数学模型第四次作业

1. (a). 将 $x(t) = lsin\theta(t)$, $y(t) = -lcos\theta(t)$ 代入 I 中有

$$I = \int \frac{1}{2}ml^2\theta'(t)^2 + mglcos\theta(t)dt := \int L(t, \theta, \dot{\theta})dt$$

由 Eular-Lagrange 方程

$$\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}}) = 0$$

即

$$-mglsin\theta(t) - ml^2\theta''(t) = 0$$

也即

$$\theta''(t) + \frac{g}{l}sin\theta(t) = 0$$

(b). 将 $y = f(x)$ 代入 I 中有

$$I = \int \frac{1}{2}mx'(t)^2[1 + f'(x(t))^2] - mgf(x(t))dt := \int L(t, x, \dot{x})$$

由 Eular-Lagrange 方程

$$\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{x}}) = 0$$

即

$$mf'(x)f''(x)x'(t)^2 - mgf'(x) - mx''(t) - mx''(t)f'(x)^2 - 2mf'(x)f''(x)x'(t)^2 = 0$$

也即

$$f'(x)f''(x)x'(t)^2 + x''(t)[1 + f'(x)^2] + gf'(x) = 0$$

(c). 将 $x(t) = r(t)cos\theta(t)$, $y(t) = r(t)sin\theta(t)$, $z(t) = r(t)$ 代入 I 中有

$$I = \int mr'(t)^2 + \frac{1}{2}mr(t)^2\theta'(t)^2 - mgr(t)dt := \int L(t, \theta, r, \dot{\theta}, \dot{r})$$

由 Eular-Lagrange 方程

$$\begin{cases} \dfrac{\partial L}{\partial \theta} - \dfrac{\partial}{\partial t}(\dfrac{\partial L}{\partial \dot{\theta}}) = 0 \\ \dfrac{\partial L}{\partial r} - \dfrac{\partial}{\partial t}(\dfrac{\partial L}{\partial \dot{r}}) = 0 \end{cases}$$

即

$$\begin{cases} -\dfrac{d}{dt}(mr(t)^2\theta'(t)) = 0 \\ mr(t)\theta'(t)^2 - mg - 2mr''(t) = 0 \end{cases}$$

也即

$$\begin{cases} \theta'(t) = \dfrac{c}{r(t)^2} \\ 2r''(t) = \dfrac{c^2}{r(t)^3} - g \end{cases}$$

2.

$$x'(t) = l\theta' cos\theta(t)'$$

$$y'(t) = -\sigma\omega\, cos\omega t + l\theta' sin\theta(t)$$

$$L = \frac{1}{2}m[x'(t)^2 + y'(t)^2] - mgy(t)$$

由于

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}(\frac{\partial L}{\partial \theta})$$

带入化简得

$$l\ddot{\theta} + gsin\theta + \omega^2\sigma\sin(\omega t)\ \sin(\theta) = 0$$

3. (a). 当 $\lambda < 0$ 时，方程的通解可表示为

$$\phi(x) = c_1 e^{\sqrt{\ }\ \lambda x} + c_2 e^{\ \sqrt{\ }\ \lambda x}$$

由边界条件知 $c_1 = c_2 = 0$, 故此时只有平凡解 $\phi(x) = 0$

(b). 当 $\lambda = 0$ 时，方程的通解可表示为

$$\phi(x) = c_1 x + c_2$$

由边界条件知 $c_1 = 0$, 故此时有解 $\phi(x) = c_2$

(c). 当 $\lambda > 0$ 时，方程的通解可表示为

$$\phi(x) = c_1 sin(\sqrt{\lambda} x) + c_2 cos(\sqrt{\lambda} x)$$

代入边值条件可得

$$\begin{cases} c_1 = 0 \\ c_2 sin(\sqrt{\lambda} L) = 0 \end{cases}$$

故有一族非平凡解, 特征值和特征函数分别为

$$\lambda_n = (\frac{n\pi}{L})^2,\ \phi_n(x) = cos(\frac{n\pi x}{L}),\ n = 0, 1, 2\cdots$$

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a. $\int_a^b \varphi_n L\varphi_m - \varphi_m L\varphi_n dx$

