数学模型第三次作业

2019 年 4 月 6 日

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1.1

考虑方程

$$\rho_t + \rho_x + 6\rho\rho_x - \rho_{xxt} = 0, \tag{1}$$

令

$$\rho(x,t) = P(x-ct) = P(s), \quad s = x - ct,$$

有

$$\rho_t = -cP', \quad \rho_x = P',$$

代入 (1),

$$(6P + 1 - c)P' + cP''' = 0. \tag{2}$$

1.2

$$\begin{aligned}
&P(s) = A\mathrm{sech}^2(Bs),\\
&P'(s) = -2AB\tanh(Bs)\mathrm{sech}^2(Bs),\\
&P''(s) = 6AB^2\tanh^2(Bs)\mathrm{sech}^2(Bs) - 2AB^2\mathrm{sech}^2(Bs),\\
&P'''(s) = 16AB^3\tanh(Bs)\mathrm{sech}^2(Bs) - 24AB^3\tanh^3(Bs)\mathrm{sech}^2(Bs).
\end{aligned} \tag{3}$$

将 (3) 代入 (2), 并利用 $\cosh^2(x) - \sinh^2(x) = 1$ 可知

$$\begin{cases}
(1+c) + 4cB^2 = 0,\\
6A + 1 - 8cB^2 - c = 0,
\end{cases}$$

解得

$$\begin{cases} A = \dfrac{1}{2}(c-1), \\ B^2 = \dfrac{c-1}{4c}. \end{cases}$$

2

2.1

考虑方程

$$\frac{\partial \rho}{\partial t} + e^{2t}\frac{\partial \rho}{\partial x} = \rho + x + t, \quad \rho(x, t=0) = \cos x,$$

设 $X = X(t)$, $P(t) = \rho(X(t), t)$, 满足

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = e^{2t}, \\ X(0) = X_0, \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = P + x + t, \\ P(0) = \cos(X_0), \end{cases}$$

求解常微分方程, 有

$$X = \frac{1}{2}\left(e^{2t} - 1\right) + X_0. \tag{4}$$

而

$$\begin{aligned} P &= \cos(X_0)e^t + e^t\int_0^t \left(\frac{1}{2}\left(e^{2t}-1\right) + X_0 + t\right)e^{-t}dt \\ &= \left(\cos(X_0) + X_0\right)e^t - \left(X_0 + t + \frac{1}{2}\right) + X_0 + \frac{1}{2}e^{2t}, \end{aligned} \tag{5}$$

将 (4) 代入 (5) 有,

$$\rho(x,t) = \left(\cos(x - \frac{1}{2}\left(e^{2t}-1\right)) + x - \frac{1}{2}\left(e^{2t}-1\right)\right)e^t - (x+t+1) + e^{2t}.$$

2.2

考虑方程

$$\frac{\partial \rho}{\partial t} + (x+4)\frac{\partial \rho}{\partial x} = -2\rho, \quad \rho(x=0,t) = \cos t, \quad \rho(x, t=0) = e^{-x}, \quad x \geqslant 0,$$

设 $X = X(t)$, $P(t) = \rho(X(t), t)$, 则

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = X + 4, \\ X(0) = X_0 \text{ or } X(t_0) = 0, \end{cases}$$

故 **1)** 当 $x > 4e^t - 4$ 时,

$$X = (4 + X_0)e^t - 4. \tag{6}$$

$P(t)$ 满足

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = -2P, \\ P(0) = e^{-X_0}, \end{cases}$$

解得

$$P = e^{-2t-X_0},$$

将 (6) 代入可得,

$$\rho(x,t) = e^{-(x+4)e^{-t}+4-2t}, \quad x > 4e^t - 4.$$

2) 当 $x < 4e^t - 4$ 时,

$$X = 4e^{t-t_0} - 4. \tag{7}$$

$P(t)$ 满足

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = -2P, \\ P(t_0) = \cos t_0, \end{cases}$$

解得

$$P = \cos t_0 e^{-2(t-t_0)},$$

将 (7) 代入可得,

$$\rho(x,t) = \cos\left(t - \ln(\frac{1}{4}x + 1)\right) e^{-2\ln(\frac{1}{4}x+1)}.$$

3

3.1

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0, \\ &\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \cdot \left(\frac{1}{\rho}\boldsymbol{j} \otimes \boldsymbol{j} + pI\right) = 0, \end{aligned} \tag{8}$$

而

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{\partial \rho}{\partial t}\boldsymbol{v} + \rho\frac{\partial \boldsymbol{v}}{\partial t} \tag{9}$$

将 (9) 代入 (8) 并利用整理 $\nabla \cdot \boldsymbol{v} = 0$ 可得

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p,$$

即

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{\nabla p}{\rho}.$$

3.2

设 $f(x) = u(x, 0)$, $X = X(t)$, $U(t) = u(X(t), t)$, 则

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = U, \\ X(0) = X_0, \end{cases} \quad \begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = 0, \\ U(0) = f(X_0), \end{cases}$$

解得

$$U(t, X_0) = f(X_0),$$
$$X = f(X_0)t + X_0.$$

将 $f(x)$ 的具体表达式代入有, 当 $t < 1$ 时

$$\begin{cases} X = X_0, & U = 0, & X_0 > 0, \\ X = -X_0 t + X_0, & U = -X_0, & -1 < X_0 < 0, \\ X = t + X_0, & U = 1, & X_0 < -1, \end{cases}$$

