

Autobidders with Budget and ROI Constraints: Efficiency, Regret, and Pacing Dynamics

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Abstract

We study a game between autobidding algorithms that compete in an online advertising platform. Each autobidder is tasked with maximizing its advertiser’s total value over multiple rounds of a repeated auction, subject to budget and return-on-investment constraints. We propose a gradient-based learning algorithm that is guaranteed to satisfy all constraints and achieves vanishing individual regret. Our algorithm uses only bandit feedback and can be used with the first- or second-price auction, as well as with any “intermediate” auction format. Our main result is that when these autobidders play against each other, the resulting expected liquid welfare over all rounds is at least half of the expected optimal liquid welfare achieved by any allocation. This holds whether or not the bidding dynamics converges to an equilibrium.

Keywords: autobidding, budget and ROI constraints, liquid welfare, regret

Acknowledgements. This research has been done while SP and MZ were graduate students at MIT and USC, resp., and (partially) while they were research interns at Microsoft Research. The authors are grateful to Bach Ha (Microsoft Bing Ads) for many conversations that informed our perspective, and to Sidharth Satya for providing research support.

Version history. January 2023: first version, with all theoretical results.

June 2024: added numerical experiments, revised presentation.

November 2024: this version, revised presentation.

Conference Publication. Presented at the 37th Conference on Learning Theory (COLT), 2024.
(Accepted as a full paper, published as an extended abstract to accommodate a journal submission.)

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1. Introduction

As the rules and algorithms governing online markets increase in complexity and scale, platforms are increasingly providing ML-powered interfaces to help users interact and navigate efficiently. A prominent example is the rise of *autobidding*, a service provided by advertising platforms to help advertisers automate their campaigns. The advertiser only needs to specify high-level objectives and constraints. A typical example might be “maximize the number of clicks received, subject to spending at most \$1000 per day, at most \$2 per click on average, and no more than \$10 for any one click.” This example encodes three different constraints on the outcome: a budget constraint, an average return-on-investment (ROI) constraint,¹ and a maximum bid. The autobidder then uses an online learning algorithm to tune a detailed advertising campaign so as to solve this optimization problem on the advertiser’s behalf. While each ad impression is sold by an auction, individual bids are managed by the autobidder.

Most major online ad platforms now feature integrated autobidding tools. This popularity is owed in part to the effectiveness of online learning methods for bid tuning, which has recently received substantial attention in the academic literature. Initial work focused on tuning bids subject to an aggregate budget constraint, a.k.a., budget pacing (Borgs et al., 2007; Balseiro and Gur, 2019; Conitzer et al., 2018, 2019). More recent work concerns ROI constraints (Feng et al., 2022; Balseiro et al., 2021; Golrezaei et al., 2021a,b; Li and Tang, 2022; Mehta and Perlroth, 2023). Thus, a variety of well-understood learning algorithms can be used by an autobidder to achieve vanishing regret in any stationary (or near-stationary) auction environment. Correspondingly, autobidder interfaces supporting both budget and ROI constraints are now ubiquitous.

Since autobidders are now predominantly competing against each other, one must worry what happens when they interact. What are the implications for the overall health of the market as a whole, in terms of aggregate objectives such as efficiency and convergence? In particular, one needs to account for unintended emergent behaviors that may arise when autobidders compete.

One solution would be to design autobidding algorithms that always converge to equilibria of the “bidding game” that they are playing against each other. This would also address efficiency, if the corresponding equilibria are sufficiently efficient. The state of the art guarantees in this respect concern *liquid welfare* (a standard notion of welfare under constraints, as discussed below), and *pacing equilibrium*, a pure Nash equilibrium of the appropriately defined single-shot bidding game with budget and/or ROI constraints (Conitzer et al., 2018, 2019). Specifically, for truthful auctions any pacing equilibrium attains expected liquid welfare at least half of the optimum, and this bound is tight (Aggarwal et al., 2019; Babaioff et al., 2021). Unfortunately, finding an equilibrium of this bidding game is PPAD-hard, even for the special case of budget-constrained second-price auctions (Chen et al., 2021). We therefore should not expect to design an algorithm that is always guaranteed to jointly converge to an equilibrium when deployed by autobidders.

This leaves us with the challenge of analyzing the joint learning dynamics of the autobidders, without relying on convergence. While this challenge may appear daunting, recent developments suggest cautious optimism. For budget-constrained advertisers, Gaitonde et al. (2023) showed that the autobidding algorithm of Balseiro and Gur (2019) does indeed generate high expected liquid welfare even when the learning dynamics does not converge. Their analysis is specific to budget constraints, and does not appear to apply to any existing multi-constraint autobidding algorithm.

1. A common alternative term is *ROAS*, Return on Ad Spend.

Nevertheless, it is natural to ask whether such results can be extended to the common scenario with both budget and ROI constraints, perhaps via new algorithms.

Our contributions. We present a novel algorithm for autobidding with budget and ROI constraints. While this by itself is not new – prior work achieves vanishing regret under such constraints – the critical new feature of our algorithm is an aggregate guarantee: when multiple autobidders all deploy our algorithm, the resulting expected liquid welfare is at least half of the optimal. This matches the best possible bound for pacing equilibria.² However, our aggregate guarantee *does not* rely on convergence to equilibrium; rather, like [Gaitonde et al. \(2023\)](#) for budget-constrained bidders, we directly analyze the learning dynamics.³

We measure market efficiency via *liquid welfare*, the maximum amount the agents are willing to pay for the allocations that they receive. This is the appropriate notion of welfare in settings like ours where each agent’s (i.e., autobidder’s) goal is to maximize value subject to monetary constraints.⁴ Liquid welfare generalizes the standard notion of welfare for agents with quasi-linear preferences in money (or, equivalently, only a constraint on the maximum willingness-to-pay in each round). It has been introduced for the special case of budget constraints in [Dobzinski and Leme \(2014\)](#), and has become a standard objective in the welfare analysis of constrained auctions since then ([Azar et al., 2017](#); [Aggarwal et al., 2019](#); [Babaioff et al., 2021](#); [Gaitonde et al., 2023](#)).

We also guarantee good performance for each individual autobidder. (Such guarantees are crucial even when the platform’s objective is overall market efficiency, since otherwise advertisers might prefer to forego autobidding and place bids themselves.) Specifically, we obtain vanishing regret in an adversarial environment, without any assumptions on the other agents. Our result holds relative to the *constraint-pacing sequence*: informally, a sequence of bids that maximize expected value in each round under the time-averaged expected constraints. Note that vanishing-regret results are impossible against the standard benchmark of best fixed bid, even for the budget constraint only ([Balseiro and Gur, 2019](#), also see [Section 2](#)). The primary reason for that is the *spend-or-save dilemma*: whether to spend the budget now or save it for later. A natural way to side-step this dilemma is to satisfy the expected constraints in each round, as in our benchmark. Specialized to a stationary environment (where the other agents’ bids are drawn from a fixed joint distribution), the benchmarks become equivalent and hence we obtain vanishing regret against the best fixed bid.

Our results hold for a broad class of auction formats, including first-price and second-price auctions, and allow impression types (which determine, e.g., click rates) to be drawn randomly in each round and potentially be correlated across agents. Our algorithm is guaranteed to satisfy all constraints ex post with probability 1 (not just with high probability or with small expected violation). Further, it only requires bandit feedback from the underlying auction (i.e., only the outcome for the actual bid submitted, not the counterfactual outcomes for the alternative bids). This is important because even when advertising platforms provide sufficient feedback to infer the

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2. The following lower bound holds for any $\lambda > \frac{1}{2}$: the liquid welfare of a pacing equilibrium cannot exceed λ fraction of the optimum for all budget-constrained bidding games and all equilibria. This holds for second-price auctions as per [Aggarwal et al. \(2019\)](#); [Babaioff et al. \(2021\)](#), and also for first-price auctions due to a simple example, see [Appendix C](#). A matching upper bound holds for all truthful auctions and all pacing equilibria.
 3. In particular, the connection to pacing equilibria is only informal. Equilibrium analysis is a common first step towards analyzing the dynamics, and equilibria results such as the lower bound in Footnote 2 are commonly interpreted as informal benchmarks for the respective dynamics. However, such lower bounds apply formally only if the dynamics can converge to a worst-case pacing equilibrium (which is not necessarily true for all learning dynamics).
 4. When/if the advertisers’ objective *is* expressible in dollars, utilitarian welfare could also be a reasonable objective. However, strong impossibility results ([Dobzinski and Leme, 2014](#)) make it less suitable for theoretical study.

counterfactual slate of ad impressions (which they do not always), it can be difficult to accurately model which ads in the slate would be clicked by a user.

Our techniques. Our algorithm uses a non-standard variation of stochastic gradient descent (SGD).⁵ When there is only a budget constraint, a common idea is to use SGD-based updates to learn the constraint-pacing bid: a bid that exactly spends the budget in expectation. With both budget and ROI constraints, one could strive to learn constraint-pacing bids for each constraint, and then aggregate these into a bidding strategy. For example, [Balseiro et al. \(2022b\)](#) employ a primal-dual framework that interpolates and places more weight on constraints that bind more tightly; [Feng et al. \(2022\)](#) apply this directly to the setting of ROI and budget constraints.

Our approach has a similar flavor, but aggregates the two per-constraint bids differently. Each round, our autobidder myopically uses the smaller of the two constraint-pacing bids, then applies an SGD step to *both* bids using the observed outcome. It may seem counter-intuitive to update the larger bid using a gradient evaluated at the smaller bid, and indeed this breaks the usual convergence guarantees of SGD. However, this approach maintains an important invariant: the multiplier for each constraint encodes the total slack (or violation) of that constraint up to the current round. This invariant lets us track outcomes while being agnostic to the details of the (potentially chaotic and non-convergent) bid dynamics. It also implies that all constraints are satisfied with certainty.

When there is only a budget constraint, our algorithm specializes to the autobidding algorithm from [\(Balseiro and Gur, 2019\)](#), and our guarantees specialize to the regret and liquid welfare guarantees from [Gaitonde et al. \(2023\)](#). Like that work, our aggregate guarantees employ an “approximate First Welfare Theorem” approach that charges any welfare loss from suboptimal allocations to revenue collected by the platform. This is a common technique for equilibrium analysis, including pacing equilibrium, but extending this approach to non-convergent learning dynamics requires new ideas; see Appendix C. While our approach to bounding liquid welfare shares a common high-level strategy with [Gaitonde et al. \(2023\)](#), handling the ROI constraint, and particularly both constraints jointly, introduces a variety of new technical challenges (discussed in [Section 5](#)) and motivates our proposed algorithm. We provide a detailed technical comparison in [Appendix B](#).

Discussion. We posit that all advertisers run the same fixed algorithm, a standard assumption in theoretical results on multi-agent learning. One motivation is that the algorithm is chosen and implemented by the platform, rather than directly controlled by the advertisers. Beyond vanishing regret, we do not explicitly consider advertisers’ incentives to use this algorithm. However, the advertisers are not likely to switch to a DIY solution without a substantial advantage: indeed, the platform-provided autobidder does not require implementation efforts and may have better access to the platform’s internal statistics and estimates.

The bids placed by an autobidder in our model scale linearly with realized impression value, following most prior work on online bidding.⁶ The linear scaling is inevitable if the impression values are not observable in advance (as in our model). It is optimal for bidders for truthful auctions under budget constraints ([Balseiro and Gur, 2019](#); [Babaioff et al., 2021](#)), but not more generally. Accordingly, our individual guarantees for the general case only compare against linear bids.

Numerical experiments. We provide numerical experiments to complement our theoretical results. We focus on a standard “multi-player” environment when all advertisers run the same algorithm. Our liquid welfare guarantees apply to this environment, but empirical performance could poten-

5. SGD is a classic algorithm in online convex optimization ([Hazan, 2015](#)).

6. Specifically, all prior work on online bidding that we are aware of, except [Balseiro et al. \(2022a\)](#).

tially be even better. Moreover, while our individual regret bounds apply to a general adversarial environment, they do *not* meaningfully specialize to the multi-player environment (see Remark 20 in Section 5.2). So, individual regret for this environment is only studied via experiments.

We consider several simulated problem instances with budget and ROI constraints. We track several metrics: constraint slackness/violation, liquid welfare, and individual regret, as well as the dynamics of the multipliers. To get a sense of the problem difficulty, we also consider other bidding algorithms (in each experiment, all bidders are assigned the same algorithm). Specifically, we consider BOMW, a “smart” algorithm from (Balseiro et al., 2022b; Feng et al., 2022), and two “naive” baselines: the greedy algorithm and the epsilon-greedy algorithm.⁷

Our high-level findings are as follows. First, our algorithm satisfies all constraints and achieves high liquid welfare relative to the baselines, whereas the “naive” baselines fail due to large constraint violations. Second, the dynamics does not always converge to a stationary state (within the timeframe considered) for our algorithm nor for BOMW. This further motivates the study of aggregate guarantees such as ours that bypass convergence, and individual guarantees such as ours that go beyond the stationary environment. We observe strong algorithm performance regardless of whether the dynamics has converged. Third, our algorithm appears to have vanishing individual regret relative to the best-in-hindsight bid, despite non-convergence of the dynamics. We empirically estimate a best-fit parameter $\alpha > 0$ such that observed regret evolves according to T^α , and we obtain an empirical estimate bounded away from 1 in all instances simulated.

2. Further Discussion of Related Work

Online bidding. Our work builds on the recent literature analyzing online algorithms for bidding under constraints. For the special case of a budget constraint, Borgs et al. (2007) analyze bidding dynamics under a multiplicative update rule and establish convergence for first-price auctions. Balseiro and Gur (2019) consider a different update rule under second-price auctions and show that it converges under some additional convexity assumptions and guarantees vanishing individual bidder regret. Balseiro et al. (2022b) consider a variation of this approach using online mirror descent (OMD) and extend the individual guarantees to repeated truthful auctions. Notably, their regret bound applies to adversarial environments, with a loss that grows with the deviation from the stationary environment.

For budget and ROI constraints, Gao et al. (2022) propose and evaluate a dual-based optimization framework, without any provable guarantees. Golrezaei et al. (2021a) achieve low regret in a stationary environment with bounds on expected constraint violations. Feng et al. (2022) extend the OMD-based approach from Balseiro et al. (2022b) to achieve vanishing regret while satisfying constraints with probability 1. We emphasize that these papers do not provide any aggregate guarantees, or any individual guarantees beyond the stationary environment. Golrezaei et al. (2021a); Feng et al. (2022) focus on, resp., repeated second-price auctions and repeated truthful auctions. They do, however, achieve better regret rates in the stationary environment: $T^{1/2}$ regret in Golrezaei et al. (2021a); Feng et al. (2022) vs. $T^{7/8}$ regret in our paper.

Equilibrium analysis. A line of work studies *pacing equilibria*: market equilibria in a single-shot game that abstracts repeated auctions when bidders have global constraints such as budgets and/or

7. Both are well-known “templates” in multi-armed bandits. In particular, the greedy algorithm ignores the need for exploration, and the epsilon-greedy algorithm (in our setting) ignores the non-stationarity and the constraints.

ROI. Conitzer et al. (2018) introduced such equilibria and characterized them for second-price auctions with budget constraints. For second-price auctions, the expected liquid welfare obtained at any pacing equilibrium is at least half of the optimal liquid welfare; this holds for a broad class of constraints including budget and ROI constraints (Aggarwal et al., 2019). Babaioff et al. (2021) provided a similar 2-approximation result in settings with “soft constraints” where agents have a separable and convex disutility for spending money. However, finding a pacing equilibrium under budget constraints is PPAD-hard (Chen et al., 2021). Recall that our analysis of liquid welfare does not rely on the equilibrium analysis, since it does not rely on convergence to equilibrium.

Conitzer et al. (2019) extended pacing equilibria to first-price auctions, showed that the equilibrium is essentially unique, and analyzed its properties. Balseiro et al. (2022a) considered an alternative equilibrium notion for a broad class of auctions, in which agents are not constrained to pacing but instead can make their bids arbitrarily contingent on realized impression values. They present a revenue-equivalence result to bound liquid welfare at any equilibrium subject to a budget constraint.

Learning theory. Repeated bidding under budget is a special case of *bandits with knapsacks* (BwK), a multi-armed bandit problem under global constraints on resource consumption (Badanidiyuru et al., 2018; Agrawal and Devanur, 2019; Immorlica et al., 2022), see Chapter 10 in (Slivkins, 2019) for a survey. BwK problems in adversarial environments do not admit vanishing regret bounds against standard benchmarks: instead, one is doomed to approximation ratios, even in relatively simple examples (Immorlica et al., 2022). A similar impossibility result is derived in (Balseiro and Gur, 2019) specifically for repeated budget-constrained bidding in second-price auctions. These results hinge on the *spend-or-save dilemma*: an algorithm does not know in advance whether to spend the budget now or save it for the future, and inevitably makes a wrong choice for some variant of the future. Prior work on BwK is not concerned with aggregate guarantees such as ours.

Convergence of learning algorithms in repeated games is extensively studied, yet not well-understood. When algorithms have vanishing regret in terms of cumulative payoffs, the *average play* (time-averaged distribution over chosen actions) converges to a (coarse) correlated equilibrium (Aumann, 1974; Moulin and Vial, 1978; Hart and Mas-Colell, 2000), and this implies welfare bounds for various auction formats in the absence of budget or ROI constraints (Roughgarden et al., 2017). In contrast, for repeated auctions with budgets, low individual regret on its own does not imply any bounded approximation for liquid welfare (Gaitonde et al., 2023). Convergence in the last iterate is even more challenging. While strong negative results are known even for two-player zero-sum games (Bailey and Piliouras, 2018; Mertikopoulos et al., 2018; Cheung and Piliouras, 2019), a recent line of work (Daskalakis et al., 2018; Daskalakis and Panageas, 2019; Golowich et al., 2020a; Wei et al., 2021) achieves last-iterate convergence under full feedback and substantial convexity-like assumptions, using two specific regret-minimizing algorithms. To the best of our understanding, these positive results do not apply to repeated auctions with budget or ROI constraints.

3. Model and Preliminaries

We study a repeated auction game played by a collection of n bidding agents (i.e., autobidders). At each round $t = 1, 2, \dots, T$, the seller (or platform) has a single unit of good available to sell. An *allocation profile* is a vector $\mathbf{x} = (x_1, \dots, x_n) \in X \subseteq [0, 1]^n$ where x_k is the quantity of the

good allocated to agent k and X is any downward-closed set of feasible allocations.⁸ An *allocation sequence* is a sequence of allocation profiles $(\mathbf{x}_1, \dots, \mathbf{x}_T)$ where $\mathbf{x}_t = (x_{1,t}, \dots, x_{n,t})$ is the allocation profile at round t . (The notation is summarized in [Appendix A](#).)

The good available for sale at a given round t has a click probability $c_{k,t} \in [0, 1]$ for each agent k .⁹ The tuple $\mathbf{c}_t = (c_{1,t}, \dots, c_{n,t})$ is drawn from a fixed distribution F_c independently across time periods. Note that F_c can be arbitrarily correlated across agents.

Auctions. At each round t , the good is allocated via an auction that proceeds as follows. Each agent k submits a bid $\beta_{k,t} \geq 0$, which can be interpreted as a bid (in dollars) per click. All agents submit bids simultaneously. The \mathbf{c}_t tuple is then realized, which determines each agent's effective bid $b_{k,t} = c_{k,t}\beta_{k,t}$.¹⁰ The auction is defined by an allocation rule \mathbf{x} and a payment rule \mathbf{p} , where $\mathbf{x}(\mathbf{b}) \in [0, 1]^n$ is the allocation profile generated under a bid profile \mathbf{b} , and $p_k(\mathbf{b}) \geq 0$ is the payment made by agent k . Allocation and payment rules are always weakly monotone in bids, meaning that $x_k(b_k, \mathbf{b}_{-k})$ and $p_k(b_k, \mathbf{b}_{-k})$ are weakly increasing in b_k for any \mathbf{b}_{-k} . Given an implied realization of bids, we will often write $x_{k,t}$ and $p_{k,t}$ to denote the allocation and payment of agent k in round t .

Auction rules satisfy the following two properties. Winners (i.e., agents with non-zero allocation) are always selected from among agents with the highest effective bid. Also, each agent's payment per unit received, $p_{k,t}/x_{k,t}$, lies between the highest and second-highest effective bids. This includes the first-price and second-price auctions, as well as anything “in between.”¹¹

Objective. Each agent k maximizes $\sum_k c_{k,t}x_{k,t}$ (i.e., expected clicks) subject to the following:

Bid constraint (θ_k). In each round t , the bid cannot exceed θ_k : $\beta_{k,t} \leq \theta_k$.

ROI constraint (w_k). The total payment cannot exceed w_k per click: $\sum_t p_{k,t} \leq w_k \sum_t c_{k,t}x_{k,t}$.

Budget constraint (B_k). The total payment cannot exceed B_k : $\sum_t p_{k,t} \leq B_k$.

All constraints bind ex post, and must be satisfied on every realization. Note there always exists an agent strategy that guarantees all constraints will be satisfied. The first constraint is always satisfied if $\beta_{k,t} \leq \theta_k$ for all t ; the second constraint is always satisfied if $\beta_{k,t} \leq w_k$ for all t ; and the third constraint is always satisfied if $\beta_{k,t} \leq B_k/T$ for all t . Bidding the minimum of these would therefore necessarily satisfy all constraints, though of course this may result in low objective value.

We write $\rho_k = B_k/T$ for agent k 's per-round budget constraint. For convenience, define $v_{k,t} = \theta_k c_{k,t}$ for the maximum allowed effective bid for agent k in round t . Since the ROI constraint would be implied by the bid constraint if $w_k > \theta_k$, we will assume without loss of generality that $w_k \leq \theta_k$, and define $\gamma_k = \theta_k/w_k \geq 1$. The ROI constraint can then be rewritten as $\gamma_k \sum_t p_{k,t} \leq \sum_t v_{k,t}x_{k,t}$. Finally, following the literature, we define multiplier $\mu_{k,t} = (\theta_k/\beta_{k,t}) - 1$ so that $\beta_{k,t} = \theta_k/(\mu_{k,t} + 1)$. We then think of agent k 's problem as choosing multiplier $\mu_{k,t} \geq 0$, where 0 corresponds to the maximum allowable bid and larger values of $\mu_{k,t}$ correspond to smaller bids.