$= (\lambda_n - \lambda_m)\int_a^b \sigma\varphi_n\varphi_m dx = 0$

又因为$\lambda_n \neq \lambda_m$

$\int_a^b \sigma\varphi_n\varphi_m dx = 0$

b $\int_a^b \varphi L\varphi + \lambda\sigma\varphi^2 dx = 0$

$\int_a^b \varphi \frac{\partial}{\partial x}\left(P\frac{\partial\varphi}{\partial x}\right) dx + \int_a^b (q+\sigma)\varphi^2 dx = 0$

$\varphi P\frac{\partial\varphi}{\partial x}|_a^b - \int_a^b \varphi P\frac{\partial\varphi^2}{\partial x} dx + \int_a^b (q+\sigma)\varphi^2 dx = 0$

可知

$\lambda = \dfrac{\varphi P\frac{\partial\varphi}{\partial x}{}_a^b + \int_a^b \varphi P\frac{\partial\varphi^2}{\partial x} dx \quad \int_a^b q\varphi^2 dx}{\int_a^b \sigma\varphi^2 dx}$

5. 由 $u(x,t)=\sum_{n=1}^{\infty}c_n(t)\Psi_n(x)$ 满足 $Hu=i\hbar u_t$, $\Psi_n(x)$ 满足 $H\Psi_n(x)=E_n\Psi_n(x)$ 可得

$$i\hbar u_t=i\hbar\sum_{n=1}^{\infty}\frac{d}{dt}c_n(t)\Psi_n(x)=Hu=\sum_{n=1}^{\infty}E_nc_n(t)\Psi_n(x)$$

由 $\{\Psi_n(x)\}$ 的单位正交性知,

$$\frac{d}{dt}c_n(t)=\frac{E_n}{i\hbar}c_n(t)$$

故

$$c_n(t)=c_n(0)e^{\frac{E_n t}{i\hbar}}$$

由 $u_0(x)=u(x,0)=\sum_{n=1}^{\infty}c_n(0)\Psi_n(x)$ 及 $\{\Psi_n(x)\}$ 的单位正交性知,

$$c_n(0)=\int_{\mathbb{R}}u_0(x)\Psi_n(x)dx$$

故

$$c_n(t)=e^{\frac{iE_n t}{\hbar}}\int_{\mathbb{R}}u_0(x)\Psi_n(x)dx$$

将其代入 $u(x,t)=\sum_{n=1}^{\infty}c_n(t)\Psi_n(x)$ 中有

$$\begin{aligned}u(x,t)=&\sum_{n=1}^{\infty}\Psi_n(x)\big(e^{\frac{iE_n t}{\hbar}}\int_{\mathbb{R}}u_0(x')\Psi_n(x')dx'\big)\\ &\int_{\mathbb{R}}u_0(x')\big(\sum_{n=1}^{\infty}e^{\frac{iE_nt}{\hbar}}\Psi_n(x)\Psi_n(x')\big)dx'\end{aligned}$$

故

$$G(x,t,x')=\sum_{n=1}^{\infty}e^{\frac{iE_nt}{\hbar}}\Psi_n(x)\Psi_n(x')$$

6. a. $\mathrm{G(x1, x2)} = \int_a^b \delta(x1 - x) G(x.x2) dx$

$\mathrm{G(x2, x1)} = \int_a^b \delta(x2 - x) G(x.x1) dx$

可得

$$
\begin{aligned}
\mathrm{G(x1, x2)} - G(x2, x1) &= \int_a^b \delta(x1 - x) G(x.x2) dx - \int_a^b \delta(x2 - x) G(x.x1) dx \\
&= \int_a^b LG(x, x1) G(x, x2) - LG(x, x2) G(x, x1) dx \\
&= 0
\end{aligned}
$$

b. u 满足 BVP

$$
\begin{cases} \ddot{u}(x) = f(x) \\ u(0) = a \quad u(L) = b \end{cases}
$$

$$
\begin{cases} \frac{d^2}{dx^2} G(x, x_s) = \delta(x - x_s) \\ G(0, x_s) = 0 \quad G(L, x_s) = 0 \end{cases}
$$

$$
\mathrm{v(x)} = u(x) - a - \frac{b\ \ a}{L} x
$$

可得

$$
\begin{cases} \ddot{v}(x) = f(x) \\ v(0) = 0 \quad u(L) = 0 \end{cases}
$$

由格林公式

$$
\mathrm{v}(x_0) = \int_a^b G(x_0, x) f(x) dx
$$

$$
\mathrm{u}(x_0) = \int_a^b G(x_0, x) f(x) dx + a + \frac{b-a}{L} x_0
$$