即

$$u(x, t) = \begin{cases} 0, & x > 0, \\ \dfrac{x}{t - 1}, & t - 1 < x < 0, \\ 1, & x < t - 1. \end{cases}$$

当 $t = 1$ 时,

$$u(x, 1) = \begin{cases} 0, & x > 0, \\ 1, & x < 0. \end{cases}$$

4

证明: 当 $x < 0$ 时，我们有 $\rho > \theta$，所以方程约化为

$$cv' + v'' + \mu(1 - v) = 0.$$

由于 $v(-\infty)=1$ (蕴含有界性)，我们得到下面的解

$$v(x)=1-(1-\theta)e^{\lambda_+ x}, \quad x<0,$$

其中

$$\lambda_+=\frac{1}{2}\left(-c+\sqrt{c^2+4\mu}\right).$$

而当 $x>0$ 时，我们有 $\rho<\theta$，所以方程约化为

$$cv'+v''=0.$$

由于 $v(+\infty)=0$ ，我们得到下面的解

$$v=\theta e^{-cx}, \quad x\geqslant 0.$$

注意到，由于 f 在 θ 处不连续，所以 v'' 在 0 处有一个间断跳跃 (这蕴含了 v' 在 0 处连续)，所以我们得到

$$(1-\theta)\lambda_+=\theta c.$$

整理可得, $-c(\theta+1)+(1-\theta)\sqrt{c^2+4\mu}=0$, 进而得到 $c>0$ 且

$$c^2(\theta+1)^2=(1-\theta)^2(c^2+4\mu),$$

解得 $c_1=(1-\theta)\sqrt{\mu/\theta}>0$, $c_2=-(1-\theta)\sqrt{\mu/\theta}<0$(舍)，故满足条件的波速 $c^*>0$ 可以被唯一确定. 对 v 求导可得

$$v'=\begin{cases} -\lambda_+(1-\theta)e^{\lambda_+ x}, & x<0, \\ -c\theta e^{-cx}, & x>0, \end{cases}$$

故 $v(x)$ 是一个单调递减函数.

5

5.1

$$
\begin{aligned}
\int_0^1 \frac{\delta J}{\delta y}(x)h(x)\,\mathrm{d}x &= \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}J[y+\epsilon h]\right]_{\epsilon=0} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_0^1 \frac{1}{2}\left(y'+\epsilon h'\right)^2 + \frac{k}{x}(y+\epsilon h)(y'+\epsilon h') + x^2(y+\epsilon h)\,\mathrm{d}x\right]_{\epsilon=0} \\
&= \int_0^1 y'h' + \frac{k}{x}hy' + \frac{k}{x}h'y + x^2 h\,\mathrm{d}x \\
&= \int_0^1 \left(-y'' + \frac{k}{x^2}y + x^2\right) h\,\mathrm{d}x,
\end{aligned}
$$

(推导过程中用到了分部积分) 故

$$
\frac{\delta J}{\delta y_*} = -y_*'' + \frac{k}{x^2}y_* + x^2 = 0.
$$

5.2

记

$$
L = \frac{1}{2}\left(y'\right)^2 + \frac{k}{x}yy' + x^2 y,
$$

由 Euler-Lagrangian 方程, y_* 满足

$$
\frac{\delta J}{\delta y} = \frac{\partial L}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y'}\right) = -y'' + \frac{k}{x^2}y + x^2 = 0.
$$

5.3

由 $k=0$ 得 $y''=x^2$, 解得

$$
y = \frac{1}{12}x^4 + C_1 x + C_2,
$$

由边界条件 $y(0)=1$, $y(1)=1$, 得 $C_1 = -\frac{1}{12}$, $C_2 = 1$.

5.4

考虑 $-y'' + \frac{2}{x^2}y = 0$, 寻找形如 $y = x^\alpha$ 的特解. 将 $y = x^\alpha$ 代入可得 $\alpha(\alpha-1)-2=0$, 即 $\alpha = 2, -1$, 故

$$
y = C_1 x^2 + C_2 x^{-1}
$$

是 $-y''+\frac{2}{x^2}y=0$ 的通解. 而 $y=\frac{1}{10}x^4$ 是 $x^2-y''+\frac{2}{x^2}y=0$ 的特解, 故

$$y=\frac{1}{10}x^4+C_1x^2+C_2x^{-1}$$

是 $x^2-y''+\frac{2}{x^2}y=0$ 的通解. 由边界条件 $y'(0)=0$, $y(1)=1$, 得 $C_1=\frac{9}{10}$, $C_2=0$, 不满足 $y(0)=1$.

6

6.1

$$\delta^2 J(h,y)=\int_0^1\left(\frac{h'^2}{2\left[1+(y')^2\right]^{3/2}}\right)\mathrm{d}x.$$

6.2

$$\delta^2 J=\int_0^1\left(\frac{h'^2}{2\left[1+(y')^2\right]^{3/2}}\right)\mathrm{d}x\geqslant 0.$$

故 $\delta^2 J=0$ 当且仅当 $h'=0$, 由边界条件 $h(0)=h(1)=0$ 知 $\delta^2 J=0$ 当且仅当 $h=0$.