Derivatives, Natural Logs and Exponential Decay

Calc 2

Atmospheric Physics, Dynamics, Chemistry, Remote Sensing … and more…

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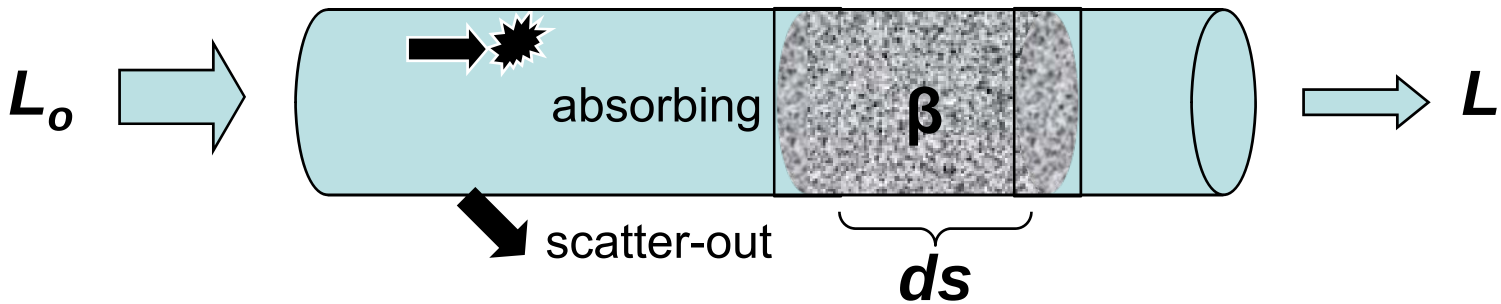
Exponential Decay

- Exponential Decay is used throughout atmospheric sciences. Some examples include
  - The slow dissipation of turbulence in the atmosphere.
  - The rate that solar energy attenuates (lessens) as it travels from the top of the atmosphere to the surface.
- Here, we'll focus on the latter

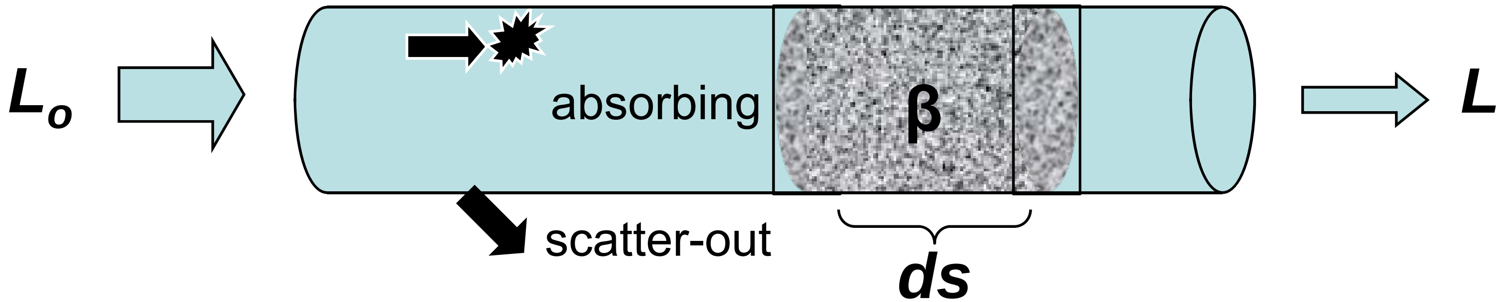
Representing Exponential Decay

- First, let's look *physically* at exponential decay.



