

Budget Pacing in Repeated Auctions: Regret and Efficiency without Convergence*

Jason Gaitonde[†] Yingkai Li[‡] Bar Light[§] Brendan Lucier[¶] Aleksandrs Slivkins^{||}

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Abstract

We study the aggregate welfare and individual regret guarantees of dynamic *pacing algorithms* in the context of repeated auctions with budgets. Such algorithms are commonly used as bidding agents in Internet advertising platforms, adaptively learning to shade bids by a tunable linear multiplier in order to match a specified budget. We show that when agents simultaneously apply a natural form of gradient-based pacing, the liquid welfare obtained over the course of the learning dynamics is at least half the optimal expected liquid welfare obtainable by any allocation rule. Crucially, this result holds *without requiring convergence of the dynamics*, allowing us to circumvent known complexity-theoretic obstacles of finding equilibria. This result is also robust to the correlation structure between agent valuations and holds for any *core auction*, a broad class of auctions that includes first-price, second-price, and generalized second-price auctions as special cases. For individual guarantees, we further show such pacing algorithms enjoy *dynamic regret* bounds for individual utility- and value-maximization, with respect to the sequence of budget-pacing bids, for any auction satisfying a monotone bang-for-buck property. To complement our theoretical findings, we provide semi-synthetic numerical simulations based on auction data from the Bing Advertising platform.

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Compared to the initial working paper and the conference version, the current version features revised presentation and expanded discussions, as well as **several major updates**: our aggregate and individual guarantees are extended to several other gradient-based pacing algorithms (Section 5, since Dec’25), the regret analysis is extended to the utility-maximization objective (since Dec’25), and numerical experiments are added (Section 6, since Aug’24, with additional baselines added in Dec’25).

[†]Duke University, Durham NC, USA. Email: jason.gaitonde@duke.edu. Many results were obtained while J. Gaitonde was a research intern at Microsoft Research New England and a graduate student at Cornell University, supported in part by NSF Award CCF-1408673 and AFOSR Award FA9550-19-1-0183.

[‡]Department of Economics, National University of Singapore, Singapore. Email: yk.li@nus.edu.sg. Research carried out while Y. Li was a graduate student at Northwestern University and a research intern at Microsoft Research NYC.

[§]Business School and Institute of Operations Research and Analytics, National University of Singapore, Singapore. Email: barlight@nus.edu.sg. Most of the research was conducted while Bar Light was a postdoc at Microsoft Research NYC.

[¶]Microsoft Research, Cambridge MA, USA. Email: brlucier@microsoft.com.

^{||}Microsoft Research, New York NY, USA. Email: slivkins@microsoft.com.

1 Introduction

Online advertising increasingly dominates the marketing landscape, accounting for 54.2% of total media ad spending in the US in 2019 (\$129 billion) [33]. Such ads are predominantly allocated by auction: advertisers submit bids to an Internet platform to determine whether they will be displayed as part of a given page view and at what price. A typical advertiser participates in many thousands of auctions each day, across a variety of possible ad sizes and formats, payment options (pay per impression, per click, per conversion, etc.), and bids tailored to a variety of signals about user demographics and intent. To further complicate the decision-making process from the advertiser’s perspective, these many auction instances are strategically linked through a *budget constraint*, the total amount of money that can be allocated to advertising. An advertiser therefore faces the daunting task of choosing how to appropriately allocate a global budget across a complex landscape of advertising opportunities, and then convert that intent into a bidding strategy.

To help address this difficulty, all major online platforms provide automated budget management services that adjust campaign parameters on an advertiser’s behalf. This is commonly achieved via *budget-pacing*: an advertiser specifies a global budget target and a maximum willingness to pay (or “value”) for different advertising opportunities, and these values are then scaled down (or “paced”) by a multiplier into bids such that the realized daily spend matches the target budget. An algorithmic bidding agent learns, online, how best to pace the advertiser’s bids as it observes auction outcomes. This campaign management service lowers the barrier to entry into the online advertising ecosystem and removes the need for the advertiser to constantly adjust their campaign in the face of changing market conditions. Moreover, the platform is often better positioned to manage the budget since they have direct access to detailed market statistics.

Bidding agents are now near-universally adopted across all mature advertising platforms, but this success raises some pressing questions about the whole-market view. What can we say about the aggregate market outcomes when nearly all advertiser spend is controlled by automated bidding agents that are simultaneously learning to pace their bids? And to what extent does this depend on the details of the underlying auction?

Central to our question is the interplay between individual learning and aggregate market efficiency, each of which has been studied on its own. For example, when each advertising opportunity is sold by a second-price auction, it is known that linear bidding strategies (i.e., mappings from maximum willingness to pay to a bid) are in fact optimal over all possible bidding strategies for both utility-maximizing and value-maximizing agents, and that gradient-based methods can be used by an agent to achieve vanishing regret relative to the best bidding strategy in hindsight [19, 13]. On the other hand, when multiple bidding agents participating in second-price auctions choose pacing factors that form a pure Nash equilibrium, the resulting outcome is known to be approximately efficient (in the sense of maximizing expected liquid welfare; more on this below) [1, 8]. At first glance this combination of results seems to address the question of aggregate performance of bidding agents in second-price auctions. But convergence of online learning algorithms to a Nash equilibrium, let alone a pure Nash equilibrium, is notoriously difficult to guarantee and should not be taken for granted. Moreover, finding a pure Nash equilibrium of the pacing game is PPAD-hard for second-price auctions [22], and hence we should not assume that bidding agents employing polynomial-time online learning strategies will efficiently converge to a pure Nash equilibrium in the full generality of second-price auctions. So the question remains: *if bidding agents do not converge, what happens to overall market performance?*

1.1 Our Contributions

We provide (classes of) bidding algorithms that simultaneously admit good *aggregate guarantees* in terms of overall market efficiency, *without relying on convergence*, while still providing good *individual guarantees* as online learning algorithms that benefit a particular advertiser. Closely related are three literatures: (i) on aggregate outcomes in single-shot budget-constrained ad auctions, without regard to bidding dynamics, (ii)

on online learning with budget constraints, without regard to the aggregate performance, and (iii) conditions under which learning agents converge to equilibrium in repeated games. With this perspective in mind, we match a state-of-art aggregate guarantee from (i), while being qualitatively on par with state-of-art individual guarantees in (ii), without relying on convergence and thereby side-stepping conditions from (iii). We accomplish this with bidding algorithms that are arguably quite natural and for a broad class of auctions.

In our model there are T rounds, each corresponding to an auction instance. The private values (i.e., maximum willingness to pay) observed by the agents are randomly drawn in each round and can be arbitrarily correlated with each other, capturing scenarios where the willingness to pay of different advertisers is correlated through characteristics of the impression.¹ In each round the bidding agents place bids on behalf of their respective advertisers. The agents operate independently of each other, interacting only through the feedback they receive from the auction.

Algorithm(s). We focus on a gradient-based pacing algorithm (Algorithm 1) that was first introduced by Balseiro and Gur [13] in the context of utility maximization in second-price auctions. They derive this algorithm via the Lagrangian dual of the (quasilinear) utility maximization problem and establish optimality of linear pacing for utility maximization. In practice, gradient-based pacing also sees widespread use in richer allocation problems (such as for multiple ad slots) and auction rules (such as first-price auctions) [2]. Even though the utility optimality guarantees do not extend to all such settings, the algorithm can nevertheless be interpreted as maximizing value subject to linearity, budget, and maximum bid constraints. Motivated by this broader usage, we extend the definition and analysis of Algorithm 1 to a richer class of allocation problems and auction formats, including first-price, second-price, and GSP auctions, to which all of our results apply.² Underlying this extension is a modified interpretation of this algorithm as stochastic gradient descent on a certain artificial objective that applies even beyond second-price auctions.

While our exposition focuses on Algorithm 1, all our guarantees extend to several other gradient-based algorithms. Essentially, Algorithm 1 invokes on an update step akin to stochastic gradient descent, and this step can be replaced with several other well-known algorithms from online convex optimization: optimistic gradient descent [52, 53], optimistic mirror descent [25, 52, 53], and optimistic FTRL [52].

