Problem Set II

1. 1. (30%) **Euler's method:** `https://en.wikipedia.org/wiki/Euler_method`

   For $\frac{dy}{dt} + 2y = 2 - e^{-4t}, y(0) = 1$,

   (a) Derive its closed-form solution.

   (b) Use Euler's Method to find the approximation to the solution at $t = \{1, 2, 3, 4, 5\}$, and compare to the exact solution in (a).

   (c) Use different step size $h = \{0.1, 0.05, 0.01, 0.005, 0.001\}$ and plot out the approximated function value.

(a) Let $y = mn$ and $\frac{\partial y}{\partial t} = n\frac{\partial m}{\partial t} + m\frac{\partial n}{\partial t}$, the partial differential equation becomes:

$$\frac{\partial y}{\partial t} + 2y = 2 - e^{-4t}$$
$$n\frac{\partial m}{\partial t} + m\frac{\partial n}{\partial t} + 2mn = 2 - e^{-4t}$$
$$n\frac{\partial m}{\partial t} + m(\frac{\partial n}{\partial t} + 2n) = 2 - e^{-4t}$$

Then, make $\frac{\partial n}{\partial t} + 2n = 0$:

$$\frac{\partial n}{\partial t} + 2n = 0$$
$$\frac{\partial n}{\partial t} = -2n$$
$$\frac{1}{n}\partial n = -2\partial t$$
$$\text{In } n = -2t + C_1$$
$$n = e^{-2t+C_1}$$
$$n = e^{C_1}e^{-2t}$$
$$n = C_2e^{-2t}$$

Take the above result back to the first formulation:

$$n\frac{\partial m}{\partial t} + m(\frac{\partial n}{\partial t} + 2n) = 2 - e^{-4t}$$
$$C_2e^{-2t}\frac{\partial m}{\partial t} = 2 - e^{-4t}$$
$$C_2\frac{\partial m}{\partial t} = 2e^{2t} - e^{-2t}$$
$$C_2\partial m = (2e^{2t} - e^{-2t})\partial t$$
$$C_2m = e^{2t} + \frac{1}{2}e^{-2t} + C_3$$
$$m = \frac{e^{2t} + \frac{1}{2}e^{-2t} + C_3}{C_2}$$

So, the formulation of y would be:

$$\begin{aligned} y &= mn \\ &= \frac{e^{2t} + \frac{1}{2}e^{-2t} + C_3}{C_2} C_2 e^{-2t} \\ &= 1 + \frac{1}{2}e^{-4t} + C_3 e^{-2t} \end{aligned}$$

Another condition for y is $y(0) = 1$:

$$\begin{aligned} y &= \frac{e^{2t} + \frac{1}{2}e^{-2t} + C_3}{C_2} C_2 e^{-2t} \\ y(0) &= 1 + \frac{1}{2} + C_2 = 1 \\ \Longrightarrow C_2 &= -\frac{1}{2} \end{aligned}$$

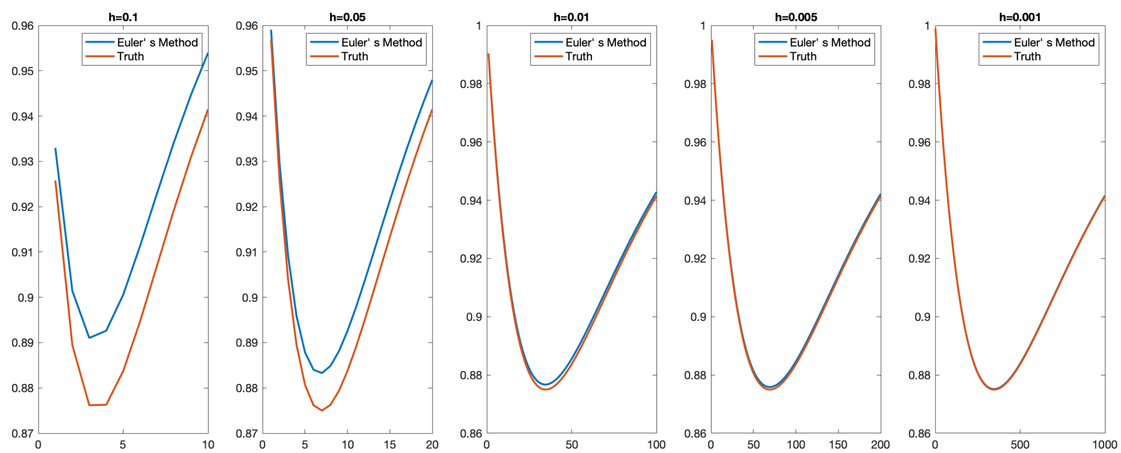
Finally, the closed-form solution is:

$$y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

(b) The result is shown as follows:

n	y_n	t_n	$f(t_n, y_n)$	h	Δy	y_{n+1}	y_{truth}
0	1.000	0	-1.000	1	-0.018	0.9816	0.94149
1	0.981	1	0.018	1	0.036	1.017	0.99101
2	1.017	2	-0.036	1	-0.035	0.982	0.99876
3	0.982	3	0.035	1	0.035	1.017	0.99983
4	1.017	4	-0.035	1	-0.035	0.982	0.99997
5	0.982	5	0.035	1	0.035	1.017	0.99999

(c) The plots are shown as follows:



BTW, the codes of sub-problem (b) and (c) are also submitted into Canvas.

2. (70%) **Geodesic shooting.** Implement geodesic shooting by the following two strategies and compare the differences between the final transformations ϕ_1 at time point $t = 1$.

(a)

$$\frac{dv_t}{dt} = K[(Dv_t)^T \cdot v_t + \text{div}(v_t v_t^T)],$$
$$\frac{d\phi_t}{dt} = v_t \circ \phi_t.$$

(b)

$$\frac{dv_t}{dt} = -K[(Dv_t)^T \cdot v_t + \text{div}(v_t v_t^T)],$$
$$\frac{d\phi_t}{dt} = -D\phi_t \cdot v_t.$$

Note: Use your code of frequency smoothing in PS1 to implement the smoothing operator K (set the truncated number of frequency as 16^2).

(c) Deform a given source image by using the transformations ϕ_1 obtained from (a) and (b).

(a) To solve the formulation in Matlab, we need to more figure out and expand some math sub-formulations. **For one specific point** (i, j) **in image**, there is two-dimension velocity eld v: $(vx(i, j), vy(i, j))$ and also two-dimension transformation ϕ: point$(\phi x(i, j), \phi y(i, j))$:

$$Dv_t(i,j) = D\begin{bmatrix} vx_t(i,j) \\ vy_t(i,j) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial vx_t(i,j)}{\partial x} & \frac{\partial vy_t(i,j)}{\partial x} \\ \frac{\partial vx_t(i,j)}{\partial y} & \frac{\partial vy_t(i,j)}{\partial y} \end{bmatrix}$$

Thus,

$$(Dv_t(i,j))^T v_t(i,j) = \begin{bmatrix} \frac{\partial vx_t(i,j)}{\partial x} & \frac{\partial vy_t(i,j)}{\partial x} \\ \frac{\partial vx_t(i,j)}{\partial y} & \frac{\partial vy_t(i,j)}{\partial y} \end{bmatrix}^T \begin{bmatrix} vx_t(i,j) \\ vy_t(i,j) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial vx_t(i,j)}{\partial x} & \frac{\partial vx_t(i,j)}{\partial y} \\ \frac{\partial vy_t(i,j)}{\partial x} & \frac{\partial vy_t(i,j)}{\partial y} \end{bmatrix} \begin{bmatrix} vx_t(i,j) \\ vy_t(i,j) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial vx_t(i,j)}{\partial x} \cdot vx_t(i,j) + \frac{\partial vx_t(i,j)}{\partial y} \cdot vy_t(i,j) \\ \frac{\partial vy_t(i,j)}{\partial x} \cdot vx_t(i,j) + \frac{\partial vy_t(i,j)}{\partial y} \cdot vy_t(i,j) \end{bmatrix}$$

