# VISCOSITY SOLUTION OF HAMILTON-JACOBI EQUATIONS

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*This paper is dedicated to my advisor.*

Abstract. This paper mainly discuss the basic property of viscosity solution of Hamiltion-Jacobi equations. It give some different definitions of viscosity solution in the history and show the equivalence of this definitions, which lead to the most simply definition. Secondly, comparsion principle is introduced to obtain uniqueness of solution and perron's method for the existence of viscosity solution. I think this two way are most simply way to ensure well-poseness of the equation.

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## 1. On The Notions of This Report

This report is a result of studying viscosity solution of Hamilton-Jacobi Equation. So it will mainly deal with the Hamilton-Jacobi Equation:

$$F(y, u, Du) = 0 \quad \text{in } \Omega \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $F$ is a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. The equation (1.1) has two sepcific forms that are familier with us.

$$H(x, u, Du) = 0 \quad \text{in } \Omega \tag{1.2}$$

for a Hamiltonian $H$ and

$$u_t + H(x, t, u, Du) \quad \text{in } \Omega \times (0, T) \tag{1.3}$$

For a time-dependent Hamiltonian $H$ and in this case, there might be some initianl conditions and boundary conditions. These give rise to Dirichlet problem and Neumann problem.

Here is some notions used in this report.

I use $C(\Omega)$ denote the continuous function on $\Omega$ and $C^1(\Omega)$ the continuously differentiable function on $\Omega$. Besides, $C(\Omega)^+$ means the function in $C(\Omega)$ whose value are nonnegative and similiar for $C(\Omega)^-$. Another notions such as $C_c(\Omega)$ means functions in $C(\Omega)$ whose support is contained in a compact subset of $\Omega$ and $C^\infty(\Omega)$

for infinitely differentiable function on $\Omega$. The set $d(\varphi)$ means all the differentiable point of $\varphi$.

## 2. On The Definition of Viscosity Solution

In this section, I will talk about some equivalent definitions of viscosity solution. In 1983, [1] put forward the concept of viscosity and gave the original definition of viscosity solution. Afterwards, [2] found that the original definition can be complicated when dealing with problems and it suggested another useful definition which is used till now.

First of all, let's see how the equations respond to the maximum point of some functions. Let $u, \varphi, \psi \in C^1(\Omega)$ and $\varphi(u-\psi)$ attains its maximum point at point $y_0$. Then

$$0 = D(\varphi(u-\psi)) = \varphi D(u-\psi) + D\varphi(u-\psi)$$

Hence

$$Du = -\frac{u-\psi}{\varphi} D\varphi + D\psi$$

Substitute $Du$ by above equation into (1.1), we obtain

$$F\left(y, u, -\frac{u-\psi}{\varphi} D\varphi + D\psi\right) = 0$$

It is a very eqaution that we need since no derivative of $u$ appears in the equation. Under this situation, $u$ need only to be continuous! Thus, the original definition comes. We use $E_+(\varphi)$ to denote the maximum point of $\varphi$ which attain positive maximum and $E_-(\varphi)$ for the minimum point of $\varphi$ which attain negetive minimum.

**Definition 2.1** (Def I)**.** A viscosity subsolution(resp. supersolution) of (1.1) is a function $u \in C(\Omega)$ such that for any function $\varphi \in C_c^\infty(\Omega)^+$ and $k \in \mathbb{R}$, if $E_+(\varphi \cdot (u-k)) \neq \emptyset$(resp. $E_-(\varphi \cdot (u-k)) \neq \emptyset$), then there exists a point $y_0 \in E_+(\varphi \cdot (u-k))$(resp. $y_0 \in E_+(\varphi \cdot (u-k))$) such that

$$F\left(y_0, u(y_0), -\frac{u(y_0)-k}{\varphi(y_0)} D\varphi(y_0)\right) \leq 0$$

(resp.

$$F\left(y_0, u(y_0), -\frac{u(y_0)-k}{\varphi(y_0)} D\varphi(y_0)\right) \geq 0$$

)

*Remark* 2.1. In the paper [5] that my teacher gives me, the subsolution is defined for upper semi-continuous function. This is a very natural generalize, but some properties may fail to be valid. So I choose to define the solution for continuous function.

Now, I give two other definitions in the paper [2].

**Definition 2.2** (Def II)**.** A viscosity solution of (1.1) is a continuous function in $\Omega$ provided for every function $\varphi \in C^1(\Omega)$

**subsolution:** if $u - \varphi$ attains a local maximum at $y_0 \in \Omega$ then

$$F(y_0, u(y_0), D\varphi(y_0)) \leq 0$$

**supersolution:** if $u - \varphi$ attains a local minimum at $y_0 \in \Omega$ then

$$F(y_0, u(y_0), D\varphi(y_0)) \geq 0$$

*Remark* 2.2. In the paper [5], it says that in the definition, $C^1$ function can be replaced by $C^\infty$ function and local maximum can be replaced by strict global maximum. And I will give a proof of it afterwards.

