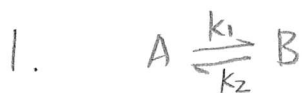数学模型

1. $A \xrightleftharpoons[k_2]{k_1} B$

(a) $\dfrac{dA}{dt} = -k_1 A + k_2 B$

$\dfrac{dB}{dt} = k_1 A - k_2 B$

$A(0) = A_0 \qquad B(0) = B_0$

由 $\dfrac{d}{dt}(A+B) = 0$ 知

$A(t) + B(t) = A_0 + B_0$

则平衡解也有 $\bar{A} + \bar{B} = A_0 + B_0$ (1)

又由反应率模型知,平衡解满足

$-k_1 \bar{A} + k_2 \bar{B} = 0 \qquad (2)$

由 (1)(2) 知 $\bar{A} = \dfrac{k_2}{k_1+k_2}(A_0+B_0)$

$\bar{B} = \dfrac{k_1}{k_1+k_2}(A_0+B_0)$

(b) 令 $f(x) = \ln x + \dfrac{1}{x} - 1$, $x > 0$

$f'(x) = \dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x+1}{x}$

则 当 $x \in (0,1)$ 时 $f'(x) < 0$

当 $x = 1$ 时 $f'(x) = 0$

当 $x > 1$ 时 $f'(x) > 0$

则 当 $x > 0$ 时 $f(x) \geqslant f(1) = 0$

则 $x > 0$ 时 $\ln x + \dfrac{1}{x} - 1 > 0$

即 $\ln x > 1 - \dfrac{1}{x}$

当 $t > 0$ 时, $A(t), B(t) > 0$, $\bar{A}, \bar{B} > 0$

则 $\bar{A} \ln\left(\dfrac{A}{\bar{A}}\right) + \bar{B} \ln\left(\dfrac{B}{\bar{B}}\right)$

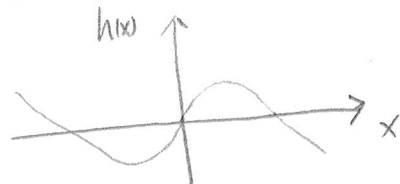$\geqslant \bar{A}\left(1 - \dfrac{\bar{A}}{A}\right) + \bar{B}\left(1 - \dfrac{\bar{B}}{B}\right)$

$= (A+B) - (\bar{A}+\bar{B}) = 0$

这里,我们用到了 (1).

证毕。

2. $\dot{x} = x - x^3 - r$

令 $h(x) = x - x^3 \qquad f(x;r) = x - x^3 - r$

令 $h(x) = 0$ 全 $x = -1, 0, 1$

$h'(x) = 1 - 3x^2 = -3\left(x^2 - \dfrac{1}{3}\right)$

令 $h'(x) = 0$ $x = \pm\sqrt{\dfrac{1}{3}}$

$h(x)$ 在 $x = \sqrt{\dfrac{1}{3}}$ 取极大值 $h\left(\sqrt{\tfrac{1}{3}}\right) = \sqrt{\dfrac{1}{3}} - \sqrt{\dfrac{1}{3}} = \dfrac{2}{3\sqrt{3}}$

在 $x = -\sqrt{\dfrac{1}{3}}$ 取极小值 $h\left(-\sqrt{\tfrac{1}{3}}\right) = -\dfrac{2}{3\sqrt{3}}$

当 $x > \sqrt{\dfrac{1}{3}}$, $h'(x) < 0$, $h(x) \searrow$

当 $-\sqrt{\dfrac{1}{3}} < x < \sqrt{\dfrac{1}{3}}$ 时, $h'(x) > 0$, $h(x) \nearrow$

当 $x < -\sqrt{\dfrac{1}{3}}$, $h'(x) < 0$, $h(x) \searrow$

注意 $h(x) = f(x;r)$, $f(x;r) = 0$ 决定平衡点.

当 $r > \dfrac{2}{3\sqrt{3}}$ 时 有一个稳定平衡点
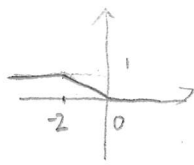
当 $r < -\dfrac{2}{3\sqrt{3}}$ 时 有一个稳定平衡点,

当 $-\dfrac{2}{3\sqrt{3}} < r < \dfrac{2}{3\sqrt{3}}$ 时, 有二个稳定,一个不稳定平衡点

当 $r = \pm\dfrac{2}{3\sqrt{3}}$ 时, 有一个稳定,一个不稳定的平衡点,此时出现了分岔。

3. (a)

$$u_t + u u_x = 0$$

$$u(x,0) = \begin{cases} 0, & x > 0 \\ -\frac{x}{2}, & -2 < x < 0 \\ 1, & x < -2 \end{cases}$$