First Result: Aggregate Market Performance. We prove that when the bidding agents employ Algorithm 1 to tune their pacing multipliers, the resulting market outcome over the full time horizon achieves at least half of the optimal expected *liquid welfare*. Crucially, this guarantee does not depend on the convergence of the algorithms’ actions to an equilibrium of the bidding game. Nevertheless, it matches the best possible guarantee even for a pure Nash equilibrium in a static truthful auction [1, 8].

Liquid welfare is the maximum amount that the agents are jointly willing to pay for a given allocation. Put differently, it is the maximum revenue that can be extracted for this allocation by an all-knowing auctioneer. Liquid welfare coincides with *compensating variation* when specialized to our setting.³ Conveniently, it is a welfare measure that applies even when agents seek to maximize *value* (e.g., number of clicks or impressions received) subject to constraints, rather than a monetary utility objective.⁴ It has become a standard objective in the analysis of budget-constrained auctions [32, 7, 1, 8].

¹An *impression* is the industry term for an “atomic” advertising opportunity: a specific slot on a specific webpage when this webpage is rendered for a specific user. Impression characteristics depend on the slot, the page, and the user.

²For non-truthful auctions the restriction to linear bidding strategies is not without loss. Our individual regret guarantees are therefore with respect to optimal *linear* pacing strategies, see our description of the individual regret guarantees later in this section.

³*Compensating variation*, a standard notion in economics, is the amount that agents would need to pay (or be paid) to return to their original utility levels after some change, such as a change in prices [51]. When interpreting liquid welfare as compensating variation, the change being considered is the allocation itself. See Appendix B.1 for further discussion.

⁴When the agents’ objective *can* be expressed in dollars, such as the objective of maximizing advertiser utility subject to the budget constraints, utilitarian welfare would be reasonable aggregate objective. However, strong impossibility results are known even in a single-shot (non-repeated) setting for a single good [32]. Thus, liquid welfare is a meaningful notion of social surplus in budgeted environments, and it specializes to utilitarian welfare when budgets are infinite.

While our discussion so far has focused mainly on second-price single-item auctions, our approximation result actually holds for a far richer set of allocation problems and auction formats, including those used in real-world advertising platforms. We allow arbitrary downward-closed constraints on the set of feasible allocations of a single divisible good,⁵ which captures single-item auctions as well as complex settings such as sponsored search auctions with multiple slots and separable click rates. Further, our result applies even when the underlying mechanism is not truthful. We accommodate any *core auction*: an auction that generates outcomes in the core, meaning that no coalition of agents could improve their joint utility by renegotiating the outcome with the auctioneer [6]. This is a well-studied class of auctions that includes both first and second-price auctions, as well as the generalized second price (GSP) auction, and has previously been studied in the context of advertising auctions [36, 40]. We emphasize, however, that the problem remains non-trivial (and almost as challenging) in a much simpler model with a repeated single-item second-price auction and constant private values.

Second Result: Individual Regret Guarantees. We have analyzed aggregate market performance for a broad class of (possibly non-truthful) auctions including first-price and GSP auctions, but is gradient-based pacing an effective learning method in those settings? Regret guarantees are known when the underlying auction is truthful and the environment is stochastic [13, 17], but what about non-truthful auctions and non-stationary environments? To address this question, we bound the regret obtained by an individual bidding agent participating in any auction format that satisfies a *monotone bang-per-buck* condition, which implies that the marginal value obtained per dollar spent weakly decreases in an agent’s bid. For example, first and second-price auctions satisfy this condition, as does the GSP auction. We first consider the *stochastic* environment where the profile of opposing bids is drawn independently from the same distribution in each round. Our benchmark is the pacing multiplier that optimally realizes the desired budget in hindsight.⁶ We obtain regret $O(T^{3/4})$ relative to this benchmark.⁷ This regret bound applies for the objective of utility maximization, value maximization, or any convex combination thereof.

When the underlying auction is truthful, our benchmark is known to be optimal over the class of all possible bidding strategies (i.e., mappings from value to bid, which may not be linear), for utility-maximizing agents [26, 13, 12]. This means that, for truthful auctions such as the second-price auction, our benchmark for regret is actually the utility-optimal bidding policy in hindsight. But even beyond truthful auctions, we argue that an appropriate benchmark is the best linear policy (i.e., best pacing multiplier). First, while we abstractly model agent values as a willingness to pay per impression, in practice the variation in values is primarily driven by click rate estimates that are internal to the platform. In such an environment, an advertiser whose bidding algorithm is external to the platform would necessarily be limited to a linear policy. Indeed, if the algorithm cannot access the platform’s click rate estimates, then a bidding “strategy” simply reduces to a single real-valued bid that would be (linearly) multiplied by the click rate; see Appendix B.3 for a more formal discussion. The benchmark therefore tracks the best performance that one could achieve in hindsight with an externally provided bid. Second, linear pacing is commonly used in practice as an algorithmic bidding policy even for non-truthful auctions [27], so from a practical perspective it is useful to focus attention on linear pacing policies. Third, linear pacing is reasonable from the online learning perspective: it is typical to choose a subset of policies as a hypothesis class (even if this class is not known to contain an optimal policy), and the set of linear policies is a common and natural class to consider.

While the discussion above is framed in a stochastic environment, we prove an even stronger individual guarantee by permitting the opposing bids to change adaptively and adversarially based on the auction

⁵I.e., a good that can be divided fractionally among the agents. Any item can be interpreted as divisible via probabilistic allocation.

⁶Pacing multipliers are defined to only adjust bids downward relative to the maximum willingness to pay. Thus, if the budget is not exhausted even when setting the bid equal to the value each round, this benchmark corresponds to a pacing multiplier of 1.

⁷A typical goal in regret minimization is regret $\tilde{O}(T^\gamma)$ for some constant $\gamma \in [\frac{1}{2}, 1)$. As a baseline, regret $O(\sqrt{T})$ is the best possible in the worst case, even in a stochastic environment with only two possible actions [4].

history. (Indeed, realistic auction environments are not necessarily stochastic from the individual bidder’s perspective, because the other agents’ bidding algorithms may be revising their bids.) In such an environment, we show that gradient-based pacing achieves vanishing regret relative to the *perfect pacing sequence*, which is the sequence of pacing multipliers such that the expected spend in each round is precisely the per-round budget.⁸ In a stochastic environment, this perfect pacing sequence is precisely the single best fixed pacing multiplier in hindsight. More generally, this sequence may not be uniform and is not necessarily the sequence that maximizes expected utility or value subject to the budget constraint. Achieving low regret against this stronger benchmark in an adversarial environment is essentially hopeless (more on this in Section 1.2). Therefore, we suggest the perfect pacing sequence as a reasonable and tractable benchmark for this problem, and one particularly suitable for our algorithm (see Theorem 4.19). In fact, following this sequence (i.e., matching a target spend rate as closely as possible across time) can be a natural and desirable goal for a budget management system.

Numerical Simulations. To complement our theoretical findings, we provide semi-synthetic numerical simulations of Algorithm 1 based on auction and campaign data from Microsoft’s Bing Advertising platform. Motivated by the impossibility of sublinear-regret guarantees against this benchmark in adversarial environments, we simulate *self-play*: the progression of a multi-player environment in which the competing bidding agents engage in simultaneous learning. We consider the utility-maximization objective for second-price payment rules, and the value-maximization objective for both first- and second-price payment rules. We focus on regret relative to the standard benchmark: the best fixed pacing multiplier in hindsight. We find numerically that simultaneous execution of Algorithm 1 yields vanishing regret in our simulations, with regret rate less than $T^{3/5}$.

Furthermore, we compare Algorithm 1 to other common online learning methods by evaluating both the (individual) empirical regret rate and (aggregate) liquid welfare obtained during self-play. Our comparators are based on Adaptive Moment Estimation (Adam) [45], Multiplicative Updating (MU) [19], and Optimistic Gradient Descent (OGD) [54]. We find that Adam has an improved regret rate at the expense of lower aggregate liquid welfare, MU is dominated by Algorithm 1 on both liquid welfare and regret rate, and OGD has comparable performance to Algorithm 1 on both measures.

Map of the paper. We detail our model in Section 2. The main results — liquid-welfare and regret guarantees for Algorithm 1 — are presented, resp., in Sections 3 and 4. Section 5 extends them to several other algorithms. In Section 6 we complement these theoretical findings with numerical simulations. Additional discussions and some details of the proofs and the simulations are relegated to the appendices.