Before write down the third definition, we need to defined the so called subdiffernetial and superdifferential.

**Definition 2.3.** Suppose $u \in C(\Omega)$ and $y_0 \in \Omega$, we define subdiffernetial $D^-(y_0)$ and superdifferential $D^+(y_0)$.

$$D^+(y_0) = \left\{ p : \overline{\lim_{y \to y_0}} \frac{u(y) - u(y_0) - p \cdot (y - y_0)}{|y - y_0|} \leq 0 \right\}$$

$$D^-(y_0) = \left\{ p : \underline{\lim_{y \to y_0}} \frac{u(y) - u(y_0) - p \cdot (y - y_0)}{|y - y_0|} \geq 0 \right\}$$

**Definition 2.4** (Def III)**.** A viscosity solution of (1.1) in $\Omega$ is a function $u \in C(\Omega)$ such that

**subsolution:**

$$F(y, u(y), p) \leq 0 \quad \text{for all } y \in \Omega \text{ and } p \in D^+(y)$$

**supersolution:**

$$F(y, u(y), p) \geq 0 \quad \text{for all } y \in \Omega \text{ and } p \in D^-(y)$$

Now we can give a theorem about these three definitions.

**Theorem 2.5.** *Three definitions of viscosity solution are equivalent.*

We need some lemmas for it.

**Lemma 2.6.** *Let $\varphi \in C(\Omega)$ be differentiable at $y_0 \in \Omega$. Then there is functions $\psi_+$ and $\psi_-$such that*

$$\psi_\pm \in C_c^1(\Omega), \psi_\pm(y_0) = \varphi(y_0), D\psi_\pm(y_0) = D\varphi(y_0)$$

*and $\psi_+ > \varphi$, $\psi_- < \varphi$ on $B_r(y_0) \setminus \{y_0\}$ for some $r > 0$.*

*Proof.* WLOG $y_0 = 0, \varphi(y_0) = 0, D\varphi(y_0) = 0$.(Otherwise let $\tilde{\varphi} = \varphi(y+y_0) - \varphi(y_0) - D\varphi(y_0) \cdot y$ )

Since $\varphi$ is differentiable at $y_0 \in \Omega$, we can suppose $\varphi(y) = |y|\rho(y)$, where $\rho \in C(\Omega)$ and $\lim_{y \to 0} \rho(y) = 0$. Define

$$w(r) = \max_{y \in B_r(0) \bigcap \Omega} \{\rho(y)\}$$

and

$$\psi_+(y) = \int_{|y|}^{|2y|} w(s) ds + |y|^2$$

Then

$$D\psi_+(y) = w(2|y|) \frac{2y}{|y|} - w(|y|) \frac{y}{|y|} + 2y$$

Hence

$$\lim_{y \to 0} D\psi_+(y) = 0 = \lim_{y \to 0} \frac{\psi_+(y)}{|y|}$$

which means $\psi_+ \in C^1(\Omega)$. Futhermore, we have

$$\begin{aligned}\psi_+(y) - \varphi(y) &= \int_{|y|}^{2|y|} w(s)ds + |y|^2 - \varphi(y) \\ &\geq |y|\rho(y) + |y|^2 - \varphi(y) = |y|^2\end{aligned}$$

Thus for general $\varphi$, we can get $\psi_+ \in C^1(\Omega)$ satisfies the properties in lemma. Moreover, we can modifies $\psi^+$ to let it in $C_c^1(\Omega)$. □

Then there comes a proof of equivalence of Def II and Def III.

*Equivalence of Def II and Def III.*

**Def II⇒ Def III:** We only prove for subsolution. Define

$$\varphi(y) = [u(y) - u(y_0) - p \cdot (y - y_0)]^+$$

for $p \in D^+(y_0)$. Then $\varphi(y_0) = 0$. By definition of $D^+(y_0)$, $\forall \epsilon > 0$, there exists $r > 0$ such that $\forall y \in B_r(y_0)$

$$\frac{u(y) - u(y_0) - p \cdot (y - y_0)}{|y - y_0|} \leq \epsilon$$

Therefore,

$$|\varphi(y) - \varphi(y_0)| = [u(y) - u(y_0) - p \cdot (y - y_0)]^+ \leq \epsilon |y - y_0|, \forall y \in B_r(y_0)$$

By arbitrariness of $\epsilon$, $D\varphi(y_0) = 0$. Hence there exists $\psi_+(y)$ with properties in Lemma 2.6. Set

$$\psi(y) = \psi_+(y) + u(y_0) + p \cdot (y - y_0)$$

Then $u - \psi$ attains maximum at point $y_0$ and $D\psi(y_0) = p$. Hence

$$F(y_0, u(y_0), p) \leq 0$$

**Def III⇒ Def II:** we can just let $p = D\varphi(y_0)$ for the test function $\varphi \in C^1(\Omega)$.

□

In order to prove the equivalence of Def I and Def II, I give a theorem in [1], which connects the discussion in the beginning and the Def I. And explains why in the definition some requirements are weakened.

