参考答案.

1. $\sum_{n=1}^{\infty}\left(\sqrt[n]{n}-1-\sin\frac{1}{n}\right)$. $\quad u_n=\sqrt[n]{n}-1-\sin\frac{1}{n}$

$$\sqrt[n]{n}=e^{\frac{1}{n}\ln n}=1+\frac{\ln n}{n}+\frac{1}{2}\left(\frac{\ln n}{n}\right)^2+o\left(\left(\frac{\ln n}{n}\right)^2\right),\ (n\to\infty)$$

$$\sin\frac{1}{n}=\frac{1}{n}+o\left(\frac{1}{n^2}\right),\ (n\to\infty);\quad u_n=\frac{\ln n-1}{n}+o\left(\frac{1}{n^2}\right)+o\left(\left(\frac{\ln n}{n}\right)^2\right),\ (n\to\infty)$$

$\therefore\ u_n\Big/\frac{\ln n-1}{n}\to 1\ (n\to+\infty)$ $\therefore\ \sum_{n=1}^{\infty}u_n$ 与 $\sum_{n=1}^{\infty}\frac{\ln n-1}{n}$ 同敛散. $\therefore\ \sum_{n=1}^{\infty}u_n$ 发散.

2. $I=\int_1^{+\infty}\sin(x^2)\arctan(x^3)\,dx$.

$$\int_1^{+\infty}\sin(x^2)\,dx\ \xlongequal[x=\sqrt{t}]{x^2=t}\ \int_1^{+\infty}\sin t\,\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_1^{+\infty}\frac{\sin t}{\sqrt{t}}\,dt$$

$\frac{1}{\sqrt{t}}\searrow 0,\ (t\to+\infty)$, $\left|\int_1^{A}\sin t\,dt\right|\le 2,\ \forall A>1$. $\therefore$ 由 Dirichlet 判. $\int_1^{+\infty}\sin(x^2)\,dx$ 收敛

又 x^3 单调递增, $\arctan(x^3)$ 单调递增 有界 $|\arctan(x^3)|\le\frac{\pi}{2},\ \forall x\in\mathbb{R}^+$.

$\therefore$ 由 Abel 判. $\int_1^{+\infty}\sin(x^2)\arctan(x^3)\,dx$ 收敛

3. ① $g(x)=g(x+T),\ \forall x\in\mathbb{R},\ T>0,\ g(x)\in C[0,T],\ \int_0^T g(x)\,dx=0$.

$g(x)\in C[0,T]\Rightarrow\exists M>0$ s.t. $|g(x)|\le M,\ \forall x\in\mathbb{R}$; $\exists n\in\mathbb{N}$, s.t. $\lambda b-\lambda a=nT+\sigma$, $\sigma\in[0,T)$.

$$\int_a^b g(\lambda x)\,dx\ \xlongequal{\lambda x=t}\ \int_{\lambda a}^{\lambda b}g(t)\frac{dt}{\lambda}=\frac{1}{\lambda}\int_{\lambda a}^{\lambda a+nT}g(t)\,dt+\frac{1}{\lambda}\int_{\lambda a+nT}^{\lambda b}g(t)\,dt$$

$$=\frac{1}{\lambda}\int_{\lambda a+nT}^{\lambda a+nT+\sigma}g(t)\,dt=\frac{1}{\lambda}\int_0^{\sigma}g(t)\,dt$$

$$\left|\int_a^b g(\lambda x)\,dx\right|\le\frac{MT}{\lambda}\quad\Rightarrow\quad\lim_{\lambda\to+\infty}\int_a^b g(\lambda x)\,dx=0.$$

3.② 令 $h(x)=g(x)-\frac{1}{T}\int_0^T g(x)dx$. 则 $\int_0^T h(x)dx=0$. $h(x)=h(x+T)$, $x\in\mathbb{R}$. 记 $M=\max\limits_{[0,T]}|h(x)|$.

$$\lim_{\lambda\to+\infty}\int_a^b g(\lambda x)f(x)dx=\frac{1}{T}\int_0^T g(x)dx\int_a^b f(x)dx \iff \lim_{\lambda\to+\infty}\int_a^b h(\lambda x)f(x)dx=0.$$

$\forall\varepsilon>0$,

$f(x)\in R[a,b] \Rightarrow$ ∃阶梯函数 $\omega(x)=\{C_i,\ x\in[x_{i-1},x_i]$, $a=x_0<x_1<\cdots<x_n=b$.

s.t. $\int_a^b |f(x)-\omega(x)|dx<\frac{\varepsilon}{2M}$.

