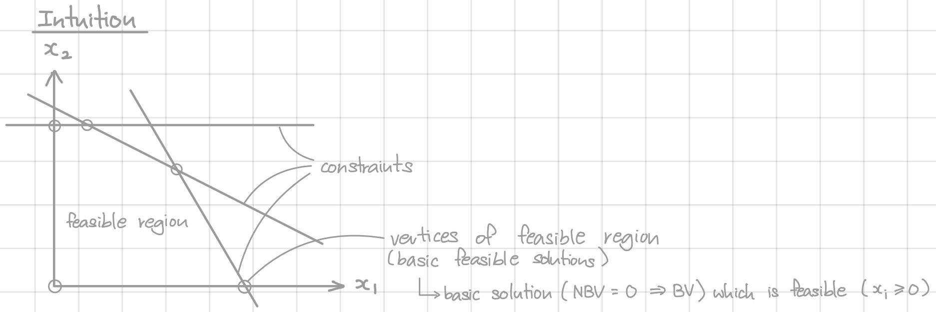
Linear Programming : The Simplex Algorithm

Section 1 : Given LP problem, solve for optimal solution
Section 2: Recognise identifying characteristics of special optimal solutions from certain tableaus
Section 3: Given set of BV, provide simplex tableau / objective value / primal solution
Section 4: Given optimal solution BV, compute sensitivity range



- ▷ The optimal solution is definitely at a vertex of the feasible region.
- ▷ The simplex algorithm starts from the origin if origin is a bfs, else some bfs. It then moves from vertex to adjacent vertex until optimal solution is obtained.
- ▷ For an LP with n variables & m constraints where $n \geq m$, any m variables can be used to form as basic solution, but only those where $x_i \geq 0$ are BFS.

Step 1 : Convert LP to standard form

Note : Add artificial variables to constraints which require NBV to be initialised to a negative value to fulfill non-negativity constraints (*)

(i) If objective is "min", convert to "max"

min $z = 3x_1 + 5x_2$ subject to $2x_1 + x_2 \leq 120$ $x_1 + 2x_2 \leq 250$ $x_2 \leq 120$ $x_1, x_2 \geq 0$	$\Rightarrow$	max $-z = -3x_1 - 5x_2$ subject to $2x_1 + x_2 \leq 120$ $x_1 + 2x_2 \leq 250$ $x_2 \leq 120$ $x_1, x_2 \geq 0$

(ii) Add artificial variables to "=" constraints

max $z = 3x_1 + 5x_2$ subject to $2x_1 + x_2 \leq 120$ $x_1 + 2x_2 \leq 250$ $x_2 = 120$ $x_1, x_2 \geq 0$	$\Rightarrow$	max $z = 3x_1 + 5x_2 - Mx_3$ subject to $2x_1 + x_2 \leq 230$ $x_1 + 2x_2 \leq 250$ $x_2 + x_3 = 120$ $x_1, x_2, x_3 \geq 0$

(iii) If x is unrestricted in sign, convert x to $x^+ + x^-$, where $x^+, x^- \geq 0$

max $z = 3x_1 + 5x_2$

subject to

$x_1 + x_2 \leq 5$

$x_1 + 2x_2 \leq 6$

$2x_1 + x_2 \leq 6$

$x_2 \geq 0$

$\Rightarrow$

max $z = 3(x_1^+ + x_1^-) + 5x_2$

subject to

$x_1^+ + x_1^- + x_2 \leq 5$

$x_1^+ + x_1^- + 2x_2 \leq 6$

$2(x_1^+ + x_1^-) + x_2 \leq 6$

$x_1^+, x_1^-, x_2 \geq 0$

(iv) Convert "$\leq$" constraints to "=" constraints by introducing slack variables

