Causal Set Research Notes

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February 28, 2014

1 Patch Volume

1.1 1+1 Dimensions

$$\begin{aligned} \mathrm{d}V &= (a\sec\eta)^{\mathrm{d}+1}\,\mathrm{d}\eta\,\mathrm{d}\Phi_d \\ &= a^2\sec^2\eta\,\mathrm{d}\eta\,\mathrm{d}\theta \end{aligned} \tag{1}$$

$$\begin{aligned} V &= a^2\int_0^{\eta_0}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\theta\,\sec^2\eta \\ &= 2\pi a^2\int_0^{\eta_0}\sec^2\eta\,\mathrm{d}\eta \\ &= 2\pi a^2\tan\eta_0 \end{aligned} \tag{2}$$

$$\begin{aligned} N &= \delta V \\ \therefore N_{1+1} &= 2\pi\delta a^2\tan\eta_0 \end{aligned} \tag{3}$$

1.2 3+1 Dimensions

$$\mathrm{d}V = a^4\sec^4\eta\,\mathrm{d}\eta\,\sin^2\phi\,\sin\chi\,\mathrm{d}\phi\,\mathrm{d}\chi\,\mathrm{d}\theta \tag{4}$$

$$\begin{aligned} V &= a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\phi\,\sec^4\eta\,\sin^2\phi\,\sin\chi \\ &= 2\pi a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{\pi}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\phi\,\sec^4\eta\,\sin^2\phi\,\sin\chi \\ &= 4\pi a^4\int_0^{\eta_0}\mathrm{d}\eta\int_0^{\pi}\mathrm{d}\phi\,\sec^4\eta\,\sin^2\phi \\ &= 2\pi^2 a^4\int_0^{\eta_0}\sec^4\eta\,\mathrm{d}\eta \\ &= \frac{2}{3}\pi^2 a^4\left(2+\sec^2\eta_0\right)\tan\eta_0 \end{aligned} \tag{5}$$

$$
\begin{aligned}
N &= \delta V \\
\therefore N_{3+1} &= \frac{2}{3}\pi^2 \delta a^4 \left(2 + \sec^2\eta_0\right) \tan\eta_0
\end{aligned} \tag{6}
$$

2 Expected Average Degrees

2.1 1+1 Dimensions

$$
\rho = \frac{\sec^2\eta}{\tan\eta_0} \tag{7}
$$

$$
V_p = \ln\sec\eta \tag{8}
$$

$$
\begin{aligned}
\langle V_p \rangle &= \int_0^{\eta_0} \frac{\sec^2\eta}{\tan\eta_0} \ln\sec\eta \, \mathrm{d}\eta \\
&= \frac{1}{\tan\eta_0} \left[\eta + \tan\eta(\ln\sec\eta - 1)\right]_0^{\eta_0} \\
&= \frac{1}{\tan\eta_0} \left[\eta_0 + \tan\eta_0(\ln\sec\eta_0 - 1)\right] \\
&= \frac{\eta_0}{\tan\eta_0} + \ln\sec\eta_0 - 1
\end{aligned} \tag{9}
$$

$$
V_f = (\eta_0 - \eta)\tan\eta_0 + \ln\left(\frac{\sec\eta}{\sec\eta_0}\right) \tag{10}
$$

$$
\begin{aligned}
\langle V_f \rangle &= \int_0^{\eta_0} \frac{\sec^2\eta}{\tan\eta_0} \left[(\eta_0 - \eta)\tan\eta_0 + \ln\left(\frac{\sec\eta}{\sec\eta_0}\right)\right] \mathrm{d}\eta \\
&= \frac{1}{\tan\eta_0} \left[\eta_0 \tan\eta_0 \int_0^{\eta_0} \sec^2\eta \, \mathrm{d}\eta - \tan\eta_0 \int_0^{\eta_0} \eta \sec^2\eta \, \mathrm{d}\eta \right. \\
&\quad \left. + \int_0^{\eta_0} \sec^2\eta \ln\sec\eta \, \mathrm{d}\eta - \ln\sec\eta_0 \int_0^{\eta_0} \sec^2\eta \, \mathrm{d}\eta \right] \\
&= \frac{1}{\tan\eta_0} \left[\eta_0 \tan^2\eta_0 - \eta_0 \tan^2\eta_0 + \tan\eta_0 \ln\sec\eta_0 \right. \\
&\quad \left. + \eta_0 + \tan\eta_0 \left(\ln\sec\eta_0 - 1\right) - \tan\eta_0 \ln\sec\eta_0 \right] \\
&= \frac{\eta_0}{\tan\eta_0} + \ln\sec\eta_0 - 1
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\therefore \langle V_p \rangle &= \langle V_f \rangle \\
\therefore \langle \bar{k} \rangle &= 2a^2\delta \left(\langle V_p \rangle + \langle V_f \rangle\right) \\
&= 4\delta a^2 \left(\frac{\eta_0}{\tan\eta_0} + \ln\sec\eta_0 - 1\right)
\end{aligned} \tag{12}
$$