1.2 Related Work

Pacing in Ad auctions. Budget-pacing is a popular approach for repeated bidding under budget constraints, both in practice and in theory [12]. Balseiro and Gur [13] attain convergence guarantees in repeated second-price auctions, under strong convexity-like assumptions. Borgs et al. [19] attain a similar result for first-price auctions, without convexity assumptions (via a different algorithm). Our emphasis on welfare guarantees *without requiring convergence* appears novel, and possibly necessary given the aforementioned PPAD-hardness result [22].

Balseiro and Gur [13] establish individual guarantees for utility-maximization in for repeated second-price auctions: \sqrt{T} regret rates for the stochastic environment and approximation guarantee for the adversarial environment, under various convexity assumptions. Balseiro et al. [17] extend similar guarantees to repeated *truthful* auctions, without convexity assumptions. They also obtain regret bounds for several (specific types of) non-stationary stochastic environments. Our individual guarantee is different from theirs in

⁸This guarantee is parameterized by the total round-to-round change in the “perfect” pacing multipliers.

several respects: (i) it applies to a much wider family of auctions, (ii) it applies to the adversarial environment, but (iii) the benchmark is the perfect pacing sequence rather than the best outcome in hindsight.

A static (single-shot) game between budget-constrained bidders who tune their pacing multipliers, a.k.a. the *pacing game*, along with the appropriate equilibrium concept was studied in Conitzer et al. [26, 27], in the context of first- and second-price auctions with quasilinear utilities, and then extended to more general payment constraints [1] and utility measures [8]. In particular, any pure Nash equilibrium of this game achieves at least half of the optimal liquid welfare when the underlying auction is truthful. In a related contextual auction setting and simultaneously with our work, Balseiro et al. [15] establishes a similar bound on liquid welfare at any (possibly non-linear) Bayes-Nash equilibrium for i.i.d. bidders, for a class of standard auctions that includes first-price and second-price auctions. In contrast to these results, our efficiency result does not rely on convergence to equilibrium, and applies to all core auctions.

A growing line of work in mechanism design targets bidding agents that maximize value or utility under spending constraints and are assumed to reach equilibrium [50, 14, 31]. In contrast, our emphasis is not on mechanism design: we take the auction specification as exogenous and focus on the learning dynamics.

Throttling (a.k.a. *probabilistic pacing*) is an alternative approach: instead of pacing their bids, agents participate in only a fraction of the auctions [12]. Very recently, Chen et al. [23] proved that throttling converges to a Nash equilibrium in the first-price auctions (albeit without any stated implications on welfare or liquid welfare). On the other hand, no such convergence is possible for second-price auctions under throttling dynamics for the same reason as pacing: a Nash equilibrium is PPAD-hard to compute [22]. The equilibria obtained by throttling and pacing dynamics in the first-price auction can differ in revenue by at most a factor of 2 [23].

Learning theory. Repeated bidding with a budget is a special case of multi-armed bandit problems with global constraints, a.k.a. *bandits with knapsacks* (BwK) [10, 3, 42] (see Chapter 10 in [58] for a survey). BwK problems in adversarial environments do not admit regret bounds: instead, one is doomed to approximation ratios, even against a time-invariant benchmark and even in relatively simple examples [42]. A similar impossibility result is derived in [13] specifically for repeated budget-constrained bidding in second-price auctions. The essential reason for this impossibility is the *spend-or-save dilemma* [42], whereby the algorithm does not know whether to spend the budget now or save it for the future.

Several known algorithms for BwK are potentially applicable to our problem, and the corresponding individual guarantees on regret may be within reach for the stochastic environment, but haven't yet been published.⁹ However, it is unclear how to derive aggregate guarantees for such algorithms. The algorithm analyzed in the present work is based upon stochastic gradient descent, which is a standard, well-understood algorithm in online convex optimization [41].

A long line of work targets convergence of learning algorithms in repeated games (not specifically focusing on ad auctions). When algorithms achieve low regret (in terms of cumulative payoffs), the *average play* (time-averaged distribution over chosen actions) converges to a (coarse) correlated equilibrium [5, 49, 39], and this implies welfare bounds for various auction formats in the absence of budget constraints [56]. In contrast, we show in Appendix B.2 that for repeated auctions with budgets, low individual regret on its own does not imply any bounded approximation for liquid welfare. Convergence in the last iterate is more challenging: strong negative results are known even for two-player zero-sum games [11, 47, 24]. Yet, a recent line of work (starting from Daskalakis et al. [30], see [29, 38, 62] and references therein) achieves last-iterate convergence under full feedback and substantial convexity-like assumptions, using two specific regret-minimizing algorithms. To the best of our understanding, these positive results do not apply to the

⁹One could run a BwK algorithm on bids discretized as multiples of some $\epsilon > 0$. Individual guarantees would depend on bounding the discretization error which is a known challenge in BwK [10, 9]. We are only aware of one such result for BwK when one has contexts and a "continuous" action set (in our case, resp., private values and bids). This result concerns a different problem: *dynamic pricing*, where actions correspond to posted prices, and only achieves regret $\tilde{O}(T^{4/5})$ in the stochastic environment [9].

setting of repeated auctions with budgets.

Subsequent work. Since the preliminary conference version of this work was made publicly available [35], several relevant papers have appeared in the literature.

The follow-up papers Fikioris and Tardos [34], Lucier et al. [46] are directly related, combining liquid-welfare and individual guarantees. Fikioris and Tardos [34] focus on the special case of repeated first-price auctions and achieve liquid welfare guarantees as long as each bidding algorithm satisfies a particular individual guarantee. Specifically, they assume multiplicative $\gamma \geq 1$ approximation relative to the best fixed pacing multiplier, and obtain multiplicative approximation ratio $R = \gamma + O(1)$ on liquid welfare. They achieve $\gamma = T/B$ and $R \approx T/B + 1/2$ plugging in a recent result on bandits with knapsacks [21]. Further, they achieve $R \approx 2.4$ if $\gamma = 1$. However, it is currently not known how to achieve $\gamma < T/B$, let alone $\gamma = 1$, with non-trivial budget-constrained bidding algorithms, e.g., such as those guaranteed to achieve vanishing regret in a stochastic environment.

Lucier et al. [46] extend our results to bidders that face return-on-investment constraints in addition to budget constraints. They achieve the same 2-approximation guarantee on liquid welfare and similar individual guarantees. Their algorithm coincides with ours when specialized to budget constraints. Their results hold for single-item allocation problems and any auction format in which the single item is sold to the highest bidder and the payment lies between the highest and second-highest bids.

Two other papers focus on individual guarantees for online bidding under budget, obtaining $\tilde{O}(\sqrt{T})$ regret under various assumptions. Wang et al. [61] consider utility maximization in a repeated first-price auction, provided that the maximum bid is revealed when an auction is lost. Balseiro et al. [16] consider repeated second-price auction, but in a non-stationary environment: they use time-varying target expenditures driven by additional data samples.

Finally, the idea to measure regret against the perfect pacing sequence, introduced in this paper to bypass the spend-or-save dilemma in the adversarial environment, has been fruitfully used (for the same purpose) in general BwK problems [59, 20].

2 Our Model and Preliminaries

The Allocation Problem. Our setting is a repeated auction with budgets. There is one seller (the platform) and n bidding agents. In each time $t = 1, \dots, T$, the seller has a single unit of a good available for sale. We will sometimes refer to the good as a (divisible) *item*. An *allocation profile* is a vector $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ where x_k is the quantity of the good allocated to agent k . There is a convex and closed set $X \subseteq [0, 1]^n$ of feasible allocation profiles, which is assumed to be downward closed.¹⁰ An allocation profile is feasible if $\mathbf{x} \in X$. An allocation sequence is a sequence of allocation profiles $(\mathbf{x}_1, \dots, \mathbf{x}_T)$ where $\mathbf{x}_t = (x_{1t}, \dots, x_{nt})$ is the allocation profile at time t .

At each time t , each agent k has a value $v_{k,t} \in [0, \bar{v}]$ per unit of the item received. By scaling values, we will assume that $\bar{v} \geq 1$. The value profile $\mathbf{v}_t = (v_{1,t}, \dots, v_{n,t})$ at time t is drawn from a distribution function F independently across different time periods. We emphasize that F is not necessarily a product distribution, so the values held by different agents can be arbitrarily correlated within each round.

