

2023春代数学2实验班期中考试

2023.4.16 上午8:00-10:30

1.(5 points) Consider the following diagram of abelian groups:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ a \downarrow & & b \downarrow & & c \downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

Suppose that the two rows are exact. If g and a are surjective and b is injective, show that c is injective.

2.(5 points) Let \mathfrak{p} be a minimal prime ideal of a ring A , show that for every $x \in \mathfrak{p}$ there is some $s \in A \setminus \mathfrak{p}$ and an integer $k \geq 0$ such that $sx^k = 0$.

3. Let A be a commutative ring. Let Σ be the set of all multiplicative closed subsets S of A such that $0 \notin S$.

(1)(5 points) Show that Σ has a maximal element. Show also that S is a maximal element of Σ if and only if $A \setminus S$ is a minimal prime ideal.

(2)(5 points) Show that for every prime ideal \mathfrak{p} there is a minimal prime ideal contained in it.

(3)(5 points) Let $f : A \rightarrow B$ be a ring homomorphism, show that for every prime ideal \mathfrak{p} of A , there is a prime ideal \mathfrak{q} of B such that $\mathfrak{p} = \mathfrak{q} \cap A$.

4. Let A be a commutative ring and M, N be A -modules. Let $u : M \rightarrow N$ be an A -module homomorphism and \mathfrak{p} be a prime ideal of A .

(1)(5 points) Let Q be a finitely generated A -module, then $Q_{\mathfrak{p}} = 0$ if and only if there is some $f \in A \setminus \mathfrak{p}$ such that $fQ = 0$.

(2)(5 points) Suppose that N is finitely generated, and $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is surjective. Show that there is some $f \in A \setminus \mathfrak{p}$ such that $u_f : M_f \rightarrow N_f$ is surjective.

(3)(5 points) Suppose furthermore that M is finitely generated, and N is finitely presented. If $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is bijective. Show that there is some $f \in A \setminus \mathfrak{p}$ such that $u_f : M_f \rightarrow N_f$ is bijective.

- 5.Let A be a commutative ring and M be a finitely presented A -module.
(1)(5 points)Show that for any A -module N and flat A -algebra B , we have

$$\text{Hom}_A(M, N) \otimes B \cong \text{Hom}_B(M \otimes B, N \otimes B)$$

- (2)(5 points)Suppose A is a local ring with maximal ideal \mathfrak{m} , and let N be a flat A -module. Suppose $x_1, \dots, x_n \in N$ are such that their images $\overline{x_1}, \dots, \overline{x_n} \in N/\mathfrak{m}N$ are linearly independent over A/\mathfrak{m} . Show that x_1, \dots, x_n are linearly independent over A .
(3)(5 points)Show that M is projective if and only if for every maximal ideal \mathfrak{m} , $M_{\mathfrak{m}}$ is free.
(4)(5 points)Show that M is projective if and only if it is flat.

- 6.Let $f : A \rightarrow B$ be a flat homomorphism.
(1)(5 points)Let N be an A -module and N_1, N_2 be two submodules. Show that we have an equality (as submodules of $N \otimes B$):

$$(N_1 \cap N_2) \otimes B = (N_1 \otimes B) \cap (N_2 \otimes B)$$

- (2)(5 points)Let $\mathfrak{a}_1, \mathfrak{a}_2$ be ideals of A , show that $(\mathfrak{a}_1 \cap \mathfrak{a}_2)B = (\mathfrak{a}_1 B \cap \mathfrak{a}_2 B)$
(3)(5 points)Suppose furthermore that \mathfrak{a}_2 is finitely generated, show that

$$(\mathfrak{a}_1 : \mathfrak{a}_2)B = \mathfrak{a}_1 B : \mathfrak{a}_2 B$$

- 7.Let M be a finitely generated A -module.
(1)(5 points)Show that $\text{Supp}(M)$ is a closed set of $\text{Spec}(A)$.
(2)(5 points)Suppose that A is Noetherian, and $\text{Supp}(M) \subseteq V(\mathfrak{a})$ for some ideal \mathfrak{a} , show that there is some $n \geq 0$ such that $\mathfrak{a}^n M = 0$.