**Theorem 2.7.** *Let $u$ be a viscosity subsolution(Def I) of 1.1, $\varphi \in C(\Omega)^+$ and $\psi \in C(\Omega)$. Then*

$$F(y, u, -\frac{u - \psi}{\varphi} D\varphi + D\psi) \leq 0 \quad \textit{in } E_+(\varphi(u - \psi)) \cap d(\varphi) \cap d(\psi)$$

*If $u$ is a viscosity supersolution(Def I), then*

$$F(y, u, -\frac{u - \psi}{\varphi} D\varphi + D\psi) \geq 0 \quad \textit{in } E_-(\varphi(u - \psi)) \cap d(\varphi) \cap d(\psi)$$

*Proof.* I divide the proof into two parts.

**Claim I:** When $\psi = k \in \mathbb{R}$, Theorem is valid.

Choose $y_0 \in E_+(\varphi \cdot (u - k)) \cap d(\varphi)$ and $\varphi \in C(\Omega)^+$. WLOG $u(y_0) - k > 0$. Then $\varphi(y_0) > 0$. By the continuity of $u$ and $\varphi$, there is $r > 0$ and $u - k > 0$ and $\varphi > 0$ on $\bar{B}_r(y_0) \subset \Omega$.

Then there is a function $\psi_- \in C_c^1(B_r(y_0))^+$ such that $\psi_-(y_0) = \varphi(y_0)$, $D\psi_-(y_0) = D\varphi(y_0)$. And $y_0$ is a strict local minimum of $\varphi - \psi_-$ on the support of $\psi_-$. Then

$$\{y_0\} = E_+(\psi_-(u-k))$$

Since $C_c^\infty(\Omega)^+$ is dense in $C_c^1(\Omega)^+$ ,there is a sequence $\{\varphi_l\}_{l=1}^\infty \subset C_c^\infty(\Omega)^+$ with support contained in $B_r(y_0)$ such that $\varphi_l \to \psi_-$ and $D\varphi_l \to D\psi_-$ uniformly. Hence for large $l$, $E_+(\varphi_l \cdot (u-k))$ is not empty and choose $y_l$ in it satisfying

$$F(y_l, u(y_l), -\frac{u(y_l)-k}{\varphi(y_l)} D\varphi(y_l))$$

Since $\bar{B}_r(y_0)$ is compact, hence sequential compact. There is a convergent subsequence, we denote it by $\varphi_l$ again. And its limit must be in $E_+(\psi_- \cdot (u-k))$ by the uniform convergence, hence

$$\lim_{l\to\infty} y_l = y_0$$

also we have

$$F(y_0, u(y_0), -\frac{u(y_0)-k}{\varphi(y_0)} D\varphi(y_0)) \leq 0$$

**Claim II:** for all $\psi \in C(\Omega)$, Theorem is valid.

Suppose $y_0 \in E_+(\varphi \cdot (u-\psi)) \cap d(\varphi) \cap d(\psi)$, then $u(y_0) \neq \psi(y_0)$ consider function $\tilde{\varphi}$ such that

$$\tilde{\varphi} \cdot (u(y) - \psi(y)) = \varphi(y)(u(y) - \psi(y))$$

i.e.

$$\tilde{\varphi}(y) = \frac{u(y)-\psi(y)}{u(y)-\psi(y_0)} \varphi(y)$$

Futhermore, we have

$$\begin{aligned} \frac{u(y)-\psi(y)}{u(y)-\psi(y_0)} &= \frac{u(y)-\psi(y_0)+\psi(y_0)-\psi(y)}{u(y)-\psi(y_0)} \\ &= 1 + \frac{\psi(y_0)-\psi(y)}{u(y)-u(y_0)} \\ &= 1 - \frac{D\psi(y_0)\cdot(y-y_0)}{u(y)-\psi(y_0)} + o(|y-y_0|) \end{aligned}$$

Hence $\tilde{\varphi}$ is differential at $y_0$. We see that $y_0 \in E_+(\tilde{\varphi} \cdot (u - \psi(y_0)))$. By Claim I,

$$F(y_0, u(y_0), -\frac{u(y_0)-\psi(y_0)}{\varphi(y_0)} D\varphi(y_0) + D\psi(y_0)) \leq 0$$

□

*Remark* 2.3. If we choose $\varphi = 1, \psi \in C^1(\Omega)$, then $E_+(\varphi \cdot (u-\psi)) = E_+(u-\psi)$ is the set of global maximum points of $u - \psi$. Hence, we need to improve the definition in Def II from replacing local maximum by global strict maximum. I fail to prove it directly but I see it can be derived from the equivalence of Def II and Def I.