Remark 1 *The ROI constraint is trivially satisfied when $\gamma_k = 1$. (This is because $p_{k,t}/x_{k,t} \leq c_{k,t}\beta_{k,t} \leq c_{k,t}\theta_k$.) Put differently, $\gamma_k = 1$ corresponds to the budget-only case.*

8. Meaning: $x \in X \Rightarrow x' \in X$ for any two vectors $x, x' \in [0, 1]^n$ with $x'_i \leq x_i \forall i \in [n]$. The canonical case is integer allocation, when X consists of all unit vectors, but X can also allow fractional allocations.

9. We could alternatively think of $c_{k,t}$ as a conversion rate, an expected revenue lift, or any other driver of realized impression value. For clarity of exposition we will use the language of clicks from now on.

10. Equivalently, the bidder observes $c_{k,t}$, but is restricted to choosing an effective bid $b_{k,t}$ that is linear in $c_{k,t}$.

11. Some of our results actually apply more generally to *core auctions*; we define these in [Section 5.1](#).

Each autobidding agent k can be equivalently thought of as maximizing $\sum_t v_{k,t} \cdot x_{k,t}$ subject to the constraints. This motivates us to refer to each $v_{k,t}$ as a *value* for agent k at round t . However, we emphasize that these “values” are independent of the advertiser’s actual utility model, to which we are agnostic. Rather, they merely parameterize the autobidders’ objective. As a shorthand, we also define *value profiles* $\mathbf{v}_t := (v_{1,t}, \dots, v_{n,t})$ for each round t , which are drawn independently from some fixed distribution F (induced by F_c). The *value sequence* is the sequence $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_T)$.

Liquid welfare is a welfare notion for agents with payment constraints which measures agents’ *maximum willingness to pay* for the received allocation. For our setting, it is defined as follows:

Definition 2 Fix a value sequence $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_T) = (v_{k,t}) \in [0, 1]^{nT}$ and a feasible allocation sequence $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_T) = (x_{k,t}) \in X^T$. The corresponding liquid value obtained by agent k is $W_k(\mathbf{x}, \mathbf{v}) = \min \left\{ B_k, \frac{1}{\gamma_k} \sum_{t=1}^T v_{k,t} x_{k,t} \right\}$, and liquid welfare is $W(\mathbf{x}, \mathbf{v}) := \sum_{k=1}^n W_k(\mathbf{x}, \mathbf{v})$.

We emphasize that liquid welfare depends on the allocations, but not on the agents’ payments. However, it follows immediately that an agent’s total payment cannot exceed her liquid value.

Observation 3 Fix any sequence of value profiles and outcomes (allocations and payments) such that the bid, budget, and ROI constraints are satisfied for a given agent k . In the notation from Defn 2, the total payment of agent k cannot exceed $W_k(\mathbf{x}, \mathbf{v})$.

Our objective of interest is the *expected* liquid welfare, over any randomness in the value sequence and the agents’ autobidding algorithms. Since the bids in one round can depend on allocations from the previous rounds, we define a mapping Φ from the entire value sequence \mathbf{v} to an allocation sequence $\mathbf{x} = \Phi(\mathbf{v})$; call Φ an *allocation-sequence rule*. (Note that it abstracts both the autobidding algorithm and the underlying auction. Treat Φ as randomized if either the algorithm or the auction are.) Then the expected liquid welfare is $W(\Phi, F) := \mathbb{E}_{\mathbf{v}} [W(\Phi(\mathbf{v}), \mathbf{v})]$, where the value sequence \mathbf{v} drawn according to distribution F .

4. Warmup: ROI Constraints and Unlimited Budget

Consider unlimited budgets, i.e., $B_k = \infty$ for all k . While bidding algorithms are already known for this problem, we present a new algorithm and emphasize some properties that will be crucial for the general setting we consider in Section 5. As we focus on a single agent, we will drop the subscripts k from our notation for the remainder of this section. In the absence of budget constraints, the agent’s goal is to maximize $\sum_t v_t x_t$ subject to the ROI constraint $\sum_t v_t x_t - \gamma \sum_t p_t \geq 0$.

Example 1 Consider a repeated second-price auction with $v_t = 1$ for every t and $\gamma = 2$. Then the agent’s ROI constraint is $\sum_t x_t - 2 \sum_t p_t \geq 0$. Suppose the competing bids are stochastic and stationary, with the highest competing bid b_{\max} either $1/4$ or $3/4$ each round with equal probability.

Suppose that the bidding agent chooses $\mu_t = 0$ for every round t , so it bids $b_t = v_t = 1$. Then the agent wins every round and pays either $1/4$ or $3/4$, for an expected payment of $1/2$. The constraint is satisfied in expectation but may be violated on some realizations. Indeed, in the (unlikely) event that $b_{\max} = 3/4$ every round, the only way to satisfy the ROI constraint would be to lose every round. However, if $b_{\max} = 1/4$ in at least half of the rounds then it is optimal to win every round.

Setting $\mu_t = \gamma - 1$ in every round (i.e., bidding v_t/γ) is guaranteed to satisfy the ROI constraint (see [Section 3](#)). However, the optimal choice of μ_t may be lower, as in [Example 1](#). Next, we show that ROI is monotone in μ_t for our auctions: higher bids result in lower average ROI. This lets us think of the agent as trying to reduce μ as much as possible subject to the ROI constraint.

Lemma 4 *Fix any auction in our class, any agent k , and any bids \mathbf{b}_{-k} of the other agents, and write $x_t(\mu)$ and $p_t(\mu)$ for the allocation and payment that results when agent k selects multiplier μ . Then $v_t x_t(\mu) - \gamma p_t(\mu)$ is weakly increasing in μ , for $\mu \in [0, \gamma - 1]$.*

This motivates us to consider [Algorithm 1](#), which initially takes the safe action $\mu_1 = \gamma - 1$ but updates μ_t online in response to auction feedback. In each round t it places bid $b_t = v_t/(1 + \mu_t)$.¹² If the observed allocation x_t and payment p_t in round t are such that $v_t x_t > \gamma p_t$ then the ROI constraint is satisfied with room to spare. This suggests the bid was lower than necessary, so the algorithm reduces μ by an amount proportional to $v_t x_t - \gamma p_t$. Likewise, if $v_t x_t < \gamma p_t$, then the ROI constraint was violated in round t so the algorithm responds by increasing μ proportionally to the violation. This can be interpreted as stochastic gradient descent (SGD): if μ_t is such that the ROI constraint is satisfied in expectation then the expected update is 0. We will make this connection with SGD more precise in [Section 5.2](#).

Algorithm 1: Bidding Under ROI Constraint

Input: ROI constraint parameter γ .
Initialization: $\mu_1 = \gamma - 1$ and learning rate $\eta > 0$.
for $t = 1, 2, \dots, T$ **do**
 | Set bid $b_t = \frac{v_t}{1 + \max\{\mu_t, 0\}}$
 | Observe allocation $x_t \in [0, 1]$ and payment p_t
 | Update the multiplier $\mu_{t+1} = \mu_t - \eta(v_t x_t - \gamma p_t)$

[Algorithm 1](#) appears myopic at first glance, always updating its bids in response to the latest outcome. However, we note that μ_t implicitly encodes the status of the aggregate ROI constraint up to round t . Indeed, an immediate implication of the update rule is that μ_t is μ_1 minus a term proportional to $\sum_{\tau < t} (v_\tau x_\tau - \gamma p_\tau)$, which is precisely the aggregate slack (or violation) of the ROI constraint up to time t . A small value of μ_t (i.e., a high bid) therefore occurs only if there is substantial slack in the ROI constraint up to round t .

What does this mean for the performance of [Algorithm 1](#)? Note that the bidding agent may receive less than the desired return on investment γ in any given round. However, as we now show, the algorithm is guaranteed, with probability 1, to satisfy the ROI constraint in aggregate over its T rounds (and indeed, over any prefix of the T rounds). Intuitively, as the ROI constraint gets closer to being violated in aggregate, μ_t gets closer to $\mu_1 = \gamma - 1$, the “safe” choice at which the ROI constraint will be satisfied each round.

Lemma 5 *Fix any (possibly adversarial) mapping from sequences of bids to sequences of allocations and payments such that $p_t \leq b_t x_t$ for every t , and suppose $\eta \leq 1/\bar{v}$. Then for the allocations and payments resulting from applying [Algorithm 1](#), we have $\sum_t v_t x_t \geq \gamma \sum_t p_t$.*

12. With one small caveat: μ_t could be negative in some rounds, in which case we treat it as 0 when setting the bid b_t .

Algorithm 2: Bidding Under ROI and Budget Constraints (for agent k)

Input: per-round budget constraint $\rho_k > 0$ and ROI constraint parameter γ_k .

Initialization: $\mu_{k,1}^R = \gamma_k - 1$, $\mu_{k,1}^B = \frac{\bar{v}}{\rho} - 1$ and learning rate $\eta_{k,R}, \eta_{k,B} > 0$.

for $t = 1, 2, \dots, T$ **do**

Calculate $\mu_{k,t} = \max\{\mu_{k,t}^R, \mu_{k,t}^B, 0\}$.

Receive value $v_{k,t}$ and set the bid $b_{k,t} = \frac{v_{k,t}}{1+\mu_{k,t}}$.

Receive the allocation $x_{k,t} \in [0, 1]$ and the payment $p_{k,t}$.

Update the ROI-multiplier $\mu_{k,t+1}^R = \mu_{k,t}^R + \eta_{k,R}(\gamma_k p_{k,t} - v_{k,t} x_{k,t})$.

Update the budget-multiplier $\mu_{k,t+1}^B = \mu_{k,t}^B + \eta_{k,B}(p_{k,t} - \rho_k)$.

An immediate corollary of [Lemma 5](#) is that, for every input sequence, the sequence of multipliers $\{\mu_t\}$ will satisfy $\mu_t \leq \gamma - 1$ for every t .¹³ As we show in the next section, this will imply high liquid welfare when multiple agents use (a generalization of) this algorithm. As it turns out, [Lemma 5](#) can also be used to show that [Algorithm 1](#) achieves vanishing regret relative to the best choice of μ in hindsight. We prove this formally in the next section in a more general setting. For now let us briefly describe the intuition. Since the ROI constraint is satisfied with probability 1 by [Lemma 5](#), any loss in value must come from bidding too low relative to the optimal fixed strategy in hindsight, say μ^* . However, the expected update to μ_t turns out to be the gradient of a function that, on the range $[\mu^*, \gamma - 1]$, is (a) convex and (b) closely related to the value function. Standard SGD analysis then bounds the total loss due to rounds where μ_t is larger than μ^* .

5. Bidding under ROI and Budget Constraints

We now turn to the general problem with both ROI and budget constraints. We study an extension of our previous algorithm to this setting, listed below as [Algorithm 2](#). The algorithm now keeps track of two multipliers, μ^R and μ^B , corresponding to the ROI and budget constraints respectively. At each round, the algorithm will place a bid using whichever of the multipliers is more conservative; i.e., whichever results in the lower bid. Each of the multipliers is then updated according to the realized allocation and payment *as if that multiplier was used to set the bid* (even if it wasn't). Multiplier $\mu_{k,t}^R$ is updated in the same way as [Algorithm 1](#). The idea behind our update rule for multiplier $\mu_{k,t}^B$ is similar: each round we compare the observed payment $p_{k,t}$ with $\rho_k = B_k/T$, the target per-round payment according to budget constraint B_k , and we update $\mu_{k,t}^B$ proportional to the difference where $\eta_{k,R}, \eta_{k,B} > 0$ is the resp. learning rate:

$$\mu_{k,t+1}^R = \mu_{k,t}^R - \eta_{k,R}(v_{k,t} x_{k,t} - \gamma_k p_{k,t}), \quad \mu_{k,t+1}^B = \mu_{k,t}^B - \eta_{k,B}(\rho_k - p_{k,t}).$$

When the budget constraint is infinite (i.e., B_k and hence ρ_k is $+\infty$), this update rule yields $\mu_{k,t}^B < 0$ for every round t . Thus, this algorithm reduces to [Algorithm 1](#). On the other hand, if $\gamma_k = 1$ (hence, no ROI constraint), then $\mu_{k,t}^R \leq 0$ for every round t (since $p_{k,t} \leq b_{k,t} x_{k,t} \leq v_{k,t} x_{k,t}$). The resulting algorithm is nearly identical to the one in [Balseiro and Gur \(2019\)](#) for bidding subject to a budget constraint. We can therefore view [Algorithm 2](#) as a generalization of both algorithms.

13. $\mu_1 - \mu_t$ is proportional to the slack in the ROI constraint up to time t , which by [Lemma 5](#) is never negative.

The key insight behind [Algorithm 2](#) is that the multipliers $\mu_{k,t}^R$ and $\mu_{k,t}^B$ encode the cumulative slack in the ROI and budget constraints, respectively, up to time t . Similar to our analysis of [Algorithm 1](#), it follows that [Algorithm 2](#) satisfies all of its constraints ex post with probability 1:

Lemma 6 *Fix any agent k and any (possibly adversarial) mapping from sequences of bids to sequences of allocations and payments such that $p_{k,t} \leq b_{k,t}x_{k,t}$ for every round t . Assume $\eta_{k,R} \leq \frac{1}{\bar{v}}$, $\eta_{k,B} \leq \min\{\frac{1}{\rho_k}, \frac{1}{\bar{v}}\}$. Then [Algorithm 2](#) satisfies $\sum_t v_{k,t}x_{k,t} \geq \gamma_k \sum_t p_{k,t}$ and $\sum_t p_{k,t} \leq B_k$.*

Also, similar to [Algorithm 1](#), another implication of this interpretation of $\mu_{k,t}^R$ and $\mu_{k,t}^B$ is that these multipliers are never higher than their “safe” levels.

Lemma 7 *[Algorithm 2](#) with learning rates $\eta_{k,R} \leq \frac{1}{\bar{v}}$ and $\eta_{k,B} \leq \frac{1}{\bar{v}}$ satisfies $\mu_{k,t}^R \leq \gamma_k - 1$ and $\mu_{k,t}^B \leq \frac{\bar{v}}{\rho_k} - 1$ or all rounds $t \in [T]$.*

The remainder of this section is dedicated to establishing our aggregate liquid welfare and individual regret guarantees for [Algorithm 2](#). In [Section 5.1](#) we prove that in expectation over the realization of values, [Algorithm 2](#) always obtains at least half of the optimal liquid welfare in hindsight. Then in [Section 5.2](#) we establish that the algorithm also satisfies strong individual regret guarantees, even in non-stochastic settings where the optimal bid sequence has bounded path length.

5.1. Liquid Welfare Analysis

We prove that when all agents employ [Algorithm 2](#) the resulting expected liquid welfare is approximately optimal. We actually show that the expected liquid welfare is at least half of the optimal *ex-ante* liquid welfare (defined in [Defn 8](#)), which is an agent’s willingness to pay for her expected allocation sequence. This is a stronger benchmark compared to the optimal expected liquid welfare due to Jensen’s inequality (see [Lemma 28](#) in [Appendix D](#)).

Definition 8 *For any distribution F over valuation profiles and any allocation rule $\mathbf{y} : [0, \bar{v}]^n \rightarrow X$, the *ex-ante* liquid welfare is $\bar{W}(\mathbf{y}, F) := \sum_{k \in [n]} \bar{W}_k(\mathbf{y}, F)$, where for each agent $k \in [n]$*

$$\bar{W}_k(\mathbf{y}, F) := T \times \min \left\{ \rho_k, \frac{1}{\gamma_k} \mathbb{E}_{\mathbf{v} \sim F} [y_k(\mathbf{v}) v_k] \right\}. \quad (5.1)$$

Recall that to this point we have focused our attention on single-item auctions. Our liquid welfare bound will actually apply to the following more general class of *core auctions*.

Definition 9 *Given any downward-closed set $X \subseteq [0, 1]^n$ of feasible allocations, an auction with allocation rule $x : [0, \bar{v}]^n \rightarrow X$ and payment rule $p : [0, \bar{v}]^n \rightarrow \mathbb{R}_{\geq 0}^n$ is a core auction if it is*

- *Welfare-maximizing:* $\mathbf{x}(\mathbf{v}) \in \arg \max_{\mathbf{x} \in X} \{\sum_i v_i(\mathbf{x})\}$
- *Individually rational:* $p_i(\mathbf{v}) \leq v_i(\mathbf{x}(\mathbf{v}))$ for all i
- *Deviation-proof:* for all $S \subseteq [n]$ and $\mathbf{y} \in X$, $\sum_{i \notin S} p_i(\mathbf{v}) \geq \sum_{i \in S} (v_i(y_i) - v_i(x_i(\mathbf{v})))$

Core auctions include first-price and second-price single-item auctions, but also more general formats like generalized second-price auctions for multiple slots and separable click rates (see [Gaitonde et al. \(2023\)](#) for further discussion). We are ready to state the main result of this subsection, using the notation of allocation-sequence rule Φ and expected liquid welfare $W(\Phi, F)$ from [Section 3](#).

Theorem 10 Fix any core auction as defined in [Defn 9](#) and any distribution F over agent value profiles. Suppose all agents bid by employing [Algorithm 2](#) with $\max\{\eta_{k,R}, \eta_{k,B}\} \leq \frac{\bar{v}}{\bar{v} + \rho_k} \frac{\sqrt{\log(\bar{v}nT)}}{\sqrt{T}}$. Write Φ for the corresponding allocation-sequence rule. Then for any allocation rule $\mathbf{y} : [0, \bar{v}]^n \rightarrow X$, the expected liquid welfare $W(\Phi, F)$ satisfies

$$W(\Phi, F) \geq \frac{1}{2} \bar{W}(\mathbf{y}, F) - \mathcal{O}\left(n\bar{v}\sqrt{T \log(\bar{v}nT)}\right). \quad (5.2)$$

Proof Intuition.¹⁴ Consider the liquid welfare (LW) obtained by some agent k over all T rounds. By definition, this LW is either the agent's budget B_k or the sum of ROI-scaled gained values $\frac{1}{\gamma_k} \sum_t v_{k,t} x_{k,t}$. The former case is easy: since B_k is an upper bound on ex-ante LW, if agent k 's LW is B_k , then this is at least as good as the benchmark. The difficulty lies in the latter case.

To bound $\frac{1}{\gamma_k} \sum_t v_{k,t} x_{k,t}$, we consider the progression of the bidding multiplier $\mu_{k,t}$ over rounds $t = 1, 2, \dots, T$. The multiplier may drift up and down over time and may not converge. We distinguish between rounds when $\mu_{k,t}$ lies above $\gamma_k - 1$ and rounds when $\mu_{k,t}$ lies below $\gamma_k - 1$.

If $\mu_{k,t} < \gamma_k - 1$ then agent k is bidding $\geq v_{k,t}/\gamma_k$ on round t . So even if agent k loses in this round, the winning bidder(s) must be paying $\geq v_{k,t}/\gamma_k$. We can charge any loss in LW against the total revenue collected, which (by [Observation 3](#)) is itself at most the LW.

On the other hand, in any round t where $\mu_{k,t} > \gamma_k - 1$, we know that $\mu_{k,t} = \mu_{k,t}^B$. This is because the ROI multiplier $\mu_{k,t}^R$ never lies above $\gamma_k - 1$ according to [Lemma 7](#). Thus, over any contiguous interval of rounds in which $\mu_{k,t} > \gamma_k - 1$, it must be the budget multiplier that is determining the bid of agent k . This allows us to employ an insight due to [Gaitonde et al. \(2023\)](#): since any such interval must begin and end close to the threshold $\gamma_k - 1$, the update rule for $\mu_{k,t}^B$ implies that the total spend over the (say) t rounds of that interval is very close to $t \times \rho_k$. As the optimal ex-ante LW cannot be more than ρ_k per round, the obtained LW must be comparable to the optimal LW over this interval. Thus, in every case, we can relate the obtained LW to the benchmark.

There are some technical challenges to formalizing this intuition. Our very first step was to condition on whether agent k 's total LW is determined by her budget or by her ROI-scaled gained value. However, this conditioning introduces correlations between rounds, and in particular it impacts our assertion that the ex-ante LW is at most ρ_k per round. We address this by explicitly bounding the impact of such correlations and arguing that they are small with high probability. This introduces the additive error term in the theorem statement.¹⁵

Another technical issue that arises is specific to handling budget and ROI constraints simultaneously. The intuition above does not carefully account for rounds in which $\mu_{k,t}$ switches from lying strictly below $\gamma_k - 1$ to strictly above $\gamma_k - 1$ or vice-versa. It turns out that these transition rounds introduce error terms that can accumulate substantially; indeed, when we said above that the total spend over an interval is very close to $t \times \rho_k$, this approximation can be off by up to $\bar{v} + \rho_k$ per round, dominating our entire approximation. We handle this by considering separately those rounds in which $\mu_{k,t}$ is very close to the boundary $\gamma_k - 1$, and directly relate the outcomes to what would occur precisely on the boundary itself. The resulting error terms are yet another source of the additive error in the theorem statement.

14. Since our approach shares common elements with a liquid welfare bound for budget constraints due to [Gaitonde et al. \(2023\)](#), we put particular emphasis on novel challenges and ideas; see [Appendix B](#) for a more detailed comparison.

15. We note that similar issues of correlation arise when analyzing the LW of the budget-pacing algorithm of [Balseiro and Gur \(2019\)](#); our solution is a variation of an idea due to [Gaitonde et al. \(2023\)](#).

5.2. Individual Regret Guarantees

In this subsection, we consider the performance of an individual autobidder k on its optimization problem when applying [Algorithm 2](#). We abstract away the bids of other agents as supplied by a (potentially adaptive) adversary. In particular, we do not assume that the other agents are controlled by any particular algorithm. Our regret bound holds w.r.t. a non-standard benchmark ([Defn 14](#)) that specializes to the standard benchmark (best fixed bid in hindsight) in stationary environments.

