SOUTH DAKOTA SCHOOL OF MINES & TECHNOLOGY

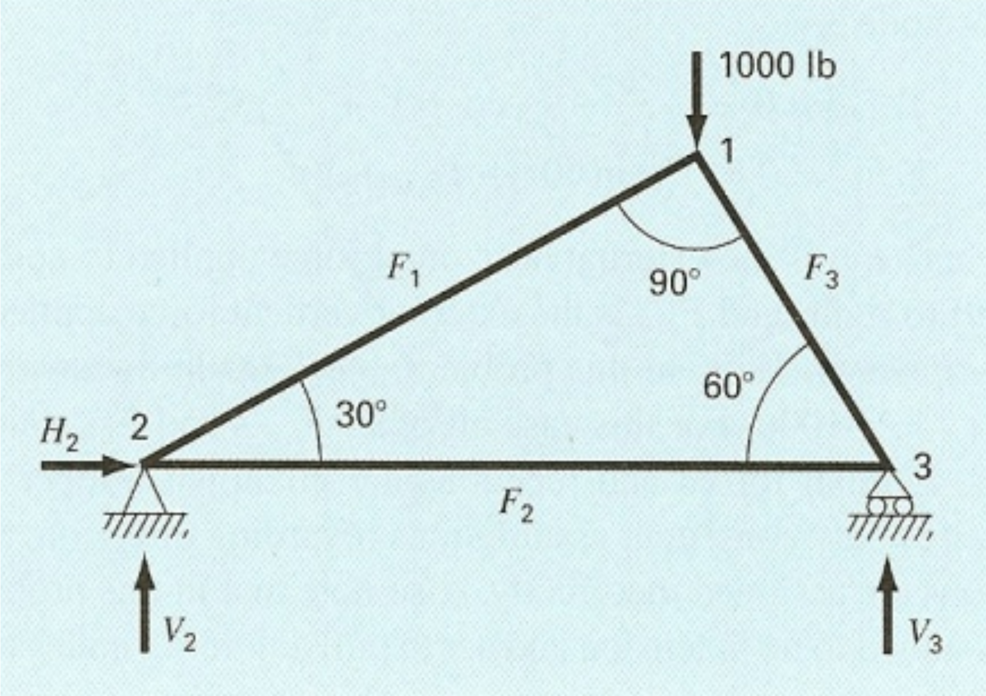
CEE 284

Session 34
Systems of Equations

Chapra, Numerical Methods;   Schowengerdt, Remote Sensing and Image Processing

Last time…

- We put together a system of equation using linear equations….

