ON THE HOMOGENIZATION OF A FRONT PROPAGATION MODEL IN OSCILLATORY ENVIRONMENTS

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Abstract. In this survey, we review the recent developments in the homogenization of a level-set Hamilton-Jacobi equation that models front propagations in oscillatory environments, where the rule determining the front movement varies in a highly heterogeneous manner in space or as well in time. We focus on two directions, one is the new tools developed to overcome the difficulties caused by environmental oscillations in time, and the other concerning finer questions beyond qualitative homogenization, such as convergence rate and inverse homogenization type problems.

Key words: Homogenization; front propagation; effective fronts; dynamic random environments, optimal rate of convergence; inverse shape theorem

Mathematics subject classification (MSC 2010): 35B40, 37J50, 49L25

1. Introduction

Environments with spatial and temporal heterogeneities are encountered in many applications in natural sciences and engineering, such as radiative transfer in atmosphere, chemical conduction and reactions in turbulent flows, etc. In such media, the governing partial differential equations (PDEs) naturally have coefficients that vary in small scales and those small scale variations are typically poorly known. An important task is then to start from reasonable modeling of the microscale structures of the environment and rigorously derive simplified model that still captures the macroscopic behavior of the original problem; this is referred as the effective medium theory or homogenization theory in applied physics. Such endeavors traces back at least to Mossotti [Mos50], Clausius [Cla79], Maxwell Garnett [MG04], and Einstein [Ein06]. The mathematical theory for homogenization has attracted enormous attentions in the applied analysis community over the past five decades; see [JKO94]. In this review we do not attempt to provide a thorough reference of the large volume of literature on homogenization of Hamilton-Jacobi equations, but only refer to those that is most relevant to (1.4).

This survey concerns mainly the homogenization of a front propagation model that is governed by a first order fully nonlinear Hamilton-Jacobi equation. To set up the problem formally, consider a typical front S_t that evolves in time according to the rule:

$$V(x,t) = a(x,t)N. \tag{1.1}$$

Here x is the spatial variable, and t is the time variable. For each point $x \in S_t$ on the front, $N = N(x,t)$ is the unit vector normal to S_t at x. The scalar function $a(x,t)$ gives the speed of the motion, and is determined completely by the environment. One way to model this geometric movement is through a level-set formulation: one wishes to find a scalar function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that the front S_t at time t is given by, say, the zero level-set of u. In other words, for each

$t \in \mathbb{R}$, the front is

$$S_t = \{x \in \mathbb{R}^n \,:\, u(x,t) = 0\}.$$

Suppose this is possible and u is a smooth function with non-vanishing gradients, then the unit normal vector N along S_t is $\frac{Du}{|Du|}$, where Du denotes the spatial gradient of u. Since $u(x(t),t)$ is conserved along any Lagrangian trajectory $x(t)$ of a point in the initial front, we get

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + V(x(t),t) \cdot Du(x(t),t) = 0. \tag{1.2}$$

Suppose the space is filled with such moving fronts that are level-sets of the same function u; then we get the following equation in the Eulerian formulation:

$$\begin{cases} \partial_t u(x,t) + a(x,t)|Du(x,t)| = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,t) = g(x), & (x,t) \in \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.3}$$

To get the equations above, we used (1.2), the velocity rule (1.1) and the aforementioned formula for N; the initial data g is assumed to be known and characterizes the initial fronts.

We consider the situation that the environmental function a is highly heterogeneous, and hence model it as $a^\varepsilon(x,t) = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$, where $\varepsilon > 0$ is a small number that will soon be sent to zero. The idea is, we think the environment a^ε as a scaled version of a model function a that is regular, in the sense that variations of a occur in unit scales in space or as well in time. For $\varepsilon \ll 1$, the environment a^ε is highly oscillatory. Equation (1.3) then becomes

$$\begin{cases} \partial_t u^\varepsilon(x,t) + a(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon}; \omega)|Du^\varepsilon(x,t)| = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,t) = g(x), & (x,t) \in \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.4}$$

Note that we secretly added another variable ω in the environmental function a^ε, as we will soon view a as a random field, i.e. stochastic process in some probability space and indexed by the space and time variables. This is natural since usually we do not have detailed information about variations in the environment, so we model them as random. The underlying idea is that, there is some self-averaging mechanism in the physical problem so that, if the model environment a satisfies certain structural assumptions such as periodicity or stationarity and ergodicity (in the random setting), the ergodicity of the environment is explored by the mechanism and the problem displays averaging effect in the large scale. We should point out, however, the effective medium is usually not a simple average of the heterogeneous environment because the mechanism is usually nonlinear with respect to the environments. A typical qualitative homogenization states as follows: under certain structural assumptions on a, as $\varepsilon \to 0$, the unique solution of (1.4) converges locally uniformly to that of the homogenized problem

$$\begin{cases} \partial_t u(x,t) + \overline{H}(Du(x,t)) = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,t) = g(x), & (x,t) \in \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.5}$$

Here, $\overline{H}$ is a deterministic function that is, in fact, positive one homogeneous. In other words, the nonlinear term above can be written as $\overline{H}(Du) = \overline{a}(Du/|Du|)|Du|$ where $\overline{a}$ is the restriction of $\overline{H}$ to the unit sphere. It is remarkable that this effective equation does not see varying environment, yet it encodes averaging of nonlinear interactions between the original equation and the environment. The precise structural assumptions and statements of convergence results will be made clear.

Our first main objective is to review the homogenization of (1.4) in dynamic environment, where a^ε has oscillations with respect to time. When those oscillations are periodic in time and random in space, then a purely PDE framework, initiated by Lions and Souganidis [LS05, LS10] and developed further by Armstrong and Souganidis [AS12], Armstrong and Tran [AT14], can be applied. We will refer it as the metric problem approach; it is also deeply connected with the metric problem in Hamiltonian dynamics, see Davini and Siconolfi [DS11]. The metric problem approach seems to be designed for static problem, but in section 3.1 we review the method of [JST16] which explores periodicity in time to adapt the metric approach to dynamic environments. This method, however, fails when the oscillations is random in time. In section 3.2 we review the framework of [JST18] for homogenization of (1.4) that is relatively new and based on a shape theorem. More precisely, it characterize the effective Hamiltonian as the support function of a deterministic compact and convex set that is the large scale average of the normalized reachable set, which is random and through which the ergodicity of the random environment propagates to the large scale.

The second objective of this paper is to present some new directions that is beyond the qualitative homogenization of (1.4). One is the important question of convergence rates; in section 4.2 we present some partial optimal convergence rate results for certain class of effective models. In section 4.1, we point also to another direction that deserves great attention, roughly phrased as inverse type problems. In particular, we review some initiative work in [JTY20] on the study of the inverse of the mapping of the effective environment, that is retrieving knowledge of the oscillations in the environment from the effective model.

2. Backgrounds and Preliminaries

The equations (1.4) and (1.5) belong to the class of fully nonlinear Hamilton-Jacobi equations, and there is a well developed viscosity solution theory for them. The Hamiltonian here is

$$H = H(x, t, p; \omega) = a(x, t; \omega)|p|.$$

In particular, H is convex in p for each fixed $(x, t; \omega)$. We refer to Lions [Lio82], and Crandall, Ishii and Lions [CIL92] for the general theory. For (1.4), the following condition guarantees its well-posedness.

(B) $a(x, t)$ is Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}$, and there exist two positive numbers λ, Λ satisfying $0 < \lambda < \Lambda$, such that

$$\lambda \le a(x, t) \le \Lambda. \tag{2.1}$$

According different situations, we assume either of the three structural assumptions.

(S1) The function $a = a(x, t, \omega)$ is $\mathbb{Z}$-periodic in t, and stationary in x with respect to $(\tau_x)_{x \in \mathbb{R}^n}$, that is, for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $(z, k) \in \mathbb{R}^n \times \mathbb{Z}$, and $\omega \in \Omega$,

$$a(x + z, t + k, \omega) = a(x, t, \tau_z \omega). \tag{2.2}$$

(S2) The function $a = a(x, t, \omega)$ is $\mathbb{Z}^n$-periodic in x, and stationary in t with respect to $(\tau_t)_{t \in \mathbb{R}}$, that is, for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $(z, s) \in \mathbb{Z}^n \times \mathbb{R}$, and $\omega \in \Omega$,

$$a(x + z, t + s, \omega) = a(x, t, \tau_s \omega). \tag{2.3}$$

(S3) The function $a = a(x,t,\omega)$ is stationary in (x,t) with respect to $(\tau_{x,t})_{(x,t)\in\mathbb{R}^{n+1}}$, that is, for every $(x,t)\in\mathbb{R}^n\times\mathbb{R}$, $(y,s)\in\mathbb{R}^n\times\mathbb{R}$, and $\omega\in\Omega$,

$$a(x+y,t+s,\omega) = a(x,t,\tau_{(y,s)}\omega). \tag{2.4}$$

We refer to [PV81, Koz79, JKO94] original set-ups of random media in linear homogenization problems. We call (S1) the temporal periodic spatial random setting, call (S2) the spatial periodic temporal random setting, and call (S3) the space-time random setting. In all of those three settings, we assume further that the groups of translations in Ω are measure preserving and ergodic. Let G be the set of index for the group actions in each case of (S1)(S2) and (S3). The assumption amounts to:

(E) For each τ_e with $e\in G$, and for all $A\in\mathcal{F}$, we have $\mathbb{P}(A) = \mathbb{P}(\tau_e^{-1}A)$. Moreover, if $A\in\mathcal{F}$ satisfies

$$\tau_e A = A, \qquad \forall e\in G,$$

then $\mathbb{P}(A)\in\{0,1\}$.