Two special cases of our model are of particular interest in the context of ad auctions. In a *single-slot* ad auction, the “item” for sale is an opportunity to display one ad to the current user. A *Multi-slot* ad auction is a natural generalization in which multiple ads can be displayed in each round. The formal details are standard, we provide them in Appendix A.1.

Auctions and Budgets. At each time t the good is allocated using an auction mechanism that we now describe. In round t , each agent $k \in [n]$ first observes her value $v_{k,t} \geq 0$. After all agents have observed

¹⁰The set $X \subseteq [0, 1]^n$ is downward closed if, for any $\mathbf{x}, \mathbf{x}' \in [0, 1]^n$ such that $\mathbf{x} \in X$ and $x'_k \leq x_k$ for all k , we have $\mathbf{x}' \in X$.

their values, each agent k then submits a bid $b_{k,t} \geq 0$ to the auction. All agents submit bids simultaneously. The auction is defined by an allocation rule \mathbf{x} and a payment rule \mathbf{p} , where $\mathbf{x}(\mathbf{b}) \in X$ is the allocation profile generated under a bid profile \mathbf{b} , and $p_k(\mathbf{b}) \geq 0$ is the payment made by agent k . All of the auction formats we consider will have weakly monotone allocation and payment rules, so we will assume this for the remainder of the paper. That is, for any k and any bids of the other agents \mathbf{b}_{-k} , both $x_k(b_k, \mathbf{b}_{-k})$ and $p_k(b_k, \mathbf{b}_{-k})$ are weakly increasing in b_k .

Each agent k has a fixed budget B_k that can be spent over the T time periods. Once an agent has run out of budget she can no longer bid in future rounds. We emphasize that this budget constraint binds *ex post*, and must be satisfied on every realization of value sequences.¹¹ An important quantity in our analysis is the *target expenditure rate* for agent k which is defined by $\rho_k \triangleq B_k/T$.

The *auction history* up to round t consists of all realized values, bids, allocations, and payments for all agents and all rounds prior to t . An agent k will observe some (but typically not all) of this history. All of our results hold under *bandit feedback* where each agent can see her own values, bids, allocations, and payments from previous rounds, but is not required to observe any other aspect of the history including the bids of other agents.

Each agent k applies a dynamic bidding strategy that, in each round t , maps (the observed part of) auction history and the realized valuation $v_{k,t}$ to a bid $b_{k,t}$. In a slight abuse of notation we'll write $\mathbf{b}(\mathbf{v})$ for the sequence of bid profiles that are generated when the sequence of value profiles is \mathbf{v} . In this paper we will focus on a particular class of bidding strategies (gradient-based pacing) described later in this section.

Given an execution of the auction over T rounds, we will typically write $x_{k,t}$ for the realized allocation obtained by agent k in time t , and $z_{k,t}$ for the realized payment of agent k in time t . This notation omits the dependency on the agents' bids when this dependency is clear from the context.

Core Auctions. Our results apply to *Core Auctions* [6], a wide class of auction mechanisms which includes standard auctions such as first-price auctions, second-price auctions for single-slot allocations, and generalized second-price (GSP) auctions for multi-slot allocations.¹² Roughly speaking, no subset of players (which may or may not include the seller) could jointly benefit by renegotiating the auction outcome among themselves. Stated more formally in our context, a core auction satisfies the following two properties:

1. The auction is *individually rational* (IR): each agent's payment does not exceed her declared welfare for the allocation received. That is, $p_k(\mathbf{b}) \leq b_k x_k(\mathbf{b})$ for every agent k and every bidding profile \mathbf{b} .
2. The seller and any subset of agents $S \subseteq [n]$ could not strictly benefit by jointly abandoning the auction and deviating to another outcome. That is, for any bidding profile \mathbf{b} and any allocation profile $\mathbf{y} \in X$,

$$\sum_k p_k(\mathbf{b}) + \sum_{k \in S} (b_k x_k(\mathbf{b}) - p_k(\mathbf{b})) \geq \sum_{k \in S} b_k y_k$$

which simplifies to

$$\sum_{k \notin S} p_k(\mathbf{b}) + \sum_{k \in S} b_k x_k(\mathbf{b}) \geq \sum_{k \in S} b_k y_k. \quad (2.1)$$

The left-hand (resp., right-hand) side of Eq. (2.1) is the total welfare obtained by the seller and the agents in S under the auction (resp., under a deviation to allocation \mathbf{y}). On both sides, the summation over agents in S accounts not only for the agent utilities, but also those agents' payments to the seller.

The second property implies that a core auction must always maximize declared welfare. Indeed, taking $S = [n]$, Eq. (2.1) states that $\sum_k b_k x_k(\mathbf{b}) \geq \sum_k b_k y_k$ for all feasible allocation profiles \mathbf{y} . The first property

¹¹All of the auctions we consider satisfy the property that each agent's payment will be at most her bid, so it is always possible to bid in such a way that does not overspend one's remaining budget.

¹²For the sake of completeness, these special cases are defined in Appendix A.2.

is simply a restatement of the core condition that no subset of buyers could jointly benefit by renegotiating an auction outcome that does not include the seller, i.e. no buyer (and hence no set of buyers) would strictly prefer to switch to the null outcome in which no goods are allocated and no payments are made.

Monotone Bang-Per-Buck. Our guarantees on individual regret require an additional property of the auction mechanism called *monotone bang-per-buck* (MBB). Roughly speaking, if increasing one’s bid leads to an increased allocation, then the increase in payment per unit of new allocation is at least the minimum bid needed for the increase. Formally, an auction mechanism satisfies MBB if for any agent k , any profile of bids \mathbf{b}_{-k} of the other agents, and any bids $b_k \leq b'_k$ of agent k , we have

$$p_k(b'_k, \mathbf{b}_{-k}) - p_k(b_k, \mathbf{b}_{-k}) \geq b_k \cdot (x_k(b'_k, \mathbf{b}_{-k}) - x_k(b_k, \mathbf{b}_{-k})). \quad (2.2)$$

One can readily check that the MBB property is satisfied by many common auction formats, such as first-price and second-price auctions as well as generalized second-price auctions (see Appendix A for further discussion). The name MBB is motivated by a simple observation that in a MBB auction each buyer’s expected payment per expected unit of allocation received is weakly increasing in their bid, for any distribution of competing bids. See, for example, inequality (4.13) in Section 4.8.

Agent Objective. To state individual regret guarantees, we must define an objective for each individual agent. Two natural, well-motivated candidates are (constrained) value maximization and utility maximization. We focus on utility maximization as a primary scenario in this paper, but also discuss value maximization below. Our *aggregate* guarantees are independent of the agent objective.

A utility-maximizing agent aims to maximize total quasi-linear utility, $\sum_t v_{k,t} x_{k,t} - z_{k,t}$, subject to the budget constraint $\sum_t z_{k,t} \leq B_k$. Implicitly, this formulation assumes that values are expressible in units of money. This objective is especially relevant in scenarios where the agents are the advertisers themselves.

For any auction that is truthful for quasi-linear bidders within a single round, such as the second-price auction, the ex-post optimal strategy for a utility-maximizing agent is to bid $v_{k,t} \times \alpha$ in each round for the largest $\alpha \in [0, 1]$ such that the budget constraint is satisfied [13]. Notably, this strategy also maximizes the total value received subject to the budget constraint and a no-overbidding condition.

Motivated by this observation, we also consider value-maximizing agents whose objective is to maximize the total value received subject to budget and maximum bid constraints. More specifically, a value-maximizing agent in our model aims to maximize the total value of the allocation obtained over all rounds, $\sum_t v_{k,t} x_{k,t}$, subject to the following two constraints. First, the total payment across all rounds must not exceed the budget: $\sum_t z_{k,t} \leq B_k$ for all k . Second, the bid placed in round t must not exceed the value of the good in round t : $b_{k,t} \leq v_{k,t}$ for all k and t . The latter can be interpreted as a no-overbidding constraint, consistent with a pacing scenario where the agent is empowered to scale bids downward but not upward. Alternatively, for all the auction formats we consider, the second constraint could equivalently be interpreted as a bound on marginal ROI: i.e., that $z_{k,t} \leq v_{k,t} x_{k,t}$ for all k and t , in which case $v_{k,t}$ is viewed as an upper bound on the maximum amount the agent is allowed to pay per unit in any round.¹³

Liquid Welfare. Liquid welfare is a measure of welfare that accounts for non-quasi-linear agent utilities. Intuitively, an agent’s liquid welfare for an allocation sequence is the agent’s *maximum willingness to pay for the allocation*. This generalizes the common notion of welfare in quasi-linear environments, and motivates the choice to measure welfare in transferable units of money. In our setting with a budget constraint that binds across rounds, liquid welfare is defined as follows.

