数学模型 HW1

2019 年 3 月 6 日

1 第一题

1.1

在 $(\bar{A}, \bar{B})$ 变化率为 0，因此

$$-\bar{A} + \bar{B} = \bar{A} - \bar{B} = 0,$$

根据

$$\bar{A} + \bar{B} = A_0 + B_0,$$

我们有

$$\bar{A} = \bar{B} = \frac{A_0 + B_0}{2}.$$

1.2

令 $f(x) = \ln x + 1/x - 1$，我们有 $f(1) = 0$,

$$f'(x) = \frac{x-1}{x^2} \begin{cases} < 0, & 0 < x < 1, \\ > 0, & x > 1. \end{cases}$$

因此 $f(x) \geq f(1) = 0$。从而 $\ln(x) \geq 1 - \frac{1}{x}$。从而

$$\begin{aligned} & A \ln\left(\frac{A}{\bar{A}}\right) + B \ln\left(\frac{B}{\bar{B}}\right) \\ \geq & A\left(1 - \frac{\bar{A}}{A}\right) + B\left(1 - \frac{\bar{B}}{B}\right) \\ = & 0. \end{aligned}$$

1.3

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}t}(1 + \ln A) + A \cdot \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\mathrm{d}B}{\mathrm{d}t}(1 + \ln B) + B \cdot \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} \\ &= (-A + B)(1 + \ln A - 1 - \ln B) \\ &\leq 0. \end{aligned}$$

最后一步分 $A > B$ 与 $A < B$ 两种情况讨论即可。

2 第二题

2.1

Conservation law:

$$u_t + \left(\frac{u^2}{2} - \mu u_x\right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0.$$

Flux function: $\frac{u^2}{2} - \mu u_x$.

2.2

Hopf-Cole 变换：

$$u = -a\frac{\partial}{\partial x}\ln\phi = -a\frac{\phi_x}{\phi}.$$

将 Hopf-Cole 变换带入到原方程，并利用 $a = 2\mu$,

$$-2\mu\frac{\phi\phi_{xt} - \phi_x\phi_t}{\phi^2} + 4\mu^2\frac{\phi\phi_x\phi_{xx} - \phi_x^3}{\phi^3} = -2\mu^2\frac{\phi^2(\phi_x\phi_{xx} + \phi\phi_{xxx} - 2\phi_x\phi_{xx}) - 2\phi\phi_x(\phi\phi_{xx} - \phi_x^2)}{\phi^4},$$

化简可得

$$-2\mu\frac{\phi\phi_{xt} - \phi_x\phi_t}{\phi^2} = -2\mu^2\frac{\phi\phi_{xxx} - \phi_x\phi_{xx}}{\phi^2},$$

即

$$\left(\frac{\phi_t}{\phi} - \mu\frac{\phi_{xx}}{\phi}\right)_x = 0.$$

因此

$$\frac{1}{\phi}\phi_t - \mu\frac{1}{\phi}\phi_{xx} = g(t)$$

是一个只关于 t 的函数。

3 第三题

3.1

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \nabla_{\boldsymbol{q}} f \cdot \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \nabla_{\boldsymbol{p}} f \cdot \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}.$$

根据守恒律，我们有

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{q}} \cdot (f\dot{\boldsymbol{q}}) + \nabla_{\boldsymbol{p}} \cdot (f\dot{\boldsymbol{p}}) = 0. \quad (\dot{\boldsymbol{p}}\text{表示}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}).$$

将上式展开，我们有

$$\frac{\partial f}{\partial t}+\nabla_{\boldsymbol{q}} f\cdot\dot{\boldsymbol{q}}+\nabla_{\boldsymbol{p}} f\cdot\dot{\boldsymbol{p}}+f\left(\nabla_{\boldsymbol{q}}\cdot\dot{\boldsymbol{q}}+\nabla_{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}}\right)=0.$$

又由

$$\dot{\boldsymbol{p}}=-\nabla_{\boldsymbol{q}} H,\quad \dot{\boldsymbol{q}}=\nabla_{\boldsymbol{p}} H,$$

因此

$$\nabla_{\boldsymbol{q}}\cdot\dot{\boldsymbol{q}}+\nabla_{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}}=\nabla_{\boldsymbol{q}}\cdot\nabla_{\boldsymbol{p}} H-\nabla_{\boldsymbol{p}}\cdot\nabla_{\boldsymbol{q}} H=\sum_i\left(\frac{\partial^2 H}{\partial q_i\partial p_i}-\frac{\partial^2 H}{\partial p_i\partial q_i}\right)=0.$$