*Equivalence of Def I and Def II.*

**Def I $\Rightarrow$ Def II:** Suppose $\psi \in C^1(\Omega)$ and $u - \psi$ attains local maximum at $y_0$. Then there is a ball $B_r(y_0) \subset \Omega$ such that $u - \psi$ attains maximum at $y_0$. Choose $\varphi \in C_c^\infty(B_r(y_0))$ such that $\varphi(y_0) = 1$ and $0 \le \varphi(y) < 1$ if $y \ne y_0$. Hence $E_+(\varphi \cdot (u - (\psi - 1))) = \{y_0\}$. Moreover, we have $D\varphi(y_0) = 0$ since $y_0$ is the maximum point of $\varphi$. Hence by Theorem 2.7,

$$\begin{aligned} 0 &\ge F(y_0, u(y_0), -\frac{u(y_0) - \psi(y_0) + 1}{\varphi(y_0)} D\varphi(y_0) + D\psi(y_0)) \\ &= F(y_0, u(y_0), D\psi(y_0)) \end{aligned}$$

**Def II $\Rightarrow$ Def I:** Suppose $y_0 \in E_+(\varphi \cdot (u - k))$, then we have

$$\varphi(y) \cdot (u(y) - k) \le \varphi(y_0) \cdot (u(y_0) - k)$$

Hence we have

$$u(y) \le k + \frac{\varphi(y_0) \cdot (u(y_0) - k)}{\varphi(y)}$$

Since right hand of the inequality is of $C^\infty$, we see that

$$F(y_0, u(y_0), -\frac{u(y_0) - k}{\varphi(y_0)} D\varphi(y_0)) \le 0$$

□

**Corollary 2.8.** In the definition of viscosity subsolution(Def II), a local maximum can be replaced by a maximum or even by a strict maximum. Besides, a $C^1(\Omega)$ test function $\varphi$ can be replaced by a $C^\infty(\Omega)$ test function $\varphi$ as well.

*Proof.* Suppose $\psi \in C(\Omega)$ and $u - \psi$ attains local maximum at $y_0$. What we know is that if $\psi$ is of $C^\infty$ and $y_0$ is strict global maximum, then

$$F(y_0, u(y_0), D\psi(y_0)) \le 0 \tag{2.1}$$

First of all, we can show if $y_0$ is just a global but not a strict global maximum, inequality (2.1) still holds. It can be done by replacing $\psi$ by $\psi + |y - y_0|^2$.

Then we see from the prove of equivalence of Def II to Def I, the conclusion in Theorem 2.7 of subsolution holds. Then from the prove of equivalence of Def I to Def II, we obtain the corollary. □

Until now, we see that all of these definitions of viscosity solution is equivalent and they may play different roles in different situation. Since In Def I, it just need the existence of one point satisfys the corresponding inequality, while in Def II it need all the maximum point to satisfy the corresponding inequality. Besides, it should be noticed that even if use the upper or lower semi-continuous to define solution, those equivalence don't change.

## 3. On The Uniqueness of Viscosity Solution

In this section, I discuss comparsion principle of viscosity solution, which implys the uniquess of viscosity solution under some special assumptions.

First of all, let us assume that $\Omega$ is a bounded open subset of $\mathbb{R}^n$, and extension definition of viscosity solution to upper semicontinuous functioins and lower semicontinuous functions, i.e. in the definition of subsolution[supersolution], $u$ only need to be upper[lower] semicontinuous function. Furthermore, assume subsolution[supersolution] $u$ is a upper[lower] semicontinuous on $\bar{\Omega}$. Denote function class by USC($\bar{\Omega}$) and LSC($\bar{\Omega}$).

**Theorem 3.1.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. $F(y,r,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and there is a $\gamma > 0$ such that:*

1. *For all $r \geq s, (x,p) \in \Omega \times \mathbb{R}^n$*
$$\gamma(r-s) \leq F(x,r,p) - F(x,s,p)$$
2. *There is a function $w : [0,\infty] \to [0,\infty]$ such that $w(0+) = 0$ and*
$$F(y,r,\alpha(x-y)) - F(x,r,\alpha(x-y)) \leq w(\alpha|x-y|^2 + |x-y|)$$
*for all $x, y \in \Omega, r \in \mathbb{R}$.*

*Suppose $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ is a subsolution and a super solution of (1.1) respectively. Furthermore, $u \leq v$ on $\partial\Omega$. Then we have $u \leq v$ in $\Omega$.*

To prove this theorem, we need a lemma.

