

# 2023春代数学2实验班期中考试

2023.4.16 上午8:00-10:30

1.(5 points)Consider the following diagram of abelian groups:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ a \downarrow & & b \downarrow & & c \downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

Suppose that the two rows are exact. If  $g$  and  $a$  are surjective and  $b$  is injective, show that  $c$  is injective.

2.(5 points)Let  $\mathfrak{p}$  be a minimal prime ideal of a ring  $A$ , show that for every  $x \in \mathfrak{p}$  there is some  $s \in A \setminus \mathfrak{p}$  and an integer  $k \geq 0$  such that  $sx^k = 0$ .

3.Let  $A$  be a commutative ring. Let  $\Sigma$  be the set of all multiplicative closed subsets  $S$  of  $A$  such that  $0 \notin S$ .

(1)(5 points)Show that  $\Sigma$  has a maximal element. Show also that  $S$  is a maximal element of  $\Sigma$  if and only if  $A \setminus S$  is a minimal prime ideal.

(2)(5 points)Show that for every prime ideal  $\mathfrak{p}$  there is a minimal prime ideal contained in it.

(3)(5 points)Let  $f : A \rightarrow B$  be a ring homomorphism, show that for every prime ideal  $\mathfrak{p}$  of  $A$ , there is a prime ideal  $\mathfrak{q}$  of  $B$  such that  $\mathfrak{p} = \mathfrak{q} \cap A$ .

4.Let  $A$  be a commutative ring and  $M, N$  be  $A$ -modules. Let  $u : M \rightarrow N$  be an  $A$ -module homomorphism and  $\mathfrak{p}$  be a prime ideal of  $A$ .

(1)(5 points)Let  $Q$  be a finitely generated  $A$ -module, then  $Q_{\mathfrak{p}} = 0$  if and only if there is some  $f \in A \setminus \mathfrak{p}$  such that  $fQ = 0$ .

(2)(5 points)Suppose that  $N$  is finitely generated, and  $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$  is surjective. Show that there is some  $f \in A \setminus \mathfrak{p}$  such that  $u_f : M_f \rightarrow N_f$  is surjective.

(3)(5 points)Suppose furthermore that  $M$  is finitely generated, and  $N$  is finitely presented. If  $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$  is bijective. Show that there is some  $f \in A \setminus \mathfrak{p}$  such that  $u_f : M_f \rightarrow N_f$  is bijective.

5. Let  $A$  be a commutative ring and  $M$  be a finitely presented  $A$ -module.  
 (1)(5 points) Show that for any  $A$ -module  $N$  and flat  $A$ -algebra  $B$ , we have

$$\operatorname{Hom}_A(M, N) \otimes B \cong \operatorname{Hom}_B(M \otimes A, N \otimes B)$$

- (2)(5 points) Suppose  $A$  is a local ring with maximal ideal  $\mathfrak{m}$ , and let  $N$  be a flat  $A$ -module. Suppose  $x_1, \dots, x_n \in N$  are such that their images  $\overline{x_1}, \dots, \overline{x_n} \in N/\mathfrak{m}N$  are linearly independent over  $A/\mathfrak{m}$ . Show that  $x_1, \dots, x_n$  are linearly independent over  $A$ .  
 (3)(5 points) Show that  $M$  is projective if and only if for every maximal ideal  $\mathfrak{m}$ ,  $M_{\mathfrak{m}}$  is free.  
 (4)(5 points) Show that  $M$  is projective if and only if it is flat.

6. Let  $f : A \rightarrow B$  be a flat homomorphism.

- (1)(5 points) Let  $N$  be an  $A$ -module and  $N_1, N_2$  be two submodules. Show that we have an equality (as submodules of  $N \otimes B$ ):

$$(N_1 \cap N_2) \otimes B = (N_1 \otimes B) \cap (N_2 \otimes B)$$

- (2)(5 points) Let  $\mathfrak{a}_1, \mathfrak{a}_2$  be ideals of  $A$ , show that  $(\mathfrak{a}_1 \cap \mathfrak{a}_2)B = (\mathfrak{a}_1 B \cap \mathfrak{a}_2 B)$   
 (3)(5 points) Suppose furthermore that  $\mathfrak{a}_2$  is finitely generated, show that

$$(\mathfrak{a}_1 : \mathfrak{a}_2)B = \mathfrak{a}_1 B : \mathfrak{a}_2 B$$

.

