Causal Set Research Notes

Will Cunningham

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1 Patch Volume

1.1 1+1 Dimensions

$$\begin{aligned} \mathrm{d}V &= (a\mathrm{sec}\eta)^{\mathrm{d}+1}\,\mathrm{d}\eta\,\mathrm{d}\Phi_d \\ &= a^2\mathrm{sec}^2\eta\,\mathrm{d}\eta\,\mathrm{d}\theta \end{aligned} \tag{1}$$

$$\begin{aligned} V &= a^2\int_0^{\eta_0}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{sec}^2\eta \\ &= 2\pi a^2\int_0^{\eta_0}\mathrm{sec}^2\eta\,\mathrm{d}\eta \\ &= 2\pi a^2\mathrm{tan}\eta_0 \end{aligned} \tag{2}$$

$$\begin{aligned} N &= \delta V \\ \therefore N_{1+1} &= 2\pi\delta a^2\mathrm{tan}\eta_0 \end{aligned} \tag{3}$$

1.2 3+1 Dimensions

$$\mathrm{d}V = a^4\mathrm{sec}^4\eta\,\mathrm{d}\eta\,\mathrm{sin}^2\phi\,\mathrm{sin}\chi\,\mathrm{d}\phi\,\mathrm{d}\chi\,\mathrm{d}\theta \tag{4}$$

$$\begin{aligned} V &= a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\phi\,\mathrm{sec}^4\eta\,\mathrm{sin}^2\phi\,\mathrm{sin}\chi \\ &= 2\pi a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{\pi}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\phi\,\mathrm{sec}^4\eta\,\mathrm{sin}^2\phi\,\mathrm{sin}\chi \\ &= 4\pi a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{\pi}\mathrm{d}\phi\,\mathrm{sec}^4\eta\,\mathrm{sin}^2\phi \\ &= 2\pi^2 a^4\int_0^{\eta_0}\mathrm{sec}^4\eta\,\mathrm{d}\eta \\ &= \frac{2}{3}\pi^2 a^4\left(2+\mathrm{sec}^2\eta_0\right)\mathrm{tan}\eta_0 \end{aligned} \tag{5}$$

$$
\begin{aligned}
N &= \delta V \\
\therefore N_{3+1} &= \frac{2}{3}\pi^2 \delta a^4 \left(2 + \sec^2\eta_0\right) \tan\eta_0
\end{aligned}
\tag{6}
$$

2 Expected Average Degrees

2.1 1+1 Dimensions

$$
\rho = \frac{\sec^2\eta}{\tan\eta_0} \tag{7}
$$

$$
V_p = \ln \sec\eta \tag{8}
$$

$$
\begin{aligned}
\langle V_p \rangle &= \int_0^{\eta_0} \frac{\sec^2\eta}{\tan\eta_0} \ln \sec\eta \, \mathrm{d}\eta \\
&= \frac{1}{\tan\eta_0} \left[\eta + \tan\eta(\ln \sec\eta - 1)\right]_0^{\eta_0} \\
&= \frac{1}{\tan\eta_0} \left[\eta_0 + \tan\eta_0(\ln \sec\eta_0 - 1)\right] \\
&= \frac{\eta_0}{\tan\eta_0} + \ln \sec\eta_0 - 1
\end{aligned}
\tag{9}
$$

$$
V_f = (\eta_0 - \eta) \tan\eta_0 + \ln \left(\frac{\sec\eta}{\sec\eta_0}\right) \tag{10}
$$

$$
\begin{aligned}
\langle V_f \rangle &= \int_0^{\eta_0} \frac{\sec^2\eta}{\tan\eta_0} \left[(\eta_0 - \eta) \tan\eta_0 + \ln \left(\frac{\sec\eta}{\sec\eta_0}\right)\right] \mathrm{d}\eta \\
&= \frac{1}{\tan\eta_0} \left[\eta_0 \tan\eta_0 \int_0^{\eta_0} \sec^2\eta \, \mathrm{d}\eta - \tan\eta_0 \int_0^{\eta_0} \eta \sec^2\eta \, \mathrm{d}\eta \right. \\
&\quad \left. + \int_0^{\eta_0} \sec^2\eta \ln \sec\eta \, \mathrm{d}\eta - \ln \sec\eta_0 \int_0^{\eta_0} \sec^2\eta \, \mathrm{d}\eta \right] \\
&= \frac{1}{\tan\eta_0} \left[\eta_0 \tan^2\eta_0 - \eta_0 \tan^2\eta_0 + \tan\eta_0 \ln \sec\eta_0 \right. \\
&\quad \left. + \eta_0 + \tan\eta_0 \left(\ln \sec\eta_0 - 1\right) - \tan\eta_0 \ln \sec\eta_0 \right] \\
&= \frac{\eta_0}{\tan\eta_0} + \ln \sec\eta_0 - 1
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\therefore \langle V_p \rangle &= \langle V_f \rangle \\
\therefore \langle \bar{k} \rangle &= 2a^2\delta \left(\langle V_p \rangle + \langle V_f \rangle\right) \\
&= 4\delta a^2 \left(\frac{\eta_0}{\tan\eta_0} + \ln \sec\eta_0 - 1\right)
\end{aligned}
\tag{12}
$$

2.2 3+1 Dimensions

$$\rho = \frac{3\sec^4\eta}{(2+\sec^2\eta_0)\tan\eta_0} \tag{13}$$

$$\begin{aligned}
V_p(\eta) &= \int_0^{\eta} \mathrm{d}\eta' \int_0^{\eta-\eta'} 2\mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\chi\, a^4 \sec^4\eta'\ \sin^2\phi\ \sin\chi \\
&\qquad \int_0^{\pi} \sin\chi\,\mathrm{d}\chi = 2 \\
&\qquad \int_0^{\pi} \sin^2\phi\,\mathrm{d}\phi = \frac{\pi}{2} \\
&\qquad \int_0^{\eta-\eta'} 2\,\mathrm{d}\theta = 2\,(\eta-\eta') \\
&= 2\pi a^4 \int_0^{\eta} (\eta-\eta')\sec^4\eta'\,\mathrm{d}\eta' \\
&\qquad \int_0^{\eta} \sec^4\eta'\,\mathrm{d}\eta' = \frac{1}{3}\tan\eta\left(2+\sec^2\eta\right) \\
&\qquad \int_0^{\eta} \eta'\sec^4\eta'\,\mathrm{d}\eta' = \frac{1}{3}\left(-2\ln\sec\eta - \frac{1}{2}\sec^2\eta + 2\eta\tan\eta + \eta\sec^2\eta\tan\eta + \frac{1}{2}\right) \\
&= 2\pi a^4\left(\frac{2}{3}\eta\tan\eta + \frac{1}{3}\eta\sec^2\eta + \frac{2}{3}\ln\sec\eta + \frac{1}{6}\sec^2\eta - \frac{2}{3}\eta\tan\eta - \frac{1}{3}\eta\sec^2\eta\tan\eta - \frac{1}{6}\right) \\
&= 2\pi a^4\left(\frac{1}{3}\eta\sec^2\eta + \frac{2}{3}\ln\sec\eta + \frac{1}{6}\sec^2\eta - \frac{1}{3}\eta\sec^2\eta\tan\eta - \frac{1}{6}\right) \\
&= \frac{2}{3}\pi a^4\left(\eta\sec^2\eta + 2\ln\sec\eta + \frac{1}{2}\sec^2\eta - \eta\sec^2\eta\tan\eta - \frac{1}{2}\right)
\end{aligned} \tag{14}$$