**Lemma 3.2.** *Suppose $\Omega$ is a subset of $\mathbb{R}^n$. $f \in USC(\Omega)$ and $g \in LSC(\Omega)$ such that $g \geq 0$. For $\alpha > 0$, define*
$$M_\alpha = \sup_{x\in\Omega}\{f(x) - \alpha g(x)\}$$
*Suppose $-\infty < \lim_{\alpha\to\infty} M_\alpha < \infty$, and for any $\alpha > 0$, $(x_\alpha) \in \Omega$ such that*
$$\lim_{\alpha\to\infty}(M_\alpha - (f(x_\alpha) - \alpha g(x_\alpha))) = 0$$
*Then we have:*

1. $\lim_{\alpha\to\infty} \alpha g(x_\alpha) = 0$
2. *if $\hat{x}$ is one of limit points of $(x_\alpha)$ then $g(\hat{x}) = 0$ and $\lim_{\alpha\to\infty} M_\alpha = f(\hat{x}) = \sup_{g(x)=0} f(x)$*

*Proof.* Observe that when $\alpha$ increases, $M_\alpha$ decreases. Let
$$\delta_\alpha = M_\alpha - (f(x_\alpha) - \alpha g(x_\alpha))$$
Then $\delta_\alpha \to 0$. What's more, we have
$$\begin{aligned} \delta_\alpha &= M_\alpha - (f(x_\alpha) - \frac{\alpha}{2} g(x_\alpha)) + \frac{\alpha}{2} g(x_\alpha) \\ &\geq M_\alpha - M_{\frac{\alpha}{2}} + \frac{\alpha}{2} g(x_\alpha) \end{aligned}$$
pass to limit, we obtain $\lim_{\alpha\to\infty} \alpha g(x_\alpha) = 0$.

Since $g \in LSC(\Omega)$, $g(\hat{x}) \leq \underline{\lim}_{\alpha\to\infty} g(x_\alpha) = 0$, hence $g(\hat{x}) = 0$. Moreover,
$$\begin{aligned} \sup_{g(x)=0}\{f(x)\} &\leq \lim_{\alpha\to\infty} M_\alpha \\ &= \lim_{\alpha\to\infty} f(x_\alpha) \leq f(\hat{x}) \\ &\leq \sup_{g(x)=0}\{f(x)\} \end{aligned}$$
Hence conclusion hold. □

Now let's prove theorem 3.1.

*Proof.* Suppose $\exists x_0 \in \Omega$ such that $u(x_0) > v(x_0)$. Denote $u(x_0) - v(x_0) = \delta$. For $\alpha > 0$, define
$$M_\alpha = \sup_{\bar{\Omega}\times\bar{\Omega}}\{u(x) - v(y) - \alpha|x-y|^2\}$$

Then $\infty > M_\alpha \geq \delta$ for all $\alpha$. Since $M_\alpha$ is non-increasing with respect to $\alpha$, we have $-\infty < \lim_{\alpha\to\infty} M_\alpha < \infty$. By supermum, there are $(x_\alpha, y_\alpha)$ such that

$$\lim_{\alpha\to\infty}(M_\alpha - (u(x_\alpha) - v(y_\alpha) - \alpha|x_\alpha - y_\alpha|^2)) = 0$$

Hence from lemma 3.2, $\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0$. Since $\bar{\Omega}$ is compact, $\{x_\alpha\}$ and $\{y_\alpha\}$ have limit point in $\bar{\Omega}$. let $\hat{x}$ and $\hat{y}$ be their limit points for the same subsequence of $\{\alpha \in \mathbb{R}\}$. Then $\hat{x} = \hat{y}$ and

$$\lim_{\alpha\to\infty} M_\alpha = u(\hat{x}) - v(\hat{x}) = \sup_{x\in\bar{\Omega}}\{u(x) - v(x)\}$$

Since $u \leq v$ on $\partial\Omega$, we see that $\hat{x} \in \Omega$. What's more, we have that for large enough $\alpha$, $x_\alpha, y_\alpha \in \Omega$ by the assumption $u \leq v$ on $\partial\Omega$ and $M_\alpha \geq \delta$.

Now condiser $\varphi(x) = v(y_\alpha) - \alpha|x - y_\alpha|^2$, $u(x) - \varphi(x)$ obtains maximum at point $x_\alpha$. Thus

$$F(x_\alpha, u(x_\alpha), 2\alpha(x_\alpha - y_\alpha)) \leq 0$$

Similarly,

$$F(y_\alpha, v(y_\alpha), 2\alpha(x_\alpha - y_\alpha)) \geq 0$$

Hence

$$\begin{aligned}
\gamma\delta &\leq \gamma(u(\hat{x}) - v(\hat{x})) \\
&\leq \lim_{\alpha\to\infty} F(x_\alpha, u(x_\alpha), 2\alpha(x_\alpha - y_\alpha)) - F(x_\alpha, v(y_\alpha), 2\alpha(x_\alpha - y_\alpha)) \\
&= \lim_{\alpha\to\infty} F(x_\alpha, u(x_\alpha), 2\alpha(x_\alpha - y_\alpha)) - F(y_\alpha, v(y_\alpha), 2\alpha(x_\alpha - y_\alpha)) \\
&\quad + F(y_\alpha, v(y_\alpha), 2\alpha(x_\alpha - y_\alpha)) - F(x_\alpha, v(y_\alpha), 2\alpha(x_\alpha - y_\alpha)) \\
&\leq \lim_{\alpha\to\infty} w(2\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|) = 0.
\end{aligned}$$

which is a contradiction. □

Now let's come to another situation when $\Omega$ is not bounded. It's easy to check following theorem.

