1. 计算极限$\lim\limits_{x\to 0}\left(\dfrac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}$.

$$\lim_{x\to 0}\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\dfrac{\frac{1}{x}\ln(1+x)-1}{x}},$$

$$\lim_{x\to 0}\frac{\frac{1}{x}\ln(1+x)-1}{x}=\lim_{x\to 0}\frac{\frac{1}{x}[x-\frac{x^2}{2}+o(x^2)]-1}{x}=\lim_{x\to 0}\frac{1-\frac{x}{2}+o(x)]-1}{x}=-\frac{1}{2}.$$

$$\lim_{x\to 0}\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}=e^{-\frac{1}{2}}.$$

2. 写出$f(x)=\dfrac{1-x+x^2}{1+x+x^2}$在$x=0$点的Peano余项型Taylor公式.

$$f(x)=\frac{1-x+x^2}{1+x+x^2}=\frac{1+x+x^2-2x}{1+x+x^2}=1-\frac{2x}{1+x+x^2}$$
$$=1-\frac{2x(1-x)}{1-x^3}=1-(2x-2x^2)\left[\sum_{k=0}^{n}x^{3k}+o(x^{3n})\right]=1-2\sum_{k=0}^{n}x^{3k+1}+2\sum_{k=0}^{n}x^{3k+2}+o(x^{3n+1}).$$

3. 设$f(x)=(\arcsin x)^2$, 求$f^{(n)}(0)$, $\ n\in\mathbb{N}$.

$$f'(x)=2\arcsin x\,\frac{1}{\sqrt{1-x^2}},\quad f'(x)\sqrt{1-x^2}=2\arcsin x,$$
$$f''(x)\sqrt{1-x^2}-\frac{xf'(x)}{\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}},\quad f''(x)(1-x^2)-xf'(x)=2,$$
$$\big(f''(x)(1-x^2)\big)^{(n-2)}-\big(xf'(x)\big)^{(n-2)}=0,$$
$$\sum_{k=0}^{n-2}C_{n-2}^{k}[(1-x^2)]^{(k)}[f''(x)]^{(n-2-k)}-\sum_{k=0}^{n-2}C_{n-2}^{k}[x]^{(k)}[f'(x)]^{(n-2-k)}=0,$$
$$(1-x^2)f^{(n)}(x)-C_{n-2}^{1}2xf^{(n-1)}(x)-C_{n-2}^{2}2f^{(n-2)}(x)-xf^{(n-1)}(x)-(n-2)f^{(n-2)}(x)=0,$$
$$f^{(n)}(0)-C_{n-2}^{2}2f^{(n-2)}(0)-(n-2)f^{(n-2)}(0)=0,\quad f^{(n)}(0)=(n-2)^2f^{(n-2)}(0).$$
$$f'(0)=0,\quad f''(0)=2,$$
$$f^{(2k+1)}(0)=0,\quad k=0,1,2,\cdots;$$
$$f^{(2k)}(0)=(2k-2)^2f^{(2k-2)}(0)=(2k-2)^2(2k-4)^2\cdots 2^2f''(0)=2((2k-2)!!)^2,\quad k=1,2,\cdots.$$

4. 非线性函数$f(x)\in C[a,b],D(a,b)$, 证明：
$\exists\xi,\eta\in(a,b)\ \ s.t.\ \ f'(\xi)>\dfrac{f(b)-f(a)}{b-a},\ \ f'(\eta)<\dfrac{f(b)-f(a)}{b-a}$.

因为$f(x)$非线性, 所以$\exists x_0\in(a,b)\ \ s.t.\ \ f(x_0)$不在过$(a,f(a))$, $(b,f(b))$的割线上.
不妨设$f(x_0)$在割线的上方. 则
$\exists\xi\in(a,x_0)\subset(a,b)\ \ s.t.\ \ f'(\xi)=\dfrac{f(x_0)-f(a)}{x_0-a}>\dfrac{f(b)-f(a)}{b-a}$;
$\exists\xi\in(a,x_0)\subset(a,b)\ \ s.t.\ \ f'(\eta)=\dfrac{f(b)-f(x_0)}{b-x_0}<\dfrac{f(b)-f(a)}{b-a}$.

