Part 1 summarized introduction to hausdorff dimension

In mathematics, Hausdorff dimension (a.k.a. fractal dimension) is a measure of *roughness* and/or chaos that was first introduced in 1918 by mathematician Felix Hausdorff.

the Hausdorff dimension is an integer agreeing with the usual sense of dimension, also known as the topological dimension.

However, formulas have also been developed that allow calculation of the dimension of other less simple objects, where, based solely on their properties of scaling and self-similarity, one is led to the conclusion that particular objects—including fractals—have non-integer Hausdorff dimensions.

The Hausdorff dimension, more specifically, is a further dimensional number associated with a given set, where the distances between all members of that set are defined. Such a set is termed a metric space.

In mathematical terms, the Hausdorff dimension generalizes the notion of the dimension of a real vector space. That is, the Hausdorff dimension of an *n*-dimensional inner product space equals *n*. This underlies the earlier statement that the Hausdorff dimension of a point is zero, of a line is one, etc.

What is measure?
Why needs measure?
How to measure?

Constructing the logical pipeline:

1.2. The Hausdorff dimension. The Hausdorff dimension and Hausdorff measure were introduced by Felix Hausdorff in 1919. Like the Minkowski dimension, Hausdorff dimension can be based on the notion of a covering of the metric space E. For the definition of the Minkowski dimension we have evaluated coverings crudely by counting the number of sets in the covering. Now we also allow infinite coverings and take the size of the covering sets, measured by their diameter, into account.

A very useful evaluation is the α-**value** of a covering. For every $\alpha \geq 0$ and covering $E_1, E_2, \ldots$ we say that the α-**value** of the covering is

$$\sum_{i=1}^{\infty} |E_i|^{\alpha}.$$

DEFINITION 4.5. *For every $\alpha \geq 0$ the α-**Hausdorff content** of a metric space E is defined as*

$$\mathcal{H}_{\infty}^{\alpha}(E) = \inf\left\{\sum_{i=1}^{\infty} |E_i|^{\alpha} \,:\, E_1, E_2, \ldots \text{ is a covering of } E\right\},$$

informally speaking the α-value of the most efficient covering. If $0 \leq \alpha \leq \beta$, and $\mathcal{H}_{\infty}^{\alpha}(E) = 0$, then also $\mathcal{H}_{\infty}^{\beta}(E) = 0$. Thus we can define

$$\dim E = \inf\left\{\alpha \geq 0 \,:\, \mathcal{H}_{\infty}^{\alpha}(E) = 0\right\} = \sup\left\{\alpha \geq 0 \,:\, \mathcal{H}_{\infty}^{\alpha}(E) > 0\right\},$$

the **Hausdorff dimension** *of the set E.* ◇

A relatable application about Hausdorff measure

Definition 4.7. *Let X be a metric space and $E \subset X$. For every $\alpha \geq 0$ and $\delta > 0$ define*

$$\mathcal{H}^\alpha_\delta(E) = \inf\Big\{ \sum_{i=1}^{\infty} |E_i|^\alpha \,:\, E_1, E_2, E_3, \ldots \text{ cover } E, \text{ and } |E_i| \leq \delta \Big\},$$

i.e. we are considering coverings of E by sets of diameter no more than δ. Then

$$\mathcal{H}^\alpha(E) = \sup_{\delta>0} \mathcal{H}^\alpha_\delta(E) = \lim_{\delta \downarrow 0} \mathcal{H}^\alpha_\delta(E)$$

*is the α-**Hausdorff measure** of the set E.* ◇

Corollary 4.16 does not make any statement about the 2-Hausdorff measure of the range, and any such statement requires more information than the Hölder exponent alone can provide, see for example Exercise 4.9. It is however not difficult to show that

$$\mathcal{H}^2\big(B([0,1])\big) < \infty \qquad \text{almost surely.} \tag{1.1}$$