**Theorem 3.3.** *Suppose $\Omega \in \mathbb{R}^n$, $F(y, r, p)$ satisfies assumptions in theorem 3.1. Suppose $u, v$ is bounded subsolution and supersolution, then comparsion principle holds for $u$ and $v$,i.e. if $u \leq v$ on $\partial\Omega$ then $u \leq v$ in $\Omega$.*

*Proof.* We only need to show $\lim_{\alpha\to\infty} M_\alpha$ exists if $u \leq v$ in $\Omega$ doesn't hold. By boundedness of $u$ and $v$, $M_\alpha$ is bounded. Besides, $M_\alpha$ is non-increasing, hence $\lim_{\alpha\to\infty} M_\alpha$ exists. Then follow the proof of theorem 3.1. □

## 4. On The Perron's Method

Consider the Dirichlet Problem

$$\begin{cases} F(y, u, Du) = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

A viscosity solution of (4.1) must satisfy the boundary condition as well as the condition for viscosity solution as follows(with some little change). I only give the definition for subsolution.

**Proposition 4.1.** *An upper semi-continuous function $u$ is a subsolution of (4.1) if $\psi \in C^1(\Omega)$ is a function such that there are $r > 0, y_0 \in \Omega$ and*

$$(u - \psi)(y) \leq -|y - y_0|^2$$

*for $y \in B_r(y_0)$, then*

$$F(y_0, u(y_0), D\psi(y_0)) \leq 0$$

It is easy to check above definition is equivalent to the definitions in Section 2.

**Theorem 4.2** (The Perron method)**.** *Let $f, g \in C(\bar{\Omega})$ be locally bounded subsolution and supersolution of (4.1), respectively. Assume that $f \leq g$ on $\Omega$, $f = g = u_0$ on $\partial\Omega$ and that*

$$\varliminf_{|p| \to \infty} F(y, r, p) > 0 \quad \textit{uniformly for } (x, r) \in \Omega \times \mathbb{R}$$

*Then, the function $u : \Omega \to \mathbb{R}$ defined by*

$$u(y) = \sup\{v(y) : f \leq v \leq g \textit{ on } \Omega, v \textit{ is a subsolution of (4.1)}\}$$

*is a viscosity solution of (4.1). Moreover, $f \leq u \leq g$ on $\Omega$.*

To prove this, we need some extra notation.

**Definition 4.3.** Suppose $u$ is a locally bounded function on $\Omega \subset \mathbb{R}^n$, define:

$$u^*(x) = \inf\{v(x) : v \in C(\Omega), v \geq u \text{ on } \Omega\}$$

and

$$u_*(x) = \sup\{v(x) : v \in C(\Omega), v \leq u \text{ on } \Omega\}$$

**Lemma 4.4.** *Suppose $u$ is a locally bounded function on $\Omega \in \mathbb{R}^n$, then $u^*$ is upper semi-continuous, $u_*$ is lower semi-continuous. Besides, if $u$ is upper[lower] semi-continuous then $u = u^*[u_*]$*

*Proof.* Here we first prove $u^*$ is upper semi-continuous. Fix $x \in \Omega$, $\forall \epsilon > 0$, there is a continuous function $v \in C(\Omega)$ such that $v \geq u$ on $\Omega$ and

$$u^*(x) > v(x) - \epsilon$$

By the continuity of $v$, there is a ball $B_r(x)$ such that for all $y \in B_r(x)$, we have

$$|v(x) - v(y)| < \epsilon$$

Hence, for such $y$,

$$u^*(y) \leq v(y) \leq v(x) + \epsilon \leq u^*(x) + 2\epsilon$$

Therefore, we obtain

$$\varlimsup_{y \to x} u^*(y) \leq u^*(x)$$

i.e. $u^*$ is upper semi-continuous. $u_*$ is lower semi-continuous is the same.

To prove the remain of lemma, we need another lemma.

**Lemma 4.5.** *Suppose $u$ is bounded and is upper semi-continuous in $\Omega$. Then there exists Lipschitz function $\{f_k\}$ such that $f_k(x) \geq u(x)$ and $\lim_{k\to\infty} f_k = u$.*

*Proof.* Define

$$f_k(x) = \sup\{u(y) - kd(x, y)\} \geq u(x)$$

Then $\{f_k\}$ are what we want. ∎

If $u$ is upper semi-continuous, then by lemma 4.5, $u$ can be approximated by Lipschitz functions, so $u \le u^* \le u$. Hence, $u = u^*$. □