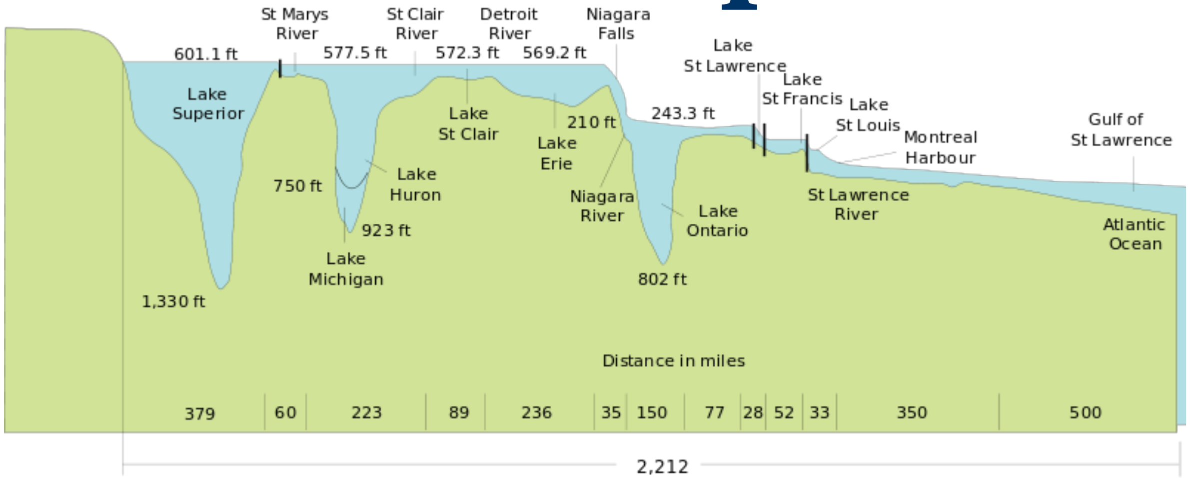


Here is another example

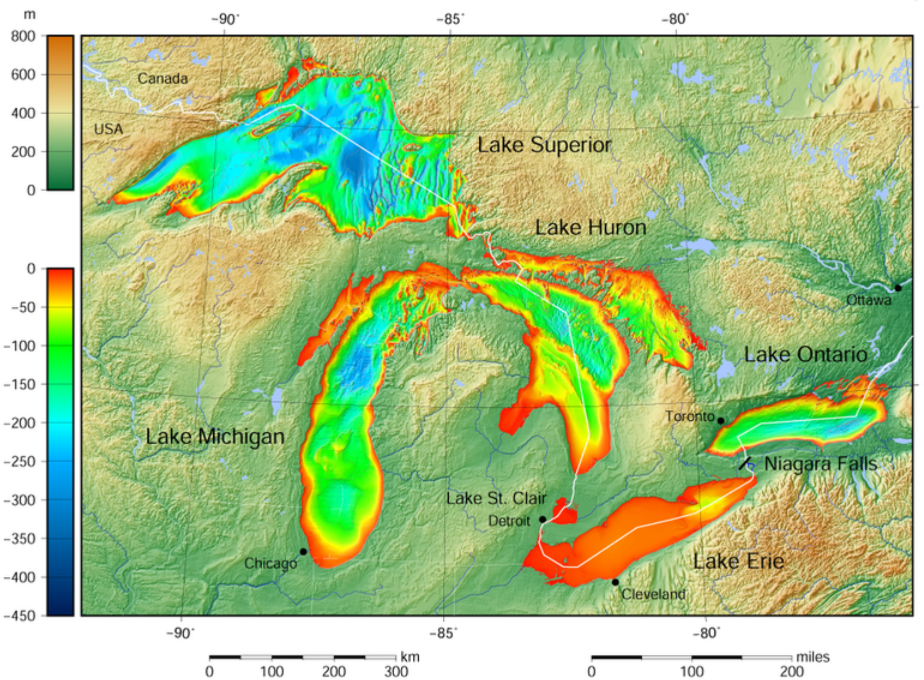
- Substance Transports
  - And a little systems modeling
  - A preview of Diff-Eq
  - And what happens when you oversolve a system…

Substance Transport

- Consider the Great Lakes system.

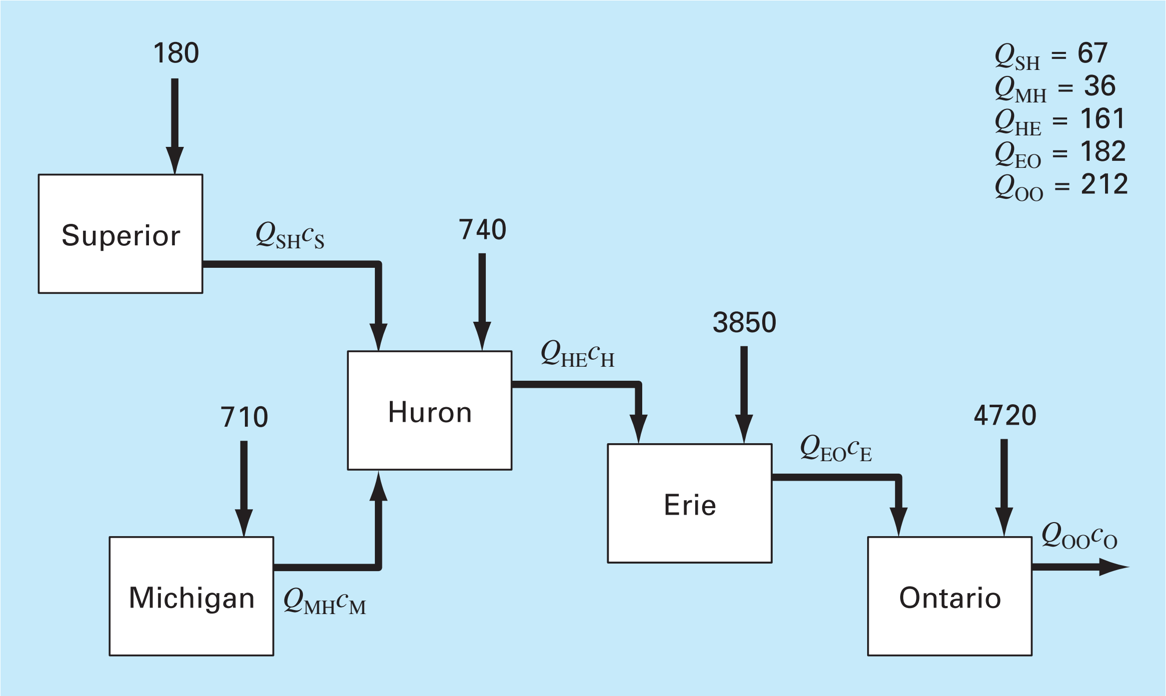


- Now consider the concentrations of a substance (chlorides) through the system.



