Causal Set Research Notes

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1 Patch Volume

1.1 1+1 Dimensions

$$\begin{aligned}
\mathrm{d}V &= (a\mathrm{sec}\eta)^{\mathrm{d}+1}\,\mathrm{d}\eta\,\mathrm{d}\Phi_d \\
&= a^2\mathrm{sec}^2\eta\,\mathrm{d}\eta\,\mathrm{d}\theta
\end{aligned} \tag{1}$$

$$\begin{aligned}
V &= a^2\int_0^{\eta_0}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{sec}^2\eta \\
&= 2\pi a^2\int_0^{\eta_0}\mathrm{sec}^2\eta\,\mathrm{d}\eta \\
&= 2\pi a^2\mathrm{tan}\eta_0
\end{aligned} \tag{2}$$

$$\begin{aligned}
N &= \delta V \\
\therefore N_{1+1} &= 2\pi\delta a^2\mathrm{tan}\eta_0
\end{aligned} \tag{3}$$

1.2 3+1 Dimensions

$$\mathrm{d}V = a^4\mathrm{sec}^4\eta\,\mathrm{d}\eta\,\mathrm{sin}^2\phi\,\mathrm{sin}\chi\,\mathrm{d}\phi\,\mathrm{d}\chi\,\mathrm{d}\theta \tag{4}$$

$$\begin{aligned}
V &= a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\phi\,\mathrm{sec}^4\eta\,\mathrm{sin}^2\phi\,\mathrm{sin}\chi \\
&= 2\pi a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{\pi}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\phi\,\mathrm{sec}^4\eta\,\mathrm{sin}^2\phi\,\mathrm{sin}\chi \\
&= 4\pi a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{\pi}\mathrm{d}\phi\,\mathrm{sec}^4\eta\,\mathrm{sin}^2\phi \\
&= 2\pi^2 a^4\int_0^{\eta_0}\mathrm{sec}^4\eta\,\mathrm{d}\eta \\
&= \frac{2}{3}\pi^2 a^4\left(2+\mathrm{sec}^2\eta_0\right)\mathrm{tan}\eta_0
\end{aligned} \tag{5}$$

$$
\begin{aligned}
N &= \delta V \\
\therefore N_{3+1} &= \frac{2}{3}\pi^2\delta a^4\left(2+\sec^2\eta_0\right)\tan\eta_0
\end{aligned} \tag{6}
$$

2 Expected Average Degrees

2.1 1+1 Dimensions

$$
\rho = \frac{\sec^2\eta}{\tan\eta_0} \tag{7}
$$

$$
V_p = \ln\sec\eta \tag{8}
$$

$$
\begin{aligned}
\langle V_p\rangle &= \int_0^{\eta_0}\frac{\sec^2\eta}{\tan\eta_0}\ln\sec\eta\,\mathrm{d}\eta \\
&= \frac{1}{\tan\eta_0}\left[\eta+\tan\eta(\ln\sec\eta-1)\right]_0^{\eta_0} \\
&= \frac{1}{\tan\eta_0}\left[\eta_0+\tan\eta_0(\ln\sec\eta_0-1)\right] \\
&= \frac{\eta_0}{\tan\eta_0}+\ln\sec\eta_0-1
\end{aligned} \tag{9}
$$

$$
V_f = (\eta_0-\eta)\tan\eta_0+\ln\left(\frac{\sec\eta}{\sec\eta_0}\right) \tag{10}
$$

$$
\begin{aligned}
\langle V_f\rangle &= \int_0^{\eta_0}\frac{\sec^2\eta}{\tan\eta_0}\left[(\eta_0-\eta)\tan\eta_0+\ln\left(\frac{\sec\eta}{\sec\eta_0}\right)\right]\,\mathrm{d}\eta \\
&= \frac{1}{\tan\eta_0}\left[\eta_0\tan\eta_0\int_0^{\eta_0}\sec^2\eta\,\mathrm{d}\eta-\tan\eta_0\int_0^{\eta_0}\eta\sec^2\eta\,\mathrm{d}\eta\right. \\
&\quad\left.+\int_0^{\eta_0}\sec^2\eta\ln\sec\eta\,\mathrm{d}\eta-\ln\sec\eta_0\int_0^{\eta_0}\sec^2\eta\,\mathrm{d}\eta\right] \\
&= \frac{1}{\tan\eta_0}\left[\eta_0\tan^2\eta_0-\eta_0\tan^2\eta_0+\tan\eta_0\ln\sec\eta_0\right. \\
&\quad\left.+\eta_0+\tan\eta_0\left(\ln\sec\eta_0-1\right)-\tan\eta_0\ln\sec\eta_0\right] \\
&= \frac{\eta_0}{\tan\eta_0}+\ln\sec\eta_0-1
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\therefore \langle V_p\rangle &= \langle V_f\rangle \\
\therefore \langle\bar{k}\rangle &= 2a^2\delta\left(\langle V_p\rangle+\langle V_f\rangle\right) \\
&= 4\delta a^2\left(\frac{\eta_0}{\tan\eta_0}+\ln\sec\eta_0-1\right)
\end{aligned} \tag{12}
$$

