1.

(1) 由转移阵 P 知,

$$P_{11} = P_{21}, \quad P_{14} = \tfrac{1}{6}, \quad P_{42} = \tfrac{1}{4}, \quad P_{23} = \tfrac{1}{3}$$

因此 由于 X 是 时齐马氏链.

$\therefore P(X_0=1, X_1=4, X_2=2, X_3=3) = P(X_0=1)P(X_1=4|X_0=1)P(X_2=2|X_1=4, X_0=1)$
$$P(X_3=3|X_2=2, X_1=4, X_0=1)$$

马氏性 $\longrightarrow P(X_0=1)P(X_1=4|X_0=1)P(X_2=2|X_1=4)P(X_3=3|X_2=2)$

时齐性 $\longrightarrow P(X_0=1) P_{14} P_{42} P_{23} = \tfrac{1}{5} \times \tfrac{1}{6} \times \tfrac{1}{4} \times \tfrac{1}{3} = \dfrac{1}{144}.$

(2) $\pi = \pi P$,

得 $\begin{cases} \pi_1 = \tfrac{1}{6}\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{3}\pi_3 + \tfrac{1}{6}\pi_4 \\ \pi_2 = \tfrac{1}{6}\pi_1 + \tfrac{1}{3}\pi_3 + \tfrac{1}{4}\pi_4 \\ \pi_3 = \tfrac{1}{6}\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{6}\pi_4 \\ \pi_4 = \tfrac{1}{6}\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{3}\pi_3 + \tfrac{1}{6}\pi_4 \end{cases}$
$\qquad$ 解得 $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$
$$= \left(\tfrac{2}{7}, \tfrac{3}{14}, \tfrac{3}{14}, \tfrac{2}{7} \right).$$

π 是 可逆分布.

因为 在 π 是 一个概率分布的 基础 上, 又有 验证 $\pi_i P_{ij} = \pi_j P_{ji}$, $i \neq j$ 即可

$i = j$ 时之式显然成立.

$\pi_1 P_{12} = \tfrac{2}{7} \times \tfrac{1}{6} = \tfrac{1}{14} \times \tfrac{3}{3} = \pi_2 P_{21} = \tfrac{1}{14}$. $\quad \pi_2 P_{23} = \tfrac{3}{14} \times \tfrac{1}{3} = \tfrac{1}{14} = \tfrac{3}{14} \times \tfrac{1}{3} = \pi_3 P_{32}$

$\pi_1 P_{13} = \tfrac{2}{7} \times \tfrac{1}{6} = \tfrac{1}{14} = \tfrac{3}{14} \times \tfrac{1}{3} = \pi_3 P_{31}$. $\quad \pi_2 P_{24} = \tfrac{3}{14} \times \tfrac{1}{3} = \tfrac{1}{14} = \dfrac{\tfrac{2}{7} \times \tfrac{1}{4}}{} = \pi_4 P_{42}$

$\pi_1 P_{14} = \tfrac{2}{7} \times \tfrac{1}{6} = \tfrac{1}{14} = \tfrac{2}{7} \times \tfrac{1}{6} = \pi_4 P_{41}$. $\quad \pi_3 P_{34} = \tfrac{3}{14} \times \tfrac{1}{3} = \tfrac{1}{14} = \tfrac{2}{7} \times \tfrac{1}{6} = \pi_4 P_{43}$

因此 π 是 可逆分布

(3) 由更一般形式的遍历定理 (函数形式).

由 X 有平稳分布 因为 有限空间, 因此 正常返
且由 P 知 所有状态 互通, 因此 不可约. 则所有状态正常返.
在以上条件下, 由更一般形式的遍历定理 (函数形式).

有 $P\!\left(\lim\limits_{n\to\infty} \tfrac{1}{n} \sum\limits_{k=0}^{n} f(X_k) = \sum\limits_{i=1}^{4} f(i)\pi_i \right) = 1.$

$\therefore \lim\limits_{n\to\infty} \tfrac{1}{n} \sum\limits_{k=0}^{n} f(X_k) = \sum\limits_{i=1}^{4} f(i)\pi_i = 2.8 \times \tfrac{2}{7} + 2.1 \times \tfrac{3}{14} + 0.7 \times \tfrac{3}{14} + 4.2 \times \tfrac{2}{7} = 2.6.$

2. $\{N_t\} \sim P_p(\lambda)$.