Since we focus on just agent k throughout this subsection, we will drop the subscript k . For each round t , we will write $x_t = x_t(\mu)$ and $p_t = p_t(\mu)$ for the allocation and payment if bidder k picks multiplier μ in round t . Note that these depend on the realized value of bidder k as well as the bids of the other auction participants. Recall also that x_t and p_t are both weakly non-increasing in μ . The expectations in this section are taken with respect to the randomness in value profiles of agent k as well as the bids supplied by the adversary. Further, we define the following quantities:

Definition 11 *For any round $t \in [T]$, define the expected budget expenditure $Z_t^B(\mu)$, the expected ROI expenditure $Z_t^R(\mu)$, the expected ROI gain $\rho_t(\mu)$, and the expected gained value $V_t(\mu)$ when the bidder chooses multiplier μ . Letting $(x)^+ = \max\{x, 0\}$,*

$$\begin{aligned} Z_t^B(\mu) &\triangleq \mathbb{E}[p_t(\mu)] \quad \text{and} \quad Z_t^R(\mu) \triangleq \mathbb{E}[(\gamma p_t(\mu) - v_t x_t(\mu))^+] , \\ \rho_t(\mu) &\triangleq \mathbb{E}[(v_t x_t(\mu) - \gamma p_t(\mu))^+] \quad \text{and} \quad V_t(\mu) \triangleq \mathbb{E}[v_t x_t(\mu)] . \end{aligned}$$

Note that $Z_t^B(\mu)$ and $Z_t^R(\mu) - \rho_t(\mu)$ are both non-increasing functions when $\mu \geq 0$ and $\mu \in [0, \gamma - 1]$, as is $V_t(\mu)$. Formally, we have the following lemma.

Lemma 12 *For any $t \in [T]$, $Z_t^B(\mu)$ is monotonically non-increasing for $\mu \geq 0$ and $Z_t^R(\mu) - \rho_t(\mu)$ is monotonically non-increasing for $\mu \in [0, \gamma - 1]$.*

Note also that $Z_t^R(\mu) - \rho_t(\mu)$ is precisely the expected value of $(\gamma p_t(\mu) - v_t x_t(\mu))$; $Z_t^R(\mu)$ captures the positive part of this random variable, and $\rho_t(\mu)$ captures the negative part.

We also assume Lipschitzness of allocations and payments with respect to μ , which implies that Z_t^B , Z_t^R , ρ_t , and V_t are all Lipschitz as well. This can be interpreted as a requirement that the allocation and payment functions are sufficiently smooth as a function of an autobidder's bid.¹⁶

Assumption 13 *$Z_t^B(\mu)$, $Z_t^R(\mu)$, $V_t(\mu)$, $\rho_t(\mu)$, $t \in [T]$ are all λ -Lipschitz, for some $\lambda \geq 0$.*

We use a non-standard regret benchmark based on *per-round pacing multipliers*.

Definition 14 *For each round $t \in [T]$, we define a value-optimizing multiplier μ_t^* , subject to the time-averaged constraints applied to this round. The formal definition is as follows:*

budget-pacing multiplier μ_t^{B*} is any $\mu \in [0, \frac{\bar{v}}{\rho} - 1]$ with $Z_t^B(\mu) = \rho$, or 0 if no such μ exists.

ROI-pacing multiplier μ_t^{R*} is any $\mu \in [0, \gamma - 1]$ with $Z_t^R(\mu) = \rho_t(\mu)$, or 0 if no such μ exists.

pacing multiplier $\mu_t^* := \max\{\mu_t^{B*}, \mu_t^{R*}\} \geq 0$.

Thus, our notion of regret is defined as follows: $\text{Reg}(T) \triangleq \sum_{t \in [T]} V_t(\mu_t^*) - \sum_{t \in [T]} V_t(\mu_t)$.

16. We note that this assumption could be satisfied by adding $O(\lambda)$ noise to the multiplier selected by any given autobidder, at a loss of welfare proportional to $1/\lambda$.

Remark 15 While not necessarily optimal globally, the pacing multipliers represent a reasonable goal for an online bidding algorithm in a complex, adversarial environment. Recall that vanishing-regret bounds with respect to the standard benchmark of the best-fixed-multiplier are impossible against an adversary, even for the special case of budget constraint only, due to the spend-or-save dilemma (Balseiro and Gur, 2019; Immorlica et al., 2022). Our benchmark side-steps this dilemma in an arguably natural way. Indeed, value-maximization subject to expected constraints in each round rules out the possibility of “saving” the budget or using the saved budget later.

Our main regret bound depends on the amount of drift in the environment, captured (in a fairly weak way) by the path-lengths of the per-round pacing multipliers:

$$P_T^R := \sum_{t \in [T]} |\mu_t^{R*} - \mu_{t+1}^{R*}| \quad \text{and} \quad P_T^B := \sum_{t \in [T]} |\mu_t^{B*} - \mu_{t+1}^{B*}|.$$

Theorem 16 Fix any distribution over the values of agent k . Posit Lipschitzness (Assumption 13). Algorithm 2 with parameters $\eta_B = \frac{1}{\sqrt{T}\rho}$ and $\eta_R = \frac{1}{\bar{v}\sqrt{T}}$ guarantees that

$$\mathbb{E}[\text{Reg}(T)] \leq \mathcal{O} \left((P_T^R + 1)^{1/4} \lambda^{3/4} (\gamma + 1) \cdot T^{7/8} + (\bar{v} + \rho)^2 \rho^{-1.5} \sqrt{\lambda(1 + P_T^B)} \cdot T^{3/4} \right).$$

When parameters $\lambda, \gamma, \bar{v}, \rho$ are all constants, we have

$$\mathbb{E}[\text{Reg}(T)] \leq \mathcal{O} \left((P_T^R + 1)^{1/4} \cdot T^{7/8} + (P_T^B + 1)^{1/2} \cdot T^{3/4} \right).$$

Remark 17 This guarantee is non-trivial when the path-lengths P_T^B, P_T^R are $O(T^\alpha)$, $\alpha < 1/2$.

Remark 18 Our guarantees match Gaitonde et al. (2023) when there is only budget constraint. Indeed, the $P_T^R + 1$ term in Theorem 16 can be improved to $P_T^R + |\mu_1^R - \mu_1^{R*}|^2$ (see Eq. (7.8)). The latter term becomes 0 without the ROI constraint (because then $\mu_1^R := \gamma - 1 = 0$ and $\mu_t^{R*} = 0$ for all rounds t). So, the $T^{7/8}$ term vanishes from the regret bound and only the $T^{3/4}$ term remains.

Let us specialize this guarantee to the *stationary-stochastic environment*, where in each round the competing bids and the agent’s value are drawn independently from a fixed joint distribution. Then the pathlengths are $P_T^B = P_T^R = 0$. Moreover, our benchmark boils down to *constraint-feasible multipliers*: $\mu \geq 0$ such that

$$\sum_{t \in [T]} \mathbb{E}[Z_t^B(\mu)] \leq B \quad \text{and} \quad \sum_{t \in [T]} \mathbb{E}[Z_t^R(\mu) - \rho_t(\mu)] \leq 0.$$

Specifically, the pacing multiplier μ_t^* in each round t is simply the constraint-pacing multiplier μ that maximizes the expected value $V(\cdot) := V_t(\cdot)$ (which is the same in all rounds).

Corollary 19 Consider the stationary-stochastic environment under Lipschitzness (Assumption 13). Assume that parameters $(\lambda, \gamma, \bar{v}, \rho)$ are all constants. Let Π be the set of all constraint-feasible multipliers. Then Algorithm 2 with the same parameters as in Theorem 16 satisfies

$$T \cdot \max_{\mu \in \Pi} V(\mu) - \sum_{t=1}^T V(\mu_t) \leq \mathcal{O}(T^{7/8}).$$

Remark 20 While [Theorem 16](#) applies to a general adversarial environment, we do not immediately obtain a meaningful corollary for the “multi-player” environment when the other agents deploy the same algorithm. Indeed, we can upper-bound the path-lengths P_T^B, P_T^R as $O((\eta_B + \eta_R)T)$, but plugging this back into [Theorem 16](#) does not yield $o(T)$ regret. This issue arises (even) for the special case of budget-only constraint, as noted in [Gaitonde et al. \(2023\)](#). That said, we do get a non-trivial guarantee when all other agents use our algorithm with different parameters η'_B, η'_R that are sufficiently small (provided that [Assumption 13](#) holds, e.g., as per [Footnote 16](#)).

Proof intuition. The high-level idea is to define auxiliary stochastic convex functions H_t^R and H_t^B that achieve their minima at $\mu_t^{R^*}$ and $\mu_t^{B^*}$ resp., and interpret the update rules for μ_t^R and μ_t^B as applying stochastic gradient descent (SGD) w.r.t. these auxiliary functions. We then relate the difference in obtained value $V_t(\mu_t^*) - V_t(\mu_t)$ by the total loss in these auxiliary functions. Then, we’d ideally use facts about SGD to bound the total loss in value relative to the optimal benchmark.

Unfortunately, this approach runs into many technical problems. The first problem is relatively straightforward. Recall that we have both a budget and an ROI constraint, but only one multiplier is used in each round; this can be either the budget-multiplier μ_t^B or the ROI-multiplier μ_t^R , whichever is larger. To avoid having to reason about which multiplier is being followed each round, we will actually bound the sum of regret experienced for both multipliers. We think of this as decomposing our experienced regret into the sum of two counterfactual regrets: one for the case where we have only the budget constraint and bid according to μ_t^B , and one for the case where we have only the ROI constraint and bid according to μ_t^R . For each of these two cases, we can bound the total loss in value V_t w.r.t. the differences $|Z_t^B(\mu_t^B) - Z_t^B(\mu_t^{B^*})|$ and $|Z_t^R(\mu_t^R) - Z_t^R(\mu_t^{R^*})|$, resp., which we can then relate to corresponding differences in our auxiliary functions.

This raises a more fundamental problem. We’d like to argue that μ_t^B evolves according to SGD on our auxiliary function H_t^B , and similarly for μ_t^R and H_t^R . However, since we only receive feedback w.r.t. the larger multiplier, the smaller multiplier (μ_t^B or μ_t^R) may not be updated according to the gradient of its corresponding loss function. One thing we do know is that, since the auxiliary functions are convex, the gradient we use to update the smaller multiplier can only be more negative, in expectation, than its “correct” gradient. At first this seems like an unacceptable source of error; if gradients are too negative, then (for example) μ_t^B could drift arbitrarily far from $\mu_t^{B^*}$ in the negative direction. However, we are saved by [Lemma 6](#): since we know that the budget and ROI constraints will necessarily be satisfied at the end of the T rounds, our algorithm will not actually suffer any loss of value due to bids being too large (and hence, multipliers being too small). We can therefore think of the evolution of one of the multipliers, say μ_t^B , as following a variant of SGD in which an adversary can, at will, perturb any given update step to be more negative; but in exchange, we only suffer losses when $\mu_t^B > \mu_t^{B^*}$. As it turns out, the usual analysis of SGD extends to this variant, so we can conclude that our total loss is not too large.

There are some additional technical challenges to handle as well. Most notably, the auxiliary loss function for the ROI-multiplier is not convex in general but only convex when $\mu \in [0, \gamma - 1]$. This requires us to handle separately the case where the budget-multiplier is greater than $\gamma - 1$, and omit such rounds from our accounting of losses w.r.t. the ROI constraint. This complicates our definition of counterfactual regret for ROI, but it turns out that the aggregate loss can still be bounded with some additional effort.

6. Proof of Theorem 10 (Liquid welfare)

We prove the theorem in three steps. We first show that with high probability, there is no significant correlation between the progression of our algorithm up to time t and the ex-ante benchmark evaluated in future rounds. We then condition on this event and bound the liquid welfare obtained on a per-instance basis. Finally, we take an expectation over realizations to obtain the desired bound on expected liquid welfare.

Step 1: Bounds on ex-ante Allocate Rules. W.l.o.g. we only consider allocation rules \mathbf{y} such that

$$\mathbb{E}_{\mathbf{v} \sim F} \left[\frac{1}{\gamma_k} y_k(\mathbf{v}) v_k \right] \leq \rho_k. \quad (6.1)$$

This is because for any y that violates this constraint, we can always decrease the allocation for agent k without affecting $\bar{W}(\mathbf{y}, F)$.

We make a stronger claim that Eq. (6.1) holds for every round t in which $\mu_{k,t} > \gamma_k - 1$. To this end, we will show that, with high probability, the ex-ante optimal allocation rule y does not generate significantly different outcomes in rounds where $\mu_{k,t} > \gamma_k - 1$ and rounds where $\mu_{k,t} \leq \gamma_k - 1$.

For each agent k , define the following quantity:

$$R_k(\mathbf{v}) \triangleq \sum_t \left[\mathbb{1}\{\mu_{k,t} \leq \gamma_k - 1\} \frac{1}{\gamma_k} y_k(\mathbf{v}) v_{k,t} + \mathbb{1}\{\mu_{k,t} > \gamma_k - 1\} \rho_k \right]. \quad (6.2)$$

We can then use the theory of concentration of martingales to establish that the following bound on $R_k(\mathbf{v})$ holds with probability at least $1 - 1/(\bar{v}nT)^2$:

$$R_k(\mathbf{v}) \leq \rho_k \cdot T + \bar{v} \sqrt{T \log(\bar{v}nT)}. \quad (6.3)$$

Eq. (6.3) is proved in Gaitonde et al. (2023) (see Lemma 26 in Appendix D). Now, taking a union bound over all agents $k \in [n]$, with probability at least $1 - 1/(\bar{v}T)^2$, we have:

$$R_k(\mathbf{v}) \leq \rho_k \cdot T + \bar{v} \sqrt{T \log(\bar{v}nT)}, \quad \forall k \in [n] \quad (6.4)$$

We say that a value realization is “good” if it satisfies Eq. (6.4).

Step 2: Liquid Welfare of “Good” Value Realizations. Fix any “good” value profile realization v . For any advertiser k whose liquid welfare is B_k , from Eq. (6.4), we know that:

$$W_k(\mathbf{v}) = B_k \geq R_k(\mathbf{v}) - \bar{v} \sqrt{T \log(\bar{v}nT)}. \quad (6.5)$$

Now we look at those agents $A \subseteq [n]$ for which the liquid welfare is strictly less than B_k :

$$W_k(\mathbf{v}) = \frac{1}{\gamma_k} \sum_{t=1}^T x_{k,t} v_{k,t} < B_k.$$

Like in Eq. (6.5), we again wish to bound $W_k(\mathbf{v})$ with respect to $R_k(\mathbf{v})$. To that end, we will derive a bound on $W_k(\mathbf{v})$ that accounts for variation in $\mu_{k,t}$. Let $\eta_k = \max\{\eta_{k,R}, \eta_{k,B}\}$. For each round t , Let $S_t \subseteq A$ denote the agents for whom $\mu_{k,t} \leq \gamma_k - 1$, and $T_t \subseteq S_t$ for the agents for whom $\gamma_k - 1 - \eta_k(\bar{v} + \rho_k) < \mu_{k,t} \leq \gamma_k - 1$. That is, S_t are the agents bidding “high enough,” and T_t are the agents in S_t that are “close to” the threshold value $\gamma_k - 1$ in round t .

Lemma 21 *The following inequality is guaranteed if all agents $k \in [n]$ apply Algorithm 2:*

$$\sum_{k \in A} \frac{1}{\gamma_k} \sum_{t=1}^T x_{k,t} v_{k,t} \geq \sum_{k \in A} \sum_{t=1}^T \left[\mathbb{1}(k \in S_t) \frac{1}{\gamma_k} x_{k,t} v_{k,t} - \mathbb{1}(k \in T_t) p_{k,t} + \mathbb{1}(\mu_{k,t} > \gamma_k - 1) \rho_k \right] \quad (6.6)$$

Proof Fix some agent $k \in A$. Divide the time interval $[1, T]$ into intervals (I_1, I_2, \dots) in the following manner: each interval $I = [t_1, t_2]$ is a minimal interval such that $\mu_{k,t_1} \leq \gamma_k - 1$ and $\mu_{k,t_2} \leq \gamma_k - 1$. That is, $\mu_{k,t} > \gamma_k - 1$ for all $t_1 < t < t_2$. Note that according to Lemma 7, we know that $\mu_{k,t} = \mu_{k,t}^B$ when $t \in (t_1, t_2)$.

We wish to bound $\frac{1}{\gamma_k} \sum_{t \in I} x_{k,t} v_{k,t}$ for each such interval I . Note that if $\mu_{k,t_1} \leq \gamma_k - 1 - \eta_k(\bar{v} + \rho_k)$, then we must have $t_2 = t_1 + 1$ (since $\mu_{k,t_1+1} \leq \mu_{t_1} + \eta_k(\bar{v} + \rho_k) \leq \gamma_k - 1$). Thus, when $\mu_{k,t_1} \leq \gamma_k - 1 - \eta_k(\bar{v} + \rho_k)$, we have $\frac{1}{\gamma_k} \sum_{t=t_1}^{t_2-1} x_{k,t} v_{k,t} = \frac{1}{\gamma_k} x_{k,t_1} v_{k,t_1}$.

On the other hand, if $\gamma_k - 1 - \eta_k(\bar{v} + \rho_k) < \mu_{k,t_1} \leq \gamma_k - 1$, we have

$$\gamma_k - 1 < \mu_{k,t_2-1} = \mu_{k,t_2-1}^B = \mu_{k,t_1}^B + \eta_k \sum_{\tau=t_1}^{t_2-2} (p_{k,\tau} - \rho_k) \leq \gamma_k - 1 + \eta_k \sum_{\tau=t_1}^{t_2-2} (p_{k,\tau} - \rho_k),$$

which means that $\sum_{\tau=t_1}^{t_2-2} (p_{k,\tau} - \rho_k) \geq 0$. Since $p_{k,t} \leq b_{k,t} x_{k,t} < \frac{1}{\gamma_k} x_{k,t} v_{k,t}$ for all $t_1 < t < t_2$, we can conclude that

$$\frac{1}{\gamma_k} \sum_{t=t_1}^{t_2-1} x_{k,t} v_{k,t} \geq \frac{1}{\gamma_k} x_{k,t_1} v_{k,t_1} + \sum_{t=t_1+1}^{t_2-1} p_{k,t} \geq \frac{1}{\gamma_k} x_{k,t_1} v_{k,t_1} - p_{k,t_1} + (t_2 - t_1 - 1) \rho_k.$$

Summing over all time steps, we conclude that

$$\begin{aligned} \frac{1}{\gamma_k} \sum_{t=1}^T x_{k,t} v_{k,t} &\geq \sum_{t=1}^T \left[\mathbb{1}(\mu_{k,t} \leq \gamma_k - 1) \frac{1}{\gamma_k} x_{k,t} v_{k,t} \right. \\ &\quad \left. - \mathbb{1}(\gamma_k - 1 - \eta_k(\bar{v} + \rho_k) < \mu_{k,t} \leq \gamma_k - 1) p_{k,t} + \mathbb{1}(\mu_{k,t} > \gamma_k - 1) \rho_k \right]. \end{aligned}$$

Summing this inequality over all agents yields Eq. (6.6). ■

Our next goal is to relate the terms in the right-hand side of Eq. (6.6) with the corresponding terms in $R_k(\mathbf{v})$. Fix some round t . We will focus on the first two terms in the expression inside the summation on the right hand side of Eq. (6.6). Consider the agents in T_t , which are the agents for whom $\gamma_k - 1 - \eta_k(\bar{v} + \rho_k) < \mu_{k,t} \leq \gamma_k - 1$. We have

$$\begin{aligned} &\sum_{k \in T_t} \left(\frac{1}{\gamma_k} x_{k,t} v_{k,t} - p_{k,t} \right) \\ &= \sum_{k \in T_t} \left(\frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \frac{1}{\gamma_k} v_{k,t} (y_k(\mathbf{v}_t) - x_{k,t}) - p_{k,t} \right) \\ &\geq \sum_{k \in T_t} \left(\frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \frac{v_{k,t}}{1 + \mu_{k,t}} (y_k(\mathbf{v}_t) - x_{k,t}) - \eta_k(\bar{v} + \rho_k)^2 - p_{k,t} \right) \\ &= \sum_{k \in T_t} \frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k \in T_t} \frac{v_{k,t}}{1 + \mu_{k,t}} (y_k(\mathbf{v}_t) - x_{k,t}) - \sum_{k \in T_t} p_{k,t} - |T_t| \eta_k(\bar{v} + \rho_k)^2, \end{aligned}$$

where the inequality follows from the definition of T_t : if $y_k(\mathbf{v}_t) \geq x_{k,t}$ we use that $\mu_{k,t} \leq \gamma_k - 1$ and hence $\frac{1}{\gamma_k} \leq \frac{1}{\mu_{k,t} + 1}$, whereas if $y_k(\mathbf{v}_t) < x_{k,t}$ we use that $\mu_{k,t} \geq \gamma_k - 1 - \eta_k(\bar{v} + \rho_k)$ and hence $\frac{1}{\gamma_k} \geq \frac{1}{\mu_{k,t} + 1} - \eta_k(\bar{v} + \rho_k)$.