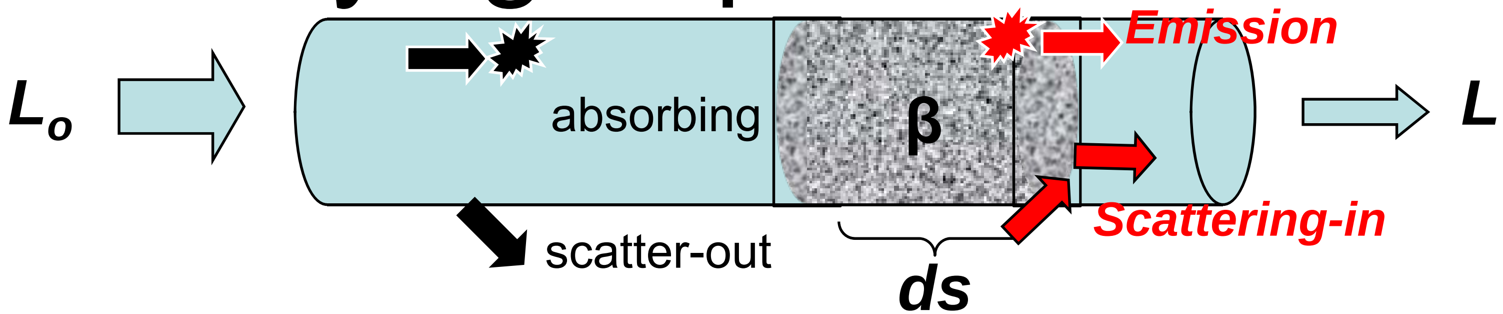
- Let's consider a "path" through a body (water, atmosphere or any other medium).
- We can characterize system by the amount of energy entering (L_o), in (L) & leaving (also L) the path, the distance along the path (ds) and amount of stuff that gets in the way, absorbing and scattering the energy (β).

Quantifying Exponential Decay



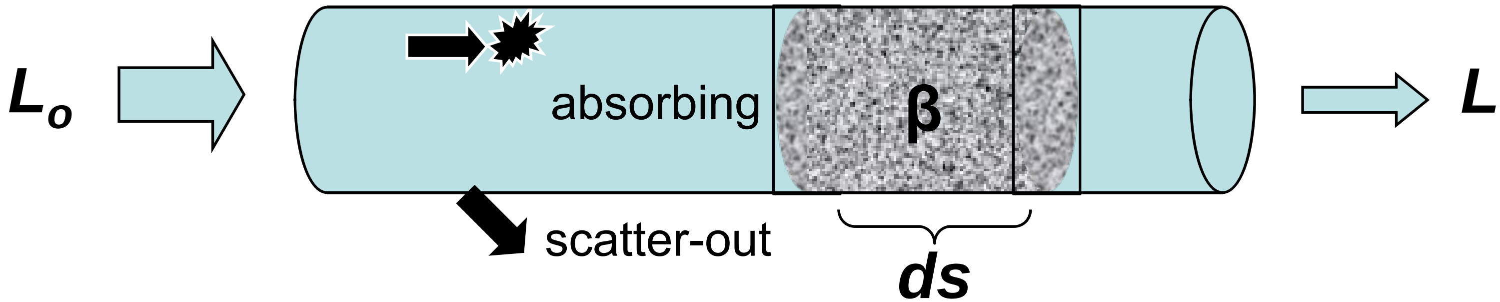
- Now let's take the first step in quantifying these units.
- Let L = Energy (for now "E" – we actually use a set of SI units for this based on watts)
- ds = length (in meters of course)
- The last one, β, is tricky. Physically, it represents the "density" of the medium with respect to the energy with which it will interact (the amount of stuff that will attenuate the beam). For this we will use a more abstract unit: amount of energy reduction per unit length (E/m).
- For simplicity, let's at least physically relate it to Stuff™ or Clutter™.

Quantifying Exponential Decay



- Let's also briefly discuss what process we are NOT including.
  - We only discuss processes that *remove* energy from the path through our cylinder.
  - We are not going to talk about energy that being "scattered-in" to the pipe.
    - When we teach this subject, we handle it separately. Normally, it goes right after we cover this part.
  - We also are not going to talk about energy that is *emitted* by material within the beam
    - this is the equivalent of the greenhouse effect and since absorption and emission are two separate processes. We also teach this separately but soon after.