用2知 N_t 有独立、平稳增量性且 $N_t \sim P(\lambda t)$. $\forall t \geq 0$. $\therefore EN_t = \lambda t$
$$\text{Var} N_t = \lambda t.$$

(1) $0 < s < t$ 时,
$$E(N_s N_t) = E(N_s(N_t - N_s) + N_s^2)$$
$$= E N_s(N_t - N_s) + E N_s^2. \qquad ①$$

其中 由于独立增量性. N_s 与 $N_t - N_s$ 独立 $(t > s > 0)$.
$$故 \quad E N_s(N_t - N_s) = E N_s \cdot E(N_t - N_s). \qquad ②$$

又由平稳增量性. $N_t - N_s$ 与 N_{t-s} 同分布.
$$\therefore E N_s \cdot E(N_t - N_s) = E N_s \cdot E N_{t-s} = (\lambda s) \cdot (\lambda(t-s)) = \lambda^2 s(t-s). \qquad ③$$

而 $E N_s^2 = (E N_s)^2 + \text{Var} N_s = (\lambda s)^2 + \lambda s = \lambda^2 s^2 + \lambda s \qquad ④$

由 ①②③④,
$$E N_s N_t = \lambda^2 s(t-s) + \lambda^2 s^2 + \lambda s = \lambda^2 st + \lambda s. \quad \forall t > s > 0$$

(2). 首先考虑 $\dfrac{N_n}{n}$, 其中 n 为正整数.

有 $\dfrac{N_n}{n} = \dfrac{(N_n - N_{n-1}) + (N_{n-1} - N_{n-2}) + \cdots + (N_2 - N_1) + N_1}{n}$, 令 $\xi_k = N_k - N_{k-1}$,
$$k = 1, 2, \cdots; n$$
$$(N_0 = 0)$$

由 $\{N_t\}$ 的独立增量性知.
$\xi_1, \cdots, \xi_n$ 相互独立.
又由 $\{N_t\}$ 的 平稳增量性. ξ_k 与 $N_1 - N_0$ 即 ξ_1, $k = 1, 2, \cdots n$
同分布 $\xi_1, \cdots \xi_n \overset{iid}{\sim} \& N_1$, 其中 $N_1 \sim P(\lambda \cdot 1) = P(\lambda)$. $\therefore$
由强大数律. $P(\lim\limits_{n \to \infty} \dfrac{N_n}{n} = \lim\limits_{n \to \infty} \dfrac{\xi_1 + \cdots + \xi_n}{n} = E N_1 = \lambda) = 1$

而 $\dfrac{N_t}{t} = \dfrac{[t] \frac{N_{[t]}}{[t]} + N_t - N_{[t]}}{t} = \dfrac{[t]}{t} \cdot \dfrac{N_{[t]}}{[t]} + \dfrac{N_t - N_{[t]}}{t} \qquad ①$

由上知 $\lim\limits_{t \to \infty} \dfrac{N_{[t]}}{[t]} = \lambda$. $\dfrac{t-1}{t} \leq \dfrac{[t]}{t} \leq \dfrac{t}{t}$ 知 $\lim\limits_{t \to \infty} \dfrac{[t]}{t} = 1$ (夹逼定理)

又 $N_t - N_{[t]} \sim P(\lambda(t - [t]))$, 有 $E(N_t - N_{[t]}) = \lambda(t - [t]) \leq \lambda$
$\therefore \lim\limits_{t \to \infty} \dfrac{N_t - N_{[t]}}{t} = 0.$ 则 $E\left(\dfrac{N_t - N_{[t]}}{t}\right) \leq \dfrac{\lambda}{t} \to$
$$\text{Var}\left(\dfrac{N_t - N_{[t]}}{t}\right) \leq \dfrac{\lambda}{t^2} \to 0$$

故由对 ① 式. 有 $\lim\limits_{t \to \infty} \dfrac{N_t}{t} = 1 \times \lambda + 0 = \lambda$

即 $P(\lim\limits_{t \to \infty} \dfrac{N_t}{t} = \lambda) = 1$.