Definition 2.1. Given a sequence of value profiles $\mathbf{v} = (v_{k,t}) \in [0, \bar{v}]^{nT}$ and any sequence of feasible

¹³We emphasize that a bound on marginal ROI is not equivalent to an average ROI constraint that binds over multiple rounds. Maximizing value subject to an average ROI constraint is a common but different model of algorithmic bidding. See Aggarwal et al. [2] for a survey and Lucier et al. [46] for follow-up work extending our analysis to scenarios with average ROI constraints.

allocations $\mathbf{x} = (x_{k,t}) \in X^T$, the *liquid value* obtained by agent k is $W_k(\mathbf{x}) = \min \left\{ B_k, \sum_{t=1}^T x_{k,t} v_{k,t} \right\}$. The *liquid welfare* of allocation sequence $\mathbf{x} \in X^T$ is $W(\mathbf{x}) = \sum_{k=1}^n W_k(\mathbf{x})$.

We emphasize that liquid welfare depends on the allocations, but not on the agents' payments.

Our objective of interest is the *expected* liquid welfare obtained by the platform over any randomness in the valuation sequence and the agents' bidding strategies (which induces randomness in the allocation sequence). Since the bid placed in one round can depend on allocations obtained in the previous rounds, we define a mapping from the entire sequence of T value profiles to an allocation sequence. An *allocation sequence rule* is a function $\mathbf{x}: [0, \bar{v}]^{nT} \rightarrow X^T$, where $x_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T)$ is the allocation obtained by agent k in round t . Then the expected liquid welfare of allocation sequence rule \mathbf{x} is

$$W(\mathbf{x}, F) := \mathbb{E}_{\mathbf{v}_1, \dots, \mathbf{v}_T \sim F} [W(\mathbf{x}(\mathbf{v}_1, \dots, \mathbf{v}_T))]. \quad (2.3)$$

Pacing Algorithm. We use a budget-pacing algorithm motivated by stochastic gradient descent that was introduced and analyzed in [13] in the context of utility maximization in second-price auctions. See Algorithm 1. Each bidder k maintains a *pacing multiplier* $\mu_{k,t} \in [0, \bar{\mu}]$ for each round t . Multiplier $\mu_{k,t}$ is determined by the algorithm before the value $v_{k,t}$ is revealed. The bid is set to $v_{k,t}/(1 + \mu_{k,t})$, or the remaining budget $B_{k,t}$ if the latter is smaller. Once the round's outcome is revealed, the multiplier is updated as per Eq. (2.4), where $P_{[a,b]}$ denotes the projection onto the interval $[a, b]$. Intuitively, the agent's goal is to keep expenditures near the expenditure rate ρ_k . Hence, if ρ_k is above (resp., below) the current expenditure $z_{k,t}$, the agent decreases (resp., increases) her multiplier, by the amount proportional to the current "deviation" from the target expenditure rate ρ_k .

Algorithm 1: Gradient-based pacing algorithm for agent k

Input: Budget B_k , time horizon T , step-size $\epsilon_k > 0$, pacing upper bound $\bar{\mu}$

1 Initialize

$\mu_{k,1} = 0$ (pacing multiplier), $B_{k,1} = B_k$ (remaining budget), $\rho_k = B_k/T$ (target spend rate).

2 **for** round $t = 1, \dots, T$ **do**

3 Observe value $v_{k,t}$, submit bid $b_{k,t} = \min \{ v_{k,t}/(1 + \mu_{k,t}), B_{k,t} \}$;

4 Observe expenditure $z_{k,t}$;

5 Update $B_{k,t+1} \leftarrow B_{k,t} - z_{k,t}$ and

$$\mu_{k,t+1} \leftarrow P_{[0, \bar{\mu}]} (\mu_{k,t} - \epsilon_k (\rho_k - z_{k,t})). \quad (2.4)$$

6 STOP if $B_{k,t} \leq 0$.

Parameters $\bar{\mu}$ and ϵ_k will be specified later. While the upper bound $\bar{\mu}$ could in general depend on the agent k , as in $\bar{\mu} = \bar{\mu}_k$, we suppress this dependence for the sake of clarity. Our bounds depend on $\max_k \bar{\mu}_k$.

A convenient property of Algorithm 1 is that it does not run out of budget too early.¹⁴

Lemma 2.2 ([13]). *Fix agent k in a core auction. Let τ_k be the stopping time of Algorithm 1 for some (possibly adaptive, randomized, adversarially generated) set of valuations and competing bids. Assume all valuations are at most \bar{v} , and the parameters satisfy $\bar{\mu} \geq \bar{v}/\rho_k - 1$ and $\epsilon_k \bar{v} \leq 1$. Then $T - \tau_k \leq \frac{\bar{\mu}}{\epsilon_k \rho_k} + \frac{\bar{v}}{\rho_k}$ almost surely.*

¹⁴This result is proven in [13] (Eq. (A-4) of Theorem 3.3) for second-price auctions, but an inspection of their proof shows this holds surely for any sequence of valuations and competing prices so long as the price is at most the bid if the agent wins.

Remark 2.3. Algorithm 1 never *overbids*, in the sense that the bids $b_{k,t}$ are always upper-bounded by the respective values $v_{k,t}$. For utility-maximizing agents, this behavior is provably optimal for second-price auctions [13]. More generally, the choice to never overbid may reduce the agent’s objective value. In such cases, we interpret no-overbidding as an additional optimization constraint motivated by the budget pacing context and/or an imposed bound on marginal ROI, as discussed in the Agent Objective paragraph above.

3 Aggregate Guarantee: Approximate Liquid Welfare

Our main result is a liquid welfare guarantee without convergence. We show that, when all agents use Algorithm 1, the expected liquid welfare (as defined in Eq. (2.3)) is at least half of the optimal minus a regret term.

Theorem 3.1. *Fix any core auction and any distribution F over agent value profiles. Suppose that each agent k employs Algorithm 1 to bid, possibly with a different step-size ϵ_k . Write \mathbf{x} for the corresponding allocation sequence rule. Then for any other allocation sequence rule \mathbf{y} we have*

$$W(\mathbf{x}, F) \geq \frac{W(\mathbf{y}, F)}{2} - O\left(n\bar{v}\sqrt{T\log(\bar{v}nT)}\right). \quad (3.1)$$

In fact, our analysis for Theorem 3.1 yields a stronger result. First, it does not invoke a full specification of the bidding rule in Algorithm 1. Instead, it applies to a wider class of algorithms that do not overbid ($b_{k,t} \leq v_{k,t}$), bid their full value when $\mu_{k,t} = 0$, and are not constrained otherwise.

Definition 3.2. Consider a measurable bidding algorithm that inputs the same parameters as Algorithm 1, internally updates multipliers $\mu_{k,t}$ in the same way, and never overbids: $b_{k,t} \leq v_{k,t}$ for all rounds t . Call it a **generalized pacing algorithm** if $(\mu_{k,t} = 0) \Rightarrow (b_{k,t} = v_{k,t})$ for all rounds t (“no unnecessary pacing”).

Second, our analysis competes against a stronger benchmark: optimal *ex ante* liquid welfare, which is each agent’s willingness to pay for her *expected* allocation sequence. It upper-bounds the expected (ex post) liquid welfare by Jensen’s inequality.

Definition 3.3. Fix distribution F over valuation profiles and allocation rule $\mathbf{y}: [0, \bar{v}]^n \rightarrow X$. The *ex ante liquid value* of each agent k is $\bar{W}_k(\mathbf{y}, F) = T \times \min \left[\rho_k, \mathbb{E}_{\mathbf{v} \sim F} [y_k(\mathbf{v}) v_k] \right]$ and the ex ante liquid welfare is $\bar{W}(\mathbf{y}, F) = \sum_{k=1}^n \bar{W}_k(\mathbf{y}, F)$.