2.2 3+1 Dimensions

$$V_p(\eta) = \int_0^{\eta} \mathrm{d}\eta' \int_0^{\eta-\eta'} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\chi\, a^4 \sec^4 \eta' \sin^2 \phi \sin \chi \tag{13}$$

$$= 2a^4 \int_0^{\eta} \mathrm{d}\eta' \int_0^{\eta-\eta'} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta\ \sec^4 \eta' \sin^2 \phi \tag{14}$$

$$= 4\pi a^4 \int_0^{\eta} \mathrm{d}\eta' \int_0^{\eta-\eta'} \mathrm{d}\phi\ \sec^4 \eta' \sin^2 \phi \tag{15}$$

$$\begin{aligned}\int_0^{\eta-\eta'} \sin^2 \phi\, \mathrm{d}\phi = \frac{1}{2} [\eta - \eta' &+ \sin \eta \cos \eta \left(1 - 2\cos^2 \eta'\right) \\ &- \sin \eta' \cos \eta' \left(1 - 2\cos^2 \eta\right)\big]\end{aligned} \tag{16}$$

$$\begin{aligned}= 2\pi a^4 \int_0^{\eta} \mathrm{d}\eta'\ \sec^4 \eta' [\eta - \eta' &+ \left(1 - 2\cos^2 \eta'\right) \sin \eta \cos \eta \\ &- \sin \eta' \cos \eta' \left(1 - 2\cos^2 \eta\right)\big]\end{aligned} \tag{17}$$

$$\int_0^{\eta} \eta \sec^4 \eta'\, \mathrm{d}\eta' = \frac{\eta}{3} \tan \eta \left(2 + \sec^2 \eta\right) \tag{18}$$

$$\begin{aligned}\int_0^{\eta} \eta' \sec^4 \eta'\, \mathrm{d}\eta' = \frac{1}{3} [&- \ln \sec^2 \eta - \frac{1}{2} \sec^2 \eta + \frac{1}{2} \\ &+ 2\eta \tan \eta + \eta \sec^2 \eta \tan \eta\big]\end{aligned} \tag{19}$$

$$\int_0^{\eta} \left(1 - 2\cos^2 \eta'\right) \sec^4 \eta' \sin \eta \cos \eta\, \mathrm{d}\eta' = -\frac{4}{3} \sin^2 \eta + \frac{1}{3} \tan^2 \eta \tag{20}$$

$$\int_0^{\eta} \sin \eta' \cos \eta' \sec^4 \eta' \left(1 - 2\cos^2 \eta\right)\ \mathrm{d}\eta' = \frac{1}{2} \left(2\cos^2 \eta + \sec^2 \eta - 3\right) \tag{21}$$

$$= \frac{2\pi a^4}{3} \left[\ln \sec^2 \eta - \sec^2 \eta + 4 - 4\sin^2 \eta + tan^2 \eta - 3\cos^2 \eta\right] \tag{22}$$

$$= \frac{2\pi a^4}{3} \left[\ln \sec^2 \eta - \sec^2 \eta + \tan^2 \eta + \cos^2 \eta\right] \tag{23}$$

