

数学模型第六次作业

1. 已知

$$\frac{d^2Y}{dT^2} = -\frac{GM_E}{(R_E+Y)^2}, \quad Y(0) = 2m, \quad Y'(0) = -V_0 m/s, \quad M_E = \frac{4}{3}\pi R_E^3 \rho_E$$

将 $Y = Ly(t)$, $T = Tt$ 代入得

$$\frac{d^2y}{dt^2} = -\frac{\Pi_1}{(1 + \Pi_2 y)^2}, \quad y(0) = \Pi_3, \quad y'(0) = -\Pi_4$$

$$\text{这里 } \Pi_1 = \frac{4\pi G \rho_E R_E T^2}{3L}, \quad \Pi_2 = \frac{L}{R_E}, \quad \Pi_3 = \frac{2}{L}, \quad \Pi_4 = \frac{V_0 T}{L}$$

(i) 当 $V_0 \rightarrow \infty$ 时, 归一化 Π_3, Π_4 , 得到 $L = 2, T = \frac{2}{V_0}, \Pi_2 = \frac{2}{R_E}$,
 $\Pi_1 = \frac{8\pi G \rho_E R_E}{3V_0^2} = \epsilon \rightarrow 0$
 故问题化为

$$\frac{d^2y}{dt^2} = -\frac{\epsilon}{(1 + \Pi_2 y)^2}, \quad y(0) = 1, \quad y'(0) = -1$$

令 $\epsilon \rightarrow 0$, 对应的 leading order problem 为

$$\frac{d^2y}{dt^2} = 0, \quad y(0) = 1, \quad y'(0) = -1$$

(ii) 当 $\rho_E \rightarrow 0$ 时, 归一化 Π_1, Π_3 , 得到 $L = 2, T = \sqrt{\frac{3}{2\pi G \rho_E R_E}}$,
 $\Pi_2 = \frac{2}{R_E}, \Pi_4 = \sqrt{\frac{3V_0^2}{8\pi G \rho_E R_E}} = \epsilon \rightarrow 0$
 故问题化为

$$\frac{d^2y}{dt^2} = -\frac{1}{(1 + \Pi_2 y)^2}, \quad y(0) = 1, \quad y'(0) = -\epsilon$$

令 $\epsilon \rightarrow 0$, 对应的 leading order problem 为

$$\frac{d^2y}{dt^2} = -\frac{1}{(1 + \Pi_2 y)^2}, \quad y(0) = 1, \quad y'(0) = 0$$

(iii) 当 $\rho_E \rightarrow 0$ 时, 归一化 Π_3, Π_4 , 得到 $L = 2, T = \frac{2}{V_0}, \Pi_1 = \frac{8\pi G \rho_E R_E}{3V_0^2} = \epsilon \rightarrow 0, \Pi_2 = \frac{2}{R_E}$
 故问题化为

$$\frac{d^2y}{dt^2} = -\frac{\epsilon}{(1 + \Pi_2 y)^2}, \quad y(0) = 1, \quad y'(0) = -1$$

令 $\epsilon \rightarrow 0$, 对应的 leading order problem 为

$$\frac{d^2y}{dt^2} = 0, \quad y(0) = 1, \quad y'(0) = -1$$

(iv) 当 $R_e \rightarrow 0$ 时, 按照上述无量纲化方式, 怎样归一化都会导致奇异, 故需改变无量纲化方式:

令 $\Pi_1 = \frac{GM_E T^2}{L^3}$, $\Pi_2 = \frac{R_E}{L}$, $\Pi_3 = \frac{2}{L}$, $\Pi_4 = \frac{V_0 T}{L}$, 则有

$$\frac{d^2y}{dt^2} = -\frac{\Pi_1}{(y + \Pi_2)^2}, \quad y(0) = \Pi_3, \quad y'(0) = -\Pi_4$$

归一化 Π_3, Π_4 , 得到 $L = 2$, $T = \frac{2}{V_0}$, $\Pi_2 = \frac{R_E}{2} = \epsilon \rightarrow 0$
故问题化为

$$\frac{d^2y}{dt^2} = -\frac{\Pi_1}{(y + \epsilon)^2}, \quad y(0) = 1, \quad y'(0) = -1$$

令 $\epsilon \rightarrow 0$, 对应的 leading order problem 为

$$\frac{d^2y}{dt^2} = -\left(\frac{GM_E}{2V_0^2}\right)/y^2, \quad y(0) = 1, \quad y'(0) = -1$$

2.

