Master's Thesis

Denseness of L-functions at $s = 1$ for cyclic extension of prime degree

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2021

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Abstract

The universality [1] asserts that some families of L-functions are dense in some set of functions. Similarly, the denseness in this thesis asserts that value of L-functions at s=1 (i.e, $L(1)$) for some families are dense in $\mathbb{R}^+$ or $\mathbb{C}$.

Some L-functions are highly related to number field. With the cyclic extensions of prime degree, corresponding L-functions has the denseness. The Dedekind zeta function $\zeta_K(s)$ is depend on field K. The set of L-functions $L(s,K) = \zeta_K(s)/\zeta(s)$ has denseness, or equivalently, the set of fields K from the $L(s,K)$ has denseness. This is first form of denseness. When the K/Q is cyclic extension of prime degree, $\zeta_K(s)$ can be written as products of Dirichlet L-functions. The set of this Dirichlet L-functions has denseness, or equivalently, set of the Dirichlet characters from the L-functions has denseness. This is second form of denseness in this thesis. Proving these two forms of denseness are main theorems.

Contents

List of Figures

I Introduction

1.1 Preliminary

Definition I.1 *(Charcter). For a group G, a function $f : G \to \mathbb{C}$ such that*

$$f(ab) = f(a)f(b)$$

*for any a,b $\in$ G, and there is $c \in G$ such that $f(c)$ is nonzero is called a **character** of G.*

Definition I.2 *(residue class). For $a \in \mathbb{Z}$, the **residue class** $\hat{a}$ modulo k is the set*

$$\hat{a} = \{x : \ x \equiv a \pmod k\}.$$

For residue classes $\hat{a}, \hat{b}$, multiplication is defined by

$$\hat{a} \cdot \hat{b} = \hat{ab}.$$

Definition I.3 *(Dirichlet character). For the group G of reduced residue classes modulo k. Corresponding to each character f of G, a function $\chi = \chi_f$ is defined by :*

$$\begin{array}{rcll} \chi(n) & = & f(\hat{n}) & \text{if } gcd(n,k) = 1, \\ \chi(n) & = & 0 & \text{if } gcd(n,k) \neq 1. \end{array}$$

*This function χ is called a **Dirichlet character** modulo k.*

If a Dirichlet character ψ has following properties

$$\psi(n) = \begin{cases} 1 & \text{if } gcd(n,k) = 1, \\ 0 & \text{if } gcd(n,k) \neq 1. \end{cases}$$

*then this ψ is called a **principal character**.*

For a Dirichlet character χ and integer N,M, if $N|M$ and there is a Dirichlet character χ' such that $\chi(n) = \chi'(n)$ for any $n \in \mathbb{Z}$ with $(n,M) = 1$. The smallest such N is called the **conductor** *of χ. If a modulus of a Dirichlet character is equals to conductor of the Dirichlet character, the Dirichlet character is called **primitive**.*

Example I.4

Table 1: Dirichlet characters modulo 3

n	0	1	2	3	4	5	6	7	8	9
$\chi_3(1,n)$	0	1	1	0	1	1	0	1	1	0
$\chi_3(2,n)$	0	1	-1	0	1	-1	0	1	-1	0

There are only two distinct Dirichlet characters modulo 3 and the Dirichlet characters modulo 3 are periodic with period 3. Generally, the following Proposition is true [2, p. 138].

Proposition I.5 *There exist $\varphi(k)$ distinct Dirichlet characters modulo k, for each χ, we have*

$$\chi(mn) = \chi(m)\chi(n) \tag{1}$$

for any m, n and

$$\chi(n+k) = \chi(n) \tag{2}$$

for any n.

Converse of this Proposition is also true.

Example I.6

Table 2: Dirichlet characters modulo 7 ($\omega = e^{2\pi i/3}$)

n	0	1	2	3	4	5	6
$\chi_7(1,n)$	0	1	1	1	1	1	1
$\chi_7(2,n)$	0	1	ω^2	ω	ω	ω^2	1
$\chi_7(3,n)$	0	1	ω	$-\omega^2$	ω^2	$-\omega$	-1
$\chi_7(4,n)$	0	1	ω	ω^2	ω^2	ω	1
$\chi_7(5,n)$	0	1	ω^2	-ω	ω	$-\omega^2$	-1
$\chi_7(6,n)$	0	1	1	-1	1	-1	-1

There are $\varphi(7) = 6$ distinct Dirichlet characters modulo 7 and the group of Dirichlet characters modulo 7 with multiplication is isomorphic to unit group of $\mathbb{Z}/7\mathbb{Z}$. i.e, $\{\chi_7(k,\cdot)|\, k \in \{1,2,3,4,5,6\}\} \cong (\mathbb{Z}/7\mathbb{Z})^{\times}$.

For $s \in \mathbb{C}$, let σ be the real part of s and t be the imaginary part of s so $s = \sigma + it$.
For $\sigma > 1$ the Riemann zeta function $\zeta(s)$ can be defined as following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3}$$

The $\zeta(s)$ can be expressed as a product, which is called "the Euler product".

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} \left(1 - \frac{1}{p}\right)^{-1}. \tag{4}$$

For $\sigma > 1$, the Dirichlet L-functions $L(s,\chi)$ defined as following series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{5}$$

where the χ is a Dirichlet character. This also can be expressed as a product, which also is "Euler product"

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p:\text{prime}} \left(1 - \frac{\chi(p)}{p}\right)^{-1}. \tag{6}$$

Actually, The Riemann zeta function $\zeta(s)$ is a special case of Dirichlet L-function. The only character modulo 1 is χ that satisfies $\chi(n) = 1$ for all integer, resulting $L(s, \chi) = \zeta(s)$.

Definition I.7 *(Hurwitz zeta function) The* ***Hurwitz zeta function*** $\zeta(s,a)$ *for* $\sigma > 1$ *can be defined as following series*

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}. \tag{7}$$

For $\sigma \leq 1$, we use a contour integral [2, p. 253]. contour integral. The contour C consists of three parts C_1, C_2 and C_3 as shown in Figure 1.

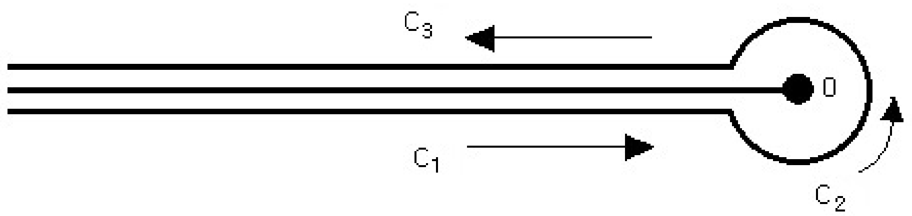

Figure 1: Contour C

Definition I.8 *(analytic function, holomorphic function)*

*For open set U, if a function f is complex differentiable for every point $z_0 \in U$, then the f is called a **holomorphic** function on U .*

For any x_0 in a open set U, if a function f can be written with the series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots,$$

*then the function f is called a **analytic** function on U.*

The Taylor series implies

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

*Hence the **holomorphic function** and **analytic function** are interchangable concepts for complex differentiable functions. Note that if a function f is complex differentiable at z_0, f is infinitely differentiable at z_0.*

Definition I.9 *(entire function)If a function f is holomorphic on any open set $U \subset \mathbb{C}$, the function f is called a **entire** function.*

Proposition I.10 *For $0 < a \leq 1$, if we define the function $I(s,a)$ with the contour integral*

$$I(s,a) = \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^{az}}{1 - e^z} dz, \tag{8}$$

where the contour C is from figure 1. Then the function $I(s,a)$ is an entire function of s. Moreover, we have

$$\zeta(s,a) = \Gamma(1-s) I(s,a) \quad \text{if } \sigma > 1. \tag{9}$$

Definition I.11 *For $\sigma \leq 1$, we can define $\zeta(s,a)$ by*

$$\zeta(s,a) = \Gamma(1-s) I(s,a). \tag{10}$$

This provides the extension of $\zeta(s,a)$ to $\sigma \leq 1$, which is called an analytic continuation of $\zeta(s,a)$.

Proposition I.12 *For $\sigma > 1$ we have*

$$\zeta(s) = \zeta(s,1) \tag{11}$$

and

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r) \zeta\left(s, \frac{r}{k}\right), \tag{12}$$

where χ is any Dirichlet character mod k.

We use (11) and (12) to extend $\zeta(s)$ and $L(s,\chi)$ for $\sigma \leq 1$ as the definitions.

Proposition I.13 *(a) The Riemann zeta function $\zeta(s)$ is analytic for every $s \neq 1$ and has a simple pole at $s = 1$ with residue 1.*
(b) If the character χ mod k is principal, the L-function $L(s,\chi)$ is analytic for every $s \neq 1$ and has a simple pole at $s = 1$ with residue $\varphi(k)/k$.
(c) If χ is not principal, $L(s,\chi)$ is an entire function of s.