**Lemma 4.6.** *For any locally bounded function $u : \Omega \to \mathbb{R}$,*

$$u^*(x) = \lim_{r\to 0} \sup_{|y-x|<r} \{u(y)\}$$

*and*

$$u_*(x) = \lim_{r\to 0} \inf_{|y-x|<r} \{u(y)\}$$

*Proof.* We only prove the $u_*$ case. Since $u_*$ is lower semi-continuous, from lemma 4.4, we have

$$u_*(x_0) \le \varliminf_{y\to x_0} u_*(y) \le \varliminf_{y\to x_0} u(y)$$

Hence

$$u_*(x_0) \le \lim_{r\to 0} \inf_{|y-x_0|<r} \{u(y)\}$$

On the other hand, Suppose $r > 0$ and $p = \inf_{|y-x_0|<r}\{u(y)\}$. Moreover, $\bar{B}_r(x_0) \subset \Omega$. Choose compact set $K_n = \bar{U}_n \subset \Omega$, where $U_n$ is a open set, such that $K_n \subset K_{n+1}$ and $K_1 = \bar{B}_r(x_0)$, $\Omega = \cup_{n=1}^{\infty} K_n$. Define

$$\bar{f}(x) = \begin{cases} p, & x \in K_1 \\ \inf_{y\in K_n} \{u(y)\}, & x \in K_{n+1} \setminus K_n, n \ge 1 \end{cases}$$

Then modify $\bar{f}$ into a continuous function and $f \le u$. Indeed, we can construct a series of functions $\{\varphi_n(x)\} \in C^\infty(\Omega)$ by following way.

Choose a compact set $K_{n-1} \subset \Omega_n \subset U_n$(set $K_0 = \emptyset$) and a $C^\infty$ function $\varphi_n$ such that $\varphi(x) = 1$ on $\Omega_n$ and $\varphi(x) = 0$ outside $U_n$. Then we define

$$f(x) = \sum_{n=1}^{\infty} \left( \left( \bar{f}(x)\chi_{(K_n\setminus K_{n-1})} - \inf_{y\in K_{n+1}} \{u(y)\} \right) \varphi_n + \inf_{y\in K_{n+1}} \{u(y)\} \right)$$

Then $f \in C^\infty(\Omega)$ and $f(x_0) = p, f(x) \le u(x)$ on $\Omega$. Hence

$$u_*(x_0) \ge \lim_{r\to 0} \inf_{|y-x_0|<r} \{u(y)\}$$

Until here we have proved the lemma. □

Now, in order to prove theorem 4.1, we first show that $u^*$ is a subsolution and $u_*$ is a supersolution. Then the continuity of $g$ shows that $u = u^*$ and the assumption of $F$ ensure the continuity of $u$, hence $u$ will be a viscosity solution of (4.1).

**Proposition 4.7.** *Let $\mathcal{P}$ be the collection of subsolutions of (4.1). Put*

$$u(x) = \sup\{v(x) : v \in \mathcal{P}\}$$

*Assume $u$ is locally bounded, then $u^*$ is a subsolution.*

*Proof.* Suppose $\varphi \in C^1(\Omega)$ and $u^* - \varphi \le -|y - y_0|^2$ for $y \in B_r(y_0)$ and some $r > 0$. Since $u^*$ is upper semi-continuous, there is a sequence $\{y_n\} \subset B(y_0, r)$ such that $y_n \to y_0$ and $(u - \varphi)(y_n) \to 0$ as $n \to \infty$. Denote $a_n = (u - \varphi)(y_n)$, then by the definition of $u$, there is $v_n \in \mathcal{P}$ such that

$$0 \ge v_n(y_n) - \varphi(y_n) > a_n - \frac{1}{n}.$$

By the semi-continuity of $v_n$, we see that $v_n - \varphi$ attains maximum in $\bar{B}_r(y_0)$ at point $y_n'$. Hence we have

$$\begin{aligned} -|y_n' - y_0|^2 &\geq u^*(y_n') - \varphi(y_n') \\ &\geq v_n(y_n') - \varphi(y_n') \\ &\geq v_n(y_n) - \varphi(y_n) > a_n - \frac{1}{n} \end{aligned}$$

Thus, if we send $n$ to infinity, we see that $y_n' \to y_0$ as $n \to \infty$. Together with $v_n(y_n') \to \varphi(y_0) = u^*(y_0)$ we shows $u^*$ is a subsolution of (4.1). $\square$

*proof of Theorem* 4.2. Define

$$\mathcal{P} = \{v(y) : f \leq v \leq g, v \text{ is a viscosity subsolution of (4.1)}\}$$

and

$$u(y) = \sup\{v(y) : v \in \mathcal{P}\}$$

Then $u^*$ is a subsolution of (4.1), moreover, since $g$ is continuous, $u^* \leq g$, we see $u^* = u$. We next prove $u_*$ is a supersolution of (4.1). Suppose it is false. Then there is a function $\psi \in C^1(\Omega)$ and $u_*(y_0) - \psi(y_0) = 0$, $u_*(y) - \psi(y) \geq |y - y_0|^2$ in $B_r(y_0)$ for some $r > 0$ such that

$$F(y_0, u_*(y_0), D\psi(y_0)) < 0$$

Claim that $u_*(y_0) < g_*(y_0)$. Otherwise $u_*(y_0) = g_*(y_0) = g(y_0)$, we see that $g - \psi$ attains minimum at $y_0$, hence

$$F(y_0, u_*(y_0), D\psi(y_0)) \geq 0$$

which is a contradiction. Hence $u_*(y_0) < g_*(y_0)$.