On the other hand, for agents in $S_t \setminus T_t$, we have

$$\begin{aligned} \sum_{k \in S_t \setminus T_t} \frac{1}{\gamma_k} x_{k,t} v_{k,t} &\geq \sum_{k \in S_t \setminus T_t} \left(\frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \frac{1}{\gamma_k} v_{k,t} (y_k(\mathbf{v}_t) - x_{k,t})^+ \right) \\ &\geq \sum_{k \in S_t \setminus T_t} \left(\frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \frac{v_{k,t}}{1 + \mu_{k,t}} (y_k(\mathbf{v}_t) - x_{k,t})^+ \right) \\ &= \sum_{k \in S_t \setminus T_t} \frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k \in S_t \setminus T_t} \frac{v_{k,t}}{1 + \mu_{k,t}} (y_k(\mathbf{v}_t) - x_{k,t})^+. \end{aligned}$$

Let U_t be the set of all agents in T_t , plus the agents in $S_t \setminus T_t$ such that $x_{k,t} \leq y_k(\mathbf{v}_t)$. Then, adding our two inequalities together gives

$$\begin{aligned} \sum_{k \in S_t} \frac{1}{\gamma_k} x_{k,t} v_{k,t} - \sum_{k \in T_t} p_{k,t} \\ \geq \sum_{k \in S_t} \frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k \in U_t} \frac{v_{k,t}}{1 + \mu_{k,t}} (y_k(\mathbf{v}_t) - x_{k,t}) - \sum_{k \in T_t} p_{k,t} - |T_t| \eta_k (\bar{v} + \rho_k)^2. \end{aligned}$$

We wish to bound the term $\sum_{k \in U_t} \frac{v_{k,t}}{1 + \mu_{k,t}} (y_k(\mathbf{v}_t) - x_{k,t})$ from the inequality above. Note that this is exactly the difference in declared value (i.e., bid) for $y_k(\mathbf{v}_t)$ and $x_{k,t}$ for agents in U_t . It is here where we use the fact that the underlying auction is a core auction. From the definition of a core auction, this difference in bids is at most the sum of payments of agents not in U_t . Therefore,

$$\begin{aligned} \sum_{k \in S_t} \left(\frac{1}{\gamma_k} x_{k,t} v_{k,t} \right) - \sum_{k \in T_t} p_{k,t} &\geq \sum_{k \in S_t} \frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k \notin U_t} p_{k,t} - \sum_{k \in T_t} p_{k,t} - |T_t| \eta_k (\bar{v} + \rho_k)^2 \\ &\geq \sum_{k \in S_t} \frac{1}{\gamma_k} y_k(\mathbf{v}_t) v_{k,t} - \sum_k p_{k,t} - |T_t| \eta_k (\bar{v} + \rho_k)^2, \end{aligned} \tag{6.7}$$

where in the second inequality we used the fact that $T_t \subseteq U_t$, so the two sums of over payments are over disjoint sets of agents. Summing up Eq. (6.7) over all rounds and substituting into Eq. (6.6) and using the definitions of $W_k(\mathbf{v})$ and $R_k(\mathbf{v})$, we conclude that

$$\sum_{k \in A} W_k(\mathbf{v}) \geq \sum_{k \in A} R_k(\mathbf{v}) - \sum_t \sum_{k=1}^n p_{k,t} - \eta_k (\bar{v} + \rho_k)^2 n T.$$

Summing over all agents $k \in [n]$, we have that for every “good” value realization v ,

$$\sum_k W_k(\mathbf{v}) \geq \sum_k R_k(\mathbf{v}) - \sum_t \sum_{k=1}^n p_{k,t} - \bar{v} n \sqrt{T \log(\bar{v} n T)} - \eta_k (\bar{v} + \rho_k)^2 n T. \tag{6.8}$$

Step 3: Bounding Expected Liquid Welfare. Recall from [Observation 3](#) that the total revenue collected over all rounds will never be greater than the liquid welfare of the allocation. In other words, $\sum_t \sum_{k=1}^n p_{k,t} \leq \sum_k W_k(\mathbf{v})$. We can therefore rearrange [Eq. \(6.8\)](#) to conclude that

$$2\sum_k W_k(\mathbf{v}) \geq \sum_k R_k(\mathbf{v}) - \bar{v}n\sqrt{T \log(\bar{v}nT)} - \eta_k(\bar{v} + \rho_k)^2 nT.$$

Taking expectations over v and conditioning on the good event, we conclude that our expected liquid welfare is at least half of the expected optimal liquid welfare with an error term that grows at a rate of $\mathcal{O}(\bar{v}n\sqrt{T \log(\bar{v}nT)})$, as we take $\eta_k \leq \frac{\bar{v}}{\bar{v} + \rho_k} \sqrt{\frac{\log(\bar{v}nT)}{T}}$. This completes the proof.

7. Proof of [Theorem 16 \(Regret\)](#)

We flesh out the intuition from [Section 5.2](#) but omit some details which are relegated to [Appendix F](#).

We begin by formalizing our interpretation of [Algorithm 2](#) as applying a form of SGD. We construct auxiliary loss functions $H_t^B(\mu) = \rho\mu - \int_0^\mu Z_t^B(\tau)d\tau$ and $H_t^R(\mu) = \int_0^\mu \rho_t(\tau) - Z_t^R(\tau)d\tau$.

Based on [Lemma 7](#), we have the following lemma, which shows that if the ROI multiplier is larger than the budget multiplier, then the ROI-multiplier is updated by applying a SGD on function $H_t^R(\mu)$, and otherwise the budget-multiplier is updated by applying SGD on function $H_t^B(\mu)$.

Lemma 22 [Algorithm 2](#) guarantees that:

- If $\mu_t^R \geq \mu_t^B$, $\mathbb{E}[\gamma p_t(\mu_t) - v_t x_t(\mu_t)] = Z_t^R(\mu_t) - \rho_t(\mu_t)$.
- If $\mu_t^R < \mu_t^B$, $\mathbb{E}[p_t(\mu_t) - \rho] = Z_t^B(\mu_t) - \rho$.

Proof For ROI-multiplier, direct calculation shows that

$$\begin{aligned} \mathbb{E}[(\gamma p_t(\mu_t) - v_t)x_t(\mu_t)] &= \mathbb{E}[(\gamma p_t(\mu_t) - v_t x_t(\mu_t))^+] - \mathbb{E}[(v_t x_t(\mu_t) - \gamma p_t(\mu_t))^+] \\ &= Z_t^R(\mu_t) - \rho_t(\mu_t). \end{aligned}$$

For budget-multiplier, direct calculation shows that $\mathbb{E}[x_t p_t(\mu_t) - \rho] = Z_t^B(\mu_t) - \rho$. Combining the above two equations completes the proof. \blacksquare

We next establish the convexity and Lipschitzness of $H_t^B(\mu)$ and $H_t^R(\mu)$.

Lemma 23 $H_t^R(\mu)$ is $(\gamma \cdot \bar{v})$ -Lipschitz and convex when $\mu \in [0, \gamma - 1]$ and $H_t^B(\mu)$ is $(\bar{v} + \rho)$ -Lipschitz and convex in $\mu \geq 0$.

Proof The result for $H_t^B(\mu)$ is proven in Lemma D.2 in [Gaitonde et al. \(2023\)](#). For the function $H_t^R(\mu)$ we have

$$\nabla H_t^R(\mu) = \rho_t(\mu) - Z_t^R(\mu) = \mathbb{E}[v_t x_t(\mu) - \gamma p_t(\mu)],$$

which we show is increasing over $\mu \in [0, \gamma - 1]$ according to [Lemma 12](#). In addition, we have for all $\mu \in [0, \gamma - 1]$, it holds that $|\nabla H_t^R(\mu)| \leq \max\{\gamma p_t(\mu), v_t\} \leq \gamma \bar{v}$. \blacksquare

We proceed to prove [Theorem 16](#). First, we decompose the overall regret into the regret with respect to budget-multiplier and ROI-multiplier as follows:

$$\begin{aligned}
 \text{Reg} &= \sum_{t=1}^T (V_t(\mu_t^*) - V_t(\mu_t)) \\
 &\leq \sum_{t=1}^T (V_t(\mu_t^*) - V_t(\mu_t)) \mathbf{1}\{\mu_t \geq \mu_t^*\} && (V_t(\mu) \text{ is decreasing in } \mu) \\
 &\leq \sum_{t=1}^T [(V_t(\mu_t^*) - V_t(\mu_t^B)) \mathbf{1}\{\mu_t^B \geq \max\{\mu_t^R, \mu_t^*\}\}] \\
 &\quad + \sum_{t=1}^T [(V_t(\mu_t^*) - V_t(\mu_t^R)) \mathbf{1}\{\mu_t^R \geq \max\{\mu_t^B, \mu_t^*\}\}] \\
 &\leq \sum_{t=1}^T [(V_t(\mu_t^*) - V_t(\mu_t^B)) \mathbf{1}\{\mu_t^B \geq \mu_t^*\}] + \sum_{t=1}^T [(V_t(\mu_t^*) - V_t(\mu_t^R)) \mathbf{1}\{\mu_t^R \geq \mu_t^*, \mu_t^B \leq \gamma - 1\}] \\
 &\leq \sum_{t=1}^T [(V_t(\mu_t^{B*}) - V_t(\mu_t^B)) \mathbf{1}\{E_t^B\}] + \sum_{t=1}^T [(V_t(\mu_t^{R*}) - V_t(\mu_t^R)) \mathbf{1}\{E_t^R\}], \tag{7.1}
 \end{aligned}$$

where E_t^B is the event that $\mu_t^B \geq \mu_t^{B*}$ and E_t^R is the event that $\mu_t^R \geq \mu_t^{R*}$ and $\mu_t^B \leq \gamma - 1$. The third inequality holds because $\mu_t^R \leq \gamma - 1$ due to [Lemma 7](#), since $\mu_t^B \leq \mu_t^R \leq \gamma - 1$ in the second term. The fourth inequality holds because $V_t(\mu)$ is non-increasing in μ .

We split the rest of the proof into four steps.

Step 1: Upper bounding the difference of V_t by the difference of Z_t^B and $Z_t^R - \rho_t$.

Using the monotonicity of $p_t(\mu)$, we show that for ROI-multiplier and budget-multiplier,

$$(V_t(\mu_t^{R*}) - V_t(\mu_t^R)) \mathbf{1}\{E_t^R\} \leq \left(\frac{\lambda}{\beta} (Z_t^R(\mu_t^{R*}) - \rho_t(\mu_t^{R*}) - Z_t^R(\mu_t^R) + \rho_t(\mu_t^R)) + \beta\lambda \right) \mathbf{1}\{E_t^R\}, \tag{7.2}$$

$$(V_t(\mu_t^{B*}) - V_t(\mu_t^B)) \mathbf{1}\{E_t^B\} \leq \left(\frac{\bar{v} + \rho}{\rho} (Z_t^B(\mu_t^{B*}) - Z_t^B(\mu_t^B)) \right) \mathbf{1}\{E_t^B\}, \tag{7.3}$$

for any $\beta > 0$ to be fixed later. The proof is in [Appendix G \(Lemma 29\)](#).

Step 2: Upper bounding the difference of Z_t^R (Z_t^B) by the difference of H_t^R (H_t^B).

Next, we relate $Z_t^R(\mu_t^{R*}) - Z_t^R(\mu_t^R)$ with $H_t^R(\mu_t^{R*}) - H_t^R(\mu_t^R)$. Direct calculation shows that

$$(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R*})) \mathbf{1}\{E_t^R\} = \int_0^{\mu_t^R - \mu_t^{R*}} (\rho_t(\tau + \mu_t^{R*}) - Z_t^R(\tau + \mu_t^{R*})) d\tau \mathbf{1}\{E_t^R\}.$$

Note that $g^R(x) = \rho_t(x + \mu_t^{R*}) - Z_t^R(x + \mu_t^{R*})$ is a non-decreasing function of x when $x \in [0, \mu_t^R - \mu_t^{R*}]$ according to [Lemma 23](#). Also, we have $g^R(0) \geq 0$. Applying a technical lemma from [Gaitonde et al. \(2023\)](#) (see [Lemma 27](#)) with $f(x) = g^R(x) - g^R(0)$, we know that

$$\begin{aligned}
 &\left[(\rho_t(\mu_t^R) - Z_t^R(\mu_t^R)) - (\rho_t(\mu_t^{R*}) - Z_t^R(\mu_t^{R*})) \right] \mathbf{1}\{E_t^R\} \\
 &\leq \sqrt{4\lambda (H_t^R(\mu_t^R) - H_t^R(\mu_t^{R*}) - (\mu_t^R - \mu_t^{R*})(\rho_t(\mu_t^{R*}) - Z_t^R(\mu_t^{R*})))} \mathbf{1}\{E_t^R\}.
 \end{aligned}$$

When $\mu_t^R \geq \mu_t^{R^*}$, we know that $(\mu_t^R - \mu_t^{R^*})(\rho_t(\mu_t^{R^*}) - Z_t^R(\mu_t^{R^*})) \geq 0$ and we can obtain

$$\left(Z_t^R(\mu_t^{R^*}) - \rho_t(\mu_t^{R^*}) - Z_t^R(\mu_t^R) + \rho_t(\mu_t^R) \right) \mathbf{1}\{E_t^R\} \leq \sqrt{4\lambda(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R^*}))} \mathbf{1}\{E_t^R\}. \quad (7.4)$$

For budget-constraint multiplier μ_t^B , we similarly define $g^B(x) = \rho - Z_t^B(\mu)$. Note that $g^B(x)$ is non-decreasing and λ -Lipschitz. Applying [Lemma 27](#) on $g^B(x) - g^B(0)$, we also have

$$\left(Z_t^B(\mu_t^{B^*}) - Z_t^B(\mu_t^B) \right) \mathbf{1}\{E_t^B\} \leq \sqrt{2\lambda(H_t^B(\mu_t^B) - H_t^B(\mu_t^{B^*}))} \mathbf{1}\{E_t^B\}. \quad (7.5)$$

Step 3-1: Upper bounding the regret with respect to H_t^R

Now we analyze the term $(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R^*})) \mathbf{1}\{E_t^R\}$ and $(H_t^B(\mu_t^R) - H_t^B(\mu_t^{B^*})) \mathbf{1}\{E_t^R\}$. For the first term, note that $(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R^*})) \mathbf{1}\{E_t^R\} \leq \langle g_t^{R,R}, \mu_t^R - \mu_t^{R^*} \rangle \mathbf{1}\{E_t^R\}$ where $g_t^{R,R} = \nabla H_t^R(\mu_t^R)$. This is because $\mu_t^{R^*} \leq \mu_t^R \leq \gamma - 1$ and $H_t^R(\mu)$ is convex when $\mu \in [0, \gamma - 1]$ according to [Lemma 23](#). According to [Lemma 22](#), if $\mu_t^R \geq \mu_t^B$, then μ_t^R is updated by a stochastic gradient with mean $g_t^{R,R} = \nabla H_t^R(\mu_t^R)$. However, note that μ_t^R may not be updated using its own stochastic gradient on $H_t^R(\mu)$, but may be updated by the gradient $g_t^{R,B} = \nabla H_t^R(\mu_t^B)$ if $\mu_t^B \geq \mu_t^R$. However, using the convexity of $H_t^R(\mu)$ when $\mu \in [0, \gamma - 1]$, we have $g_t^{R,B} \geq g_t^{R,R}$ as $\mu_t^R \leq \mu_t^B \leq \gamma - 1$. Therefore, let g_t^R be the gradient that is used to update μ_t^R and we have

$$(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R^*})) \mathbf{1}\{E_t^R\} \leq \langle g_t^{R,R}, \mu_t^R - \mu_t^{R^*} \rangle \mathbf{1}\{E_t^R\} \leq \langle g_t^R, \mu_t^R - \mu_t^{R^*} \rangle \mathbf{1}\{E_t^R\}, \quad (7.6)$$

since when $\mu_t^B \geq \mu_t^R$, $g_t^R > g_t^{R,R}$ and $\mu_t^R \geq \mu_t^{R^*}$. Let \hat{g}_t^R be the empirical gradient of μ_t^R at round t with $\mathbb{E}[\hat{g}_t^R] = g_t^R$. Then, applying the analysis of online gradient descent,

$$\begin{aligned} \mathbb{E} \left[\sum_{t=1}^T (H_t^R(\mu_t^R) - H_t^R(\mu_t^{R^*})) \mathbf{1}\{E_t^R\} \right] &\leq \mathbb{E} \left[\langle \hat{g}_t^R, \mu_t^R - \mu_t^{R^*} \rangle \mathbf{1}\{E_t^R\} \right] \\ &\leq \sum_{t=1}^T \left(\frac{|\mu_t^R - \mu_t^{R^*}|^2 - |\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2}{2\eta_R} + \frac{\eta_R}{2}(\gamma + 1)^2 \bar{v}^2 + \frac{(\gamma - 1)|\mu_t^{R^*} - \mu_{t+1}^{R^*}|}{\eta_R} \right) \mathbf{1}\{E_t^R\}, \end{aligned} \quad (7.7)$$

where the last inequality is because $|\hat{g}_t^R| \leq (\gamma + 1)\bar{v}$ and

$$\begin{aligned} |\mu_{t+1}^R - \mu_t^{R^*}|^2 &\leq |\mu_t^R - \eta_R \hat{g}_t^R - \mu_t^{R^*}|^2 = |\mu_t^R - \mu_t^{R^*}|^2 - 2\eta_R \langle \hat{g}_t^R, \mu_t^R - \mu_t^{R^*} \rangle + \eta_R^2 |\hat{g}_t^R|^2, \\ |\mu_{t+1}^R - \mu_t^{R^*}|^2 &= |\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2 + 2 \langle \mu_{t+1}^R - \mu_{t+1}^{R^*}, \mu_t^{R^*} - \mu_{t+1}^{R^*} \rangle + |\mu_t^{R^*} - \mu_{t+1}^{R^*}|^2 \\ &\geq |\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2 - 2|\mu_{t+1}^R - \mu_{t+1}^{R^*}| \cdot |\mu_t^{R^*} - \mu_{t+1}^{R^*}| \\ &\geq |\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2 - 2(\gamma - 1)|\mu_t^{R^*} - \mu_{t+1}^{R^*}|. \end{aligned}$$

While generally online gradient descent gives $\mathcal{O}(\sqrt{T})$ regret, the challenge in bounding the terms on the right hand side of [Eq. \(7.7\)](#) is that with the condition $\mathbf{1}\{E_t^R\}$, the term $|\mu_t^R - \mu_t^{R^*}|^2 -$

$|\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2$ cannot be telescoped after summation. Fortunately, with a more careful analysis on the dynamic of μ_t^R , we show that

$$\sum_{t=1}^T \frac{|\mu_t^R - \mu_t^{R^*}|^2 - |\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2}{2\eta_R} \cdot \mathbf{1}\{E_t^R\} \leq \mathcal{O}\left(\frac{|\mu_1^R - \mu_1^{R^*}|^2 + (\gamma + 1)P_T^R}{\eta_R} + \eta_R(\gamma + 1)^2\bar{v}^2T\right). \quad (7.8)$$

(The proof is in [Appendix G](#).) Combining [Eq. \(7.8\)](#) and [Eq. \(7.7\)](#) with $\eta_R = \frac{1}{\sqrt{T}\bar{v}}$ gives

$$\begin{aligned} \mathbb{E}\left[\sum_{t \in [T]} \left(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R^*})\right) \mathbf{1}\{E_t^R\}\right] &\leq \mathcal{O}\left(\frac{(\gamma + 1)^2(P_T^R + 1)}{\eta_R} + \eta_R(\gamma + 1)^2\bar{v}^2T\right) \\ &= \mathcal{O}\left((\gamma + 1)^2\bar{v}(P_T^R + 1)\sqrt{T}\right). \end{aligned} \quad (7.9)$$

Step 3-2: Upper bounding the regret with respect to H_t^B

Next, we consider $H_t^B(\mu_t^B) - H_t^B(\mu_t^{B^*})$. Let $g_t^{B,B} = \nabla H_t^B(\mu_t^B)$ and $g_t^{B,R} = \nabla H_t^B(\mu_t^R)$. Similar to [Step 3-1](#), because of the convexity of $H_t^B(\mu)$ when $\mu \in [0, +\infty)$, we know that if $\mu_t^B \leq \mu_t^R$, we have $g_t^{B,R} \geq g_t^{B,B}$. Therefore, we have the following inequality similar to [Eq. \(7.6\)](#):

$$\left(H_t^B(\mu_t^B) - H_t^B(\mu_t^{B^*})\right) \mathbf{1}\{E_t^B\} \leq \langle g_t^B, \mu_t^B - \mu_t^{B^*} \rangle \mathbf{1}\{E_t^B\}, \quad (7.10)$$

where $g_t^B = \nabla H_t^B(\mu_t)$. Then, similar to [Eq. \(7.7\)](#), using the fact that $\mu_t^{B^*} \leq \frac{\bar{v}}{\rho}$ for all t , we have

$$\begin{aligned} \mathbb{E}\left[\sum_{t=1}^T \left(H_t^B(\mu_t^B) - H_t^B(\mu_t^{B^*})\right) \mathbf{1}\{E_t^B\}\right] &\leq \sum_{t=1}^T \left(\frac{|\mu_t^B - \mu_t^{B^*}|^2 - |\mu_{t+1}^B - \mu_{t+1}^{B^*}|^2}{2\eta_B} + \frac{\eta_B(\rho + \bar{v})^2}{2} + \frac{\bar{v}|\mu_t^{B^*} - \mu_{t+1}^{B^*}|}{\rho\eta_B}\right) \mathbf{1}\{E_t^B\}, \end{aligned} \quad (7.11)$$

where $(\rho + \bar{v})$ is a universal upper bound of the empirical gradient for μ_t^B . Again, different from the standard online gradient descent analysis, the term $|\mu_t^B - \mu_t^{B^*}|^2 - |\mu_{t+1}^B - \mu_{t+1}^{B^*}|^2$ cannot be telescoped and we prove (similarly to [Eq. \(7.8\)](#)), that

$$\sum_{t=1}^T \frac{|\mu_t^B - \mu_t^{B^*}|^2 - |\mu_{t+1}^B - \mu_{t+1}^{B^*}|^2}{2\eta_B} \cdot \mathbf{1}\{E_t^B\} \leq \mathcal{O}\left(\frac{|\mu_1^B - \mu_1^{B^*}|^2}{\eta_B} + \frac{\bar{v}P_T^B}{\rho\eta_B} + \eta_B(\bar{v} + \rho)^2T\right). \quad (7.12)$$

(The proof is in [Appendix G](#).) Combining [Eq. \(7.12\)](#) and [Eq. \(7.11\)](#) with $\eta_B = \frac{1}{\sqrt{T}\rho}$ gives

$$\begin{aligned} \mathbb{E}\left[\sum_{t=1}^T \left(H_t^B(\mu_t^B) - H_t^R(\mu_t^{B^*})\right) \mathbf{1}\{E_t^B\}\right] &\leq \mathcal{O}\left(\frac{|\mu_1^B - \mu_1^{B^*}|^2}{\eta_B} + \frac{\bar{v}P_T^B}{\rho\eta_B} + \eta_B(\bar{v} + \rho)^2T\right) \\ &\leq \mathcal{O}\left(\frac{\bar{v}^2}{\eta_B\rho^2} + \frac{\bar{v}P_T^B}{\rho\eta_B} + \eta_B(\bar{v} + \rho)^2T\right) \\ &\leq \mathcal{O}\left(\frac{(\bar{v} + \rho)^2}{\rho}(1 + P_T^B)\sqrt{T}\right). \end{aligned} \quad (7.13)$$

Step 4: Combining all the above analysis. Finally, we combine Eq. (7.1), Eq. (7.2), Eq. (7.3), Eq. (7.4), Eq. (7.5), Eq. (7.9), Eq. (7.13) and obtain the following

$$\begin{aligned}
& \mathbb{E} \left[\sum_{t=1}^T (V_t(\mu_t^*) - V_t(\mu_t)) \right] \\
& \leq \mathbb{E} \left[\sum_{t=1}^T \left(\frac{\gamma}{\beta} \left(Z_t^R(\mu_t^{R*}) - \rho_t(\mu_t^{R*}) - Z_t^R(\mu_t^R) + \rho_t(\mu_t^R) \right) + \beta\lambda \right) \mathbf{1}\{E_t^R\} \right] \\
& \quad + \mathbb{E} \left[\sum_{t=1}^T \frac{\bar{v} + \rho}{\rho} \left(Z_t^B(\mu_t^{B*}) - Z_t^B(\mu_t^B) \right) \mathbf{1}\{\mu_t^B \geq \mu_t^{B*}\} \right] \quad (\text{Eq. (7.2), Eq. (7.3)}) \\
& \leq \mathbb{E} \left[\sum_{t=1}^T \left(\frac{\gamma}{\beta} \sqrt{4\lambda(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R*}))} + \beta\lambda \right) \mathbf{1}\{E_t^R\} \right] \\
& \quad + \mathbb{E} \left[\sum_{t=1}^T \frac{\bar{v} + \rho}{\rho} \sqrt{2\lambda(H_t^B(\mu_t^B) - H_t^B(\mu_t^{B*}))} \mathbf{1}\{\mu_t^B \geq \mu_t^{B*}\} \right] \quad (\text{Eq. (7.4), Eq. (7.5)}) \\
& \leq \mathbb{E} \left[\left(\frac{\gamma}{\beta} \sqrt{4T \sum_{t=1}^T \lambda(H_t^R(\mu_t^R) - H_t^R(\mu_t^{R*}))} + \beta\lambda \right) \mathbf{1}\{E_t^R\} \right] \\
& \quad + \mathbb{E} \left[\frac{\bar{v} + \rho}{\rho} \sqrt{2T\lambda \sum_{t=1}^T (H_t^B(\mu_t^B) - H_t^B(\mu_t^{B*}))} \mathbf{1}\{\mu_t^B \geq \mu_t^{B*}\} \right] \quad (\text{Jensen's inequality}) \\
& \leq \mathcal{O} \left(\frac{\gamma}{\beta} \sqrt{\lambda T^{1.5}(\gamma+1)^2 \bar{v}(P_T^R + 1)} + \beta\lambda T + \frac{(\bar{v} + \rho)^2}{\rho^{1.5}} \sqrt{T^{1.5}\lambda(1 + P_T^B)} \right) \\
& \quad (\text{Eq. (7.9), Eq. (7.13)}) \\
& \leq \mathcal{O} \left((P_T^R + 1)^{\frac{1}{4}} \lambda^{\frac{3}{4}} (\gamma+1) T^{\frac{7}{8}} + (\bar{v} + \rho)^2 \rho^{-1.5} \sqrt{\lambda(1 + P_T^B)} T^{\frac{3}{4}} \right),
\end{aligned}$$

where we obtain the last inequality by picking the optimal $\beta > 0$. This completes the proof.

