北京大学数学科学学院期末试题参考答案

2020 – 2021 学年第 1 学期

考试科目：拓扑学　　　考试时间：2021 年 1 月 13 日

1. 设 $f, g : X \to Y$ 是连续映射，其中 Y 是 Hausdorff 空间。证明 $\{x \in X \mid f(x) = g(x)\}$ 是 X 中闭集。

Proof. $A := \{x \in X \mid f(x) = g(x)\}$ is the compliment of $B := \{x \in X \mid f(x) \neq g(x)\}$. It suffices to show that B is open. For any $x \in B$, $f(x)$ and $g(x)$ are distinct and hence has disjoint open neighborhoods U and V since Y is hausdorff. $W = f^{-1}(U) \cap g^{-1}(V)$ is an open neighborhood of x. For any $z \in W$, we have $f(z) \in U$ and $g(z) \in V$, so $f(z) \neq g(z)$ and $z \in B$. We have $z \in W \subset B$. Hence $z \in B^o$. It follows that B is open and A is closed. □

2. 设 X 是可数的拓扑空间（即 X 作为集合是可数的）。若 X 是正则空间，证明 X 是正规空间。

Proof. Let A and B be disjoint closed subsets in X. For any x in A, the open subset B^c contains x. Since X is regular, there exists an open neighborhood U of x such that $\overline{U} \subset B^c$. Since A is (at most) countable, we have countably many open subsets $U_1, U_2, \cdots$ such that

$$A \subset \bigcup_{n=1}^{\infty} U_n, \qquad \overline{U}_n \subset B^c$$

By same arguments, there are countably many open subsets $V_1, V_2, \cdots$ such that

$$B \subset \bigcup_{n=1}^{\infty} V_n, \qquad \overline{V}_n \subset A^c$$

We then have the collection of open sets

$$U' = U_n \setminus \bigcup_{k=1}^{n} \overline{V}_k \qquad \text{and} \qquad V' = V_n \setminus \bigcup_{k=1}^{n} \overline{U}_k$$

Since A is disjoint from each $\overline{V}_k$, we have $A \cap U'_n = A \cap U_n$, and therefore

$$A \subset U := \bigcup_{n=1}^{\infty} U'_n \qquad \text{and similarly} \qquad B \subset V := \bigcup_{n=1}^{\infty} V'_n$$

If $U \cap V$ is nonempty and $x \in U \cap V$, then x belongs to some U'_m and some V'_n. If $m \leqslant n$, then

$$x \in U'_n \subset U_n \qquad \text{and} \qquad x \in V'_n = V_n \setminus \bigcup_{k=1}^{n} \overline{U}_k$$

which is clearly a contradiction; we reach a similar contradiction if $n \leqslant m$. Hence U and V are disjoint, and so X is normal. □

Remark. One may want to show that X is second countable, and then apply Lindelöf theorem to show the normality of X. However, a countable topological space is not necessarily second countable. For example, the one point compactification of rationals is not second countable, and it is even not first countable.

3. 设连续映射 $f: S^2 \to S^2$ 对 $\forall x \in S^2$ 满足 $f(x) \neq f(-x)$，证明 f 是满射。

Proof. Suppose, on the contrary, that f is not surjective. Let $x_0 \in S^2 \setminus f(S^2)$. The stereographic projection p through x_0 gives a homeomorphism from $S^2 \setminus \{x_0\}$ to $\mathbb{E}^2$. Since $f(S^2) \subset S^2 \setminus \{x_0\}$, $p \circ f : S^2 \to \mathbb{E}^2$ is well-defined and continuous. The Borsuk-Ulam theorem implies that there exists $x, -x \in S^2$ such that $p \circ f(x) = p \circ f(-x)$. Since p is a homeomorphism, this implies that $f(x) = f(-x)$, contradicting the assumption that $f(x) \neq f(-x)$ for all x in S^2. Hence f is surjective. □

4. 计算 $\mathbb{E}^3$ 中除去过原点的 n 条不同的直线的基本群。

Proof. Let X be the compliment of n distinct lines through the origin in $\mathbb{E}^3$. The homotopy

$$H(x,t) = ((1-t)\|x\| + t)\frac{x}{\|x\|}$$

gives a strong deformation retraction of X to the unit sphere with $2n$ points removed. The $2n$ points are the intersections of the lines with the unit sphere, and the deformation retraction is along the rays from the origin.

