MATH 3015 Mathematics of Finance Essay

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Abstract

This essay gives an introduction to Haudorff dimension.

1 Introduction to Hausdorff dimension

A measurement is an action of measuring something to give us the better understanding of the world around us. In other words, the outcome of measurement can also reshape the way we think and help us make more rational and logical decisions. To measure the roughness or chaos, mathematician Felix Hausdorff introduced Hausdorff dimension in 1918. Specifically, Hausdorff dimension is a dimensional number associated with a given set and the distances between the elements of the set are defined.
Since Hausdorff dimension is based on the notion of a covering of the matrix space, before formally introducing Hausdorff dimension, let us consider the infinite coverings of the matrix space first where each covering is measured by its diameter.
let X be a matrix space and $E \subset X$, the α-value of a covering is defined as $\sum_{i=1}^{\infty} |E_i|^\alpha$ where $\{E_i\}$ is a covering of the matrix space E, for each E_i we take the diameter into account, denoted $|\cdot|$.
Based on α-value, it is defined that for every $\alpha > 0$, α-Hausdorff content of a matrix space is:

$$\mathcal{H}^\alpha_\infty(E) = inf\{\sum_{i=1}^{\infty} |E_i|^\alpha \colon \{E_i\} \text{ is a covering on matrix space E}\}$$

Now, the Hausdorff dimension of a set E is defined as:

$$\text{Dim}E = inf\{\alpha \geq 0 \colon \mathcal{H}^\alpha_\infty(E) = 0\} = sup\{\alpha \geq 0 \colon \mathcal{H}^\alpha_\infty(E) > 0\}$$

Noticeably, this definition (based on either inf or sup) reflects the essential idea of measuring dimension. In other words, we are looking for the valua α at which the "jump" from ∞ to 0 occurs.
If there's constraint on the diamater of covering such that for $\delta > 0$, $E_i \leq \delta$, we coucl define:

$$\mathcal{H}^\alpha_\delta(E) = inf\{\sum_{i=1}^{\infty} |E_i|^\alpha \colon \{E_i\} \text{ is a covering on matrix space E and } |E_i| \leq \delta \text{ for each i}\}$$

and

$$\mathcal{H}^\alpha(E) = sup_{\delta > 0}\mathcal{H}^\alpha_\delta(E) = lim_{\delta \downarrow 0}\mathcal{H}^\alpha_\delta(E)$$

where $\mathcal{H}^\alpha(E)$ is called α-Hausdorff measure.

The remarkable points about Hausdorff dimension:
1: Hausdroff dimension is taking variations of the size in the different sets in a covering into account;
2: Hausdorff dimension captures finer details of the set

Now, we do a quick proof of an interesting proposition which I introduced in my talk:

$$\mathcal{H}^\alpha(B([0,1])) < \infty \text{ almost surely}$$

First, it is required that we find the coving of $B([0,1])$, therefore we could construct a set of balls:

$$\{\mathcal{B}(B(\frac{k}{n}), \max_{k/n \leq t \leq (k+1)/n} |B(t) - B(\frac{k}{n})|) : k \in \{0, ..., n-1\}\}$$

Now we consider the diameter of each ball by taking the expectation:

$$E[(\max_{k/n \leq t \leq (k+1)/n} |B(t) - B(\frac{k}{n})|)^2] \leq E[(\max_{0 \leq t \leq 1/n} |B(t)|)^2] \text{ (by switching the subinterval)}$$

Using Brownian Scalling, we get:

$$E[(\max_{0 \leq t \leq 1/n} |B(t)|)^2] = \frac{1}{n} \cdot E[(\max_{0 \leq t \leq 1} |B(t)|)^2]$$

Therefore, the expected 2-value of the nth covering:

$$E[\sum_{k=1}^{n}(\max_{k/n\leq t\leq (k+1)/n}|B(t)-B(\frac{k}{n})|)^2]$$

is bounded from above by

$$n\cdot\frac{1}{n}\cdot E[(\max_{0\leq t\leq 1}|B(t)|)^2]=E[(\max_{0\leq t\leq 1}|B(t)|)^2]$$

Noticeably, the right is finite, hence by Fatou's lemma:

$$E[\liminf_{n\to\infty}\sum_{k=1}^{n}(\max_{k/n\leq t\leq (k+1)/n}|B(t)-B(\frac{k}{n})|)^2] \text{ is finite}$$

We are done with the proof.

2 Introduction to Hölder Condition

Geberally, A function f that satisfies Hölder γ is defined as:

$$|f(x)-f(y)|\leq M\cdot|x-y|^{\gamma} \text{ for all x, y } \subset E$$

where E is a metrix space and M is a constant.