此时 $$\int_a^b h(\lambda x)f(x)dx = \int_a^b h(\lambda x)\omega(x)dx+\int_a^b h(\lambda x)[f(x)-\omega(x)]dx$$

$$\left|\int_a^b h(\lambda x)[f(x)-\omega(x)]dx\right|\le M\int_a^b|f(x)-\omega(x)|dx< M\cdot\frac{\varepsilon}{2M}=\varepsilon$$

$$\left|\int_a^b h(\lambda x)\omega(x)dx\right| = \left|\sum_{i=1}^n\int_{x_{i-1}}^{x_i} h(\lambda x)C_i\,dx\right| = \left|\sum_{i=1}^n C_i\int_{x_{i-1}}^{x_i}h(\lambda x)dx\right|$$

由①, $\lim\limits_{\lambda\to+\infty}\int_{x_{i-1}}^{x_i}h(\lambda x)dx=0$, $i=1,2,\cdots,n$.

∴ $\exists\Lambda_i>0$ s.t. $\lambda>\Lambda_i$ 时 $\left|\int_{x_{i-1}}^{x_i}h(\lambda x)dx\right|\le\frac{\varepsilon}{2nM_0}$. 其中 $M_0=\max\limits_{1\le i\le n}|C_i|$

取 $\Lambda=\max\limits_{1\le i\le n}\Lambda_i$. 则 $\lambda>\Lambda$ 时.

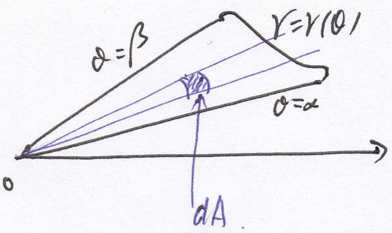
$$\left|\int_a^b h(\lambda x)\omega(x)dx\right|\le\sum_{i=1}^n|C_i|\left|\int_{x_{i-1}}^{x_i}h(\lambda x)dx\right|\le\sum_{i=1}^n|C_i|\frac{\varepsilon}{2nM_0}\le\frac{\varepsilon}{2}.$$

从而, $\forall\varepsilon>0$. $\exists\Lambda>0$ s.t. $\lambda>\Lambda$ 时, $\left|\int_a^b h(\lambda x)f(x)dx\right|<\varepsilon$.

∴ $\lim\limits_{\lambda\to+\infty}\int_a^b h(\lambda x)f(x)dx=0$. ※

4.

[法一]



根据古鲁金第二定理，$V = A\,2\pi\bar{y}$.

其中A是平面图形的面积，$\bar{y}$是平面图形的重心的纵坐标.

极坐标中面积微元为 $dA = r\,dr\,d\theta$. 对 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$A = \int dA = \int_\alpha^\beta d\theta \left(\int_0^{r(\theta)} r\,dr\right) = \frac{1}{2}\int_\alpha^\beta r^2(\theta)\,d\theta$$

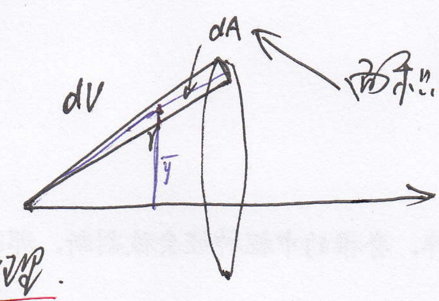
$$\therefore \bar{y} = \frac{\int y\,dA}{\int dA},$$

$$= \frac{1}{A}\int_\alpha^\beta d\theta \int_0^{r(\theta)} r\sin\theta\, r\,dr = \frac{1}{A}\int_\alpha^\beta \sin\theta \cdot \frac{r^3(\theta)}{3}\,d\theta = \frac{1}{3A}\int_\alpha^\beta r^3(\theta)\sin\theta\,d\theta$$

$$\therefore V = A2\pi\bar{y} = \frac{2\pi}{3}\int_\alpha^\beta r^3(\theta)\sin\theta\,d\theta$$

[法二]

$V = \int dV$.



面积微元. $dA = \frac{1}{2}r^2(\theta)\,d\theta$

对 dV 使用古鲁金第二定理.

$dV = dA\,2\pi\bar{y}$，其中$\bar{y}$是dA的重心的纵坐标.

等腰三角形的重心在其高的$\frac{2}{3}$处

$\therefore \bar{y} = \frac{2}{3}r(\theta)\sin\theta$ $\quad\therefore dV = 2\pi\cdot\frac{2}{3}r(\theta)\sin\theta\,\frac{1}{2}r^2(\theta)\,d\theta = \frac{2\pi}{3}r^3(\theta)\sin\theta\,d\theta$

$$\therefore V = \frac{2\pi}{3}\int_\alpha^\beta r^3(\theta)\sin\theta\,d\theta$$

5. $I=\int_0^1 \frac{\ln x}{1-x}dx \xlongequal{1-x=t} \int_1^0 \frac{\ln(1-t)}{t}(-dt) = \int_0^1 \frac{\ln(1-t)}{t}dt$

$\ln(1-t) = -\int_0^t \frac{ds}{1-s} = -\int_0^t \left(\sum_{n=0}^\infty s^n\right) ds = -\sum_{n=0}^\infty \frac{t^{n+1}}{n+1}$ $\quad t\in[-1,1)$

$\sum_{n=0}^\infty \frac{t^{n+1}}{n+1}$ 于 $[0,A]$ 一致收敛 $\forall A\in(0,1)$.