Substance Transport

- The lakes exchange water through open flow and systems of canals that transport chlorides between them

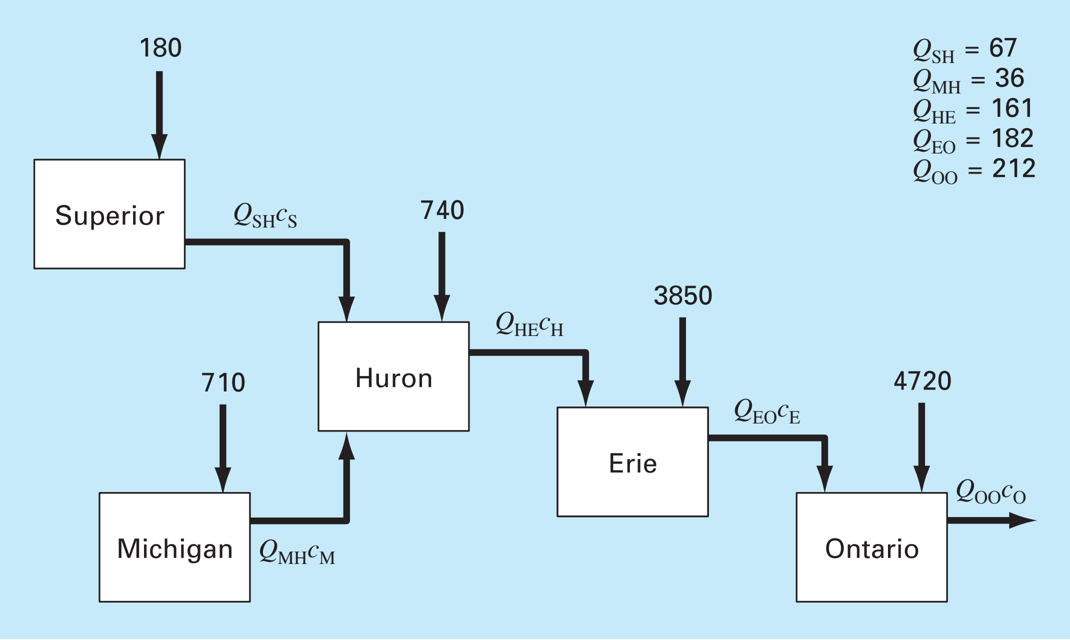


- Each lake has local inputs if chlorides

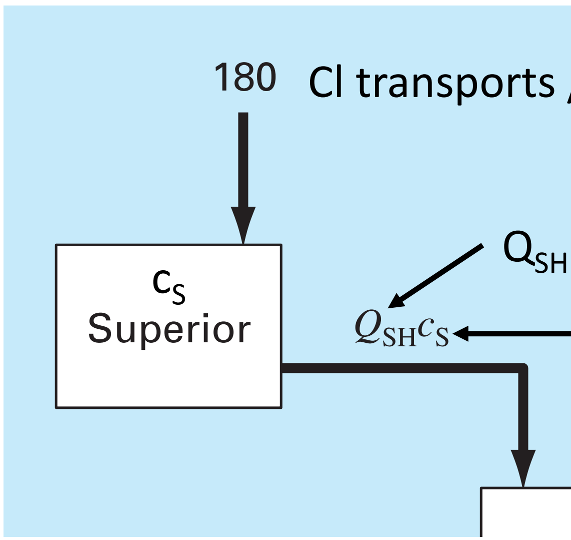
- After an extended period of time, hypothetically, the chloride concentrations in the lake system reaches a steady state (i.e., the change in the local concentrations go to zero and their levels are constant.

How to get the balance state? One lake at a time…

- Attack strategy:
  - Create a balance equation for each lake, ***do each one at a time…***
  - You know the transport rates between the lakes
    - Q_{LaLb}
  - … and their local input rate of chlorides



Budget for Lake Superior (S)



180 Cl transports / unit Time

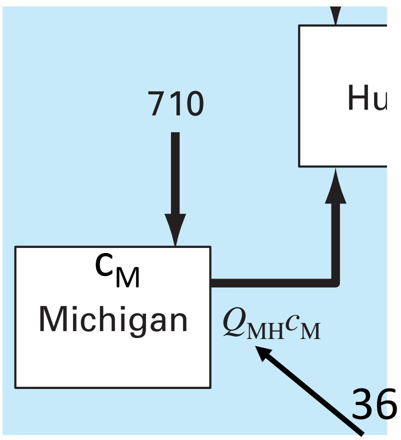
Q_{SH}=67 unit lake volume / unit Time

unit CL / per unit volume

$$\frac{dc_S}{dt} = \underbrace{\overbrace{180}^{q_S}}_{IN} - \underbrace{Q_{SH}c_s}_{OUT}$$

$$c_S(t_o + \Delta t) \approx c_S(t_o) + \frac{dc_S}{dt}\Delta t$$

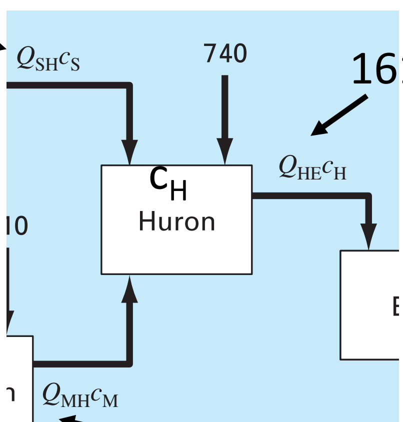
Michigan (M) & Huron (H)



$$\frac{dc_M}{dt} = \overbrace{\underbrace{710}_{IN}}^{q_M} - \underbrace{Q_{MH}c_M}_{OUT}$$

