1 .

PROPOSITION 3. If $L(s,\rho)$ is entire and is zero-free in the rectangle $[\alpha, 1] \times [-x, x]$, where $x = (\log N)^{\beta}$, $\beta(1-\alpha) > 2$, and N is the conductor of ρ, then

$$L(1,\rho) = \prod_{p<x} \prod_{i=1}^{d} (1-\alpha_i p^{-1})^{-1} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

If $L(1,\rho)$ satisfies GRH, then

$$L(1,\rho) = \prod_{p<x} \prod_{i=1}^{d} (1-\alpha_i p^{-1})^{-1} \left(1 + O\left(\frac{1}{\log\log N}\right)\right).$$

On C_q setting, where q is a prime, for prime p,

$$\prod_{i=1}^{q-1} (1-\alpha_i p^{-1})^{-1} = \begin{cases} \prod (1-p^{-1})^{-(q-1)} & if\ \ p\ \ split\ \ completely. \\ \prod \frac{(1-p^{-q})^{-1}}{(1-p^{-1})^{-1}} & if\ \ p\ \ inert \end{cases}$$

Claim) for prime p and q,

$$\frac{(1-p^{-q})^{-1}}{(1-p^{-1})^{-1}} \leq (1-p^{-1})^{-(q-1)}$$

equivalently,

$$(1-p^{-q})^{-1} \leq (1-p^{-1})^{-q}$$

or

$$(1-p^{-1})^{q} \leq (1-p^{-q})$$

this is also equivalent to

$$(p-1)^q \leq (p^q - 1)$$

this is true because

$$(p-1)^q < p^q$$

Now, we can say that

$$\prod \frac{(1-p^{-q})^{-1}}{(1-p^{-1})^{-1}} \leq \prod (1-\alpha_i p^{-1})^{-1} \leq \prod (1-p^{-1})^{-(q-1)} \tag{2}$$

for $n \geq 2$,

$$\prod_{p \leq y} (1-p^{-n})^{-1} = \zeta(n)(1 + O(\frac{1}{y \log y})) \tag{3}$$

So, under GRH, prop3 with (3) and the Mertens' theorem to (2) gives

$$\left(\frac{1}{2} + o(1)\right) \frac{\zeta(q)}{e^{\gamma} \log\log N} \leq L(1,\rho) \leq (2 + o(1))^{(q-1)} (e^{\gamma} \log\log N)^{(q-1)}$$