2.2 3+1 Dimensions

$$\rho = \frac{3\sec^4\eta}{(2+\sec^2\eta_0)\tan\eta_0} \tag{13}$$

$$\begin{aligned}
V_p(\eta) &= \int_0^{\eta} \mathrm{d}\eta' \int_0^{\eta-\eta'} 2\mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\chi\, a^4 \sec^4\eta'\ \sin^2\phi\ \sin\chi \\
&\qquad \int_0^{\pi} \sin\chi\,\mathrm{d}\chi = 2 \\
&\qquad \int_0^{\pi} \sin^2\phi\,\mathrm{d}\phi = \frac{\pi}{2} \\
&\qquad \int_0^{\eta-\eta'} 2\,\mathrm{d}\theta = 2\left(\eta-\eta'\right) \\
&= 2\pi a^4 \int_0^{\eta} (\eta-\eta')\sec^4\eta'\,\mathrm{d}\eta' \\
&\qquad \int_0^{\eta} \sec^4\eta'\,\mathrm{d}\eta' = \frac{1}{3}\tan\eta\left(2+\sec^2\eta\right) \\
&\qquad \int_0^{\eta} \eta'\sec^4\eta'\,\mathrm{d}\eta' = \frac{1}{3}\left(-2\ln\sec\eta - \frac{1}{2}\sec^2\eta + 2\eta\tan\eta + \eta\sec^2\eta\tan\eta + \frac{1}{2}\right) \\
&= 2\pi a^4 \left(\frac{2}{3}\eta\tan\eta + \frac{1}{3}\eta\sec^2\eta + \frac{2}{3}\ln\sec\eta + \frac{1}{6}\sec^2\eta - \frac{2}{3}\eta\tan\eta - \frac{1}{3}\eta\sec^2\eta\tan\eta - \frac{1}{6}\right) \\
&= 2\pi a^4 \left(\frac{1}{3}\eta\sec^2\eta + \frac{2}{3}\ln\sec\eta + \frac{1}{6}\sec^2\eta - \frac{1}{3}\eta\sec^2\eta\tan\eta - \frac{1}{6}\right) \\
&= \frac{2}{3}\pi a^4 \left(\eta\sec^2\eta + 2\ln\sec\eta + \frac{1}{2}\sec^2\eta - \eta\sec^2\eta\tan\eta - \frac{1}{2}\right)
\end{aligned} \tag{14}$$

$$\langle \bar{k_o} \rangle = \int_0^{\eta 0} \rho \delta V_p(\eta)\,\mathrm{d}\eta \tag{15}$$

$$= \frac{2\pi\delta a^4}{(2+\sec^2\eta_0)\tan\eta_0} \int_0^{\eta_0} \left(\eta \sec^6\eta + \sec^4 \ln\sec^2\eta + \frac{1}{2}\sec^6\eta \right. \\ \left. -\eta\sec^6\eta\tan\eta - \frac{1}{2}\sec^4\eta \right)\,\mathrm{d}\eta \tag{16}$$

$$\int_0^{\eta_0} \eta\sec^6\eta\,\mathrm{d}\eta = -\frac{8}{15}\ln\sec\eta_0 - \frac{2}{15}\sec^2\eta_0 - \frac{1}{20}\sec^4\eta_0 + \frac{8}{15}\eta_0\tan\eta_0 \\ + \frac{4}{15}\eta_0\sec^2\eta_0\tan\eta_0 + \frac{1}{5}\eta_0\sec^4\eta_0\tan\eta_0 + \frac{11}{60} \tag{17}$$

$$\int_0^{\eta_0} \sec^4\eta\ln\sec\eta\,\mathrm{d}\eta = \frac{2}{3}\eta_0 - \frac{5}{9}\tan\eta_0 - \frac{1}{9}\sec^2\eta_0\tan\eta_0 \\ + \frac{1}{3}\ln\sec\eta_0\sec^2\eta_0\tan\eta_0 + \frac{2}{3}\ln\sec\eta_0\tan\eta_0 \tag{18}$$

$$\int_0^{\eta_0} \sec^6\eta\,\mathrm{d}\eta = \frac{8}{15}\tan\eta_0 + \frac{4}{15}\sec^2\eta_0\tan\eta_0 \\ + \frac{1}{5}\sec^4\eta_0\tan\eta_0 \tag{19}$$

$$\int_0^{\eta_0} \eta\sec^6\eta\tan\eta\,\mathrm{d}\eta = \frac{1}{6}\eta_0\sec^6\eta_0 - \frac{4}{45}\tan\eta_0 \\ - \frac{2}{45}\sec^2\eta_0\tan\eta_0 - \frac{1}{30}\sec^4\eta_0\tan\eta_0 \tag{20}$$