Proposition I.14 *For all s we have*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \tag{13}$$

or, equivalently,

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s), \tag{14}$$

where Γ *is the gamma function*

Proposition I.15 *(Functional equation for Dirichlet L-functions) If* χ *is a primitive character mod k then for any s,*

$$L(1-s,\chi) = \frac{k^{s-1}\Gamma(s)}{(2s\pi)^s}\{e^{-\pi is/2}+\chi(-1)e^{\pi is/2}\}G(1,\chi)L(s,\overline{\chi}), \tag{15}$$

where $G(m,\chi)$ *is the Gauss sum with* χ*,*

Definition I.16 *(Dedekind domian) An integral domain A is* ***Dedekind domain*** *if every nonzero proper ideal of A has a unique factorization into prime ideals.*

The for C_l field $K/\mathbb{Q}$ and we consider O_K, which is the ring of integer of K. A prime number p can be represented by the ideal $(p) = \{px | x \in \mathbb{Z}\}$ or $(p) = \{px | x \in O_K\}$. In $\mathbb{Z}$, the ideal (p) is a prime ideal of $\mathbb{Z}$ but in O_K, (p) is not always a prime ideal. Now we consider the ideal factorization of (p) in O_K.

$$(p) = \mathfrak{p}_1^{e_1}\dots\mathfrak{p}_2^{e_g}$$

For the positive integers $\{e_1,\dots,e_g\}$, we say that the (p) is **ramified** if $e_i > 1$ for some i. If $e_i = 1$ for all i, we say that a prime (p) is **split** . If $g = e = 1$, we say that (p) is **inert**.

We also say that a prime number p is **ramified** if (p) is ramified, p is **split** if (p) is split and p is **inert** if (p) is inert.

For the case of cyclic extension of prime degree l, there are only 3 types of factorization, totally ramify($e_1 > 1$,$g = 1$), totally split, and inert.

Example I.17 *Let* $\alpha_1 = \zeta_7^1+\zeta_7^{-1}$*,* $\alpha_2 = \zeta_7^2+\zeta_7^{-2}$ *and* $\alpha_3 = \zeta_7^3+\zeta_7^{-3}$*. Then* $\alpha_1,\alpha_2,\alpha_2$ *are solutions of* $x^3+x^2-2x-1=0$*.*

$\alpha_1^2 = \alpha_2+2$ *implies* $\alpha_2 \in \mathbb{Q}(\alpha_1)$ *and* $[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}] = 3$*. This implies* $[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}(\alpha_1)] = 1$*.Hence,* $\mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_1,\alpha_2)$*. From* $\alpha_1+\alpha_2+\alpha_3 = -1$*, we can show that* $\mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_1,\alpha_2) = \mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)$*. With similar computations, it gives the result*

$$\mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_2) = \mathbb{Q}(\alpha_3) = \mathbb{Q}(\alpha_1,\alpha_2,\alpha_3).$$

$\{\sigma_{i,j}:\alpha_i \to \alpha_j\}$ *is automorphisms that fixes* $\mathbb{Q}$ *in the field* $\mathbb{Q}(\alpha_1)$*.Now we have* $\sigma_{1,2} = \sigma_{2,3} = \sigma_{3,1}$ *and* $\sigma_{2,1} = \sigma_{3,2} = \sigma_{1,3}$*,* $\sigma_{i,j}^2 = \sigma_{j,i}$ *and* $\sigma_{i,j}^3 = \sigma_{i,i}$*. By these facts, we can conclude that* $Gal(\mathbb{Q}(\alpha_1)/\mathbb{Q}) \cong C_3$*.*

Example I.18 *Let* α *be a solution of* $x^3+x^2-2x-1=0$*.*
The ideals $(\alpha),(\alpha+2)$ *and* $(\alpha-1)$ *are equal to* (1) *because* $\alpha(\alpha+2)(\alpha-1) = 1$ *by the original polynomial.*

Let $K = \mathbb{Q}(\alpha)$ and O_K be the ring of integer of $\mathbb{Q}(\alpha)$. Then the ideal (7) is ramifies in O_K because

$$(\alpha-2)^3 = ((\alpha-2)^3 - (\alpha^3+\alpha^2-2\alpha-1))$$

$$= (-7(\alpha^2-2\alpha+1)) = (7)(\alpha-1)^2 = (7).$$

The second equality can be done by $\alpha^3+\alpha^2-2\alpha-1=0$.

The ideal (13) is split in O_K. The explicit factorization is

$$(13) = (13,\alpha+3)(13,\alpha+5)(13,\alpha+6).$$

Actually, $(13,\alpha+5) = (2\alpha-3)$. First, $2(\alpha+5)-13 = 2\alpha-3$ implies $2\alpha-3 \in (13,\alpha+5)$. Second,

$$(2\alpha-3)(\alpha^2-4\alpha+5) = 2(\alpha-2)^3+(\alpha-1)^2 = -14(\alpha-1)+(\alpha-1)^2 = -13(\alpha-1)^2,$$

so $13 \in (2\alpha-3)$. Moreover, $\alpha+5 = -6(2\alpha-3)+13(\alpha-1)$. The factorization is now

$$(13) = (2\alpha-3)(13,\alpha+3)(13,\alpha+6).$$

The ideal (2) is inert in O_K since (2) is a prime ideal of O_K.

Definition I.19 *(Dedekind zeta function) For an algebraic number field K, let O_K be the ring of integer of K. Then the* ***Dedekind zeta function*** *ζ_K is the series*

$$\zeta_K(s) = \sum_{I\text{:non-zero ideal of } O_K} \frac{1}{[O_K : I]^s} \quad \text{for } \sigma > 1. \tag{16}$$

The Dedekind zeta function has an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}\text{:prime ideal of } O_K} \left(1 - \frac{1}{[O_K:\mathfrak{p}]^s}\right)^{-1} \quad \text{for } \sigma > 1.$$

The the Riemann zeta function is just a special case of the Dedekind zeta function. When $K = \mathbb{Q}$,

$$\zeta_Q(s) = \zeta(s).$$

For $\sigma < 1$, we can use following proposition as the definition of the Dedekind zeta function.

Proposition I.20 *Let K be a number field, If $Gal(K/\mathbb{Q})$ is a cyclic group C_l of prime degree l, then*

$$\zeta_K(s) = \prod_{p\text{:ramified}} \left(1-\frac{1}{p^s}\right)^{-1} \prod_{p\text{:split}} \left(1-\frac{1}{p^s}\right)^{-l} \prod_{p\text{:inert}} \left(1-\frac{1}{p^{ls}}\right)^{-1} = \zeta(s)\prod_{k=1}^{l-1} L(s,\chi^k) \tag{17}$$

for some Dirichlet character χ with conductor N.

Or equivalently,

$$\frac{\zeta_K(s)}{\zeta(s)} = \prod_{p\text{:split}} \left(1-\frac{1}{p^s}\right)^{-(l-1)} \prod_{p\text{:inert}} \frac{\left(1-\frac{1}{p^{ls}}\right)^{-1}}{\left(1-\frac{1}{p^s}\right)^{-1}} = \prod_{k=1}^{l-1} L(s,\chi^k). \tag{18}$$

The conductor N of χ is equal to **conductor** of K so the conductor of χ and conductor of K is interchangeable for the cyclic extension of prime degree. Further, the number N^{l-1} is equal to the **discriminant** [3, p. 172] of K.

Definition I.21 *(Discriminant) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is an n-tuple of elements of L/K we define the **discriminant** $\Delta(\alpha_1, \ldots, \alpha_n)$ to be $det(Tr_{L/K}(\alpha_i \alpha_j))$.*

The Discriminant also represented by integral basis and their embeddings, which a square of determinant of a matrix as following example.

Example I.22 *Let $\alpha_1, \alpha_2, \alpha_3$ be the solutions of $x^3 + x^2 - 2x - 1 = 0$. Then $\mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_2) = \mathbb{Q}(\alpha_3)$ for this special field and the discriminant is*

$$det \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \alpha_2 \end{pmatrix}^2 = (3\alpha_1\alpha_2\alpha_3 - \alpha_1^3 - \alpha_2^3 - \alpha_3^3)^2$$

The discriminant of the field is 49.