By stereographic projection through one of the $2n$ points, the unit sphere with $2n$ points removed is homeomorphic to $\mathbb{E}^2$ with $2n-1$ points removed, which can strongly deformation retracts to a wedge of $(2n-1)$ circles. By the Seifert-van Kampen theorem, we have $\pi_1(X) \cong F_{2n-1}$, the group freely generated by $2n-1$ elements. □

5. 设 $X = \{a, b, c, d\}$，取如下拓扑基生成的拓扑

$$\Big\{ \{a\}, \{c\}, \{a,b,c\}, \{a,c,d\} \Big\}$$

证明 X 道路连通但不单连通。

Proof. Define

$$\alpha : [0,1] \to X, \qquad \alpha(t) = \begin{cases} a, & 0 \leqslant t < 1, \\ b, & t = 1. \end{cases}$$

$$\beta : [0,1] \to X, \qquad \alpha(t) = \begin{cases} b, & t = 0, \\ c & 0 < t \leqslant 1. \end{cases}$$

$$\gamma : [0,1] \to X, \qquad \alpha(t) = \begin{cases} c, & 0 \leqslant t < 1, \\ d, & t = 1. \end{cases}$$

$$\delta : [0,1] \to X, \qquad \alpha(t) = \begin{cases} d, & t = 0, \\ a & 0 < t \leqslant 1. \end{cases}$$

For the continuity of these paths, it suffices to show that the preimages of the base open sets are open for each path, which is clear from the following table of preimages

	$\{a\}$	$\{c\}$	$\{a,b,c\}$	$\{a,c,d\}$
α	$[0,1)$	$\emptyset$	$[0,1]$	$[0,1)$
β	$\emptyset$	$(0,1]$	$[0,1]$	$(0,1]$
γ	$\emptyset$	$[0,1)$	$[0,1)$	$[0,1]$
δ	$(0,1]$	$\emptyset$	$(0,1]$	$[0,1]$

It follows that X is path connected.

The loop $\alpha.\beta.\gamma.\delta$ is not null homotopic, but it is not easy to prove. We prove the nontriviality of $\pi_1(X)$ via covering spaces.

Define $\widetilde{X} = \{a_n, b_n, c_n, d_n \,|\, n \in \mathbb{Z}\}$, with topology induced by the basis

$$\Big\{\{a_n\}, \{c_n\}, \{a_n, b_n, c_n\}, \{c_{n-1}, d_{n-1}, a_n\} \;\Big|\; n \in \mathbb{Z}\Big\}$$

By the same reason as above, $\widetilde{X}$ is path connected. The function $p : \widetilde{X} \to X$ with

$$p(a_n) = a, \quad p(b_n) = b, \quad p(c_n) = c, \quad p(d_n) = d, \qquad \forall n \in \mathbb{Z}$$

is surjective and continuous. p is actually a covering space. It is regular and $\mathcal{D}(\widetilde{X}) = \mathbb{Z}$. By the classification of covering spaces, we have $\mathbb{Z} \cong \mathcal{D}(\widetilde{X}) \cong N(H)/H$, where $H = p_*(\pi_1(\widetilde{X})) < \pi_1(X)$. In particular, the normal closure $N(H)$ of H is nontrivial, which implies $\pi_1(X)$ is nontrivial. Therefore X is not simply connected. $\square$

Remark. A further argument can show that $\widetilde{X}$ is contractible and is the universal cover of X, and hence $\pi_1(X) \cong \mathbb{Z}$. X is actually the identification space of S^1 by setting $(x_1, y_1) \sim (x_1, y_1)$ if they are equal or $y_1 y_2 > 0$. However, it is not a quotient space of a CW complex by a subcomplex. X does not satisfy the T_1 axiom.

6. 证明 $2T^2$ 不是 T^2 的复迭空间。

Proof. Suppose there is a covering map $p : 2T^2 \to T^2$. Then $p_* : \pi_1(2T^2) \to \pi_1(T^2)$ is injective. We have

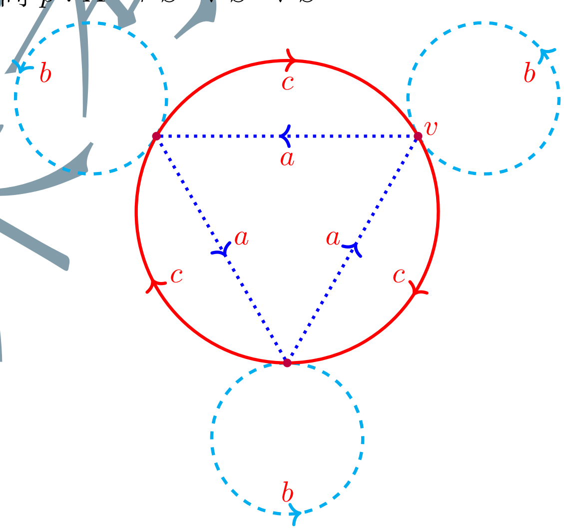
$$\pi_1(2T^2) \cong \big\langle a_1, b_1, a_2, b_2 \,\big|\, [a_1, b_1][a_2, b_2] = 1 \big\rangle, \qquad \pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