Indeed, for any $n \in \mathbb{N}$, we look at the covering of $B([0,1])$ by the closure of the balls

$$\mathcal{B}\Big(B(\tfrac{k}{n}), \max_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} \big|B(t) - B(\tfrac{k}{n})\big|\Big), \quad k \in \{0, \ldots, n-1\}.$$

By the uniform continuity of Brownian motion on the unit interval, the maximal diameter in these coverings goes to zero, as $n \to \infty$. Moreover, we have

$$\mathbb{E}\Big[\Big(\max_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} \big|B(t) - B(\tfrac{k}{n})\big|\Big)^2\Big] \leq \mathbb{E}\Big[\Big(\max_{0 \leq t \leq \frac{1}{n}} |B(t)|\Big)^2\Big] = \frac{1}{n}\,\mathbb{E}\Big[\Big(\max_{0 \leq t \leq 1} |B(t)|\Big)^2\Big],$$

using Brownian scaling. The expectation on the right is finite by Theorem 2.39. Hence the expected 2-value of the nth covering is bounded from above by

$$4\mathbb{E}\Big[\sum_{k=1}^{n} \Big(\max_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} \big|B(t) - B(\tfrac{k}{n})\big|\Big)^2\Big] \leq 4\,\mathbb{E}\Big[\Big(\max_{0 \leq t \leq 1} |B(t)|\Big)^2\Big],$$

which implies, by Fatou's lemma, that

$$\mathbb{E}\Big[\liminf_{n \to \infty} 4 \sum_{k=1}^{n} \sum_{k=1}^{n} \Big(\max_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} \big|B(t) - B(\tfrac{k}{n})\big|\Big)^2\Big] < \infty.$$

Hence the liminf is almost surely finite, which proves (1.1).

Why hausdorff dimension?

1.1. The Minkowski dimension. How can we capture the dimension of a geometric object? One requirement for a useful definition of dimension is that it should be *intrinsic*. This means that it should be independent of an embedding of the object in an ambient space like $\mathbb{R}^d$. Intrinsic notions of dimension can be defined in arbitrary metric spaces.

Suppose E is a bounded metric space with metric ρ. Here bounded means that the diameter $|E| = \sup\{\rho(x,y) : x,y \in E\}$ of E is finite. The example we have in mind is a bounded subset of $\mathbb{R}^d$. The definition of Minkowski dimension is based on the notion of a *covering* of the metric space E. A **covering** of E is a finite or countable collection of sets

$$E_1, E_2, E_3, \ldots \text{ with } E \subset \bigcup_{i=1}^{\infty} E_i \,.$$

Define, for $\varepsilon > 0$,

$$M(E,\varepsilon) = \min\Big\{k \geq 1 : \text{there exists a finite covering} \\ E_1, \ldots, E_k \text{ of } E \text{ with } |E_i| \leq \varepsilon \text{ for } i = 1, \ldots, k\Big\},$$

where $|A|$ is the diameter of a set $A \subset E$. Intuitively, when E has dimension s the number $M(E,\varepsilon)$ should be of order ε^{-s}. This can be verified in simple cases like line segments, planar squares, etc. This intuition motivates the definition of *Minkowski dimension.*

Remark 4.4. There is an unpleasant limitation of Minkowski dimension: Observe that singletons $S = \{x\}$ have Minkowski dimension 0, but we shall see in Exercise 4.2 that the set

$$E := \big\{\tfrac{1}{n} : n \in \mathbb{N}\big\} \cup \big\{0\big\}$$

has positive dimension. Hence the Minkowski dimension does not have the **countable stability property**

$$\dim \bigcup_{k=1}^{\infty} E_k = \sup\big\{\dim E_k : k \geq 1\big\}.$$

This is one of the properties we expect from a reasonable concept of dimension. There are two ways out of this problem.