Example I.23 *Let α be a solution of $x^3 + x^2 - 2x - 1 = 0$. Then $Gal(\mathbb{Q}(\alpha)/\mathbb{Q}) \cong C_3$ and the discriminant of $\mathbb{Q}(\alpha)$ is $49 (= 7^2)$ and the conductor $N = 7$. The (18) become*

$$\frac{\zeta_{\mathbb{Q}(\alpha)}(s)}{\zeta(s)} = \prod_{p \equiv 1,6 (mod\, 7)} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \equiv 2,3,4,5 (mod\, 7)} \frac{\left(1 - \frac{1}{p^{3s}}\right)^{-1}}{\left(1 - \frac{1}{p^s}\right)^{-1}}$$

$$= \prod_{p \equiv 1,6 (mod\, 7)} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \equiv 2,3,4,5 (mod\, 7)} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

$$= \prod_p \left(1 - \frac{\chi_7(2,p)}{p^s}\right)^{-1} \prod_p \left(1 - \frac{\chi_7(4,p)}{p^s}\right)^{-1}$$

$$= L(s, \chi_7(2,\cdot)) \cdot L(s, \chi_7(4,\cdot)),$$

where $\chi_7(2,\cdot)$ and $\chi_7(4,\cdot)$ are the Dirichlet characters modulo 7 in Example I.6. $\chi_7(2,\cdot)^2 = \chi_7(4,\cdot)$ and $\chi_7(4,\cdot)^2 = \chi_7(2,\cdot)$.
When $p \equiv 1,6 (mod\, 7)$, the prime p is split and $\chi_7(2,p) = \chi_7(4,p) = 1$ so

$$\left(1 - \frac{\chi_7(2,p)}{p^s}\right)^{-1} \left(1 - \frac{\chi_7(4,p)}{p^s}\right)^{-1} = \left(1 - \frac{1}{p^s}\right)^{-2}.$$

When $p \equiv 2,3,4,5 (mod\, 7)$, the prime p is inert, $\chi_7(2,p) + \chi_7(4,p) = -1$ and $\chi_7(2,p)\chi_7(4,p) = 1$ so

$$\left(1 - \frac{\chi_7(2,p)}{p^s}\right)^{-1} \left(1 - \frac{\chi_7(4,p)}{p^s}\right)^{-1} = \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}}\right)^{-2},$$

$$\zeta_{\mathbb{Q}(\alpha)}(s) = \zeta(s) L(s, \chi_7(2,\cdot)) L(s, \chi_7(4,\cdot)),$$

$$\left(1-\frac{1}{7^s}\right)\zeta_{\mathbb{Q}(\alpha)}(s) = L(s,\chi_7(1,\cdot))L(s,\chi_7(2,\cdot))L(s,\chi_7(4,\cdot)).$$

$X^2 \equiv a(mod\,7)$ have solutions if and only if $a \equiv 1(mod\,7)$ or $a \equiv 2(mod\,7)$ or $a \equiv 4(mod\,7)$. Equivalently, the solutions of $X^3 \equiv 1(mod\,7)$ are 1,2, and 4.

For a number field K, proposition I.20 says that the function ζ_K/ζ is a product of Dirichlet L-functions. By Proposition I.13 (c), each factor of the Dirichlet L-function is entire. Now we define

$$L(s,K) := \prod_{k=1}^{l-1} L(s,\chi^k) = \frac{\zeta_K(s)}{\zeta(s)}. \tag{19}$$

It shows that $L(s,K)$ is entire function so we can define $L(1,K)$. The values of $L(1,K)$ is various and actually $\{L(1,K)|Gal(K/\mathbb{Q}) \cong C_l\}$ is dense in $\mathbb{R}^+$. Moreover, the set of all of corresponding Dirichlet L-functions is dense in $\mathbb{C}$. We prove them in section IV.

1.2 Historical background

Bohr and Courant showed that for any fixed $\sigma \in \mathbb{R}$ with $1/2 < \sigma \leq 1$, $\{\zeta(\sigma+it)|t \in \mathbb{R}\}$ is dense in $\mathbb{C}$ [4]. In 1975, Voronin [1] obtained the result that is called **universality** of the Riemann zeta function. For any $\{\zeta(s+i\tau)|0<\tau<T\}$ $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\left|\{0<\tau<T|\max_{x\in C}|\zeta(s+i\tau)-h(s)|<\varepsilon\}\right| > 0,$$

where $h(s)$ is a holomorphic and $h(s) \neq 0$ on $D = \{s=\sigma+it|1/2<\sigma<1\}$ and C is a compact set in D.

Mishou and Nagoshi [5] showed **universality** of $L(s,\chi_d)$ and they also showed analogue **denseness** theorem, which says that $\{L(1,\chi_d)|d \in D^\gamma\}$ is dense in $\mathbb{R}^+$. In other words, for any $x_0 \in \mathbb{R}^+$ and $\varepsilon > 0$ we have

$$\liminf_{X\to\infty} \frac{1}{\#D_X^\gamma}\#\{d \in D_X^\gamma : |L(1,\chi_d)-x_0|<\varepsilon\}. \tag{20}$$

Cho and Kim [6] generalize Mishou and Nagoshi's result to number fields of higher degree for the family $L(X)^{(r2)} = \{K \mid K : S_{d+1}-\text{field with signature } (r_1,r_2),\ \frac{X}{2} < |d_k| < X\}$, $d+1 = 3,4$, and 5. The Artin conjecture is assumed for S_5 case.

We show the analogous **denseness** theorems for cyclic fields of prime degree. Note that Mishou and Nagoshi's result is some $S_2(=C_2)$ case. We generalize the Mishou and Nagoshi's result to $C_3, C_5, C_7, \ldots$ while Cho and Kim to S_3, S_4 and S_5.

As in Proposition I.20, the Dedekind zeta function $\zeta_K(s)$ can be expressed with the following decomposition:

$$\zeta_K(s) = \zeta(s)L(s,K).$$

Then the $L(s,K)$ has the decomposition :

$$L(s,K) = \prod_{k=1}^{l-1} L(s,\chi^k). \tag{21}$$

Now define

$$D_l = \{L(s,\chi) | L(s,\chi) \text{ is a factors of } L(s,K), K: C_l\text{-field of prime degree } l \geq 3\} \tag{22}$$

$$D_l(X) = \{L(s,\chi) \in D(X) | \chi \text{ of order l}, N_\chi \leq X\}, \tag{23}$$

$$L_l(X) = \{K | K: C_l\text{-field with discriminant } D_K \leq X^{l-1}\}. \tag{24}$$

where N_χ is the conductor of χ and D_K is the discriminant of K. Actually, $D_K = N_\chi^{l-1}$ for (21) and this implies

$$|D_l(X)| = (l-1)|L_l(X)|. \tag{25}$$

The set D_l can be seen as set of characters χ instead of set of L-functions, and then D_l also can be seen as set of characters χ. The set L_l also can be seen as set of L-functions $L(s,K)$ instead of set of C_l-fields K.

II Counting number field

In [7], Mäki showed that for any $\varepsilon > 0$, the number of abelian number fields over $\mathbb{Q}$ is given by

$$N_{con}(X) = XP(\log X) + O\left(X^{1-\frac{3}{\nu_0+6}+\varepsilon}\right). \tag{26}$$

In [8], Wood showed that the probability of a fixed unramified prime p completely splitting in a random field extension, is the same as the Chebotarev density of a random prime p in a fixed field extension. She also proved the independence of local splitting conditions. From the Mäki and Wood's result, we can predict followings.

(a). Let $C_l(X)$ be the number of cyclic extensions K of degree l such that $D_K \leq X^{l-1}$ (equivalently, $f_K \leq X$). Then, we have

$$C_l(X) = \mathfrak{C}_l X + O\left(X^{\frac{l+2}{l+5}}\right), \tag{27}$$

where

$$\mathfrak{C}_l = \frac{1}{l}\left(1+\frac{l-1}{l^2}\right)\prod_{p\equiv 1 \bmod l}\left(1+\frac{l-1}{p}\right)\left(1-\frac{1}{p}\right)\prod_{p\not\equiv 1, p\not\equiv 0 \bmod l}\left(1-\frac{1}{p}\right). \tag{28}$$

(b). Let $C_l(X,\Sigma)$ be the number of C_l-fields K of degree l such that $f_K \leq X$ and K satisfies all the local conditions in Σ. Then, we have

$$C_l(X,\Sigma) = \prod_{LC_q \in \Sigma} |LC_q| \mathfrak{C}_l X + O\left(X^{\frac{l+2}{l+5}}\right). \tag{29}$$

For (a), let $c_l(n)$ be the number of C_l-fields with conductor n. Define

$$F(s) := \sum_{n=1}^{\infty} \frac{c_l(n)}{n^s} = \frac{1}{l-1}\left(1+\frac{1}{l^{2s}}\right)\prod_{p\equiv 1 \bmod l}\left(1+\frac{l-1}{p^s}\right),$$

then

$$C_l(X) = \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} \frac{F(s)}{s} X^s dX.$$