5. 设函数$f(x)$在$[-1,1]$上三阶可导, $f(0)=f'(0)=f(-1)=0,\quad f(1)=1$.
求证: $\exists \xi\in(-1,1)\quad s.t.\quad f'''(\xi)=3$.

$$0=f(-1)=f(0)+f'(0)(-1)+\frac{f''(0)}{2}+\frac{f'''(\xi_1)}{6}(-1)^3,\quad \xi_1\in(-1,0);$$
$$1=f(1)=f(0)+f'(0)(1)+\frac{f''(0)}{2}+\frac{f'''(\xi_2)}{6}(1)^3,\quad \xi_2\in(0,1).$$
$$1=\frac{1}{6}(f'''(\xi_1)+f'''(\xi_2))=\frac{1}{3}f'''(\xi),\quad \xi\in(\xi_1,\xi_2)\subset(a,b).$$

6. 设有界函数$f(x)$在$(-\infty,+\infty)$上二次可微, 证明: $\exists x_0\in(-\infty,+\infty)\quad s.t.\quad f''(x_0)=0$.

【反证法】假设$\not\exists x_0\in(-\infty,+\infty)\quad s.t.\quad f''(x_0)=0$, 则$f''(x)$与$(-\infty,+\infty)$定号, 不妨设$f''(x)\geq 0,\quad x\in(-\infty,+\infty)$. 则$f'(x)$单调上升.
(1)若$f'(x)\equiv 0,\quad x\in(-\infty,+\infty)$, 则$f''(x)\equiv 0$, 与假设不符.
(2)若$\exists x_0\in(-\infty,+\infty)\quad s.t.\quad f'(x_0)>0$, 则$f'(x)\geq f'(x_0)>0,\quad x\in[x_0,+\infty)$
$\Rightarrow\quad f(x)-f(x_0)=f'(\xi)(x-x_0)\geq f'(x_0)(x-x_0),\quad \forall x\in[x_0,+\infty)$
$\Rightarrow\quad f(x)\geq f'(x_0)(x-x_0)+f(x_0),\quad \forall x\in[x_0,+\infty)$
$\Rightarrow\quad \lim\limits_{x\to+\infty}f(x)=+\infty$, 与$f(x)$有界矛盾.
(3)若$\exists x_0\in(-\infty,+\infty)\quad s.t.\quad f'(x_0)<0$, 则$f'(x)\leq f'(x_0)<0,\quad x\in(-\infty,x_0]$
$\Rightarrow\quad f(x)-f(x_0)=f'(\xi)(x-x_0)\geq f'(x_0)(x-x_0),\quad \forall x\in(-\infty,x_0]$
$\Rightarrow\quad f(x)\geq f'(x_0)(x-x_0)+f(x_0),\quad \forall x\in(-\infty,x_0]$
$\Rightarrow\quad \lim\limits_{x\to-\infty}f(x)=+\infty$, 与$f(x)$有界矛盾.
综上所述, 假设不成立. 所以$\exists x_0\in(-\infty,+\infty)\quad s.t.\quad f''(x_0)=0$.

7. $$\int\frac{1+\sin x+\cos x}{1+\cos^2 x}dx=\int\left[\frac{1}{1+\cos^2 x}+\frac{\sin x}{1+\cos^2 x}+\frac{\cos x}{1+\cos^2 x}\right]dx$$
$$=\int\frac{1}{\sec^2 x+1}\frac{dx}{\cos^2 x}+\int\frac{-d\cos x}{1+\cos^2 x}+\int\frac{d\sin x}{2-\sin^2 x}$$
$$=\int\frac{d\tan x}{2+\tan^2 x}-\arctan(\cos x)+\int\frac{1}{2\sqrt{2}}\left[\frac{1}{\sqrt{2}-\sin x}+\frac{1}{\sqrt{2}+\sin x}\right]d\sin x$$
$$=\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)-\arctan(\cos x)+\frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{2}+\sin x}{\sqrt{2}-\sin x}\right|+c$$

