c_H - Q_{EO} \cdot c_E \\ q_O + Q_{EO} \cdot c_E - Q_{OO} \cdot c_O \end{bmatrix}$$

$$c_{arrow} := dCdt(c_S, c_M, c_H, c_E, c_O) \xrightarrow{solve, c_S, c_M, c_H, c_E, c_O} \begin{bmatrix} \frac{180}{67} & \frac{355}{18} & \frac{1630}{161} & \frac{2740}{91} & \frac{2550}{53} \end{bmatrix}$$

Since I don't arrays going lengthwise, I can use the transpose of my answer to make it better looking

$$c_{arrow}{}^{T} = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

And there you go! We have demonstrated three ways to solve a simulatneous system of equations...

Finally you can test your answer by plugging things back into your original equation form. If all works out, you should have zero (or almost zero) for all values...

$$Q \cdot c_{matrix} - q = \begin{bmatrix} 2.842 \cdot 10^{-14} \\ 0 \\ 2.274 \cdot 10^{-13} \\ -9.095 \cdot 10^{-13} \\ 9.095 \cdot 10^{-13} \end{bmatrix}$$

Now for an intermission.

If you have had Diff-Eqs the next page will show you how to work through an example where we take this system of equations and use them to form a dynamic solution that changes over time.

Once again, the idea here is to show that this system will converge to a steady state solution eventually. And that solution SHOULD be the same as your matrix solution.

If you have not had Diff-Eqs, don't panic over it. You may want to come back after surviving ODE.

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Now for those of you with Diff Eq under your belts, we can have some fun (and for those of you who are not.... Sooooooon.......). You can find guidance here in Chapter 13 of the Mathcad Text

Let's start by setting up a solve block to create a *dynamic or "transient"* (not a steady state) solution starting with empty values of "c" and taking forward in time up to a final time of "0.5" arbitrary units $t_{final} := 0.5$

Guess Values

$This\_Space\_For\_Rent := 0$ <- This keeps this unused section small.

Constraints

Here are our equations written "diff-eq" style and our initial conditons at time "0"

$$\frac{d}{dt}c_S(t) = q_S - Q_{SH} \cdot c_S(t)$$

$$\frac{d}{dt}c_E(t) = q_E + Q_{HE} \cdot c_H(t) - Q_{EO} \cdot c_E(t)$$

$$\frac{d}{dt}c_M(t) = q_M - Q_{MH} \cdot c_M(t)$$

$$\frac{d}{dt}c_O(t) = q_O + Q_{EO} \cdot c_E(t) - Q_{OO} \cdot c_O(t)$$

$$\frac{d}{dt}c_H(t) = q_H + Q_{SH} \cdot c_S(t) + Q_{MH} \cdot c_M(t) - Q_{HE} \cdot c_H(t)$$

$$c_S(0) = 0 \quad c_M(0) = 0 \quad c_H(0) = 0 \quad c_E(0) = 0 \quad c_O(0) = 0$$

Solver

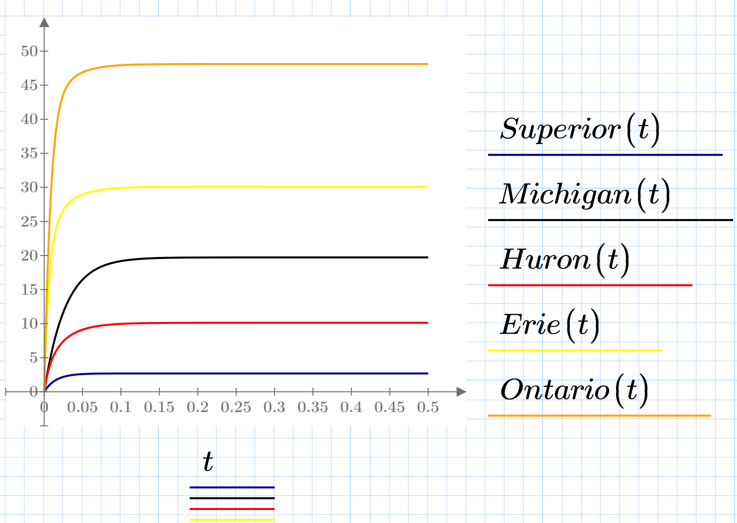
$$c_{odesolve} := \text{odesolve}\left(\begin{bmatrix} c_S(t) \\ c_M(t) \\ c_H(t) \\ c_E(t) \\ c_O(t) \end{bmatrix}, t_{final}\right)$$