- 8.(10 points)Let A be a Noetherian ring and M be a finitely generated A -module. Prove that the following are equivalent.

- (1) M is of finite length;
- (2)Every prime ideal $\mathfrak{p} \in \text{Ass}(M)$ is a maximal ideal;
- (3)Every prime ideal $\mathfrak{p} \in \text{Supp}(M)$ is a maximal ideal.

- 9.(5 points)Let A be a ring whose prime ideals are all finitely generated. Show that A is Noetherian.

Hints and Solutions (not official):

For 4(3), try to prove the following lemma:

Let M be finitely generated and N be finitely presented. Suppose $u : M \rightarrow N$ is surjective, then $\text{Ker}(u)$ is finitely generated.

For 6(3), first reduce to the case that \mathfrak{a}_2 is generated by a single element a , then it's clear that $(\mathfrak{a}_1 : (a))B \subseteq \mathfrak{a}_1 B : aB$. Conversely, let $b \in B$ such that $ab = \sum a_i b_i \in \mathfrak{a}_1 B$, by flatness, we have the relation $b = \sum x_j y_j$ and $b_i = \sum x_{ij} y_j$, then $ax_j = \sum a_i x_{ij} \in \mathfrak{a}_1$, hence $x_j \in (\mathfrak{a}_1 : (a))$ and $b \in (\mathfrak{a}_1 : (a))B$.

1. Let γ be any element in $\text{Ker}(c)$, we can find some $\beta \in B$ such that $g(\beta) = \gamma$. Set $\beta' = b(\beta)$, then $g'(\beta') = c(\gamma) = 0$, and by exactness we have $\beta' \in \text{Ker}(g') = \text{Im}(f')$, say $\beta' = f'(\alpha')$. Since a is surjective, there is some $\alpha \in A$ such that $a(\alpha) = \alpha'$. Now note that $b(f(\alpha)) = f'(a(\alpha)) = \beta' = b(\beta)$ and by injectivity, $f(\alpha) = \beta$, then $c = g \circ f(\alpha) = 0$ by exactness.

2. We localize at \mathfrak{p} , then $\mathfrak{p}A_{\mathfrak{p}}$ is the only prime ideal of $A_{\mathfrak{p}}$ by minimality, hence is the nilradical. Now for every $x \in \mathfrak{p}$, $(\frac{x}{1})^k = 0$ for some integer $k \geq 0$, it follows our conclusion.

3.(1) The first claim is a direct corollary from Zorn's lemma. For the latter one, note that $S^{-1}A$ is not the zero ring so there is some prime ideal which doesn't intersect S . Also note that for any prime ideal \mathfrak{p} , $A \setminus \mathfrak{p} \in \Sigma$.

(2) Localize at \mathfrak{p} and use (1)

(3) Localize at \mathfrak{p} , then $B_{\mathfrak{p}}$ is not a zero ring. Take a maximal prime ideal \mathfrak{q}' of $B_{\mathfrak{p}}$ and take $\mathfrak{q} = \mathfrak{q}' \cap B$.

4.(1) Write $Q = Aq_1 + \cdots + Aq_n$, and for each q_i there is some $f_i \in A \setminus \mathfrak{p}$ such that $f_i q_i = 0$. Then set $f = \prod f_i$.

(2) By exactness of localization, $\text{Coker}(u)_{\mathfrak{p}} = 0$ and $\text{Coker}(u)$ is finitely generated. Then there is some $f \in A \setminus \mathfrak{p}$ such that $f \text{Coker}(u) = 0$. It follows that $\text{Coker}(u)_f = 0$ and u_f is surjective.

(3) By (2), there is some $f_c \in A \setminus \mathfrak{p}$ such that u_{f_c} is surjective. Replace A by A_{f_c} , we may assume that u itself is surjective. We claim that $\text{Ker}(u)$ is finitely generated, then a similar argument shows that there is some $f_k \in A \setminus \mathfrak{p}$ such that $\text{Ker}(u)_{f_k} = 0$ and hence u_{f_k} is bijective. The claim is due to the following

lemma.