因此我们有

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\nabla_{\boldsymbol{q}} f\cdot\dot{\boldsymbol{q}}+\nabla_{\boldsymbol{p}} f\cdot\dot{\boldsymbol{p}}=0.$$

3.2

注意到

$$\dot{\boldsymbol{q}}=\frac{\boldsymbol{p}}{m},\quad \dot{\boldsymbol{p}}=-\nabla_{\boldsymbol{q}} V(\boldsymbol{q}),$$

记 $H=\frac{|\boldsymbol{p}|^2}{2m}+V(\boldsymbol{q})$, 从而

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\boldsymbol{p}\cdot\dot{\boldsymbol{p}}}{m}+\frac{\partial V(\boldsymbol{q})}{\partial t}=\frac{\boldsymbol{p}\cdot\dot{\boldsymbol{p}}}{m}+\nabla_{\boldsymbol{q}} V(q)\cdot\dot{\boldsymbol{q}}=0.$$

$$\nabla_{\boldsymbol{p}} H=\frac{\boldsymbol{p}}{m},\quad \nabla_{\boldsymbol{q}} H=\nabla_{\boldsymbol{q}} V(q).$$

对于任意区域 Ω,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Hf(t,\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\\
=&\int_\Omega H\frac{\partial f}{\partial t}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\\
=&-\int_\Omega H\left(\nabla_{\boldsymbol{q}} f\cdot\frac{\boldsymbol{p}}{m}-\nabla_{\boldsymbol{p}} f\cdot\nabla_{\boldsymbol{q}} V(\boldsymbol{q})\right)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\\
=&-\int_\Omega H\left(\nabla_{\boldsymbol{q}} f\cdot\nabla_{\boldsymbol{p}} H-\nabla_{\boldsymbol{p}} f\cdot\nabla_{\boldsymbol{q}} H\right)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\\
=&\int_\Omega\nabla\cdot(f\nabla_{\boldsymbol{q}}(\frac{1}{2}H^2),-f\nabla_{\boldsymbol{p}}(\frac{1}{2}H^2))\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p},\quad(\text{此处}\nabla\cdot(\boldsymbol{\alpha},\boldsymbol{\beta})=\nabla_{\boldsymbol{p}}\cdot\boldsymbol{\alpha}+\nabla_{\boldsymbol{q}}\cdot\boldsymbol{\beta})\\
=&\int_{\partial\Omega}(f\nabla_{\boldsymbol{q}}(\frac{1}{2}H^2),-f\nabla_{\boldsymbol{p}}(\frac{1}{2}H^2))\cdot\boldsymbol{n}\,\mathrm{d}S.
\end{aligned}$$

当 Ω 包含了 f 的支集时，在 $\partial\Omega$ 上 f 恒为 0. 根据上式我们可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int Hf(t,\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}=0.$$

4 第四题

4.1

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2 \, \mathrm{d}x &= \int \frac{\partial}{\partial t}|u|^2 \, \mathrm{d}x \\
&= \int (u_t\bar{u} + u\bar{u}_t) \, \mathrm{d}x \\
&= \int \left(\frac{\hat{H}}{i\hbar}u\bar{u} - u\frac{\hat{H}}{i\hbar}\bar{u}\right) \mathrm{d}x
\end{aligned}
$$

注意到 $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$,

$$
\begin{aligned}
\text{上式} &= \frac{1}{i\hbar}\int (\bar{u}\hat{H}u - u\hat{H}\bar{u}) \, \mathrm{d}x \\
&= -\frac{\hbar}{2mi}\int (\bar{u}\Delta u - u\Delta\bar{u}) \, \mathrm{d}x \\
&= -\frac{\hbar}{2mi}(\bar{u}\nabla u - u\nabla\bar{u})|_{-\infty}^{\infty} + \frac{\hbar}{2mi}\int (\nabla\bar{u}\nabla u - \nabla u\nabla\bar{u}) \, \mathrm{d}x \\
&= 0.
\end{aligned}
$$

4.2

由上题我们可知

$$
\begin{aligned}
\rho_t &= u_t\bar{u} + u\bar{u}_t \\
&= \frac{1}{i\hbar}(\bar{u}\hat{H}u - u\hat{H}\bar{u}) \\
&= -\frac{\hbar}{2mi}(\bar{u}\Delta u - u\Delta\bar{u}) \\
&= -\frac{\hbar}{2mi}\nabla(\bar{u}\nabla u - u\nabla\bar{u})
\end{aligned}
$$

因此

$$
J = \frac{\hbar}{2mi}(\bar{u}\nabla u - u\nabla\bar{u}).
$$