And here is our transient solution valid for a range between t = 0 and t_final

You can break them up for graphing and printing with the [subscript method.

$Superior := c_{odesolve_0}$ $Michigan := c_{odesolve_1}$ $Huron := c_{odesolve_2}$

$Erie := c_{odesolve_3}$ $Ontario := c_{odesolve_4}$



$Superior(t_{final}) = 2.687$
$Michigan(t_{final}) = 19.722$
$Huron(t_{final}) = 10.124$
$Erie(t_{final}) = 30.11$
$Ontario(t_{final}) = 48.113$

$$c_{matrix} = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

And once again! We have a bench once the time series steady out over time and we have a balanced system.

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Now let's step back to a more "interesting scenario". Let's go back our five equations.

$$\frac{d}{dt}c_S = q_S - Q_{SH} \cdot c_S$$

$$\frac{d}{dt}c_M = q_M - Q_{MH} \cdot c_M$$

$$\frac{d}{dt}c_H = q_H + Q_{SH} \cdot c_S + Q_{MH} \cdot c_M - Q_{HE} \cdot c_H$$

$$\frac{d}{dt}c_O = q_O + Q_{EO} \cdot c_E - Q_{OO} \cdot c_O$$

In a steady state, the left-hand-sides (LHSs) of the equations will all be zero.

However, let's throw a wrench in the mix. In this case it's not quite a good example physically unless we go into extreme "Mysterious Unknown Force" ("MUF" - another Space 1999 joke) that typically plagues victims of endless science fiction shows....

Let's assume that there is a Deus ex Machina force that requires all of total concentrations of c to be equal to 100 units. For example there is ONLY 100 units available to the Lake System.

We will implement it by adding a SIXTH equation.

$$c_{total} := 100. \qquad c_{totalB} := \sum c_{matrix} = 110.756$$

$$c_{total} = c_S + c_M + c_H + c_E + c_O$$

SIX equations... FIVE unknowns. This is new territory. This is an OVERSOLVED equation. We will not be able to exactly solve for this of c_total is not naturally the total of the balanced steady state values of c from earlier.

But first, let's first set up our matrices.

$$Q_2 := \begin{bmatrix} -Q_{SH} & 0 & 0 & 0 & 0 \\ 0 & -Q_{MH} & 0 & 0 & 0 \\ Q_{SH} & Q_{MH} & -Q_{HE} & 0 & 0 \\ 0 & 0 & Q_{HE} & -Q_{EO} & 0 \\ 0 & 0 & 0 & Q_{EO} & -Q_{OO} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad q_{2}\_and\_c_{total} := -\begin{bmatrix} q_S \\ q_M \\ q_H \\ q_E \\ q_O \\ -c_{total} \end{bmatrix}$$

That last line seems to break the "look and feel" of our equation system from 5 transport equations to 5 transport equation plus one mass balance equation. With the exception that we are "oversolved" or "overclosed" that is not really a major issue. An equation is an equation.