In either of the three cases, and for each realization of the random environment, because $a(x,t)$ satisfies (B), the solution (1.4) is well-posed and a unique solution u^ε can be found. The homogenization problem concerns the limiting behavior of u^ε as ε goes to zero.

2.1. **Classical PDE approach.** Homogenization problem for Hamilton-Jacobi equation was first studied by P.L. Lions, Papanicolaou and Varadhan [LPV87], in the periodic setting with Hamiltonian that is static, i.e. not varying in time. Take for example a general Hamilton-Jacobi equation of the form

$$\begin{cases} \partial_t u^\varepsilon + H(\frac{x}{\varepsilon}, Du^\varepsilon) = 0, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u^\varepsilon(x,0) = c + p\cdot x, & (x,t)\in\mathbb{R}^n\times\{0\}, \end{cases}$$

that is the Cauchy problem with affine initial data, where $c\in\mathbb{R}$ and $p\in\mathbb{R}^n$. It is natural to seek for a solution u^ε with the ansatz

$$u^\varepsilon(x,t) = c + p\cdot x - \overline{H}(p)t + \varepsilon v(\frac{x}{\varepsilon})$$

where $u = c + p\cdot x - \overline{H}(p)t$ takes care of the large scale mean profile, which is a plane wave type solution, and v encodes the small scale perturbative profile. For this ansatz, one would get

$$H(y, p + Dv(y)) = \overline{H}(p), \qquad y\in\mathbb{T}^n. \tag{2.5}$$

Here one should think y as a replacement of $\frac{x}{\varepsilon}$ and it varies in the unit period $\mathbb{T}^n = [0,1]^n$. The above equation is known as the cell problem; note both the function $v\in C(\mathbb{T}^n,\mathbb{R})$ and the real number $\overline{H}(p)$, for each fixed p, are unknowns. Suppose $\overline{H}(p)$ and v satisfy the equation, then the formula above for u^ε hold and, from maximum principle, one easily proves the homogenization result and, in fact, with optimal convergence rate $\|u^\varepsilon - u\|_{L^\infty}\le C\varepsilon$ on any compact set in space and time. It is shown in [LPV87] under certain structural assumption of H that for each p, there exists a unique real number $\overline{H}(p)$ so that (2.5) has a continuous viscosity solution. The method proposed to solve (2.5) is through a "regularization" by adding a positive term which results in the approximate cell problem

$$\delta v^\delta + H(y, p + Dv^\delta(y)) = 0, \qquad y\in\mathbb{T}^n. \tag{2.6}$$

Here $\delta > 0$ and for viscosity solution it yields certain monotonicity and then comparison can be invoked to get a unique solution for each $\delta > 0$. Then by establishing uniform (in δ) estimates for $\|\delta v^\delta\|_{L^\infty}$ and for $\|Dv^\delta\|_{L^\infty}$, a subsequence can be chosen along which $v^\delta - v^\delta(0)$ converges to a solution of (2.5) with $\overline{H}(p)$ given by the limit of $-\delta v^\delta(0)$. On the other hand, there can exist at most one such right hand side so that (2.5) has a continuous solution. The uniqueness of $\overline{H}$ and the effective Hamiltonian function is then settled.

A remarkable fact discovered by Evans in [Eva92] is, in fact, to prove homogenization it suffices to establish the following locally uniform convergence for the family of approximate problem (2.6),

$$\limsup_{\delta\to 0} \sup_{y\in B_{\frac{R}{\delta}}} \left|\delta v^\delta(y;p) - \overline{H}(p)\right| = 0, \tag{2.7}$$

for all $R \geq 1$ and for all $p \in \mathbb{R}^n$. Evans showed that given these condition, one can prove homogenization by a perturbative test function argument. In the first step, one shows uniform (in ε) estimates on the heterogeneous solutions u^ε so that along subsequences u^ε converges. In the second step, one shows the only accumulating point must be the solution to (1.5) by a contradiction argument: if u touches a test function φ from above (or below) but φ satisfies an inequality of (1.5) in the wrong direction, one can then use a properly rescaled version of (2.6) to perturb the test function to φ^ε, compare the equations for φ^ε and u^ε in carefully chosen domains, and eventually obtain a contradiction with (2.7).

The advantage of the homogenization criterion (2.7) above is, it can be easily generalized to other settings. For example, in the stationary ergodic setting, the cell problem (2.5) is posed on the whole space and we seek for a pair $(v(\cdot,p;\omega), \overline{H}(p))$ with v being stationary with respect to the translation group of the probability space. In particular, the equation is posed on the whole space. Very often, it is impossible or not clear to obtain a solution to the cell problem, but the regularized problem is always well posed. In this case, as long as the locally uniform convergence (2.7) holds almost surely, the perturbed test function argument goes through and we obtain homogenization result almost surely. This is the approach of [Ish00] for the setting of almost periodic Hamiltonian. For the stochastic setting, such PDE approach based on (2.7) was initiated by Lions and Souganidis [LS03], and further developed, for example, by Armstrong and Souganidis [AS12], and Armstrong and Tran [AT14].

In the stochastic setting, to obtain (2.7), a key step is to identify $\overline{H}(p)$ not from (2.6) but from other functions or concepts that are fundamental to the equations considered. One candidate is the so-called fundamental solution which explores the control formulation of the H-J equation assuming the Hamiltonian is convex. Another is the so-called metric problem. Let us only explain the ideas for (1.4) and in the special case of a being static. The metric function in this setting will be the minimal travel time between two points x to y, and it satisfies the equation

$$a(y,\omega)|D_y m(y,x;\omega)| = 1, \qquad y \in \mathbb{R}^n \setminus \{x\} \qquad \text{and} \qquad m(x,x) = 0. \tag{2.8}$$

Either from the above PDE satisfied by this metric function, or from the property of m being the travel time, it can be shown that $m(x,y;\omega)$ is a subadditive function in the sense that

$$m(x,z;\omega) \leq m(x,y;\omega) + m(y,z;\omega)$$

for any $x,y,z \in \mathbb{R}^n$. On the other hand, it is clear that if a satisfies (A1), $|Dm|$ is uniformly bounded, which provides proper integrability for m. Then we can apply the subadditive ergodic theory (see e.g. [AK81]) and get the following almost sure convergence:

$$\frac{1}{t}m(tx,ty;\omega) \xrightarrow{t\to\infty} \overline{m}(x-y), \tag{2.9}$$

where $\overline{m}:\mathbb{R}^n\to\mathbb{R}$ is a deterministic convex positive one homogeneous function, and the convergence holds locally uniformly in the physical space and almost surely in the probability space. It turns out that if $\overline{H}$ is defined as the support function of the one-level set of $\overline{m}$, that is

$$\overline{H}(p) = \sup_{y\in D} p\cdot y, \qquad \text{where} \quad D := \{y\in\mathbb{R}^n \,:\, \overline{m}(y)\le 1\},$$

then the criterion (2.7) can be checked. As a result, u^ε of (1.4) converges almost surely and locally uniformly to u of (1.5), with $\overline{H}$ defined as above.

The outline above fails when the environment $a(\cdot;\omega)$ is oscillatory with respect to time. In this setting, either for periodic or random temporal oscillations, the cell problem (2.5), the approximate cell problem (2.6) and the metric problem (2.8) all have to be modified by adding the time derivatives $\partial_s v(\cdot,p;\omega)$, $\partial_s v^\delta$ and $\partial_s m(\cdot;\omega)$ where s is the microscopic time variable that plays the role of $\frac{t}{\varepsilon}$; it lives in the unit cell for the periodic setting or in the whole space $\mathbb{R}$ for the random setting. In those cases, because the added term is linear in those time derivatives and, hence, no coercivity can be used to get uniform (in ε or δ) estimates on the time derivative. In the special case of (1.4), in fact, we do not have (available at hands) any uniform modulus of continuity for u^ε or v^δ. This is the main difficulty when homogenization of (1.4) in dynamic oscillatory environment is concerned. Let us emphasize that, even for the space-time periodic setting, the homogenization of (1.4) remained open until [JST18].