$$\int_0^{\eta_0} \sec^4\eta\,\mathrm{d}\eta = \frac{2}{3}\tan\eta_0 + \frac{1}{3}\sec^2\eta_0\tan\eta_0 \tag{21}$$

$$= \frac{2\pi\delta a^4}{(2+\sec^2\eta_0)\tan\eta_0} I \tag{22}$$

$$I = -\frac{8}{15}\ln\sec\eta_0 - \frac{2}{15}\sec^2\eta_0 - \frac{1}{20}\sec^4\eta_0 + \frac{8}{15}\eta_0\tan\eta_0 + \frac{4}{15}\eta_0\sec^2\eta_0\tan\eta_0 \\ + \frac{1}{5}\eta_0\sec^4\eta_0\tan\eta_0 + \frac{11}{60} + \frac{4}{3}\eta_0 - \frac{49}{45}\tan\eta_0 - \frac{19}{90}\sec^2\eta_0\tan\eta_0 \\ + \frac{2}{3}\ln\sec\eta_0\sec^2\eta_0\tan\eta_0 + \frac{4}{3}\ln\sec\eta_0\tan\eta_0 + \frac{2}{15}\sec^4\eta_0\tan\eta_0 \\ - 16\eta_0\sec^6\eta_0 \tag{23}$$

$$
\begin{aligned}
\langle \bar{k} \rangle &= 2\langle \bar{k_o} \rangle \qquad (24) \\
&= \frac{4}{9}\frac{\pi\delta a^4}{2+\sec^2\eta_0}\Bigg[12\left(\eta_0\cot\eta_0+\ln\sec\eta_0\right)+\left(6\ln\sec\eta_0-\frac{19}{10}\right)\sec^2\eta_0 \\
&\quad -\frac{49}{5}-\frac{24}{5}\ln\sec\eta_0\cot\eta_0-\frac{6}{5}\sec^2\eta_0\cot\eta_0-\frac{9}{20}\sec^4\eta_0\cot\eta_0 \\
&\quad +\frac{24}{5}\eta_0+\frac{12}{5}\eta_0\sec^2\eta_0+\frac{9}{5}\eta_0\sec^4\eta_0+\frac{33}{20}\cot\eta_0+\frac{6}{5}\sec^4\eta_0 \\
&\quad -\frac{3}{2}\eta_0\sec^6\eta_0\cot\eta_0\Bigg] \qquad (25)
\end{aligned}
$$

3 Expected Isolated Nodes

3.1 1+1 Dimensions

The discrete Poisson distribution is given by

$$
P(X=x)=\frac{(\delta V)^x}{x!}e^{-\delta V} \tag{26}
$$

which describes the probability X points fall in the volume V. The respective volumes of the past and future light cones at conformal time η are given by

$$
\begin{aligned}
V_p(\eta) &= 2a^2\ln\sec\eta \\
V_f(\eta) &= 2a^2\left[(\eta_0-\eta)\tan\eta_0+\ln\left(\frac{\sec\eta}{\sec\eta_0}\right)\right] \\
V &= 2\pi a^2\tan\eta_0
\end{aligned} \tag{27}
$$

where V is the total volume of the spacetime patch on the de Sitter manifold. The volume of the region outside a given node's past and future light cones is described by

$$
\begin{aligned}
V_o(\eta) &= V-\left(V_p(\eta)+V_f(\eta)\right) \qquad (28) \\
&= 2a^2\left[\pi\tan\eta_0-2\ln\sec\eta+(\eta-\eta_0)\tan\eta_0\right. \\
&\qquad \left.+\ln\sec\eta_0\right] \qquad (29) \\
&= 2a^2\left[\left(\eta\tan\eta_0-\ln\sec^2\eta\right)+(\pi\tan\eta_0\right. \\
&\qquad \left.-\eta_0\tan\eta_0+\ln\sec\eta_0)\right] \qquad (30) \\
&= 2a^2\left(\mu(\eta)+\xi\right) \qquad (31) \\
\delta V_o(\eta) &= 2a^2\delta\left(\mu(\eta)+\xi\right) \qquad (32)
\end{aligned}
$$

where we have defined the functions

$$
\begin{aligned}
\mu(\eta) &\equiv \eta\tan\eta_0-\ln\sec^2\eta \\
\xi &\equiv \pi\tan\eta_0-\eta_0\tan\eta_0+\ln\sec\eta_0
\end{aligned} \tag{33}
$$