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Notice also that our New Q, Q2, is no longer square. We won't be able to get a traditional inverse. We must do something else. This new structure is called a PSUEDOINVERSE

$$Q2_{pseudo} := \left(Q_2^{\mathrm{T}} \cdot Q_2\right)^{-1} \cdot Q_2^{\mathrm{T}}$$

$$Q2_{pseudo} = \begin{bmatrix} -0.015 & 2.061 \cdot 10^{-5} & 7.658 \cdot 10^{-6} & 4.762 \cdot 10^{-6} & 2.2 \cdot 10^{-6} & 4.663 \cdot 10^{-4} \\ 3.836 \cdot 10^{-5} & -0.028 & 2.01 \cdot 10^{-5} & 1.25 \cdot 10^{-5} & 5.772 \cdot 10^{-6} & 0.001 \\ -0.006 & -0.006 & -0.006 & 5.814 \cdot 10^{-6} & 2.686 \cdot 10^{-6} & 5.694 \cdot 10^{-4} \\ -0.005 & -0.005 & -0.005 & -0.005 & 2.64 \cdot 10^{-6} & 5.596 \cdot 10^{-4} \\ -0.005 & -0.005 & -0.005 & -0.005 & -0.005 & 5.026 \cdot 10^{-4} \end{bmatrix}$$

We are still a non-square matrix but it will allow us to solve our system.

$$Q^{-1} = \begin{bmatrix} -0.015 & 0 & 0 & 0 & 0 \\ 0 & -0.028 & 0 & 0 & 0 \\ -0.006 & -0.006 & -0.006 & 0 & 0 \\ -0.005 & -0.005 & -0.005 & -0.005 & 0 \\ -0.005 & -0.005 & -0.005 & -0.005 & -0.005 \end{bmatrix}$$

And now we solve.

$$c_{oversolved} := Q2_{pseudo} \cdot q_{2}\_and\_c_{total} = \begin{bmatrix} 2.682 \\ 19.709 \\ 10.118 \\ 30.104 \\ 48.108 \end{bmatrix}$$

If we plug in our new values into our linear system to check if our answer creates all zeros (as it would if we were a neatly closed system...), we get the following....

$$Q_2 \cdot c_{oversolved} - q_{2}\_and\_c_{total} = \begin{bmatrix} 0.336 \\ 0.474 \\ 0.176 \\ 0.109 \\ 0.051 \\ 10.72 \end{bmatrix}$$

Here we should have values that are "near" zero but not quite getting there... As with the Least Squares Method, this is a compromise to create a "least-bad" solution.

Note that if you go up and change teh value for c_total to something closer to the balanced total of values, you will get a

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Now that our "MUF" case is demonstrated, let's get a little more realistic.

Let's fix the above case by still having a regulatory requirement that we reduce the total concentration in all lakes to a given amount.... For example, once again, 100 units.

Our equatons stay the same but we can fix our problem. We will lay the burden of balancing our regulatory requirement to the managers of Lake Erie in honor of a classic Saturday Night Live Sketch (Lake Erie is doing a lot better than in the past). Lake Erie will need to change its concentrations to satisfy requirement. This its q_E will have to be turned into a variable. The larger equation system when written out root-style or diff-eq style may be similar as before. But our matrix structure will change. Our "C" matrix will now hold an extra value -- q_E. While q_E's previous place in the q matrix will be set to zero.

$$Q_3 := \begin{bmatrix} -Q_{SH} & 0 & 0 & 0 & 0 & 0 \\ 0 & -Q_{MH} & 0 & 0 & 0 & 0 \\ Q_{SH} & Q_{MH} & -Q_{HE} & 0 & 0 & 0 \\ 0 & 0 & Q_{HE} & -Q_{EO} & 0 & 1 \\ 0 & 0 & 0 & Q_{EO} & -Q_{OO} & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad c\_and\_q_E = \begin{bmatrix} c_S \\ c_M \\ c_H \\ c_E \\ c_O \\ q_E \end{bmatrix} \quad q_3\_and\_c_{total} := -\begin{bmatrix} q_S \\ q_M \\ q_H \\ 0 \\ q_O \\ -c_{total} \end{bmatrix}$$

$$c\_and\_q_E := Q_3^{-1} \cdot q_3\_and\_c_{total} = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 24.322 \\ 43.145 \\ 2796.666 \end{bmatrix} \qquad c_{matrix} = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

$$c\_and\_q_{E_5} - q_E = -1053.334$$ Local Input Reduction of Chlorides

And our loss in quality beverages is the Great Lake water qualitiy's gain.