8. Numerical Experiments

We provide numerical experiments on several simulated problem instances with budget and ROI constraints, so as to illustrate our algorithm's performance in “multi-player” environments when all advertisers run the same algorithm. In particular, individual regret in this environment is only studied via experiments, as we cannot meaningfully upper-bound it with our theory (see Remark 20 in Section 5.2). We also consider a few other algorithms, to get a sense of the problem difficulty and verify that our problem instances are indeed non-trivial. (In each experiment, all bidders are assigned the same algorithm.) All plots are deferred to Appendix H.

Our algorithm(s). We consider a variant of Algorithm 2 with the budget-multiplier initialized as $\mu_{k,1}^B = \frac{1}{2\rho_k}$ instead of $\mu_{k,1}^B = \frac{\bar{v}}{\rho_k} - 1$, the latter being too conservative in practice. This variant enjoys the same aggregate and individual guarantees as Algorithm 2, except that the budget constraint is violated by $\leq O(\sqrt{T})$ with high probability. (All proofs carry over with minimal modifications.)

Moreover, we consider an *optimistic variant* of our algorithm, in line with optimistic variants of online gradient descent (OGD) and online mirror descent (OMD) which have been prominent in recent prior work on repeated multi-agent games [Syrgkanis et al. \(2015\)](#); [Daskalakis et al. \(2021\)](#); [Golowich et al. \(2020b\)](#); [Wei et al. \(2021\)](#). These variants have lead to provable guarantees in terms of regret and convergence in several scenarios when the original OGD and OMD do not appear to be amenable to analysis.¹⁷ The general template for round t of “optimistic OGD” is as follows:

$$\text{action}_t \leftarrow \text{action}_t + \text{update}_t + (\text{update}_t - \text{update}_{t-1}).$$

Accordingly, the optimistic variant of our algorithm updates the multipliers as

$$\mu_{k,t+1}^R = \mu_{k,t}^R + 2\eta_R(x_{k,t}(\gamma_k p_{k,t} - v_{k,t})) - \eta_R(x_{k,t-1}(\gamma_k p_{k,t-1} - v_{k,t-1})), \quad (8.1)$$

$$\mu_{k,t+1}^B = \mu_{k,t}^B + 2\eta_B(x_{k,t}p_{k,t} - \rho_k) - \eta_B(x_{k,t-1}p_{k,t-1} - \rho_k). \quad (8.2)$$

This variant is a *heuristic*, with no provable guarantees.

An algorithm from prior work. We consider an algorithm from [Feng et al. \(2022\)](#), which in turn is based on an algorithm from [Balseiro et al. \(2022b\)](#) (for budget constraints only). We call this algorithm *Best-Of-Many-Worlds (BOMW)*, following the title of [Balseiro et al. \(2022b\)](#). The algorithm is based on dual mirror descent. The analysis focuses on the stationary-stochastic environment, achieving $O(\sqrt{T})$ regret with no constraint violations.¹⁸ No aggregate guarantees for this algorithm are provided, and no individual guarantees beyond the stationary-stochastic environment.

“Naive” baselines. We also consider two “naive” baselines: the greedy algorithm and the epsilon-greedy algorithm. Both are standard “templates” in multi-armed bandits. The greedy algorithm *exploit* in each round, i.e., chooses the best action given the current observations. The epsilon-greedy algorithm explores uniformly with some fixed probability $\varepsilon > 0$, and exploits otherwise. In particular, the greedy algorithm ignores the need for exploration, and the epsilon-greedy algorithm (in our setting) ignores the non-stationarity of the multi-player environment and the constraints. So, we expect these algorithms to fail, and our experiments confirm this is indeed the case.

The two algorithms are implemented in the following unified way. For computational efficiency, we divide the rounds in batches of M rounds each, where M is a hyper-parameter, and update the multipliers only in the beginning of each such batch; we use $M = \lfloor \sqrt{T} \rfloor$. In each round t when we do update the multipliers, each algorithm calculates the budget- and ROI-pacing multipliers $\mu_t^{B^*}$, $\mu_t^{R^*}$ from [Defn 14](#) based on the observations collected so far,¹⁹ uses them to update the multipliers μ_t^B , μ_t^R for budget and ROI constraint, resp., and outputs $\mu_t = \max\{\mu_t^B, \mu_t^R\}$.

The “greedy” update sets μ_t^B , μ_t^R to the resp. pacing multipliers: $\mu_t^B \leftarrow \mu_t^{B^*}$ and $\mu_t^R \leftarrow \mu_t^{R^*}$.

The epsilon-greedy update is as follows. With probability $1 - \varepsilon$, follow the greedy algorithm. Else, explore uniformly, namely: choose μ_t^B and μ_t^R independently and uniformly at random within their respective ranges: $(0, \frac{\bar{v}}{\rho_k} - 1)$ and $[0, \gamma - 1]$. We use $\varepsilon = 0.1$.

Problem Instances. We strived for a variety of problem instances, while keeping the experiments manageable. Recall that a problem instance in our model is defined by the per-round auction, the

17. However, this prior work does not directly consider neither repeated auctions nor scenarios in which the players have global constraints, to the best of our knowledge.

18. Specifically, we follow Algorithm 5.1 in [Feng et al. \(2022\)](#). Algorithm 5.2 therein avoids constraint violations altogether due to an additional warm-up phase, which appears less suitable to be multi-player environment.

19. Specifically, we interpret the history as an “empirical distribution” for the stationary-stochastic environment, and define the pacing multipliers w.r.t. this distribution.

number of agents K , constraint parameters $\rho_k = B_k/T$ and γ_k for each agent $k \in [K]$, and the distribution F from which the value profiles are drawn. We endowed all players with the same constraint parameters: $\rho_k = \rho$ and $\gamma_k = \gamma$. We kept K and γ the same throughout all experiments. We considered the following variations: first-price or second-price auctions, two different values for ρ , and three different distributions F (allowing both independence and correlation across bidders). Thus, our space of experiments is as follows: $\{2\text{ auctions}\} \times \{\text{high/low budgets}\} \times \{3\text{ choices for }F\}$, for the total of $2 \times 2 \times 3 = 12$ choices. We represent them as 4×3 matrix of plots, so that all experiments are laid out on the same page.²⁰ The numerical choices are as follows: $K = 16$ (neither “too small” nor “too large”), $\rho \in \{0.15, 0.25\}$, $\gamma = 1.5$, and time horizon $T = 9000$.

Each problem instance is repeated $N = 8$ times, with results averaged across the runs.

The three distributions F that we consider are as follows:

- draw each value v_k independently and uniformly at random from some interval, namely $[0, 1]$.
- draw each value v_k independently from some fixed Gaussian distribution, clipping all values within range $[0, 1]$. Namely, use mean 0.4 and variance 0.2.
- draw the entire value profile from a correlated Gaussian distribution, clipping all values within range $[0, 1]$. Namely, use mean 0.4 for each agent, and covariance matrix $\Sigma = AA^\top$ where A is a random matrix with each entry uniformly chosen from $[-0.5, 0.5]$.²¹

Performance metrics target individual and aggregate guarantees as well as convergence:

Multiplier dynamics Focusing on agent 1, we plot multiplier $\mu_{1,t}$ (averaged over runs) as a function of time t ([Figure 1](#)).

Constraint slackness Time-averaged slackness in, resp., budget and ROI constraint, for agent k :

$$\frac{1}{T} \sum_{t=1}^T (p_{k,t} - \rho) \quad \text{and} \quad \frac{1}{T} \sum_{t=1}^T (x_{k,t} v_{k,t} - \gamma p_{k,t}). \quad (8.3)$$

Positive value means no violation in the respective constraint. For each constraint, we plot the empirical CDF of the slackness over all agents and all runs. The results are in [Figure 2](#) and [Figure 3](#) for the budget constraint, and in [Figure 4](#) and [Figure 5](#) for the ROI constraint.

Liquid welfare Time-averaged liquid welfare up to a given round t :

$$\overline{\text{LW}}_t = \sum_{\text{agents } k \in [K]} \min \left\{ \rho, \frac{1}{\gamma t} \sum_{\text{rounds } \tau \leq t} x_{k,\tau} v_{k,\tau} \right\}. \quad (8.4)$$

We average $\overline{\text{LW}}_t$ over the $N = 4$ runs, and plot the result over time ([Figure 6](#)).

Time-averaged Regret Focus on agent 1. We consider regret in terms of value received, with respect to the best-in-hindsight, call it *time-averaged static regret*. In a formula:

$$\text{StaticReg}_1(T) := \sum_{t \in [T]} v_t (x_t(\mu^*) - x_t(\mu_t)), \quad (8.5)$$

where μ^* is the smallest multiplier (i.e., the multiplier leading to largest bids) that does not violate the constraints given the realized history. We consider time-averaged static regret $\bar{R}(t) = \frac{1}{t} \text{StaticReg}_1(t)$, average it over the runs, and plot this average vs time t ([Figure 7](#)).

20. This desiderata to visualize all experiments on the same page was one reason to limit the space of experiments.

21. Matrix A is generated once for each of the $N = 4$ runs, and then kept the same throughout all experiments. Hence, only 4 different matrices have been generated.

Remark 24 Eq. (8.5) is a standard notion of regret from adversarial bandits. As such, this is an important metric to consider for a multi-player environment, even though one cannot guarantee vanishing regret in the adversarial environment.²² Recall that our own regret bounds are relative to a different benchmark: pacing multipliers (Defn 14). However, it is unclear how to compute this benchmark in a computationally efficient way.

Our findings. We make the following observations from the figures:

- The optimistic variant of our algorithm performed indistinguishably from the original variant, as far as our plots are concerned. Therefore we omit it from all plots.
- Both our algorithm and BOMW are OK on the constraints. Our algorithm satisfies them exactly, while BOMW exhibits small constraint violations.
- Both naive baselines *fail*, in that they exhibit large violations on at least one of the constraints in every experiment. (This is unsurprising, as discussed before.)
- The dynamics does not always converge to a stationary state, at least not within the (fairly large) timeframe that we considered, neither for our algorithm nor for BOMW.
- All algorithms achieve similar liquid welfare. Hence, the advantage of our algorithm (at least on these problem instances) lies in not violating the constraints.
- Time-averaged static regret $\bar{R}(t)$ decreases over time in a substantial way and appears to tend to 0, across all experiments, both for our algorithms and for BOMW. The latter performs somewhat better as far as $\bar{R}(t)$ is concerned.

We zoom in on the static regret of our algorithm and fit it to the familiar shape $\text{StaticReg}_1(T) = T^\alpha$ for some $\alpha > 0$. To this end, Figure 8 shows the log-log plot of for the cumulative static regret. We fit the curve by a linear function and show the corresponding slope and R-square value in Table 1, showing a strong empirical fit for 10 out of 12 instances. In the remaining two instances, which are both first-price auctions with $\rho = 0.25$, the static regret is non-monotone but small relative to the other cases. We omit these entries from Table 1 due to the poor fit, but see Figure 8 for a visualization.

Table 1: The rate $\alpha > 0$ when we express the (cumulative) static regret of our algorithm as T^α . We obtain α as a linear fit of the log-log curve. Each table entry is of the form “ α (R-square value)”.

| | i.i.d Uniform | i.i.d Gaussian | Correlated Gaussian |
|-----------------------------|----------------|----------------|---------------------|
| First-price, $\rho = 0.15$ | 0.743 (0.9997) | 0.768 (0.9989) | 0.739 (0.9992) |
| First-price, $\rho = 0.25$ | / | 0.340 (0.9733) | / |
| Second-price, $\rho = 0.15$ | 0.799 (0.9996) | 0.811 (0.9994) | 0.800 (0.9995) |
| Second-price, $\rho = 0.25$ | 0.621 (0.9958) | 0.606 (0.9998) | 0.668 (0.9981) |

22. Recall that in the adversarial environment, any algorithm suffers from an approximation ratio in the worst case, even for the special case of budget constraints (Balseiro and Gur, 2019).

Main findings (summary). *First*, our algorithm performs well on the metrics considered: no constraint violations, liquid-welfare performance same as BOMW, and vanishing static regret. Our problem instances appear non-trivial, given the performance of BOMW and the naive baselines.

Second, the optimistic variant does not appear to improve performance of our algorithm. This is somewhat surprising, given its theoretical superiority in some other scenarios.

Third, the dynamics does not always converge to a stationary state. This further motivates the study of aggregate guarantees such as ours that bypass convergence, and individual guarantees such as ours that go beyond the stationary environment. We observe strong algorithm performance regardless of whether the dynamics has converged.

Fourth, our algorithm appears to have vanishing static regret, despite non-convergence of the dynamics. In fact, the observed regret fits well into the standard T^α shape, for some fixed $\alpha > 0$ that we estimated empirically to be at most 0.82 in all instances we simulated (and at most 0.75 in most instances). We interpret this finding as further motivation for studying “beyond the worst case” individual guarantees for multi-player environments.

9. Conclusions and Open Problems

We consider the problem of online bidding with both budget and ROI constraints, under a broad class of auction formats including first-price and second-price auction. We set out to achieve both aggregate and individual guarantees, as expressed, resp., by liquid welfare and vanishing regret. We accomplish this with a novel variant of constraint-pacing, achieving (i) the best possible guarantee in expected liquid welfare, (ii) vanishing individual regret against an adversary, and (iii) satisfying the budget and ROI constraints with probability 1. The regret bound holds against a non-standard (albeit reasonable) benchmark, side-stepping impossibility results from prior work.

Our work opens up several directions for future work. First, one would like to obtain similar results for other algorithms or classes thereof. Second, one would like to improve regret rates while maintaining a similar aggregate guarantee. This is a non-standard question for the literature on online bidding. Particularly interesting are regret bounds that go beyond a stationary stochastic environment (since the auction environment is often/typically not stationary in practice). Another open direction is to analyze other aggregate market metrics such as platform revenue. Here, it would be helpful to have a more complete understanding of the interaction between autobidders and tunable parameters like reserve prices.

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Appendix A. Notation Summary

Let us list the notations used throughout the paper.

| | |
|-------------------------------------|--|
| $k \in [n]$ | agents |
| $t \in [T]$ | rounds |
| $X \in [0, 1]^n$ | the set of feasible allocation profiles |
| $\mathbf{x}_t \in X$ | allocation profile at round t , $\mathbf{x}_t = (x_{1,t}, \dots, x_{n,t})$ |
| $\mathbf{v}_t \in [0, \bar{v}]^n$ | value profile at round t , $\mathbf{v}_t = (v_{1,t}, \dots, v_{n,t})$ |
| \mathbf{b}_t | (effective) bid profile at round t , $\mathbf{b}_t = (b_{1,t}, \dots, b_{n,t})$ |
| \mathbf{p}_t | payment profile at round t , $\mathbf{p}_t = (p_{1,t}, \dots, p_{n,t})$ |
| γ_k | ROI constraint parameter for agent k |
| $B_k = \rho_k \cdot T$ | total budget for agent k |
| $W(\mathbf{x})$, $W_k(\mathbf{x})$ | liquid welfare and agent k 's liquid value for allocation sequence \mathbf{x} (Defn 2) |
| $\overline{W}(\mathbf{x}, F)$ | ex-ante liquid welfare of allocation rule \mathbf{x} and value distribution F (Defn 8) |
| $\mu_{k,t}^B$, $\mu_{k,t}^R$ | resp., budget-multiplier and ROI-multiplier for agent k at round t |
| $\eta_{k,B}$, $\eta_{k,R}$ | learning rate for $\mu_{k,t}^B$ and $\mu_{k,t}^R$ |
| Z_t^B , Z_t^R | expected budget- and ROI-expenditure at round t (Defn 11) |
| ρ_t , V_t | expected ROI and value gain at round t (Defn 11) |
| $\mu_t^{B^*}$, $\mu_t^{R^*}$ | budget- and ROI-pacing multiplier at round t (Defn 14) |
| μ_t^* | the pacing multiplier at round t (Defn 14) |
| P_T^B | path-length of budget-pacing multiplier, $P_T^B = \sum_{t=1}^{T-1} \mu_t^{B^*} - \mu_{t+1}^{B^*} $ |
| P_T^R | path-length of ROI-pacing multiplier, $P_T^R = \sum_{t=1}^{T-1} \mu_t^{R^*} - \mu_{t+1}^{R^*} $ |

Appendix B. Detailed Comparison to Gaitonde et al. (2023) (Budgets Only)

Our work builds upon results from [Gaitonde et al. \(2023\)](#) on an autobidding with budget constraints but not ROI constraints. Like [Gaitonde et al. \(2023\)](#), our work also bounds the liquid welfare and individual regret obtained over the dynamics of an autobidding algorithm, and we employ a similar high-level strategy to obtain such bounds. However, new technical challenges arise due to (a) fundamental differences between ROI constraints and budget constraints, and (b) extra complications due to handling multiple constraints simultaneously. These challenges necessitate a new algorithm, changes to the way liquid welfare bounds are established, and a different methodology for establishing regret properties. We elaborate on the similarities and differences below.

Algorithm. [Gaitonde et al. \(2023\)](#) studied an existing algorithm based on stochastic gradient descent (SGD), due to [Balseiro and Gur \(2019\)](#) (henceforth BG). This is our algorithm restricted to the special case of budget constraints only. To handle multiple constraints we re-interpret the BG algorithm as maintaining a certain invariant whereby the choice of multiplier in round t encodes the total slack in the budget constraint up to time t . Our algorithm extends this interpretation to multiple constraints, which we view as distinct from standard multi-dimensional gradient descent. We emphasize that SGD-based autobidding algorithms for budget and ROI constraints already exist in the literature, as discussed in [Section 2](#), but these algorithms do not appear amenable to our liquid welfare analysis.

Welfare analysis. Our liquid welfare analysis makes use of the fact that all constraints hold ex post with probability 1, since this enables us to compare revenue and liquid welfare. This is easy to guarantee for budget constraints, but not for ROI. Prior algorithms for ROI (that we are aware of) do not have this property, and (as we discuss in [Section 4](#)) after just one round it is possible to reach a state from which the ROI constraint cannot be satisfied. This challenge is further compounded with multiple constraints; a common strategy for multiple constraints is to estimate the likelihood each constraint will ultimately bind, but such methods tend to have some probability of failure.

Once we have established that all constraints are satisfied, the remainder of the liquid welfare analysis itself (Theorem 4.2) shares a high-level approach with ([Gaitonde et al., 2023](#)), but enacts this approach differently. The approach common to both papers is as follows: condition on the constraint that determines the liquid welfare for a given agent, split rounds into intervals according to that agent’s bid, and then bound the liquid welfare contributions from each interval separately. These liquid welfare bounds work by charging any loss in welfare (relative to a benchmark) to revenue collected by the mechanism, either from the given agent or from other competing agents. When doing this charging argument, it is important to take care when handling correlations introduced by the conditioning, and both papers handle this in a similar fashion.