Example 1 : Positive RHS

max $z = 3x_1 + 5x_2$

subject to

$2x_1 + x_2 \leq 230$

$x_1 + 2x_2 \leq 250$

$x_2 \leq 120$

$x_1, x_2 \geq 0$

$\Rightarrow$

max $z = 3x_1 + 5x_2$

subject to

$2x_1 + x_2 + s_1 = 230$

$x_1 + 2x_2 + s_2 = 250$

$x_2 + s_3 = 120$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Example 2 : Negative RHS

max $z = 4x_1 + 2x_2$

subject to

$x_1 + x_2 \leq 4$

$2x_1 + x_2 \leq 5$

$x_1 - 4x_2 \leq -2$

$x_1, x_2 \geq 0$

$\Rightarrow$

$x_1 - 4x_2 + s_1 = -2$

$\Downarrow$

$-x_1 + 4x_2 - s_1 = 2$

$\Downarrow (*)$

$-x_1 + 4x_2 - s_1 + s_2 = 2$

max $z = 4x_1 + 2x_2 - Ms_4$

subject to

$x_1 + x_2 \leq 4$

$2x_1 + x_2 \leq 5$

$-x_1 + 4x_2 - s_3 + s_4 = 2$

$x_1, x_2, s_3, s_4 \geq 0$

artificial variable

slack variable

(v) Convert "$\geq$" constraints to "=" constraints by introducing surplus variables (and artificial variables)

max $z = 3x_1 + 5x_2$

subject to

$2x_1 + x_2 \leq 230$

$x_1 + 2x_2 \geq 250$

$x_1, x_2 \geq 0$

$\Rightarrow$

$-x_1 - 2x_2 \leq -250$

$\Downarrow$

$-x_1 - 2x_2 + s_1 = -250$

$\Downarrow$

$x_1 + 2x_2 - s_1 = 250$

$\Downarrow (*)$

$x_1 + 2x_2 - s_1 + x_3 = 250$

max $z = 3x_1 + 5x_2 - Mx_3$

subject to

$2x_1 + x_2 \leq 230$

$x_1 + 2x_2 - s_1 + x_3 = 250$

$x_1, x_2, s_1, x_3 \geq 0$

artificial variable

surplus variable

Step 2 : Solve for initial basic feasible solution

(i) Initialise set of basic variables to the last (num-row) variables

Matrix representation

$$\begin{array}{ll} \max & z = c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \Rightarrow \begin{array}{ll} \max & z = c_B^T x_B + c_{NB}^T x_{NB} \\ \text{s.t.} & [B \ NB] \begin{pmatrix} x_B \\ x_{NB} \end{pmatrix} = b \\ & x_B, \ x_{NB} \geq 0 \end{array} \xRightarrow{x_{NB} = 0} \begin{array}{ll} \max & z = c_B^T x_B \\ \text{s.t.} & B x_B = b \\ & x_B \geq 0 \end{array}$$

(ii) Generate initial simplex tableau

Basic variables	Coefficient of						RHS
	z	x_1	x_2	s_1	s_2	s_3	
z	1	-3	-5	0	0	0	0
s_1	0	-2	1	1	0	0	230
s_2	0	1	2	0	1	0	250
s_3	0	0	1	0	0	1	120

Objective function

e.g. $z = 3x_1 + 5x_2 \Rightarrow 1z - 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

Constraints

e.g. $-2x_1 + x_2 + s_1 = 230 \Rightarrow 0z - 2x_1 + 2x_2 + 1s_1 + 0s_2 + 0s_3 = 230$

Matrix representation : $\begin{pmatrix} 1 & \overset{-c_B^T}{\underset{\sim}{0}} & -c_{NB}^T \\ 0 & \underset{B}{I} & \underset{NB}{A} \end{pmatrix} \begin{pmatrix} z \\ x_B \\ x_{NB} \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$