In this definition, the restriction that the same allocation rule \mathbf{y} is used in every round is without loss of generality. Indeed, given any allocation sequence rule there is a single-round allocation rule with the same ex ante liquid welfare. This property is formally stated and proved as Lemma C.1 in Appendix C.

To sum up, the following is the generalization of Theorem 3.1 that we actually prove:

Theorem 3.4. *Fix any core auction and any distribution F over agent value profiles. Suppose that each agent k employs a generalized pacing algorithm to bid, possibly with a different step-size ϵ_k . Write $\mathbf{x}: [0, \bar{v}]^{nT} \rightarrow X^T$ for the corresponding allocation sequence rule. Then for any allocation rule $\mathbf{y}: [0, \bar{v}]^n \rightarrow X$ we have*

$$W(\mathbf{x}, F) \geq \frac{\bar{W}(\mathbf{y}, F)}{2} - O\left(n\bar{v}\sqrt{T\log(\bar{v}nT)}\right). \quad (3.2)$$

Remark 3.5. In the proof, it is not necessary for each advertiser k to employ a fixed step-size ϵ_k throughout the entire time horizon. Instead, ϵ_k can change whenever the corresponding pacing multiplier becomes 0, and needs to stay fixed till the next time this happens.

The rest of this section proves Theorem 3.4. We start with a proof sketch to provide intuition, then continue with a formal proof.

3.1 Proof Sketch and Intuition

One easy observation is that since B_k is an upper bound on the liquid welfare obtainable by agent k , any agents who are (approximately) exhausting their budgets in our dynamics are achieving optimal liquid welfare. We therefore focus on agents who do not exhaust their budgets. We'd like to argue that such agents are often bidding very high, frequently choosing pacing multipliers equal to 0 (i.e., bidding their values). This would be helpful because our auction is assumed to be a core auction, which implies that either the high-bidding agents are winning (and generating high liquid welfare) or other, budget-exhausting agents are generating high revenue for the seller (which likewise implies high liquid welfare).

Why should agents who are underspending their budget be placing high bids in many rounds? While it's true that the pacing dynamics increases the next-round bid whenever spend is below the per-round target, this is only a local adjustment and does not depend on total spend. We make no convergence assumptions about the dynamics and, as we show in Appendix B.2, regret bounds do not directly imply a bound on liquid welfare. So how do we analyze bidding patterns in aggregate across rounds?

It is here where we use the fact that agents use generalized pacing algorithms. For each k , the evolution of the multipliers $\mu_{k,t}$ has the following convenient property: over any contiguous range of time steps $[t_1, t_2]$ such that the multiplier is never 0 or $\bar{\mu}$, the total amount spent from time t_1 to time t_2 is determined by $\mu_{k,t_2} - \mu_{k,t_1}$. More precisely,

$$\sum_{t=t_1}^{t_2-1} z_{k,t} = (t_2 - t_1 - 1)\rho_k + \frac{1}{\epsilon_k}(\mu_{k,t_2} - \mu_{k,t_1}) \quad (3.3)$$

which follows immediately from the update rule of $\mu_{k,t}$ on line 5 of Algorithm 1: if the multipliers are never 0 or $\bar{\mu}$ then the projection operator is never invoked, so we have $\mu_{k,t+1} = \mu_{k,t} - \epsilon_k(\rho_k - z_{k,t})$ and (3.3) follows.

Motivated by this observation, we introduce the notion of an *epoch*: essentially, a maximal contiguous sequence of rounds in which an agent's pacing multiplier is strictly greater than 0. In Lemma 3.8 we show that an agent k 's total spend over an epoch of length t must be (approximately) $t\rho_k$. In other words, the average spend over an epoch approximately matches the target per-round spend. This is a direct implication of the update rule for the pacing multiplier: since multiplier increases and decreases balance out (approximately) over the course of an epoch, the budget deficits and surpluses must balance out as well. An immediate implication is that an agent whose total spend is much less than $T\rho_k$ must often have her pacing multiplier set equal to 0.

To summarize, each agent either spends most of her budget by time T or spends many rounds bidding her value. It might seem that by combining these two cases we should obtain a constant approximation factor, and not just in expectation but for every realized value sequence \mathbf{v} . But this is too good to be true. Indeed, this proof sketch misses an important subtlety: whether an agent exhausts her budget or not depends on the realization of the value sequence, which is also correlated with the benchmark allocation \mathbf{y} . For example, what if an agent under-spends her budget precisely on those value sequences where it would have been optimal (according to the benchmark \mathbf{y}) for her liquid welfare to equal her budget? This could result in a situation where, conditional on $\mu_{k,t} = 0$, the value obtained by the benchmark is much higher than expected and cannot be approximated. To compare against the liquid welfare of \mathbf{y} we must control the extent of this correlation. To this end we employ a variation of the Azuma-Hoeffding inequality (Lemma 3.9) to argue that since value realizations are independent across time, the pacing sequence and the benchmark allocation cannot be too heavily correlated.

3.2 Proof Preliminaries

Before getting into the details of the proof of Theorem 3.4, we first introduce some notation and define an epoch more formally. Write $\mu_{k,t} = \flat$ if at time t , the agent's algorithm has stopped, i.e., if the agent is out of money or if $t = T + 1$. For $t_1 \leq t_2 \in \mathbb{N}$, we slightly abuse notation and write $[t_1, t_2] \triangleq \{t_1, \dots, t_2\}$ to be the set of integers between them, inclusive, when the meaning is clear. Similarly, we write $[t_1, t_2) \triangleq \{t_1, \dots, t_2 - 1\}$ for the half-open set of integers between them and analogously for $(t_1, t_2]$ and (t_1, t_2) .

Definition 3.6. Fix an agent k , time horizon T , and a sequence of pacing multipliers $\mu_{k,1}, \dots, \mu_{k,T}$. A half-open interval $[t_1, t_2)$ is an **epoch** with respect to these multipliers if it holds that $\mu_{t_1} = 0$ and $\mu_t > 0$ for each $t_1 < t < t_2$ and t_2 is maximal with this property.

Remark 3.7. Because we initialized any generalized pacing algorithm at $\mu_{k,1} = 0$ for all $k \in [n]$, the epochs completely partition the set of times the agent is bidding. Moreover, if the pacing multipliers at times t_1 and $t_1 + 1$ are both zero, then $[t_1, t_1 + 1)$ is an epoch; we refer to this as a trivial epoch.

The following lemma shows that an agent's total spend over a maximal epoch can be bounded from below by an amount roughly equal to the target spend rate ρ_k times the epoch length, plus an adjustment for the first round of the epoch. This lemma is crucial in proving Theorem 3.4 and is proved in Appendix C.

Lemma 3.8. Fix an agent k , fix any choice of core auction and any sequence of bids of other agents $(\mathbf{b}_{-k,1}, \dots, \mathbf{b}_{-k,T})$. Fix any realization of values $(v_{k,1}, \dots, v_{k,T})$ for agent k and suppose that $\mu_{k,1}, \dots, \mu_{k,T}$ is the sequence of multipliers generated by the generalized pacing algorithm given the auction format and the other bids. Then for any epoch $[t_1, t_2)$ where $\mu_{k,t_2} \neq \flat$, we have

$$\sum_{t=t_1}^{t_2-1} x_{k,t} v_{k,t} \geq x_{k,t_1} v_{k,t_1} - z_{k,t_1} + \rho_k \cdot (t_2 - t_1 - 1). \quad (3.4)$$

Next, motivated by the intuition in Section 3.1, we introduce a concentration inequality that will be helpful for our analysis. Roughly speaking, we will use this lemma to show that the sequence of values obtained by agent k in the benchmark on rounds in which $\mu_{k,t} = 0$ are not “far from expectation,” in the sense that the total expected value obtained over such rounds is not much greater than ρ_k per round.

Lemma 3.9. Let Y_1, \dots, Y_T be random variables and $\mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_T$ be a filtration such that:

1. $0 \leq Y_t \leq \bar{v}$ with probability 1 for some parameter $\bar{v} \geq 0$ for all t .
2. $\mathbb{E}[Y_t] \leq \rho$ for some parameter $\rho \geq 0$ for all t .
3. For all t , Y_t is \mathcal{F}_t -measurable but is independent of \mathcal{F}_{t-1} .