We now discuss some differences. One major distinction is that because we have multiple constraints, we must track which one to follow at any given round. However, this is not uniquely implied by the bids. So our division into intervals does not perfectly correspond to the constraints to follow in our analysis. We show that a certain bid threshold serves as an (imperfect on one side) proxy. Unfortunately, whenever bids cross the threshold this generates errors that, by accumulating over rounds, have total magnitude that can be as large as the liquid welfare itself. These errors would significantly degrade the approximation ratio. To handle this problem we need a different accounting of the rounds, which makes it possible to use a charging argument for ROI constraints but introduces new additive errors that must be controlled.

Regret analysis. Finally, our new algorithm requires a new regret analysis. The challenges and intuition are described in [Section 5.2](#); but we summarize them again here.

First, since we only have (bandit) feedback for the larger multiplier, it is unclear why this provides a reasonable update for the smaller multiplier. We resolve this by noticing that the gradient we use to update the smaller multiplier can only be more negative than the gradient we would have obtained by bidding with the smaller multiplier, and establishing loss bounds that are robust to these one-sided gradient errors.

Second, the auxiliary loss function for the ROI-multiplier is not convex in general, but only in certain parameter regimes. We handle this by accounting for per-constraint losses in a way that lets us omit rounds and constraints where the ROI-multiplier may lie in a non-convex region.

Third, since the algorithm transitions between budget-binding and ROI-binding time intervals, trivially adding up the regret in each interval leads to $\Theta(T)$ regret bound, since there can be $\Theta(T)$ intervals. We handle this using a careful analysis on the value of each multiplier at the beginning and the end of each interval.

Appendix C. Details on Pacing Equilibria

An example for first-price auctions. We provide a simple example to lower-bound liquid welfare for first-price auctions (proving the statement in [Footnote 2](#)).

Claim 25 *Consider pacing equilibria in budget-constrained first-price auctions. For any $\lambda > \frac{1}{2}$, there exists a pacing equilibrium whose liquid welfare does not exceed the λ fraction of the optimum liquid welfare for the corresponding budget-constrained bidding game.*

Proof Consider the following simple example. There is a single divisible item, one agent with (large) value K per unit and budget 1, and a second agent with value 1 per unit and infinite budget. There is a pacing equilibrium where both agents bid 1, and the first agent wins the entire item. This solution has liquid welfare 1, whereas liquid welfare $2 - 1/K$ is possible: give the first agent $1/K$ of the item and the second agent $1 - 1/K$ of the item. ■

Comparing the techniques. Our liquid welfare bound shares a common high-level proof strategy with the corresponding results for pacing equilibria (as well as with those for prophet inequalities and other "smooth pricing" methods), but with some important differences.

All these results stem from the following "approximate first-welfare theorem" approach: if an agent is obtaining much less (liquid) welfare at equilibrium than in an optimal allocation, this must be because they face high effective prices, which must be supported by some other agents' payments. One can therefore charge any lost welfare against the total revenue collected by the platform, which is itself bounded by the (liquid) welfare. This proof strategy typically yields a 2- approximation. In a single-shot game, this charging argument can be applied directly to equilibrium bids.

The additional challenge in dynamic learning scenarios like ours is that, since bids are not necessarily in equilibrium, one cannot always charge lost welfare to high payments in the same round. To handle this, a now-standard idea is to amortize across rounds, by interpreting a low-regret sequence as a coarse correlated equilibrium (CCE). But unfortunately this doesn't work in our setting: CCE is not strong enough to imply a constant PoA for liquid welfare; see [Gaitonde et al. \(2023\)](#) for an example. So instead of appealing to low regret and CCE we analyze the dynamic sequence directly, separating "high bids" (where the ROI constraint might bind) from "low bids"

(where only the budget constraint can bind) and using different amortization strategies for each. Many of our technical challenges come from setting up the learning algorithm to enable this analysis of the learning dynamics.

Appendix D. Auxiliary Lemmas from prior work

Lemma 26 (Lemma 3.7 of Gaitonde et al. (2023)) *Let Y_1, \dots, Y_T be random variables and $\mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_T$ be a filtration such that:*

1. $0 \leq Y_t \leq \bar{v}$ with probability 1 for some parameter $\bar{v} \geq 0$ for all t .
2. $\mathbb{E}[Y_t] \leq \rho$ for some parameter ρ for all t .
3. For all t , Y_t is \mathcal{F}_t -measurable but is independent of \mathcal{F}_{t-1} -measurable. Then:

$$\Pr\left(\sum_{t \in [T]} X_t Y_t + (1 - X_t)\rho \geq \rho \cdot T + \theta\right) \leq \exp\left(-2\theta^2 / (T\bar{v}^2)\right).$$

Lemma 27 (Lemma 4.12 of Gaitonde et al. (2023)) *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing λ -Lipschitz function such that $f(0) = 0$. Let $R = \int_0^x f(y)dy$ for some $x \in \mathbb{R}$. Then $|f(x)| \leq \sqrt{2\lambda R}$.*

Lemma 28 (Lemma B.1 of Gaitonde et al. (2023)) *Let Φ be an arbitrary allocation-sequence rule. Then there exists a (single-round) allocation rule $y : [0, \bar{v}]^n \rightarrow X$ such that*

$$\begin{aligned} \widetilde{W}(\Phi, F) &:= \sum_{k=1}^n \min \left\{ B_k, \frac{1}{\gamma_k} \mathbb{E}_{\mathbf{v} \sim F} \left[\sum_{t=1}^T \Phi_{k,t}(\mathbf{v}) v_{k,t} \right] \right\} \\ &= \sum_{k=1}^n T \cdot \min \left\{ \rho_k, \frac{1}{\gamma_k} \mathbb{E}_{\mathbf{v} \sim F} [y_k(\mathbf{v}) v_k] \right\} = \overline{W}(\mathbf{y}, F). \end{aligned}$$

Appendix E. Warm-up proofs (Section 4)

PROOF OF [LEMMA 4](#): MONOTONICITY OF $v_t x_t(\mu) - \gamma p_t(\mu)$

Let $d = \max_{j \neq k} b_j$, and let μ' be such that $v_t/(1 + \mu') = d$. Then for all $\mu > \mu'$ we have $x_t(\mu) = p_t(\mu) = 0$ (which is weakly increasing in μ), and for all $\mu < \mu'$ we have that $x_t(\mu) = 1$ and $p_t(\mu)$ is weakly decreasing in μ , so $v_t x_t(\mu) - \gamma p_t(\mu)$ is weakly increasing.

It only remains to establish what happens at the threshold $\mu = \mu'$, and then only when $\mu' \leq \gamma - 1$. Note however that when $\mu = \mu'$, the first and second highest bids are equal, so the payment of agent k is determined to be $x_t(\mu)v_t/(1 + \mu) \geq x_t(\mu)v_t/\gamma$. This implies $v_t x_t(\mu') - \gamma p_t(\mu') \leq 0$, and hence $v_t x_t(\mu) - \gamma p_t(\mu) \leq 0$ for all $\mu < \mu'$ as well. Since $v_t x_t(\mu) - \gamma p_t(\mu) = 0$ for all $\mu > \mu'$, we conclude that the difference is monotone in μ as claimed.

PROOF OF [LEMMA 5](#): NEVER VIOLATING THE ROI CONSTRAINT

We prove this using induction. The base case follows trivially (since the multiplier is initialized to $\gamma - 1$). Now, suppose this is true for all time up to $t - 1$, i.e.,

$$\sum_{\tau \in [t']} v_\tau x_\tau \geq \gamma \sum_{\tau \in [t']} p_\tau, \quad \forall t' \leq t - 1$$

Consider time t . From the update rule, we have

$$\mu_t \geq \mu_0 + \eta \Lambda, \quad \text{where } \Lambda := \sum_{\tau \in [t-1]} v_\tau x_\tau - \gamma p_\tau.$$

We consider two cases. First, suppose $\eta \Lambda < \gamma - 1$. Then $p_t \leq b_t \leq v_t/(\gamma - \eta \Lambda)$. Using the fact that $\eta p_t \leq 1$ since $\eta < 1/\bar{v}$, we have:

$$\gamma p_t + \sum_{\tau \in [t-1]} p_\tau \leq v_t x_t + \sum_{\tau \in [t-1]} v_\tau x_\tau,$$

which gives us the required claim. The second case is $\eta \Lambda > \gamma - 1$. Then $p_t \leq b_t \leq v_t$ which implies $\sum_{\tau \in [t]} v_\tau x_\tau - \gamma p_\tau > 0$. This completes the proof.

Appendix F. Missing proofs from Section 5

F.1. Proof of [Lemma 6](#): never violating the constraints

We first handle the ROI constraint, similarly to the proof of [Lemma 5](#). We prove the following:

$$\sum_{t \in [T]} v_{k,t} x_{k,t} \geq \gamma_k \sum_{t \in [T]} p_{k,t}. \tag{F.1}$$

We use induction on t . We consider a fixed individual bidder and omit the subscript k what follows. The base case still follows trivially, since the multiplier μ_1 is initialized to $\max\{\gamma - 1, \frac{\bar{v}}{\rho} - 1\}$. Now, suppose this is true for all time up to $t - 1$, i.e.,

$$\sum_{\tau \in [t']} v_\tau x_\tau \geq \gamma \sum_{\tau \in [t']} p_\tau, \quad \forall t' \leq t - 1.$$

Now, consider time t . From the update rule, we have $\mu_t^R \geq \mu_0^R + \eta_R \sum_{\tau=1}^{t-1} (\gamma p_\tau - v_\tau x_\tau)$. We split the proof into two parts.

Suppose $\eta_R \sum_{\tau=1}^{t-1} (v_\tau x_\tau - \gamma p_\tau) < \gamma - 1$. This gives us:

$$p_t \leq b_t x_t = \frac{v_t x_t}{1 + \mu_t} \leq \frac{v_t x_t}{1 + \mu_t^R} \leq \frac{v_t x_t}{\gamma - \eta_R \sum_{\tau=1}^{t-1} (v_\tau x_\tau - \gamma p_\tau)}.$$

Now, using the fact that $\eta_R p_t \leq \eta_R v_t \leq \eta \bar{v} \leq 1$, we have:

$$\gamma \sum_{\tau=1}^t p_\tau = \gamma \left(p_t + \sum_{\tau=1}^{t-1} p_\tau \right) \leq \eta_R p_t \sum_{\tau=1}^{t-1} (x_\tau v_\tau - \gamma p_\tau) + v_t x_t + \gamma \sum_{\tau=1}^{t-1} p_\tau \leq \sum_{\tau=1}^t x_\tau v_\tau,$$

where the last inequality uses the induction hypothesis.

Now, if $\eta_R \sum_{\tau=1}^{t-1} (v_\tau x_\tau - \gamma p_\tau) > \gamma - 1$, we have $\sum_{\tau=1}^{t-1} (v_\tau x_\tau - \gamma p_\tau) \geq (\gamma - 1) \bar{v} \geq (\gamma - 1) v_t x_t$ and $p_t \leq b_t \leq v_t$, which means that

$$\gamma \sum_{\tau=1}^t p_\tau \leq \gamma \sum_{\tau=1}^{t-1} p_\tau + \gamma v_t x_t \leq \sum_{\tau=1}^t v_\tau x_\tau.$$

Combining the above two claims finishes the proof for Eq. (F.1).

For budget constraint, we prove that $\sum_{\tau \in [t]} p_\tau \leq \rho t$ using induction on the round t . The base case holds as $p_1 \leq b_1 \leq \frac{v_1}{1 + \mu_1^B} \leq \rho$. Suppose that the claim holds up to time $t - 1$. According to the update rule, we know that $\mu_t^B \geq \mu_0^B + \eta_B \sum_{\tau=1}^{t-1} (x_\tau p_\tau - \rho)$. We also split the proof into two parts. Suppose $\eta_B \sum_{\tau=1}^{t-1} (x_\tau p_\tau - \rho) > 1 - \frac{\bar{v}}{\rho}$. Then we have

$$p_t \leq b_t x_t = \frac{v_t x_t}{1 + \mu_t} \leq \frac{v_t x_t}{1 + \mu_t^B} \leq \frac{v_t x_t}{\eta_B \sum_{\tau=1}^{t-1} (x_\tau p_\tau - \rho) + \frac{\bar{v}}{\rho}}.$$

Therefore, we have:

$$\sum_{\tau=1}^t p_\tau = \sum_{\tau=1}^{t-1} p_\tau + \frac{\rho}{\bar{v}} \left(v_t x_t + \eta_B p_t \sum_{\tau=1}^{t-1} (\rho - p_\tau) \right) \leq \sum_{\tau=1}^{t-1} p_\tau + \rho + \eta_B \rho \sum_{\tau=1}^{t-1} (\rho - p_\tau) = \rho t.$$

If $\eta_B \sum_{\tau=1}^{t-1} (x_\tau p_\tau - \rho) < 1 - \frac{\bar{v}}{\rho}$, then we have $p_t \leq b_t \leq v_t$, which gives us

$$\sum_{\tau \in [t]} x_\tau p_\tau \leq \rho(t - 1) + \bar{v} \leq \rho t.$$

Combining the above two inequalities finishes the proof.

F.2. Proof of Lemma 7: $\mu_t^R \leq \gamma - 1$

We prove this by induction. For conciseness, we omit the subscript of the agent index k . Base case trivially holds. Suppose that up to round t , $\mu_t^R \leq \gamma - 1$ and $\mu_t^B \leq \frac{\bar{v}}{\rho} - 1$. First, consider μ_t^R . At round $t + 1$, if the bidder does not win an auction, then $\mu_{t+1}^R = \mu_t^R$. Otherwise, we have

$$\begin{aligned} \mu_{t+1}^R &= \mu_t^R + \eta_R (\gamma p_t(\mu_t) - v_t) \\ &\leq \mu_t^R + \eta_R (\gamma p_t(\mu_t^R) - v_t) \quad (\mu_t = \max\{\mu_t^B, \mu_t^R, 0\} \text{ and } p_t(\mu) \text{ is non-increasing in } \mu) \\ &\leq \mu_t^R + \eta_R \left(\gamma \frac{v_t}{1 + \mu_t^R} - v_t \right) \quad (\text{payment does not exceed bid}) \\ &\leq \mu_t^R + \eta_R \frac{(\gamma - 1 - \mu_t^R)v_t}{1 + \mu_t^R} \leq \mu_t^R + \gamma - 1 - \mu_t^R = \gamma - 1, \end{aligned}$$

where the last inequality uses the fact that $\eta_R \leq \frac{1}{\bar{v}}$. This proves the result for μ_t^R .

Consider the budget-multiplier μ_t^B . At round $t + 1$, similarly, if the bidder does not win an auction, then $\mu_{t+1}^B = \mu_t^R - \eta_B \rho \leq \frac{\bar{v}}{\rho} - 1$. Otherwise, we have

$$\begin{aligned}\mu_{t+1}^B &= \mu_t^B + \eta_B (p_t(\mu_t) - \rho) \\ &\leq \mu_t^B + \eta_B \left(\frac{v_t}{1 + \mu_t^B} - \rho \right) && (p_t(\mu_t) \leq \frac{v_t}{1 + \mu_t} \leq \frac{v_t}{1 + \mu_t^B}) \\ &\leq 1 + \mu_t^B + \frac{\eta_B \bar{v}}{1 + \mu_t^B} - \rho \eta_B - 1 \leq \frac{\bar{v}}{\rho} + \frac{\eta_B \bar{v}}{\bar{v}/\rho} - \rho \eta_B - 1 \leq \frac{\bar{v}}{\rho} - 1,\end{aligned}$$

where the third inequality is because $\eta_B \bar{v} \leq 1$ and $h(x) = x + \frac{\eta_B \bar{v}}{x}$ is increasing for $x \geq 1$.

F.3. Proof of Lemma 12: monotonicity

For any $t \in [T]$, since $p_t(\mu)$ is non-increasing in μ , $Z_t^B(\mu)$ is non-increasing for $\mu \geq 0$. Next, consider $\rho_t(\mu) - Z_t^R(\mu) = \mathbb{E}[v_t x_t(\mu) - \gamma p_t(\mu)]$. Let d_t denote the competing bid. Also note that the bid $b_t = v_t/(1 + \mu) > v_t/\gamma$. We split the proof into three cases:

1. If $d_t < v_t/\gamma$. We have $x_t(\mu) = 1$ (since $b_t > d_t$) and $p_t(\mu)$ is decreasing in μ .
2. If $v_t/\gamma \leq d_t \leq v_t$. Suppose the competing bid is $d_t = v_t/(1 + \mu')$ for some constant $\mu' \in [1, \gamma - 1]$. Then $x_t(\mu) = 1$ for $\mu \in [0, \mu' - 1]$, and $v_t - \gamma p_t(\mu) < 0$ and increasing. At $\mu = \mu'$, we have $b_t = d_t$, and so $x_t(\mu) \leq 1$. Furthermore, we have $p_t(\mu) = x_t(\mu)v_t/(1 + \mu')$. Therefore $v_t x_t(\mu) - \gamma p_t(\mu) = x_t(\mu)(v_t - \gamma v_t/(1 + \mu')) \leq 0$, and also greater than $v_t x_t(\mu) - \gamma p_t(\mu)$, $\mu < \mu'$ (since $p_t(\mu)$ decreases with μ and $x(\mu') \leq 1$). This shows that there will be an increase in $v_t x_t(\mu) - \gamma p_t(\mu)$ when we move from $\mu < \mu'$ to $\mu = \mu'$. Finally, for $\mu \in (\mu', \gamma - 1]$, $v_t x_t(\mu) - \gamma p_t(\mu)$ will identically equal to zero. Therefore, we have $v_t x_t(\mu) - \gamma p_t(\mu)$ is increasing in this case.
3. If $d_t > v_t$. In this case, since $b_t < d_t$, $x_t(\mu)$ and $p_t(\mu)$ will both be zero.

Combining these three cases, and taking expectations gives the desired result.

F.4. Proof of Corollary 19: Regret in stationary-stochastic Environment

Focus on the stationary-stochastic environment. Define $Z^B(\mu) \triangleq Z_t^B(\mu)$, $Z^R(\mu) \triangleq Z_t^R(\mu)$, $\rho(\mu) \triangleq \rho_t(\mu)$, and $V(\mu) \triangleq V_t(\mu)$ for all $t \in [T]$. Also, we have for all $t \in [T]$, $\mu_t^{R^*} = \mu^{R^*}$ and $\mu_t^{B^*} = \mu^{B^*}$. Here μ^{R^*} is any $\mu \in [0, \gamma - 1]$ such that $\mathbb{E}[Z^R(\mu) - \rho(\mu)] = 0$, or 0 if no such μ exists; μ^{B^*} is any $\mu \in [0, \frac{\bar{v}}{\rho} - 1]$ such that $\mathbb{E}[Z^B(\mu) - \rho] = 0$, or 0 if no such μ exists.

Now consider any $\mu \in \Pi$. As $\mathbb{E}[Z^B(\mu)] \leq \frac{B}{T} = \rho$ and $\mathbb{E}[Z^R(\mu) - \rho(\mu)] \leq 0$, according to the monotonicity of $Z^B(\mu)$ and $Z^R(\mu) - \rho(\mu)$ proven in Lemma 12, there exists $\mu^{B^*} \in [0, \frac{\bar{v}}{\rho} - 1]$ and $\mu^{R^*} \in [0, \gamma - 1]$ such that $\mu \geq \mu^{B^*}$ and $\mu \geq \mu^{R^*}$. Therefore, according to Theorem 16 and the monotonicity of $V(\mu)$, we know that

$$\sum_{t \in [T]} (V(\mu) - V(\mu_t)) \leq \sum_{t \in [T]} (V(\max\{\mu_t^{B^*}, \mu_t^{R^*}\}) - V(\mu_t)) \leq \mathcal{O}(T^{\frac{7}{8}}),$$

where the last inequality is because Theorem 16 and $P_T^R = P_T^B = 0$.

