다음 식의 증명을 정리해둠.

$$\frac{\zeta(3)}{4\pi^2} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^2 \ln k\right) - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12}$$

Thm1. $\frac{1}{2}\ln 2\pi = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\ln k\right) - \left(n+\frac{1}{2}\right)\ln n + n$ (stirling's approximation.)

(증명생략)

Thm2. $\ln A = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k\ln k\right) - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}$

(A : Glasher-Kinkelin constant.)

(증명생략)

thm3. $\Gamma(x) = \lim_{n\to\infty}\frac{1}{x(x+1)}\frac{2}{(x+2)}\frac{3}{(x+3)}\cdots\frac{n}{(x+n)}n^x$

(증명생략)

Thm4. Let $f(x)=\Gamma(x+1)$ (f(x) = x!)

then $\ln f(x) = -\gamma x + \frac{\zeta(2)}{2}x^2 - \frac{\zeta(3)}{3}x^3 + \ldots$ (|x|<1)

pf) thm3에 의해, $f(x) = \lim_{n\to\infty}\frac{1}{(x+1)}\frac{2}{(x+2)}\frac{3}{(x+3)}\cdots\frac{n}{(x+n)}n^x$

$$\ln(f(x)) = \lim_{n\to\infty}\ln\left(\frac{1}{(x+1)}\frac{2}{(x+2)}\frac{3}{(x+3)}\cdots\frac{n}{(x+n)}n^x\right)$$

(f(x)는 |x|<1에서 연속이므로 가능)

$$\ln(f(x)) = \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{x+k}\right)+x\ln n\right]$$

x=0에서 테일러 급수를 취하면 됨. □ (thm8증명 첫 줄도 참고할 것.)

$$\text{Thm5. } -\ln(2\sin(\pi x)) = -\ln(2\pi x) + 2[\frac{\zeta(2)}{2}x^2 + \frac{\zeta(4)}{4}x^4 + ...]$$

pf)Euler's reflection formula에 의해

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \Leftrightarrow f(x-1)f(-x) = \frac{\pi}{\sin(\pi x)}$$

여기서 f(x)는 thm4이 f(x)와 같다. (f(x) = x!)

$$f(x)f(-x) = \frac{\pi x}{\sin(\pi x)}$$ (양변에 x 곱하고)

$$\ln(f(x)f(-x)) = \ln(\frac{\pi x}{\sin(\pi x)})$$ (양변에 자연로그)

위 식에 thm4 를 적용하면 증명 끝.□

$$\text{Thm6. } -\ln|2(\sin\pi x)| = \cos 2\pi x + \frac{1}{2}\cos 4\pi x + \frac{1}{3}\cos 6\pi x + ...$$

(정수 x를 제외한 모든 실수에서 성립.)

pf)

|y|<1인 복소수 y에 대해서,

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + ... \Rightarrow -\ln(1-y) = y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + ...$$

$y = re^{2\pi xi}$로 치환 (0<r<1) 그리고 실수부만 취하면

$$Re[-\ln(1-re^{2\pi xi})] = rcos(2\pi x) + \frac{1}{2}r^2\cos(4\pi x) + \frac{1}{3}r^3\cos(6\pi x) + ...$$

실수 x에 대해, x가 정수만 아니면, r(0<r<1)에 대해 연속이고, $\sum_{n=1}^{\infty}\frac{1}{n}\cos(2n\pi x)$이 수렴하므로,

abel's theorem에 의해

$$\lim_{r\to 1^-} Re[-\ln(1-re^{2\pi xi})] = \cos(2\pi x) + \frac{1}{2}\cos(4\pi x) + \frac{1}{3}\cos(6\pi x) + ...$$

실수 r(0<r<1)에 대해서 r=1일 때 연속??(<-이 부분 보충 필요)

$$\lim_{r\to 1^-} Re[-\ln(1-re^{2\pi xi})] = Re(\lim_{r\to 1^-}[-\ln(1-re^{2\pi xi})]) = Re[-\ln(1-e^{2\pi xi})]$$

여기서,

$$1-e^{2\pi xi}=1-\cos(2\pi x)-i\sin(2\pi x)=2\sin^2(\pi x)-2i\sin(\pi x)\cos(\pi x)$$
$$=2\sin(\pi x)\,[\sin(\pi x)-icos(\pi x)]\;=\;2\sin(\pi x)\,e^{\pi xi}$$