Also, $v_t(i,j)v_t^T(i,j) = \begin{bmatrix} vx_t(i,j) & vy_t(i,j) \end{bmatrix} \begin{bmatrix} vx_t(i,j) \\ vy_t(i,j) \end{bmatrix} = vx_t^2(i,j) + vy_t^2(i,j)$

$$\text{div}(v_t(i,j)) = \frac{\partial(vx_t^2(i,j) + vy_t^2(i,j))}{\partial x} + \frac{\partial(vx_t^2(i,j) + vy_t^2(i,j))}{\partial y}$$

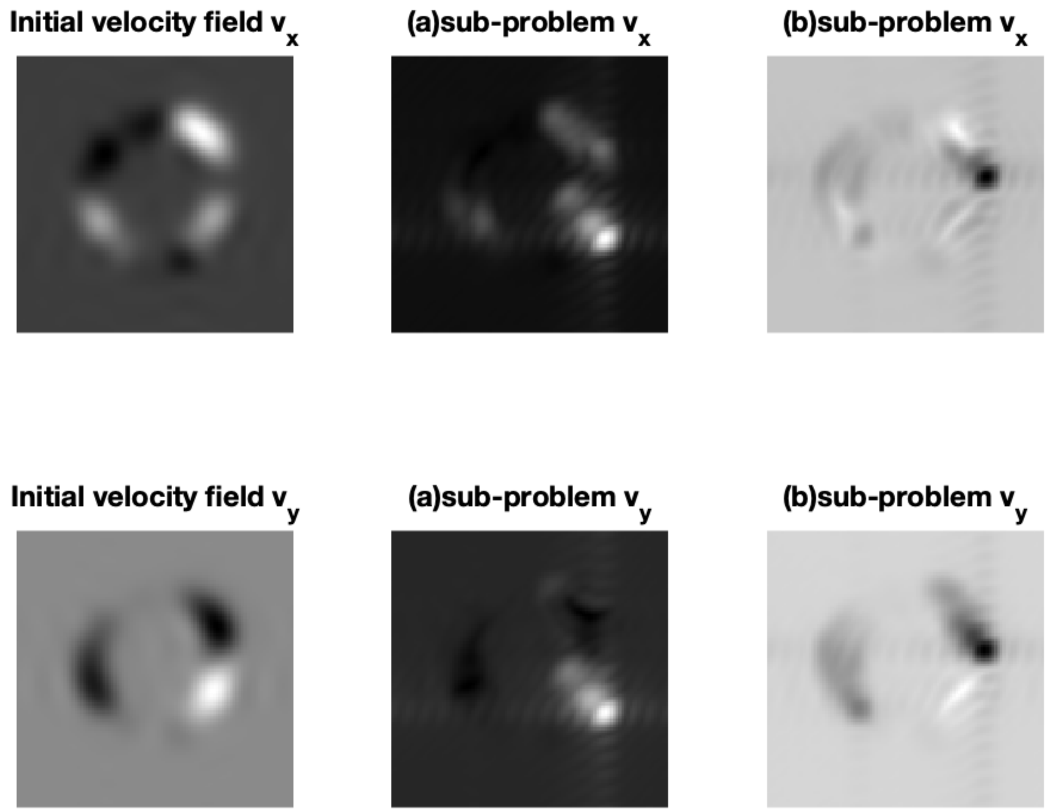


Figure 1: Comparison of Velocity Field

(b) In this problem, we also need to specify formulation of $D\phi_t \cdot v_t$:

$$
\begin{aligned}
D\phi_t \cdot v_t &= D \begin{bmatrix} \phi x_t(i,j) \\ \phi y_t(i,j) \end{bmatrix} \cdot v_t \\
&= \begin{bmatrix} \frac{\partial \phi x_t(i,j)}{\partial x} & \frac{\partial \phi y_t(i,j)}{\partial x} \\ \frac{\partial \phi x_t(i,j)}{\partial y} & \frac{\partial \phi y_t(i,j)}{\partial y} \end{bmatrix} \begin{bmatrix} vx_t(i,j) \\ vy_t(i,j) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \phi x_t(i,j)}{\partial x} vx_t(i,j) + \frac{\partial \phi y_t(i,j)}{\partial x} vy_t(i,j) \\ \frac{\partial \phi x_t(i,j)}{\partial y} vx_t(i,j) + \frac{\partial \phi y_t(i,j)}{\partial y} vy_t(i,j) \end{bmatrix}
\end{aligned}
$$

The results from both (a) and (b) problem are shown as figure (1) and (2):

(c) The results are shown as figure(3).