36 unit lake volume / unit Time

67 unit lake volume / unit Time

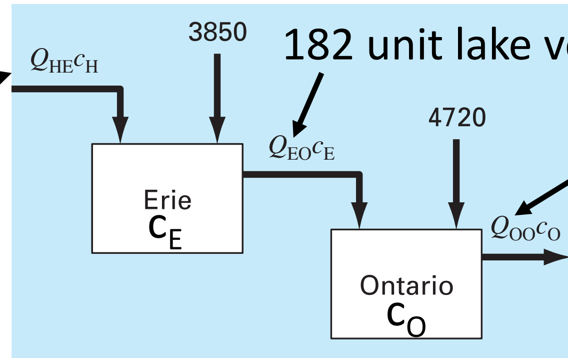


161 unit lake volume / unit Time

$$\frac{dc_H}{dt} = \underbrace{\overbrace{710}^{q_H} + Q_{SH}c_s + Q_{MH}c_M}_{IN} - \underbrace{Q_{HE}c_H}_{OUT}$$

36 unit lake volume / unit Time

Erie (E) & Ontario (O)



182 unit lake volume / unit Time

212 unit lake volume / unit Time

161
unit lake volume / unit Time

$$\frac{dc_E}{dt} = \underbrace{\overbrace{3850}^{q_E} + Q_{HE}c_H}_{IN} - \underbrace{Q_{EO}c_E}_{OUT}$$

$$\frac{dc_O}{dt} = \underbrace{\overbrace{4720}^{q_O} + Q_{EO}c_E}_{IN} - \underbrace{Q_{OO}c_O}_{OUT}$$

And all together! 5 Equations

$$\frac{dc_S}{dt} = q_S - Q_{SH}c_S$$

$$\frac{dc_M}{dt} = q_M - Q_{MH}c_M$$

$$\frac{dc_H}{dt} = q_H + Q_{SH}c_s + Q_{MH}c_M - Q_{HE}c_H$$

$$\frac{dc_E}{dt} = q_E + Q_{HE}c_H - Q_{EO}c_E$$

$$\frac{dc_O}{dt} = q_O + Q_{EO}c_E - Q_{OO}c_O$$

- We have five ***"prognostic"*** equations
- Each showing how a lake's concentration changes over time.

And all together! 5 Unknowns!

$$\frac{dc_S}{dt} = q_S - Q_{SH}c_S$$

$$\frac{dc_M}{dt} = q_M - Q_{MH}c_M$$

$$\frac{dc_H}{dt} = q_H + Q_{SH}c_S + Q_{MH}c_M - Q_{HE}c_H$$

$$\frac{dc_E}{dt} = q_E + Q_{HE}c_H - Q_{EO}c_E$$

$$\frac{dc_O}{dt} = q_O + Q_{EO}c_E - Q_{OO}c_O$$

- We have five ***unknown variables***
- This means that our system is closed
- And those we need to solve for when the system is in equilibrium
- When the change over time is zero for all of the lakes

And if you have OCD

$$\frac{dc_S}{dt} = q_S - Q_{SH}c_S$$

$$\frac{dc_M}{dt} = q_M - Q_{MH}c_M$$

$$\frac{dc_H}{dt} = q_H + Q_{SH}c_S + Q_{MH}c_M - Q_{HE}c_H$$

$$\frac{dc_E}{dt} = q_E + Q_{HE}c_H - Q_{EO}c_E$$

$$\frac{dc_O}{dt} = q_O + Q_{EO}c_E - Q_{OO}c_O$$

And if all our d/dt's are 0 (steady state)

$$\begin{bmatrix} -Q_{SH} & 0 & 0 & 0 & 0 \\ 0 & -Q_{MH} & 0 & 0 & 0 \\ +Q_{SH} & +Q_{MH} & -Q_{HE} & 0 & 0 \\ 0 & 0 & +Q_{HE} & -Q_{EO} & 0 \\ 0 & 0 & 0 & +Q_{EO} & -Q_{OO} \end{bmatrix} \begin{bmatrix} c_S \\ c_M \\ c_H \\ c_E \\ c_O \end{bmatrix} = \begin{bmatrix} -q_S \\ -q_M \\ -q_H \\ -q_E \\ -q_E \end{bmatrix}$$