Now,we can find the constant $\mathfrak{C}_l$ explicitly

$$\begin{aligned}
\mathfrak{C}_l &= \lim_{X\to\infty}\frac{C_l(X)}{X} = \lim_{s\to 1}(s-1)F(s) = \lim_{s\to 1}\frac{(s-1)F(s)}{(s-1)\zeta(s)} = \lim_{s\to 1}\frac{F(s)}{\zeta(s)} \\
&= \lim_{s\to 1}\frac{1}{l-1}\left(1+\frac{l-1}{l^{2s}}\right)\prod_{p\equiv 1 \bmod l}\left(1+\frac{l-1}{p^s}\right)\prod_{p}\left(1-\frac{1}{p^s}\right) \\
&= \lim_{s\to 1}\frac{1}{l-1}\left(1+\frac{l-1}{l^{2s}}\right)\left(1-\frac{1}{l^s}\right)\prod_{p\equiv 1 \bmod l}\left(1+\frac{l-1}{p^s}\right)\left(1-\frac{1}{p^s}\right)\prod_{p\not\equiv 1, p\not\equiv 0 \bmod l}\left(1-\frac{1}{p^s}\right) \\
&= \frac{1}{l-1}\left(1+\frac{l-1}{l^2}\right)\left(1-\frac{1}{l}\right)\prod_{p\equiv 1 \bmod l}\left(1+\frac{l-1}{p}\right)\left(1-\frac{1}{p}\right)\prod_{p\not\equiv 1, p\not\equiv 0 \bmod l}\left(1-\frac{1}{p}\right).
\end{aligned}$$

For (b), we can consider three kinds of local conditions LC_q at prime q, which are $Split_q$, $Inert_q$, and TR_q. A cyclic extension K satisfy $Split_q$, $Inert_q$ or TR_q means the prime q is totally split, inert, or totally ramified in K respectively. Let's define the probability of local conditions:

$$\left|TR_q\right| = \begin{cases} \frac{l-1}{l^2+l-1} & \text{if } q = l, \\ \frac{l-1}{q+l-1} & \text{if } q \equiv 1 \bmod l, \\ 0 & \text{otherwise.} \end{cases} \quad \left|Split_q\right| = \begin{cases} \frac{l}{l^2+l-1} & \text{if } q = l, \\ \frac{q}{l(q+l-1)} & \text{if } q \equiv 1 \bmod l, \\ \frac{1}{l} & \text{otherwise.} \end{cases} \tag{30}$$

We also can consider a local condition $Frob_q = C$, which means that the Frobenius automorphism of q is C as a subcondition of $Inert_q$ or $Split_q$. In fact, $|Frob_q = C| = |split_q|$ and $\left|Inert_q\right| = (l-1)\left|Split_q\right|$.

Let Σ be a finite collection of local conditions at different primes q. We can count cyclic extensions K of degree l with $f_K \leq X$ which satisfies all the local conditions in Σ.

III Zero free region and bound

Under GRH, Littlewood [9] obtained the bound

$$\left(\frac{1}{2}+o(1)\right)\frac{\zeta(2)}{e^{\gamma}\log\log|D|} \leq L(1,\chi_D) \leq (2+o(1))e^{\gamma}\log\log|D|, \tag{31}$$

where the γ is the Euler-Mascheroni constant and χ_D is the quadratic character with fundamental discriminant D of the quadratic extension $K=\mathbb{Q}(\sqrt{D})$.

Cho and Kim [10] obtained analogue proposition for $S_{d+1}(s=2,3,4)$ fields. We can prove the analogue proposition for extension of prime degree $l \geq 3$.

For $\sigma > 1$, $L(s,K)$ can be expressed as the Euler product:

$$L(s,K)=\prod_p \prod_{i=i}^{l-1}\left(1-\frac{\alpha_i(p)}{p^s}\right)^{-1}. \tag{32}$$

For $\sigma > 1$,

$$\log L(s,K)=\sum_{n=2}^{\infty}\frac{\Lambda(n)a_K(n)}{n^s\log n}, \tag{33}$$

where $a_K\ (p^k)=\alpha_1(p)^k+\cdots+\alpha_l(p)^k$.

Proposition III.1 *Let N be the conductor of K(equivalently, conductor of some Dirichlet character χ), $x=(\log N_\chi)^\beta$ and $\beta(1-\alpha)>2$. If $L(s,K)$ is entire and it has no zeros in $[\alpha,1]\times[-x,x]$ then*

$$\log L(1,K)=\sum_{n<x}\frac{\lambda(n)a_K(n)}{n\log n}+O((\log N)^{-1}). \tag{34}$$

The $L(s,K)$ is entire for $K=C_l$ cases. Some of $L(s,K)$ might not be always zero-free in the rectangle but at least, we can prove that it is mostly zero free. More explicitly, the number of zeros of L-functions on $[\alpha,1]\times[-(\log X)^\beta,(\log X)^\beta]$ is $O(X^k)$ with $k<1$, since this is the case of $A=n=d=1$, $T\leq(\log X)^\beta$, and $\alpha\geq 3/4$ can be chosen satisfying $\frac{c_0+1}{c_0+2}<\alpha$ so that $c_0(\frac{1-\alpha}{2\alpha-1})<1$(see Theorem 2 in [11]). If $f(X)<1$ is an arbitrary positive function, then $\frac{c_0+f(X)}{c_0+2f(X)}<\alpha$ gives $c_0(\frac{1-\alpha}{2\alpha-1})<f(X)$. Let K_1 and K_2 be distinct two fields. Note that $\zeta_{K_1}(s)/\zeta(s)=L(s,\chi_1)\cdots L(s,\chi_1^{l-1})$ and $\zeta_{K_2}(s)/\zeta(s)=L(s,\chi_2)L(s,\chi_2^2)\cdots L(s,\chi_2^{l-1})$, so if $L(s,\chi_1^m)=L(s,\chi_2^n)$, since the characters form a multiplicative cyclic group of order l, then $\zeta_{K_1}(s)$ and $\zeta_{K_2}(s)$ coincide each other. So the number of zeros of L-functions cannot be accumulated further by sharing the same specific L-function. Thus, the number of zero-non-free functions are still bounded by $O(X^k)$ with $k<1$.

Proposition III.2 *Let N_χ be the conductor of χ, $x=(\log N_\chi)^\beta$ and $\beta(1-\alpha)>2$.*
If $L(s,K)=L(s,\chi)\cdots L(s,\chi^{l-1})$ has no zeros in $[\alpha,1]\times[-x,x]$, then

$$L(1,K)=\prod_{p<x}\prod_{i=1}^{l-1}(1-\alpha_i p^{-1})^{-1}\left(1+O\left(\frac{1}{\log x}\right)\right). \tag{35}$$

If $L(1,K)$ satisfies GRH, then

$$L(1,K)=\prod_{p<(\log N)^{2+\varepsilon}}\prod_{i=1}^{l-1}(1-\alpha_i p^{-1})^{-1}\left(1+O\left(\frac{1}{\log\log N_\chi}\right)\right). \tag{36}$$

Lemma III.3 *For prime p and l,*

$$\frac{(1-p^{-l})^{-1}}{(1-p^{-1})^{-1}} \le (1-p^{-1})^{-(l-1)}$$

Proof. *Equivalently,*

$$(1-p^{-l})^{-1} \le (1-p^{-1})^{-l}$$

or

$$(1-p^{-1})^{l} \le (1-p^{-l}).$$

This is also equivalent to

$$(p-1)^{l} \le (p^{l}-1)$$

since we have $(p-1)^l < p^l$.

□

On C_l setting, where l is a prime, for prime p,

$$\prod_{i=1}^{l-1}(1-\alpha_i p^{-1})^{-1} = \begin{cases} \prod(1-p^{-1})^{-(l-1)} & \text{if } p \text{ split completely.} \\ \prod \frac{(1-p^{-l})^{-1}}{(1-p^{-1})^{-1}} & \text{if } p \text{ inert.} \end{cases} \tag{37}$$

by Lemma III.3,

$$\prod \frac{(1-p^{-l})^{-1}}{(1-p^{-1})^{-1}} \le \prod(1-\alpha_i p^{-1})^{-1} \le \prod(1-p^{-1})^{-(l-1)} \tag{38}$$

for $n \ge 2$,

$$\prod_{p\le y}(1-p^{-n})^{-1} = \zeta(n)(1+O(\frac{1}{y\log y})). \tag{39}$$

So, under the GRH, Proposition III.2 with (39) and the Mertens' theorem to (38) gives

$$\left(\frac{1}{2}+o(1)\right)\frac{\zeta(l)}{e^{\gamma}\log\log N} \le L(1,K) \le (2+o(1))^{(l-1)}(e^{\gamma}\log\log N_{\chi})^{(l-1)}. \tag{40}$$

Proposition III.4 *Let* $y = c_1 \log X$ *and* $r \le c_2 \frac{\log X}{\log\log X}$ *for constants* $c_1, c_2 > 0$. *Then,*

$$\sum_{K\in M_l(X)}\left(\sum_{y<p<x}\frac{a_\rho(p)}{p}\right)^{2r} \ll 2^{2r-1}(l-1)^{2r}\frac{(2r)!}{r!}X\frac{2^{2r}}{(y\log y)^r}$$

where

$$M_l(X): \textit{the set of } C_l \textit{ fields } K \textit{ with } X/2 \le |D_K| \le X. \tag{41}$$

