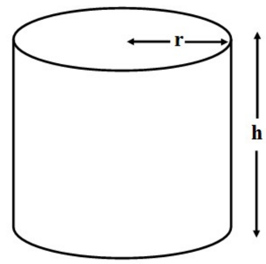
Error Propagation Sandbox

Let's consider a simple formula from our days in Mrs. Gentry's Geometry Class.

Our volume is a function of two dependant variables: Height and Radius. (Notice also that the Radius is Squared). [Actually, we can count *three* if we presume that we are rounding pi to values they felt safe to let use in Middle School or Junior High.]

$$V = \pi r^2 h$$



Let's draft a formula for Volume:

$$V(r,h) := \pi \cdot r^2\ h$$

Let's also presume that we have an uncertainty our measurements of r and h.

We'll derive that error by assuming that we are using a simple ruler that is accurate to a quarter of an inch. (And I'll throw in a rounding error of PI as 3.14).

$$\Delta h := \frac{0.25}{2}\ in \qquad \Delta r := \frac{0.25}{2}\ in \qquad \Delta \pi := |3.14 - \pi| = 0.002$$

If follows, that if we have uncertainty in our measured values height and radius, those errors will "propagagte" into our volume calculations. Now to quantify these this propogation we are going to use the Swiss Army Knife of Numerical Methods: The Taylor Series:

$$f(x) =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

The good news is that we are only going to be working as far as the "first-order" term. We will assume here that "x" above is the real value of our dependant variable. While "a" is our estimate. Our measurement error is (x-a). With a little bit of "Algebra Fu" and only using the zero order and first order terms (and ignoring signs...

$$f(x_{true}) = f(x_{est}) + \frac{df(x_{est})}{dx}\underbrace{(x_{true} - x_{est})}_{\Delta x}\ldots\{\text{we ignore the higher order terms}\}$$

$$\underbrace{|f(x_{true}) - f(x_{est})|}_{\Delta F} = \left|\frac{df(x_{est})}{dx}\underbrace{(x-a)}_{\Delta x}\right|$$

$$\Delta f = \frac{df(x)}{dx}\Delta x$$

Now at this point, you may notice that we have more than one dependant variable. Not to worry, there is a Taylor Expansion for that too. And not to worry again, since as with the single variable case, we won't use all of it!

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \left[f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y\right] + \frac{1}{2!}\left[(\Delta x)^2 f_{xx}(x,y) + 2\,\Delta x\,\Delta y\, f_{xy}(x,y) + (\Delta y)^2 f_{yy}(x,y)\right] + \frac{1}{3!}\left[(\Delta x)^3 f_{xxx}(x,y) + 3(\Delta x)^2\,\Delta y\, f_{xxy}(x,y) + 3\,\Delta x\,(\Delta y)^2 f_{xyy}(x,y) + (\Delta y)^3 f_{yyy}(x,y)\right] + \ldots$$

[the subscripts on the "f's" are the derivatives by variable (fx is the first derivative of f by x, fxy is the second derivative of f by x and y, etc...)]

Before you panic, not to worry again... we are only taking the **zero** and **first** order terms of the taylor series. (And for higher numbers of variables the first two terms on the Right Hand Side follow the same model as the above case).

$$f(x_{true},y_{true}) = f(x_{est},y_{est}) + \frac{df(x_{est},y_{est})}{dx}\underbrace{(x_{true}-x_{est})}_{\Delta x} + \frac{df(x_{est},y_{est})}{dy}\underbrace{(y_{true}-y_{est})}_{\Delta y}$$

$$\underbrace{\left|f(x_{true},y_{true}) - f(x_{est},y_{est})\right|}_{\Delta F} = \left|\frac{df(x_{est},y_{est})}{dx}\underbrace{(x_{true}-x_{est})}_{\Delta x} + \frac{df(x_{est},y_{est})}{dy}\underbrace{(y_{true}-y_{est})}_{\Delta y}\right|$$

$$\Delta f = \frac{df(x,y)}{dx}\Delta x + \frac{df(x,y)}{dy}\Delta y$$

So let's go back to our original volume problem (and throwing out some values for h & r...

$$\Delta V = \Delta r\frac{dV(r,h,[\pi])}{dr} + \Delta h\frac{dV(r,h,[\pi])}{dh}\left[+\Delta\pi\frac{dV(r,h,[\pi])}{d\pi}\right]$$

We get the below values for the base-value for Volume and our potential error! where r = 10 inches and h = 5 inches

$$V(r,h) := \pi \cdot r^2\ h \qquad \Delta V(r,h) := \frac{d}{dr}V(r,h)\cdot\Delta r + \frac{d}{dh}V(r,h)\ \Delta h$$

$$V(10\ in, 5\ in) = (1.571\cdot 10^3)\ in^3 \qquad \Delta V(10\ in, 5\ in) = 78.54\ in^3$$

On your own, do this by hand in your notebook by hand. Also can you add pi? (You'll need to patch the volume formula and use the "symbol" value of Pi rather than the constant.) .

Now let's consider how our Volume's error propagation changes with its dependant values and also relative to the Volume itself. We call the latter, "Sensitivity."