其对应的 Lagrangian ODE 为

$$\begin{cases} \dfrac{dX}{dt} = U(t; X_0) & X(0) = X_0 \\ \dfrac{dU}{dt} = 0, & U(0; X_0) = u(X_0, 0) \end{cases}$$

当 $X_0 > 0$ 时　　$U(0; X_0) = u(X_0, 0) = 0$

$$U(t; X_0) = U(0; X_0) = 0$$

则　$\dfrac{dX}{dt} = 0 \Rightarrow X = X_0$

即　$u(x,t) = 0, \quad x > 0$

当 $-2 < X_0 < 0$ 时

$$U(t; X_0) = U(0; X_0) = u(X_0, 0) = -\frac{X_0}{2}$$

则　$\dfrac{dX}{dt} = -\dfrac{X_0}{2}$

$$\Rightarrow X = -\frac{X_0}{2} t + X_0 = (1 - \frac{t}{2}) X_0$$

$$\Rightarrow X_0 = \frac{1}{1 - \frac{t}{2}} X = \frac{2X}{2-t}$$

则　$u(x,t) = -\dfrac{x}{2-t}, \quad t-2 < x < 0$

当　$X_0 < -2$ 时

$$U(t; X_0) = u(X_0, 0) = 1$$
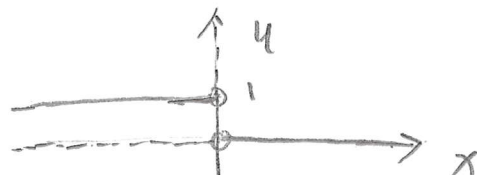
则　$\dfrac{dX}{dt} = 1$

$$\Rightarrow X = t + X_0$$

$$\Rightarrow X_0 = x - t$$

$$u(x,t) = 1, \qquad x < t-2.$$

则 $0 \leq t < 2$ 时

$$u(x,t) = \begin{cases} 1, & x < t-2 \\ \dfrac{x}{t-2}, & t-2 < x < 0 \\ 0, & x > 0 \end{cases}$$
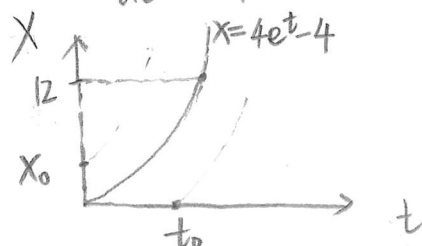
当 $t = 2$ 时，$u(x,2)$ 如下

(b) 　$p_t + (x+4) p_x = -2p \quad x \geq 0$

$$p(0,t) = 1, \qquad p(x,0) = 0$$

对应的 Lagrangian ODE 为

(1) $\dfrac{dX}{dt} = X + 4 \quad X(0) = X_0$ 或 $X(t_0) = 0$

(2) $\dfrac{dp}{dt} = -2p, \quad ($令 $p(t) = p(X,t))$

解(1)得　当 $X_0 > 0$ 时

$$X(t) = (X_0 + 4) e^t - 4$$

当 $t_0 > 0$ 时

$$X(t) = 4 e^{t - t_0} - 4$$

两个区域的分割线为

$$X(t) = 4 e^t - 4$$

则 当 $x > 4e^t - 4$ 时

$$p(x,t) = 0$$

当 $x < 4e^t - 4$ 时

$$p(x,t) > 0$$

令 $\bar{X} = 4e^b - 4$，得

$$t = \ln 4$$

则 从 $t = \ln 4$ 开始，$\bar{X} = 12$ 开始接收信号

考虑 $t_0 = 0^+$ 时

$$\frac{dP}{dt} = -2P, \quad P(0, t_0^+) = P(0, 0^+) = 1$$

$$\Rightarrow P(t) = P(0) e^{-2t} = e^{-2t}$$

$$P(\ln 4) = 4^{-2} = \frac{1}{16}$$

则 刚接收信号时，强度为 $\frac{1}{16}$

4. (a)