In this part, we will be introducing two corollaries related to Hölder Condition and Hausdorff dimension.

1: $\mathcal{H}^{\beta}(f(E))\leq M^{\beta}\cdot\mathcal{H}^{\alpha}(E)$
2: $Dim_{\mathcal{H}}(f(E))\leq Dim_{\mathcal{H}}(E)/\gamma$

where function f($\cdot$) satisfies α Hölder continuity and E is a compact set.

Proof of 1:
First suppose we have got a family of sets: $\{E_k\}$ such that: compact set E is covered by $\{E_k\}$.
Then, by the study of Analysis 1, we know the set $\{f(E\cap E_k)\}$ is covering $f(E)$.
Therefore,

$$Diameter(f(E\cap E_k))\leq M\cdot(Diameter(E_k))^{\gamma}$$

since function f($\cdot$) satisfies α Hölder continuity.
Therefore,

$$\sum_k(Diameter(f(E\cap E_k)))^{\alpha/\gamma}\leq M^{\alpha/\gamma}\cdot\sum_k(Diameter(E_k))^{\alpha}$$

This simply implies the second corollary.
But here I would like to give another proof of the second corollary.

Proof of 2:
First suppose:

$$Dim_{\mathcal{H}}(E)<\alpha<\infty$$

and

$$\exists E_1, E_2, .., E_i \text{ such that } E\subset\cup_i E_i$$

By the study of Analysis 1, we know that

$$\{f(E_i)\} \text{ is a covering of } f(E)$$

Noticeably, function f(·) satisfies α Hölder continuity, therefore,

$$|f(E_i)| \leq C \cdot |E_i|^{\gamma}$$

Hence:

$$\sum_i |f(E_i)|^{\alpha/\gamma} \leq C^{\alpha/\gamma} \cdot \sum_i |E_i|^{\alpha} \leq C^{\alpha/\gamma} \cdot \epsilon$$

$$\text{as } \epsilon \to 0, \ \sum_i |f(E_i)|^{\alpha/\gamma} \to 0,$$

$$Dim_{\mathcal{H}}(f(E)) \leq \alpha/\gamma \leq Dim_{\mathcal{H}}(E)/\gamma$$

3 Upper bound of Hausdorff dimension

For a function $f : E \to R^d$, first we define the *Graph*:

$$Graph_f = \{(t, f(t)) : t \in E\} \subset R^{d+1} \tag{1}$$

where $E \subset [0, \infty]$

Now we introduce the proporsition about about upper bound of Hausdorff dimension of $Graph_f$:

$$dim(Graph_f) \leq 1 + (1 - \alpha) \cdot min\{d, 1/\alpha\} \tag{2}$$

where $f : E \to R^d$ satisfies α Hölder continuity.

Proof:

First we consider f satisfies α Hölder continuity, thus, if t, s $\in$ [0, 1] with $|t - s| \leq \epsilon$, we have

$$|f(t) - f(s)| \leq C \cdot \epsilon^{\alpha}$$

Therefore, we divide the interval $[0, 1]$ into $\lfloor 1/\epsilon \rfloor$ subintervals with length ϵ

Now, the image of each subinterval is contained in a ball with diameter $C \cdot \epsilon^{\alpha}$

In order to cover each image, we have two ways here:

- we will be using the balls with diameter ϵ to cover it, thus we need $C \cdot \epsilon^{n\alpha - n}$ balls where C is a constant

(e.g. if two dimensional, we could have $\pi(C\epsilon^{\alpha}/2)^2/\pi(\epsilon/2)^2$ = a constant multiple of $\epsilon^{2\alpha-2}$)

- Since the image of the subinterval of length $(\epsilon/C)^{1/\alpha}$ has diameter ϵ,

Now we need $\epsilon/(\epsilon^{1/\alpha}/C^{1/\alpha})$ = a constant multiple of $\epsilon^{1-1/\alpha}$ balls

Therefore, we observe that to cover the graph consisting of the product of intervals and corresponding balls in $[0, 1] \cdot R$ of diameter ϵ, the first way needs a multiple constant of $\epsilon^{n\alpha - n - 1}$ product sets, and the second way needs a multiple constant of $\epsilon^{1-(1/\alpha)-1} = \epsilon^{-(1/\alpha)}$ product sets.

Based on the definition of Hausdorff dimension, we have obtained the desired upper bounds.