Representing Exponential Decay



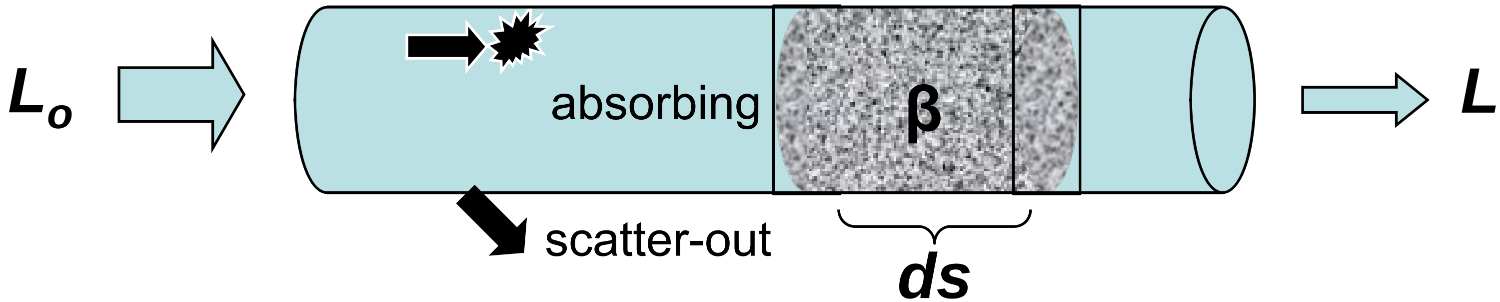
- L = Energy (well call the units for this "E")
- ds = length interval (units are length or meters, m)
- β = "The Extinction Rate (or Extinction Coefficient)" (E/m)

- Notice that we haven't created any formulas yet
- Rather, we have assembled the key processes and parameters

- Now lets think about what's going on.

1. It's fairly easy to assume that the energy leaving the "pipe" is going to be less than what came into it.
2. The rate in the reduction of energy is going to be proportional to the amount of energy at that point (L) the amount of stuff in the medium (β) and the distance along which we measure (ds).

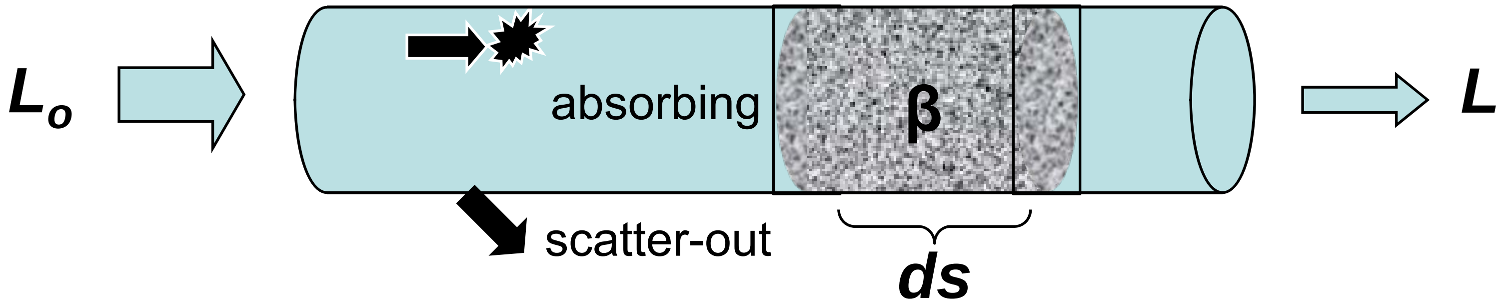
Quantifying Exponential Decay



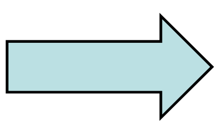
- Therefore let's start setting up an expression that shows the reduction of energy (L) over distance (s).

  - dL = ? ds

Quantifying Exponential Decay



- dL = ? ds
- But how do we fill what's in the question mark?
- Lets start by reviewing the rules we established a few slides ago.



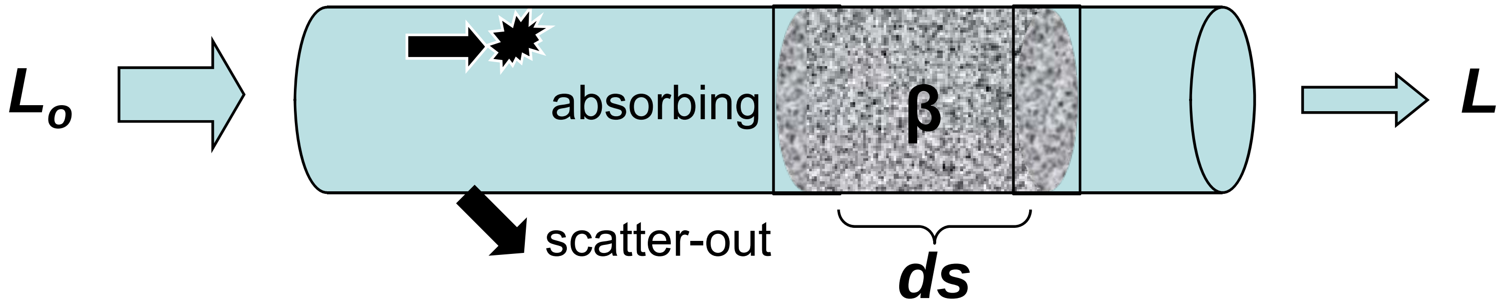
- Hint: the best way to start is to get the sign right!