Suppose that $X_1, \dots, X_n \in [0, 1]$ are random variables such that X_t is \mathcal{F}_{t-1} -measurable. Then

$$\Pr \left(\sum_{t=1}^T X_t Y_t + (1 - X_t) \rho \geq \rho \cdot T + \theta \right) \leq \exp \left(\frac{-2\theta^2}{T\bar{v}^2} \right). \quad (3.5)$$

The proof of Lemma 3.9 appears in Appendix C. We are now ready to prove Theorem 3.4.

3.3 Proof of Theorem 3.4

We first introduce some notations. Fix the generalized pacing dynamics algorithm used by each agent. We will then write $\mathbf{x} = \{x_{k,t}\}_{k \in [n], t \in [T]}$ for the random variable corresponding to the allocation obtained under these bidding dynamics, given the values $\{v_{k,t}\}$. We also write $\mu_{k,t}$ for the pacing multiplier of agent k

in round t , and $z_{k,t}$ for the realized spend of agent k in round t , which are likewise random variables. For notational convenience we will write $\text{WEL}_{\text{GPD}}(\mathbf{v})$ for the liquid welfare (where GPD stands for ‘‘Generalized Pacing Dynamics’’) given valuation sequence \mathbf{v} . That is,

$$\text{WEL}_{\text{GPD}}(\mathbf{v}) \triangleq \sum_{k=1}^n \min \left\{ B_k, \sum_{t=1}^T x_{k,t} v_{k,t} \right\}. \quad (3.6)$$

We also write $\text{WEL}_{k,\text{GPD}}(\mathbf{v})$ for the liquid welfare obtained by agent k , and $\text{WEL}_{\text{GPD}}(F)$ for the total expected liquid welfare $\mathbb{E}_{\mathbf{v} \sim F^T} [\text{WEL}_{\text{GPD}}(\mathbf{v})]$.

We claim that to prove Theorem 3.4, it is sufficient to show that inequality (3.2) holds for any allocation rule \mathbf{y} such that

$$\mathbb{E}_{\mathbf{v} \sim F} [y_k(\mathbf{v}) v_k] \leq \rho_k \quad \text{and} \quad \overline{W}_k(\mathbf{y}, F) = T \cdot \mathbb{E}_{\mathbf{v} \sim F} [y_k(\mathbf{v}) v_k] \leq \rho_k \cdot T \quad \forall k \in [n]. \quad (3.7)$$

This is sufficient because if one of the above conditions is violated, one can always decrease the allocation for agent k , which maintains the feasibility (since we assume that the set of feasible allocations X is downward closed) without affecting the ex ante liquid welfare $\overline{W}(\mathbf{y}, F)$. We will therefore assume without loss that \mathbf{y} satisfies (3.7).

Preliminaries completed, we now prove Theorem 3.4 in three steps. First, we will define a ‘‘good’’ event in which the benchmark allocations are not too heavily correlated with the pacing multipliers of the generalized pacing dynamics, and show that this good event happens with high probability. Second, for all valuation sequences \mathbf{v} that satisfy the good event, we bound the liquid welfare obtained by the pacing dynamics in terms of the benchmark allocation and the payments collected by the auctioneer. In the third and final step we take expectations over all valuation profiles to bound the expected liquid welfare.

Step 1: A Good Event. For each agent $k \in [n]$, we define the following quantity, whose significance will become apparent shortly:

$$R_k(\mathbf{v}) \triangleq \sum_{t=1}^T [\mathbf{1}\{\mu_{k,t} = 0\} y_k(\mathbf{v}) v_{k,t} + \mathbf{1}\{\mu_{k,t} \neq 0\} \rho_k].$$

$R_k(\mathbf{v})$ is the total value obtained by agent k under allocation rule \mathbf{y} , except that in any round in which $\mu_{k,t} \neq 0$ this value is replaced by the target spend ρ_k .

We would like to apply Lemma 3.9 to bound $R_k(\mathbf{v})$. Some notation: we will write $Y_t = y_k(\mathbf{v}_t) v_{k,t}$ and $X_t = \mathbf{1}\{\mu_{k,t} = 0\}$ with $\mathcal{F}_t = \sigma(\mathbf{v}_1, \dots, \mathbf{v}_t)$. Then because $\mu_{k,t}$ is \mathcal{F}_{t-1} measurable and the sequence $\{\mathbf{v}_j\}$ is a sequence of independent random variables, Lemma 3.9 implies that with probability at least $1 - 1/(\bar{v}nT)^2$, we have

$$\sum_{t=1}^T [\mathbf{1}\{\mu_{k,t} = 0\} y_k(\mathbf{v}) v_{k,t} + \mathbf{1}\{\mu_{k,t} \neq 0\} \rho_k] \leq \rho_k \cdot T + \bar{v} \sqrt{T \log(\bar{v}nT)}.$$

Taking a union bound over $k \in [n]$, with probability at least $1 - 1/(\bar{v}nT)^2$ over the randomness in the sequence $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_T)$, we have that

$$R_k(\mathbf{v}) \leq \rho_k \cdot T + \bar{v} \sqrt{T \log(\bar{v}nT)}, \quad \forall k \in [n]. \quad (3.8)$$

We will write E_{GOOD} for the event in which (3.8) holds. Going back to the intuition provided before the statement of Lemma 3.9, E_{GOOD} is the event that the value each agent obtains in the benchmark allocation \mathbf{y} is not ‘‘too high’’ on rounds in which their pacing multipliers are 0.

Step 2: A Bound on Liquid Welfare For ‘‘Good’’ Value Realizations. Fix any realized sequence $\mathbf{v}_1, \dots, \mathbf{v}_T$ such that (3.8) holds. We will now proceed to derive a lower bound on the liquid welfare of the agents

(under allocation \mathbf{x}) by considering the two different possible cases for $\text{WEL}_{k,\text{GPD}}(\mathbf{v})$. Recall that the liquid welfare $\text{WEL}_{k,\text{GPD}}(\mathbf{v})$ of any agent k is either $B_k = \rho_k \cdot T$ or is $\sum_{t=1}^T x_{k,t} v_{k,t}$. For any k such that $\text{WEL}_{k,\text{GPD}}(\mathbf{v}) = B_k$, we obtain via (3.8) that

$$\text{WEL}_{k,\text{GPD}}(\mathbf{v}) = B_k \geq R_k(\mathbf{v}) - \bar{v} \sqrt{T \log(\bar{v} n T)}. \quad (3.9)$$

We now consider all of the remaining agents. Let $A \subseteq [n]$ be the set of agents k such that $\sum_{t=1}^T x_{k,t} v_{k,t} < B_k$ on this realized sequence \mathbf{v} , and hence $\text{WEL}_{k,\text{GPD}}(\mathbf{v}) = \sum_{t=1}^T x_{k,t} v_{k,t}$. That is, each of their contributions to the liquid welfare on this realized sequence is uniquely determined by their true value for winning the items in the auction. Note that this further implies that no agent in A runs out of budget early because the generalized pacing algorithm does not allow overbidding (i.e., bidding above the value). Thus for all $k \in A$, $\mu_{k,T} \neq \flat$ and $\mu_{k,t} \geq 0$ surely for all t . We claim that

$$\sum_{k \in A} \text{WEL}_{k,\text{GPD}}(\mathbf{v}) = \sum_{k \in A} \sum_{t=1}^T x_{k,t} v_{k,t} \geq \sum_{k \in A} R_k(\mathbf{v}) - \sum_{k \in [n]} \sum_{t=1}^T z_{k,t}. \quad (3.10)$$

To show that the inequality holds, we partition the interval $[1, T]$ into maximal epochs for each agent k and bound the value obtained by agent k on each maximal epoch separately. Fix any agent $k \in A$ and suppose that $[t_1, t_2]$ is a maximal epoch inside $[1, T]$. Since agent k does not run out of budget on the entire interval $[1, T]$, she does not run out of budget on this epoch in particular. We can therefore apply Lemma 3.8 to obtain

$$\sum_{t=t_1}^{t_2-1} x_{k,t} v_{k,t} \geq x_{k,t_1} v_{k,t_1} - z_{k,t_1} + \rho_k \cdot (t_2 - t_1 - 1).$$