2

3. X 的状态空间为 $\{0,1\}$

(1) X 在 0 处寿命(停时) I_0 满足 $P(I_0 > t | X_0 = 0) = e^{-\lambda t}$.

因此 $I_0 \sim \mathcal{E}(\lambda)$. 同理 X 在 1 处寿命 I_1 服从 $\mathcal{E}(\mu)$ 指数分布.

因此在很短的时间 h 内.

发生两次转移的概率 $P(I_0 + I_1 < h) = \iint_{x+y<h} \lambda e^{-\lambda x} \mu e^{-\mu y} dx dy = \int_0^h dx \int_0^{h-x} \lambda e^{-\lambda x} \mu e^{-\mu y} dy$

$= 1 - e^{-\lambda h} - e^{-\mu h} \frac{\lambda}{\mu - \lambda}(e^{(\mu - \lambda)h} - 1)$

$= \lambda h + o(h) - (1 - \mu h + o(h))(\frac{\lambda}{\mu - \lambda}) \cdot (\mu - \lambda) h + o(h))$

$= \lambda h - \frac{\lambda}{\mu - \lambda}(\mu - \lambda) h + o(h) = o(h)$.

而 $P(\text{发生两次以上转移}) \leq P(I_0 + I_1 < h) = o(h)$

从 0 出发. 发生一次转移的概率为:

$P(\text{发生转移}) - P(\text{发生两次及以上次转移})$

$= P(I_0 < h) + o(h) = 1 - e^{-\lambda h} + o(h) = \lambda h + o(h)$. 即 $P_{01}(h) = \lambda h + o(h)$

从 0 出发不转移概率为 $P(I_0 \geq h) = e^{-\lambda h} = 1 - \lambda h + o(h)$. 即 $P_{00}(h) = 1 - \lambda h + o(h)$

同理 $P_{11}(h) = 1 - \mu h + o(h)$. $P_{10}(h) = \mu h + o(h)$.

由 $g_{ij}(h) = \lim_{h \to 0} \frac{P_{ij}(h) - \delta_{ij}}{h}$ 得. $g_{00} = -\lambda$. $g_{01} = \lambda$. $g_{10} = \mu$. $g_{11} = -\mu$.

∴ 转移速率矩阵 $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$.

由 Kolmogorov 前进方程. $P'(t) = P(t) Q$.

有 $P_{00}'(t) = -\lambda P_{00}(t) + \mu P_{01}(t) = -\lambda P_{00}(t) + \mu(1 - P_{00}(t)) = -(\lambda + \mu) P_{00}(t) + \mu$

$P_{11}'(t) = \lambda P_{10}(t) - \mu P_{11}(t) = \lambda(1 - P_{11}(t)) - \mu P_{11}(t) = -(\lambda + \mu) P_{11}(t) + \lambda$.

(2) 由 (1) 中 ODE.

$\frac{d P_{00}(t)}{-(\lambda + \mu) P_{00}(t) + \mu} = dt$, $\Rightarrow$ $\ln(-(\lambda + \mu) P_{00}(t) + \mu) = -(\lambda + \mu) t + C$

$\Rightarrow$ $P_{00}(t) = \frac{\mu - C \cdot e^{-(\lambda + \mu)t}}{\lambda + \mu}$ 又 $P_{00}(0) = 1$ 得 $C = -\lambda$.

∴ $P_{00}(t) = \frac{\mu + \lambda e^{-(\lambda + \mu)t}}{\lambda + \mu}$.

同理 $P_{11}(t) = \frac{\lambda + \mu e^{-(\lambda + \mu)t}}{\lambda + \mu}$ (λ 和 μ 的对称性)

而 $P_{01}(t) = 1 - P_{00}(t) = \frac{\lambda - \lambda e^{-(\lambda + \mu)t}}{\lambda + \mu}$. $P_{10}(t) = 1 - P_{11}(t) = \frac{\mu - \mu e^{-(\lambda + \mu)t}}{\lambda + \mu}$.