At any stage, basic feasible solution : NBV = 0 , BV = RHS

Note: Interpretting simplex tableau

BV	Coefficient of					RHS
	z	x_1	x_2	x_3	x_4	
z	1	0	0	$-\frac{1}{5}$	$\frac{7}{5}$	$\frac{19}{5}$
x_1	0	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{9}{5}$
x_2	0	0	1	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{8}{5}$

$$\begin{pmatrix} z \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{19}{5} \\ \frac{9}{5} \\ \frac{8}{5} \end{pmatrix} + \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{pmatrix} x_3 + \begin{pmatrix} -\frac{7}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix} x_4$$

Step 3 : While solution is not optimal, optimise the solution

Matrix representation : $\begin{pmatrix} 1 & C_B^T B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 1 & -C_B^T & -C_{NB}^T \\ 0 & B & NB \end{pmatrix} \begin{pmatrix} z \\ x_B \\ x_{NB} \end{pmatrix} = \begin{pmatrix} 1 & C_B^T B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$

$$\begin{pmatrix} 1 & \underset{\sim}{0} & -C_B^T + C_B^T B^{-1} NB \\ 0 & \underset{\sim}{I} & B^{-1} NB \end{pmatrix} \begin{pmatrix} z \\ x_B \\ x_{NB} \end{pmatrix} = \begin{pmatrix} C_B^T B^{-1} b \\ B^{-1} b \end{pmatrix}$$

(i) Conduct optimality test

↳ All coefficients in z-row ≥ 0 (∵ NBV take value 0. ↑ any NBV ⇒ ↓ z)

While the solution fails the optimality test :

DEFAULT METHOD

(for cases w/o "M")

(ii) Determine entering BV and leaving BV

↳ Entering BV : pivot column —— column with most negative coefficient in z-row

Break ties arbitrarily.

↳ Leaving BV : pivot row —— minimum ratio test (ratio = $\frac{\text{RHS}}{\text{value in pivot column}}$ if value in pivot column >0 (not $\geq$), else ∞) . Break ties arbitrarily —— When there is a tie, there will be BV(s) with RHS = 0 after updating ; they are degenerate BVs

↳ pivot number is the intersection of pivot row and pivot column

(iii) Update the simplex tableau (essentially Gauss-Jordan Elimination)

↳ pivot number $A_{r,c} = 1$

↳ pivot row $A_{r,j} = \frac{A_{r,j}}{A_{r,c}}$

↳ pivot column $A_{i,c} = 0$

↳ remaining elements $A_{i,j} \mathrel{-}= \frac{A_{i,c} \times A_{r,j}}{A_{r,c}}$

BV	Coefficient of: z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
z	1	-3	-5	0	0	0	0	
s_1	0	2	1	1	0	0	230	$\frac{230}{1} = 230$
s_2	0	1	2	0	1	0	250	$\frac{250}{2} = 125$
s_3	0	0	1	0	0	1	120	$\frac{120}{1} = 120$

⟶

BV	Coefficient of: z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-3	0	0	0	5	600
s_1	0	2	0	1	0	-1	110
s_2	0	1	0	0	1	-2	10
x_2	0	0	1	0	0	1	120

BIG-M METHOD

(for cases w/ M)

Step 1: Convert initial tableau to proper initial tableau

- All values in BV columns must be 0, except one 1

BV	Coefficients of: z	x_1	x_2	s_1	s_2	x_3	RHS
z	1	-3	-5	0	0	M	0
s_1	0	2	1	1	0	0	230
s_2	0	1	2	0	1	0	250
x_3	0	0	1	0	0	1	120

$\xrightarrow{\text{row}_1 \mathrel{-}= M \cdot \text{row}_4}$

BV	Coefficients of: z	x_1	x_2	s_1	s_2	x_3	RHS
z	1	-3	-5-M	0	0	0	-120M
s_1	0	2	1	1	0	0	230
s_2	0	1	2	0	1	0	250
x_3	0	0	1	0	0	1	120

Step 2 : Optimise as per default method, treating M as an infinitely large number

TWO-PHASE METHOD

(for cases w/ M)

E.g. max $z = 4x_1 + 2x_2 + 3x_3 + 5x_4$

subject to

$2x_1 + 3x_2 + 4x_3 + 2x_4 = 300$

$8x_1 + x_2 + x_3 + 5x_4 = 300$

$x_1, x_2, x_3, x_4 \geq 0$

$\longrightarrow$

max $z = 4x_1 + 2x_2 + 3x_3 + 5x_4 + Ms_1 + Ms_2$

subject to

$2x_1 + 3x_2 + 4x_3 + 2x_4 + s_1 = 300$

$8x_1 + x_2 + x_3 + 5x_4 + s_2 = 300$

$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0$

Phase I : Find a basic feasible solution.