Proof. Using the multinomial formula, we can expand the left hand side as

$$\sum_{K\in M_l(X)}\sum_{u=1}^{2r}\frac{1}{u!}\sum\nolimits'_{r_1,\cdots,r_u}\frac{(2r)!}{r_1!\cdots r_u!}\sum\nolimits''_{p_1,\cdots,p_u}\frac{a_\rho(p_1)^{r_1}\cdots a_K(p_u)^{r_u}}{p_1{}^{r_1}\cdots p_u{}^{r_u}}.$$

Note that $\sum'$ is over the ordered u-tuples $(r_1, \cdots, r_u)$ such that $r_1 + \cdots + r_u = 2r$, and $\sum''$ is over the u-tuples $(p_1, \cdots, p_u)$ of distinct primes such that $y < p_i < x$ for each i. Also, it is

$$\sum_{u=1}^{2r} \sum{}' \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum{}'' \frac{1}{p_1{}^{r_1} \cdots p_u{}^{r_u}} \left(\sum_{L(s,K) \in M_l(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right). \tag{42}$$

Now we'll focus on the inner summation, considering the several cases of the ordered partition of $2r$. For the first case, assume that $r_i \geq 2$ for all i. Then we can bound the summation by the trivial bound of the character sum,

$$\sum{}'' \frac{1}{p_1{}^{r_1} \cdots p_u{}^{r_u}} \left(\sum_{L(s,K) \in M_l(X)} a_K(p_1)^{r_1} \cdots a_K(p_u)^{r_u} \right) \ll (l-1)^{2r} X \prod_{k=1}^{u} \sum_{y < p_k < x} \frac{1}{p_k{}^{r_k}},$$

and it is bounded by

$$(l-1)^{2r} X \frac{2^{2r}}{(y \log y)^r} \left(\frac{\log y}{y} \right)^{r-u}. \tag{43}$$

Note that $r \leq c_2 \frac{\log X}{\log\log X}$, so for sufficiently small c_1,

$$\frac{r!}{u! r_1! \cdots r_u!} \leq \frac{r!}{u!} \leq r^{r-u} \leq \left(\frac{y}{\log y} \right)^{r-u}. \tag{44}$$

So (42) is bounded by

$$\sum_{u=1}^{2r} \sum{}' (l-1)^{2r} \frac{(2r)!}{r!} X \frac{2^{2r}}{(y \log y)^r}. \tag{45}$$

Since the number of ordered partition of $2r$ is 2^{2r-1}, this case ends.
For the second case, assume that $r_i = 1$ for some i. Then we can write that $r_1 + \cdots + r_m + r_{m+1} + \cdots + r_u = 2r$, with $r_i = 1$ for $1 \leq i \leq m$ and $r_j > 1$ for $m+1 \leq j \leq u$. And note that $a_\rho(p) = 0$ if p is a ramified prime. So we can assume, further that all primes $p_1, \cdots, p_u$ are unramified(call this condition S_{un}). Let's consider the innermost summation of (42). Let C be the conjugacy class of Frob_p, then that is nothing but

$$\sum_C a_K(p_1) \cdots a_K(p_m) a_K(p_{m+1})^{r_{m+1}} \cdots a_K(p_u)^{r_u} \sum_{L(s,K) \in L(X; S_{un})} 1. \tag{46}$$

If we fix one p_i, the inner sum is

$$\frac{|C|}{3 + f(p_i)} A(p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_u) X + O\left((\prod_{p \in S} p)^\kappa X^\delta \right) \tag{47}$$

but $\sum_C |C| a_K(p_i) = 0$ by orthogonality. So we just need to consider the error term. Then, (42) is

$$\begin{aligned}
&\ll (l-1)^{2r} \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} X^\delta \sum{}'' \left(p_1{}^{\kappa-1} \cdots p_m{}^{\kappa-1} p_{m+1}{}^{\kappa - r_{m+1}} \cdots p_u{}^{\kappa - r_u} \right) \\
&\ll (l-1)^{2r} X^\delta \frac{(2r)!}{r!} \left(\frac{y}{\log y} \right)^{r-u} y^m \frac{x^{u\kappa}}{(\log x)^u} \\
&\ll (l-1)^{2r} X^\delta \frac{(2r)!}{r!} (\log X)^{2r(\kappa\beta+1)}
\end{aligned} \tag{48}$$

(48) can be obtained by the Lemma 4.3 from the paper [10]. Note that $r \le c_2 \frac{\log X}{\log\log X}$, and take $c_2 = \frac{1-\delta}{2(2\kappa\beta+1)}$ sufficiently small. Then

$$\ll (l-1)^{2r} \frac{(2r)!}{r!} X \frac{2^{2r}}{(y \log y)^r}. \tag{49}$$

So all cases are done.

□

IV Proof of denseness theorems

Theorem IV.1 *Assume (27) and (29). For cyclic field K of prime degree $l \geq 3$, suppose that the $L(s,\chi)$ satisfies (21). Then the set $\{L(1,\chi) : L(s,\chi) \in D_l(X)\}$ is dense in $\mathbb{C}$, equivalently, for any $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, we have*

$$\liminf_{X\to\infty} \frac{1}{|D_l(X)|} \#\{L(s,\chi) \in D_l(X) : |L(1,\chi) - z_0| < \varepsilon\} > 0. \tag{50}$$

Theorem IV.2 *Assume (27) and (29). Let l be a prime. For C_l-fields, the set $\{L(1,K) : K \in L_l(X)\}$ is dense in $\mathbb{R}^+$, equivalently, for any $x_0 \in \mathbb{R}^+$ and $\varepsilon > 0$, we have*

$$\liminf_{X\to\infty} \frac{1}{|L_l(X)|} \#\{K \in L_l(X) : |L(1,K) - x_0| < \varepsilon\} > 0. \tag{51}$$

If $l = 2$, $L(s,\chi) = L(s,K)$. However, the value at $s = 1$ should be real for quadratic case. The second Theorem is generalization of $L(s,K)$ form of $l = 2$ case to arbitrary odd prime l, while the first Theorem is the generalization of $L(s,\chi)$ case.

4.1 Proof of Theorem IV.1

To prove Theorem IV.1, we use analogous Lemmas and Propositions in [5]. First, we show that any $z \in \mathbb{C}$ can be approximated by the set of $\sum \frac{\zeta_l^k}{p_n}$, where p_n is the n-th prime and $\zeta_k \neq 1$ is a primitive l-th root of unity. In other words, the set of sums of the form $\sum \frac{\zeta_l^k}{p_n}$ is dense in $\mathbb{C}$.

Lemma IV.3 *Let p_n be the n-th prime and y is fixed. Then $\{\sum_{y\leq k\leq \nu} \frac{a_k}{p_k} |\ a_k \in \{1, \zeta_l, \zeta_l^2, ..., \zeta_l^{l-1}\}$ for $y \leq k \leq \nu\}$ is dense in $\mathbb{C}$ for odd prime l.*

Proof. *This lemma is saying that for any $z \in \mathbb{C}$ and $\varepsilon > 0$, there exist $\nu \geq y$ and $c_p \in \{1, \zeta_l, \zeta_l^2, \cdots, \zeta_l^{l-1}\}$ for each prime p with $y \leq p \leq \nu$ such that $\left|z - \sum_{y\leq p\leq \nu} \frac{c_p}{p}\right| \leq \varepsilon$. To prove this, for $\varepsilon > 0$ and for given $A, B \in \mathbb{R}$, we find $a_k \in \{1, \zeta_l, \zeta_l^2, ..., \zeta_l^{l-1}\}$ that satisfy $\left|A + iB - \sum_{y\leq k\leq \nu} \frac{a_k}{p_k}\right| < \varepsilon$.*
First, we consider the imaginary part to find ν' and $a_k \in \{1, \zeta_l, \zeta_l^2, ..., \zeta_l^{l-1}\}$ that satisfy

$$\left| B - \mathrm{Im}\left(\sum_{y\leq k\leq \nu'} \frac{a_k}{p_k}\right)\right| < \varepsilon/2 \tag{52}$$

We can find such a_k's by Lemma 6.3 in [5] with $a_k \in \{\zeta_l^n, \zeta_l^{l-n}\}$ for $y \leq k \leq \nu'$. Note that $\mathrm{Im}(\zeta_l^{l-n}) = -\mathrm{Im}(\zeta_l^n)$.
Now we consider the real part.

$$\left| A - \mathrm{Re}\left(\sum_{y\leq k\leq \nu'} \frac{a_k}{p_k}\right) - \mathrm{Re}\left(\sum_{\nu'\leq k\leq \nu} \frac{1}{p_k}\right)\right| < \varepsilon/2 \tag{53}$$

There exist at least one positive integer n that satisfies $\mathrm{Re}(\zeta_l^n) < 0$ and $0 < n < l$ since $\sum_{k=0}^{l-1} \zeta_l^k = -1$. We can find $a_k \in \{\zeta_l^n, \zeta_l^{l-n}\}$ with $y < k \leq \nu'$ that satisfy (53).