(i) One can use a notion of dimension taking variations of the size in the different sets in a covering into account. This captures finer details of the set and leads to the notion of *Hausdorff dimension.*

(ii) One can enforce the countable stability property by subdividing every set in countably many bounded pieces and taking the maximal dimension of them. The infimum over the numbers such obtained leads to the notion of *packing dimension.*

We follow the first route now, but come back to the second route later in the book. ◇

Part2 Holder Stuff

1. Definition + Proof

Definition 4.11. *A function $f\colon (E_1, \rho_1) \to (E_2, \rho_2)$ between metric spaces is called α-Hölder continuous if there exists a (global) constant $C > 0$ such that*

$$\rho_2\big(f(x), f(y)\big) \le C\, \rho_1(x, y)^{\alpha} \qquad \text{for all } x, y \in E_1\,.$$

The constant C is sometimes called the **Hölder constant**. ◇

A function f defined on a subset E of $\mathbb{R}^d$ satisfies a **Lipschitz condition** on E if there exists $M > 0$ such that

$$|f(x) - f(y)| \le M|x - y| \qquad \text{for all } x, y \in E.$$

More generally, a function f satisfies a **Lipschitz condition with exponent** γ (or is **Hölder** γ) if

$$|f(x) - f(y)| \le M|x - y|^{\gamma} \qquad \text{for all } x, y \in E.$$

The only interesting case is when $0 < \gamma \le 1$. (See Exercise 3.)

Lemma 2.2 *Suppose a function f defined on a compact set E satisfies a Lipschitz condition with exponent γ. Then*

(i) $m_\beta(f(E)) \leq M^\beta m_\alpha(E)$ *if* $\beta = \alpha/\gamma$.

Proof. Suppose $\{F_k\}$ is a countable family of sets that covers E. Then $\{f(E \cap F_k)\}$ covers $f(E)$ and, moreover, $f(E \cap F_k)$ has diameter less than $M(\text{diam } F_k)^\gamma$. Hence

$$\sum_k (\text{diam } f(E \cap F_k))^{\alpha/\gamma} \leq M^{\alpha/\gamma} \sum_k (\text{diam } F_k)^\alpha,$$

(b) *and, for any $A \subset [0,1]$, we have* $\dim f(A) \leq \frac{\dim A}{\alpha}$.

Proof of (b). Suppose that $\dim(A) < \beta < \infty$. Then there exists a covering $A_1, A_2, A_3, \ldots$ such that $A \subset \bigcup_j A_j$ and $\sum_j |A_j|^\beta < \varepsilon$. Then $f(A_1), f(A_2), \ldots$ is a covering of $f(A)$, and $|f(A_j)| \leq C|A_j|^\alpha$, where C is the Hölder constant. Thus,

$$\sum_j |f(A_j)|^{\beta/\alpha} \leq C^{\beta/\alpha} \sum_j |A_j|^\beta < C^{\beta/\alpha}\varepsilon \to 0$$

as $\varepsilon \downarrow 0$, and hence $\dim f(A) \leq \beta/\alpha$. ∎

Link to “later important stuff”

Part3: Hausdorff Upper Bound

COROLLARY 1.20. *If $\alpha < 1/2$, then, almost surely, Brownian motion is everywhere locally α-Hölder continuous.*

We now take a first look at dimensional properties of Brownian motion and harvest the results from our general discussion so far. We have shown in Corollary 1.20 that linear Brownian motion is everywhere locally α-Hölder for any $\alpha < 1/2$, almost surely. This extends obviously to d-dimensional Brownian motion, and this allows us to get an upper bound on the Hausdorff dimension of its range and graph.

Quick definition:

DEFINITION 4.13. *For a function $f\colon A \to \mathbb{R}^d$, for $A \subset [0,\infty)$, we define the* **graph** *to be*

$$\mathsf{Graph}_f = \big\{(t, f(t)) \,:\, t \in A\big\} \subset \mathbb{R}^{d+1},$$

and the **range** *or* **path** *to be*

$$\mathsf{Range}_f = f(A) = \big\{f(t) \,:\, t \in A\big\} \subset \mathbb{R}^d. \qquad \diamond$$

Two theorems about upper bound:

PROPOSITION 4.14. *Suppose $f\colon [0,1] \to \mathbb{R}^d$ is an α-Hölder continuous function. Then*

(a) $\dim(\mathsf{Graph}_f) \le 1 + (1-\alpha)\left(d \wedge \frac{1}{\alpha}\right)$,
(b) *and, for any $A \subset [0,1]$, we have* $\dim f(A) \le \frac{\dim A}{\alpha}$.