- ▷ Step 1 : Make z-row coefficient of artificial variable 0
- ▷ Step 2 : Optimise as per default method

PHASE I

Step 1

min $z = s_1 + s_2 \Rightarrow$ max $-z = -s_1 - s_2$ OR max $z = -s_1 - s_2$

(both are alright because by the end of phase I, $-s_1 - s_2 = 0$)

BV	Coefficient of							Right side
	z	x_1	x_2	x_3	x_4	s_1	s_2	
z	1	0	0	0	0	1	1	0
s_1	0	2	3	4	2	1	0	300
s_2	0	8	1	1	5	0	1	300

$\downarrow$ (z row) − (s_1 row) − (s_2 row)

BV	Coefficient of							Right side
	z	x_1	x_2	x_3	x_4	s_1	s_2	
z	1	−10	−4	−5	−7	0	0	−600
s_1	0	2	3	4	2	1	0	300
s_2	0	8	1	1	5	0	1	300

Step 2

BV	Coefficient of							Right side
	z	x_1	x_2	x_3	x_4	s_1	s_2	
z	1	0	0	0	0	0	1	0
x_3	0	0	$\frac{11}{15}$	1	$\frac{1}{5}$	$\frac{4}{15}$	$-\frac{1}{15}$	60
x_1	0	1	$\frac{1}{30}$	0	$\frac{3}{5}$	$-\frac{1}{30}$	$\frac{2}{15}$	30

PHASE II: Find the optimal solution.

- Step 1: Construct tableau without artificial variables
- Step 2: Convert tableau to proper (make z-row coefficient of pivot columns 0)
- Step 3: Optimise as per default method

PHASE II

Step 1

$$\max z = 4x_1 + 2x_2 + 3x_3 + 5x_4$$

BV	z	x_1	x_2	x_3	x_4	Right side
z	1	-4	-2	-3	-5	0
x_3	0	0	$\frac{11}{15}$	1	$\frac{1}{5}$	60
x_1	0	1	$\frac{1}{30}$	0	$\frac{3}{5}$	30

Step 2

$(z \text{ row}) + 4(x_3 \text{ row}) + 3(x_1 \text{ row})$

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	$\frac{1}{3}$	0	-2	300
x_3	0	0	$\frac{11}{15}$	1	$\frac{1}{5}$	60
x_1	0	1	$\frac{1}{30}$	0	$\frac{3}{5}$	30

Step 3

BV	z	x_1	x_2	x_3	x_4	Right side
z	1	$\frac{10}{3}$	$\frac{4}{9}$	0	0	400
x_3	0	$-\frac{1}{3}$	$\frac{13}{18}$	1	0	50
x_4	0	$\frac{5}{3}$	$\frac{1}{18}$	0	1	50

$\therefore$ Solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 50$, $x_4 = 50$, $z = 400$

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Type I: Multiple optimal solutions

In final tableau, at least one of the NBV in the z row has coefficient of 0. Increasing any such variable would not change the value of z, but provides alternative basic feasible optimal solutions.

set as pivot column

Basic variable	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS	"Ratio"
z	1	0	0	0	2	0	M	10	
s_1	0	0	0	1	$-\frac{5}{9}$	$\frac{1}{9}$	$-\frac{1}{9}$	1	9
x_1	0	1	0	0	$\frac{4}{9}$	$\frac{1}{9}$	$-\frac{1}{9}$	2	18
x_2	0	0	1	0	$\frac{1}{9}$	$-\frac{2}{9}$	$\frac{2}{9}$	1	∞