ν in (53) can be arbitrary big enough and ν is depending on ν' and ν' is depending on y so $\{a_k\}_{k=y}^{\nu'}$ can satisfy both (52) and (53) at the same time for some ν.

□

Proposition IV.4 *Let $z \in \mathbb{C}$ and $\nu_1 \geq 2$. Then for any $\varepsilon > 0$, there exist $\nu \geq \nu_1$ and $b(p)$ such that $\chi(p) = \zeta_l^m$ for $m \in \mathbb{Z}$, $b(p) = \chi(p)$ and by choosing $\chi(p) \in \{1, \zeta_l, \ldots, \zeta_l^{l-1}\}$ as the Dirichlet character,*

$$\left| z - \log \prod_{p \leq \nu} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right| < \varepsilon \tag{54}$$

where

$$\log \prod_{p \leq \nu} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = \sum_{p \leq \nu} \sum_{n=1}^{\infty} \frac{\chi(p^n)}{np^n}. \tag{55}$$

Proof. *Let y be a real number that satisfy $1/y < \varepsilon$. Then*

$$\sum_{p \geq y} \sum_{n=2}^{\infty} \frac{1}{np^n} \ll \sum_{p \geq y} \frac{1}{p^2} \ll \frac{1}{y} < \varepsilon. \tag{56}$$

By Lemma IV.3, there are $\nu \geq y$ and c_p for $y \leq p \leq \nu$ for which

$$\left| \left(z - \sum_{p < y} \sum_{n=1}^{\infty} \frac{\chi(p)}{np^n} \right) - \sum_{y \leq p \leq \nu} \frac{c_p}{p} \right| \leq \varepsilon. \tag{57}$$

For all prime p for $p \leq \nu$, we set

$$b(p) = \begin{cases} \chi(p) & \text{if } p < y, \\ c_p & \text{if } y \leq p \leq \nu. \end{cases} \tag{58}$$

Then

$$\left| z - \log \prod_{p \leq \nu} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right| = \left| z - \sum_{p < y} \sum_{n=1}^{\infty} \frac{\chi(p)}{np^n} - \sum_{y \leq p \leq \nu} \frac{c_p}{p} - \sum_{y \leq p \leq \nu} \sum_{n=2}^{\infty} \frac{\chi(p)}{p} \right|$$

$$\leq \left| z - \sum_{p < y} \sum_{n=1}^{\infty} \frac{\chi(p)}{np^n} - \sum_{y \leq p \leq \nu} \frac{c_p}{p} \right| + \left| \sum_{y \leq p \leq \nu} \sum_{n=2} \frac{\chi(p^n)}{np^n} \right| \leq \varepsilon + \sum_{p \leq y} \sum_{n=2}^{\infty} \frac{1}{np^n} \ll \varepsilon$$

□

Lemma IV.5 *Set $h_X = exp((\log\log X)^{\frac{3}{4}})$. Let N_χ be the conductor of χ. Then*

$$\sum_{N_\chi \leq X} \left| L(1, \chi) - \prod_{p \leq h_X} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right|^2 \ll \frac{X}{(\log h_X)^2} \tag{59}$$

Proof.

$$\prod_{p < x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = L(1, \chi) + O\left(\frac{1}{\log x}\right) \tag{60}$$

A well-known result [12] that approximates $L(1, \chi)$ gives the inequality

$$\left| L(1, \chi) - \prod_{p \leq h_X} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right|^2 \ll \frac{1}{(\log h_X)^2} \tag{61}$$

The theorem follows with (25) and (27).

$$\sum_{N_\chi \leq X} 1 = (l-1) \sum_{f_K \leq X} 1 = (l-1)\mathfrak{C}_l X + O(X^{\frac{l+2}{l+5}}). \tag{62}$$

□

Proposition IV.6 *Define*

$$A_X(\varepsilon) = \left\{ L(s,\chi) \in D_l(X) \middle| \left| L(1,\chi) - \prod_{p \le h_X} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right| < \varepsilon \right\}, \tag{63}$$

Then $\frac{\#A_X(\varepsilon)}{\#D_l(X)} > 1 - \varepsilon$.

Proof. *The product* $\prod \left(1 - \frac{\chi(p)}{p}\right)$ *converges to* $L(1,\chi)$ *as* $X \to \infty$, *so there exists* X_0 *large enough such that for every* $X > X_0$,

$$\left| L(1,\chi) - \prod_{p \le h_X} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right|^2 < \#D_l(X)\varepsilon^3. \tag{64}$$

We assume that there exist $X > X_0$ *such that*

$$\#(D_l(X) - A_X) \ge \varepsilon \#D_l(X).$$

This assumption denies the conclusion of this Proposition. When we divide $D_l(X)$ *both sides, the inequality gives* $\frac{\#A_X(\varepsilon)}{\#D_l(X)} \le 1 - \varepsilon$.

Now, for this X *we have,*

$$\sum_{L(s,\chi) \in D_l} \left| L(1,\chi) - \prod \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right|^2 \tag{65}$$

$$\ge \sum_{L(s,\chi) \in D_l - A_X} \left| L(1,\chi) - \prod \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right|^2 \tag{66}$$

$$\ge \varepsilon \#D_l(X)\varepsilon^2 = \varepsilon^3 \#D_l(X) \tag{67}$$

This contradicts (64). Therefore, for any $X > X_0$,

$$\#(D_l - A_X) < \varepsilon \#D_l(X)$$

we can show $\frac{\#A_X(\varepsilon)}{\#D_l(X)} > 1 - \varepsilon$.

□

Lemma IV.7 *let* $\nu \ge 3$ *be fixed and* $a_p \in \{1, \zeta_l, \zeta_l^2, \ldots \zeta_l^{l-1}\}$. *Define* $D_{l,\nu}(X, \{a_p\})$ *by*

$$D_{l,\nu}(X) := \{L(s,\chi) \in D_l(X) | \chi(p) = a_p \text{ for any odd prime } p \le \nu\} \tag{68}$$

Then there exist X_0 *such that for* $X > X_0$, *we have*

$$\frac{\#D_{l,\nu}(X)}{\#D_l(X)} > C_{l,\nu} + O(X^{-\frac{3}{l+5}}) \tag{69}$$

where $C_{l,\nu}$ *is a constant depends on* l *and* ν

$$C_{l,\nu} > \frac{1}{\mathfrak{C}_l} \prod_{3 \le p \le \nu} \frac{1}{l^{2\pi(\nu)}} > 0$$

and $\mathfrak{C}_l$ is a constant from (30).

Proof. *For any cases, $\left|Split_q\right| > \frac{1}{l^2}$ and $\left|Inert_q\right| = (l-1)\left|Split_q\right| > (l-1)\frac{1}{l^2}$.*

$$\left|Frob_q = C\right| = \frac{1}{l-1}\left|Inert_q\right| > \frac{1}{l^2}. \tag{70}$$

Let $A_{l,\nu}$ be a constant depending on l and ν,

$$A_{l,\nu} = \prod_{3\le p\le \nu} \frac{1}{l^{2\pi(\nu)}}.$$

where π is the Prime counting function. By the counting cyclic extensions of a prime degree l with these local conditions with (30),

$$\#D_{l,\nu}(X) > A_{l,\nu}X + O(X^{\frac{l+2}{l+5}}). \tag{71}$$

Hence,

$$\frac{\#D_{l,\nu}(X)}{\#D_l(X)} > \frac{A_{l,\nu}X + O(X^{\frac{l+2}{l+5}})}{\mathfrak{C}_l X + O(X^{\frac{l+2}{l+5}})} = \frac{A_{l,\nu}}{\mathfrak{C}_l} + O(X^{-\frac{3}{l+5}}). \tag{72}$$

□

.

Intuitively, it is saying about probability of $D_{l,\nu}$. It means that choosing $\chi(p)$ in $\{1, \zeta_l, \ldots, \zeta_l^{l-1}\}$ as we want until ν is always possible. The "always possible" means that $D_{l,\nu}$ has positive probabilities for any l and ν in this case. We can see the constant C_l, ν is positive.