$$\mathbf{Q}\,\vec{c} = \vec{q}$$

$$\vec{c} = \mathbf{Q}^{-1}\,\vec{q}$$

And on to Mathcad

Our Transport Rates

$Q_{SH} := 67 \quad Q_{MH} := 36 \quad Q_{HE} := 161$

$Q_{EO} := 182 \quad Q_{OO} := 212$

Our Local Chloride Contributions

$q_S := 180 \quad q_M := 710 \quad q_H := 740$

$q_E := 3850 \quad q_O := 4720$

$$Q := \begin{bmatrix} -Q_{SH} & 0 & 0 & 0 & 0 \\ 0 & -Q_{MH} & 0 & 0 & 0 \\ Q_{SH} & Q_{MH} & -Q_{HE} & 0 & 0 \\ 0 & 0 & Q_{HE} & -Q_{EO} & 0 \\ 0 & 0 & 0 & Q_{EO} & -Q_{OO} \end{bmatrix} \quad c = \begin{bmatrix} c_S \\ c_M \\ c_H \\ c_E \\ c_O \end{bmatrix} \quad q := -\begin{bmatrix} q_S \\ q_M \\ q_H \\ q_E \\ q_O \end{bmatrix}$$

Our Steady-State Solution Matricies (c is just for show)

$$c_{matrix} := Q^{-1} q = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

And our steady state solution

Or if you'd rather…

$$\frac{d}{dt}c_S = q_S - Q_{SH}\cdot c_S$$

$$\frac{d}{dt}c_M = q_M - Q_{MH}\cdot c_M$$

$$\frac{d}{dt}c_H = q_H + Q_{SH}\cdot c_S + Q_{MH}\cdot c_M - Q_{HE}\cdot c_H$$

$$\frac{d}{dt}c_E = q_E + Q_{HE}\cdot c_H - Q_{EO}\cdot c_E$$

$$\frac{d}{dt}c_O = q_O + Q_{EO}\cdot c_E - Q_{OO}\cdot c_O$$

Guess Values

$c_S := 0 \quad c_M := 0 \quad c_H := 0$

$c_E := 0 \quad c_O := 0$

Constraints

$$0 = q_S - Q_{SH}\cdot c_S$$
$$0 = q_M - Q_{MH}\cdot c_M$$
$$0 = q_H + Q_{SH}\cdot c_S + Q_{MH}\cdot c_M - Q_{HE}\cdot c_H$$
$$0 = q_E + Q_{HE}\cdot c_H - Q_{EO}\cdot c_E$$
$$0 = q_O + Q_{EO}\cdot c_E - Q_{OO}\cdot c_O$$

Solver

$$c_{block} := \text{find}\left(c_S, c_M, c_H, c_E, c_O\right) = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

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… and yet again…

And our steady state solution using a solve arrow... Also "root-style"

$$dCdt\left(c_S, c_M, c_H, c_E, c_O\right) := \begin{bmatrix} q_S - Q_{SH} \cdot c_S \\ q_M - Q_{MH} \cdot c_M \\ q_H + Q_{SH} \cdot c_S + Q_{MH} \cdot c_M - Q_{HE} \cdot c_H \\ q_E + Q_{HE} \cdot c_H - Q_{EO} \cdot c_E \\ q_O + Q_{EO} \cdot c_E - Q_{OO} \cdot c_O \end{bmatrix}$$

$$c_{arrow} := dCdt\left(c_S, c_M, c_H, c_E, c_O\right) \xrightarrow{solve, c_S, c_M, c_H, c_E, c_O} \begin{bmatrix} \frac{180}{67} & \frac{355}{18} & \frac{1630}{161} & \frac{2740}{91} & \frac{2550}{53} \end{bmatrix}$$

Since I don't arrays going lengthwise, I can use the transpose of my answer to make it better looking

$$c_{arrow}{}^{T} = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

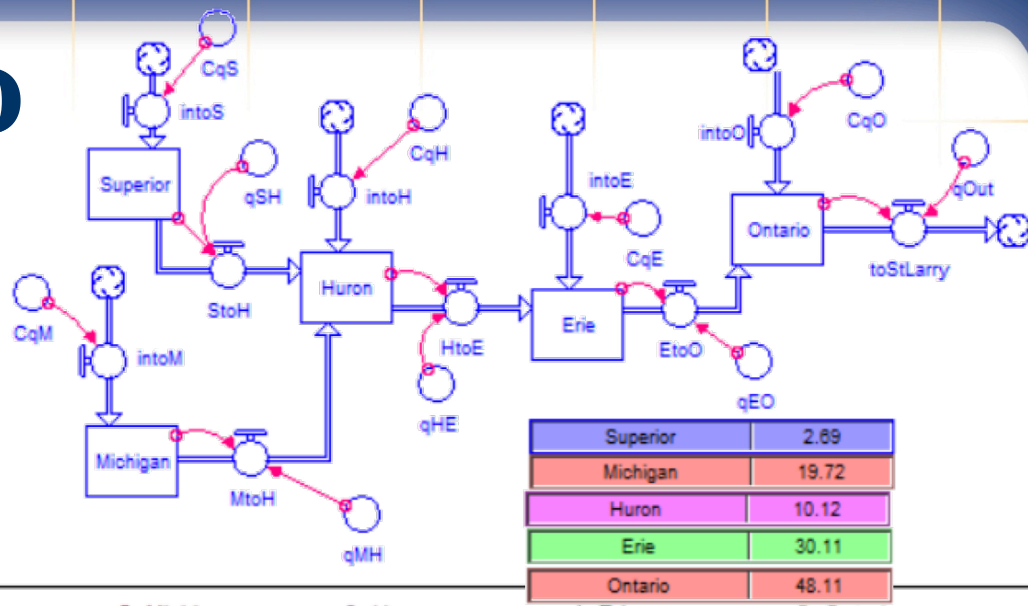
How to check our work?