4 Introduction to Energy Method

Compared to computing the upper bound, it is more complex to compute the upper bound of a Hausdorff dimension. Thus, we will be introducing Energy Method to give us a viable solution

Basically, Energy Method allows us find the lower bound by showing finiteness of a single integral

Before formally defining Energy Method, let us introduce Potential theoretical methods first

For $\alpha \geq 0$, the α-potential at a point x of R^n is defined as

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{|x-y|^\alpha}$$

Noticeably, here's μ denotes a Mass Distribution which is being supported on the on the matrix space and $0 < \mu < \infty$

Then, the α-energy of μ is defined as

$$I_\alpha(\mu) = \int \phi_\alpha(x) d\mu(x) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}$$

Hence, if there is a mass dirtribution with $I_\alpha(\mu) < \infty$, the mass is spread out so that at each point the concentration is small enough to overcome the singularity of integrand.

Now we prove that if there is a mass distribution on a set E which has finite α-energy, then E has dimension at least α.

The book provided on wattle has already given us a fantastic proof. But here I would like to consider another proof:

First we suppose that the α-energy is finite for some mass distribution μ with support contained in E

Now we define: $E' = \{x \in E : \overline{\lim_{r\to 0}} \mu(B_r(x))/r^\alpha > 0\}$

Therefore, if x $\in E'$, we could find a sequence $\{r_i\}$ decreasing to 0 and an $\epsilon > 0$ such that

$$\mu(B_r(x)) \geq \epsilon \cdot r_i^\alpha$$

Now we select a small enough q_i such that $0 < q_i < r_i$ and $\mu(B_{r_i}(x)/B_{q_i}(x)) \geq \frac{1}{4}\epsilon r_i^\alpha$

Noticeably, this step is based on the continuity of μ and we could assmu that $r_{i+1} < q_i$ if necessary

Now, let us compute the α-potential for $x \in E'$

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{|x-y|^\alpha} \geq \sum_{i=1}^{\infty} \cdot \int_{A_i} \frac{d\mu(y)}{|x-y|^\alpha} \geq \sum_{i=1}^{\infty} \frac{1}{4}\epsilon r_i^\alpha \cdot r_i^{-\alpha} = \infty$$

Now it leads to an interesting point that, since $I_\alpha(\mu) = \int \phi_\alpha(x) d\mu(x) < \infty$, $\phi_\alpha(x) < \infty$ for all x

Therefore, we may conclude that $\mu(E') = 0$

since $\overline{\lim_{r\to 0}} \mu(B_r(x))/r^\alpha = 0$ if $x \in E \setminus E'$

We now have $\mathcal{H}^\alpha(E) \geq \mathcal{H}^\alpha(E \setminus E') \geq \mu(E \setminus E')/c \geq (\mu(E) - \mu(E'))/c = \mu(E)/c$ for all $c > 0$

Hence, $\mathcal{H}^\alpha(E) = \infty$

If $I_\alpha(\mu) < \infty$ then $\mathcal{H}^\alpha(E) = \infty$ and $dimE \geq \alpha$

5 Hausdorff dimension and Brownian motion

In this part, we will be mainly computing the Hausdorff dimension of one dimensional Brownian motion. Specifically, we aim to show that one dimensional Brownian motion has 3/2 as its Hausdorff dimension. Thus, we need to compute 3/2 as both its lower bound and upper bound.

5.1 the upper bound of one dimensional Brownian motion

First recall that: If $\alpha < 1/2$, then, almost surely, Brownian motion is everywhere locally α Hölder continuous

Thus, by the corollary we just proved: $dim(Graph_f) \leq 1 + (1-\alpha) \cdot min\{d, 1/\alpha\}$

Therefore, after we substitute α with 1/2, d with 1, we can easily get its upper bound: 3/2

5.2 the lower bound of one dimensional Brownian motion

Now we are interested in computing its lower bound, here we need to utilise the Energy Method we just proved
Therefore, our goal is going to find the value of α such that the expectation of the α-energy is finite:

$$\begin{aligned} &E(\int\int |x-y|^{-\alpha} d\mu_x(x) d\mu_x(y)) \\ &= \int_0^1 \int_0^1 E((|X(t)-X(\mu)|^2 + |t-\mu|^2)^{-\alpha/2}) dt d\mu \\ &\leq \int_0^1 \int_0^1 C \cdot |t-\mu|^{1/2-\alpha} dt d\mu \\ &\leq \infty \text{ (if } \alpha < 3/2) \end{aligned}$$

where X is the one dimensional Brownian motion graph, μ_x is the mass distribution supported on graph X and C is just a constant
Hence, if $\alpha < 3/2$, with probability 1, the mass distribution μ_x supported on Graph X has finite α-energy, the Hausdorff dimension of one dimensional Brownian motion is at least 3/2.

Combine 5.1 and 5.2, we may conclude that one dimensional Brownian motion has 3/2 as its Hausdorff dimension.