7. Let  $M$  be a finitely generated  $A$ -module.

- (1)(5 points) Show that  $\operatorname{Supp}(M)$  is a closed set of  $\operatorname{Spec}(A)$ .  
 (2)(5 points) Suppose that  $A$  is Noetherian, and  $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$  for some ideal  $\mathfrak{a}$ , show that there is some  $n \geq 0$  such that  $\mathfrak{a}^n M = 0$ .

8.(10 points) Let  $A$  be a Noetherian ring and  $M$  be a finitely generated  $A$ -module. Prove that the following are equivalent.

- (1)  $M$  is of finite length;
- (2) Every prime ideal  $\mathfrak{p} \in \operatorname{Ass}(M)$  is a maximal ideal;
- (3) Every prime ideal  $\mathfrak{p} \in \operatorname{Supp}(M)$  is a maximal ideal.

9.(5 points) Let  $A$  be a ring whose prime ideals are all finitely generated. Show that  $A$  is Noetherian.

Hints and Solutions (not official):

For 4(3), try to prove the following lemma:

Let  $M$  be finitely generated and  $N$  be finitely presented. Suppose  $u : M \rightarrow N$  is surjective, then  $\text{Ker}(u)$  is finitely generated.

For 6(3), first reduce to the case that  $\mathfrak{a}_2$  is generated by a single element  $a$ , then it's clear that  $(\mathfrak{a}_1 : (a))B \subseteq \mathfrak{a}_1B : aB$ . Conversely, let  $b \in B$  such that  $ab = \sum a_i b_i \in \mathfrak{a}_1B$ , by flatness, we have the relation  $b = \sum x_j y_j$  and  $b_i = \sum x_{ij} y_j$ , then  $ax_j = \sum a_i x_{ij} \in \mathfrak{a}_1$ , hence  $x_j \in (\mathfrak{a}_1 : (a))$  and  $b \in (\mathfrak{a}_1 : (a))B$ .

1. Let  $\gamma$  be any element in  $\text{Ker}(c)$ , we can find some  $\beta \in B$  such that  $g(\beta) = \gamma$ . Set  $\beta' = b(\beta)$ , then  $g'(\beta') = c(\gamma) = 0$ , and by exactness we have  $\beta' \in \text{Ker}(g') = \text{Im}(f')$ , say  $\beta' = f'(\alpha')$ . Since  $a$  is surjective, there is some  $\alpha \in A$  such that  $a(\alpha) = \alpha'$ . Now note that  $b(f(\alpha)) = f'(a(\alpha)) = \beta' = b(\beta)$  and by injectivity,  $f(\alpha) = \beta$ , then  $c = g \circ f(\alpha) = 0$  by exactness.

2. We localize at  $\mathfrak{p}$ , then  $\mathfrak{p}A_{\mathfrak{p}}$  is the only prime ideal of  $A_{\mathfrak{p}}$  by minimality, hence is the nilradical. Now for every  $x \in \mathfrak{p}$ ,  $(\frac{x}{1})^k = 0$  for some integer  $k \geq 0$ , it follows our conclusion.

3.(1) The first claim is a direct corollary from Zorn's lemma. For the latter one, note that  $S^{-1}A$  is not the zero ring so there is some prime ideal which doesn't intersect  $S$ . Also note that for any prime ideal  $\mathfrak{p}$ ,  $A \setminus \mathfrak{p} \in \Sigma$ .

(2) Localize at  $\mathfrak{p}$  and use (1)

(3) Localize at  $\mathfrak{p}$ , then  $B_{\mathfrak{p}}$  is not a zero ring. Take a maximal prime ideal  $\mathfrak{q}'$  of  $B_{\mathfrak{p}}$  and take  $\mathfrak{q} = \mathfrak{q}' \cap B$ .

4.(1) Write  $Q = Aq_1 + \cdots + Aq_n$ , and for each  $q_i$  there is some  $f_i \in A \setminus \mathfrak{p}$  such that  $f_i q_i = 0$ . Then set  $f = \prod f_i$ .

(2) By exactness of localization,  $\text{Coker}(u)_{\mathfrak{p}} = 0$  and  $\text{Coker}(u)$  is finitely generated. Then there is some  $f \in A \setminus \mathfrak{p}$  such that  $f \text{Coker}(u) = 0$ . It follows that  $\text{Coker}(u)_f = 0$  and  $u_f$  is surjective.

(3) By (2), there is some  $f_c \in A \setminus \mathfrak{p}$  such that  $u_{f_c}$  is surjective. Replace  $A$  by  $A_{f_c}$ , we may assume that  $u$  itself is surjective. We claim that  $\text{Ker}(u)$  is finitely generated, then a similar argument shows that there is some  $f_k \in A \setminus \mathfrak{p}$  such that  $\text{Ker}(u)_{f_k} = 0$  and hence  $u_{f_k}$  is bijective. The claim is due to the following

lemma.