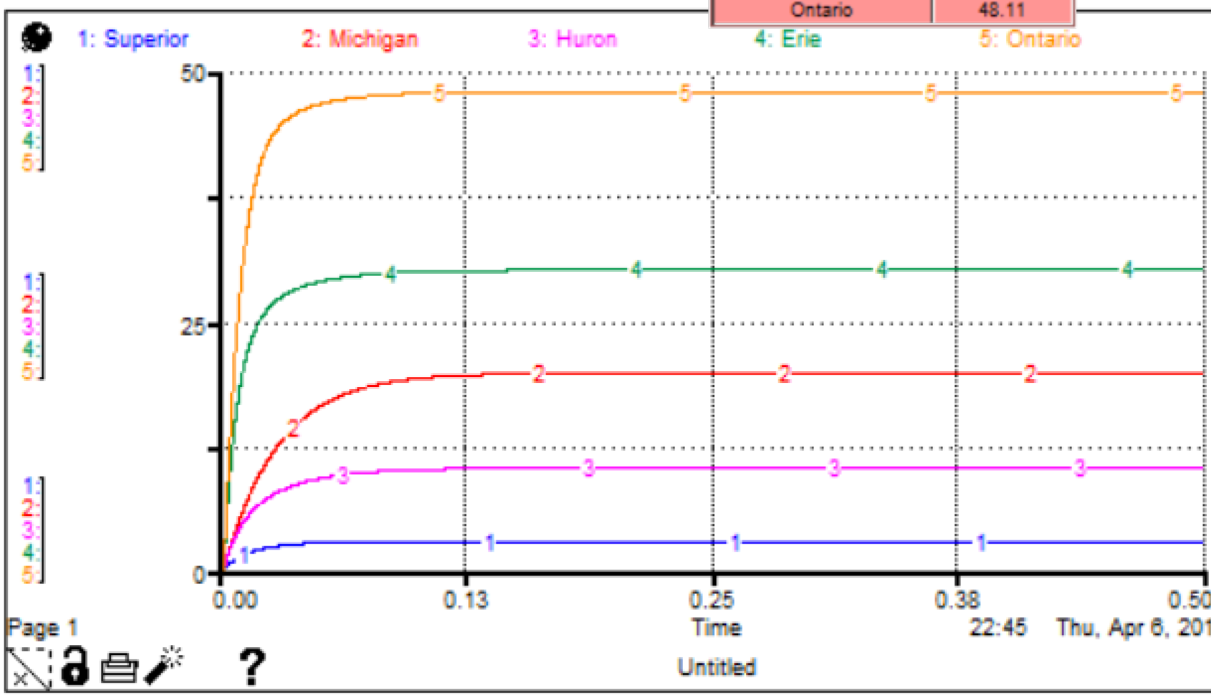
- If all went well, your plugging in your newly known unknowns into your linear equation will give you a collection of zeros.

$$Q \cdot c_{matrix} - q = \begin{bmatrix} 2.842 \cdot 10^{-14} \\ 0 \\ 2.274 \cdot 10^{-13} \\ -9.095 \cdot 10^{-13} \\ 9.095 \cdot 10^{-13} \end{bmatrix}$$