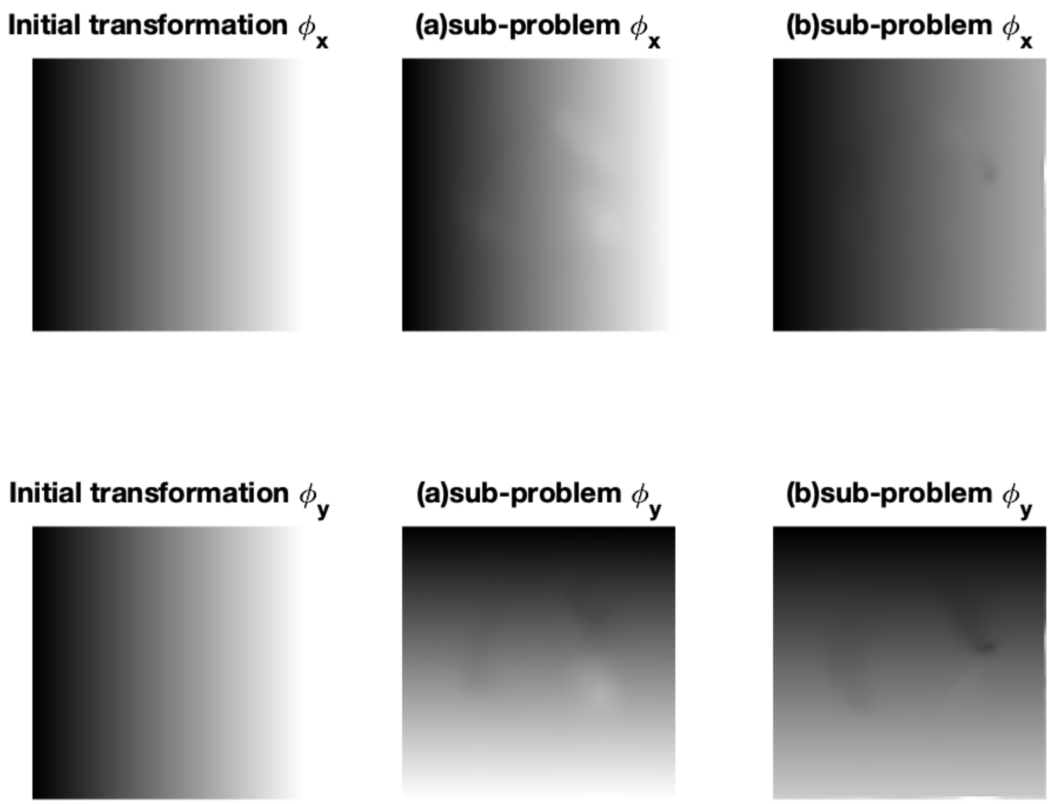


Figure 2: Comparison of Transform

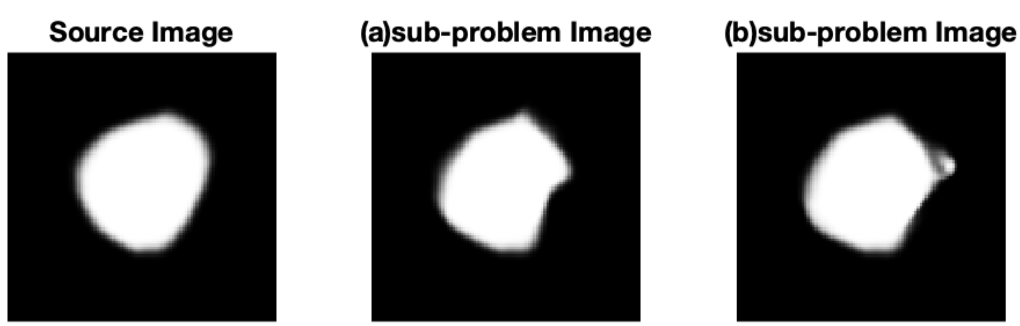


Figure 3: Comparison of Image Results