Exam 2

1. a. $J=\frac{1}{2}\int_0^T (x(t)\dot{y}(t)-y(t)\dot{x}(t))dt$

$$G=\int_0^T\sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2}\,dt-L$$

$$=\int_0^T\left(\sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2}-\frac{L}{T}\right)dt$$

$$\max_{x(t),\,y(t)} J,\quad \text{subject to } G=0$$

令 $I=\int_0^T \mathcal{L}(x,\dot{x},y,\dot{y},\lambda)\,dt$

这里 $\mathcal{L}=\frac{1}{2}(x(t)\dot{y}(t)-y(t)\dot{x}(t))-\lambda\left(\sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2}-\frac{L}{T}\right)$.

为增广的 Lagrangian function

$$\frac{\partial\mathcal{L}}{\partial x}=\frac{1}{2}\dot{y}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}}=-\frac{1}{2}y-\frac{\lambda\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}$$

$$\frac{\partial\mathcal{L}}{\partial y}=-\frac{1}{2}\dot{x}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{y}}=\frac{1}{2}x-\frac{\lambda\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}$$

$$\Rightarrow\begin{cases}\frac{1}{2}\dot{y}-\frac{d}{dt}\left(-\frac{1}{2}y-\frac{\lambda\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\right)=0 & ①\\ -\frac{1}{2}\dot{x}-\frac{d}{dt}\left(\frac{1}{2}x-\frac{\lambda\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\right)=0 & ②\\ \int_0^T\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt=L & ③\end{cases}$$

b. 由①②知

$$\frac{d}{dt}\left(y+\frac{\lambda\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\right)=0$$

$$\frac{d}{dt}\left(x-\frac{\lambda\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\right)=0$$

则 $y+\frac{\lambda\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}=C_1$

$$x-\frac{\lambda\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}=C_2$$

易得 $(x-C_2)^2+(y-C_1)^2$

$$=\lambda^2\left(\frac{(\dot{x})^2}{(\dot{x})^2+(\dot{y})^2}+\frac{(\dot{y})^2}{(\dot{x})^2+(\dot{y})^2}\right)$$

$$=\lambda^2$$

再由③知 $L=2\pi\lambda$

即有 $(x-C_2)^2+(y-C_1)^2=r^2,\ r=\frac{L}{2\pi}$

说明面积最大时，区域为一个圆.

2. a. 由 $G(x_1,x_2)=\int_a^b \delta(x-x_1)G(x,x_2)dx$

$$G(x_2,x_1)=\int_a^b \delta(x-x_2)\,G(x,x_1)dx$$

再由格林公式

$$\int_a^b[\mathcal{L}G(x,x_1)G(x,x_2)-G(x,x_1)\mathcal{L}G(x,x_2)]dx=0$$

所以 $G(x_1,x_2)-G(x_2,x_1)$

$$=\int_a^b[\delta(x-x_1)G(x,x_2)-\delta(x-x_2)G(x,x_1)]dx$$

$$=\int_a^b[\mathcal{L}G(x,x_1)G(x,x_2)-\mathcal{L}G(x,x_2)G(x,x_1)]dx$$

$$=0$$

b. 考虑 $\hat{u}(x)=a+\frac{b-a}{L}x$,

易知 $\hat{u}(0)=a$ $\hat{u}(L)=b$, $\hat{u}''=0$

令 $v(x)=u(x)-\hat{u}(x)$

易知 $v(x)$ 满足

$$\begin{cases} v''(x)=f(x) \\ v(0)=0, \quad v(L)=0 \end{cases}$$

易知 $v(x)=\int_a^b G(x',x)f(x')dx'$

则 $u(x)=v(x)+\hat{u}(x)$

$$=\int_a^b G(x',x)f(x')dx'+a+\frac{b-a}{L}x.$$

3. a (I) 解齐次问题 $(m=2)$

$$\phi_0''+4\phi_0=0 \quad \phi(0)=0,\ \phi(\pi)=0$$

$$\phi_0(x)=C_1\sin(2x)+C_2\cos(2x)$$

由 B.C. 知 $C_2=0$

则 $\phi_0(x)=C_1\sin(2x)$

由 F.A. 知，若原B.V.P. 有解

则有 $\int_0^\pi(\beta+x)\sin(2x)dx=0$.