And also this…

- Same system using STELLA Systems Modeling Software





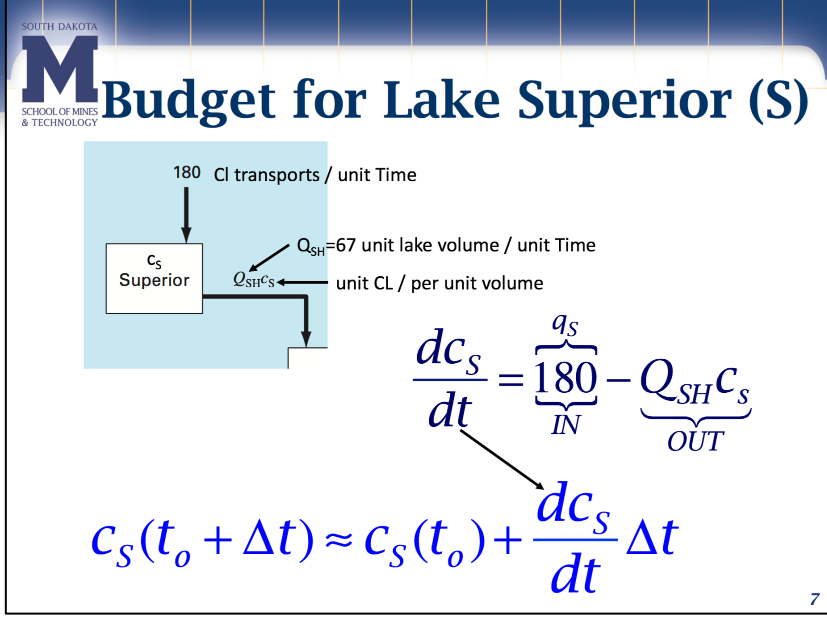
Common Directory:
F:/Dept/IAS/STELLA7

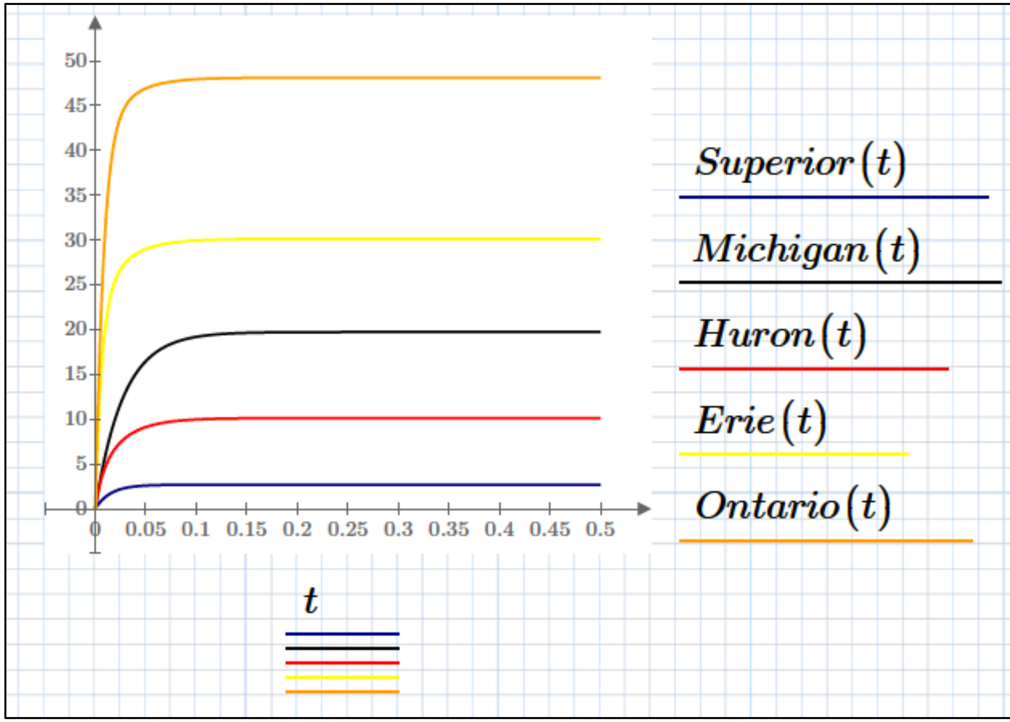
Program
Stella7.exe

File:
CEE_284__L34_Stella_Great_Lakes_Example.STM

But wait… There's more!

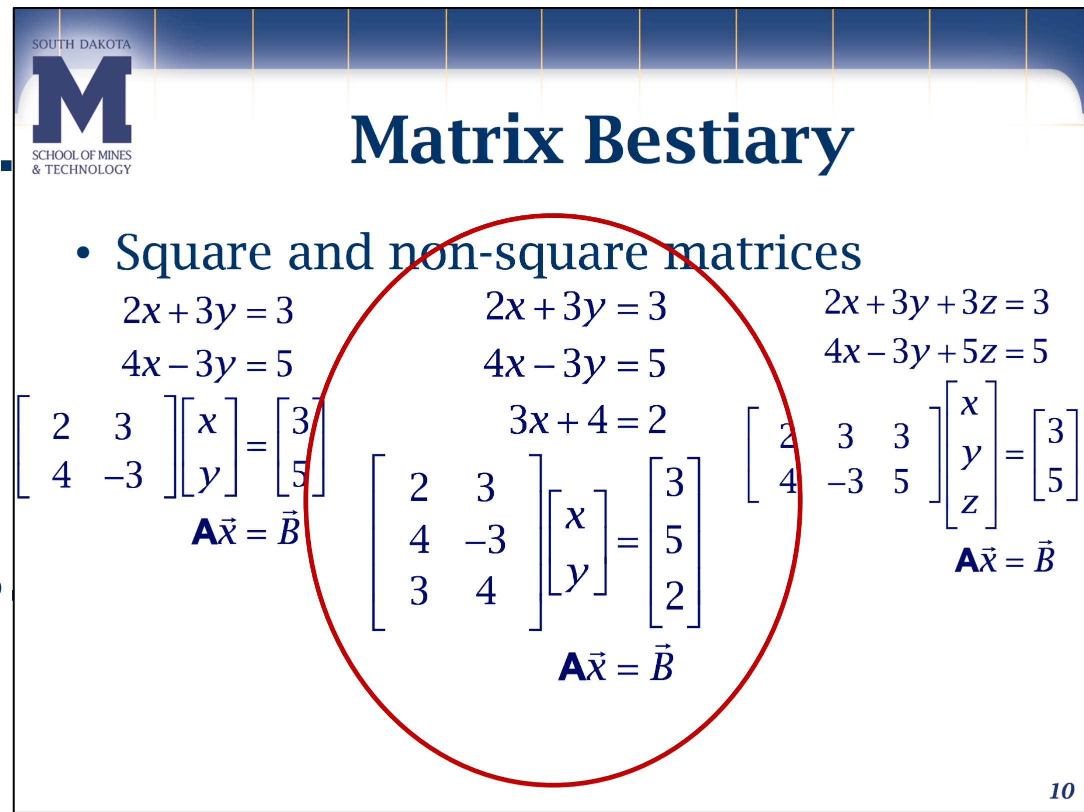
- We also have created a framework to actually watch the steady state come into being.
- For those of you in Diff Eqs (and those of you going *into* Diff Eqs consult the "sandbox" on today's D2L