Thus, the continuous Poissonian distribution is

$$P\left(N,\eta\right)=\frac{\left(2a^2\delta\right)^N}{\Gamma\left(N+1\right)}\left(\mu\left(\eta\right)+\xi\right)^N e^{-2a^2\delta\left(\mu\left(\eta\right)+\xi\right)} \tag{34}$$

which describes the probability N points lie outside the light cones at conformal time η. Therefore, the expected number of isolated nodes is simply

$$\langle N\left(0\right)\rangle = N\int_0^{\eta_0}\rho\left(\eta\right)P\left(N-1,\eta\right)\,\mathrm{d}\eta \tag{35}$$

To evaluate the integral, we must expand the term $\left(\mu\left(\eta\right)+\xi\right)^N$ using a Binomial expansion. This is allowed if N is large enough. Note that we use $N-1$ because the point whose light cones we are considering is not included in the Poisson point process. To do this trick, consider $\alpha\equiv\frac{\pi}{2}-\eta$.

$$\begin{aligned}\left(\mu\left(\eta\right)+\xi\right)^N &= x-m\left(\alpha\right)\\ &= x\left(1-\frac{m\left(\alpha\right)}{x}\right)\end{aligned} \tag{36}$$

where we have defined the new quantities

$$\begin{aligned} m &\equiv \alpha\tan\eta_0+\ln\csc^2\alpha\\ x &\equiv \frac{3\pi}{2}\tan\eta_0-\eta_0\tan\eta_0+\ln\sec\eta_0\end{aligned} \tag{37}$$

Now, we may use the binomial expansion:

$$\begin{aligned}\left(1-\frac{m\left(\alpha\right)}{x}\right)^N &\approx 1-\frac{N}{x}m\left(\alpha\right)+\frac{N\left(N-1\right)}{2}\left(\frac{m\left(\alpha\right)}{x}\right)^2-\frac{N\left(N-1\right)\left(N-2\right)}{6}\left(\frac{m\left(\alpha\right)}{x}\right)^3\\ m\left(\alpha\right) &= \left(\frac{\pi}{2}-\eta\right)\tan\eta_0+\ln\sec^2\eta\\ &\approx 1-\frac{N}{x}\left[\left(\frac{\pi}{2}-\eta\right)\tan\eta_0+\ln\sec^2\eta\right]\end{aligned} \tag{38}$$

and use the following approximations:

$$\begin{aligned} m &\approx \frac{\pi}{2}\tan\eta_0-\eta\tan\eta_0\\ m^2 &\approx \pi\tan\eta_0\left(\frac{\pi}{4}\tan\eta_0-\eta_0\right)+\eta^2\tan^2\eta_0\\ x &\approx \left(\frac{3\pi}{2}-\eta_0\right)\tan\eta_0\end{aligned} \tag{39}$$

so that the distribution is now

$$P\left(N,\eta\right)=\frac{\left[2a^2\delta\left(\frac{3\pi}{2}-\eta_0\right)\tan\eta_0\right]^N}{\Gamma\left(N+1\right)}e^{-2a^2\delta(\pi-\eta_0)\tan\eta_0}$$
$$\left[1-N\left(\frac{\pi-2\eta}{3\pi-2\eta_0}\right)+\frac{N\left(N-1\right)}{2}\frac{\pi\left(\frac{\pi}{4}-\eta_0\cot\eta_0\right)+\eta^2}{\left(\frac{3\pi}{2}-\eta_0\right)^2}\right]e^{-2a^2\delta\eta\tan\eta_0} \tag{40}$$
$$=\Xi\left(N\right)\left[1-N\left(\frac{\pi-2\eta}{3\pi-2\eta_0}\right)+\frac{N\left(N-1\right)}{2}\frac{\pi\left(\frac{\pi}{4}-\eta_0\cot\eta_0\right)+\eta^2}{\left(\frac{3\pi}{2}-\eta_0\right)^2}\right]e^{-2a^2\delta\eta\tan\eta_0} \tag{41}$$
$$\left\langle N\left(0\right)\right\rangle=\frac{N}{\tan\eta_0}\Xi\left(N\right)\int_0^{\eta_0}\sec^2\eta e^{-2a^2\delta\eta\tan\eta_0}$$
$$\left[1-N\left(\frac{\pi-2\eta}{3\pi-2\eta_0}\right)+\frac{N\left(N-1\right)}{2}\frac{\pi\left(\frac{\pi}{4}-\eta_0\cot\eta_0\right)+\eta^2}{\left(\frac{3\pi}{2}-\eta_0\right)^2}\right]\,\mathrm{d}\eta \tag{42}$$

where the function $\Xi\left(N\right)$ has been defined to be

$$\Xi\left(N\right)\equiv\frac{\left[2a^2\delta\left(\frac{3\pi}{2}-\eta_0\right)\tan\eta_0\right]^N}{\Gamma\left(N+1\right)}e^{-2a^2\delta(\pi-\eta_0)\tan\eta_0} \tag{43}$$