Appendix G. Missing proofs from Section 7

Lemma 29 (Implies Eq. (7.2), Eq. (7.3)) *For any $0 \leq \mu_1 \leq \mu_2 \leq \gamma - 1$ and any $\beta > 0$, we have*

$$V_t(\mu_1) - V_t(\mu_2) \leq \frac{\gamma}{\beta} (Z_t^R(\mu_1) - \rho_t(\mu_1) - Z_t^R(\mu_2) + \rho_t(\mu_2)) + \beta\lambda, \quad (\text{G.1})$$

where $\lambda > 0$ is the Lipschitz constant from [Assumption 13](#). Moreover, for any $0 \leq \mu_1 \leq \mu_2 \leq \frac{\bar{v}}{\rho} - 1$,

$$V_t(\mu_1) - V_t(\mu_2) \leq \frac{\bar{v}}{\rho} (Z_t^B(\mu_1) - Z_t^B(\mu_2)). \quad (\text{G.2})$$

Proof For any $\beta > 0$, as $V_t(\mu)$ is λ -Lipschitz based on [Assumption 13](#), we have

$$\begin{aligned} V_t(\mu_1) - V_t(\mu_2) &= V_t(\mu_1) - V_t(\min\{\mu_2, \gamma - 1 - \beta\}) + V_t(\min\{\mu_2, \gamma - 1 - \beta\}) - V_t(\mu_2) \\ &\leq V_t(\mu_1) - V_t(\min\{\mu_2, \gamma - 1 - \beta\}) + \beta\lambda. \end{aligned} \quad (\text{G.3})$$

Now we show that

$$\begin{aligned} V_t(\mu_1) - V_t(\min\{\mu_2, \gamma - 1 - \beta\}) \\ \leq \frac{\gamma}{\beta} (Z_t^R(\mu_1) - \rho_t^R(\mu_1) - Z_t^R(\min\{\mu_2, \gamma - 1 - \beta\}) + \rho_t(\min\{\mu_2, \gamma - 1 - \beta\})). \end{aligned}$$

For any $\mu \in [\mu_1, \gamma - 1 - \beta]$,

$$\begin{aligned} \nabla[Z_t^R(\mu) - \rho_t(\mu)] &= \nabla\mathbb{E}[((\gamma p_t(\mu) - v_t)x_t(\mu))] \\ &= \mathbb{E}[\gamma\nabla p_t(\mu)x_t(\mu)] + \nabla\mathbb{E}[\gamma p_t(\mu)x_t(\mu)] - \nabla\mathbb{E}[v_t x_t(\mu)] \\ &\leq \nabla\mathbb{E}\left[\gamma \frac{v_t}{1+\mu} x_t(\mu)\right] - \nabla V_t(\mu) \\ &\quad (\nabla p_t(\mu) \leq 0, \nabla x_t(\mu) \leq 0 \text{ and } p_t(\mu) \leq \frac{v_t}{1+\mu}) \\ &= \frac{\gamma - \mu - 1}{1 + \mu} \nabla V_t(\mu). \end{aligned}$$

In addition, note that $Z_t^R(\mu_1) - \rho_t(\mu_1) - Z_t^R(\mu) + \rho_t(\mu) = \int_{\mu}^{\mu_1} \nabla(Z_t^R(\tau) - \rho_t(\tau))d\tau$ and $V_t(\mu_1) - V_t(\mu) = \int_{\mu}^{\mu_1} \nabla V_t(\tau)d\tau$. Therefore, we have

$$\begin{aligned} V_t(\mu_1) - V_t(\min\{\mu_2, \gamma - 1 - \beta\}) \\ &= \int_{\mu_1}^{\min\{\mu_2, \gamma - 1 - \beta\}} -\nabla V_t(\tau)d\tau \\ &\leq \frac{1 + \min\{\mu_2, \gamma - 1 - \beta\}}{\gamma - \min\{\mu_2, \gamma - 1 - \beta\} - 1} \int_{\mu_1}^{\min\{\mu_2, \gamma - 1 - \beta\}} -\nabla(Z_t^R(\tau) - \rho_t(\mu))d\tau \\ &= \frac{1 + \min\{\mu_2, \gamma - 1 - \beta\}}{\gamma - \min\{\mu_2, \gamma - 1 - \beta\} - 1} \\ &\quad ((Z_t^R(\mu_1) - Z_t^R(\min\{\mu_2, \gamma - 1 - \beta\}) - \rho_t(\mu_1) + \rho_t(\min\{\mu_2, \gamma - 1 - \beta\})) \\ &\leq \frac{\gamma}{\beta} (Z_t^R(\mu_1) - \rho_t(\mu_1) - Z_t^R(\mu_2) + \rho_t(\mu_2)). \\ &\quad (Z_t^R(\mu) - \rho_t(\mu) \text{ is non-increasing in } \mu \in [0, \gamma - 1]) \end{aligned}$$

Plugging the above into Eq. (G.3) gives

$$V_t(\mu_1) - V_t(\mu_2) \leq \frac{\gamma}{\beta} (Z_t^R(\mu_1) - \rho_t(\mu_1) - Z_t^R(\mu_2) + \rho_t(\mu_2)) + \beta\lambda, \quad (\text{G.4})$$

which finishes the proof of Eq. (G.1). For Eq. (G.2), for any $\mu \in [0, \frac{\bar{v}}{\rho} - 1]$ we have

$$\begin{aligned} \nabla Z_t^B(\mu) &= \nabla \mathbb{E}[p_t(\mu)x_t(\mu)] \\ &= \mathbb{E}[\nabla p_t(\mu)x_t(\mu)] + \mathbb{E}[p_t(\mu)\nabla x_t(\mu)] \\ &\leq \nabla \mathbb{E}\left[\frac{v_t}{1+\mu}x_t(\mu)\right] \quad (\nabla p_t(\mu) \leq 0, \nabla x_t(\mu) \leq 0 \text{ and } p_t(\mu) \leq \frac{v_t}{1+\mu}) \\ &= \frac{1}{1+\mu} \nabla V_t(\mu) \leq \frac{\rho}{\bar{v}} \nabla V_t(\mu). \\ V_t(\mu_1) - V_t(\mu_2) &= \int_{\mu_1}^{\mu_2} -\nabla V_t(\tau)d\tau \leq \frac{\bar{v}}{\rho} \int_{\mu_1}^{\mu_2} -\nabla Z_t^B(\tau)d\tau = \frac{\bar{v}}{\rho} (Z_t^B(\mu_1) - Z_t^B(\mu_2)) \\ &\leq \frac{\bar{v}}{\rho} (Z_t^B(\mu_1) - Z_t^B(\mu_2)). \quad \blacksquare \end{aligned}$$

Proof of Eq. (7.8). We decompose the T rounds into S time intervals $I_1 = [1, e_1], \dots, I_S = [w_S, e_S]$, where each time interval is a maximal sequence of consecutive rounds such that $\mu_t^R \geq \mu_t^{R^*}$ and $\mu_t^B \leq \gamma - 1$. Then we have

$$\sum_{t=1}^T \frac{|\mu_t^R - \mu_t^{R^*}|^2 - |\mu_{t+1}^R - \mu_{t+1}^{R^*}|^2}{2\eta_R} \cdot \mathbf{1}\{E_t^R\} \leq \sum_{s=1}^S \frac{|\mu_{w_s}^R - \mu_{w_s}^{R^*}|^2 - |\mu_{e_s+1}^R - \mu_{e_s+1}^{R^*}|^2}{2\eta_R}. \quad (\text{G.5})$$

For $s \geq 2$, consider the most recent round σ_s before round w_s such that $\mu_{\sigma_s}^B \leq \gamma - 1$. As $\mu_t^R \leq \gamma - 1$, we know that $\mu_t^B \geq \mu_t^R$ when $t \in [\sigma_s + 1, w_s - 1]$. In addition, according to the update rule of μ_t^R , we know that when $t \in [\sigma_s + 1, w_s - 1]$ we have

$$\mu_{t+1}^R = \mu_t^R + \eta_R (\gamma p_t(\mu_t) - v_t) x_t \leq \mu_t^R + \eta_R \left(\frac{\gamma v_t}{1 + (\gamma - 1)} \right) x_t \leq \mu_t^R.$$

Next, consider the round σ_s . If σ_s belongs to some interval I_i , according to the definition of σ_s , σ_s must be the end of I_{s-1} (i.e. $\sigma_s = e_{s-1}$). In this case, we have

$$\begin{aligned} |\mu_{w_s}^R - \mu_{w_s}^{R^*}|^2 &\leq |\mu_{\sigma_s+1}^R - \mu_{w_s}^{R^*}|^2 \quad (\mu_{w_s}^{R^*} \leq \mu_{w_s}^R \leq \mu_{\sigma_s+1}^R) \\ &= |\mu_{\sigma_s+1}^R - \mu_{\sigma_s+1}^{R^*}|^2 + 2(\mu_{\sigma_s+1}^R - \mu_{\sigma_s+1}^{R^*})(\mu_{\sigma_s+1}^{R^*} - \mu_{w_s}^{R^*}) + |\mu_{\sigma_s+1}^{R^*} - \mu_{w_s}^{R^*}|^2 \\ &\leq |\mu_{e_{s-1}+1}^R - \mu_{e_{s-1}+1}^{R^*}|^2 + 3(\gamma - 1) \left(\sum_{t \in [e_{s-1}+1, w_s-1]} |\mu_t^{R^*} - \mu_{t+1}^{R^*}| \right). \end{aligned}$$

Otherwise, σ_s is outside the interval and $\mu_{\sigma_s}^R < \mu_{\sigma_s}^{R^*}$. From the update of μ_t^R , we know that $\mu_{\sigma_s+1}^R \leq \mu_{\sigma_s}^R + \eta_R(\gamma+1)\bar{v} < \mu_{\sigma_s}^{R^*} + \eta_R(\gamma+1)\bar{v}$. Therefore we know that

$$\begin{aligned} |\mu_{w_s}^R - \mu_{w_s}^{R^*}|^2 &\leq |\mu_{\sigma_s+1}^R - \mu_{w_s}^{R^*}|^2 \\ &\leq |\mu_{\sigma_s}^{R^*} + \eta_R(\gamma+1)\bar{v} - \mu_{w_s}^{R^*}|^2 \\ &\leq |\mu_{\sigma_s}^{R^*} - \mu_{w_s}^{R^*}|^2 + 2\eta_R(\gamma+1)\bar{v} \cdot \sum_{\tau \in [\sigma_s, w_s-1]} |\mu_{\tau}^{R^*} - \mu_{\tau+1}^{R^*}| + \eta_R^2(\gamma+1)^2\bar{v}^2 \\ &\leq (\gamma-1)|\mu_{\sigma_s}^{R^*} - \mu_{w_s}^{R^*}| + 2\eta_R(\gamma+1)\bar{v} \cdot \sum_{\tau \in [\sigma_s, w_s-1]} |\mu_{\tau}^{R^*} - \mu_{\tau+1}^{R^*}| + \eta_R^2(\gamma+1)^2\bar{v}^2 \\ &\leq \mathcal{O}\left((\gamma+1) \sum_{\tau \in [\sigma_s, w_s-1]} |\mu_{\tau}^{R^*} - \mu_{\tau+1}^{R^*}| + \eta_R^2(\gamma+1)^2\bar{v}^2\right). \end{aligned}$$

Combining the above two cases and noticing that in the second case, σ_s does not belong to an interval I_i , we have for any $\eta_R \leq \min\{1, \frac{1}{\bar{v}}\}$,

$$\begin{aligned} &\sum_{s=1}^S \left(\frac{|\mu_{w_s}^R - \mu_{w_s}^{R^*}|^2 - |\mu_{e_s+1}^R - \mu_{e_s+1}^{R^*}|^2}{\eta_R} \right) \\ &\leq \frac{|\mu_1^R - \mu_1^{R^*}|^2}{\eta_R} + \sum_{s=1}^{S-1} \left(\frac{|\mu_{w_{s+1}}^R - \mu_{w_{s+1}}^{R^*}|^2 - |\mu_{e_{s+1}}^R - \mu_{e_{s+1}}^{R^*}|^2}{\eta_R} \right) \\ &\leq \frac{|\mu_1^R - \mu_1^{R^*}|^2}{\eta_R} + \sum_{s=1}^{S-1} \left(\frac{(\gamma+1) \sum_{\tau=e_s+1}^{w_{s+1}} |\mu_{\tau}^{R^*} - \mu_{\tau+1}^{R^*}|}{\eta_R} + \eta_R(\gamma+1)^2\bar{v}^2 \right) \\ &\leq \mathcal{O}\left(\frac{|\mu_1^R - \mu_1^{R^*}|^2 + (\gamma+1)P_T^R}{\eta_R} + \eta_R(\gamma+1)^2\bar{v}^2T\right). \quad \blacksquare \end{aligned}$$

Proof of Eq. (7.12). Similar to the proof of Eq. (7.8), we decompose the total horizon $[T]$, into S_b intervals $I_1 = [1, e'_1], \dots, I'_{S_b} = [w'_{S_b}, e'_{S_b}]$, where each interval I contains a maximal sequence of consecutive rounds such that $\mu_t^B \geq \mu_t^{B^*}$. Then we have

$$\sum_{t=1}^T \frac{|\mu_t^B - \mu_t^{B^*}|^2 - |\mu_{t+1}^B - \mu_{t+1}^{B^*}|^2}{2\eta_B} \cdot \mathbf{1}\{E_t^B\} \leq \sum_{s=1}^{S_b} \frac{|\mu_{w'_s}^B - \mu_{w'_s}^{B^*}|^2 - |\mu_{e'_s+1}^B - \mu_{e'_s+1}^{B^*}|^2}{2\eta_B}. \quad (\text{G.6})$$

For $s \geq 2$, Note that we have $\mu_{w'_s-1}^B \leq \mu_{w'_s-1}^{B^*}$ and according to the update of μ_t^B , we also have $\mu_{w'_s}^B \leq \mu_{w'_s-1}^B + \eta_B(\bar{v} + \rho)$. Combining the fact that $\mu_{w'_s}^B \geq \mu_{w'_s}^{B^*}$, we have

$$\begin{aligned} |\mu_{w'_s}^B - \mu_{w'_s}^{B^*}|^2 &\leq |\mu_{w'_s-1}^B + \eta_B(\bar{v} + \rho) - \mu_{w'_s}^{B^*}|^2 \leq |\mu_{w'_s-1}^{B^*} + \eta_B(\bar{v} + \rho) - \mu_{w'_s}^{B^*}|^2 \\ &\leq \left(\frac{\bar{v}}{\rho} + 2\eta_B(\bar{v} + \rho)\right) |\mu_{w'_s-1}^{B^*} - \mu_{w'_s}^{B^*}| + \eta_B^2(\bar{v} + \rho)^2. \end{aligned}$$

Combining the above inequality with Eq. (G.6) finishes the proof. \blacksquare

Appendix H. Numerical Experiments: plots

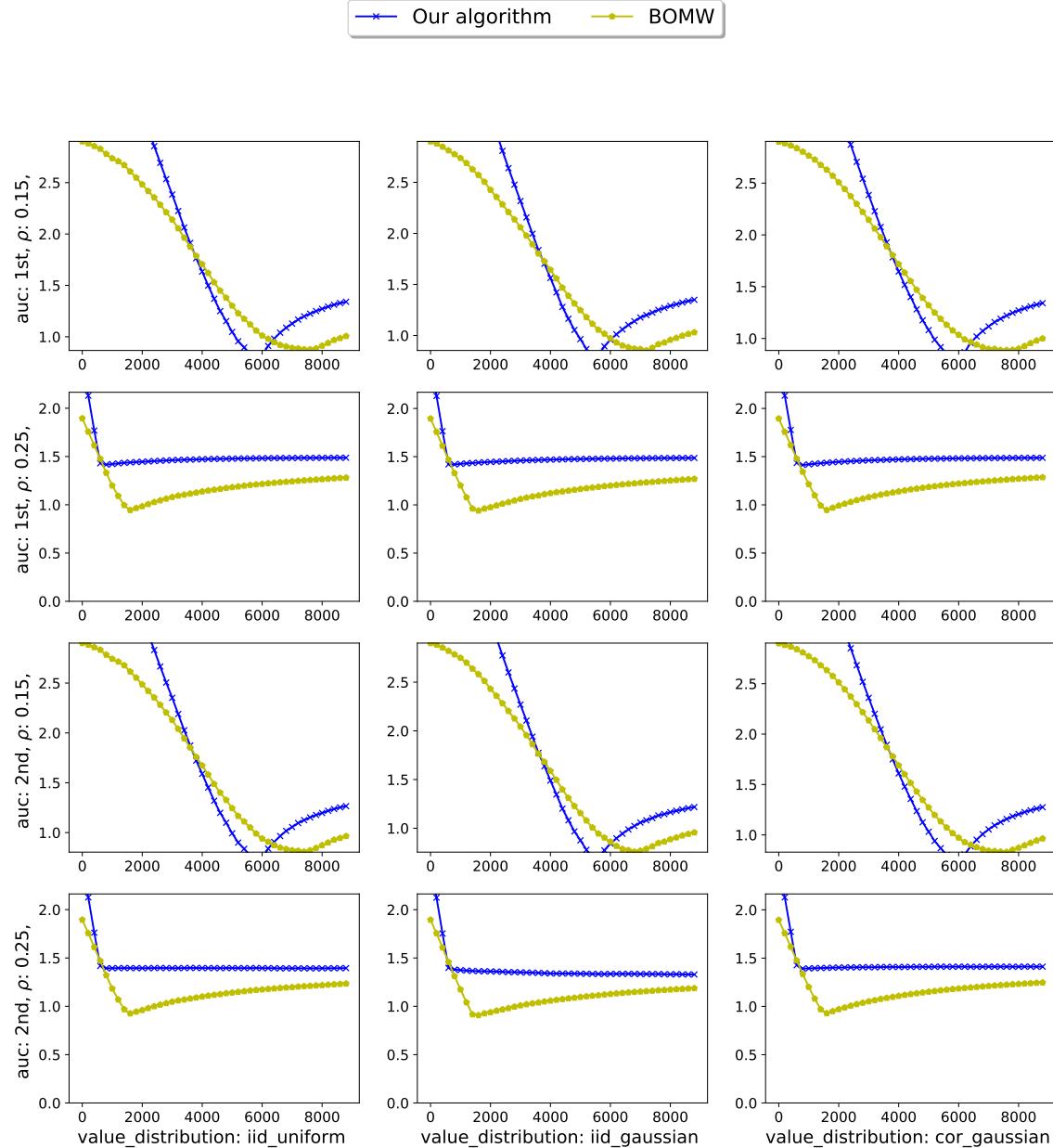


Figure 1: Multiplier dynamics for $\mu_{1,t}$, as a function of time t .

The plots show that the trajectory of $\mu_{1,t}$ is very stable across runs (else, there would have been a noticeable shaded area), and **that non-convergence is a typical outcome**.

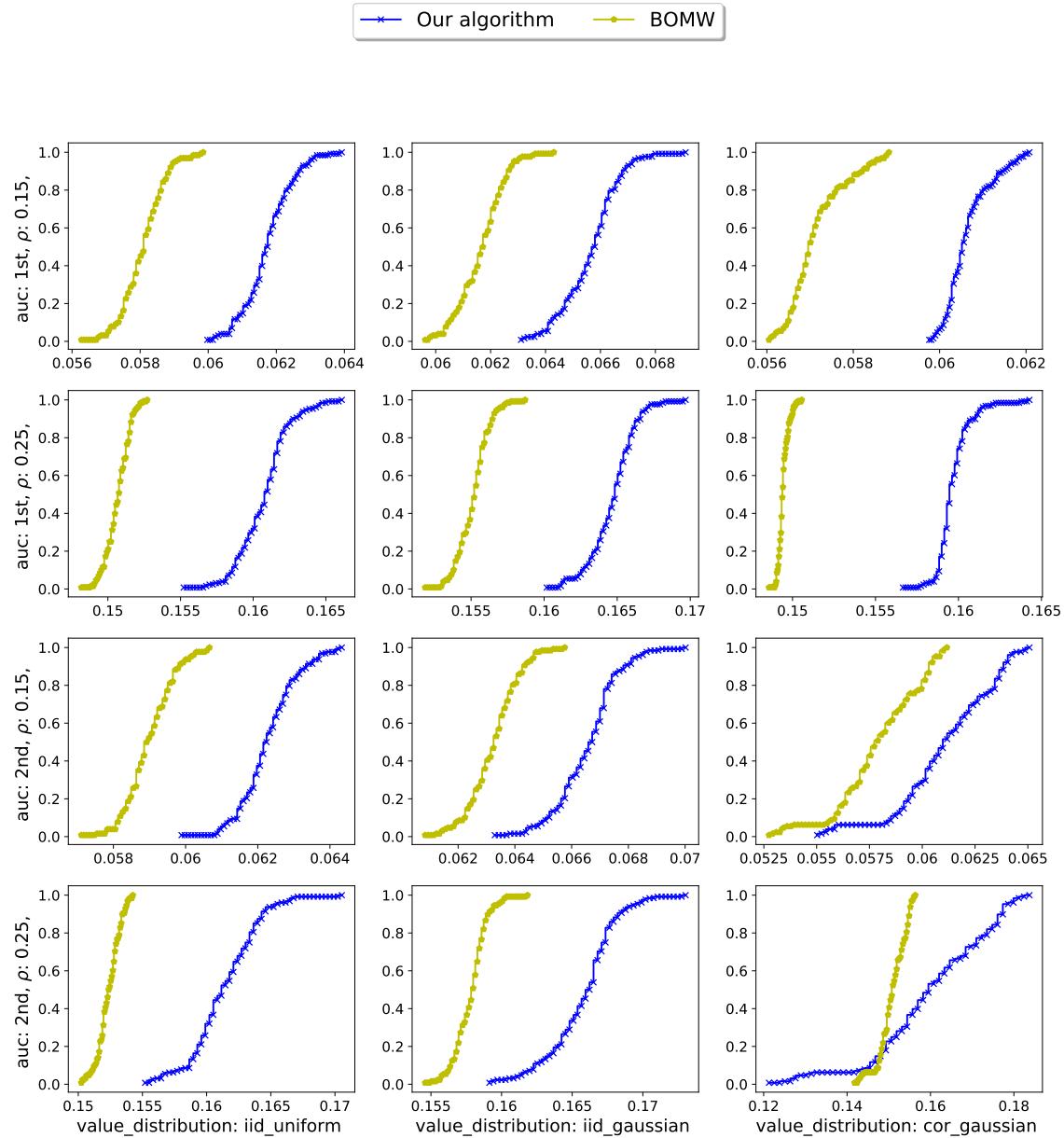
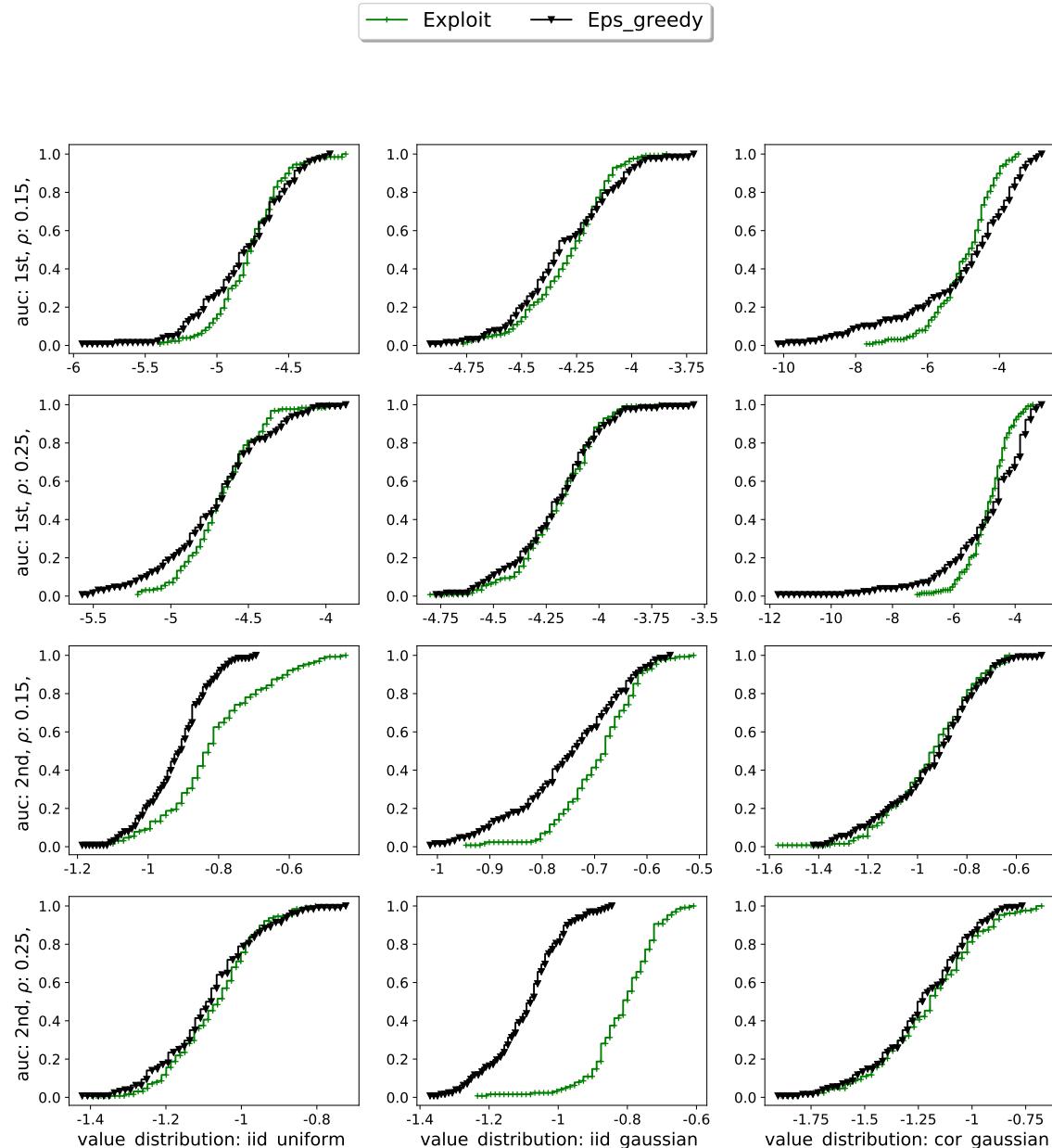


Figure 2: Budget Slackness (8.3): empirical CDF over all agents and all runs.
Both our algorithm and BOMW (Feng et al., 2022) do not violate the budget constraint.