The Rules

1. It's fairly easy to assume that the energy leaving the "pipe" is going to be less than what came into it.

2. The rate in the reduction of energy is going to be proportional to the amount of energy at that point (L) the amount of stuff in the medium (β) and the distance along which we measure (ds).

Quantifying Exponential Decay



- dL = ? ds

- Rule One says that we'll have a loss.

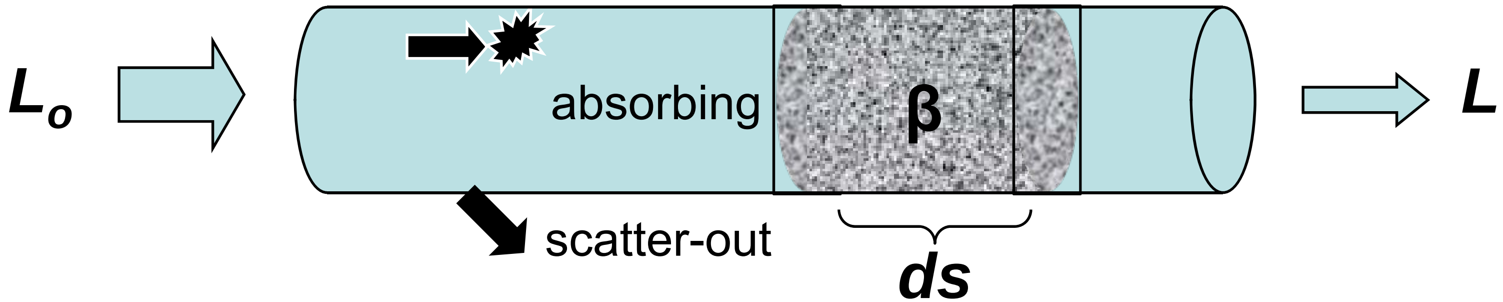
- OK … that implies that we will have a negative sign in front.

dL = – ? ds

The Rules

1. It's fairly easy to assume that the energy leaving the "pipe" is going to be less than what came into it.
2. The rate in the reduction of energy is going to be proportional to the amount of energy at that point (L) the amount of stuff in the medium (β) and the distance along which we measure (ds).

Quantifying Exponential Decay



- dL = – ? ds

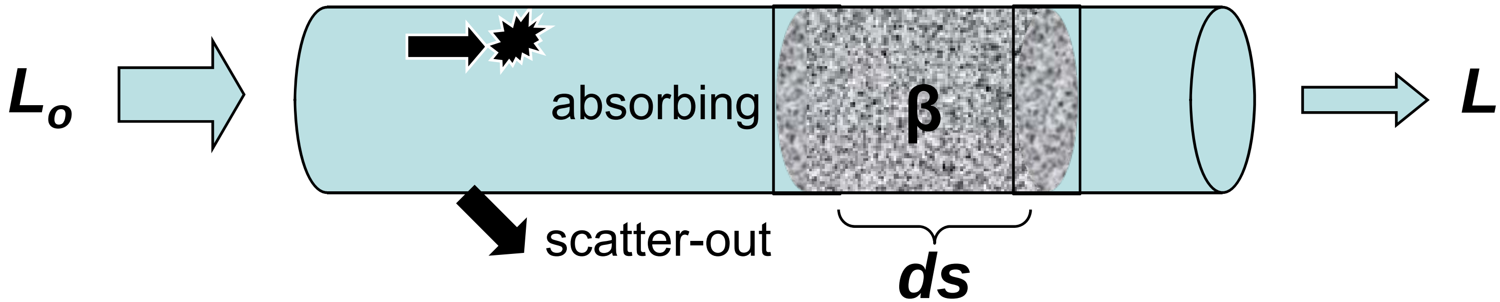
- Rule Two says that the more energy (L) moving into the "pipe", the greater the magnitude of loss. I.e., you can't take something from nothing.
- Likewise, the more scatters (ds) and the greater their ability to scatter (β), the greater the loss.
- That, in turn, implies the following

$$dL = -L\,\beta\,ds$$

The Rules

1. It's fairly easy to assume that the energy leaving the "pipe" is going to be less than what came into it.
2. The rate in the reduction of energy is going to be proportional to the amount of energy at that point (L) the amount of stuff in the medium (β) and the distance along which we measure (ds). (Because if nothing's coming in, nothing's coming out anyway!)

