Problem Set I

1. Prove the properties of convolution. For all continuous function f, g, and h, the following axioms hold:

(a) Associativity: $(f * g) * h = f * (g * h)$:

$$\begin{aligned}
((f*g)*h)(t) &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(u_1)g(t-u_1)du_1\right)h(t-u_2)du_2 && \text{Obtain from definition}\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u_1)g(t-u_1)h(t-u_2)du_1du_2\\
&= \int_{-\infty}^{+\infty} f(u_1)du_1\int_{-\infty}^{+\infty} g(t-u_1)h(t-u_2)du_2 && \text{Using Fubini's theorem}\\
&= \int_{-\infty}^{+\infty} f(u_1)du_1\int_{-\infty}^{+\infty} g(t-u_1)h(t-u_1+u_1-u_2)du_2\\
&= \int_{-\infty}^{+\infty} f(u_1)du_1\int_{-\infty}^{+\infty} g(t-u_1)h(t-u_1-u_3)du_3 && \text{Let } u_3 = u_2 - u_1\\
&= \int_{-\infty}^{+\infty} f(u_1)du_1(g*h)(t-u_1)\\
&= \int_{-\infty}^{+\infty} f(u_1)(g*h)(t-u_1)du_1\\
&= (f*(g*h))(t)
\end{aligned}$$

(b) Distributivity: $f * (g + h) = f * g + f * h$:

$$\begin{aligned}
f*(g+h)(t) &= \int_{-\infty}^{+\infty} f(u)(g+h)(t-u)du\\
&= \int_{-\infty}^{+\infty} f(u)(g(t-u)+h(t-u))du && \text{Obtain from definition}\\
&= \int_{-\infty}^{+\infty} f(u)(g(t-u)du + \int_{-\infty}^{+\infty} f(u)h(t-u))du && \text{Linearity of the integral}\\
&= (f*g+f*h)(t)
\end{aligned}$$

(c) Differentiation rule: $(f * g)' = f' * g = f * g'$:

$$\begin{aligned}
(f*g)'(t) &= \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} f(u)g(t-u)du\\
&= \int_{-\infty}^{+\infty} f(u)\frac{\partial}{\partial t}g(t-u)du\\
&= \int_{-\infty}^{+\infty} f(u)g'(t-u)du
\end{aligned}$$

(d) Convolution theorem: $\mathcal{F}(g * h) = \mathcal{F}(g)\mathcal{F}(h)$:

The definition of Fourier Transform is:

$$\mathcal{F}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x \xi} dx$$

Then,

$$\begin{aligned}
g * h(t) &= \int_{-\infty}^{+\infty} g(u) h(t-u) du \\
\mathcal{F}(g * h)(\xi) &= \int_{-\infty}^{+\infty} e^{-2\pi i x \xi} \int_{-\infty}^{+\infty} g(u) h(x-u) du \ dx \quad \text{Obtain from definition} \\
&= \int_{-\infty}^{+\infty} e^{-2\pi i x \xi} h(x-u) dx \int_{-\infty}^{+\infty} g(u) du \\
&= \int_{-\infty}^{+\infty} e^{-2\pi i x \xi} h(x-u) dx \int_{-\infty}^{+\infty} g(u) du \\
&= \mathcal{F}(h)(\xi - u) \int_{-\infty}^{+\infty} g(u) du \\
&= \mathcal{F}(h)(\xi) e^{-2\pi i u \xi} \int_{-\infty}^{+\infty} g(u) du \qquad \text{Shift Theorem} \\
&= \mathcal{F}(h)(\xi) \cdot \mathcal{F}(g)(\xi)
\end{aligned}$$

2. Frequency smoothing:

> (a) Compute Fourier transform of the given image lenaNoise. PNG by using fft2 function in Matlab and then center the low frequencies (fftshift).
>
> (b) Keep different number of low frequencies (e.g., 10^2 , 20^2 , 40^2 and up to the full dimension), but set all other high frequencies to 0.
>
> (c) Reconstruct the original image (ifft2) by using the new generated frequencies in step (b).
>
> Submit the code and include the restored images with different number of low frequencies in your report.

In the problem, the basic idea is to:

- Firstly compute fourier transform of input image, called as I, by fft2 and fftshift command.
- Then, to construct a mask matrix, called as M, with all value equal to 0, except a square area in the center of matrix.
- After that, multiplying the mask M and I, we can get the result in fourier domain.
- Finally, using ifft2 command, to get the result in real domain.

The images results are shown as figure [1].