**Figure 3:** Budget Slackness (8.3): empirical CDF over all agents and all runs.

The naive baselines exhibit large constraint violations.

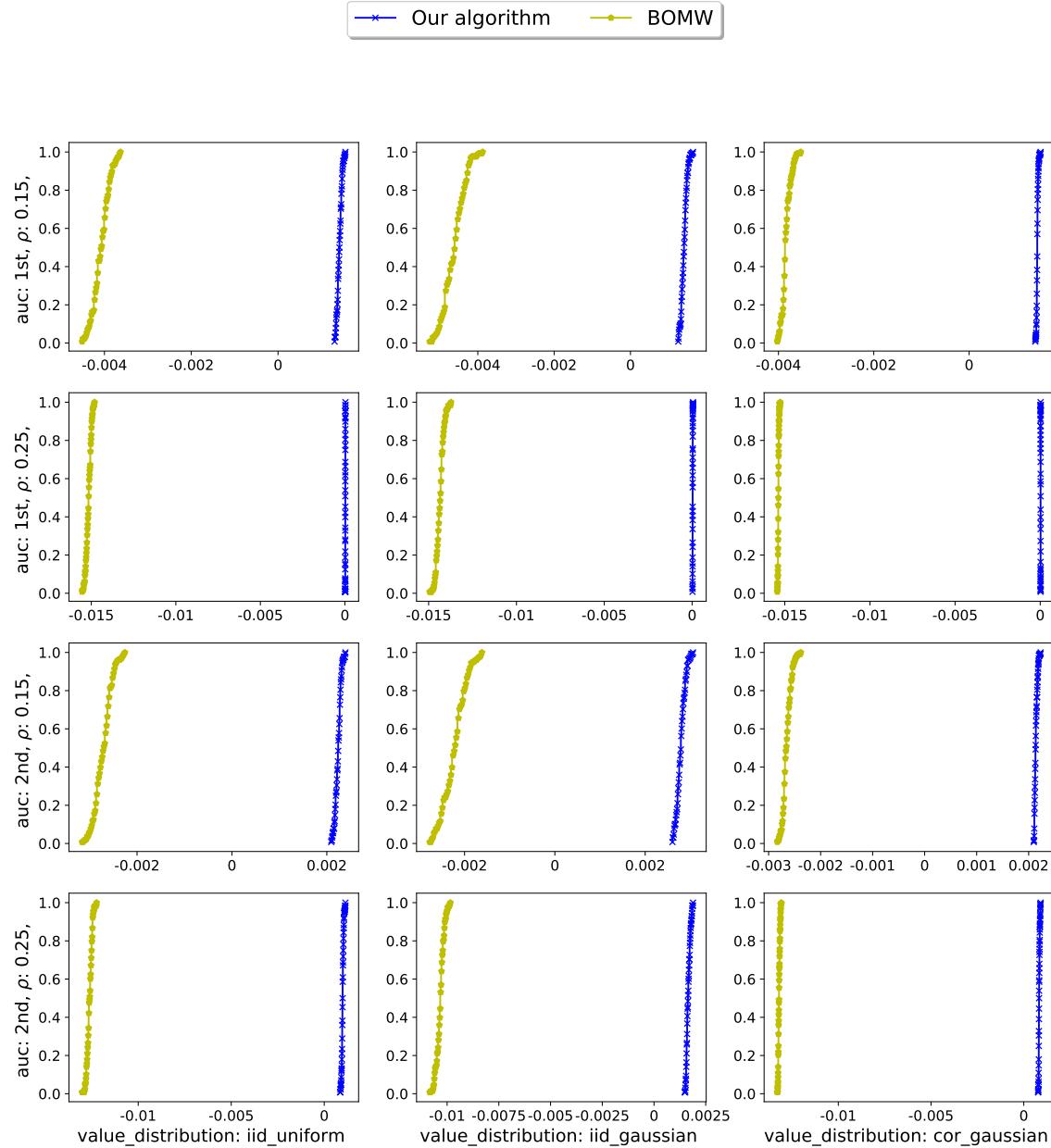


Figure 4: ROI Slackness (8.3): empirical CDF over all agents and all runs.
Our algorithm does not violate the ROI constraint, while BOMW's violations are very small.

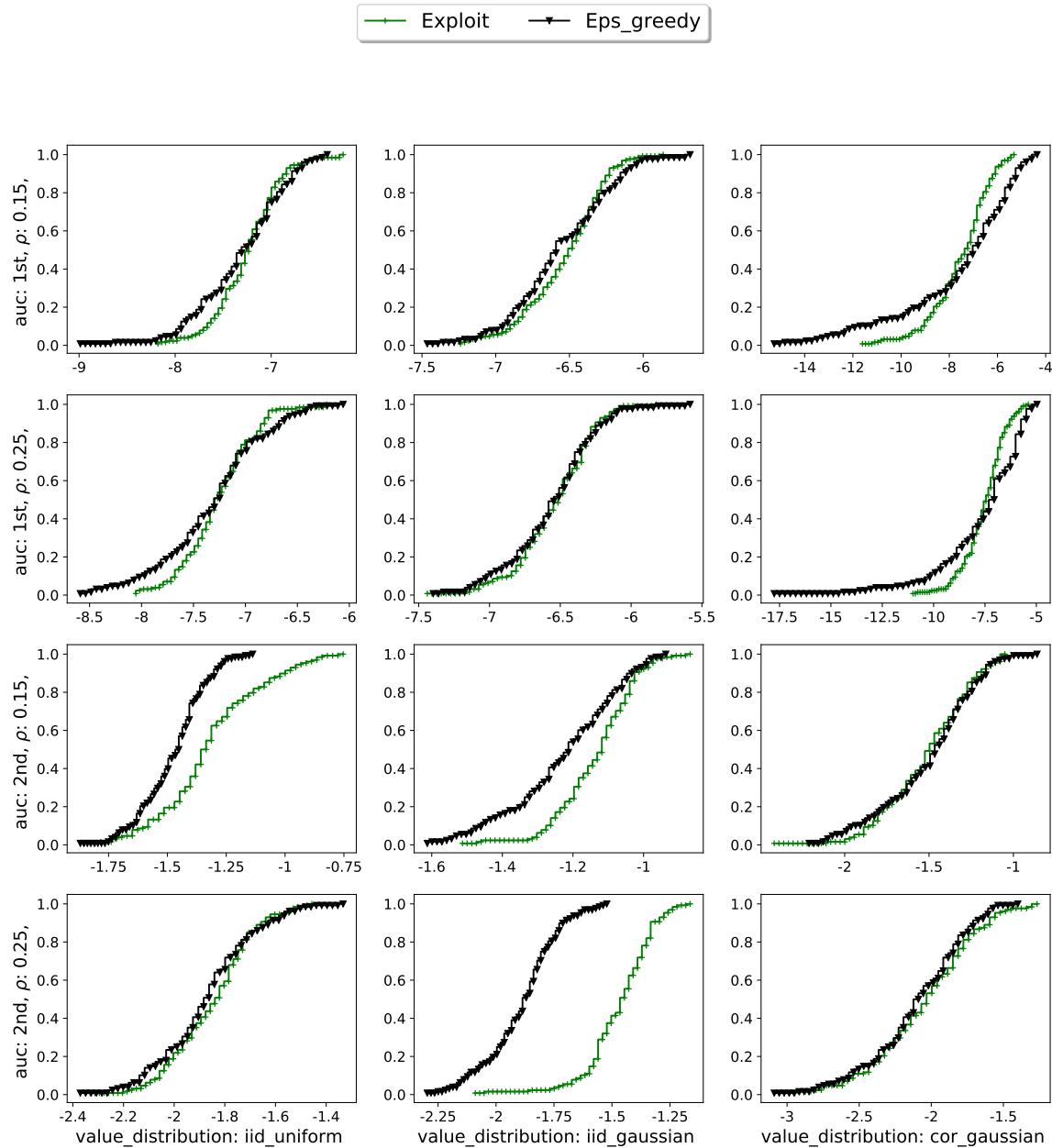


Figure 5: ROI Slackness (8.3): empirical CDF over all agents and all runs.
The naive baselines exhibit large constraint violations.

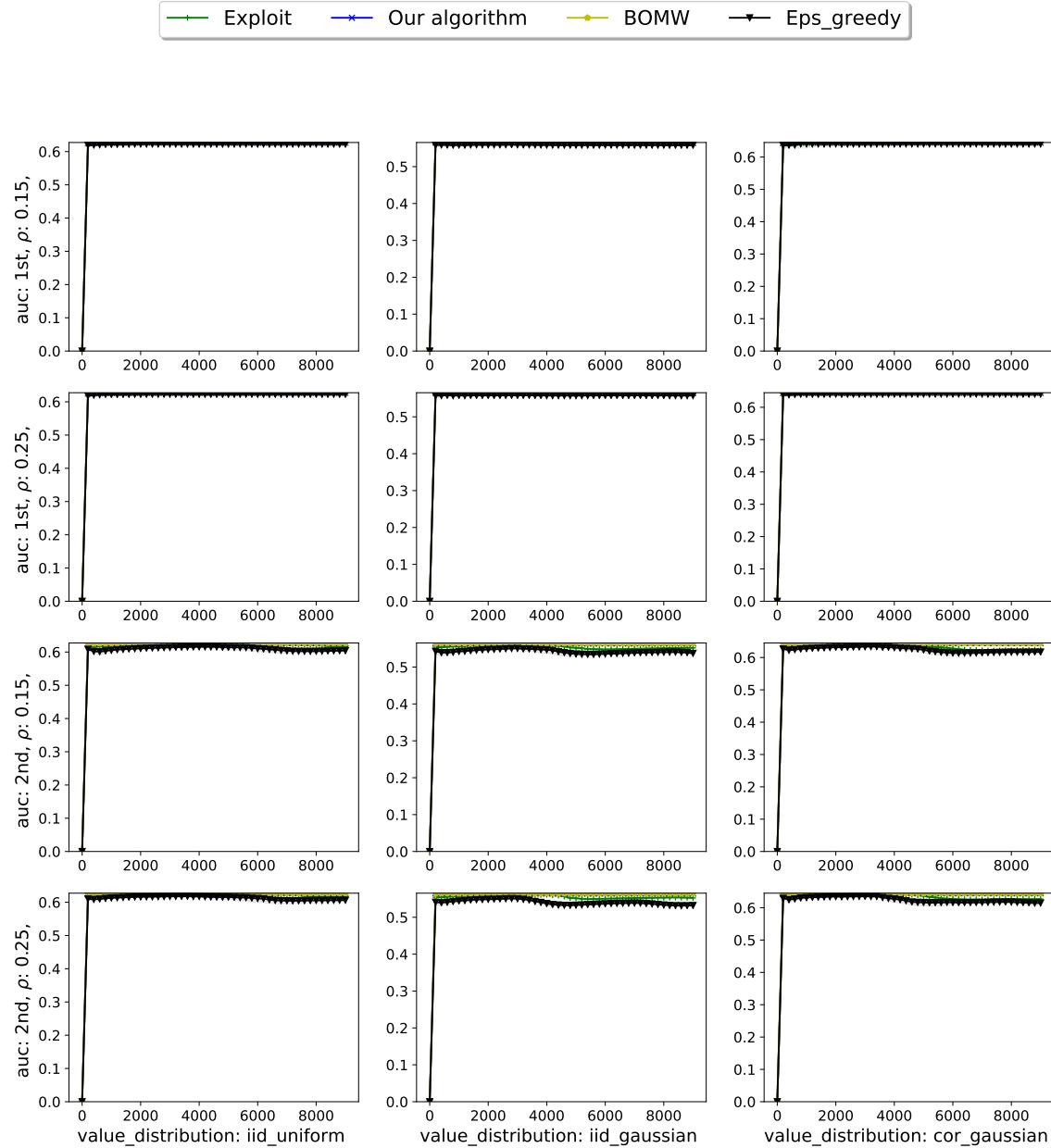


Figure 6: Time-averaged liquid welfare \bar{LW}_t as a function of time t .
All implemented algorithms enjoy similar liquid welfare value.

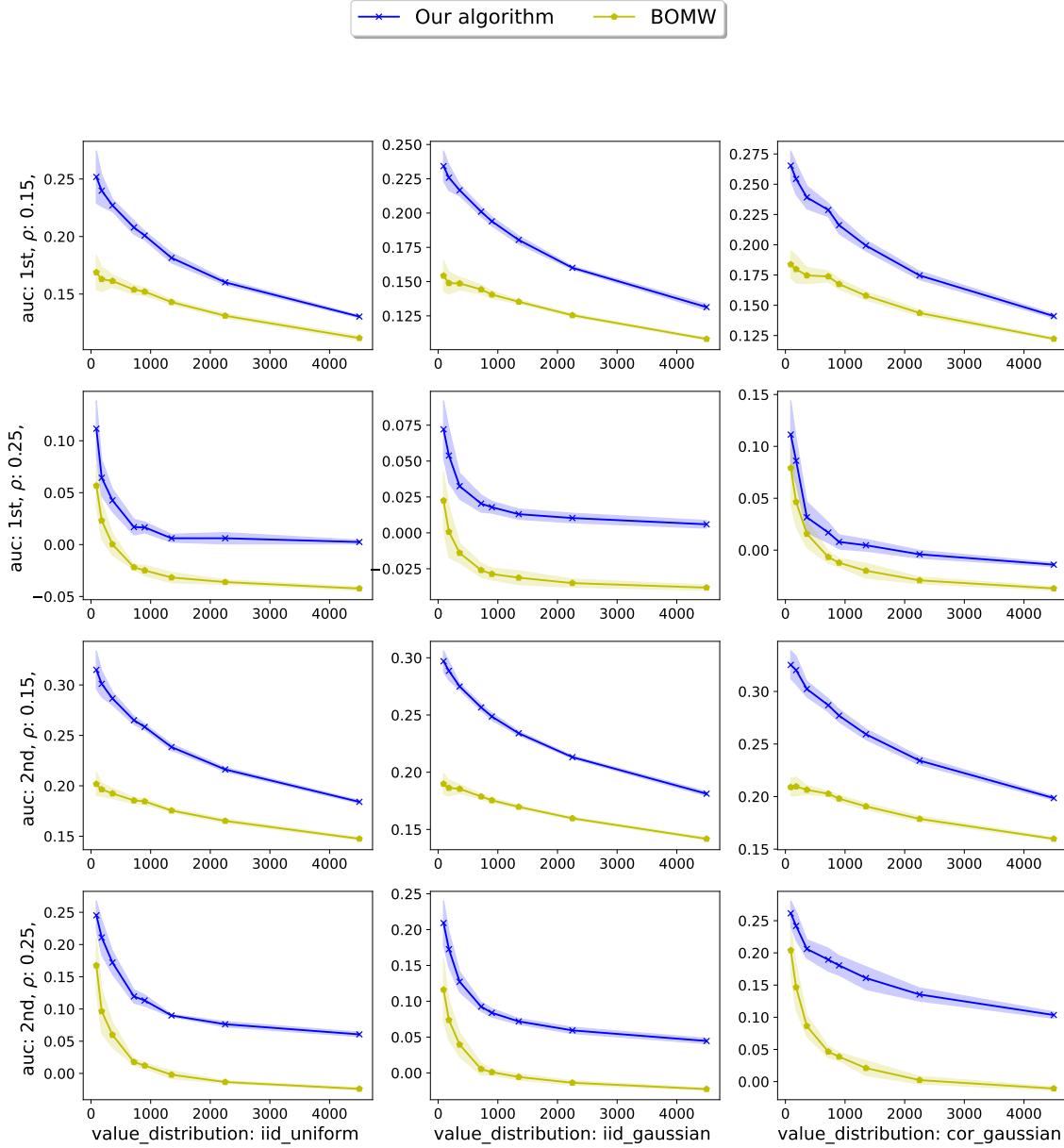
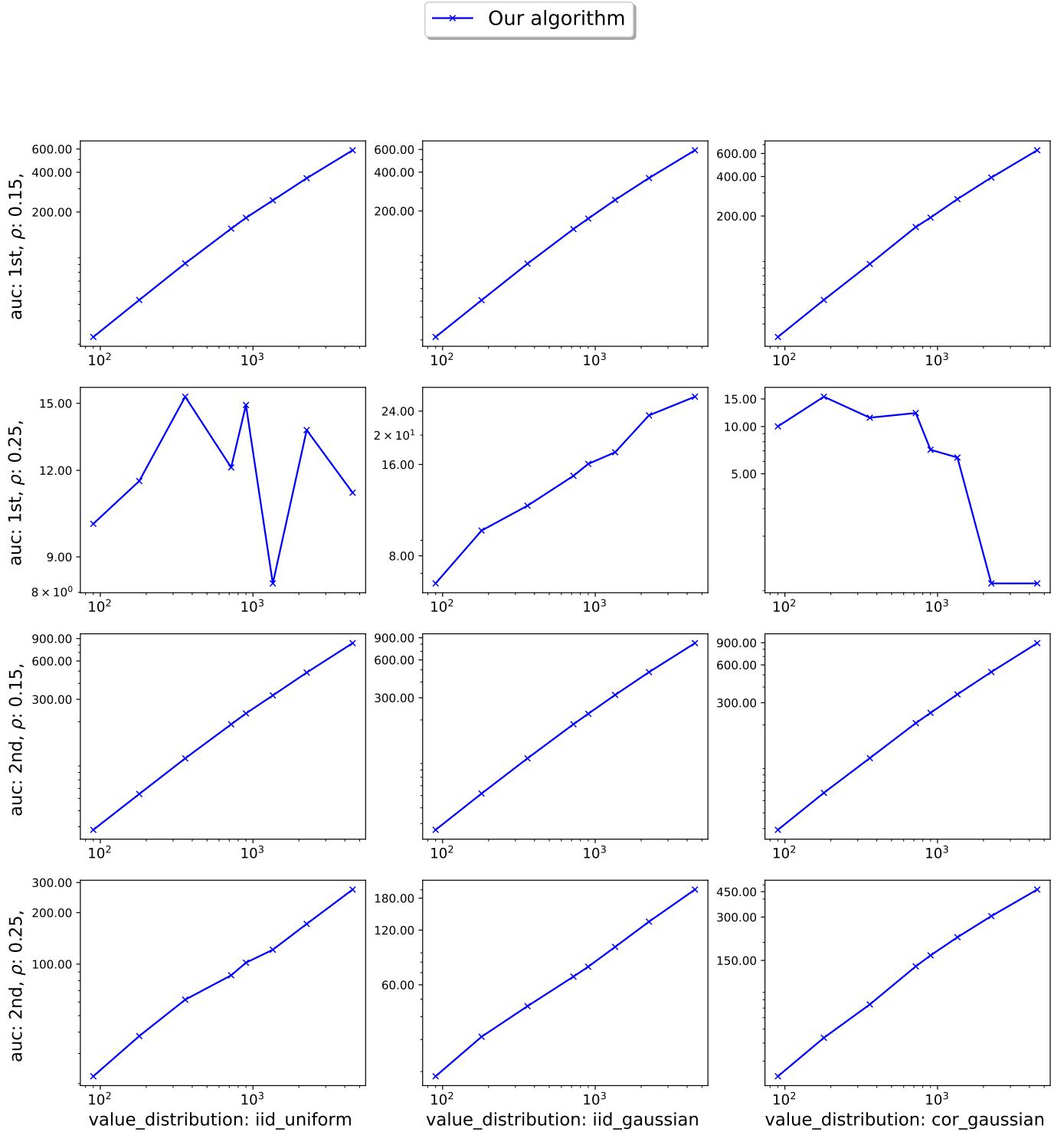


Figure 7: The time-averaged static regret for agent 1, $\bar{R}(t) = \frac{1}{t} \text{StaticReg}_1(t)$, vs time t . $\bar{R}(t)$ is averaged over runs, with shaded area indicating the (small) discrepancy between the runs. $\bar{R}(t)$ substantially decreases over time and appears to tend to 0, for both algorithms. BOMW performs somewhat better, as far as $\bar{R}(t)$ is concerned.

**Figure 8:** Static regret for agent 1, $R(t) = \text{StaticReg}_1(t)$, as a function of t .

The plot is presented in log-log axes, namely $\log(\max(R(t), 0))$ vs $\log(t)$.

10 out of 12 plots are near-straight lines, indicating a good fit to $R(T) = T^\alpha$ for some $\alpha \in (0, 1)$.

AUTOBIDDERS WITH BUDGET AND ROI CONSTRAINTS