OK now for some danger time

- Recall how we discussed the idea of an *oversolved matrix.*
- More equations than unknowns.
- The answer is a compromise between all the other equations.



Solving Oversolved Systems

- One way to do this is to use something called a "Pseudo Inverse"

$$\mathbf{A}\vec{x} = \vec{B}$$

$$\mathbf{A}^{\dagger} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$$

$$\vec{x} = \mathbf{A}^{\dagger}\vec{B}$$

Solving Oversolved Systems

- One way to do this is to use something called a "Pseudo Inverse"

$$A\vec{x} = \vec{B}$$

$$A^{\dagger} = (A^{T}A)^{-1}A^{T}$$

$$\vec{x} = A^{\dagger}\vec{B}$$

$$A := \begin{bmatrix} 2 & 3 \\ 4 & -3 \\ 3 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad B := \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\text{geninv}(A) = \begin{bmatrix} 0.053 & 0.162 & 0.082 \\ 0.079 & -0.117 & 0.103 \end{bmatrix}$$

$$x := \text{geninv}(A) \cdot B = \begin{bmatrix} 1.133 \\ -0.141 \end{bmatrix}$$

$$A \cdot x - B = \begin{bmatrix} -1.158 \\ -0.046 \\ 0.834 \end{bmatrix}$$

(not *quite* zero, think of this as your Sr)

Applying it to our case…

- Imagine a [bizarre] case where the whole system is forced to have 100 *total* units total of Chlorides.
- This gives is a SIXTH equation!
- But still FIVE unknowns…

$$\frac{dc_S}{dt} = q_S - Q_{SH}c_S$$

$$\frac{dc_M}{dt} = q_M - Q_{MH}c_M$$

$$\frac{dc_H}{dt} = q_H + Q_{SH}c_S + Q_{MH}c_M - Q_{HE}c_H$$

$$\frac{dc_E}{dt} = q_E + Q_{HE}c_H - Q_{EO}c_E$$

$$\frac{dc_O}{dt} = q_O + Q_{EO}c_E - Q_{OO}c_O$$

$$100 = +c_S + c_M + c_H + c_E + c_O$$

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Applying it to our case…

- And our matrix system looks now like this:

$$\begin{bmatrix} -Q_{SH} & 0 & 0 & 0 & 0 \\ 0 & -Q_{MH} & 0 & 0 & 0 \\ +Q_{SH} & +Q_{MH} & -Q_{HE} & 0 & 0 \\ 0 & 0 & +Q_{HE} & -Q_{EO} & 0 \\ 0 & 0 & 0 & +Q_{EO} & -Q_{OO} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_S \\ c_M \\ c_H \\ c_E \\ c_O \end{bmatrix} = \begin{bmatrix} -q_S \\ -q_M \\ -q_H \\ -q_E \\ -q_E \\ +100 \end{bmatrix}$$

Back to Mathcad….

$$c_{oversolved} := Q2_{pseudo} \cdot q_{2\_}and\_c_{total} = \begin{bmatrix} 2.682 \\ 19.709 \\ 10.118 \\ 30.104 \\ 48.108 \end{bmatrix}$$

$$:Q^{-1} q = \begin{bmatrix} 2.687 \\ 19.722 \\ 10.124 \\ 30.11 \\ 48.113 \end{bmatrix}$$

Close but not quite the same (the closer the total available chlorates comes to the balanced amount the less the combined error.

$$Q_2 \cdot c_{oversolved} - q_{2\_}and\_c_{total} = \begin{bmatrix} 0.336 \\ 0.474 \\ 0.176 \\ 0.109 \\ 0.051 \\ 10.72 \end{bmatrix}$$

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Next Time

Play with MATLAB!!!