3

由 图此 $P(t) = \cancel{Rght+} \begin{pmatrix} P_{00}(t) & P_{01}(t) \\ P_{10}(t) & P_{11}(t) \end{pmatrix}$

$$= \begin{pmatrix} \dfrac{\mu + \lambda e^{-(\lambda+\mu)t}}{\lambda+\mu} & \dfrac{\lambda - \lambda e^{-(\lambda+\mu)t}}{\lambda+\mu} \\[4mm] \dfrac{\mu - \mu e^{-(\lambda+\mu)t}}{\lambda+\mu} & \dfrac{\lambda + \mu e^{-(\lambda+\mu)t}}{\lambda+\mu} \end{pmatrix}.$$

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4. (1) 考虑 $P(X_{t+s}=j \mid X_t=i, X_{t_1}=i_1, \cdots, X_{t_n}=i_n)$

$= P(Y_{N_{t+s}}=\delta \mid Y_{N_t}=i, Y_{N_{t_1}}=i_1, \cdots, Y_{N_{t_n}}=i_n)$, $\quad 0 < t_1 < t_2 < \cdots < t_n < t < t+s.$

$= \dfrac{P(Y_{N_{t+s}}=\delta, Y_{N_t}=i, Y_{N_{t_1}}=i_1, \cdots, Y_{N_{t_n}}=i_n)}{P(Y_{N_t}=i, Y_{N_{t_1}}=i_1, \cdots, Y_{N_{t_n}}=i_n)}.$ ①

(由于 N_t �增的,)

分母 $P(Y_{N_t}=i, Y_{N_{t_1}}=i_1, \cdots, Y_{N_{t_n}}=i_n)$

$= \displaystyle\sum_{0 < k_1 < k_2 \cdots < k_n < k} P(Y_k=i, Y_{k_1}=i_1, \cdots, Y_{k_n}=i_n) \cdot P(N_t=k, N_{t_1}=k_1, \cdots, N_{t_n}=k_n)$

由 Y 和过程 N 的 独立性 $\quad \displaystyle\sum_{0<k_1<\cdots<k_n<k} \big[P(Y_k=i \mid Y_{k_n}=i_n) P(Y_{k_n}=i_n \mid Y_{k_{n-1}}=i_{n-1}) \cdots P(Y_{k_1}=i_1) \big] \cdot$

$\big[P(N_t=k \mid N_{t_n}=k_n) P(N_{t_n}=k_n \mid N_{t_{n-1}}=k_{n-1}) \cdots P(N_{t_1}=k_1) \big]$

$= \displaystyle\sum_{0<k_1<\cdots<k_n<k} P(Y_{k-k_n+k_n}=i \mid Y_{k_n}=i_n) P(N_{t-t_n}=k-k_n) \big[P(Y_{k_n}=i_n \mid Y_{k_{n-1}}=i_{n-1}) \cdots P(Y_{k_1}=i_1) P(N_t=k \mid N_{t_{n-1}}=k_{n-1}) \cdots P(N_{t_1}=k_1) \big]$ ②

因为 Y 为时齐的. $\therefore P(Y_{k-k_n+k_n}=i \mid Y_{k_n}=i_n) = P(Y_{k-k_n}=i \mid Y_0=i_n).$

上式又用到 N_t 的平稳增量性. $P(N_t=k \mid N_{t_n}=k_n) = P(N_t-N_{t_n}=k-k_n \mid N_{t_n}=k_n) \xrightarrow{独立} P(N_t-N_{t_n}=k-k_n) \xrightarrow{平稳} P(N_{t-t_n}=k-k_n)$

因此 对②式 作 替换 $k^* = k - k_n.$

则有 ② $= \displaystyle\sum_{0<k_1<\cdots<k_n} \sum_{k^*} P(Y_{k^*}=i \mid Y_0=i_n) P(N_{t-t_n}=k^*) \big[P(Y_{k_n}=i_n \mid Y_{k_{n-1}}=i_{n-1}) \cdots P(Y_{k_1}=i_1) P(N_{t_n}=k_n \mid N_{t_{n-1}}=k_{n-1}) \cdots P(N_{t_1}=k_1) \big]$

$= \displaystyle\sum_{k^*} P(Y_{k^*}=i \mid Y_0=i_n) P(N_{t-t_n}=k^*) \cdot \sum_{0<k_1<\cdots<k_n} \big[P(Y_{k_n}=i_n \mid Y_{k_{n-1}}=i_{n-1}) \cdots \cdots P(N_{t_1}=k_1) \big]$ ③

