Assaulting Time-Dependent Problems with Mathcad and a little Diff-E-Fu.

Using USCS Units
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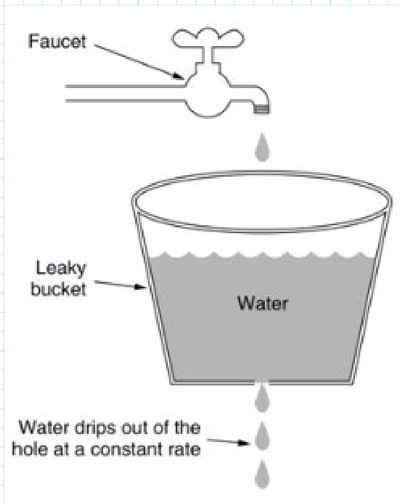
When working with advanced programming on some coursework you will need to some of the more advanced features in Mathcad.

You can explore some of this Chapter 13 of the Mathcad text. Today we are going to explore time dependent functions similar to what we did with the Great Lakes case.

Let's start with a classic case: the Leaky Bucket Problem. (The Great Lakes Case was actually a series of Leaking Buckets).

Consider a bucket with a hole at the bottom.

- We pour water into the bucket at a constant rate.
- The water flows out at a rate that includes the size of the hole and also the weight of the water in the bucket.



We can make a simple inflow-outflow budget to represent the volume (V) of the bucket (we can assume that the bucket is an upright cylinder)

$$\frac{d}{dt}V = q_{in} - q_{out}$$

qin can be set to a constant for now.

qout is represented by the following equation you may recognize it as Torricelli's Law.

$$q_{out} = \sqrt{2\ g \cdot h} \cdot A_h$$

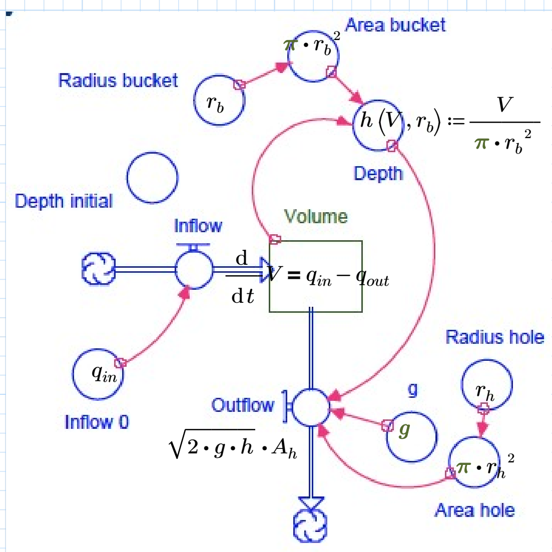
Where g is gravity, h is the height of water in the budget, and Ah is (the area of the hole).

We can if we have a radius of the bucket (rb) and radius of the hole (rh) we can fill in the equation...

$$V_h(h, r_b) := \pi \cdot r_b^{\ 2}\ h$$

$$h(V, r_b) := \frac{V}{\pi \cdot r_b^{\ 2}} \qquad A_h(r_h) := \pi \cdot r_h^{\ 2}$$

Also using symbolic cartoons (we do this in Systems Modeling) we can sketch out our system's relationships ------------------------------>



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If we just apply a little Algebra-Fu (and we'll use Mathcad to do the work because... at Mathcad...), we can create a working equation to represent our system. (A close eye will show that I'm cheating with bold equals and a text equal... right now we're just making the equation...

$$\frac{d}{dt}V = q_{in} - q_{out}$$

$$\frac{d}{dt}V \quad = \quad q_{in} - \sqrt{2 \cdot g \cdot h(V, r_b)} \cdot A_h(r_h) \xrightarrow{simplify} q_{in} - r_h^{\ 2} \cdot \sqrt{\frac{2 \cdot \pi \cdot V \cdot g}{r_b^{\ 2}}}$$

I like to keep my equations organized with my independent values separated from any constants so I am going to do some very basic algebra-fu to the right-hand-side of the equation

$$\frac{d}{dt}V(t) = q_{in} - \frac{r_h^{\ 2}}{r_b} \cdot \sqrt{2 \cdot \pi \cdot g} \cdot V(t)^{\frac{1}{2}}$$

Stripping things down, this gives a general form of

$$\frac{d}{dt}V(t) = c_1 - c_2 \cdot V(t)^{\frac{1}{2}}$$

Depending on your Math Prof, you may be asked to write it like this:

$$\frac{d}{dt}V(t) - q_{in} + \frac{r_h^{\ 2}}{r_b} \cdot \sqrt{2 \cdot \pi \cdot g} \cdot V(t)^{\frac{1}{2}} = 0$$

In any respect you now have a time-dependent differential equation

So let's solve a classic Differential Equations Problem.