이므로

$$Re[-\ln(1-e^{2\pi xi})]\;=\;Re[-\ln(2\sin(\pi x)e^{\pi xi})]=-\ln|2\sin(\pi x)|\;\square$$

Thm7. (이거 굳이 필요 없는 듯)

$$-\frac{\zeta(3)}{4\pi^2}-\frac{1}{2}(\ln 2\pi)x^2-\frac{1}{2}x^2\ln x+\frac{3}{4}x^2+2[\frac{\zeta(2)}{(2)(3)(4)}x^4+\frac{\zeta(4)}{(4)(5)(6)}x^6+...]$$
$$=\frac{-1}{4\pi^2}\,[\cos(2\pi x)+\frac{1}{2^3}\cos(4\pi x)+\frac{1}{3^3}\cos(6\pi x)+...\,],$$ (0<x<1)

thm5 와 thm6에서,

$$-\ln(2\pi)-\ln x+2[\frac{\zeta(2)}{2}x^2+\frac{\zeta(4)}{4}x^4+\frac{\zeta(6)}{6}x^6+...]$$
$$=\cos(2\pi x)+\frac{1}{2}\cos(4\pi x)+\frac{1}{3}\cos(6\pi x)+...$$

위 식은 (0<x<1에서 항상 성립함.) 양변을 x에 대해 적분(0<x<1에서는 연속이므로
([a,b] 0<a,b<1 에서 적분 가능)

$$C-\ln(2\pi)x-x\ln x+x+2[\frac{\zeta(2)}{(2)(3)}x^3+\frac{\zeta(4)}{(4)(5)}x^5+\frac{\zeta(6)}{(6)(7)}x^7+...]$$
$$=\frac{1}{2\pi}[\sin(2\pi x)+\frac{1}{2^2}\sin(4\pi x)+\frac{1}{3^2}\sin(6\pi x)+...]$$

양변을 $\int_a^x$ 로 적분하면 위와 같이 식이 정리된다. 0<a<x<1일 때, 항상 성립하고 연속이므로, x가 0-로 극한을 취할 때도 양변이 성립해야 함. 따라서 C=0이다.(또는 a를 0-로 극한)

$$-\ln(2\pi)x-x\ln x+x+2[\frac{\zeta(2)}{(2)(3)}x^3+\frac{\zeta(4)}{(4)(5)}x^5+\frac{\zeta(6)}{(6)(7)}x^7+...]$$
$$=\frac{1}{2\pi}[\sin(2\pi x)+\frac{1}{2^2}\sin(4\pi x)+\frac{1}{3^2}\sin(6\pi x)+...]$$

(0<x<1)

위 식을 이런 식으로 한 번 더 적분하면 나옴.□

Thm8.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{6}(k+x)^3\ln(k+x)-\frac{1}{2}\frac{1}{3}(\frac{1}{2}+\frac{1}{3})(k+x)^3$$

$$-\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{6}(k-x)^3\ln(k-x)-\frac{1}{2}\frac{1}{3}(\frac{1}{2}+\frac{1}{3})(k-x)^3$$

$$-\frac{1}{3}(\sum_{k=1}^{n}(\ln k))x^3-\frac{1}{3}nx^3-2(\sum_{k=1}^{n}\frac{1}{2}k^2\ln k-\frac{1}{4}k^2)x$$

$$=-\frac{1}{6}(\ln 2\pi)x^3-\frac{1}{6}x^3(\ln x)+\frac{1}{3!}(1+\frac{1}{2}+\frac{1}{3})x^3$$

$$-\frac{\zeta(3)}{4\pi^2}x+\frac{1}{8\pi^3}[\sin(2\pi x)+\frac{1}{2^4}\sin(4\pi x)+\frac{1}{3^4}\sin(6\pi x)+...]$$

pf) $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+...\qquad(|x|<1)$

$$\ln(1+\frac{x}{k})=\frac{1}{k}x-\frac{1}{2}\frac{1}{k^2}x^2+\frac{1}{3}\frac{1}{k^3}x^3-\frac{1}{4}\frac{1}{k^4}x^4+...\qquad(|\frac{x}{k}|<1)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln(1+\frac{x}{k})+(\ln n)x=\gamma x-\frac{1}{2}\zeta(2)x^2+\frac{1}{3}\zeta(3)x^3-...\quad(|x|<1)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln(k+x)-\sum_{k=1}^{n}\ln k+(\ln n)x=\gamma x-\frac{1}{2}\zeta(2)x^2+\frac{1}{3}\zeta(3)x^3-...$$

(thm3과 thm4에서도 유도 가능)

x대신-x를 넣은 식과 더한다.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln(k+x)+\sum_{k=1}^{n}\ln(k-x)-2\sum_{k=1}^{n}\ln k=-2[\frac{1}{2}\zeta(2)x^2+\frac{1}{4}\zeta(4)x^4+...]$$

$$=-\ln 2\pi-\ln x-[\cos(2\pi x)+\frac{1}{2}\cos(4\pi x)+\frac{1}{3}\cos(6\pi x)+...]$$

(thm5와 thm6에 의한 것.)