Solution 1: $z = 10$, $x_1 = 2$, $x_2 = 1$

↓

Basic variable	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	1	0	0	0	2	0	M	10
s_3	0	0	0	9	-5	1	-1	9
x_1	0	1	0	1	$-\frac{1}{9}$	0	$-\frac{2}{9}$	1
x_2	0	0	1	-1	$-9\frac{8}{9}$	0	0	3

Solution 2: $z = 10$, $x_1 = 1$, $x_2 = 3$

$\therefore$ Any point on the line connecting these vertices is also a solution

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\quad \underbrace{0 \leq \lambda \leq 1}$

↳ if $\lambda < 0$ or $\lambda > 1$, solution is infeasible

→ This can be extended to when there are >1 columns with 0 z-row coefficient.
⇒ Solution is always the set of convex combinations of all solutions

Type II : Unbounded linear problem

▷ Ill-posed problem, usually due to misspecification of constraints (profits can ↑ indefinitely)

When there is no leaving basic variable, ie all elements of $B^{-1}A_j \leq 0$, the entering basic variable x_j can be increased indefinitely without giving negative values to any of the current basic variables.

max $\quad z = 8x_1 + 5x_2$

subject to

$\quad x_1 \leq 120$

BV		Coefficient of			RHS
	z	x_1	x_2	s_1	
z	1	$\boxed{-8}$	-5	0	0
s_1	0	$\boxed{0}$	1	1	120

Obviously, there is no leaving BV. LP is unbounded.

Intuitively x_1 can be ↑ indefinitely w/o Δ optimal solution

BV			Coefficient of			Right side
	z	x_1	x_2	x_3	x_4	
z	1	0	0	$-\frac{1}{5}$	$\frac{7}{5}$	$\frac{19}{5}$
x_1	0	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{9}{5}$
x_2	0	0	1	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{8}{5}$

−ve ⇒ can't be leaving BV

Suppose x_1 and x_2 are selected as initial BV (ie x_3 & x_4 are NBV)

$$\begin{pmatrix} z \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{19}{5} \\ \frac{9}{5} \\ \frac{8}{5} \end{pmatrix} + \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{pmatrix} x_3 + \begin{pmatrix} -\frac{7}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix} x_4$$

We can set $x_4 = 0$ and ↑ x_3 arbitrarily large and stay in the feasible region

⇒ LP is unbounded

Type III . No feasible solution

If the problem has no feasible solution (ie is infeasible), the Big M method or phase I of two phase method yields a final solution with at least one artificial variable ≥ 0.

BV				Coefficient of			Right side
	z	x_1	x_2	s_1	s_2	x_3	
z	1	0	$2+M$	0	M	0	$3-M$
x_1	0	1	2	1	0	0	3
x_3	0	0	-1	-1	-1	1	1

All ≥ 0

Artificial variable $x_3 = 1 > 0$

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Step 1: Obtain B, NB, C_B, C_{NB}, b

E.g. max $z = \boxed{3x_1 + 5x_2}\ \boxed{+0s_1 + 0s_2 + 0s_3}$

subject to

$$\begin{aligned} 2x_1 + x_2 + s_1 \quad\quad\quad\quad &= 230 \\ x_1 + 2x_2 \quad + s_2 \quad\quad &= 250 \\ x_2 \quad\quad\quad + s_3 &= 120 \end{aligned}$$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Given: $X_B = \{s_3, x_1, x_2\}$ $X_{NB} = \{s_1, s_2\}$

$B = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ $NB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $C_B = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$ $C_{NB} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $b = \begin{pmatrix} 230 \\ 250 \\ 120 \end{pmatrix}$