2.2. **Optimal control formula.** For (1.4) the Hamiltonian function $H(x,t,p)=a(x,t)|p|$, when $\varepsilon=1$. Although it is only linearly growing in $|p|$, H is convex in the momentum variable p. As a result, the control theory representation of H-J equation applies. Define the Lagrangian $L(x,t,v)$ to be

$$L(x,t,v) = \sup_{p\in\mathbb{R}^n} v\cdot p - H(x,t,p).$$

Then we find, for each $v\in\mathbb{R}^n$, the formula

$$L(x,t,v) = \begin{cases} 0 & |v|\le a(x,t),\\ +\infty & |v|>a(x,t). \end{cases} \tag{2.10}$$

For general $\varepsilon>0$, by a proper scaling, it can be checked that u^ε, is then given by

$$u^\varepsilon(x,t) = \inf_{y\in\mathbb{R}^n}\left\{ g(y) + \inf_{\gamma\in\mathcal{A}} \int_0^{\frac{t}{\varepsilon}} L(\gamma(s),s,\dot\gamma(s))ds \right\},$$

where $\mathcal{A}$ consists of Lipschitz (in time) paths that start from $\frac{y}{\varepsilon}$ at time zero and goes to $\frac{x}{\varepsilon}$ at time $\frac{t}{\varepsilon}$. Recall that L is either zero or infinity; the above minimization must can be rewritten as

$$u^\varepsilon(x,t) = \inf\left\{ g(y) \,:\, \frac{x}{\varepsilon}\in\mathcal{R}_{\frac{t}{\varepsilon}}(\frac{y}{\varepsilon},0\;\omega)\right\} \tag{2.11}$$

where $\mathcal{R}_t(y,s;\omega)$ is the reachable set by time $t>s$ starting from the point $y\in\mathbb{R}^n$ at time $s\in\mathbb{R}$, and it is defined by

$$\mathcal{R}_t(y,s;\omega) = \{x\in\mathbb{R}^n \,:\, \exists\gamma\in W^{1,\infty}([s,t],\mathbb{R}^n), \text{ such that for all } r,\ |\dot\gamma(r)|\le a(\gamma(r),r;\omega)\}. \tag{2.12}$$

We call a path satisfying the velocity constraint above an admissible path from during time s and t, and denote the set of all such paths as $\mathcal{A}_{s,t}$. This control formula turn out to be very useful for the homogenization of (1.4) in the aforementioned situations that cannot be treated by the classical PDE approach, and it also gives useful insights on further studies such as the convergence rates and inverse type problems; see below.

2.3. **Notations and useful results.** We end this section by clarifying some notations used in this paper, and some useful tools from convex analysis and ergodic theory.

The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$, and $\mathbb{T}^n$ denotes the flat torus $[0,1]^n$. We use $B_r(x)$ to denote the open ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$, and use $Q_r(x,t)$ denote the cylinder $B_r(x) \times (t-r, t]$ centered at $(x,t) \in \mathbb{R}^{n-1}$. The collection of non-empty compact sets of $\mathbb{R}^n$ is denoted by $\mathcal{C}$, and it is augmented with the Hausdorff metric ρ_{H}, that is, for $A, B \in \mathcal{C}$,

$$\rho_{\mathrm{H}}(A,B) := \max\{\max_{x\in A} d(x,B), \max_{y\in B} d(y,A)\}, \tag{2.13}$$

where $d(x,B)$ is the Euclidean distance of a point x from a set B. We will also use various results from convex analysis. Recall that, for a compact convex set $D \subseteq \mathbb{R}^n$, we say $p \in \partial D$ is an exposed point if there is an affine plane normal to a unit vector n, so that it touches D only at p. In other words, $n \cdot (y-p) > 0$ for all $y \in D \setminus \{p\}$. We say $p \in \partial D$ is an extreme point if it cannot be written as a convex combination of other points in D with coefficients in $(0,1)$. We refer to [Roc70] for a comprehensive treatment of convex analysis.

3. Homogenization in dynamic environments

3.1. **The temporal-periodic and spatial-random setting.** In this subsection, we always assume (B)(S1) and (E). In particular, the environment function a, before rescaling, is periodic in time and stationary ergodic in space. Due to the time oscillations, the cell problem should be: for each $p \in \mathbb{R}^n$, find the unique $\overline{H}(p)$ such that the problem has a continuous viscosity solution,

$$\partial_s v(s,y;\omega) + a(s,y;\omega)|p + D_y v| = \overline{H}(p), \qquad (s,y) \in [0,1] \times \mathbb{R}^n, \tag{3.1}$$

with the requirements that v is periodic in time, Dv is stationary and v has sublinear growth as $|x|$ goes to infinity. Unlike the periodic cell problem (2.5), there is no coercivity to control $|\partial_s v|$ and hence no control of $|D_y v|$ either, so the construction method before that yield $\overline{H}(p)$ fails. Due to the same reason, an attempt through the approximate cell problem like before would also fail. Let us emphasize again that, even in the space-time periodic setting, the homogenization of (1.4) was open until [JST18] due to this difficulty.

What prevails in this setting turns out to be the metric problem approach. Due to the positive one-homogeneity in p of $H(x,t,p)$, the metric problem for (1.4) is the dynamic analog of (2.8) and reads

$$\begin{cases} \partial_t m(y,t,x;\omega) + a(y,t;\omega)|D_y m(y,t,x;\omega)| = 1, & (y,t) \in (\mathbb{R}^n \setminus \{x\}) \times \mathbb{R}, \\ m(y,t,x) = 0, & (y,t) \in \{x\} \times \mathbb{R}. \end{cases} \tag{3.2}$$

It turns out the metric problem solution still has the meaning of travel time, but now it depends on the time variable t which plays the role of the terminating time (or the starting time depending on the point of views). More precisely,

$$m(y,t,x;\omega) = \inf\{T > 0 \,:\, y \in \mathcal{R}_t(x, t-T)\}. \tag{3.3}$$

In other words, $m(y,t,x;\omega)$ is the minimal time to travel from x and get to y at time t. Note that $m(\cdot,t,\cdot;\omega)$ is a dynamic metric because it depends on the terminal time t. This dynamic feature might cause many difficulties but it turns out to be very manageable in the time periodic setting. Indeed, because the environment is time periodic, the travel time m is also periodic in t. Therefore, we can define the minimum travel time to be the minimization over all possible terminal times; we hence define

$$\theta(y,x;\omega) = \inf_{t\in\mathbb{R}} m(y,t,x;\omega). \tag{3.4}$$

Because m is periodic in time, the minimization can be taken over $[0,1]$, that is the unit time period, instead and the minimizer can be achieved. Then we can check that θ satisfies essential properties and can be used as the static metric function. In fact, the following properties hold.

Proposition 3.1. *Assume (B)(S1). Then for every $\omega \in \Omega$, the following statements hold:*

(i) *for every $x,y,z\in\mathbb{R}^n$ and $k\in\mathbb{Z}$,*

$$m(x,t+k,y,\tau_z\omega) = m(x+z,t,y+z,\omega), \tag{3.5}$$

and,

$$\beta^{-1}|x-y| \le m(x,t,y,\omega) \le \alpha^{-1}|x-y|. \tag{3.6}$$

(ii) *the oscillations of $m(x,\cdot,y,\omega)$ are uniformly controlled as follows:*

$$\operatorname*{osc}_{t\in\mathbb{R}} m(x,t;0,\omega) = \sup_{t\in\mathbb{R}} m(x,t,0,\omega) - \inf_{s\in\mathbb{R}} m(x,s,0,\omega) \le 1 \tag{3.7}$$

(iii) *the difference between m and θ is uniformly controlled as follows:*

$$\sup_{t\in\mathbb{R}} |m(x,t,y;\omega) - \theta(x,y;\omega)| \le 1 \tag{3.8}$$

(iv) *the function $m(\cdot,\cdot,y;\omega)$ solves* (3.2).

Property (i) is due to the stationarity of the environment and the upper and lower bound of a; item (ii) is due to time periodicity, and so is (iii). The fourth property can be proved by using the standard dynamic programming principle.

In view of (i) and the definition of θ, the function θ is stationary in the sense that

$$\theta(x,y;\tau_z\omega) = \theta(x+z,y+z;\omega), \qquad \forall x,y,z\in\mathbb{R}^n,\ \forall\omega\in\Omega.$$

Although θ is not a metric, it satisfies, for all $x,y,z\in\mathbb{R}^n$ and for all $\omega\in\Omega$,

$$\theta(x,y;\omega) \le \theta(x,z;\omega) + \theta(z,y;\omega) + 1.$$

This is due to the metric property of m and the control of $m-\theta$. From this and (3.6) we also verify that θ satisfies the bound

$$|\theta(x,y;\tau_z\omega) - \theta(x,y;\omega)| \le C(1+|z|), \tag{3.9}$$

for some constant C independent of z and ω. This control of the random fluctuations of θ is very important later.

