1. (1) $\Omega_0 = \left\{ \begin{pmatrix} 1&1&1\\0&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&1\\0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\1&1&0\\0&1&1 \end{pmatrix} \right\}$. (2) $\begin{pmatrix} 1&&\\1&1&\\&1&1 \end{pmatrix}$, $\begin{pmatrix} 0&&\\&1&\\&&2 \end{pmatrix}$, $\begin{pmatrix} 2&&\\&e^{\frac{\pi i}{3}}&\\&&e^{-\frac{\pi i}{3}} \end{pmatrix} = \begin{pmatrix} 2&&\\&\frac{1+\sqrt{3}i}{2}&\\&&\frac{1-\sqrt{3}i}{2} \end{pmatrix}$.

(1)的证明: 记$\mathbb{C}^{3\times 1}$的标准基为$\{\epsilon_1, \epsilon_2, \epsilon_3\}$. 若$A \in \Omega$的某列有两个0, 则存在指标$1, 2, 3$的排列$i, j, k$, 满足以下两个条件之一:
- $A\epsilon_i = \epsilon_i$, $A\epsilon_j = \epsilon_i + \epsilon_j$, $A\epsilon_k = \epsilon_i + \epsilon_j + \epsilon_k$, 此时$A$相似于$\Omega_0$中的第一个矩阵;
- $A\epsilon_i = \epsilon_i$, $A\epsilon_j = \epsilon_j + \epsilon_k$, $A\epsilon_k = \epsilon_i + \epsilon_j + \epsilon_k$, 此时$A$相似于$\Omega_0$中的第二个矩阵.

若$A \in \Omega$的某行有两个0, 则A^t的某列有两个0. 由于A相似于A^t, 所以相似于Ω_0中的前两个矩阵之一. 若A的每行每列均恰好有一个0, 则存在指标$1, 2, 3$的排列r, s, t, 满足$A\epsilon_r = \epsilon_r + \epsilon_s$, $A\epsilon_s = \epsilon_s + \epsilon_t$, $A\epsilon_t = \epsilon_t + \epsilon_r$. 此时$A$相似于$\Omega_0$中的第三个矩阵. □

2. (1) 取V的有序基$\mathcal{B}$使$[T]_{\mathcal{B}} = \begin{pmatrix} a_1 && \\ & \ddots & \\ && a_n \end{pmatrix}$, 其中$n = \dim V$. 对$i, j \in \{1, \dots, n\}$, 取$S_{ij} \in \mathcal{L}(V)$满足$[S_{ij}]_{\mathcal{B}} = E_{ij}$. 则$[\Phi_T(S_{ij})]_{\mathcal{B}} = [T]_{\mathcal{B}}[S_{ij}]_{\mathcal{B}}[T]_{\mathcal{B}} = a_i a_j [S_{ij}]_{\mathcal{B}}$. 这说明$\Phi_T(S_{ij}) = a_i a_j S_{ij}$. 特别地, S_{ij}为Φ_T的特征向量. 由于$\{S_{ij} \mid i, j \in \{1, \dots, n\}\}$为$\mathcal{L}(V)$的基, 所以$\Phi_T$可对角化.

(2) 我们需要证明: 存在线性无关的$S_1, S_2 \in \mathcal{L}(V)$满足$TS_iT = \det(T)S_i$. 设$T$在$V$的某有序基下矩阵为$A = \begin{pmatrix} a&b\\c&d \end{pmatrix}$. 则只需证明: 存在线性无关的$B_1, B_2 \in F^{2\times 2}$满足$AB_iA = \det(A)B_i$. 不妨设$A$与$I_2$线性无关. 考虑$\operatorname{adj}(A) = \begin{pmatrix} d&-b\\-c&a \end{pmatrix}$. 由于

$$\begin{pmatrix} &1\\-1& \end{pmatrix} \operatorname{adj}(A) \begin{pmatrix} &1\\-1& \end{pmatrix}^{-1} = A^t,$$