Lemma. Let A be a ring, M, N be A -modules, and $u : M \rightarrow N$ be a surjective homomorphism. Suppose M is finitely generated and N is finitely presented. Then $\text{Ker}(u)$ is finitely generated.

Proof. Let $0 \rightarrow K' \xrightarrow{i'} A^s \xrightarrow{p'} N \rightarrow 0$ and $A^r \xrightarrow{\tilde{p}} M \rightarrow 0$ be exact sequences where K' is finitely generated. Since free modules are projective, we have the following liftings:

$$\begin{array}{ccc} M & \xrightarrow{u} & N \longrightarrow 0 \\ \nwarrow p & \nearrow (\tilde{p} \circ u) \oplus p' & \\ A^r \oplus A^s & & \end{array} \quad \begin{array}{ccc} A^s & \xrightarrow{p'} & N \longrightarrow 0 \\ \nwarrow \tilde{v} & \nearrow \tilde{p} \circ u & \\ A^r & & \end{array}$$

Let $v = \tilde{v} \oplus id : A^r \otimes A^s \rightarrow A^n$, which is a surjective map. Consider the embedding $i : K = \text{Ker}(p) \rightarrow A^r \oplus A^s$, one can verify that the image of the composition $v \circ i$ has to lie in the kernel of p' , then we get a homomorphism $w : K \rightarrow K'$. It follows that we have commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & A^r \oplus A^s & \xrightarrow{p} & M \longrightarrow 0 \\ & & \downarrow w & & \downarrow v & & \downarrow u \\ 0 & \longrightarrow & K' & \xrightarrow{i'} & A^s & \xrightarrow{p'} & N \longrightarrow 0 \end{array}$$

Then by snake lemma, we have an exact sequence $\text{Ker}(v) \rightarrow \text{Ker}(u) \rightarrow \text{Coker}(w) \rightarrow 0$ since v is surjective. Note that K' is finitely generated, then $\text{Coker}(w)$ is finitely generated. It remains to show that $\text{Ker}(v)$ is finitely generated. This is because the following exact sequence splits:

$$0 \longrightarrow \text{Ker}(v) \longrightarrow A^r \oplus A^s \xrightarrow{v} A^s \longrightarrow 0$$

then we have a projection $A^r \oplus A^s \rightarrow \text{Ker}(v)$.

5.(1) Choose a finite presentation of M :

$$A^r \rightarrow A^s \rightarrow M \rightarrow 0$$

then the following sequences are exact:

$$0 \rightarrow \text{Hom}_A(M, N) \otimes B \rightarrow \text{Hom}_A(A^s, N) \otimes B \rightarrow \text{Hom}_A(A^r, N) \otimes B$$

$$0 \rightarrow \text{Hom}_B(M \otimes B, N \otimes B) \rightarrow \text{Hom}_B(A^s \otimes B, N \otimes B) \rightarrow \text{Hom}_B(A^r \otimes B, N \otimes B)$$

Then we use the canonical isomorphisms

$$\text{Hom}_A(A^n, N) \otimes B \cong N^n \otimes B \cong (N \otimes B)^n$$

$$\text{Hom}_B(A^n \otimes B, N \otimes B) \cong \text{Hom}_B(B^n, N \otimes B) \cong (N \otimes B)^n$$

(2) See notes in class. (Use induction on n .)

(3) “ \Rightarrow ” Since M is projective, M is flat, so $M_{\mathfrak{m}}$ is flat for every maximal ideal \mathfrak{m} . Then it is free since it is finitely generated over a local ring.
“ \Leftarrow ” Let $N \rightarrow N'$ be any surjective map, it suffices to show that $\text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N')$ is surjective. Take $B = A_{\mathfrak{m}}$ in (1), we see that $(\text{Hom}_A(M, N))_{\mathfrak{m}} = \text{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}})$, so by local property, it remains to show that $\text{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \rightarrow \text{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N'_{\mathfrak{m}})$ surjective for each \mathfrak{m} . This follows from that $A_{\mathfrak{m}}$ is free hence projective.