Then the function $\theta + 1$ is then stationary and sub-additive. Together with the ergodicity assumption (E), we can use the subadditive ergodic theorem, and prove that, for each $e \in \mathbb{S}^{n-1}$,

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{t} m(te,0,0;\omega) = \lim_{t\to\infty} \frac{1}{t}\theta(te,0;\omega)\\ &= \lim_{t\to\infty} \frac{1}{t}\left[\theta(te,0;\omega)+1\right] = \overline{m}(e), \qquad \text{almost surely in } \Omega.\end{aligned} \tag{3.10}$$

In particular, the first equality is due to the uniform control of $m - \theta$. The number $\overline{m}(e)$ can be verified to be deterministic; we refer for example to [AS12, Proposition 6.9] for the subtle details. It is not hard then to verify that if we replace e by re, then the limit will be $r\overline{m}(e)$. In other words, the mapping $v \mapsto \overline{m}(v) := \overline{m}(v/|v|)|v|$ is the large scale limit of $\frac{1}{t}m(tv,0,0;\omega)$. It can be checked that $v \mapsto \overline{m}(v)$ is convex. Just like the metric problem approach in [AS12, AT14], the effective Hamiltonian $\overline{H}$ should be defined as the support function of the one sub-level-set of $\overline{m}$. That is,

$$\overline{H}(p) = \sup_{y\in D} p\cdot y, \qquad \text{where} \quad D := \{y \in \mathbb{R}^n \,:\, \overline{m}(y) \le 1\}.$$

For the front propagation model considered in this review, the above definitions have clear physical importance. We should view $\overline{m}(y)$ as the effective travel time, say, from the origin 0 to a point y. The unit time reachable set, that is the points that can be reached at time one starting from the origin, is precisely the set D. If we define the effective Lagrangian as

$$\overline{L}(v) = \begin{cases} 0 & v \in D,\\ +\infty & v \in \mathbb{R}^n \setminus D,\end{cases} \tag{3.11}$$

then $\overline{H}$ is again the Legendre transform of $\overline{L}$. With the effective Hamiltonian defined as above, we can prove the following homogenization result.

Theorem 3.2. *Assume (B)(S1) and (E), and $\overline{H}$ be defined as above. Let u^ε and u be, respectively, the solutions to* (1.3) *and* (1.5)*. Then there exists a measurable set $\widetilde{\Omega} \subseteq \Omega$ with full probability measure, such that for every $\omega \in \widetilde{\Omega}$ and for every $T \ge 0, R > 0$, we have*

$$\lim_{\varepsilon\to 0} \sup_{(x,t)\in B_R\times[0,T]} |u^\varepsilon(x,t,\omega) - u(x,t)| = 0. \tag{3.12}$$

In other words, the theorem says the solution u^ε of the heterogeneous equation converges, locally uniformly and almost surely in Ω, to the solution u of the effective equation. To prove this homogenization result, we just need to establish the Evans' criterion (2.7). This time, the approximate cell problem should read

$$\delta v^\delta(y,t;p,\omega) + \partial_t v^\delta + a(y,t;\omega)|p + D_y v^\delta| = 0, \qquad (y,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.13}$$

Moreover, the convergence in (2.7) should hold almost surely in Ω, and locally uniformly in space-time, that is

$$\limsup_{\delta\to 0} \sup_{(y,t)\in Q_{\frac{R}{\delta}}} \left|\delta v^\delta(y,t;p,\omega) + \overline{H}(p)\right| = 0. \tag{3.14}$$

This reduction of the homogenization proof to proving the above Evans' criterion is, as before, due to the perturbation test function argument; the presence of the time dependence in the Hamiltonians and the presence of the randomness play essentially no role. We refer to [AS12], or [AT14, JST17] for the details.

In the following, to demonstrate the main steps in the whole picture of the homogenization proof, we review two important ingredients in the verification of the criterion (3.14). The first is a locally uniform version of the large scale limit of the metric function m, and the second converting the large scale convergence of m to the locally uniform convergence in (3.14).

The local uniform convergence for the metric problem. We note that the convergence in (3.10) is for a fixed direction $e \in \mathbb{S}^{n-1}$, and it holds on a full measure set $\Omega_e \in \mathcal{F}$ that depend on e. To get locally uniform convergence in (3.14) it is importance to upgrade (3.10) to

Lemma 3.3. *Assume (B)(S1) and (E). Then there exists a deterministic convex Lipschitz continuous and positive 1-homogeneous function $\overline{m} : \mathbb{R}^n \to \mathbb{R}$, and a set $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\tilde{\Omega}) = 1$, such that for all $\omega \in \tilde{\Omega}$ and for all $R > 1$,*

$$\lim_{t\to\infty} \sup_{x,y\in B_R} \sup_{s\in\mathbb{R}} \left| \frac{1}{t} m(tx, s, ty; \omega) - \overline{m}(x-y) \right| = 0. \tag{3.15}$$

Compare the above with (3.10); there are several points to stress. First, the “vertex” of $m(tx, s, y; \omega)$ in (3.10) is $(0, 0)$ but here it is (s, ty). Secondly, the full measure set in (3.10) depends on the direction e (which would correspond to $\frac{x-y}{|x-y|}$ here) but now it is uniform in those direction variables. Thirdly, the vertices (s, y) are allowed to be in any large box and the convergence is uniform with respect those vertices. Upgrading from (3.10) to (3.15) is often referred to by experts as “making the vertices flow” and it is an important ingredient that might easily be overlooked.

To prove Lemma 3.3, we first note that in (3.10) the convergence holds in a set Ω_e of full measure that depends on the direction e. It is natural to let $\Omega' := \cap_{p\in\mathbb{Q}^n}\Omega_p$; then Ω' indeed has full measure and the convergence (3.10) now holds in Ω' for all rational p's. To show this holds actually for all $p \in \mathbb{R}^n$, we need a density argument to prove that it holds for all $p \in \mathbb{R}^n$; the key is to show that $\overline{m}$ is in fact Lipschitz, and this turns out to be true thanks to (3.9).

Next, to make the vertices flow, one uses the standard argument usually attributed to Varadhan: first for each $1/k$, we use Egoroff's theorem to identify a set $E_k \in \mathcal{F}$ with $\mathbb{P}(E_k) \geq 1 - 1/k^n$ so that

$$Z_t(\omega) := \sup_{x\in B_R, s\in\mathbb{R}} |t^{-1}m(tx, s, 0; \omega) - \overline{m}(x)| < 1/k.$$

Then using ergodic theorem, and find R_k large and a set $\tilde{\Omega}^k$ with full probability measure such that, for all $L > R_k$

$$|\{z \in B_L \,:\, \tau_z\omega \in E_k\}| > (1 - k^{-n})|B_L|, \qquad \forall \omega \in \tilde{\Omega}^k.$$

This is due to the fact that the spatial average of the indicator function function of E_k over B_L converges to $\mathbb{P}(E_k)$ as $L \to \infty$. Finally, we can check that (3.15) hold for $\tilde{\Omega} := \cap_{k\in\mathbb{N}}\tilde{\Omega}^k$, which is still of full measure. Indeed, fix k first. For any fixed $y \in B_R$, when t is sufficiently large (with the threshold depending on R but not on $y \in B_R$), we can find $\hat{y}$ close to y and $\tau_{t\hat{y}}$ falls in the good set E_k, so that $Z_t(\tau_{t\hat{y}}\omega)$ is controlled. Using (3.9) again, we can control $Z_t(\tau_{t\hat{y}}\omega) - Z_t(\tau_{ty}\omega)$, and as a result we get a control $Z_t(\tau_{ty}\omega)$. Moreover, this control is uniform with respect to $y \in B_R$. Letting t goes to infinity, we get (3.15). We refer to [JST17, JST16] for instance for the details.

From the metric problem to the Evans criterion. Another main ingredient of the metric approach is to use the locally uniform convergence of the large scale average of the metric function to establish the criterion (3.14). This is done by a so-called *reversed* perturbed test function argument.

The argument is purely deterministic, so we fix an $\omega \in \tilde{\Omega}$ so that (3.15) hold; furthermore, we omit ω for notational simplicity. We need to prove that for $R \geq 1$,

$$\limsup_{\varepsilon \to 0} \sup_{(x,t)\in Q_R} v_\varepsilon(x,t) + \overline{H}(p) \leq 0, \qquad \text{and} \qquad \liminf_{\varepsilon \to 0} \inf_{(x,t)\in Q_R} v_\varepsilon(x,t) + \overline{H}(p) \geq 0. \tag{3.16}$$

Here, $v_\varepsilon = v^\varepsilon(\cdot/\varepsilon, \cdot/\varepsilon; \omega)$ is the rescaled version of (3.1). It solves the rescaled equation

$$v_\varepsilon(x,t) + \partial_t v_\varepsilon + a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}; \omega)|p + Dv_\varepsilon| = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.17}$$

The idea is to the above inequalities by contradiction. To illustrate the ideas, let us provide some details for the first inequality which is relatively easier.