$$\rho = \frac{3 \sec^4 \eta}{(2 + \sec^2 \eta_0) \tan \eta_0} \tag{24}$$

$$
\begin{aligned}
\langle \bar{k_o} \rangle &= \delta \int_0^{\eta_0} \rho\left(\eta\right) V_p\left(\eta\right)\, \mathrm{d}\eta &(25)\\
&= \frac{2\pi\delta a^4}{\left(2+\sec^2\eta_0\right)\tan\eta_0} \int_0^{\eta_0} \left[\sec^4\eta \ln\sec^2\eta - \sec^6\eta + \tan^2\eta\sec^4\eta + \sec^2\eta\right]\, \mathrm{d}\eta &(26)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\eta_0} \sec^4\eta\ln\sec^2\eta\,\mathrm{d}\eta =& \frac{4}{3}\eta_0 - \frac{10}{9}\tan\eta_0 + \frac{4}{3}\tan\eta_0\ln\sec\eta_0 \\
&- \frac{2}{9}\sec^2\eta_0\tan\eta_0 + \frac{2}{3}\sec^2\eta_0\tan\eta_0\ln\sec\eta_0
\end{aligned} \tag{27}
$$

$$
\int_0^{\eta_0} \sec^6\eta\,\mathrm{d}\eta = \frac{8}{15}\tan\eta_0 + \frac{4}{15}\tan\eta_0\sec^2\eta_0 + \frac{1}{5}\tan\eta_0\sec^4\eta_0 \tag{28}
$$

$$
\int_0^{\eta_0} \tan^2\eta\sec^4\eta\,\mathrm{d}\eta = -\frac{2}{15}\tan\eta_0 - \frac{1}{15}\tan\eta_0\sec^2\eta_0 + \frac{1}{5}\tan\eta_0\sec^4\eta_0 \tag{29}
$$

$$
\int_0^{\eta_0} \sec^2\eta\,\mathrm{d}\eta = \tan\eta_0 \tag{30}
$$

$$
= \frac{2\pi\delta a^4}{2+\sec^2\eta_0}\left[\frac{4}{3}\frac{\eta_0}{\tan\eta_0} - \frac{7}{9} + \frac{4}{3}\ln\sec\eta_0 - \frac{5}{9}\sec^2\eta_0 + \frac{2}{3}\sec^2\eta_0\ln\sec\eta_0\right] \tag{31}
$$

$$
\begin{aligned}
\langle \bar{k} \rangle &= 2\langle \bar{k_o} \rangle \\
&= \frac{4}{9}\frac{\pi\delta a^4}{2+\sec^2\eta_0}\left[12\left(\frac{\eta_0}{\tan\eta_0} + \ln\sec\eta_0\right) + \left(6\ln\sec\eta_0 - 5\right)\sec^2\eta_0 - 7\right]
\end{aligned} \tag{32}
$$

3 Expected Isolated Nodes

3.1 1+1 Dimensions

Poisson point process:

$$
P(x) = \frac{\left(\delta V\right)^x}{x!}e^{-\delta V} \tag{33}
$$

The probability a node is isolated at conformal time η is given by this expression, where $x = 0$ is the expected number of nodes in the sum of the light cone volumes $V = V_p\left(\eta\right) + V_f\left(\eta\right)$. Manipulating this expression yields

$$
\begin{aligned}
P\left(0\right) &= e^{-2\delta a^2\left[\left(\eta_0-\eta\right)\tan\eta_0 + \ln\sec^2\eta - \ln\sec\eta_0\right]} \\
&= e^{-2\delta a^2\left[\eta_0\tan\eta_0 - \ln\sec\eta_0\right]}e^{-2\delta a^2\left[\ln\sec^2\eta - \eta\tan\eta_0\right]} \\
&= \xi e^{-2\delta a^2\left[\ln\sec^2\eta - \eta\tan\eta_0\right]}
\end{aligned} \tag{34}
$$

Then, the expected number of isolated nodes is given by

$$\begin{aligned}
\langle N\left(0\right)\rangle &= N\int_0^{\eta_0}\rho\left(\eta\right)P\left(0\right)\,\mathrm{d}\eta \\
&= \frac{N\xi}{\tan\eta_0}\int_0^{\eta_0}\sec^2\eta e^{-2\delta a^2\left[\ln\sec^2\eta-\eta\tan\eta_0\right]}\,\mathrm{d}\eta \\
&\quad e^{-2\delta a^2\ln\sec^2\eta}=\left(\cos\eta\right)^{4\delta a^2} \\
&= \frac{N\xi}{\tan\eta_0}\int_0^{\eta_0}\left(\cos\eta\right)^{4\delta a^2-2}e^{\left(2\delta a^2\tan\eta_0\right)\eta}\,\mathrm{d}\eta
\end{aligned} \tag{35}$$