(a) 把 $X(T) = Xx(t)$, $Y(T) = Yy(t)$, $Z(T) = Zz(t)$, $T = Tt$ 带入并比较可得

$$\frac{X}{T} \frac{dx}{dt} = -BYy + A$$

$$\frac{Y}{T} \frac{dy}{dt} = -DZz + CXx$$

$$\frac{Z}{T} \frac{dz}{dt} = -HZz + EYy - FY^2y^2 + GY^3y^3$$

可依次求出:

$$X = DE^2/(HFC)$$

$$Y = E/F$$

$$Z = E^2/(HFC)$$

$$T = DE/(HBC)$$

$$\alpha = AF/(BE)$$

$$\beta = H^2 BC / (D^2 E^2)$$

$$\gamma = BC / (DE)$$

$$\delta = 3GE/F^2$$

$$\mu = HFCX_0 / DE^2$$

$$\sigma = FY_0 / E$$

$$\omega = HFZ_0 / E^2$$

(b) $\gamma = 0$ 时有

$$z = y - y^2 + \frac{1}{3}\delta y^3$$

初值满足

$$\omega = \sigma - \sigma^2 + \frac{1}{3}\delta\sigma^3$$

问题化简为

$$\frac{dx}{dt} = \alpha - y, \quad x(0) = \mu$$

$$\beta \frac{dy}{dt} = x - y + y^2 - \frac{1}{3}\delta y^3 \quad y(0) = \sigma$$

(c) 当 β 等于 0 时 $x=z$ 化简可得

$$x + \gamma(\alpha - y) = y - y^2 + \frac{1}{3}\delta y^3$$

将初值带入

$$\mu + \gamma(\alpha - \sigma) = \sigma - \sigma^2 + \frac{1}{3}\delta\sigma^3$$

再对 t 求导有：

$$\alpha - y = (1 + \gamma - 2y - \delta y^2) \frac{dy}{dt}$$

3.

$$(x - 3)^3 = 24\epsilon x^2$$

由 $\delta_0 x_0$ 满足 $(\delta_0 x_0 - 3)^3 = 0$ 可知 $\delta_0 = 1$, $x_0 = 3$
设 $x \sim 3 + \delta_1(\epsilon)x_1$, 将其代入 $(x - 3)^3 = 24\epsilon x^2$ 得

$$\delta_1^3 x_1^3 - 216\epsilon - 144\epsilon \delta_1 x_1 - 24\epsilon \delta_1^2 x_1^2 = 0$$

由主项平衡原理, 只有 $\delta_1^3 x_1^3$ 与 216ϵ 为主项时才不违背主项平衡原理,
故有

$$\delta_1^3 x_1^3 = 216\epsilon$$

由此可得到

$$\delta_1 = \epsilon^{\frac{1}{3}}, \quad x_1 = 6e^{\frac{i2\pi k}{3}} \quad k = 0, 1, 2$$

再进一步, 设 $x \sim 3 + 6e^{\frac{i2\pi k}{3}}\epsilon^{\frac{1}{3}} + \delta_2(\epsilon)x_2$, ($k = 0, 1, 2$), 代入原方程并
由主项平衡原理得

$$3\delta_2 x_2 (\delta_1 x_1)^2 = 48\epsilon \delta_0 x_0 \delta_1 x_1$$

整理得

$$\delta_2 = \epsilon^{\frac{2}{3}}, \quad x_2 = 8e^{\frac{i4\pi k}{3}}, \quad k = 0, 1, 2$$

综上

$$x \sim 3 + 6e^{\frac{i2\pi k}{3}}\epsilon^{\frac{1}{3}} + 8e^{\frac{i4\pi k}{3}}\epsilon^{\frac{2}{3}}, \quad k = 0, 1, 2$$

4.