定理 ① 中分子有

$$P(Y_{N_{t+s}}=j,\ Y_{N_t}=i,\ \cdots,\ Y_{N_{t_n}}=i_n)$$

$$=\sum_{0<k_n<\cdots<k_n<k<k'} P(Y_{k'}=j,\ Y_{k_1}=i,\cdots,Y_{k_n}=i_n,\ Y_k=i)\,P(N_{t+s}=k',\ N_t=k,\ N_{t_1}=k_1,\cdots N_{t_n}=k_n)$$

由独立性，
将 $k_1\sim k_n$ 与 k,k' 分离，

$$\sum_{k^*<k'} P(Y_{k'}=j\mid Y_{k^*}=i)\,P(Y_{k^*}=i\mid Y_0=i_n)\,P(N_{t+s}=k'\mid N_t=k^*)\,P(N_{t-t_n}=k^*)$$

$$\times \sum_{0<k_1<\cdots<k_n}\big[\,P(Y_{k_n}=i_n\mid Y_{k_{n-1}}=i_{n-1})\cdots P(N_{t_1}=k_1)\,\big]$$

⑭

而由 ③④，代入 ①，有

$$P(X_{t+s}=j\mid X_t=i,\ X_{t_1}=i_1,\cdots,X_{t_n}=i_n)=\frac{\displaystyle\sum_{k^*<k'} P(Y_{k'}=j\mid Y_{k^*}=i)\,P(Y_{k^*}=i\mid Y_0=i_n)\,P(N_{t+s}=k'\mid N_t=k^*)\,P(N_{(t-t_n)}=k^*)}{\displaystyle\sum_{k^*} P(Y_{k^*}=i\mid Y_0=i_n)\,P(N_{(t-t_n)}=k^*)}$$

$$=\frac{P(Y_{N_{t+s}}=j,\ Y_{N_t}=i)}{P(Y_{N_t}=i)}=P(X_{t+s}=j\mid X_t=i)\qquad (\text{因为 } k_1\sim k_n \text{ 被约掉})$$

因此 X 是马氏链。

而 $P(X_{t+s}=j\mid X_t=i)=P(Y_{N_{t+s}}=j\mid Y_{N_t}=i)=P(Y_{N_{t+s}-N_t+N_t}=j\mid Y_{N_t}=i)$

Y 的齐次性 $\longrightarrow P(Y_{N_{t+s}-N_t}=j\mid Y_0=i)\ \xrightarrow[\ N_{t+s}-N_t\ \overset{d}{=}\ N_s\ ]{\text{由 } N \text{ 的独立增量性}}\ P(Y_{N_s}=j\mid Y_0=i)$ 与 t 无关

因此 X 是时齐的。

$$P_{ij}(t)=P(X_t=j\mid X_0=i)=P(Y_{N_t}=j\mid Y_0=i)$$

对 N_t 全概率展开

$$=\sum_{k=0}^{\infty} P(Y_k=j\mid Y_0=i)\,P(N_t=k)$$

$$=\sum_{k=0}^{\infty} P_{ij}(k)\cdot P(N_t=k)\qquad \text{其中 } P_{ij}(k) \text{ 为 } k \text{ 步转移概率}$$

由 Kolmogorov 理论，k 步转移矩阵为 $\hat P^k$，即 $P_{ij}(k)=\hat P^k_{(i,j)}$

则 $P_{ij}(t)=\Big[\sum_{k=0}^{\infty}\hat P^k\,P(N_t=k)\Big]_{(i,j)}$

因此 $P(t)=\sum_{k=0}^{\infty}\hat P^k\cdot\frac{(\lambda t)^k}{k!}\,e^{-\lambda t}=\sum_{k=0}^{\infty}\frac{(\lambda t\hat P)^k}{k!}\,e^{-\lambda t I}$ ⑤

$$=e^{\lambda t(\hat P-I)}$$

其中 指数 $e^A \overset{def}{=} \sum_{k=0}^{\infty} \frac{A^k}{k!}$

而 $Q = P'(t)|_{t=0} = \lambda(\hat{P}-I) \cdot e^{\lambda t(\hat{P}-I)}|_{t=0} = \lambda(\hat{P}-I)$

$\therefore Q = \lambda(\hat{P}-I)$ $(q_{ij} = \lambda(\hat{P}_{ij}-\delta_{ij}))$

(2). 若 $\vec{\pi}$ 有不变分布 $\vec{\pi}$.