Choose $\delta > 0$ such that

$$F(y_0, \psi(y_0) + \frac{\delta^2}{2}, D\psi(y_0)) \leq 0, \quad \psi(y) + \frac{\delta^2}{2} < g(y) \text{ in } B_{2\delta}(y_0)$$

Then we see that $\psi + \frac{\delta^2}{2}$ is a viscosity solution in $B_{2\delta}(y_0)$ and define

$$w(y) = \begin{cases} \max\left\{\psi(y) + \dfrac{\delta^2}{2}, u(y)\right\}, & y \in B_\delta(y_0) \\ u(y), & y \in \Omega \setminus B_\delta(y_0) \end{cases}$$

Then $w$ is upper semi-continuous and a viscosity subsolution of (4.1). Moreover, Since

$$\psi(y_0) = u_*(y_0) = \lim_{r \to 0} \inf_{|y - y_0| < r} \{u(y)\}$$

We see there is $y_1 \in B_\delta(y_0)$ and $u(y_1) < \psi(y_0) < \psi(y_1) + \frac{\delta^2}{2}$, hence $w(y) > u(y)$ for some $y \in \Omega$, which is a contradiction.

Finally, we need to show $u_* = u$. Or $u$ is continuous. By the hypothesis, there is an $R > 0$ such that

$$F(y, r, p) > 0 \text{ if } (x, r, p) \in \Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus B_R(0))$$

Fix $x_0 \in \Omega$ and $u$ is bounded in $B_{2\delta}(x_0)$. Choose $h \in C^1(\mathbb{R})$ such that $h(s) = Rs$ if $0 \leq s \leq \delta$ and $h'(s) \geq R$ for $s \geq 0$, $h(2\delta) > 2\sup\{|u(y)| : y \in B_{2\delta}(x_0)\} + 1$. Define

$$\theta(x) = u(x_0) + h(|x - x_0|)$$

Then $\theta(x)$ is continuous and $\theta(x_0) = u(x_0)$, $\theta(x) > \sup\{|u(y)| : y \in B_{2\delta}(x_0)\}$ on $\partial B_{2\delta}(x_0)$. If $\theta < u$ at some point in $B_{2\delta}(x_0)$. Then $u - \theta$ attains maximum at point $\bar{x} \in \bar{B}_{2\delta}(x_0)$. Since $u(x_0) - \theta(x_0) = 0$ and $u(x) - \theta(x) < 0$ on $\partial B_{2\delta}(x_0)$, $\bar{x} \in B_{2\delta}(x_0) \setminus \{x_0\}$. Hence

$$F(\bar{x}, u(\bar{x}), D\theta(\bar{x})) \leq 0$$

However,

$$|D\theta(\bar{x})| = h'(|\bar{x} - x_0|)|D(|\bar{x} - x_0|)| \geq R$$

Then by assumption, $F(\bar{x}, u(\bar{x}), D\theta(\bar{x})) \geq 0$, which is a contradiction. Hence $\theta \geq u$ in $B_{2\delta}(x_0)$. We thus have

$$u(x) - u(x_0) \leq h(|x - x_0|) = R|x - x_0| \text{ in } B_\delta(x_0)$$

Hence $u$ is continuous. Set $u = u_0$ on $\partial\Omega$, then $u$ is the viscosity solution of (4.1). □

*Remark* 4.1. Form the proof of theorem 4.2, we see that if we exclud assumption for $F$, we can only obtain a function $u$ such that $u^*$ is subsolution of equation 4.1 while $u_*$ is a supersolution of 4.1, which is really near to be a solution to 4.1. If we suppose further that comparison principle holds for equation then it is obvious that $u_* \geq u^* \geq u_*$, which leads to conclusion that $u$ is continuous. Hence we obtain a new theorem.

**Theorem 4.8.** *Let $f, g \in C(\bar{\Omega})$ be locally bounded subsolution and supersolution of (4.1), respectively. Assume that $f \leq g$ on $\Omega$, $f = g = u_0$ on $\partial\Omega$ and comparsion principle holds for equation. Then, the function $u : \Omega \to \mathbb{R}$ defined by*

$$u(y) = \sup\{v(y) : f \leq v \leq g \text{ on } \Omega, v \text{ is a subsolution of (4.1)}\}$$

*is a viscosity solution of (4.1). Moreover, $f \leq u \leq g$ on $\Omega$.*

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