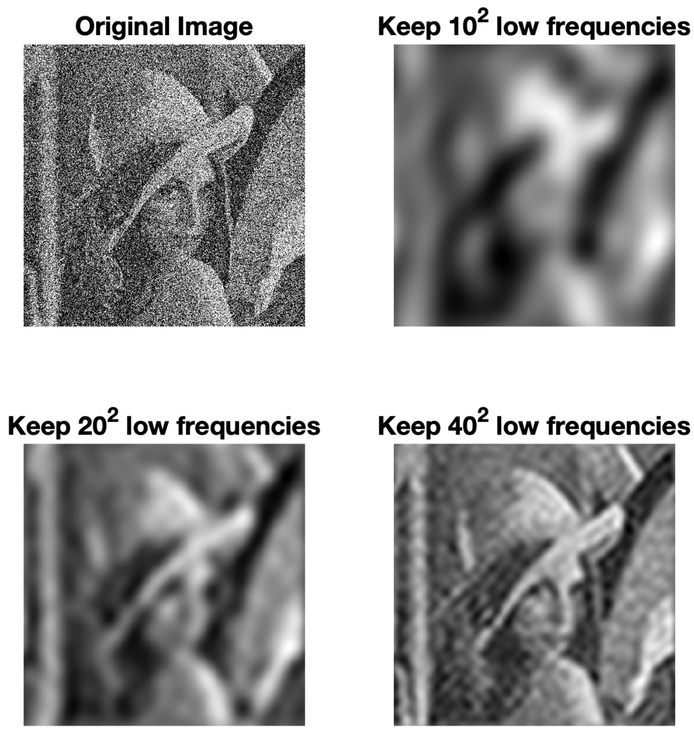

Figure 1: Result of Problem2

3. Implement gradient decent algorithm for ROF model with total variation minimization. All codes and a two-page report including problem description, your solution, and experimental results (denoised image, convergence graph, etc.) with discussions should be submitted.

Based on the slide of class, the formulation of ROF model is:

$$E(u) = \lambda||f-u||_2^2 + ||\nabla u||$$

where f is input noised image. The discrete formulation of $||\nabla u||$ is $\sqrt{u_x^2 + u_y^2}$ where $u_x = u_{i,j} - u_{i+1,j}$ and $u_y = u_{i,j} - u_{i,j+1}$.

The gradient of $E(u)$ is:

$$\nabla E(u) = \lambda(f-u) + \frac{\mathrm{div}(\nabla u)}{||\nabla u||}$$

where $\mathrm{div}(\nabla u)$ is indicated in class as $\frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y$. The discrete formulation of $\frac{\partial}{\partial x}u_x$ is:

$$\frac{\partial}{\partial x}u_x = 2 \cdot u_{i,j} - u_{i-1,j} - u_{i+1,j}$$

And similarly,

$$\frac{\partial}{\partial y}u_y = 2 \cdot u_{i,j} - u_{i,j-1} - u_{i,j+1}$$

To optimize $E(u)$, gradient descend is applied and the iterate formulation is:

$$u = u - \alpha \cdot \nabla E(u)$$

where α is learning rate.

Experiment results is shown as Figure [2] and Figure [3]. In the experiment, iterate epoch is 200, λ is 0.01 and α is 0.5.

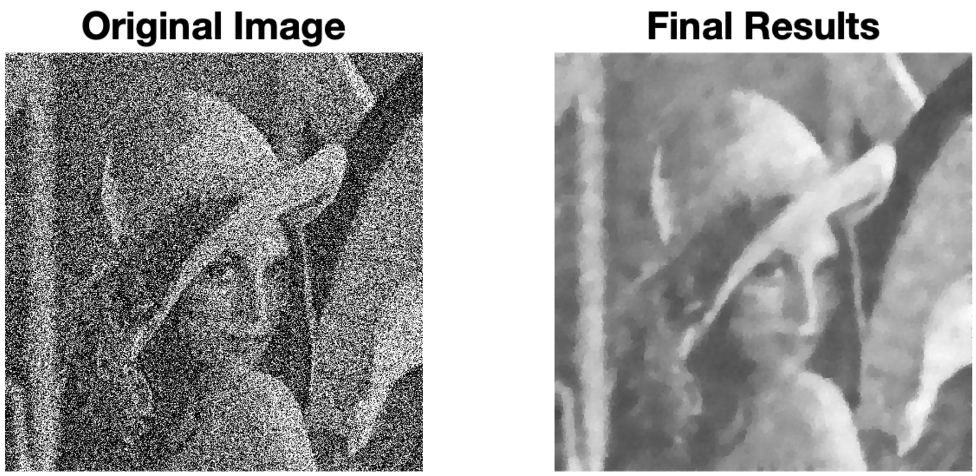

Figure 2: Images Result of Problem3

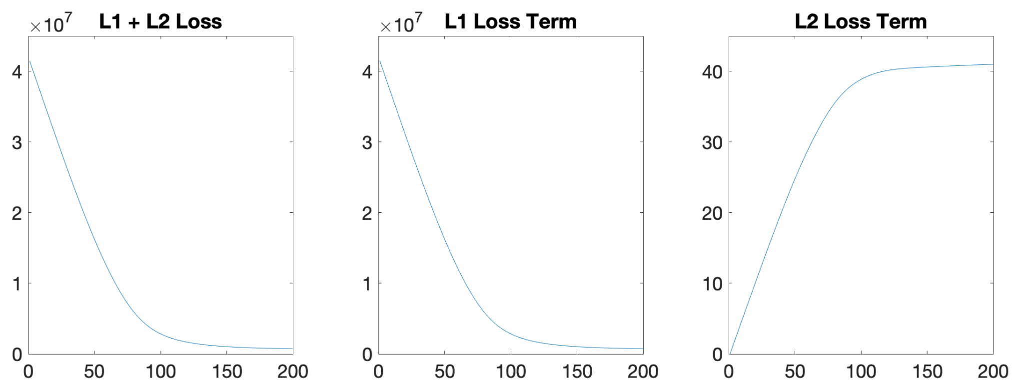

Figure 3: Plots Result of Problem3. In these plots, x-axis means epoch and y-axis means value of loss term. The right plot indicates the $L2 - Norm$: $\lambda||f - u||_2^2$. The center plot indicates the $L1 - Norm$: $||\nabla u||$. The left plot indicates sum of $L2 - Norm$ and $L1 - Norm$.