Arguing by contradiction, we assume the inequality fails, so there exist $R > 1$, a sequence still denoted by $\varepsilon \to 0$, a sequence $\{(x_\varepsilon, t_\varepsilon)\} \subseteq Q_R$ and a positive number $\theta > 0$, such that

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) + \overline{H}(p) \geq \theta > 0.$$

Then consider the function

$$w_\varepsilon(x,t) := v_\varepsilon(x,t) - v_\varepsilon(x_\varepsilon, t_\varepsilon) - c\theta(\sqrt{1 + |x - x_\varepsilon|^2} - 1) - c\theta(s - t_\varepsilon),$$

where the small number c is chosen so that on the domain

$$U_\theta := \left\{ (x,t) \in \mathbb{R}^n \times (-\infty, t_\varepsilon] \, : \, w_\varepsilon \geq -\frac{\theta}{4} \right\},$$

we can prove

$$\partial_t w_\varepsilon + a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon})|p + Dw_\varepsilon| \leq \overline{H}(p) - \frac{\theta}{4}.$$

This claim is easily checked by touching w_ε from above in the carefully designed region U_θ and by choosing c small. It is important to notice that U_θ is non-empty and contains certain cylinder centered at $(x_\varepsilon, t_\varepsilon)$.

Then we compare w_ε with a function modified from the metric function, which is given by

$$\phi_\varepsilon = \overline{H}(p) \left[\varepsilon m \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}; \frac{x_\varepsilon - re}{\varepsilon} \right) - \varepsilon m \left(\frac{x_\varepsilon}{\varepsilon}, \frac{t_\varepsilon}{\varepsilon}; \frac{x_\varepsilon - re}{\varepsilon} \right) \right] - p \cdot (x - x_\varepsilon).$$

Here, the vertex is carefully chosen to be $\frac{x_\varepsilon - re}{\varepsilon}$ where $e \in \mathbb{S}^{n-1}$ is a direction such that p is a sub-differential of $y \mapsto \overline{H}(p)\overline{m}(y)$ at re for all $r > 0$; in other words,

$$\overline{H}(p)\overline{m}(y) - \overline{H}(p)\overline{m}(re) - p \cdot (y - re) \geq 0, \qquad \forall\, y \in \mathbb{R}^n.$$

The existence of such a direction e is due to the convexity and positive one-homogeneity of the mapping $y \mapsto \overline{H}(p)\overline{m}(y)$. Moreover, if p is an exposed point of the $\overline{H}(p)$-level set of $\overline{H}$, then we can find a direction $e \in \mathbb{S}^{n-1}$ such that $p = \overline{H}(p) D\overline{m}(e)$.

By construction, ϕ_ε is a solution (in particular a super-solution) to

$$\partial_t \phi_\varepsilon + a\left(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}\right) |p + D\phi_\varepsilon| = \overline{H}(p)$$

away from the pole $\frac{x_\varepsilon - re}{\varepsilon}$. We check that as long as r is sufficiently large, the pole is outside U_θ, and then we can compare w_ε with ϕ_ε. This comparison yields

$$0 = (w_\varepsilon - \phi_\varepsilon)(x_\varepsilon, t_\varepsilon) \le \sup_{U_\theta}(w_\varepsilon - \phi_\varepsilon) = \sup_{\partial U_\theta}(w_\varepsilon - \phi_\varepsilon) = -\frac{\theta}{4} - \inf_{\partial U_\theta} \phi_\varepsilon.$$

Above, in the second relation we noticed that $(x_\varepsilon, z_\varepsilon)$ is in U_θ; in the last relation we used the definition of U_θ. Then we can find certain $R' = R'(\theta, p)$ such that

$$\inf_{Q_{R'}(x_\varepsilon, t_\varepsilon)} \overline{H}(p)\left[\varepsilon m\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{x_\varepsilon - re}{\varepsilon}\right) - \varepsilon m\left(\frac{x_\varepsilon}{\varepsilon}, \frac{t_\varepsilon}{\varepsilon}, \frac{x_\varepsilon - re}{\varepsilon}\right)\right] - p\cdot(x - x_\varepsilon) \le -\frac{\theta}{4}.$$

Note that by assumption, the sequence $\{(x_\varepsilon, t_\varepsilon)\}$ is bounded in Q_R, so they must accumulate at some point (z, s). In view of the convergence (3.15), we get

$$\inf_{Q_{R'}(z)} \left(\overline{H}(p)\left[\overline{m}(x + re) - \overline{m}(re)\right] - p\cdot x\right) \le -\frac{\theta}{4}.$$

This is a contradiction with the fact that p is sub-differential of $\overline{H}(p)\overline{m}(y)$ along re. This establishes the first half of the inequalities in (3.16).

For the second half of (3.16), we need a similar argument that treats ϕ_ε as a subsolution in the comparison. We then conclude that, if (z, s) is an accumulating point for a sequence $\{(x_\varepsilon, t_\varepsilon)\}$ such that the second half of (3.16) is violated, then

$$\sup_{Q_{R'}(z)} \left(\overline{H}(p)\left[\overline{m}(x + re) - \overline{m}(re)\right] - p\cdot x\right) \ge \frac{\theta}{4}.$$

This won't be a contradiction if p is merely a sub-differential of $\overline{H}(p)\overline{m}$ at re. Nevertheless, if we know that p is an exposed point of the $\overline{H}(p)$-level set of $\overline{H}$, then $p = \overline{H}(p)D\overline{m}(e)$ and then we can send $r \to \infty$ and get a contradiction.

To establish the second half of (3.16) for more general p, we need to use more convex analysis facts and the stability of the viscosity solution, to reduce the problems to extremal p and then to exposed p established above. Then the proof is complete. We refer to [JST17] for the details of the reversed perturbed test function argument; see also [AT14] for the static setting when a second order derivative term is present.

3.2. **The spatial-periodic and temporal-random setting.** In this subsection, we always assume (B)(S2) and (E). In particular, the environment function a, before rescaling, is periodic in space and stationary ergodic in time.

The evident contrast of (S2) with the previous setting (S1) is, the virations in the environment a is not periodic but random in time. On the one hand, the problem (1.4) and the related cell problem (3.1) share the difficulty of the lack of uniform modulus of continuity. On the other hand, because the variations in time is not periodic, we cannot rely on the periodicity in time to reduce the problem essentially to a static metric problem (unless $n = 1$ in which case the roles of time and space can be switched); see (3.4). We hence need new ideas to deal the random oscillations in time.

Due to the above reasons, a pure PDE approach seems prohibitive, so we explore and rely on the optimal control formulation. According to (2.11), the solution of (1.4) (its value at (x, t)) is the value function of minimizing the initial cost $g(y)$ over *admissible* paths that starts from $\frac{y}{\varepsilon}$ and reaches $\frac{x}{\varepsilon}$ at time $\frac{t}{\varepsilon}$. Note that, morally speaking,

$$\frac{x}{\varepsilon} \in \mathcal{R}_{\frac{t}{\varepsilon}}(\frac{y}{\varepsilon}) \qquad \text{is equivalent to} \quad x - y \in \varepsilon \mathcal{R}_{\frac{t}{\varepsilon}}(0) = t\frac{1}{t/\varepsilon}\mathcal{R}_{t/\varepsilon}(0).$$

Here, we shift the starting point to the origin and omitted the starting time which is at zero. For fixed $t > 0$, the last item suggests us to study the long time average of the reachable set starting from the origin; previously, the concept of reachability was used also in [XY10]. This idea is fruitful and we can prove the following *shape theorem*.

Theorem 3.4 (The shape theorem)**.** *Assume (B)(S2) and (E). Then there exists a compact and convex set $D \subset \mathbb{R}^n$, a set $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ so that, for all $\omega \in \tilde{\Omega}$ and for any $x \in \mathbb{R}^n$,*

$$\lim_{t\to\infty} \frac{\mathcal{R}_t(x;\omega)}{t} = D \qquad \textit{in the space } \ (\mathcal{C}, \rho_{\mathrm{H}}).$$

This theorem says the limit of the normalized reachable set, the latter being random and not necessarily convex at each time, is a deterministic and convex compact set. Using this theorem, actually its loaclly uniform version

$$\lim_{t\to\infty} \sup_{x\in\mathbb{T}^n} \rho_{\mathrm{H}}\left(\frac{\mathcal{R}_t(x;\omega)}{t}, D\right) = 0, \qquad \forall\, \omega \in \tilde{\Omega}, \tag{3.18}$$

we can obtain the qualitative homogenization of the second setting. Given D, we define the effective Lagrangian by (3.11), and define $\overline{H}$ as the Legendre transform of $\overline{L}$. Equivalently, given D, we define $\overline{H}$ as the support function of D.

Theorem 3.5. *Assume (B)(S2) and (E). Let D and $\tilde{\Omega} \in \mathcal{F}$ be as in Theorem 3.4 and let $\overline{H}$ be the support function of D. Let u^ε and u be, respectively, the solutions to* (1.3) *and* (1.5)*. Then for every $\omega \in \widetilde{\Omega}$ and for every $T \geq 0, R > 0$, the convergence* (3.12) *holds.*

The route from Theorem 3.4 to Theorem 3.5 is very smooth. On the one hand, we have the representation formula (2.11) for u^ε. On the other hand, the solution u to (1.5) has a similar representation:

$$u(x,t) = \inf\{g(y) \,:\, x - y \in tD\}.$$

By the argument above Theorem 3.4, it is clear that, provided g has some modulus of continuity, we can transfer the distance of $u^\varepsilon - u$ to the distance of $\varepsilon\mathcal{R}_{t/\varepsilon}$ to tD. The latter is precisely taken care of by the shape theorem.