Proof of a:

Proposition 5.4. *Let $f : [0,1] \mapsto \mathbb{R}^n$ be an α-Hölder continuous function. Then*

$$\dim(\mathrm{Graph_f}) \leq 1 + (1-\alpha)\min\{\mathrm{n}, 1/\alpha\}.$$

Proof. Since f is α-Hölder continuous, there exists a constant C such that, if $s, t \in [0,1]$ with $|t-s| \leq \epsilon$, then $|f(t) - f(s)| \leq C\epsilon^\alpha$. Now, cover $[0,1]$ by no more than $\lfloor 1/\epsilon \rfloor$ intervals of length ϵ. The image of each such interval is contained in a ball of diameter $C\epsilon^\alpha$. One can now

- *either* cover each ball by no more than a constant multiple of $\epsilon^{n\alpha-n}$ balls of diameter ϵ,
- *or* use the fact that subintervals of length $(\epsilon/C)^{1/\alpha}$ in the domain are mapped into balls of diameter ϵ to cover the image inside the ball by a constant multiple of $\epsilon^{1-1/\alpha}$ balls of radius ϵ.

In both cases, look at the cover of the graph consisting of the product of intervals and corresponding balls in $[0,1] \times \mathbb{R}^n$ of diameter ϵ. The first construction needs a constant multiple of $\epsilon^{n\alpha-n-1}$ product sets, the second uses $\epsilon^{-1/\alpha}$ product sets, all of which have diameter of order ϵ. These coverings give the desired upper bounds. □

Proof of b:

Proof of (b). Suppose that $\dim(A) < \beta < \infty$. Then there exists a covering $A_1, A_2, A_3, \ldots$ such that $A \subset \bigcup_j A_j$ and $\sum_j |A_j|^\beta < \varepsilon$. Then $f(A_1), f(A_2), \ldots$ is a covering of $f(A)$, and $|f(A_j)| \leq C|A_j|^\alpha$, where C is the Hölder constant. Thus,

$$\sum_j |f(A_j)|^{\beta/\alpha} \leq C^{\beta/\alpha} \sum_j |A_j|^\beta < C^{\beta/\alpha}\varepsilon \to 0$$

as $\varepsilon \downarrow 0$, and hence $\dim f(A) \leq \beta/\alpha$. ■

Part 4: Energy method

Theorem 2.4. **(Mass distribution principle)** *Suppose E is a metric space and $\alpha \geq 0$. If there is a mass distribution μ on E and constants $C, \delta > 0$ such that $\mu(V) \leq C\,|V|^\alpha$ for all closed sets V with $|V| \leq \delta$, then $\mathcal{H}_\alpha^\delta(E) \geq \mu(E)/C > 0$, and hence* $\dim E \geq \alpha$.

Proof. Let $\{F_i\}$ be a covering of E. WLOG we can take the F_i to be closed since $|\overline{F_i}| = |F_i|$. We have $0 < \mu(E) \leq \mu(\bigcup F_i) \leq \sum \mu(F_i) \leq C \sum |F_i|^\alpha$, and the statement follows. □

The mass distribution principle requires to spread a positive finite mass over a set such that local concentration is bounded from above. The next technique, the *energy method*, is essentially a computational means of measuring the local concentration of the mass.

Alpha potential and alpha energy

4.3 Potential theoretic methods

In this section we introduce a technique for calculating Hausdorff dimensions that is important both in theory and in practice. This replaces the need for estimating the mass of a large number of small sets by a single check for the convergence of a certain integral.