양변을 적분.

$$\lim_{n\to\infty}\sum_{k=1}^{n}(k+x)\ln(k+x)-\sum_{k=1}^{n}(k-x)\ln(k-x)-2nx-2(\sum_{k=1}^{n}\ln k)x$$

$$= -(\ln 2\pi)x-x\ln x+x-\frac{1}{2\pi}[\sin(2\pi x)+\frac{1}{2^2}\sin(4\pi x)+\frac{1}{3^2}\sin(6\pi x)+\dots]$$

한 번 더 적분.

$$\lim_{n\to\infty}\sum_{k=1}^{n}[\frac{1}{2}(k+x)^2\ln(k+x)-\frac{1}{4}(k+x)^2]-\sum_{k=1}^{n}[\frac{1}{2}(k-x)^2\ln(k-x)-\frac{1}{4}(k-x)^2]$$

$$-(\sum_{k=1}^{n}\ln k)x^2-2(\sum_{k=1}^{n}\frac{1}{2}k^2\ln k-\frac{1}{4}k^2)-nx^2$$

$$= -\frac{1}{2}(\ln 2\pi)x^2-\frac{1}{2}x^2\ln x+\frac{3}{4}x^2-\frac{\zeta(3)}{4\pi^2}$$

$$+\frac{1}{4\pi^2}[\cos(2\pi x)+\frac{1}{2^3}\cos(4\pi x)+\frac{1}{3^3}\cos(6\pi x)+\dots]$$

여기서 한 번 더 적분하면 끝.□

Thm8에서 x=1을 대입하면(1에서 좌극한을 취하면) 아래와 같은 식이 된다.

$$\lim_{n\to\infty}\frac{1}{6}(n+1)^3\ln(n+1)-\frac{1}{2}\frac{1}{3}(\frac{1}{2}+\frac{1}{3})(n+1)^3+[\frac{1}{6}n^3\ln(n)-\frac{1}{2}\frac{1}{3}(\frac{1}{2}+\frac{1}{3})n^3]$$

$$+\frac{5}{36}-\frac{1}{3}(\sum_{k=1}^{n}(\ln k))-\frac{1}{3}n-2(\sum_{k=1}^{n}\frac{1}{2}k^2\ln k-\frac{1}{4}k^2)$$

$$=-\frac{1}{6}(\ln 2\pi)+\frac{11}{36}-\frac{\zeta(3)}{4\pi^2}$$

(이 식에 thm1(스털링 근사)을 적용시키고, 마지막으로 thm9를 적용시키면 끝난다.)
일단 스털링 근사를 적용하면 이렇게 됨.(+식을 조금 정리)

$$\lim_{n\to\infty}\frac{1}{6}(n+1)^3\ln(n+1)+\frac{1}{6}n^3\ln(n)-\frac{5}{36}(n+1)^3-\frac{5}{36}n^3+\frac{5}{36}$$

$$-\frac{1}{3}(\frac{1}{2}\ln 2\pi+\frac{1}{2}\ln n+n\ln n-n)-\frac{1}{3}n-2(\sum_{k=1}^{n}\frac{1}{2}k^2\ln k-\frac{1}{4}k^2)$$

$$=-\frac{1}{6}(\ln 2\pi)+\frac{11}{36}-\frac{\zeta(3)}{4\pi^2}$$

$$\lim_{n\to\infty}\frac{1}{6}(n+1)^3\ln(n+1)+\frac{1}{6}n^3\ln(n)-\frac{5}{36}(2n^2+3n+3n)$$

$$-\frac{1}{3}(\frac{1}{2}\ln n+n\ln n)-(\sum_{k=1}^{n}k^2\ln k)+\frac{1}{6}n^3+\frac{1}{4}n^2+\frac{1}{12}n$$

$$=\frac{11}{36}-\frac{\zeta(3)}{4\pi^2}$$

Thm9.