The inequality

$$|L(1,\chi) - z_0| \le \left| L(1,\chi) - \prod_{3\le p\le \nu}\left(1 - \frac{\chi(p)}{p}\right)^{-1}\right| + \left| z_0 - \prod_{3\le p\le \nu}\left(1 - \frac{\chi(p)}{p}\right)^{-1}\right| \tag{73}$$

implies that

$$\#\{L(1,\chi) \in D_l(X) || L(1,\chi) - z_0| < \varepsilon\} \ge \#(A_X \cap D_{l,\nu})$$

by Proposition IV.4, Proposition IV.6 and Lemma IV.7. Now we have

$$\begin{aligned}\#(A_X \cap D_{l,\nu}) &\ge \#D_l - \#(A_X - D_{l,\nu}) - \#(D_{l,\nu} - A_X) \ge \#D_l - \#(D_l - D_{l,\nu}) - \#(D_l - A_X) \\ &\ge \#D_l - \#D_l + \#D_{l,\nu} - \#D_l + \#A_X = \#A_X + \#D_{l,\nu} - \#D_l. \end{aligned} \tag{74}$$

We choose $\varepsilon_1 < \min\{\varepsilon, C_{l,\nu}\}$.

$$\#(A_X(\varepsilon_1) \cap D_{l,\nu}(X)) \ge \left(C_{l,\nu} - \varepsilon_1 + O\left(X^{-\frac{3}{l+5}}\right)\right)\#D_l(X). \tag{75}$$

$$\frac{\#(A_X(\varepsilon_1) \cap D_{l,\nu}(X))}{\#D_l(X)} \ge \frac{\left(C_{l,\nu} - \varepsilon_1 + O\left(X^{-\frac{3}{l+5}}\right)\right)}{\#D_l(X)}. \tag{76}$$

Taking $\liminf X \to \infty$ both sides provides the proof of Theorem IV.1.

4.2 Proof of Theorem IV.2

To show Theorem IV.2. we use analogous Lemmas and Propositions in [6].

Lemma IV.8 *Let y be fixed. Then for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\nu \geq y$ and $c_p \in \{-1, l-1\}$, for each prime p with $y \leq p \leq \nu$, such that*

$$\left| x_0 - \sum_{y \leq p \leq \nu} \frac{c_p}{p} \right| < \varepsilon. \tag{77}$$

Proof. *Fix integer $y_0 > 0$ and let p_n be the n-th prime. Then for integer $\nu_0 > 0$, there are $2^{\nu_0 - y_0 + 1}$ cases of the sum $\sum_{y_0 \leq n \leq \nu_0} c_{p_n}/p_n$. The sum closest to x_0 satisfies*

$$\left| x_0 - \sum_{y_0 \leq n \leq \nu_0} \frac{c_{p_n}}{p_n} \right| \leq (l-1)/p_{\nu_0} \tag{78}$$

for sufficiently large ν_0.

□

Proposition IV.9 *Let $x \in \mathbb{R}$ and $\nu_1 \geq 2$. Then for any $\varepsilon > 0$, there exist $\nu \geq \nu_1$ and $b(p) \in \{-1, l-1\}$ for $p \leq \nu$ such that $b(p) = a(p)$, and by choosing the l-th roots of unity $\alpha_i(p)$, i=1,...,l-1 by the equation (32),*

$$\left| x - \log \prod_{p \leq \nu} \prod_{i=1}^{l-1} \left(1 - \frac{\alpha_i(p)}{p}\right)^{-1} \right| < \varepsilon, \tag{79}$$

where

$$\log \prod_{p \leq \nu} \prod_{i=1}^{l-1} \left(1 - \frac{\alpha_i(p)}{p}\right)^{-1} = \sum_{p \leq \nu} \sum_{n=1}^{\infty} \frac{a_\rho(p^n)}{np^n}. \tag{80}$$

Proof. *Choose a large $y > \nu_1$ for which $1/y < \varepsilon$. Then*

$$\sum_{p \geq y} \sum_{n=2}^{\infty} \frac{l-1}{np^n} \ll \sum_{p \geq y} \frac{1}{p^2} \ll \frac{1}{y} < \varepsilon. \tag{81}$$

By Lemma IV.8, there exist $\nu \geq y$ and c_p for $y \leq p \leq \nu$ for which

$$\left| \left(x - \sum_{p < y} \sum_{n=1}^{\infty} \frac{l-1}{np^n} \right) - \sum_{y \leq p \leq \nu} \frac{c_p}{p} \right| \leq \varepsilon. \tag{82}$$

For each prime p for $p \leq \nu$, we set

$$b(p) = \begin{cases} l-1 & \text{if } p < y, \\ c_p & \text{if } y \leq p \leq \nu. \end{cases} \tag{83}$$

Then

$$\begin{aligned} \left| x - \log \prod_{p \leq \nu} \prod_{i=1}^{l-1} \left(1 - \frac{\alpha_i(p)}{p}\right)^{-1} \right| &= \left| x - \sum_{p < y} \sum_{n=1}^{\infty} \frac{l-1}{np^n} - \sum_{y \leq p \leq \nu} \frac{c_p}{p} - \sum_{y \leq p \leq \nu} \sum_{n=2}^{\infty} \frac{a_\rho(p^n)}{np^n} \right| \\ &\leq \left| x - \sum_{p < y} \sum_{n=1}^{\infty} \frac{l-1}{np^n} - \sum_{y \leq p \leq \nu} \frac{c_p}{p} \right| + \left| \sum_{y \leq p \leq \nu} \sum_{n=2}^{\infty} \frac{a_\rho(p^n)}{np^n} \right| \leq \varepsilon + \sum_{p \leq y} \sum_{n=2}^{\infty} \frac{l-1}{np^n} \ll \varepsilon. \end{aligned} \tag{84}$$

$\square$

Lemma IV.10 *Set $h_X = exp((\log\log)^{\frac{3}{4}})$. Then*

$$\sum_{K\in M_l(X)} \left| L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \ll \frac{X(\log h_X)^{2l-3}}{h_X}. \tag{85}$$

where the set M_l is in (41).

Proof. *We first show*

$$\sum_{K\in M_l(X)} \left| \log L(1,K) - \log\prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \ll \frac{X}{h_X\log h_X}. \tag{86}$$

Suppose $L(s,K)$ has the no zeros in the region in the Section III. Then

$$\log(1,K) = \log\prod_{p\le H_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1} + O\left(\frac{1}{\log X}\right), \tag{87}$$

where $H_X = (\log X)^{\beta}$. Since

$$\log\prod_{h_X<p\le H_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1} = \sum_{h_X<p\le H_X}\frac{a_\rho(p)}{p} + O\left(\frac{1}{h_X}\right),$$

$$\begin{aligned}\left|\log L(1,K) - \log\prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right| &= \left|\log\prod_{h_X<p\le H_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right) + O\left(\frac{1}{\log X}\right)\right| \\ &= \left|\sum_{h_X<p\le H_X}\frac{a(p)}{p} + O\left(\frac{1}{h_X}\right)\right|. \end{aligned}\tag{88}$$

Let $M_l'(X)\subset M_l(X)$ be the set that satisfies (40). For L-functions in $M_l'(X)$ which might have some zeros on the region in the Section III, we use the trivial trivial bounds :

$$|L(1,K)| \ll X^{\varepsilon}, \tag{89}$$

$$\left|\prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right| \ll (\log h_X)^{l-1} \ll (\log\log X)^{\frac{3(l-1)}{4}}. \tag{90}$$

The second inequality is from $\left|1-\frac{\alpha_i(p)}{p}\right|^{-1} \le \left(1-\frac{1}{p}\right)^{-1}$ and Mertens' theorem: $\prod_{p\le x}(1-\frac{1}{p})^{-1} \ll \log x$.

Now define $\widehat{M_l}(X)$ as the subset of $M_l'(X)$, which consists of L-functions with the zero-free region in the section III. Then,

$$\begin{aligned}&\sum_{K\in\widehat{M_l}(X)} \left|\log L(1,K) - \log\prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \\ &= \sum_{K\in\widehat{M_l}(X)}\left|\sum_{h_X<p\le H_X}\frac{a_K(p)}{p} + O\left(\frac{1}{h_X}\right)\right|^2 \ll \sum_{K\in M_l(X)}\left|\sum_{h_X<p\le H_X}\frac{a_K(p)}{p}\right|^2 + \frac{X}{h_X^2}.\end{aligned}\tag{91}$$

With the proposition III.4, we can show that

$$\sum_{K\in M_l(X)}\left|\sum_{h_X<p\le H_X}\frac{a_K(p)}{p}\right|^2 \ll \frac{X}{h_X\log\log h_X}. \tag{92}$$

Hence, there exist X_1 such that for $X > X_1$,

$$\sum_{K\in\widehat{M_l}(X)} \left|\log L(1,K) - \log \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \ll \frac{X}{h_X \log h_X}. \tag{93}$$

From the above inequality

$$\begin{aligned}
&\left|L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right| \\
&= \frac{1}{\left|\prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|}\left|\exp\left(\log L(1,K) - \log \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right)-1\right| \\
&\ll \left|\prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)\right|\left|\left(\log L(1,\rho) - \log \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right)\right| \\
&\ll \left(\frac{\log h_X}{\zeta(2)}\right)^{l-1}\left|\log L(1,K) - \log \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|,
\end{aligned} \tag{94}$$

where the last inequality comes from $\left|1-\frac{\alpha_i(p)}{p}\right| \le 1+\frac{1}{p} = \frac{1-1/p^2}{1-1/p}$.