The ideas of potential and energy will be familiar to readers with a knowledge of gravitation or electrostatics. For $s \geqslant 0$ the *s-potential* at a point x of $\mathbb{R}^n$ due to the mass distribution μ on $\mathbb{R}^n$ is defined as

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s}. \tag{4.12}$$

(If we are working in $\mathbb{R}^3$ and $s = 1$ then this is essentially the familiar Newtonian gravitational potential.) The *s-energy* of μ is

$$I_s(\mu) = \int \phi_s(x)\, d\mu(x) = \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^s}. \tag{4.13}$$

The following theorem relates Hausdorff dimension to seemingly unconnected potential theoretic ideas. In particular, if there is a mass distribution on a set F which has finite s-energy, then F has dimension at least s.

The idea of the energy method is that mass distributions with $I_\alpha(\mu) < \infty$ spread the mass so that at each point the concentration is sufficiently small to overcome the singularity of the integrand.

THEOREM 4.27 (Energy method). *Let $\alpha \geq 0$ and μ be a mass distribution on a metric space E. Then, for every $\varepsilon > 0$, we have*

$$\mathcal{H}_\varepsilon^\alpha(E) \geq \frac{\mu(E)}{\iint_{\rho(x,y)<\varepsilon} \frac{d\mu(x)\,d\mu(y)}{\rho(x,y)^\alpha}}.$$

Hence, if $I_\alpha(\mu) < \infty$ then $\mathcal{H}^\alpha(E) = \infty$ and, in particular, $\dim E \geq \alpha$.

Proof

(*a*) Suppose that $I_s(\mu) < \infty$ for some mass distribution μ with support contained in F. Define

$$F_1 = \left\{ x \in F : \overline{\lim_{r \to 0}}\, \mu(B_r(x))/r^s > 0 \right\}.$$

If $x \in F_1$ we may find $\varepsilon > 0$ and a sequence of numbers $\{r_i\}$ decreasing to 0 such that $\mu(B_r(x)) \geqslant \varepsilon r_i^s$. Unless $\mu(\{x\}) > 0$ (in which case it is clear that $I_s(\mu) = \infty$) it follows from the continuity of μ that, by taking q_i $(0 < q_i < r_i)$ small enough, we get $\mu(A_i) \geqslant \frac{1}{4}\varepsilon r_i^s (i = 1, 2, \ldots)$, where A_i is the annulus $B_{r_i}(x) \backslash B_{q_i}(x)$. Taking subsequences if necessary, we may assume that $r_{i+1} < q_i$ for all i, so that the A_i are disjoint annuli centred on x. Hence for $x \in F_1$

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s} \geqslant \sum_{i=1}^{\infty} \int_{A_i} \frac{d\mu(y)}{|x-y|^s}$$

$$\geqslant \sum_{i=1}^{\infty} \tfrac{1}{4} \varepsilon r_i^s r_i^{-s} = \infty$$

since $|x-y|^{-s} \geqslant r_i^{-s}$ on A_i. But $I_s(\mu) = \int \phi_s(x)\, d\mu(x) < \infty$, so $\phi_s(x) < \infty$ for μ-almost all x. We conclude that $\mu(F_1) = 0$. Since $\overline{\lim}_{r \to 0} \mu(B_r(x))/r^s = 0$ if $x \in F \backslash F_1$, the Proposition 4.9(*a*) tells us that, for all $c > 0$, we have

$$\mathscr{H}^s(F) \geqslant \mathscr{H}^s(F \backslash F_1) \geqslant \mu(F \backslash F_1)/c \geqslant (\mu(F) - \mu(F_1))/c = \mu(F)/c.$$

Hence $\mathscr{H}^s(F) = \infty$.