(4) Recall that M is flat if and only if $M_{\mathfrak{m}}$ is flat, that is free, for every maximal ideal \mathfrak{m} .

6.(1) From $0 \rightarrow N_i \rightarrow N \rightarrow N/N_i \rightarrow 0$, we have

$$0 \rightarrow N_i \otimes B \rightarrow N \otimes B \rightarrow N/N_i \otimes B \rightarrow 0$$

then $N \otimes B/N_i \otimes B \cong (N/N_i) \otimes B$. Now use the canonical exact sequence $0 \rightarrow N_1 \cap N_2 \rightarrow N \rightarrow (N/N_1) \times (N/N_2)$ and get

$$0 \rightarrow N_1 \cap N_2 \otimes B \rightarrow N \otimes B \rightarrow (N/N_1) \times (N/N_2) \otimes B$$

Since $(N/N_1) \times (N/N_2) \otimes B = (N \otimes B/N_1 \otimes B) \times (N \otimes B/N_2 \otimes B)$, then $(N_1 \cap N_2) \otimes B = (N_1 \otimes B) \cap (N_2 \otimes B)$.

(2) Recall that since B is flat, $\mathfrak{a} \otimes B \cong \mathfrak{a}B$.

(3) Since $(\mathfrak{a}_1 : (a) + (b)) = (\mathfrak{a}_1 : (a)) \cap (\mathfrak{a}_1 : (b))$, by (2), we may assume \mathfrak{a}_2 is generated by a single element a . If $am \in \mathfrak{a}_1$ for some $m \in A$, it is clear that $amb \in \mathfrak{a}_1 B$ for every $b \in B$ hence $(\mathfrak{a}_1 : (a))B \subseteq \mathfrak{a}_1 B : ab$. Conversely, let $b \in B$ such that $ab = \sum a_i b_i \in \mathfrak{a}_1 B$, by flatness, we have the relations $b = \sum x_j y_j$ and $b_i = \sum x_{ij} y_j$ with $x_j, x_{ij} \in A$, then $ax_j = \sum a_i x_{ij} \in \mathfrak{a}_1$, hence $x_j \in (\mathfrak{a}_1 : (a))$ and $b \in (\mathfrak{a}_1 : (a))B$.

7.(1) Suppose $M = Am_1 + \cdots + Am_n$, then $\text{Supp}(M) = \bigcup \text{Supp}(Am_i) = \bigcup V(\text{Ann}(m_i))$ is closed.

(2) By the given condition, $\sqrt{\text{Ann}(m_i)} \supseteq \sqrt{I}$. Hence

$$I \subseteq \sqrt{I} \subseteq \bigcap \sqrt{\text{Ann}(m_i)} = \sqrt{\bigcap \text{Ann}(m_i)}$$

Since A is noetherian, we can assume I is generated by (a_1, \dots, a_k) and $a_j^{r_j} \in \bigcap \text{Ann}(m_i)$. Hence $I^{\sum r_j} M = 0$. (or there is some positive integer n such that $I^n \subseteq \sqrt{\bigcap \text{Ann}(m_i)}^n \subseteq \bigcap \text{Ann}(m_i)$.)

8.(1) \Rightarrow (2) As a submodule of M , A/\mathfrak{p} is of finite length for each $\mathfrak{p} \in \text{Ass}(M)$. Note that each A -submodule of A/\mathfrak{p} is an A/\mathfrak{p} -module, A/\mathfrak{p} is both Noetherian and Artinian as a ring, so $\dim(A/\mathfrak{p}) = 0$ and (0) is the only prime ideal of A/\mathfrak{p} , so \mathfrak{p} is maximal.

(2) \Rightarrow (3) Every minimal element in $\text{Supp}(M)$ is in $\text{Ass}(M)$

(3) \Rightarrow (2) $\text{Ass}(M) \subseteq \text{Supp}(M)$

(2) \Rightarrow (1) By definition of associated primes, we have a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each subquotient $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \text{Ass}(M)$. Note that A/\mathfrak{p}_i is a field hence is a simple A -module, this gives a finite composition series of M .

9. This is exactly the first exercise in chapter 7 whose hints give enough details.