By setting $b_1 = b_2 = 1$, we get the quotient group G of $\pi_1(2T^2)$

$$G = \big\langle a_1, b_1, a_2, b_2 \,\big|\, b_1, b_2, [a_1, b_1][a_2, b_2] \big\rangle = \langle a_1, a_2 \rangle$$

which is a free group of 2 generators. This implies that $\pi_2(2T^2)$ is not abelian. Since $\pi_1(T^2)$ is abelian and any subgroup of an abelian group is abelian, there exists no injective homomorphism from $\pi_2(2T^2)$ to $\pi_1(T^2)$. This contradicts that p_* is an injective homomorphism. Hence there exists no covering map from $2T^2$ to T^2. $\square$

7. 考虑如下图所示的复迭空间 $p : X \to S^1 \vee S^1 \vee S^1$



设 $x_0 = p(v)$。给出 $\pi_1(S^1 \vee S^1 \vee S^1, x_0) = \langle a, b, c \rangle$ 的子群 $H = p_*(\pi_1(X, v))$ 的自由生成元，并给出 H 的尽可能多的性质。

Proof. A properly chosen neighborhood of X can (strongly) deformation retracts to X and to a wedge of 7 circles. It follows that X is homotopy equivalent to a wedge of 7 circles, whose fundamental group is the group freely generated by 7 generators. As based loops in X, they are given by the following paths:

$$b, a^{-1}ba, aba^{-1}, a^3, ac, ca, ac^{-1}a$$

Hence

$$H = \langle b, a^{-1}ba, aba^{-1}, a^3, ac, ca, ac^{-1}a \rangle$$

H is a subgroup of $\langle a, b, c\rangle$ of index 3, because X is a 3-sheeted covering space of $S^1 \vee S^1 \vee S^1$.

There are covering transformations of X consisting of rotations by $\frac{2\pi}{3}$ in the figure above. The rotations preserves the labeling, and hence they commutes with the covering map. The rotations can take v to any other vertex, and hence $p : X \to S^1 \vee S^1 \vee S^1$ is regular covering. Therefore H is a normal subgroup of $\langle a, b, c\rangle$. □

8. 设 G 是一个群，乘法记为 $\circ$，单位元记为 e。称 G 是一个拓扑群，若赋予 G 一个拓扑，使得如下两个函数

$$m : G \times G \to G, \qquad m(g, h) = g \circ h$$
$$i : G \to G, \qquad i(g) = g^{-1}$$

都连续。

(a) 若 $\{e\}$ 是 G 的闭子集，证明 G 是 Hausdorff 空间。

(b) 证明 $\pi_1(G)$ 是交换群。

Proof. Let $\Delta = \big\{(g, h) \in G \times G \,\big|\, g = h\big\}$ be the diagonal in $G \times G$. For any $g, h \in G$, we have

$$g = h \quad \Leftrightarrow \quad g \circ h^{-1} = e \quad \Leftrightarrow \quad m(g, i(h)) = e$$

Hence

$$\Delta = (\mathbb{1} \times i)^{-1}(m^{-1}(e))$$

is a closed subset of $G \times G$ since $\mathbb{1} \times i$ and m are both continuous. It follows that $G \setminus \Delta$ is open. For any distinct g and h in G, we have $(g, h) \in (G \times G) \setminus \Delta$. By the product topology, there exists open subsets U and V of G such that $(g, h) \in U \times V \subset G \times G \setminus \Delta$. If $U \cap V \neq \emptyset$, then $(U \times V) \cap \Delta \neq \emptyset$. U and V are disjoint open neighborhood of g and h. Hence G is Hausdorff.

Let α and β are two loops at e. Then we have

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) \circ e, & 0 \leqslant t \leqslant \frac{1}{2}, \\ e \circ \beta(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \qquad \text{and} \qquad (\beta.\alpha)(t) = \begin{cases} e \circ \beta(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ \alpha(2t-1) \circ e, & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Define $\alpha(t) = e$ and $\beta(t) = e$ for $t < 0$ and $t > 1$. We can write

$$(\alpha.\beta)(t) = \alpha(2t) \circ \beta(2t-1) \qquad \text{and} \qquad (\beta.\alpha)(t) = \alpha(2t-1) \circ \beta(2t)$$