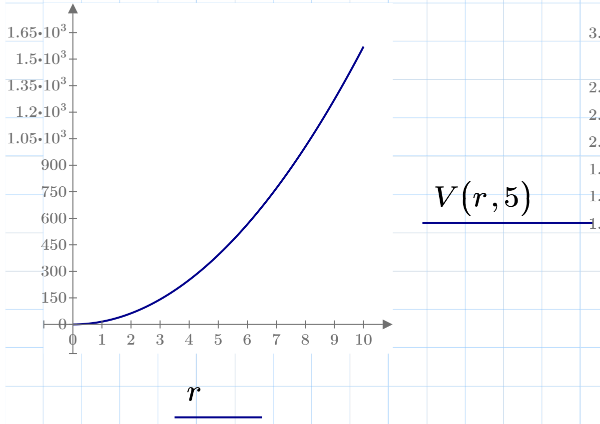
$$S(r,h) := \frac{\Delta V(r,h)}{V(r,h)} \qquad S(10\ in, 5\ in) = 0.05$$

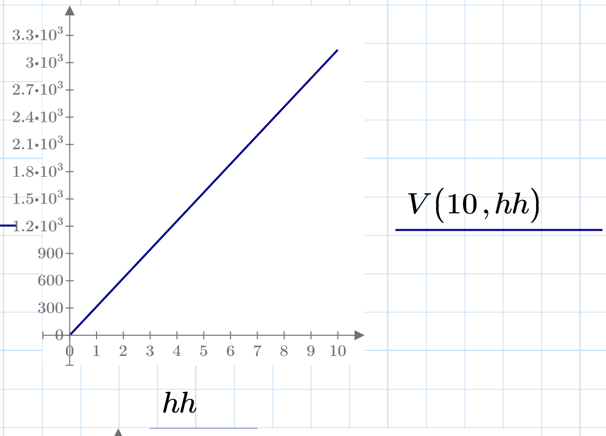
We can also graph this and for simplicity we are removing any units as they can be problematic with some of Mathcad's graphing features.

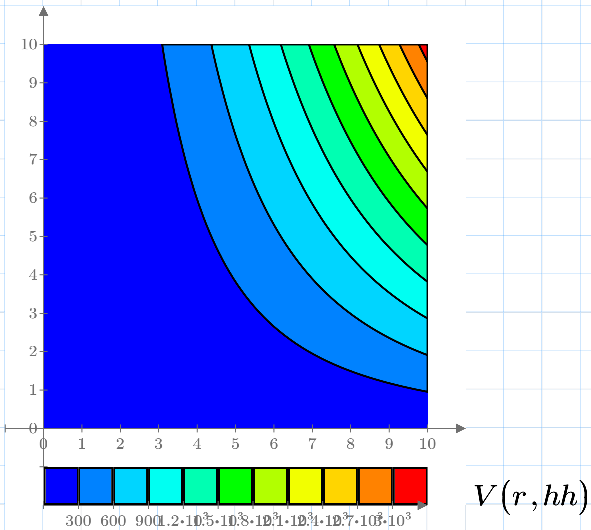
$\Delta h := 0.125$

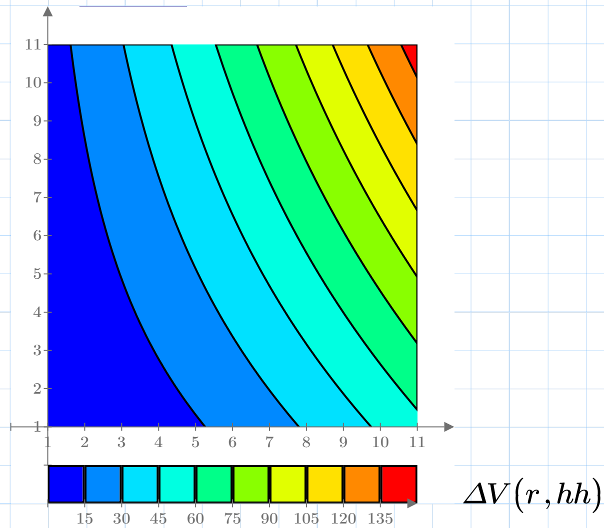
$\Delta r := 0.125$

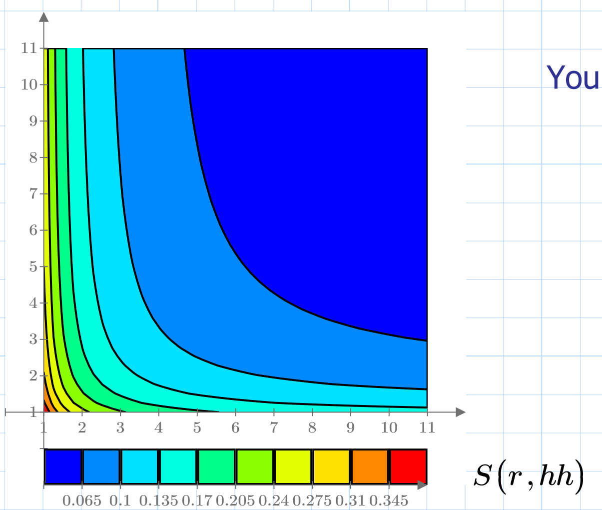
$$\Delta V(r,h) := \frac{d}{dr} V(r,h) \cdot \Delta r + \frac{d}{dh} V(r,h)\ \Delta h \qquad S(r,h) := \frac{\Delta V(r,h)}{V(r,h)}$$











You can find graph methods in Chapter 8

Notice that the overall sensitivity is greatest at the low values of measurements of r and h