Quantifying Exponential Decay



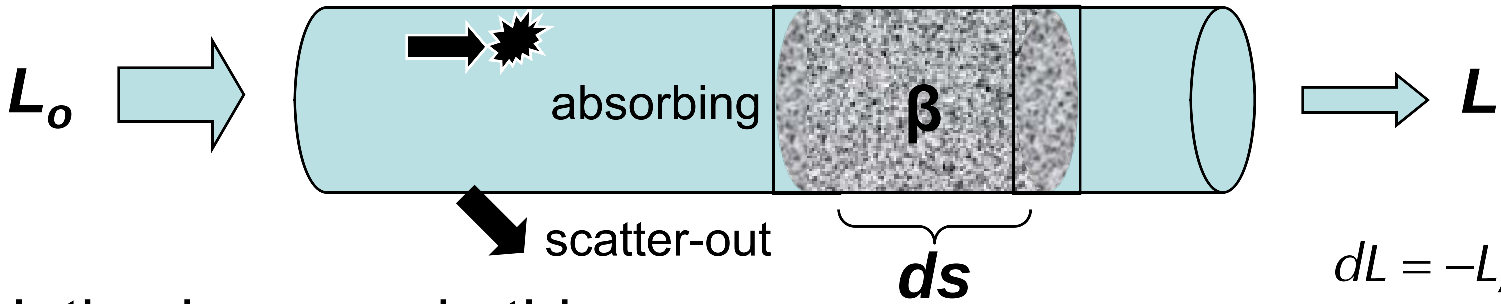
- This creates the relationship dL = -L βds
  or with some quick algebra -> $\frac{dL}{L} = -\beta ds$
- If you have gone far enough in Calc 2, you should recognize this as a setup for the natural logarithm
- We also call this relationship in radiative transfer, “Beer’s Law”
  - Or to recognize all the people who developed the principle at the same time, the “Beers-Buoyott-Lambert” Law. But we often just call it “Beer’s Law”

Intermission

- The approach here also demonstrates how to construct other relationships that leverage exponential growth and decay such as population models
- It also demonstrates how to quantitatively merge the concepts of a SWAG* into a numerical estimation.
- This is an incredibly useful skill that you should add to your problem assault skill toolbox!

*Scientific Wild … Guess

Quantifying Exponential Decay



$$dL = -L\beta ds$$

- Notice how we do this
  - We get the L's on the same side (where you see the setup for the natural log!)
  - When integrating BOTH sides of the equation – we do it in parallel. As rule, we "start" at the bottom of the integral sign and "end" at the top!
  - The Left Hand Side (LHS) has us starting with our input amount of radiation (L_o). On the Right Hand Side (RHS), we start at the beginning of the path (a distance of 0), and at the end, we are at where were want to measure our outgoing radiation, L(s).
  - The s' in the ds' keeps us honest. In the real atmosphere as you go up, β changes, so it would appear in the formula as function of distance (or height): β(s')ds', when put into the solution integral. That way we don't confuse our … ahem… 's's later on.