$$P_t + (PV)_x = 0 \cdots (1)$$

$$V = X^2 e^{-3t}$$

$(1) \Rightarrow \quad P_t + V P_x = -P V_x$

即 $\quad P_t + X^2 e^{-3t} P_x = -2x e^{-3t} P \cdots (2)$

(2) 即为一个半线性(一阶)波方程

对应的 Lagrangian ODE 为

$$\frac{dX}{dt} = X^2 e^{-3t} \quad X(0) = X_0 \cdots (3)$$

$$\frac{dP}{dt} = -2X e^{-3t} P, \quad (P(t; X_0) = p(X, t))$$

为了求 $X_1(t), X_2(t)$，只需当 $X_0 = 1, X_0 = 2$ 时

求解 $X(t)$

由 (3) 得 $\quad -\frac{1}{X} = -\frac{1}{3} e^{-3t} + C$

代入两个初值得

$$X_1(t) = \frac{1}{\frac{2}{3} + \frac{1}{3} e^{-3t}}$$

$$X_2(t) = \frac{1}{\frac{1}{6} + \frac{1}{3} e^{-3t}}$$

(b) 注意 $\frac{dX_1}{dt} = v(x, t)\big|_{x = X_1}$

$$\frac{dX_2}{dt} = v(x, t)\big|_{x = X_2}$$

则 $\frac{d}{dt} m(t)$

$$= \frac{d}{dt} \int_{X_1(t)}^{X_2(t)} P(x, t) \, dx$$

$$= \int_{X_1}^{X_2} \frac{\partial}{\partial t} P \, dx$$

$$- P(X_1, t) \frac{dX_1}{dt} + P(X_2, t) \frac{dX_2}{dt}$$

$$= \int_{X_1}^{X_2} \frac{\partial}{\partial t} P \, dx$$

$$+ P(x, t) v(x, t) \big|_{x = X_1}^{x = X_2}$$

$$= \int_{X_1}^{X_2} \left[\frac{\partial}{\partial t} P + \frac{\partial}{\partial x} (p v) \right] dx$$

即推导了 Reynolds Transport Thm.

又由 (1) 知

$$\frac{d}{dt} m(t) = \int_{X_1}^{X_2} 0 \, dx = 0$$

证毕.

5. $E = \int_{\mathbb{R}} H(\rho)\,dx + \int_{\mathbb{R}} \rho V\,dx + \frac{1}{2}\iint W(|x-y|)\,\rho(x)\rho(y)\,dx\,dy$

(a) 令 $h(x)$ 为扰动函数

$$\int_{\mathbb{R}} \frac{\delta E}{\delta \rho} h\,dx = \left[\frac{d}{d\varepsilon}\int_{\mathbb{R}} H(\rho+\varepsilon h)\,dx + \int_{\mathbb{R}}(\rho+\varepsilon h)V\,dx + \frac{1}{2}\iint W(|x-y|)(\rho(x)+\varepsilon h(x))(\rho(y)+\varepsilon h(y))\,dx\,dy\right]\Big|_{\varepsilon}$$

$$= \int_{\mathbb{R}} H'(\rho)\,h\,dx + \int V h\,dx + \frac{1}{2}\iint W(|x-y|)(\rho(x)h(y)+\rho(y)h(x))\,dx\,dy$$

$$= \int_{\mathbb{R}}\left(H'(\rho(x))+V(x)+\int W(|x-y|)\rho(y)\,dy\right)h(x)\,dx$$

则 $\xi = \dfrac{\delta E}{\delta \rho} = H'(\rho) + V + \int_{\mathbb{R}} W(|x-y|)\rho(y)\,dy$

(b) $\dfrac{dE}{dt} = \int_{\mathbb{R}} H'(\rho)\,\rho_t\,dx + \int_{\mathbb{R}} V\rho_t + \frac{1}{2}\iint W(|x-y|)\left(\rho_t(x,t)\rho(y,t)+\rho(x,t)\rho_t(y,t)\right)\,dx\,dy$

$$= \int_{\mathbb{R}}\left(H'(\rho(x))+V(x)+\int_{\mathbb{R}} W(|x-y|)\rho(y)\,dy\right)\rho_t(x,t)\,dx$$

$$= \int_{\mathbb{R}} \xi\,\frac{\partial}{\partial x}\left(\rho\,\frac{\partial \xi}{\partial x}\right)\,dx$$

$$= -\int_{\mathbb{R}}\left(\frac{\partial \xi}{\partial x}\right)^2 \rho\,dx$$

上一步分部积分的边界项为 0 是因为 ρ 只在有限区域不为 0

由假设 $\rho(x,t) \geqslant 0$，则 $\left(\dfrac{\partial \xi}{\partial x}\right)^2 \rho \geqslant 0$

则 $\dfrac{dE}{dt} = -\int_{\mathbb{R}}\left(\frac{\partial \xi}{\partial x}\right)^2 \rho\,dx \leq 0$.

证毕.