但是 $\forall\beta$, $\int_0^\pi(\beta+x)\sin(2x)dx=-\frac{\pi}{2}\neq 0$

则 $m=2$ 时 BVP 一定无解

(II) 当 $m=3$ 时，类似可求

$$\phi_0=C\sin(3x)$$

由 F.A., 若 BVP 有解

则 $\int_0^\pi(\beta+x)\sin(3x)dx=\frac{1}{3}(2b+\pi)=0$

则 $b=-\frac{\pi}{2}$ 时, B.V.P. 有(无穷多)解

3. <u>b.</u> $(-\mathcal{L}^0-\varepsilon\mathcal{L}^1)(\psi_n^0+\varepsilon\psi_n^1+o(\varepsilon))$

$$=(\lambda_n^0+\varepsilon\lambda_n^1+o(\varepsilon))(\psi_n^0+\varepsilon\psi_n^1+o(\varepsilon))$$

$O(\varepsilon^0)$ 方程为 $-\mathcal{L}^0\psi_n^0=\lambda_n^0\psi_n^0$

$O(\varepsilon^1)$ 方程为 $-\mathcal{L}^0\psi_n^1-\mathcal{L}^1\psi_n^0$

$$=\lambda_n^0\psi_n^1+\lambda_n^1\psi_n^0$$

<u>整理可得</u>

$$(-\mathcal{L}^0-\lambda_n^0)\psi_n^1=\mathcal{L}^1\psi_n^0+\lambda_n^1\psi_n^0 \quad (*)$$

易知 $\mathcal{L}^0+\lambda_n^0$ 也是 Regular S.L. 算子.

而且 $(\mathcal{L}^0+\lambda_n^0)\phi_0=0$ 有非平凡解

$$\phi_0=c\,\psi_n^0$$

则若 ψ_n^1 有(非平凡)解

$$\int_a^b(\psi_n^0\mathcal{L}^1\psi_n^0+\lambda_n^1(\psi_n^0)^2)\,dx=0$$

$$\Rightarrow \quad \lambda_n^1=-\frac{\int_a^b\psi_n^0\mathcal{L}^1\psi_n^0\,dx}{\int_0^b(\psi_n^0)^2dx}.$$

c. 由 $\{\psi_n^0\}$ 的"完备性"

$$\psi_k^1=\sum_{m=1}^{\infty}c_m\psi_m^0$$

代入 $(*)$ 得

$$\sum_{m=1}^{\infty}(\lambda_m^0-\lambda_k^0)c_m\psi_m^0=\mathcal{L}^1\psi_k^0+\lambda_k^1\psi_k^0$$

再乘以某个 ψ_m^0 $(m\neq k)$ 并积分得

$$(\lambda_m^0-\lambda_k^0)c_m\int_a^b(\psi_m^0)^2dx$$

$$=\int_a^b\psi_m^0\mathcal{L}^1\psi_k^0\,dx$$

$$\Rightarrow \quad c_m=\frac{\int_a^b\psi_m^0\mathcal{L}^1\psi_k^0\,dx}{(\lambda_m^0-\lambda_k^0)\int_a^b(\psi_m^0)^2dx}$$