$$\frac{dL}{L} = -\beta ds$$

$$\int_{L_o}^{L(s)} \frac{dL}{L} = -\int_0^s \beta ds'$$

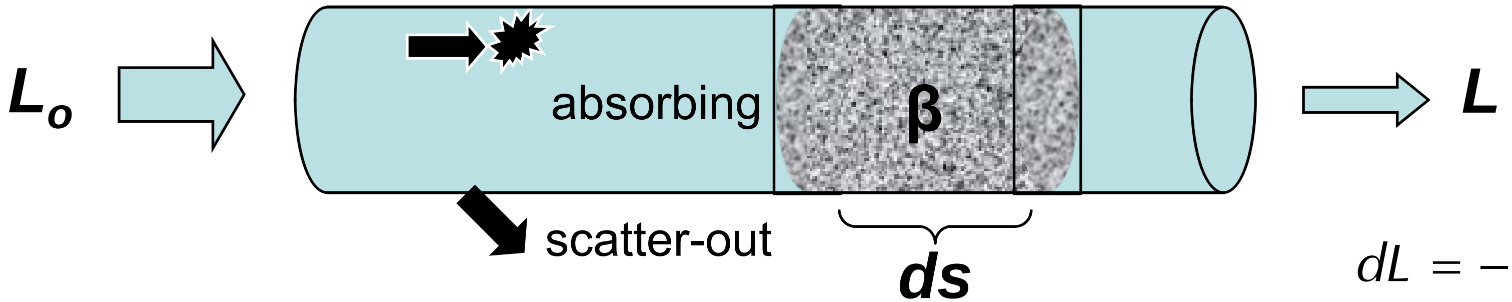
$$\ln \frac{L(s)}{L_o} = -\beta s$$

$$\exp\left[\ln \frac{L(s)}{L_o}\right] = \exp[-\beta s]$$

$$\frac{L(s)}{L_o} = e^{-\beta s}$$

$$L(s) = L_o e^{-\beta s}$$

Quantifying Exponential Decay



- Once we solve the integral, we have a ℓn on the LHS but we want to have an expression that says L(s) = *whatever.*
- Therefore we use the inverse of the natural log, the exponential, to cancel out the ℓn.
- Notice when the equations get messy, we use the function notation, exp(), rather than the raising things to a power of "e". We have to do this when we write computer programs to solve these problems! The same goes for spreadsheets
- And from there, we solve it with a little algebra!

$$dL = -L\beta ds$$

$$\frac{dL}{L} = -\beta ds$$

$$\int_{L_o}^{L(s)} \frac{dL}{L} = -\int_0^s \beta ds'$$

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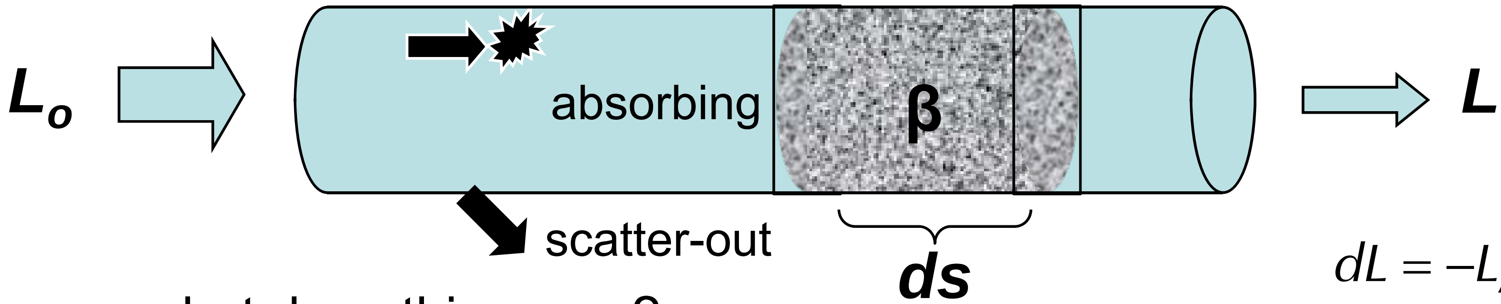
Quantifying Exponential Decay



- No now, what does this mean?
- Basically, it means that we have an exponential decay of energy as it passes through a medium.