8. 计算$\int\frac{x^3\arcsin x}{\sqrt{1-x^2}}dx$.
因为$\int\frac{x^3}{\sqrt{1-x^2}}dx=\frac{1}{2}\int\frac{x^2d(x^2)}{\sqrt{1-x^2}}\overset{1-x^2=u}{=}\frac{1}{2}\int\frac{(u-1)du}{\sqrt{u}}$
$$=\frac{1}{2}\int(-\frac{1}{\sqrt{u}}+\sqrt{u})du=\frac{1}{2}(-2\sqrt{u}+\frac{2}{3}u^{\frac{3}{2}})+c=-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}+c.$$
所以,
$$\int\frac{x^3\arcsin x}{\sqrt{(1-x^2)}}dx=\int\arcsin x\ d\left(-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right)$$
$$=\arcsin x\ (-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}})-\int\left[-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]\frac{dx}{\sqrt{1-x^2}}$$

$$= \arcsin x\ (-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}}) + x - \frac{1}{3}\int(1-x^2)dx$$
$$= \arcsin x\ (-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}}) + \frac{2x}{3} + \frac{x^3}{9} + c.$$

9. $\int \sqrt{\frac{e^x-1}{e^x+1}}dx.$

【法一】:

令$\sqrt{\frac{e^x-1}{e^x+1}} = t$, 则$e^x - 1 = t^2e^x + t^2$, $e^x = \frac{1+t^2}{1-t^2} = \frac{2}{1-t^2} - 1$, $e^x dx = \frac{4t}{(1-t^2)^2}dt$,

$$dx = \frac{1-t^2}{1+t^2}\frac{4t}{(1-t^2)^2}dt = \frac{4t}{(1+t^2)(1-t^2)}dt.$$
$$\int \sqrt{\frac{e^x-1}{e^x+1}}dx = \int t\ \frac{4t}{(1+t^2)(1-t^2)}dt = 2\int\left(\frac{1}{1-t^2} - \frac{1}{1+t^2}\right)dt$$
$$= 2\int\left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt - 2\arctan t + c = \ln\left|\frac{1+t}{1-t}\right| - 2\arctan t + c.$$
$$\frac{1+t}{1-t} = \frac{\sqrt{\frac{e^x-1}{e^x+1}}+1}{\sqrt{\frac{e^x-1}{e^x+1}}-1} = \frac{\sqrt{e^x-1}+\sqrt{e^x+1}}{\sqrt{e^x+1}-\sqrt{e^x-1}}$$
$$= \frac{1}{2}\left[\sqrt{e^x-1}+\sqrt{e^x+1}\right]^2 = \frac{1}{2}\left[2e^x + 2\sqrt{e^{2x}-1}\right] = e^x + \sqrt{e^{2x}-1}.$$
$$\int \sqrt{\frac{e^x-1}{e^x+1}}dx = \ln\left(e^x + \sqrt{e^{2x}-1}\right) - 2\arctan\sqrt{\frac{e^x-1}{e^x+1}} + c.$$

【法二】:

$\int \sqrt{\frac{e^x-1}{e^x+1}}dx = \int \frac{e^x-1}{\sqrt{e^{2x}-1}}dx.$ 原题显示$x > 0$.

令$e^x = \sec\theta$, 则$e^x dx = \sec\theta\tan\theta d\theta$, $dx = \tan\theta d\theta$.

$$\int \frac{e^x-1}{\sqrt{e^{2x}-1}}dx = \int \frac{\sec\theta - 1}{\tan\theta}\tan\theta d\theta \quad = \int \sec\theta d\theta - \theta + c$$
$$= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \theta + c = \frac{1}{2}\ln\left|\frac{1+\frac{\sqrt{e^{2x}-1}}{e^x}}{1-\frac{\sqrt{e^{2x}-1}}{e^x}}\right| - \arccos(e^{-x}) + c$$
$$= \frac{1}{2}\ln\left|\frac{e^x+\sqrt{e^{2x}-1}}{e^x-\sqrt{e^{2x}-1}}\right| - \arccos(e^{-x}) + c = \ln(e^x + \sqrt{e^{2x}-1}) - \arccos(e^{-x}) + c.$$

10. 设函数$f(x)$在$x = 0$处连续, 且$\exists a > 1\ \ s.t.\ \ \lim\limits_{x\to 0}\frac{f(ax)-f(x)}{x}$存在, 证明$f(x)$在$x = 0$处可导.