Proof. Suppose that $\{A_n : n = 1, 2, \ldots\}$ is a pairwise disjoint covering of E by sets of diameter $< \varepsilon$. Then

$$\iint_{\rho(x,y)<\varepsilon} \frac{d\mu(x)\, d\mu(y)}{\rho(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \iint_{A_n \times A_n} \frac{d\mu(x)\, d\mu(y)}{\rho(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^\alpha}.$$

Moreover, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mu(E) &\leq \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} |A_n|^{\frac{\alpha}{2}} \frac{\mu(A_n)}{|A_n|^{\frac{\alpha}{2}}} \\
&\leq \sum_{n=1}^{\infty} |A_n|^\alpha \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^\alpha} \leq \mathcal{H}^\alpha_\varepsilon(E) \iint_{\rho(x,y)<\varepsilon} \frac{d\mu(x)\, d\mu(y)}{\rho(x,y)^\alpha}.
\end{aligned}$$

Dividing both sides by the integral gives the stated inequality. If $\mathbb{E} I_\alpha(\mu) < \infty$ the integral converges to zero, so that $\mathcal{H}^\alpha_\varepsilon(E)$ diverges to infinity. ■

REMARK 4.28. To get a lower bound on the dimension from this method it suffices to show finiteness of a single integral. In particular, in order to show for a random set E that $\dim E \geq \alpha$ almost surely, it suffices to show that $\mathbb{E} I_\alpha(\mu) < \infty$ for a (random) measure on E. $\diamond$

Part5 Compute the hausdorff dimension of first dimensional Brownian motion

Theorem 16.4

With probability 1, the graph of a Brownian sample function $X:[0,1]\to\mathbb{R}$ has Hausdorff and box dimension $1\frac{1}{2}$.

Upper bound <=> Lower bound

Proof. From the Hölder condition (16.5) and Corollary 11.2(*a*) it is clear that, with probability 1, graph X has Hausdorff dimension and upper box dimension at most $2-\lambda$ for every $\lambda<\frac{1}{2}$, so has dimensions at most $1\frac{1}{2}$. For the lower estimate, as in the proof of Theorem 16.2,

$$\begin{aligned}
\mathsf{E}((|X(t+h)-X(t)|^2+h^2)^{-s/2}) &= \int_0^\infty (r^2+h^2)^{-s/2}\,\mathrm{d}p(r)\\
&= ch^{-1/2}\int_0^\infty (r^2+h^2)^{-s/2}\exp\left(\frac{-r^2}{2h}\right)\mathrm{d}r\\
&= \tfrac{1}{2}c\int_0^\infty (uh+h^2)^{-s/2}u^{-1/2}\exp\left(\frac{-u}{2}\right)\mathrm{d}u\\
&\leqslant \tfrac{1}{2}c\int_0^h (h^2)^{-s/2}u^{-1/2}\,\mathrm{d}u\\
&\quad+\tfrac{1}{2}c\int_h^\infty (uh)^{-s/2}u^{-1/2}\,\mathrm{d}u\\
&\leqslant c_1 h^{1/2-s}
\end{aligned}$$

So the dimension is at most 3/2

on splitting the range of integration and estimating the integral in two ways. We may lift Lebesgue measure from the t axis to get a mass distribution μ_f on the graph of a function f given by $\mu_f(A) = \mathscr{L}\{t: 0 \leqslant t \leqslant 1 \text{ and } (t, f(t)) \in A\}$. Using Pythagoras's Theorem,

$$\begin{aligned}\mathsf{E}\left(\iint |x-y|^{-s}\,\mathrm{d}\mu_X(x)\,\mathrm{d}\mu_X(y)\right) &= \int_0^1 \int_0^1 \mathsf{E}(|X(t)-X(u)|^2 + |t-u|^2)^{-s/2})\,\mathrm{d}t\,\mathrm{d}u \\ &\leqslant \int_0^1 \int_0^1 c_1 |t-u|^{1/2-s}\,\mathrm{d}t\,\mathrm{d}u \\ &< \infty\end{aligned}$$

if $s < 1\frac{1}{2}$. With probability 1, the mass distribution μ_X on graph X is positive and finite and has finite s-energy, so Theorem 4.13(a) gives $\dim_{\mathrm{H}} \operatorname{graph} X \geqslant 1\frac{1}{2}$. □