Let's assume that an empty bucket is 15 inches high, 12 inches in diameter and hole is half-an-inch in diameter. Let's start filling the bucket at a rate of 10 gal/min.

$$r_h := 0.5\,\frac{in}{2} \qquad r_b := 12\,\frac{in}{2} \qquad h_b := 15\ in \qquad g := g = 32.174\,\frac{ft}{s^2} \qquad q_{in} := 10\,\frac{gal}{min}$$

So for today, let's ask this:

1. At what volume does the bucket's inflow = outflow?
2. Will the bucket overflow. (Hint, the answer will be yes because otherwise we will have to stop having fun)?
3. If it does overflow... when will it spill over?

So how do we attack this problem? Note that we are using a mixture of units so some of our ways to go after this problem will be off limits.

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1) Get the equilibrium [Steady State] Volume of between the qin and qout.

If we put our system together neatly we have the following formula from above.

$$\frac{d}{dt}V(t) = q_{in} - \frac{r_h^2}{r_b} \cdot \sqrt{2 \cdot \pi \cdot g} \cdot V(t)^{\frac{1}{2}} \quad \text{where} \quad \frac{d}{dt}V(t) = 0$$

Notice that I have writen it as a "root equation". So good, this should be in our confort zone. Let's use a solve block!

Guess Values

$V := 0\ gal$ First Guess for V

$r_b := 12\ \frac{in}{2}$ $q_{in} := 10\ \frac{gal}{min}$ $r_h := 0.5\ \frac{in}{2}$ our other parameters

Constraints

$0 = q_{in} - r_h^2 \cdot \sqrt{\frac{2 \cdot \pi \cdot V \cdot g}{r_b^2}}$ Root Equation valid when dVdt(t) == 0

Solver

$V_{eq} := \text{find}(V) = 24.377\ gal$ And we close the deal

2) Is that bucket going to overflow?

That should be easy... what's the bucket capacity? $V_h(h_b, r_b) = 7.344\ gal$

And again, our equilibrium capacity where qin=qout? $V_{eq} = 24.377\ gal$

$$(V_h(h_b, r_b) > V_{eq}) = 0$$
$$(7.344\ gal > 24.377\ gal) = 0$$

for logical [boolean] operators, 1=true, 0=false[

Yes, it *is* going to overflow...

3) WHEN does it overflow?

Now it gets fun: We are going to need our time-dependent relationship for V(t)

$$\frac{d}{dt}V(t) = q_{in} - \frac{r_h^2}{r_b} \cdot \sqrt{2 \cdot \pi \cdot g} \cdot V(t)^{\frac{1}{2}}$$

This is a non-linear ordinary differential equation. Solving it by hand is a nasty affair, but we are living in the amazing futuristic 21st century and while we don't have the flying cars we were promised we DO have Mathcad.

There are a couple functions that will do this for us but the easist one is going to be the ODESOLVE() function. We can use it in a solve block as in the example below.

We WILL need to know a good guess for when it overflows. Since it overflows at 7.344 gal, and it's filling at 10 gal per minute, accounting for the hole, let's take our guessing range out to 1.5 minutes. We can always change it if we've over/underestimated. Don't overshoot too far though.

Guess Values

$t_0 := 0\ min \quad t_f := 1.5\ min$

$r_b := 12\ \frac{in}{2} \quad q_{in} := 10\ \frac{gal}{min} \quad r_h := 0.5\ \frac{in}{2}$

You don't really need these items here but if you don't use the "guess value" part of the solve block can get really big and look messy

Constraints

We need our differential equation...

$$\frac{d}{dt}V(t) - q_{in} + \frac{r_h^{\ 2}}{r_b} \cdot \sqrt{2 \cdot \pi \cdot g} \cdot V(t)^{\frac{1}{2}} = 0$$

...and a *known* boundary condition for a given time
(we're starting with an empty bucket so the volume is 0 at t=t0)

$V(t_0) = 0\ gal$

For ODESOLVE(), the arguments are the variable for which you want to solve (the first argument), and the final time extent starting from zero.

Solver

$v := \text{odesolve}\left(V(t), t_f\right)$ I'm explorting to the newly made complex function"v")

Unfortunately if you want to look at v(t), Mathcad won't oblige.
But v(t)'s behavior is embedded in the object.

