

Linear programs  
in two variables  
(3.2 and 3.3)

Recall that a **linear program** is an optimization problem where the objective function (what we are trying to maximize or minimize) is a linear function, and the constraints are linear equations and inequalities.

Any vector which satisfies all the constraints is a **feasible solution**. Any feasible solution which optimizes the objective function is an **optimal solution**, and the value of the objective function at one of these points is the **optimal value**.

There can be multiple optimal solutions, but only one optimal value.

$$\text{Max } x+y$$

$$\text{s.t. } 2x+y \leq 6$$

$$x+2y \leq 4$$

$$x, y \geq 0$$

$$\text{Min } x + 3y$$

$$\text{s.t. } 2x + y \geq 6$$

$$x + 2y \geq 4$$

$$x, y \geq 0$$

$$\text{Min } x + 2y$$

$$\text{s.t. } 2x + y \geq 6$$

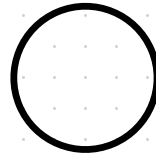
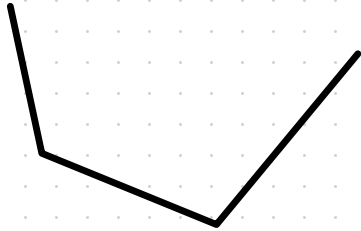
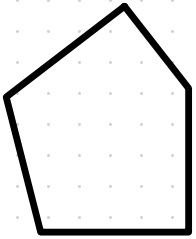
$$x + 2y \geq 4$$

$$x, y \geq 0$$

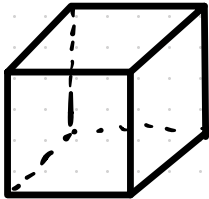
The feasible region of a linear program is a **convex polyhedron**.

**Convex**: If two points are feasible, then any point on the line segment between them is feasible.

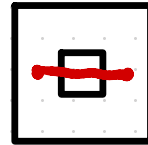
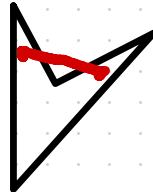
**Polyhedron**: A shape with "flat sides".



not  
polyhedra

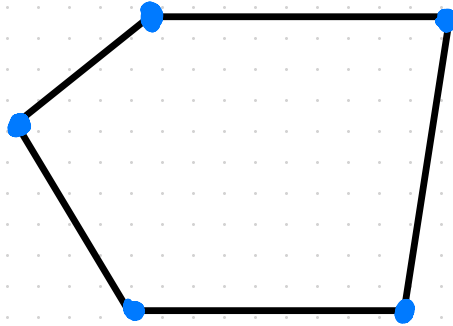


convex  
polyhedra



not  
convex

A "corner" of a polyhedron is called a **vertex**.



Thm: If a LP is bounded and the feasible region has vertices then it has an optimal solution at a vertex.

A farmer must decide how many acres of corn and wheat to plant this year.

- An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week.
- An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week.
- Wheat is sold at \$4 / bushel, corn at \$3 / bushel.
- 7 acres of land and 40 hours/week of labor are available.
- Government requires at least 30 bushels of corn to be produced.

What should the farmer do to maximize profit?



$$\text{Max } 100w + 30c$$

$$\text{s.t. } w + c \leq 7$$

$$10w + 4c \leq 40$$

$$10c \geq 30$$

$$w, c \geq 0$$

If there are more than two variables, then the feasible region lives in higher-dimensional space. In theory, both the level set method and check-vertices method will work.

$$\text{Max } x+y+z$$

$$\text{s.t. } 2x+3y+z \leq 6$$

$$x, y, z \geq 0$$

In practice, it's not reasonable to solve LP's with more than two variables by hand. (Typical LP's in applications have thousands of variables.) We instead use computer algorithms.