**Lemma.** Let  $A$  be a ring,  $M, N$  be  $A$ -modules, and  $u : M \rightarrow N$  be a surjective homomorphism. Suppose  $M$  is finitely generated and  $N$  is finitely presented. Then  $\text{Ker}(u)$  is finitely generated.

**Proof.** Let  $0 \rightarrow K' \xrightarrow{i'} A^s \xrightarrow{p'} N \rightarrow 0$  and  $A^r \xrightarrow{\tilde{p}} M \rightarrow 0$  be exact sequences where  $K'$  is finitely generated. Since free modules are projective, we have the following liftings:

$$\begin{array}{ccc} M & \xrightarrow{u} & N \longrightarrow 0 \\ \nwarrow p & & \uparrow (\tilde{p} \circ u) \oplus p' \\ & A^r \oplus A^s & \end{array} \quad \begin{array}{ccc} A^s & \xrightarrow{p'} & N \longrightarrow 0 \\ \nwarrow \tilde{v} & & \uparrow \tilde{p} \circ u \\ & A^r & \end{array}$$

Let  $v = \tilde{v} \oplus id : A^r \otimes A^s \rightarrow A^r \otimes A^s$ , which is a surjective map. Consider the embedding  $i : K = \text{Ker}(p) \rightarrow A^r \oplus A^s$ , one can verify that the image of the composition  $v \circ i$  has to lie in the kernel of  $p'$ , then we get a homomorphism  $w : K \rightarrow K'$ . It follows that we have commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & A^r \oplus A^s & \xrightarrow{p} & M \longrightarrow 0 \\ & & \downarrow w & & \downarrow v & & \downarrow u \\ 0 & \longrightarrow & K' & \xrightarrow{i'} & A^s & \xrightarrow{p'} & N \longrightarrow 0 \end{array}$$

Then by snake lemma, we have an exact sequence  $\text{Ker}(v) \rightarrow \text{Ker}(u) \rightarrow \text{Coker}(w) \rightarrow 0$  since  $v$  is surjective. Note that  $K'$  is finitely generated, then  $\text{Coker}(w)$  is finitely generated. It remains to show that  $\text{Ker}(v)$  is finitely generated. This is because the following exact sequence splits:

$$0 \longrightarrow \text{Ker}(v) \longrightarrow A^r \oplus A^s \xrightarrow{v} A^s \longrightarrow 0$$

then we have a projection  $A^r \oplus A^s \rightarrow \text{Ker}(v)$ .

5.(1) Choose a finite presentation of  $M$ :

$$A^r \rightarrow A^s \rightarrow M \rightarrow 0$$

then the following sequences are exact:

$$0 \rightarrow \text{Hom}_A(M, N) \otimes B \rightarrow \text{Hom}_A(A^s, N) \otimes B \rightarrow \text{Hom}_A(A^r, N) \otimes B$$

$$0 \rightarrow \text{Hom}_B(M \otimes B, N \otimes B) \rightarrow \text{Hom}_B(A^s \otimes B, N \otimes B) \rightarrow \text{Hom}_B(A^r \otimes B, N \otimes B)$$

Then we use the canonical isomorphisms

$$\text{Hom}_A(A^n, N) \otimes B \cong N^n \otimes B \cong (N \otimes B)^n$$

$$\mathrm{Hom}_B(A^n \otimes B, N \otimes B) \cong \mathrm{Hom}_B(B^n, N \otimes B) \cong (N \otimes B)^n$$

(2) See notes in class. (Use induction on  $n$ .)

(3) “ $\Rightarrow$ ” Since  $M$  is projective,  $M$  is flat, so  $M_{\mathfrak{m}}$  is flat for every maximal ideal  $\mathfrak{m}$ . Then it is free since it is finitely generated over a local ring. “ $\Leftarrow$ ” Let  $N \rightarrow N'$  be any surjective map, it suffices to show that  $\mathrm{Hom}_A(M, N) \rightarrow \mathrm{Hom}_A(M, N')$  is surjective. Take  $B = A_{\mathfrak{m}}$  in (1), we see that  $(\mathrm{Hom}_A(M, N))_{\mathfrak{m}} = \mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}})$ , so by local property, it remains to show that  $\mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \rightarrow \mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N'_{\mathfrak{m}})$  surjective for each  $\mathfrak{m}$ . This follows from that  $A_{\mathfrak{m}}$  is free hence projective.