$$dL = -L\beta ds$$

$$\frac{dL}{L} = -\beta ds$$

$$\int_{L_o}^{L(s)} \frac{dL}{L} = -\int_0^s \beta ds'$$

$$\ln\frac{L(s)}{L_o} = -\beta s$$

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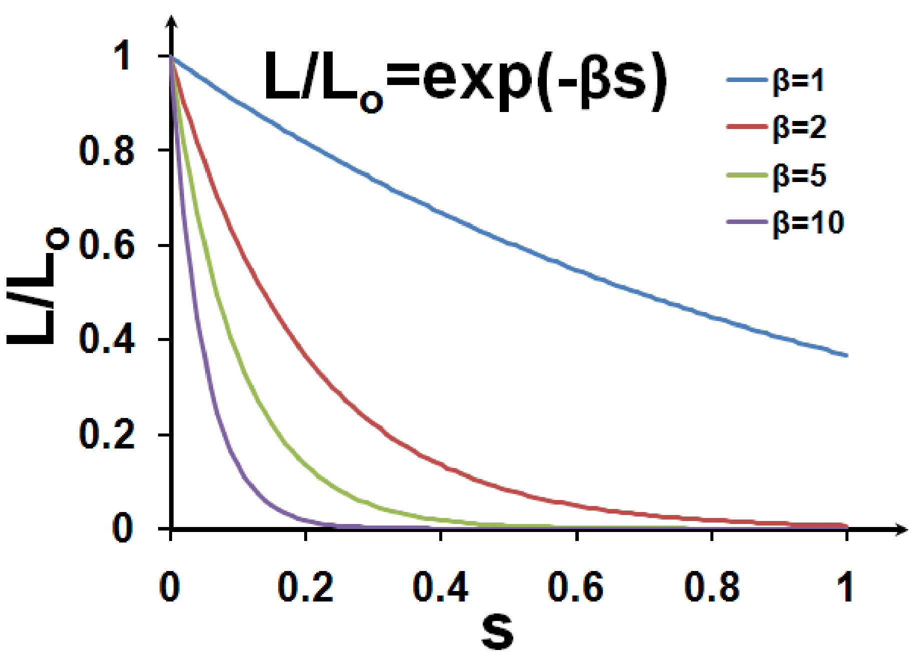
Relative decay of radiation over distance for various levels of "soupiness" As told by β



Another commonly used example of exponential decay and natural log

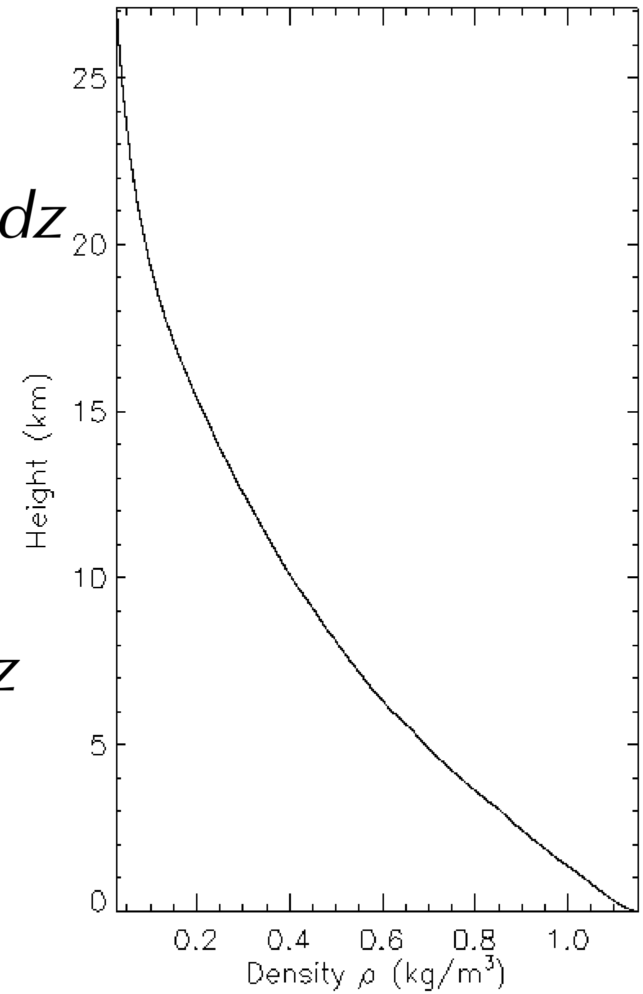
- The other major use of exponential decay and the natural log is the reduction of density and pressure with height
- Also notice the alternative notation for the differential on the LHS that emerges. This is common notation in many meteorology derivations
- It also demonstrates the use of the “ln-p” vertical coordinate that is comparable to the “z” height coordinate, where plots in both spaces have a similar “look-and-feel.”
- Try this on your own

$$dp = -g\rho dz$$

$$dp = -g \underbrace{\left[\rho = \frac{p}{RT} \right]}_{\text{Equation of State}} dz$$

$$\frac{dp}{p} = -\frac{g}{RT(z)} dz$$

$$d(\ln p) = -\frac{g}{RT(z)} dz$$



More Questions

- Talk to:
  - Dr Helsdon and Dr Detwiler: Atmopsheric Physics and Thermo
  - Dr Capehart: Remote Sensing