Step 2. Obtain solution / objective value / dual multiplier / simplex tableau

Simplex tableau at any stage: ①

	z	BV	NBV	RHS
z	0	$\underset{\sim}{0}$	$-C_{NB}^T + C_B^T B^{-1} NB$	$C_B^T B^{-1} b$
BV	1	$\underset{\sim}{I}$	$B^{-1} NB$	$B^{-1} b$

objective value

solution

optimality check (optimal if all $x_i > 0$)

② dual multiplier $= y^T = C_B^T B^{-1}$

③ Alternative optimality check: dual feasibility
$A^T y - c \geq 0$? Optimal : sub-optimal

④ Value of any column(s) $= B^{-1} \times$ ([column(s) in standard form LP])

At optimal solution, $y \geq \underset{\sim}{0}$

↳ $NB^T y \geq C_{NB}$
↳ $B^T y = C_B$

$\Big\}$ $A^T y \geq \underset{\sim}{c}$ (y is feasible for dual problem)

↓

$w = b^T y = y^T b = C_B^T B^{-1} b = C_B^T x_B = z$

$\therefore$ By strong duality, y is optimal solution for dual problem & shadow price for primal problem

Objective function value $= y^T b = c^T x$

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If dual for a constraint, $y_i > 0$, constraint is binding.
If slack for a constraint is a BV, constraint is non-binding.

Sensitivity range

Type I: Allowable range for changing RHS of constraint w/o changing BV set

Suppose $B^{-1}b = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}\begin{pmatrix} 230 \\ 250 \\ 120 \end{pmatrix} = \begin{pmatrix} 30 \\ 70 \\ 90 \end{pmatrix}$

Effect of changing right side of constraint 1 from 230 to $(230 - \Delta b_1)$

→ objective function value changes

→ The set of BV will not change if

$$B^{-1}\begin{pmatrix} 230-\Delta b_1 \\ 250 \\ 120 \end{pmatrix} = \begin{pmatrix} 30 \\ 70 \\ 90 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}\begin{pmatrix} -\Delta b_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 30 \\ 70 \\ 90 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}\Delta b_1 \\ -\frac{2}{3}\Delta b_1 \\ \frac{1}{3}\Delta b_1 \end{pmatrix} \boxed{\geq 0}$$

∴ adhere to optimality test

→ $30 - \frac{1}{3}\Delta b_1 \geq 0$ $70 - \frac{2}{3}\Delta b_1 \geq 0$ $90 + \frac{1}{3}\Delta b_1 \geq 0$

$\Delta b_1 \leq 90$ $\Delta b_1 \leq 115$ $\Delta b_1 \geq -30$

$\therefore -30 \leq \Delta b_1 \leq 90$

Variation of this qn type: If ↑ RHS by 1 unit, will obj fn value ↑ by optimal dual value for that constraint?

- Check feasibility: $B^{-1}b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow x_1, x_2, x_3 \geq 0$, or
- Check allowable range

Type II: Allowable range for changing objective function coefficient of a BV w/o changing BV set = allowable range for dual solution

Suppose dual solution is: $y^T = (y_1 \;\; y_2 \;\; y_3) = c_B{}^T B^{-1}$, where $BV = \{s_3, x_1, x_2\}$

$$= (0 \;\; 3 \;\; 5)\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

Effect of changing coefficient of x_1 from 3 to $3 + \Delta c_1$

→ $y^T = (0 \;\; 3+\Delta c_1 \;\; 5)\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$

$= (\frac{1}{3} \;\; 2\frac{1}{3} \;\; 0) + (\frac{2}{3}\Delta c_1 \;\; -\frac{1}{3}\Delta c_1 \;\; 0)$

→ max Δc_1 for $y \geq 0$:

$\frac{1}{3} + \frac{2}{3}\Delta c_1 \geq 0$ $2\frac{1}{3} - \frac{1}{3}\Delta c_1 \geq 0$