Proof of the shape theorem. We review the main ingredients to prove the shape theorem, especially the uniform version (3.18), as the latter is needed for the locally uniform convergence in Theorem 3.5. We see from the definition (2.12) that a path is admissible means the velocity along it at (x,t) does not exceeds the bound $a(x,t)$. In view of (B), the reachable set can be bounded from above and from below by

$$\overline{B}_{\lambda t} \subseteq \mathcal{R}_t(x;\omega) - x \subseteq \overline{B}_{\Lambda t}.$$

The reachable set hence has a growth rate proportional to t; this gives another explanation of the normalization $t^{-1}\mathcal{R}_t$.

Since the environment a is periodic in space and stationary ergodic in time, we verify that the reachable set satisfies, for all $x \in \mathbb{R}^n, t, s \in \mathbb{R}$ and for all $\omega \in \Omega$,

$$\mathcal{R}_t(x, 0; \omega) = \mathcal{R}_t(\hat{x}; \omega) + [x], \qquad \mathcal{R}_{t+s}(x, s; \omega) = \mathcal{R}_t(x, 0; \tau_s \omega).$$

Note that the variable s in $\mathcal{R}_t(x, s; \omega)$ means the starting time, and it is usually omitted when $s = 0$. The second property above can be understood as follows: the random environment starting from s is as the translated environment starting from 0, and hence the above relation.

The reachable set $\mathcal{R}_t$ also enjoys a subadditive property which essentially controls the fluctuations of its average. The precise statement is:

Lemma 3.6. *Assume (B) and (S2). Then for any $t \in \mathbb{R}$, $s \in \mathbb{N}$ such that $t \geq s$, and for all $\omega \in \Omega$, we have*

$$\mathcal{R}_t(Y; \omega) \subseteq \mathcal{R}_s(Y; \omega) + \mathcal{R}_{t-s}(Y; \tau_s \omega) + \tilde{Y}. \tag{3.19}$$

Here and in the sequel, $Y = [0, 1]^n$ is the unit cell and $\tilde{Y} = -Y$. The reachable set $\mathcal{R}_t(Y, s; \omega)$ is the union of $\mathcal{R}_t(x, s; \omega)$ with $x \in Y$. This lemma can be proved, for each reachable point on the left, by constructing and connecting admissible paths allowed on the right to reach the target point. Periodicity plays a role in the construction, and in the guarantee that the translation of environment involved is measure-preserving; the latter is a particularly subtle point. As a result, $\mathcal{R}_t(Y; \omega) + \tilde{Y}$ is a stationary subadditive family of closed sets. Note also if we replace the $\mathcal{R}_t$'s above by their convex hulls co$\mathcal{R}_t$'s, then the subadditive relation remains.

One would like to use the set-valued subadditive ergodic theorem to conclude the shape theorem. However, such theorems require the random sets to be compact and convex; see [HH00, Sch93, AH85]. We can prove that

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{co}\mathcal{R}_t(Y; \omega) = D, \qquad \text{in } (\mathcal{C}, \rho_{\mathrm{H}}) \text{ almost surely in } \Omega. \tag{3.20}$$

The above can be established first along integer sequences for co$\mathcal{R}_t(Y; \omega) + \tilde{Y}$. The compactness of $\tilde{Y}$ allows the removal of this term, and the growth bound of $\mathcal{R}_t$ in t allows control of $\mathcal{R}_t$ by its value along integer times.

Removing the convex hull. We need to show that the convex hull "co" can be removed and the uniform convergence (3.18) holds. In view of the definition of ρ_{H} and the fact that the convex hull co$\mathcal{R}_t$ contains $\mathcal{R}_t$ itself, the problem is reduced to show

$$d\left(x, \frac{1}{m}\mathcal{R}_m(Y; \omega)\right) \xrightarrow{m \to \infty} 0, \qquad \forall\, x \in D. \tag{3.21}$$

Note the uniform convergence between compact sets is reduced to pointwise convergence of distance between a point and the averaged reachable set, thanks to the known compactness and convexity of D. We achieve this goal in three steps.

First, for exposed points, (3.21) holds, essentially because the convex hull is determined by the exposed points and (3.20) implies the desired result for such points.

Then, for extreme points which, by convex analysis, are limits of exposed point, the desired result follow from a density argument and is implied by the previous results for exposed points.

Finally, for other points, say x in D, we note that it can be written as a convex combination of at most $n + 1$ extreme points $y_1, y_2, \cdots, y_{n+1}$, say with combination coefficients $q_1, \cdots, q_{n+1}$. The idea then is to construct admissible path reaching to tx (x is rescaled by time t) by connecting admissible paths that start from Y to tq_1y_1, then making the increments tq_2y_2, tq_3y_3 and so on until reaching $tx = tq_1y_1 + \cdots + tq_{n+1}y_{n+1}$. The existences of those admissible path segments are guaranteed by the previous results for extreme points.

Examining this plan carefully, one should notice that to construct the admissible segment with increment tq_iy_i, we should apply (3.21) for x being the extreme point y_i but with respect to a shifted environment $\tau_{s_i}\omega$ where s_i should be the time accumulated by the previous segments. Therefore, we need to upgrade the convergence (3.21) to

$$d\left(y, \frac{1}{m}\mathcal{R}_m(Y;\tau_{sm}\omega)\right) \xrightarrow{m\to\infty} 0, \qquad \forall\, y \in \mathcal{E}(D),\ \forall\, s \in \mathbb{N}.$$

Here $\mathcal{E}(D)$ denotes the set of extreme points of D. The importance here is, again, the vertex sm (which should be thought as the shifted starting time) now flows. This can be achieved by repeating the "making the vertex flow" argument; we refer to [JST18] for the details of the argument.

Connections to Poincaré's rotation numbers. We end the review for the spatial periodic temporal random setting by a connection of the shape theorem with the rotation number of Poincaré. When the spatial dimension is one, $\mathcal{R}_t(x_0;\omega)$ is clearly an interval $[R_{\text{left}}(t), R_{\text{right}}(t)]$ that grows with respect to t. In fact, they satisfies the differential equations

$$\begin{cases} \dot{R}_{\text{left}}(t) = -a(R_{\text{left}}(t), t;\omega), \\ R_{\text{left}}(0) = x_0, \end{cases} \qquad \begin{cases} \dot{R}_{\text{right}}(t) = a(R_{\text{right}}(t), t;\omega), \\ R_{\text{right}}(0) = x_0, \end{cases}$$

If further a is periodic in space and in time, then the above can be viewed as dynamical systems on the torus. A classical result of Poincaré says no matter where we start, the numbers $\frac{1}{t}R_{\text{left}}(t)$ and $\frac{1}{t}R_{\text{right}}(t)$ have limits that are independent of x_0; they are the rotation numbers associated to the two systems on the torus. See also the Denjoy theory on further properties of the rotation number. We refer to [Arn88] for an extended description of the work of Poincaré and Denjoy. When a is periodic in one variable and random stationary ergodic in the other, the existence of deterministic rotation number (almost surely in Ω) was established in [LL08]. The shape theorem is in some sense a generalization, although it is not about the rotation number (or vectors) itself.

A remark on the space-time-random setting. We end this section by some comments on the space-time random setting, that is under the assumptions (B)(S3) and (E). Now the variations in the environments are random both with respect to space and time; we cannot use periodicity to reduce the dynamic metric function to a static one like in section 3.1, and we cannot get the stationary subadditive property (3.19). Indeed, we need to replace the translation $\tau_s\omega$ in (3.19) by $\tau_{x_\omega,s}\omega$ where x_ω is the new origin for the second reachable set. Such translations may not preserve measure and we are prevented from using the subadditive ergodic theorem directly.

In a recent work [BIN19], Burago, Ivanov and Novikov studied the G-equation in dynamic random environment. They proved a homogenization result assuming that the background conducting velocity field has finite time range dependence, which, in our context, roughly means the following:

> The σ-algebra generated by $\{a(x,t) \,:\, t \ge t_1, x \in \mathbb{R}^n\}$ is independent with that generated by $\{a(x,t) \,:\, t \le t_2, x \in \mathbb{R}^n\}$ as long as $t_1 - t_2 \ge 1$.

Under this very strong assumption, they managed to prove qualitative homogenization using both the metric function, i.e. the dynamic travel time function, and the support function of the reachable sets; their method combines ideas from the previous sections with new arguments that explore the finite time dependence of the random environments. We refer to [BIN19] for the details.