(4) Recall that  $M$  is flat if and only if  $M_{\mathfrak{m}}$  is flat, that is free, for every maximal ideal  $\mathfrak{m}$ .

6.(1) From  $0 \rightarrow N_i \rightarrow N \rightarrow N/N_i \rightarrow 0$ , we have

$$0 \rightarrow N_i \otimes B \rightarrow N \otimes B \rightarrow N/N_i \otimes B \rightarrow 0$$

then  $N \otimes B/N_i \otimes B \cong (N/N_i) \otimes B$ . Now use the canonical exact sequence  $0 \rightarrow N_1 \cap N_2 \rightarrow N \rightarrow (N/N_1) \times (N/N_2)$  and get

$$0 \rightarrow N_1 \cap N_2 \otimes B \rightarrow N \otimes B \rightarrow (N/N_1) \times (N/N_2) \otimes B$$

Since  $(N/N_1) \times (N/N_2) \otimes B = (N \otimes B/N_1 \otimes B) \times (N \otimes B/N_2 \otimes B)$ , then  $(N_1 \cap N_2) \otimes B = (N_1 \otimes B) \cap (N_2 \otimes B)$ .

(2) Recall that since  $B$  is flat,  $\mathfrak{a} \otimes B \cong \mathfrak{a}B$ .

(3) Since  $(\mathfrak{a}_1 : (a) + (b)) = (\mathfrak{a}_1 : (a)) \cap (\mathfrak{a}_1 : (b))$ , by (2), we may assume  $\mathfrak{a}_2$  is generated by a single element  $a$ . If  $am \in \mathfrak{a}_1$  for some  $m \in A$ , it is clear that  $amb \in \mathfrak{a}_1 B$  for every  $b \in B$  hence  $(\mathfrak{a}_1 : (a))B \subseteq \mathfrak{a}_1 B : aB$ . Conversely, let  $b \in B$  such that  $ab = \sum a_i b_i \in \mathfrak{a}_1 B$ , by flatness, we have the relations  $b = \sum x_j y_j$  and  $b_i = \sum x_{ij} y_j$  with  $x_j, x_{ij} \in A$ , then  $ax_j = \sum a_i x_{ij} \in \mathfrak{a}_1$ , hence  $x_j \in (\mathfrak{a}_1 : (a))$  and  $b \in (\mathfrak{a}_1 : (a))B$ .

7.(1) Suppose  $M = Am_1 + \cdots + Am_n$ , then  $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(Am_i) = \bigcup V(\mathrm{Ann}(m_i))$  is closed.

(2) By the given condition,  $\sqrt{\mathrm{Ann}(m_i)} \supseteq \sqrt{I}$ . Hence

$$I \subseteq \sqrt{I} \subseteq \bigcap \sqrt{\mathrm{Ann}(m_i)} = \sqrt{\bigcap \mathrm{Ann}(m_i)}$$

Since  $A$  is noetherian, we can assume  $I$  is generated by  $(a_1, \dots, a_k)$  and  $a_j^{r_j} \in \bigcap \mathrm{Ann}(m_i)$ . Hence  $I^{\sum r_j} M = 0$ . (or there is some positive integer  $n$  such that  $I^n \subseteq \sqrt{\bigcap \mathrm{Ann}(m_i)}^n \subseteq \bigcap \mathrm{Ann}(m_i)$ .)

8.(1) $\Rightarrow$ (2) As a submodule of  $M$ ,  $A/\mathfrak{p}$  is of finite length for each  $\mathfrak{p} \in \text{Ass}(M)$ . Note that each  $A$ -submodule of  $A/\mathfrak{p}$  is an  $A/\mathfrak{p}$ -module,  $A/\mathfrak{p}$  is both Noetherian and Artinian as a ring, so  $\dim(A/\mathfrak{p}) = 0$  and  $(0)$  is the only prime ideal of  $A/\mathfrak{p}$ , so  $\mathfrak{p}$  is maximal.

(2) $\Rightarrow$ (3) Every minimal element in  $\text{Supp}(M)$  is in  $\text{Ass}(M)$

(3) $\Rightarrow$ (2)  $\text{Ass}(M) \subseteq \text{Supp}(M)$

(2) $\Rightarrow$ (1) By definition of associated primes, we have a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each subquotient  $M_i/M_{i-1} \cong A/\mathfrak{p}_i$  for some  $\mathfrak{p}_i \in \text{Ass}(M)$ . Note that  $A/\mathfrak{p}_i$  is a field hence is a simple  $A$ -module, this gives a finite composition series of  $M$ .

9. This is exactly the first exercise in chapter 7 whose hints give enough details.