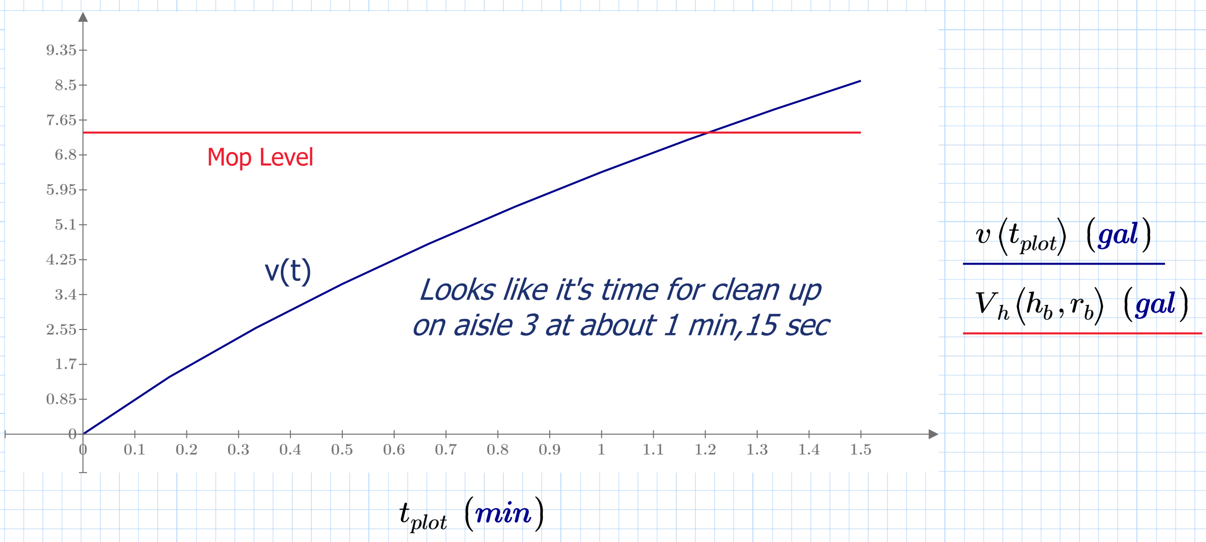
$v(t) \rightarrow$ "Symbolic result is an invalid Prime expression"

But we CAN access the contents of v(t) by ...

$v(60\ sec) = 6.383\ gal$ ...Involing it like any other function

...Plotting it...
we'll need to give it a period to plot through

$t_{plot} := 0\ s, 10\ s .. 90\ s$



...and... as a traditional solve block!

Guess Values

$t := 0\ s$

(notice I'm using just setting my logical test to be true when we reach the volumn of a full bucket)

Constraints

$v(t) = V_h(h_b, r_b)$

Solver

$t_{spill} := \text{find}(t) = 72.293\ sec$

Bucket should spill over at 1 minute, and 12 seconds

And now to our next step: Modeling

We can apply this equation to a more brute force approach to representing our systems. The method below is Euler's Method. It works as follows and it should look familiar to you.

$$dVdt(V) := q_{in} - \frac{{r_h}^2}{r_b} \cdot \sqrt{2 \cdot \pi \cdot g} \cdot V^{\frac{1}{2}}$$

Our Volume Rate of change as a function fo Volume.

And a function that should give you a case of Déjà-Fu... uh.. Vu.

$$V_{euler}(dVdt, V_0, t_0, t_f, \Delta t) := \left\| \begin{array}{l} N \leftarrow \dfrac{(t_f - t_0)}{\Delta t} + 1 \\ V_{out} \leftarrow V_0 \\ t \leftarrow t_0 \\ \text{for } n \in 2..N \\ \quad \left\| \begin{array}{l} t \leftarrow t + \Delta t \\ V_{out} \leftarrow V_{out} + dVdt(V_{out}) \cdot \Delta t \end{array} \right. \\ \text{return } V_{out} \end{array} \right.$$

$v(72\ sec) = 7.322\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 0.5\ sec) = 7.337\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 1\ sec) = 7.353\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 2\ sec) = 7.386\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 3\ sec) = 7.42\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 4\ sec) = 7.455\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 6\ sec) = 7.528\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 8\ sec) = 7.605\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 9\ sec) = 7.644\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 12\ sec) = 7.767\ gal$

$V_{euler}(dVdt, 0\ gal, 0\ sec, 72\ sec, 18\ sec) = 8.036\ gal$

As with our PI pseudocode, the more increments we use the more accurate the solution

This is the method we will use in our next section where we will explore the interaction of several interacting "stocks," or "variables,"

... or as we will explore:

sick, healthy and recovered people