$\Delta c_1 \geq -\frac{1}{2}$ $\Delta c_1 \leq 7$

$\therefore -\frac{1}{2} \leq \Delta c_1 \leq 7$

Type III : Allowable range for changing objective function coefficient of an NBV w/o changing BV set

$$-c_{NB}^T + c_B^T B^{-1} N B = (0 \quad 0) + (0 \quad 3 \quad 5)\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \left(\frac{1}{3} \quad 2\tfrac{1}{3}\right)$$

$\hookrightarrow \Delta s_1 \leqslant \frac{1}{3}, \quad \Delta s_2 \leqslant 2\frac{1}{3}$

Type IV : Introduce new constraint ⇒ range of RHS for which optimal solution remains unchanged

E.g. Initially,

$$\max \quad z = 25x_1 + 40x_2 + 20x_3 \quad \text{for products } 1, 2, 3$$

subject to

$$\begin{aligned} 1x_1 + 5x_2 + 1x_3 &\leqslant 80 && \text{for resource 1} \\ 1x_2 + 2x_3 &\leqslant 120 && \text{for resource 2} \\ 2x_1 + 2x_2 + 1x_3 &\leqslant 100 && \text{for resource 3} \\ x_1,\ x_2,\ x_3 &\geqslant 0 \end{aligned}$$

⇒ optimal solution $\begin{pmatrix} 3 \\ 3 \\ 11 \end{pmatrix}$

- A new product, product 4, is being considered. To produce 1 unit of product 4, amount of resources 1, 2, 3 required are 0.5, 1, 2 respectively.

⇒ · To produce product 4, current solution must be sub-optimal. $(A^T y - c)_{new} < 0 \Rightarrow A^T y - c_{new} < 0$

$c_{new} > A^T y = \begin{pmatrix} 0.5 \\ 1 \\ 2 \end{pmatrix}^T \cdot \begin{pmatrix} 3 \\ 3 \\ 11 \end{pmatrix} = 26.5$

Type V : Tightening of lower bound

- ▷ If constraint is a BV, in optimal solution, new production level for that product will be at lower bound. Minus off resources required to produce those product and recompute problem w/o that product.
- ▷ If constraint is a NBV, include it as a new row in final tableau and pivot about it

E.g. Add $x_1 \geqslant 100$

BV	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
z	1	$\frac{13}{2}$	0	0	0	4	$\frac{1}{4}$	2050
s_1	0	-1	0	0	1	$-\frac{1}{2}$	$-\frac{1}{4}$	60
x_3	0	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_2	0	$\frac{1}{4}$	1	0	0	0	$\frac{1}{8}$	250
x_1	0	1	0	0	0	0	0	100

↓

BV	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	
z	1	0	0	0	0	4	$\frac{1}{4}$	1400	⇒ optimal π
s_1	0	0	0	0	1	$-\frac{1}{2}$	$-\frac{1}{4}$	160	
x_3	0	0	0	1	0	$\frac{1}{2}$	0	80	⇒ solution
x_2	0	0	1	0	0	0	$\frac{1}{8}$	80	
x_1	0	1	0	0	0	0	0	100	

shadow price

Method 1

To find shadow price for second constraint:

initial $B^{-1}b = \frac{1}{27}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix}\begin{pmatrix} 180 \\ 270 \\ 180 \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \\ 50 \end{pmatrix}$ $\Rightarrow z = 6(50) + 8(30) + 9(50) = 990$

new $B^{-1}b = \frac{1}{27}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix}\begin{pmatrix} 180 \\ 270+1 \\ 180 \end{pmatrix} = \begin{pmatrix} 49\frac{8}{9} \\ 30\frac{1}{3} \\ 50\frac{8}{9} \end{pmatrix}$ $\Rightarrow z = 6(49\frac{8}{9}) + 8(30\frac{1}{3}) + 9(50\frac{8}{9}) = 991$

$\therefore$ Shadow price $= 991 - 990 = 1$

Method 2

$y^T = c_B^T B^{-1}$

$= \begin{pmatrix} \frac{4}{3} & 1 & \frac{8}{3} \end{pmatrix}$

shadow price for second constraint