所以$\operatorname{adj}(A)$与A^t相似, 从而也与A相似, 即存在可逆矩阵$B_1 \in F^{2\times 2}$满足$B_1AB_1^{-1} = \operatorname{adj}(A)$. 因此

$$AB_1A = A(B_1AB_1^{-1})B_1 = A\operatorname{adj}(A)B_1 = \det(A)B_1.$$

取$B_2 = AB_1$. 则$AB_2A = \det(A)B_2$, 并且由B_1可逆和A与I_2的线性无关性, 易得B_1与B_2线性无关. □

(2) 证明*2*. 只需证明: 对$A, B \in F^{2\times 2}$, 若$\operatorname{tr}(AB) = \operatorname{tr}(B) = 0$, 则$ABA = \det(A)B$. 由Cayley-Hamilton定理, 对任意$X \in F^{2\times 2}$有$X^2 - \operatorname{tr}(X)X + \det(X)I_2 = 0$. 所以对任意$A, B \in F^{2\times 2}$有

$$ABAB = \operatorname{tr}(AB)AB - \det(A)\det(B)I_2 = \operatorname{tr}(AB)AB - \det(A)(\operatorname{tr}(B)B - B^2) = \operatorname{tr}(AB)AB - \det(A)\operatorname{tr}(B)B + \det(A)B^2.$$

这推出当B可逆时, 有

$$ABA = \operatorname{tr}(AB)A - \det(A)\operatorname{tr}(B)I_2 + \det(A)B. \tag{1}$$

由于(1)式两边关于B线性, 而可逆矩阵的集合线性生成$F^{2\times 2}$, 所以(1)式对任意$A, B \in F^{2\times 2}$总成立. 特别地, 当$\operatorname{tr}(AB) = \operatorname{tr}(B) = 0$时, 有$ABA = \det(A)B$. □

3. 记$R = F[x]$. 先证明: 若$W = R\alpha$为V的循环子空间, p为素多项式, 则

$$\dim \operatorname{Ker}(p(T_W)) = \begin{cases} \deg p, & p \mid p_{T_W}, \\ 0, & p \nmid p_{T_W}. \end{cases}$$

- 假设$p \mid p_{T_W}$. 记$f = p_{T_W}/p$, $\beta = f\alpha$. 我们断言$\operatorname{Ker}(p(T_W)) = R\beta$. 首先, 若$\gamma \in \operatorname{Ker}(p(T_W))$, 设$\gamma = g\alpha$, 则$pg\alpha = p\gamma = 0$, 所以$p_\alpha \mid pg$, 而$p_\alpha = p_{T_W} = pf$, 因此$f \mid g$, 这推出$\gamma = g\alpha = (g/f)\beta \in R\beta$. 反过来, 若$\delta \in R\beta$, 设$\delta = h\beta = hf\alpha$, 则$p\delta = phf\alpha = hp_{T_W}\alpha = 0$, 所以$\delta \in \operatorname{Ker}(p(T_W))$. 这就验证了断言. 另一方面有$p_\beta = p$. 因此$\dim \operatorname{Ker}(p(T_W)) = \dim R\beta = \deg p_\beta = \deg p$.
- 假设$p \nmid p_{T_W}$. 则存在$a, b \in R$满足$ap + bp_{T_W} = 1$. 设$\gamma \in \operatorname{Ker}(p(T_W))$, 则$p\gamma = 0 = p_{T_W}\gamma$, 从而$\gamma = ap\gamma + bp_{T_W}\gamma = 0$. 所以$\operatorname{Ker}(p(T_W)) = \{0\}$.

现在证明原题. 设$W \subset V$为不变子空间. 考虑循环分解$W = \bigoplus_{i=1}^k W_i$, 其中$T_{W_i}$循环, 并且$p_{T_{W_k}} | \cdots | p_{T_{W_1}}$. 取$p_{T_{W_k}}$的素因式$p$. 则

$$\dim \operatorname{Ker}(p(T_W)) = \dim \bigoplus_{i=1}^k \operatorname{Ker}(p(T_{W_i})) = \sum_{i=1}^k \dim \operatorname{Ker}(p(T_{W_i})) = k \deg p.$$

另一方面, 设$V = \bigoplus_{i=1}^m V_i$, 其中每个$V_i$为循环子空间, 则

$$\dim \operatorname{Ker}(p(T)) = \dim \bigoplus_{i=1}^m \operatorname{Ker}(p(T_{V_i})) = \sum_{i=1}^m \dim \operatorname{Ker}(p(T_{V_i})) \leqslant m \deg p.$$

由于$\operatorname{Ker}(p(T_W)) \subset \operatorname{Ker}(p(T))$, 所以$k \leqslant m$. □