4. Beyond qualitative homogenization

The previous section concerns qualitative behavior of equations in heterogeneous media. The homogenization result identifies an effective problem with homogeneous environment whose solution captures the macroscopic behavior of the heterogeneous problem. In this section, we address a couple of questions beyond qualitative homogenization. For simplicity, we focus only on the setting of periodic and static media. In other words, we impose the following assumptions:

(P) $a = a(x)$, and the mapping $x \mapsto a(x)$ is $[0,1]^n$-periodic. Furthermore, a is bounded from above and below by positive real numbers.

4.1. **Inverse shape theorem.** We may view the periodic assumption (P) as a special case of (B)(S2) and (E); we may also check the method in section 3.2 applies under assumption (P). The homogenization result through the shape theory goes through; in fact, the proofs can be simplified. In particular, the shape theorem holds: there exists a compact and convex set $D \subset \mathbb{R}^n$ such that

$$\lim_{t\to\infty} \frac{1}{t}\mathcal{R}_t(Y) = D_a \qquad \text{in } \ (\mathcal{C}, \rho_{\mathrm{H}}). \tag{4.1}$$

The limit shape D_a then determines the effective Hamiltonian and the homogenized problem. Therefore, the shape theorem, in some sense, is the backbone of the homogenization of the front propagation model (1.4). A natural question to ask is:

(Q) Given a shape D, does there exists a periodic environment a, so that it is the limit shape of the normalized reachable set associated to the heterogeneous front propagation model

$$\begin{cases} \partial_t u^\varepsilon(x,t) + a(\frac{x}{\varepsilon})|Du^\varepsilon(x,t)| = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = g(x), & (x,t) \in \mathbb{R}^n \times \{0\}. \end{cases}$$

Note that we emphasize through the subscript of D_a that the limit shape is associated to the environment function a.

The question above is a kind of "inverse homogenization" question, which seeks for information about the microscopic structures of the environment from the macroscopic effective environment. Such inverse problems find many applications, for instance, in terrain detection, obstacle detection, imaging sciences and so on. Of course, the precise question above is only a first step toward such directions. We shall restrict to continuous a. Because a is time independent, the metric function, i.e. the travel time function, is symmetric with respect to the starting and terminal points; so D must be symmetric about the origin. Therefore, we also restrict D in the set of convex compact centro-symmetric sets with non-empty interior.

In [JTY20], we establish the following result:

Theorem 4.1. *Assume that the spatial dimension $n \geq 3$. Let P be a compact convex centro-symmetric polytope with rational vertices and with non-empty interior. Then there exists an environment function $a \in C^\infty(\mathbb{T}^n, (0,\infty))$ such that $D_a = P$.*

This theorem should be compared with Theorem 3.4 and be viewed as an inverse shape theorem. It partially answers the questions of what shapes can be a limit shape of the front propagation model in the periodic setting.

This inverse shape problem is closely related to the rigidity of stable norms of periodic Riemannian metrics on $\mathbb{R}^n$. Indeed, given such a metric, the distance function ρ determined by the metric admit a large scale limit as follows: for every $v \in \mathbb{R}^n$,

$$\|v\| := \lim_{\lambda\to\infty} \frac{\rho(0,\lambda v)}{\lambda}$$

exists and $\|\cdot\|$ is a norm on $\mathbb{R}^n$ termed the stable norm, or the limit norm or the asymptotic norm of the periodic Riemannian metric. The one sub-level-set $B_{\|\cdot\|}$ of $\|\cdot\|$ is also a compact convex set with non-zero interior. There is a natural inverse problem of what set B can be the unit sphere of the stable norm of a periodic metric. There is a clear connection between this problem with ours, with B playing the role of the limit shape D. In fact, a translation can be established for the two problems, and, as a result, we are not the first to establish a result like Theorem 4.1. See, for instances, [Hed32] and [BB06, Jot09] in the context of stable norms. We also refer to [Ban94] and [JTY17] for other inverse type questions concerning the strict convexity of the limit shapes.

We briefly review the proof of Theorem 4.1. The novelty of the proof is that, compared with the aforementioned references in the geometry context, it is a PDE approach based on known results of homogenization, and, hence, quite simple in a sense.

Let the vertices of P be $\pm q_1, \pm q_2, \ldots, \pm q_m$, which are rational vectors; they are mutually non-parallel. We start with $\pm q_1$, let L_1 be the line passing 0 and q_1, and let $L_1 + \mathbb{Z}^n$ be the periodic array formed by translating L_1. Similarly, we can construct a periodic array of lines in the direction of q_2, denoted by $x_2 + L_2 + \mathbb{Z}^n$, where x_2 is carefully chosen so the lines do not intersect with the array of q_1; note that here we use the dimension assumption $n \geq 3$. Iterating this process, we can construct in an inductive manner a periodic array $x_i + L_i + \mathbb{Z}^n$ in each of the $\{q_1, \cdots, q_m\}$ directions, and the arrays do not intersect each other. Moreover, because those directions are rational, the projection of those arrays in the torus $\mathbb{T}^n$ form a closed orbit, i.e. there are only finitely many segments. As a result, the distances between those segment has a positive lower bound, say $\delta > 0$, which depend only on P through its vertices. Then we can *widen* those arrays a bit and make them arrays of channels and, still, there is a positive lower bound, still denoted by δ, for the distances between those arrays. We denote by $T_{\delta,i}$ the union of the array of channels in direction q_i.

To construct the periodic environmental function a, we will assign very high value to a on $T_{\delta,i}$. Hence, those channels are the fast track where admissible paths can travel in very fast speed. On the other hand, outside the union $\cup_{i=1}^m T_{\delta,i}$, we assign very low value to a. Then the travel time between points, especially when they are far away, will be realized by admissible paths that favor the fast tracks and only get off to the slow regions for necessary connections.

We need to design the values on the fast tracks $T_{\delta,i}$ well so that the resulting limit shape is P. We can find a smooth $\mathbb{Z}^n$-periodic function $a_A : \mathbb{R}^n \to (0,\infty)$, with the parameter A large to be chosen, such that

$$\begin{cases} a_A(x) = A|q_i| & \text{on} \quad x_i + L_i + \mathbb{Z}^n, \ i = 1, 2, \cdots, m, \\ a_A(x) \in [1, A|q_i|] & \text{on} \quad T_{\delta,i}, \ i = 1, 2, \cdots, m, \\ a_A(x) = 1 & \text{on} \quad \mathbb{R}^n \setminus \cup_{i=1}^m T_{\delta,i}. \end{cases} \tag{4.2}$$

Here we set the slow region to have speed limit one, and the maximum speed limit on the fast tracks are proportional to $|q_i|$ on each $T_{\delta,i}$ with the factor A, which is large, to be set. The final environmental function a that yields the limit shape P will be determined later after A is chosen.

For the environmental function a_A above, with A properly chosen, we can check that the effective Hamiltonian is

$$\overline{H}_A(p) = A \max_{1\le i\le m} |q_i \cdot p| = A \max_{q\in P} q\cdot p, \qquad \forall p \in \mathbb{R}^n. \tag{4.3}$$

We appeal to standard results about the cell problem (2.5). One is the inf-sup representation (see e.g. [Tra]) of $\overline{H}$ which says, in our setting,

$$\overline{H}(p) = \inf_{\phi\in C^1} \max_{y\in\mathbb{T}^n} a(y)|p + D\phi(y)|. \tag{4.4}$$

Given the fact that the regions $\{T_{\delta,i}\}_i$ are well separated, for each fixed p on the unit sphere, there exists a function $\varphi \in C^1(\mathbb{T}^n)$ such that $D\varphi(x) = -p_i^\perp$ on $T_{\delta,i}$, for each i, where $p_i^\perp$ is the component that is perpendicular to q_i. More importantly, we can make sure that $\|D\varphi\|_{L^\infty} \le C_\delta$ with C_δ depending only on δ and $\{\pm q_i\}_i$ but not on $p \in S^{n-1}$. We then A large depending only on C_δ and $\{\pm q_i\}_i$, so that the maximization in (4.4) is obtained along the fast tracks $\cup_i(x_i + L_i + \mathbb{Z}^n)$. Then we get

$$\overline{H}_A(p) \le \max_{1\le i\le m} A|q_i||p - p_i^\perp| = A \max_{1\le i\le m} |q_i \cdot p|.$$

For the reversed direction to hold, we take a solution v_p to the cell problem (2.5) with environment a_A. Along the fast lines $x_i + L_i + \mathbb{Z}^n$, the following equality holds:

$$A|q_i||p + Dv_p(x)| = \overline{H}(p).$$

Consider the function $u(x) = p\cdot x + v_p(x)$ for $x \in \mathbb{R}^n$. Because q_i is rational, so mq_i belongs to $\mathbb{Z}^n$ for some positive integer m. Then we get

$$u(x_i + mq_i) - u(x_i) = p\cdot(mq_i).$$

We hence get

$$\begin{aligned} m|p\cdot q_i| = |u(x_i + mq_i) - u(x_i)| &\le m|q_i| \max_{y\in x_i+L_i} |Du(y)| \\ &= m|q_i| \max_{y\in x_i+L_i} |p + Dv_p(y)| = \frac{m}{A}\overline{H}(p). \end{aligned}$$

This establish the claim (4.3). Then divide a_A by A, and by the scaling property of $\overline{H}$, we check that the effective Hamiltonian associated to $\frac{1}{A}a_A$ is the support function of P. That is equivalent to say, the limit shape associated to a is P. Theorem 4.1 is hence established.