则有 $\vec{\pi} = \vec{\pi}\hat{P}$.   同理 $\vec{\pi}\hat{P}^k = (\vec{\pi}\hat{P})\cdot\hat{P}^{k-1} = \vec{\pi}\hat{P}^{k-1} = \cdots = \vec{\pi}\hat{P} = \vec{\pi}$.

⑥

又同理 $\vec{\pi}P(t) \overset{由⑤}{\Longrightarrow} \vec{\pi}\cdot\sum_{k=0}^{\infty}\frac{(\lambda t\hat{P})^k}{k!}e^{-\lambda t I}$

$= \vec{\pi}\sum_{k=0}^{\infty}\frac{(\lambda t)^k \cdot \vec{\pi}\hat{P}^k}{k!}e^{-\lambda t I}$

$\overset{由⑥}{\Longrightarrow} \sum_{k=0}^{\infty}\frac{(\lambda t)^k \cdot \vec{\pi}}{k!}e^{-\lambda t} = \vec{\pi}\cdot\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t} = \vec{\pi}e^{\lambda t}\cdot e^{-\lambda t}$

$= \vec{\pi}$

同理 $\vec{\pi}$ 满足 $\vec{\pi} = \vec{\pi}P(t)$, $P(t)$ 为 X 的转移概率矩阵.

$\therefore \vec{\pi}$ 是 X 的不变分布.

___

5. (1) $\propto r < s < t$. 由 独立增量性.

$B_r, B_s - B_r, B_t - B_s$ 相互独立.

因 $B_t \sim N(0,t)$, $B_r \sim N(0,r)$, $B_s - B_r \sim N(0,s-r)$, $B_t - B_s \sim N(0,t-s)$

同理 $(B_r, B_s - B_r, B_t - B_s)$ 的 联合概率密度 为 3个高斯分布的乘积:

$$p(a,b,c) = \frac{1}{\sqrt{2\pi r}}e^{-\frac{a^2}{2r}}\cdot\frac{1}{\sqrt{2\pi(s-r)}}e^{-\frac{b^2}{2(s-r)}}\cdot\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{c^2}{2(t-s)}}$$

而 (B_r, B_s, B_t) 为 (B_r, B_s-B_r, B_t-B_s) 的线性变换, 仍为 高斯分布.

因且 $(B_r, B_s, B_t) = (B_r, B_s-B_r, B_t-B_s)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

而 雅可比矩阵 行列式 为 $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$.

同理变量替换后.

(B_r, B_s, B_t) 的 联合密度 为: $p(x,y,z) = \frac{1}{\sqrt{8\pi^3 r(s-r)(t-s)}}e^{-\frac{x^2}{2r}}\cdot e^{-\frac{(y-x)^2}{2(s-r)}}\cdot e^{-\frac{(z-y)^2}{2(t-s)}}$

B_t 是标准布朗运动. 同理 $EB_t=0$. $EB_tB_s=s$, $t\geq s\geq 0$

(2). $EW_t = t\cdot EB_{\frac{1}{t}} = 0$.

同时由书之定理. $\lim_{t\to\infty}\frac{B_t}{t}=0$. (由 Borel-Cantelli 定理及 B_t 独立增量性 易证)

同理 $EW_t = t\cdot EB_{\frac{1}{t}} = 0$. ①

$\overline{EW_tW_s}$

$t\geq s>0$ 时 $EW_tW_s = ts\, EB_{\frac{1}{t}}B_{\frac{1}{s}} \xrightarrow{\frac{1}{t}\leq\frac{1}{s}} ts\cdot\frac{1}{t} = s$. ②

同时在 0 处.

$\lim_{t\to 0} W_t = \lim_{t\to 0} t B_{\frac{1}{t}} = \lim_{t\to +\infty}\frac{B_t}{t} = 0 = W_0$

在 $t>0$ 时 由 B_t 关于 t 连续. (布朗运动性质)

同理 W_t 关于 t 连续.