4. 证明*1*. 考虑外直和$W := \bigoplus_{i=1}^k W_i$, 并定义$S \in \mathcal{L}(W)$为$S(\alpha_1, \dots, \alpha_k) = (T\alpha_1, \dots, T\alpha_k)$. 由于$\operatorname{Ker}(S) = \bigoplus_{i=1}^k (\operatorname{Ker}(T) \cap W_i)$, 所以只需证$\dim \operatorname{Ker}(T) \leqslant \dim \operatorname{Ker}(S)$. 考虑满映射$\Phi \in \mathcal{L}(W, V)$, $\Phi(\alpha_1, \dots, \alpha_k) = \sum_{i=1}^k \alpha_i$. 则$\Phi S = T\Phi$. 考虑映射$S|_{\operatorname{Ker}(\Phi S)} : \operatorname{Ker}(\Phi S) \to W$. 它的核为$\operatorname{Ker}(S)$, 像为$\operatorname{Ker}(\Phi) \cap \operatorname{Im}(S)$, 所以

$$\dim \operatorname{Ker}(\Phi S) = \dim \operatorname{Ker}(S) + \dim(\operatorname{Ker}(\Phi) \cap \operatorname{Im}(S)).$$

类似地,

$$\dim \operatorname{Ker}(T\Phi) = \dim \operatorname{Ker}(\Phi) + \dim(\operatorname{Ker}(T) \cap \operatorname{Im}(\Phi)) = \dim \operatorname{Ker}(\Phi) + \dim \operatorname{Ker}(T).$$

由于$\Phi S = T\Phi$, 所以

$$\dim \mathrm{Ker}(T) = \dim \mathrm{Ker}(S) + \dim(\mathrm{Ker}(\Phi) \cap \mathrm{Im}(S)) - \dim \mathrm{Ker}(\Phi) \leqslant \dim \mathrm{Ker}(S).$$

□

证明2. 由归纳法, 不妨设$k = 2$. 则$T(W_1) + T(W_2) = T(V)$, $T(W_1) \cap T(W_2) \subset W_1 \cap W_2$. 于是

$$\begin{aligned} &\dim(\mathrm{Ker}(T) \cap W_1) + \dim(\mathrm{Ker}(T) \cap W_2) - \dim \mathrm{Ker}(T) \\ = &(\dim W_1 - \dim T(W_1)) + (\dim W_2 - \dim T(W_2)) - (\dim V - \dim T(V)) \\ = &(\dim W_1 + \dim W_2 - \dim V) - (\dim T(W_1) + \dim T(W_2) - \dim T(V)) \\ = &\dim(W_1 \cap W_2) - \dim(T(W_1) \cap T(W_2)) \\ \geqslant &0. \end{aligned}$$

□

证明3. 由归纳法, 不妨设$k = 2$. 取$W_1 \cap W_2$的有序基$\mathcal{B}_0$, 并扩充为W_i的有序基$\mathcal{B}_0 \cup \mathcal{B}_i$, $i = 1, 2$. 则$\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$为$V$的有序基. 设$[T_{W_1 \cap W_2}]_{\mathcal{B}_0} = A$. 则$[T_{W_i}]_{\mathcal{B}_0 \cup \mathcal{B}_i}$形如$\left(\begin{smallmatrix} A & B_i \\ 0 & C_i \end{smallmatrix}\right)$, 进而$[T]_{\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2} = \left(\begin{smallmatrix} A & B_1 & B_2 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{smallmatrix}\right)$. 于是

$$\begin{aligned} &\dim(\mathrm{Ker}(T) \cap W_1) + \dim(\mathrm{Ker}(T) \cap W_2) - \dim \mathrm{Ker}(T) \\ = &(\dim W_1 - \mathrm{rank}[T_{W_1}]_{\mathcal{B}_0 \cup \mathcal{B}_1}) + (\dim W_2 - \mathrm{rank}[T_{W_2}]_{\mathcal{B}_0 \cup \mathcal{B}_2}) - (\dim V - \mathrm{rank}[T]_{\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2}) \\ = &|\mathcal{B}_0| + \mathrm{rank} \left(\begin{smallmatrix} A & B_1 & B_2 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{smallmatrix}\right) - \mathrm{rank} \left(\begin{smallmatrix} A & B_1 \\ 0 & C_1 \end{smallmatrix}\right) - \mathrm{rank} \left(\begin{smallmatrix} A & B_2 \\ 0 & C_2 \end{smallmatrix}\right) \\ \geqslant &|\mathcal{B}_0| + \mathrm{rank}(C_2) - \mathrm{rank} \left(\begin{smallmatrix} A & B_2 \\ 0 & C_2 \end{smallmatrix}\right) \\ \geqslant &0. \end{aligned}$$

□