$$\lim_{x\to 0}\frac{f(ax)-f(x)}{x} = m \ \Rightarrow\ \forall\varepsilon > 0,\ \exists\delta > 0\ \ s.t.\ \left|\frac{f(ax)-f(x)}{x} - m\right| < \varepsilon,\ \forall x \in (-\delta, +\delta).$$
$$\forall x \in (0, +\delta),\ \ -\varepsilon < \frac{f(ax)-f(x)}{x} - m < \varepsilon,\ \ (m-\varepsilon)x < f(ax) - f(x) < (m+\varepsilon)x.$$
$$(m-\varepsilon)x < f(ax) - f(x) < (m+\varepsilon)x$$
$$(m-\varepsilon)\frac{x}{a} < f(x) - f(\frac{x}{a}) < (m+\varepsilon)\frac{x}{a}$$
$$(m-\varepsilon)\frac{x}{a^2} < f(\frac{x}{a}) - f(\frac{x}{a^2}) < (m+\varepsilon)\frac{x}{a^2}$$
$$\cdots \quad \cdots \quad \cdots$$
$$(m-\varepsilon)\frac{x}{a^n} < f(\frac{x}{a^{n-1}}) - f(\frac{x}{a^n}) < (m+\varepsilon)\frac{x}{a^n}$$
$$(m-\varepsilon)x\frac{1-\frac{1}{a^{n+1}}}{1-\frac{1}{a}} < f(ax) - f(\frac{x}{a^n}) < (m+\varepsilon)x\frac{1-\frac{1}{a^{n+1}}}{1-\frac{1}{a}}$$

Let $n \to +\infty$, $(m-\varepsilon)x \dfrac{a}{a-1} \le f(ax) - f(0) < (m+\varepsilon)x \dfrac{a}{a-1}$

$\dfrac{m-\varepsilon}{a-1} < \dfrac{f(ax)-f(0)}{ax} < \dfrac{m+\varepsilon}{a-1} \quad \Rightarrow \quad \left|\dfrac{f(ax)-f(0)}{ax} - \dfrac{m}{a-1}\right| < \dfrac{\varepsilon}{a-1}, \quad \forall x \in (-\delta, +\delta).$

所以,$f'_+(0) = \dfrac{m}{a-1}$. 同理可证$f'_-(0) = \dfrac{m}{a-1}$. 所以$f'(0) = \dfrac{m}{a-1}$.

11. 证明：有限长度开区间上的有界凸函数一致连续.

【pku-week-13-1】设有界函数$f(x)$于有限区间(a,b)凸, 证明: $\lim\limits_{x\to b-0} f(x)$, $\lim\limits_{x\to a+0} f(x)$都存在.

任意固定一点$x_0 \in (a,b)$, 记$x_1 = \dfrac{x_0+b}{2}$, $M = \sup\limits_{x\in(a,b)} f(x)$.

$\dfrac{f(x)-f(x_0)}{x-x_0} \le \dfrac{M-f(x_0)}{x_1-x_0}, \quad \forall x \in (x_1, b).$

$f(x)$于(a,b)凸 $\Rightarrow$ $\dfrac{f(x)-f(x_0)}{x-x_0}$于$(x_1,b)$单调上升有上界$\dfrac{M-f(x_0)}{x_1-x_0}$.

所以$\exists A \in \mathbb{R}$ $s.t.$ $\lim\limits_{x\to b-0} \dfrac{f(x)-f(x_0)}{x-x_0} = A$.

从而$f(x) = \dfrac{f(x)-f(x_0)}{x-x_0}(x-x_0) + f(x_0)$

$\Rightarrow \lim\limits_{x\to b-0} f(x) = \lim\limits_{x\to b-0}\left[\dfrac{f(x)-f(x_0)}{x-x_0}(x-x_0) + f(x_0)\right] = A(b-x_0) + f(x_0).$

同理可证$\lim\limits_{x\to a+0} f(x)$的存在性.

有限长度开区间上的有界凸函数可以连续延拓为闭区间上的凸函数.
有限长度开区间上的有界凸函数可以延拓为闭区间上的连续函数, 从而是开区间/闭区间上的一致连续函数.