$\epsilon=0$ 时能得到 $60=0$ 无解, 所以原方程没有非奇异解

将 $x = x_0 \delta_0$ 带入

$$\epsilon^6 \delta_0^3 x_0^3 - 5\epsilon^3 \delta_0^2 x_0^3 - 20\epsilon x_0 \delta_0 + 60 = 0$$

在可能的平衡中 δ_0 中 ϵ 一定是负幂次, 根据主项平衡原理, 只有三组可行, 分别是:

1,2	$x \sim 5\epsilon^{-3}$
2,3	$x \sim -4\epsilon^{-2}$
3,4	$x \sim 3\epsilon^{-1}$

5. (a). 需要展开 2 项, 设 $x \sim x_0(t) + \epsilon x_1(t) + O(\epsilon^2)$, 将其代入方程并由

$$-\frac{1}{(1 + \epsilon x)^2} = -1 + 2\epsilon x - 3\epsilon^2 x^2 + O(\epsilon^3)$$

得到

$$\frac{d^2 x}{dt^2} = x_0''(t) + \epsilon x_1''(t) + O(\epsilon^2) = -1 + 2\epsilon x_0 + \epsilon^2 (2x_1 - 3x_0^2) + O(\epsilon^3)$$

对初值进行展开:

$$\begin{aligned}x_0(0) + \epsilon x_1(0) + O(\epsilon^2) &= 1 \\x'_0(0) + \epsilon x'_1(0) + O(\epsilon^2) &= 3\epsilon\end{aligned}$$

故有

$$\begin{aligned}x''_0(t) &= -1, \quad x_0(0) = 1, \quad x'_0(0) = 0 \\x''_1(t) &= 2x_0, \quad x_1(0) = 0, \quad x'_1(0) = 3\end{aligned}$$

即

$$x_0(t) = 1 - \frac{1}{2}t^2, \quad x_1(t) = -\frac{1}{12}t^4 + t^2 + 3t$$

由 $t^{max} = t_0 + \epsilon t_1 + O(\epsilon^2)$ 满足 $x'(t^{max}) = 0$, 并将其代入 x 表达式比较同阶系数知

$$t_0 = 0, \quad t_1 = 3, \quad t^{max} = 3\epsilon + O(\epsilon^2)$$

(b). $x'(0) = \frac{4}{\epsilon}$ 时, 若按照 (a) 中方法展开并比较同阶系数

$$x'_0(0) + \epsilon x'_1(0) + O(\epsilon^2) = \frac{4}{\epsilon} \Rightarrow 0 = 4$$

矛盾!

设 $x(t) = \frac{X(t)}{\epsilon} \sim X_0(t)/\epsilon + X_1(t)$, i.e. $X(t) = X_0(t) + X_1(t)\epsilon + O(\epsilon^2)$

故有

$$\begin{aligned}\epsilon \frac{d^2x}{dt^2} &= \frac{d^2X}{dt^2} \\ \frac{1}{(1+\epsilon x)^2} &= \frac{1}{(1+X)^2} = 1 - 2X + 3X^2 + O(X^3)\end{aligned}$$

将 $X(t)$ 及 $1/(1+X)^2$ 展开式代入 $\frac{1}{\epsilon} \frac{d^2X}{dt^2} = -\frac{1}{(1+X)^2}$ 并比较同阶系数知

$$X''_0(t) = 0, \quad X''_1(t) = -\frac{1}{1+X_0^2}$$

由初值条件

$$\begin{aligned}X(0) &= X_0(0) + \epsilon X_1(0) + O(\epsilon^2) = \epsilon \\X'(0) &= X'_0(0) + \epsilon X'_1(0) + O(\epsilon^2) = 4\end{aligned}$$

比较同阶系数可得初值条件, 由此可解出

$$\begin{aligned}X_0(t) &= 4t \\X_1(t) &= \frac{1}{16} \ln(1+4t) - \frac{1}{4}t + 1\end{aligned}$$

故

$$x(t) \sim \frac{4t}{\epsilon} + \frac{1}{16} \ln(1+4t) - \frac{1}{4}t + 1$$

6.

(a) 由 $v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + O(\epsilon^3)$ 可以得到

$$\epsilon v^2 = \epsilon v_0^2 + 2\epsilon^2 v_0 v_1 + O(\epsilon^3)$$

$$v' = v'_0 + \epsilon v'_1 + \epsilon^2 v'_2 + O(\epsilon^3)$$

带回原式可依次解得

$$v_0 = -\frac{1}{2}t^2, \quad v_1 = -\frac{1}{20}t^5, \quad v_2 = -\frac{1}{160}t^8$$

所以

$$v(t) \sim -\frac{1}{2}t^2 - \frac{1}{20}\epsilon t^5 - \frac{1}{160}\epsilon^2 t^8$$

(b)

因为 $O(t^2) \gg O(\epsilon t^5) \gg O(\epsilon^2 t^8)$

所以 $O(t^{-3}) \gg O(\epsilon)$

可知 $0 < t \ll O(\epsilon^{-\frac{1}{3}})$