Hence,

$$\sum_{K\in\widehat{M_l}(X)} \left|L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \ll \frac{X(\log h_X)^{2l-3}}{h_X}. \tag{95}$$

The bounds (89) and (90) imply that

$$\sum_{K\in M_l'(X)} \left|L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \ll \frac{X(\log h_X)^{2l-3}}{h_X}. \tag{96}$$

Therefore,

$$\sum_{K\in M_l(X)} \left|L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 \ll \frac{X(\log h_X)^{2l-3}}{h_X}. \tag{97}$$

□

Proposition IV.11 *Define*

$$A_X(\varepsilon) = \{K\in L(X) | \left|L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right| < \varepsilon\} \tag{98}$$

Then

$$\frac{\#A_X(\varepsilon)}{\#L(X)} > 1-\varepsilon. \tag{99}$$

Proof. *Use the Similar idea with Proposition IV.6. Since as $X\to\infty$, there exist a large number X_0 such that for every $X > X_0$,*

$$\left|L(1,K) - \prod_{p\le h_X}\prod_{i=1}^{l-1}\left(1-\frac{\alpha_i(p)}{p}\right)^{-1}\right|^2 < \#L(X)\varepsilon^3. \tag{100}$$

assume that there exist $X > X_0$ such that

$$\#(L(X) - A_X) \geq \varepsilon \# D_l(X).$$

to deny the conclusion of this Proposition.

Then for this X we have,

$$\sum_{K \in M_l(X)} \left| L(1,K) - \prod\prod \left(1 - \frac{\alpha_i(p)}{p}\right)^{-1} \right|^2 \geq \varepsilon^3 \# L(X) \tag{101}$$

as in proposition IV.6 and this contradicts (100).Hence for all $X > X_0$ we have

$$\#(M_l(X) - A_X) < \varepsilon \# M_l(X).$$

We can show $\frac{\#\widetilde{\mathbf{A}}_X(\varepsilon)}{\#M_l(X)} > 1 - \varepsilon$.

□

Lemma IV.12 *let $\nu \geq 3$ be fixed and $a_p \in \{-1, l-1\}$. Define $M_{l,\nu}(X, \{a_p\})$ by*

$$M_{l,\nu}(X) := \{K \in M_l(X) | a_i(p) = a_p \textit{ for all odd prime } p \leq \nu\}.$$

Then there exist X_l such that for $X > X_l$, we have

$$\lim_{X\to\infty} \frac{\#M_{l,\nu}(X)}{\#M_l(X)} > C_{l,\nu},$$

where $C_{l,\nu} > 0$ is a constant depends on l and ν.

Proof. *define $L_{l,\nu}(X, \{a_p\})$ by*

$$L_{l,\nu}(X) := \{K \in L_l(X) | a_i(p) = a_p \textit{ for all odd prime } p \leq \nu\}.$$

Then by the lemma IV.7 and (25) we have

$$\frac{\#L_{l,\nu}(X)}{\#L_l(X)} > C'_{l,\nu} + O\left(X^{-\frac{3}{l+5}}\right)$$

for a positive constant $C'_{l,\nu}$. From the relation between $M_l(X)$ and $L_l(X)$ we have

$$\lim_{X\to\infty} \frac{\#M_l(X)}{\#L_l(X)} = A_{l,\nu},$$

$$\lim_{X\to\infty} \frac{\#M_{l,\nu}(X)}{\#L_{l,\nu}(X)} = B_{l,\nu}.$$

for a positive constant A_ν and B_ν. Hence,

$$\lim_{X\to\infty} \frac{\#M_{l,\nu}(X)}{\#M_l(X)} = \lim_{X\to\infty} \frac{\#M_{l,\nu}(X)/\#L_{l,\nu}(X)}{\#M_l(X)/\#L_l(X)} \cdot \frac{\#L_{l,\nu}(X)}{\#L_l(X)} > \frac{B_\nu}{A_\nu} C'_\nu.$$

□

The inequality

$$|L(1,K)-x_0| \leq \left|L(1,K)-\prod_{3\leq p\leq v}\prod_{i=1}^{l-1}\left(1-\frac{a_i(p)}{p}\right)^{-1}\right| + \left|x_0-\prod_{3\leq p\leq v}\prod_{i=1}^{l-1}\left(1-\frac{a_i(p)}{p}\right)^{-1}\right| \tag{102}$$

implies that

$$\#\{\rho \in L(X) | |L(1,K)-x_0| < \varepsilon\} \geq \#A_X.$$

Choose $\varepsilon_2 < \min\{\varepsilon, C_{l,v}\}$. We have

$$\#(A_X(\varepsilon_2) \cap M_{l,v}(X)) \geq (C_{l,v}-\varepsilon_2+o(1))\#M_l(X)$$

by proposition IV.9 , proposition IV.11 and lemma IV.12. Hence,

$$\frac{\#(A_X(\varepsilon_2)\cap M_{l,v}(X))}{M_l(X)} \geq \frac{(C_{l,v}-\varepsilon_2+o(1))}{\#M_l(X)}$$

Taking $\liminf X \to \infty$ and the set inclusion $M_l(X) \subset L_l(X)$ provides the proof of theorem IV.2.

V Conclusions

We proved denseness of $L(1,\chi)$ and $L(1,K)$. As a further work, now we can consider the universality of $L(s,\chi)$ and $L(s,K)$ when K is a number field and cyclic extension of prime degree l as stated in the following conjectures.

Conjecture V.1 *Let $\Omega \in D$ be a simply connected region. Suppose that Ω is symmetric with respect to the real axis. Let $h(s) : \Omega \to \mathbb{C}$ be a holomorphic function. Suppose that the $h(s)$ has positive real value on $\Omega \cap \mathbb{R}$. For $\varepsilon > 0$ and a compact set $C \in \Omega$,*

$$\liminf_{X\to\infty} \frac{1}{|L_l(X)|} \# \left\{ K \in L_l(X) | \max_{s\in C} |L(s,K) - h(s)| < \varepsilon \right\} > 0.$$

Conjecture V.2 *Let $\Omega \in D$ be a simply connected region. Suppose that Ω is symmetric with respect to the real axis. Let $h(s) : \Omega \to \mathbb{C}$ be a holomorphic function. For $\varepsilon > 0$ and a compact set $C \in \Omega$,*

$$\liminf_{X\to\infty} \frac{1}{|D_l(X)|} \# \left\{ L(s,\chi) \in D_l(X) | \max_{s\in C} |L(s,\chi) - h(s)| < \varepsilon \right\} > 0.$$

When we try to prove Conjecture V.2, the same processes from [5] and [6] are hard to be used because they use holomorphic function $h(s)$ whose values on $\Omega \cap \mathbb{R}$ are real but the Conjecture V.2 use the arbitrary holomorphic function.

We can predict the generalization of denseness and universality from cyclic extension of prime degree to the arbitrary cyclic extension. Further, we can predict the generalization to arbitrary abelian extension. To do the generalization of denseness or universality, we may use the bounds of L-function as like in the Section III and counting fields as like in the Section II.

We can try to the generalization of counting fields in the Section II to any arbitrary cyclic group first and to arbitrary abelian groups. We also think about the generalizations of bound in the Section III to the any arbitrary cyclic group and abelian groups. In the Section III, the bound is for $L(1,K)$ so we tried to find the bound of $L(1,\chi)$. If $l = 3$, the $L(1,K)$ is just multiplication of two Dirichlet L-functions $L(1,\chi)$ and $L(1,\chi^2)$. For any integer n, $\chi(n) = \overline{\chi^2(n)}$ so $|\chi(n)| = \left|\chi^2(n)\right|$, hence $L(1,\chi) = \overline{L(1,\chi^2)}$ and $|L(1,\chi)| = \left|L\left(1,\chi^2\right)\right|$, the (40) implies that

$$\left(\frac{1}{\sqrt{2}} + o(1)\right) \frac{\sqrt{\zeta(l)}}{e^{\gamma/2}\sqrt{\log\log N_\chi}} \leq |L(1,\chi)| \leq (2 + o(1))(e^\gamma \log\log N_\chi).$$

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Acknowledgements

I am sincerely thankful to my advisor Peter Jae-Hyun Cho, who has provided mathematical knowledge, mentorships and many other helps. Thank you for all other my dissertation reading committee members, Hae-Sang Sun and Chol Park. I also thanks to Jae Hun Park who has studied related topic together.