$$\lim_{n\to\infty}\frac{1}{6}(n+1)^3\ln(n+1)+\frac{1}{6}n^3\ln n-\frac{5}{36}(2n^3+3n^2+3n)-\frac{1}{3}(\frac{1}{2}\ln n+n\ln n)$$

$$+(\frac{n^3}{6}+\frac{n^2}{4}+\frac{n}{12})(1-2\ln n)+\frac{n^3}{9}-\frac{n}{12}=\frac{11}{36}$$

pf)

$$\lim_{n\to\infty}\frac{1}{6}(n+1)^3\ln(n+1)+\frac{1}{6}n^3\ln n-\frac{5}{36}(2n^3+3n^2+3n)-\frac{1}{3}(\frac{1}{2}\ln n+n\ln n)$$

$$+(\frac{n^3}{6}+\frac{n^2}{4}+\frac{n}{12})(1-2\ln n)+\frac{n^3}{9}-\frac{n}{12}$$

$$=\lim_{n\to\infty}(\frac{1}{6}n^3+\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{6})(\ln(n+1)-\ln n)$$

$$(\frac{-10+6+4}{36})n^3+(\frac{-5+3}{12})n^2+(\frac{-5+1-1}{12})n$$

$$=\lim_{n\to\infty}(\frac{1}{6}n^3+\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{6})(\ln(1+\frac{1}{n}))-\frac{1}{6}n^2-\frac{5}{12}n$$

$$=\lim_{n\to\infty}(\frac{1}{6}n^3+\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{6})(\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\frac{1}{3}\frac{1}{n^3}-\ldots)-\frac{1}{6}n^2-\frac{5}{12}n$$

$$=\lim_{n\to\infty}\frac{1}{6}n^2+(\frac{-1+6}{12})n+(\frac{1}{18}-\frac{1}{4}+\frac{1}{2})+()\frac{1}{n}+()\frac{1}{n^2}\ldots-\frac{1}{6}n^2-\frac{5}{12}n$$

$$=\lim_{n\to\infty}\frac{11}{36}+()\frac{1}{n}+()\frac{1}{n^2}+\ldots=\frac{11}{36}\quad\square$$

$$\frac{1}{2}\ln 2\pi = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\ln k\right) - \left(n+\frac{1}{2}\right)\ln n + n$$

$$\ln A = \lim_{n\to\infty}\left(\sum_{k=1}^{n}k\ln k\right) - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}$$

$$\frac{\zeta(3)}{4\pi^2} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}k^2\ln k\right) - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\ln n + \frac{n^3}{9}-\frac{n}{12}$$

$$-\ln B = \lim_{n\to\infty}\left(\sum_{k=1}^{n}k^3\ln k\right) - \left(\frac{n^3}{4}+\frac{n^2}{2}+\frac{n}{4}-\frac{1}{120}\right)\ln n + \frac{n^3}{16}-\frac{n^2}{12}$$

$$-\frac{3\zeta(5)}{4\pi^4} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}k^4\ln k\right) - \left(\frac{n^3}{5}+\frac{n^2}{2}+\frac{n}{3}-\frac{1}{30}\right)\ln n + \frac{n^3}{25}-\frac{n^2}{12}+\frac{13}{360}n$$

B는 편의상 정한 상수이며, 음수인 것을 강조하기 위해서 –lnB로 표기.

$\frac{1}{2}\ln 2\pi = -\zeta'(0)$	$\frac{1}{2}\ln 2\pi = -\zeta'(0)-(0)\zeta(0)$
$\ln A = -\zeta'(-1)+\frac{1}{12}$	$\ln A = -\zeta'(-1)-(1)\zeta(-1)$
$\frac{\zeta(3)}{4\pi^2} = -\zeta'(-2)$	$\frac{\zeta(3)}{4\pi^2} = -\zeta'(-2)-(1+\frac{1}{2})\zeta(-2)$
$-\ln B = -\zeta'(-3)-\frac{11}{720}$	$-\ln B = -\zeta'(-3)-(1+\frac{1}{2}+\frac{1}{3})\zeta(-3)$
$-\frac{3\zeta(5)}{4\pi^4} = -\zeta'(-4)$	$-\frac{3\zeta(5)}{4\pi^4} = -\zeta'(-4)-(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})\zeta(-4)$

$\zeta(0) = -\frac{1}{2}$, $\zeta(-2)=\zeta(-4)=\zeta(-6)=\ldots=0$

$\zeta(-1) = -\frac{1}{12}$, $\zeta(-3) = \frac{1}{120}$