由 ①② 及 轨道连续性. W 也是标准布朗运动.

(3). 考虑 分割 $0=t_0 = t_1 < t_2 < \cdots < t_n = t$. $\lambda = \max \Delta t_k$

设 $\Delta t_k = t_k - t_{k-1}$, $k=1,2,\cdots,n$. $\lambda = \max \Delta t_k \to 0$

u_k 为 (t_{k-1}, t_k) 为中的 任意取值.

则 X_t 写作 $\lim_{\lambda\to 0}\sum_{k=1}^{n} u_k(B_{t_k} - B_{t_{k-1}})$

全由 $\sum_{k=1}^{n} u_k(B_{t_k}-B_{t_{k-1}})$ 为高斯分布的线性组合 因此仍为高斯分布

在 $\lambda\to 0$ 极限下也是高斯分布.

由 $B_{t_k}-B_{t_{k-1}} \sim N(0, t_k-t_{k-1})$, 且由独立增量性, 不同的 $B_{t_k}-B_{t_{k-1}}$ 独立

∴ $EX_t = \lim_{\lambda\to 0}\sum_{k=1}^{n} u_k E(B_{t_k}-B_{t_{k-1}}) = 0$.

$Var X_t = \sum_{k=1}^{n} u_k^2 Var(B_{t_k}-B_{t_{k-1}}) = \sum_{k=1}^{n} u_k^2(t_k-t_{k-1}) = \sum_{k=1}^{n} u_k^2 \Delta t_k$, $t_{k-1}\leq u_k\leq t_k$

$\lambda\to 0$ 时, 上式可看作 $\int_0^t s^2 ds$.

∴ $X_t \sim N(0, \int_0^t s^2 ds)$.

而协方差函数 $cov(X_t, X_s) = EX_t X_s$.

当 $t > s > 0$ 时.

考虑分割 Δ': $0 = t_0 < t_1 < t_2 < \cdots < t_k = s < t_{k+1} < \cdots < t_n = t$.

则 $cov(X_t, X_s) =$

$EX_t X_s = E\left(\sum_{k=1}^{k} u_k^2 \Delta t_k + \sum_{k=s+1}^{t} u_k^2 \Delta t_k \right)$

$E\left(\left[\sum_{i=1}^{k} u_k (B_{t_k} - B_{t_{k-1}}) + \sum_{i=k+1}^{n} u_k (B_{t_k} - B_{t_{k-1}}) \right] \times \sum_{i=1}^{k} u_k (B_{t_k} - B_{t_{k-1}}) \right)$

由独立增量性

$\longrightarrow$

后一项中当 $k \neq k'$ 时 $E(B_{t_{i_1}} - B_{t_{i_1}-1})(B_{t_{i_2}} - B_{t_{i_2}-1}) = 0$.

$= E\left(\sum_{i=1}^{k} u_k (B_{t_k} - B_{t_{k-1}}) \right)^2 = Var X_s = \int_0^s u^2 du$.

$\therefore cov(X_t, X_s) = \int_0^s u^2 du$. $t > s > 0$.

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(4). 随机积分. 考虑 $f(t, x) = tx^2$.

则 $\frac{\partial f}{\partial t} = x^2$. $\frac{\partial f}{\partial x} = 2tx$ $\frac{\partial^2 f}{\partial x^2} = 2t$.

而 $dB_t = 0 \cdot dt + 1 \cdot dB_t$. $\therefore b = 0$. $\sigma = 1$.

得 $df(t, B_t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot b + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 \right)^{(t, B_t)} dt + \left(\frac{\partial f}{\partial x} \sigma \right)^{(t, B_t)} \cdot dB_t$

$= (x^2 + t)(t, B_t) dt + (2tx)(t, B_t) dB_t$

$= (B_t^2 + t) dt + 2t B_t dB_t$

故 $t B_t^2 - 0 \cdot B_0^2 = \int_0^t (B_s^2 + s) ds + 2 \int_0^t s B_s dB_s$

$= \int_0^t (B_s^2 + s) ds + 2 Y_t$

$\therefore Y_t = \frac{1}{2}\left[t B_t^2 - \int_0^t (B_s^2 + s) ds \right]$

8.