$I=\lim_{A\to 1-0}\int_0^A \frac{\ln(1-t)}{t}dt = \lim_{A\to 1-0}\int_0^A \left(-\sum_{n=0}^\infty \frac{t^{n+1}}{n+1}\right)\frac{dt}{t}$

$=\lim_{A\to 1-0} -\sum_{n=0}^\infty \int_0^A \frac{t^n}{n+1}dt$ ← 一致收敛

$\because \sum_{n=0}^\infty \frac{x^{n+1}}{(n+1)^2}$ 于 $[0,1]$ 一致收敛.

$=\lim_{A\to 1-0} \sum_{n=0}^\infty \frac{-A^{n+1}}{(n+1)^2} = \sum_{n=0}^\infty \lim_{A\to 1-0} \frac{-A^{n+1}}{(n+1)^2} = \sum_{n=0}^\infty \frac{-1}{(n+1)^2} = -\sum_{n=1}^\infty \frac{1}{n^2}.$

也可以. $I=\lim_{A\to 1-0}\int_0^A \ln x \sum_{n=0}^\infty x^n dx = \cdots\cdots$ 只是此时 $\int_0^A x^n \ln x\, dx$ 计算略繁.

6. ① 收敛级数的Cauchy乘积不一定收敛. 见课本例9.4.1 (Page 160).

$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^\alpha}$, $\alpha\in(0,\frac{1}{2}]$. 则自乘的Cauchy乘积发散.

② 如果 $\sum_{n=1}^\infty a_n=A$, $\sum_{n=1}^\infty b_n=B$, $\sum_{n=1}^\infty c_n=C$, 其中 $c_n=\sum_{i+j=n} a_ib_j$. 则 $C=AB$.

见增补习题课107内容第8页.

对幂级数 $\sum_{n=1}^\infty a_nx^n$, $\sum_{n=1}^\infty b_nx^n$, $\sum_{n=1}^\infty c_nx^n$ 来说,

$\because \sum_{n=1}^\infty a_n$, $\sum_{n=1}^\infty b_n$, $\sum_{n=1}^\infty c_n$ 收敛, ∴ 它们的收敛域都包含 $[0,1]$. 从而它们都在 $[0,1]$ 一致收敛, 并在 $[0,1)$ 绝对收敛.

$\therefore \sum_{n=1}^\infty c_nx^n = \sum_{n=1}^\infty a_nx^n \sum_{n=1}^\infty b_nx^n$, $\quad x\in[0,1)$. 又根据一致收敛对极限运算的连续性, $\lim_{x\to 1-0}\sum_{n=1}^\infty c_nx^n = \lim_{x\to 1-0}\sum_{n=1}^\infty a_nx^n \lim_{x\to 1-0}\sum_{n=1}^\infty b_nx^n$.

即 $\sum_{n=1}^\infty c_n = \sum_{n=1}^\infty a_n \sum_{n=1}^\infty b_n$. i.e. $C=AB$. (极限运算与求和交换顺序).

7. $f(x)=\sum_{n=1}^{\infty}\frac{e^{-n}}{\sqrt{n}}\sin nx,\ x\in\mathbb{R}.$

① $f'(x)\ \exists\ x\in\mathbb{R}$

② $\underbrace{f'(x_0)}_{\exists x_0\ \text{s.t.}}\quad |f'(x_0)|\geq\frac{e}{\sqrt{2}(e^2-1)}.$

① $f'(x)=\sum_{n=1}^{\infty}\sqrt{n}e^{-n}\cos nx,\quad x\in\mathbb{R}\quad\Leftarrow\quad$ U.C.

② $\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)^2\,dx=\sum_{n=1}^{\infty}ne^{-2n}.$

记 $g(t)=\sum_{n=1}^{\infty}nt^n$, $|t|<1$. 则 $g(t)=t\left(\sum_{n=1}^{\infty}nt^{n-1}\right)=t\left(\sum_{n=1}^{\infty}t^n\right)'$

$=t\cdot\left(\frac{t}{1-t}\right)'=t\left(\frac{1}{1-t}-1\right)'=t/(1-t)^2\qquad t\in(-1,1).$

$\therefore\ g(e^{-2})=\frac{e^{-2}}{(1-e^{-2})^2}=\frac{e^2}{(e^2-1)^2}.$

$f'(x)\in C[-\pi,\pi].\quad\exists\ x_0\ \text{s.t.}\ |f'(x_0)|=\max.$

$\frac{1}{\pi}\cdot|f'(x_0)|^2\cdot2\pi\geq\frac{e^2}{(e^2-1)^2}\Rightarrow|f'(x_0)|\geq\frac{e}{\sqrt{2}(e^2-1)}.$