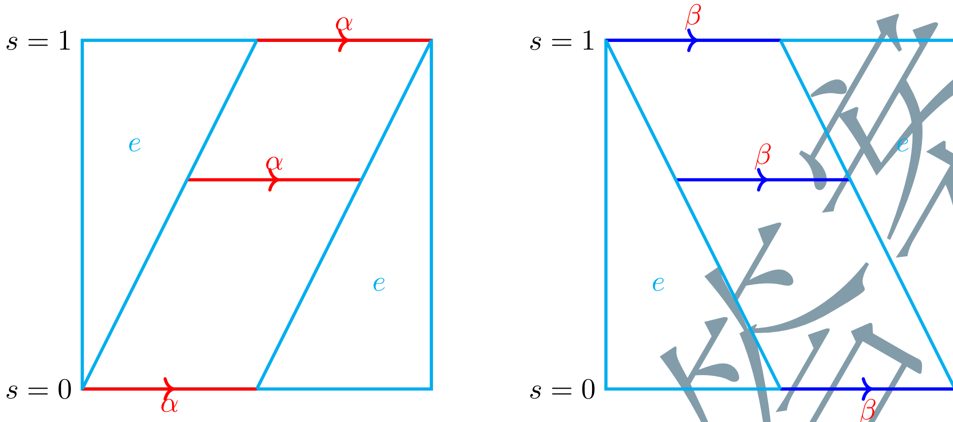
Define

$$H : [0, 1] \times [0, 1] \to G, \qquad H(t, s) = \alpha(2t - s) \circ \beta(2t + s - 1)$$

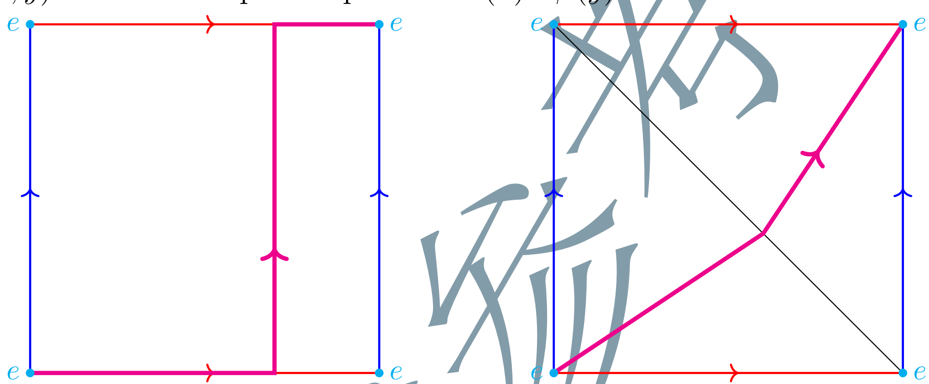
Then H is a relative homotopy from $\alpha.\beta$ to $\beta.\alpha$. Hence $\pi_1(G, e)$ is abelian.

If G is path connected, $\pi_1(G)$ does not depends on the choice of the basepoint and hence $\pi(G)$ is abelian. If G is not path connected, then any path component is homeomorphic to the one containing e and the conclusion follows. □

Remark. The homotopy is the product of two homotopies illustrated in the following figure



There can also be other relative homotopies between $\alpha.\beta$ and $\beta.\alpha$, as illustrated in the following figure, in which a point (x,y) in the unit square represents $\alpha(x)\circ\beta(y)$.



The first homotopy is

$$H_1(s,t)=\begin{cases}\alpha(2t), & 2t\leqslant 1-s,\\ \alpha(1-s)\circ\beta(2t+s-1), & 1-s\leqslant 2t\leqslant 2-s,\\ \alpha(2t+s-2), & 2-t\leqslant 2t.\end{cases}$$

and the second homotopy is

$$H_2(s,t)=\begin{cases}\alpha(2t(1-s))\circ\beta(2ts), & 0\leqslant t\leqslant \frac{1}{2},\\ \alpha((2t-1)s+(1-s))\circ\beta((2t-1)(1-s)+s), & \frac{1}{2}\leqslant t\leqslant 1.\end{cases}$$

9. 设 $A\subset\mathbb{E}^2$ 同胚于 $[0,1]$。证明 A 是 $\mathbb{E}^2$ 的形变收缩核。

Proof. Let $h:A\to[0,1]$ be a homeomorphism. A is compact as the continuous image of the compact interval $[0,1]$ and hence closed as $\mathbb{E}^2$ is Hausdorff. By the Tietze extension theorem, we can extend h to a continuous function $f:\mathbb{E}^2\to[0,1]$. $r=h^{-1}\circ f$ is then a contraction of $\mathbb{E}^2$ to A. Since $\mathbb{E}^2$ is contractible, r and the identity map are homotopic. The homotopy

$$F:\mathbb{E}^2\times[0,1]\to\mathbb{E}^2,\qquad F(x,t)=(1-t)x+tr(x)$$

is the desired (strong) deformation contraction from $\mathbb{E}^2$ to A. □

Remark. The deformation retraction can also be obtained as an extension of the map

$$F:X\to\mathbb{E}^2,\qquad F(x,t)=\begin{cases}x, & t=0\text{ or }x\in A,\\ r(x), & t=1.\end{cases}$$

on the closed subset $(\mathbb{E}^2\times\{0,1\})\cup(A\times[0,1])\subset\mathbb{E}^2\times[0,1]$ by another appeal of the Tietze extension theorem.