即有

$$\psi_k^1(x)=\sum_{m\geq 1,\,m\neq k}\frac{\int_a^b\psi_m^0\mathcal{L}^1\psi_k^0\,dy}{(\lambda_m^0-\lambda_k^0)\int_a^b(\psi_m^0)^2dy}\psi_m^0(x)$$

$$+c_k\psi_k^0(x).$$

4. $(x-3)^3 = 24\varepsilon x^2$

$x \sim \delta_0 x_0$，易得 $\delta_0=1$ $x_0=3$（重根）

再令 $x\sim 3+\delta_1 x_1$，则有

$$\underbrace{\delta_1^3 x_1^3}_{(1)} = \underbrace{24\varepsilon\delta_1^2 x_1^2}_{(2)} + \underbrace{144\varepsilon\delta_1 x_1}_{(3)} + \underbrace{216\varepsilon}_{(4)}$$

由主项平衡原理，只有(1)(4)平衡符合

则 $\delta_1^3=\varepsilon$，即 $\delta_1=\varepsilon^{\frac{1}{3}}$

而由 $x_1^3=216$ 得

$$x_1 = 6e^{\frac{i2\pi k}{3}},\quad k=0,1,2$$

最后，令 $x\sim 3+6e^{\frac{i2\pi k}{3}}\varepsilon^{\frac{1}{3}}+\delta_2 x_2$ $(k=0,1,2)$

由主项平衡原理知

$$3\delta_2 x_2(\delta_1 x_1)^2 = 48\varepsilon\delta_0 x_0\delta_1 x_1$$

整理得 $\delta_2=\varepsilon^{\frac{2}{3}}$，$x_2=8e^{\frac{i4\pi k}{3}}$，$k=0,1,2$

于是有 $x\sim 3+6e^{\frac{i2\pi k}{3}}\varepsilon^{\frac{1}{3}}+8e^{\frac{i4\pi k}{3}}\varepsilon^{\frac{2}{3}}$，

其中 $k=0,1,2$.

$\varepsilon\to 0$ 时，代数方程的阶数没有退化，

故有3个非奇异解.

5. 当 $n<n_0$ 时，显然 $P_n(t)=0$ $\forall t\geq 0$

当 $n=n_0$ 时

$$P_{n_0}(t+\Delta t) = P_{n_0-1}(t)b_{n_0-1}\Delta t + P_{n_0}(t)(1-b_{n_0}\Delta t)+o(\Delta t)$$

$\Rightarrow$ （注意 $P_{n_0-1}(t)=0$）

$$\frac{d}{dt}P_{n_0}(t) = -\lambda n_0 P_{n_0}(t)$$

当 $n>n_0$ 时

$$P_n(t+\Delta t) = P_{n-1}(t)b_{n-1}\Delta t + P_n(t)(1-b_n\Delta t)+o(\Delta t)$$

$$\Rightarrow \frac{d}{dt}P_n(t) = \lambda(n-1)P_{n-1}(t) - \lambda n P_n(t)$$

则由 $E(t)=\sum_n nP_n(t)$

$$=\sum_{n=n_0}^{+\infty} nP_n(t)$$

$$\frac{d}{dt}E(t) = -\lambda n_0^2 P_{n_0}(t) + \sum_{n=n_0+1}^{+\infty}\left[\lambda n(n-1)P_{n-1}(t) - \lambda n^2 P_n(t)\right]$$

$$= -\sum_{n=n_0}^{+\infty}\lambda n^2 P_n(t) + \sum_{n=n_0}^{+\infty}\lambda(n+1)nP_n(t)$$

$$=\lambda\sum_{n=n_0}^{+\infty} nP_n(t)$$

$$=\lambda E(t)$$

又由 $E(0)=n_0$

则 $E(t)=n_0e^{\lambda t}$.

6. 附加：（Cáceres, Carrillo, Perthame, 2011）

$$\begin{cases} \partial_t p + \partial_v[(-v+I(t))p] - \frac{a}{2}\partial_{vv}p = \delta(v-V_R)N(t) \\ N(t) = -\frac{a}{2}\partial_v p(V_F,t) \\ p(v,0) = p_0(v) \\ p(-\infty,t) = p(V_F,t) = 0 \end{cases}$$