4.2. **Optimal convergence rates.** Another question beyond the qualitative homogenization concerns the rate of convergence. In practice, one typical encounters heterogeneous media that have small but non-vanishing scales for the variations of the media, but the qualitative homogenization only predicts the effective medium in the limit that those scales vanish. Therefore, to make sure the effective media is a good approximation for given media with fixed small scales, it is of fundamental importance to quantify the error of approximations.

Due to the full nonlinearity, quantitative estimates for homogenization of Hamilton-Jacobi (H-J) equations are very difficult. For periodic homogenization of H-J equations in the classical setting, Capuzzo-Dolcetta and Ishii [CDI01] developed a method for quantitative estimates and established

$O(\varepsilon^{\frac{1}{3}})$ bounds for the homogenization error $u^\varepsilon - u$, although the optimal rate should be of order $O(\varepsilon)$ as a formal expansion would show. Very recently, Mitake, Tran and Yu [MTY19] developed a powerful new method for quantitative estimates of periodic homogenization of H-J equations, based on deep connections between the fine properties of the effective Hamiltonian $\overline{H}$ and the underlying dynamical system and optimal control formulations of the equations.

For the front propagation problem (1.4), which partially falls in the framework of [MTY19], their results can be applied to establish optimal convergence rate, that is of order $O(\varepsilon)$, when the dimension is two; see Theorem 1.2 of [MTY19]. For $n \geq 3$, we can prove the following optimal rate of convergence result.

Theorem 4.2. *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope with non-empty interior. Assume that the limit shape associated to a periodic environment function a is P. Then there exists $C > 0$ depending on P and a such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} \leq C\varepsilon. \tag{4.5}$$

From the control formula (2.11), the error estimate (4.5) follows if the convergence to the limit shape in Theorem 3.4 is quantified; this is given in the next result.

Lemma 4.3. *Let P be the limit shape associated to a periodic environment function $a \in C(\mathbb{T}^n; \mathbb{R}_+)$. Then there exists a constant $C > 0$ depending only on P and a, such that for all $t > 0$, we have*

$$\rho_{\mathrm{H}}\left(\frac{\mathcal{R}_t(Y)}{t}, P\right) \leq \frac{C}{t}.$$

We need to find a constant C so that for all $t \geq 1$, the following relations hold:

$$\frac{\mathcal{R}_t(Y)}{t} \subseteq P + B_{\frac{C}{t}}, \qquad \text{and} \qquad P \subseteq \frac{\mathcal{R}_t(Y)}{t} + B_{\frac{C}{t}}. \tag{4.6}$$

The upper bound for reachable sets. The first inclusion is the relatively easier one to establish. For each $z \in \partial P$, we provide an upper bound, at each $t > 0$, for the projection of $\mathcal{R}_t(Y)$ in the direction of z.

To this end, suppose $\lambda z \in \mathcal{R}_t(Y)$, then there should be an admissible path γ such that $\gamma(0) \in Y$ and $\gamma(t) = \lambda z$. We show that λ must have a bound that is proportional to t. More precisely, for each $p \in \mathbb{R}^n$, let $(v_p, \overline{H}(p))$ be the solution to the cell problem (2.5). Then almost everywhere along the path of γ,

$$\overline{H}(p) = a(\gamma(s))|p + Dv_p(\gamma(s))| \geq \dot{\gamma}(s) \cdot (p + Dv_p(\gamma(s))).$$

This bound is due to Cauchy-Schwarz and that γ is admissible and, hence, $|\dot{\gamma}(\cdot)|$ bounded by $a(\gamma(\cdot))$. Integrate in time over $[0, t]$, we get

$$p \cdot \frac{\lambda z}{t} - \overline{H}(p) \leq \frac{C}{t},$$

where C can be bounded by $|p| + \|Dv_p\|_{L^\infty}$. Note that although v_p may not be unique for each fixed p, the norm $\|Dv_p\|_{L^\infty}$ enjoys bounds that only depend on p and on the lower bound of a. From the inequality above and the relationship between P and $\overline{H}$, we get

$$\frac{\lambda}{t} \leq 1 + \frac{C}{t},$$

where C above is defined by an optimization of certain C_p over $p \in P$; in particular, it depends only on P and a.

The lower bound for reachable sets. It remains to prove the second half of (4.6), which says that $\mathcal{R}_t(Y)$ contains tP with with an error of B_C. Due to the convexity of P, it suffices to show that all exposed points of tP is contained in $\mathcal{R}_t(Y)+B_C$. Furthermore, because tP only has finitely many exposed points, we only need to show tv is reached for each vertex v of P. The reachability of ty, for other points $y \in P$, can be proved by convex combinations, by repeating the idea of constructions in the proof of (3.21).

Now fix a vertex z of P, for all $t \ge 1$, we should be able to find an admissible path ξ that starts from Y and satisfies that $|\xi(t) - tz| \le C$ for some C depending only on P and a. We can construct such ξ using the so-called backward characteristics. The concept of backward characteristics, also called calibrated curves, is very important in the weak KAM theory. In the language of cell problems, for each fixed $p \in \mathbb{R}^n$, and a solution v_p to the cell problem (2.5) with right hand side $\overline{H}(p)$, a curve $\xi : (-\infty, 0] \to \mathbb{R}^n$ is called a backward characteristics of v_p if it satisfies

$$p \cdot \xi(t_1) + v_p(\xi(t_1)) - (p \cdot \xi(t_2) + v_p(\xi(t_2))) = \int_{t_2}^{t_1} L(\xi(t), \dot{\xi}(t)) + \overline{H}(p)\, dt,$$

for all $t_2 < t_1 \le 0$. Given the singular structure (2.10) of the Lagrangian, in the context of front propagation problem, a backward characteristic of v_p satisfies

$$\begin{cases} \xi(s) \in \mathcal{R}_{|s|}(y), & s < 0, \\ p \cdot \xi(t_1) + v_p(\xi(t_1)) - (p \cdot \xi(t_2) + v_p(\xi(t_2))) = (t_1 - t_2)\overline{H}(p), & t_2 < t_1 \le 0. \end{cases}$$

The existence of backward characteristics is proved in [MTY19] and the results there can be adapted to our setting.

For the vertex z of P, we can find $p \in \mathbb{R}^n$ so that $\overline{H}$ is differentiable at p with $D\overline{H}(p) = z$. Let v_p be a solution to the cell problem associated to vector p, and let $\xi : (-\infty, 0] \to \mathbb{R}^n$ be a backward characteristics. Then

$$p \cdot \xi(0) - p \cdot \xi(t) + v_p(\xi(0)) - v_p(\xi(t)) = |t|\overline{H}(p), \qquad \forall t < 0.$$

Further, we know that $\xi(t) \in \mathcal{R}_{|t|}(\xi(0))$. On the other hand, for other $\tilde{p} \in \mathbb{R}^n$, and for a solution $v_{\tilde{p}}$ of the cell problem with vector $\tilde{p}$, we have the inequality

$$\tilde{p} \cdot \xi(0) - \tilde{p} \cdot \xi(t) + v_{\tilde{p}}(\xi(0)) - v_{\tilde{p}}(\xi(t)) \le \int_t^0 L(\xi(s), \dot{\xi}(s)) + \overline{H}(\tilde{p})\, ds = |t|\overline{H}(\tilde{p}).$$

Subtracting these equations, we get

$$\overline{H}(\tilde{p}) - \overline{H}(p) \ge (\tilde{p} - p) \cdot \frac{\xi(t) - \xi(0)}{t} - \frac{C(1 + |p|)}{|t|}.$$

Next we realize that $\overline{H}$ is linear in a neighborhood of p. So the left hand side above can be written as $D\overline{H}(p) \cdot (\tilde{p} - p)$. By choosing $\tilde{p}$ close to p and that $\tilde{p} - p$ is is in the same half space of w, with

$$w = \frac{\xi(t) - \xi(0)}{t} - D\overline{H}(p) = \frac{\xi(t) - \xi(0)}{t} - z,$$

we get the desired estimate

$$\left| \frac{\xi(t) - \xi(0)}{t} - z \right| \le \frac{C_p}{|t|}.$$

Because there are only finitely many vertices to consider, the constant can be made uniform with respect to the vertices. This establish the reachability, with correct rate, of the vertices of P, and this completes the proof of Lemma 4.3.

Acknowledgments

The author is very grateful to P.E. Souganidis, H.V. Tran and Y. Yu for fruitful collaborations on the subjects discussed here. This work is partially supported by the National Natural Science Foundation of China under Grants No. 11701314 and No. 11871300, and by the the Recruitment Program of Global Experts of China.

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