Since $[1, T]$ can be partitioned into maximal epochs for each agent $k \in A$, we can sum over all time periods and apply the definition of a maximal epoch to conclude that

$$\sum_{t=1}^T x_{k,t} v_{k,t} \geq \sum_{t=1}^T [\mathbf{1}\{\mu_{k,t} = 0\} (x_{k,t} v_{k,t} - z_{k,t}) + \mathbf{1}\{\mu_{k,t} \neq 0\} \cdot \rho_k].$$

Summing over all $k \in A$ and changing the order of the summations implies that

$$\sum_{k \in A} \sum_{t=1}^T x_{k,t} v_{k,t} \geq \sum_{t=1}^T \sum_{k \in A} [\mathbf{1}\{\mu_{k,t} = 0\} (x_{k,t} v_{k,t} - z_{k,t})] + \sum_{k \in A} \sum_{t=1}^T \mathbf{1}\{\mu_{k,t} \neq 0\} \cdot \rho_k. \quad (3.11)$$

We will now use the assumption that the auction is a core auction. For any $t \in [1, T]$, let $S \subseteq A$ be the set of agents in A for which $\mu_{k,t} = 0$. We have that

$$\begin{aligned} \sum_{k \in A} [\mathbf{1}\{\mu_{k,t} = 0\} (x_{k,t} v_{k,t} - z_{k,t})] &= \sum_{k \in S} (x_{k,t} v_{k,t} - z_{k,t}) \\ &\geq \sum_{k \in S} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k=1}^n z_{k,t} = \sum_{k \in A} \mathbf{1}\{\mu_{k,t} = 0\} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k=1}^n z_{k,t}. \end{aligned}$$

The inequality follows from the definition of a core auction and the no unnecessary pacing condition (which implies $b_{k,t} = v_{k,t}$ for all $k \in S$), by considering the deviation in which the agents in S jointly switch to allocation $\{y_k(\mathbf{v}_t)\}$. Substituting the above inequality into (3.11) and rearranging yields (3.10).

Summing over (3.9) for each $k \notin A$ and combining it with (3.10), we obtain that as long as inequality (3.8) holds (i.e., event E_{GOOD} occurs), we have

$$\sum_{k \in [n]} \text{WEL}_{k,\text{GPD}}(\mathbf{v}) \geq \sum_{k \in [n]} R_k(\mathbf{v}) - \sum_{k \in [n]} \sum_{t \in [T]} z_{k,t} - n \bar{v} \sqrt{T \log(\bar{v} n T)}. \quad (3.12)$$

Step 3: A Bound on Expected Liquid Welfare. Since the liquid welfare is nonnegative, we can take expectations over $\mathbf{v}_1, \dots, \mathbf{v}_T$ to conclude from (3.12) that

$$\begin{aligned}\mathbb{E} \left[\sum_{k=1}^n \text{WEL}_{k,\text{GPD}}(\mathbf{v}) \right] &\geq \mathbb{E} \left[\mathbf{1}\{E_{\text{GOOD}}\} \cdot \sum_{k=1}^n \text{WEL}_{k,\text{GPD}}(\mathbf{v}) \right] \\ &\geq \mathbb{E} \left[\mathbf{1}\{E_{\text{GOOD}}\} \cdot \sum_{k \in [n]} R_k(\mathbf{v}) \right] - \mathbb{E} \left[\sum_{k=1}^n \sum_{t=1}^T z_{k,t} \right] - n\bar{v} \sqrt{T \log(\bar{v}nT)},\end{aligned}\quad (3.13)$$

where the last inequality holds via (3.12).

It remains to analyze the expectations on the right side of the inequality. First, note that

$$\begin{aligned}\mathbb{E} \left[\mathbf{1}\{E_{\text{GOOD}}\} \sum_{k \in [n]} R_k(\mathbf{v}) \right] &= \mathbb{E} \left[\sum_{k \in [n]} R_k(\mathbf{v}) \right] - \mathbb{E} \left[(1 - \mathbf{1}\{E_{\text{GOOD}}\}) \sum_{k \in [n]} R_k(\mathbf{v}) \right] \\ &\geq \mathbb{E} \left[\sum_{k \in [n]} R_k(\mathbf{v}) \right] - \frac{n\bar{v}T}{n\bar{v}T^2} = \mathbb{E} \left[\sum_{k \in [n]} R_k(\mathbf{v}) \right] - 1/T.\end{aligned}$$

The inequality holds due to the fact that $R_k(\mathbf{v}) \leq \bar{v}T$ as well as our bound on the probability that event (3.8) does not hold. Let $q_{k,t}$ be the unconditional probability that $\mu_{k,t} = 0$. Then

$$\begin{aligned}\mathbb{E} [R_k(\mathbf{v})] &= \sum_{t=1}^T \mathbb{E} [\mathbf{1}\{\mu_{k,t} = 0\} \mathbb{E}[y_k(v_{k,t})v_{k,t} | \mathcal{F}_{t-1}] + \mathbf{1}\{\mu_{k,t} \neq 0\} \rho_k] \\ &= \sum_{t=1}^T [q_{k,t} \mathbb{E}[y_k(v_k)v_k] + (1 - q_{k,t})\rho_k] \geq \sum_{t=1}^T \mathbb{E}[y_k(v_k)v_k] = \bar{W}_k(\mathbf{y}, F).\end{aligned}$$

The first inequality uses the conditional independence of $y_k(v_{k,t})v_{k,t}$ on $\mu_{k,t}$, as this is already determined by time $t - 1$. The inequality holds since $\mathbb{E}[y_k(v_k)v_k] \leq \rho_k$ according to our assumption on \mathbf{y} . Substituting the inequalities into (3.13), we obtain that

$$\text{WEL}_{\text{GPD}}(F) \geq \sum_{k=1}^n \bar{W}_k(\mathbf{y}, F) - \mathbb{E} \left[\sum_{k=1}^n \sum_{t=1}^T z_{k,t} \right] - n\bar{v} \sqrt{T \log(\bar{v}nT)} - 1/T. \quad (3.14)$$

Recall that

$$\mathbb{E} \left[\sum_{k=1}^n \sum_{t=1}^T z_{k,t} \right] = \sum_{k=1}^n \sum_{t=1}^T \mathbb{E} [z_{k,t}] = \sum_{k=1}^n \mathbb{E}[P_k] \leq \text{WEL}_{\text{GPD}}(F), \quad (3.15)$$

where P_k is the total expenditure of agent k , and this is upper bounded by the liquid value they obtain. Substituting into (3.14) and rearranging the terms, then noting that $\text{WEL}_{\text{GPD}}(F)$ is precisely the left-hand side of (3.2), we conclude that inequality (3.2) holds.

4 Individual Guarantee: Vanishing Regret for MBB Auctions

In this section, we supplement our aggregate welfare guarantees with bounds on the individual performance of Algorithm 1 when used in any auction that satisfies the monotone bang-for-buck (MBB) condition. We focus on a particular single agent k , henceforth called simply *the agent*. The agent faces an online bidding problem subject to the budget constraint. The action set consists of pacing multipliers $\mu \geq 0$ like in Algorithm 1. That is, the agents are restricted to linear values-to-bids policies without overbidding. We show that under mild conditions on the environment, the agent achieves regret bounds for utility maximization; in fact, we show that the algorithm achieves regret bounds for any combination of utility and value maximization.

4.1 Environment and Notation

We make no explicit assumptions on the individual behavior of the other agents. Their bidding strategies \mathbf{b}_{-k} can depend arbitrarily on the realized values and the observed history. This is desirable since the agents may be reluctant to follow a particular bidding algorithm, and even when they do, the aggregate behavior is not well-understood.

In each round t , let G_t be the joint distribution of value $v_{k,t}$ and other agents' bids $\mathbf{b}_{-k,t}$ given the history in the previous rounds. Thus, one can think of pair $(v_{k,t}, \mathbf{b}_{-k,t})$ as being drawn from a distribution G_t that could depend on the history up to time t . In particular, G_t could depend on the previous distributions G_1, \dots, G_{t-1} and the agent's past bids.

We can think of the sequence G_1, \dots, G_T as specifying the environment that the agent operates in. Our main guarantee in Theorem 4.8 considers an *adversarial environment*, i.e., an arbitrary sequence. We also consider important special cases, particularly the *stochastic environment* where G_t is the same in all rounds.

The dependence of the agent's problem on G_t is captured via the following notation. Let $V_t(\mu_{k,t})$ and $Z_t(\mu_{k,t})$ be, respectively, the agent's expected value and expected payment in round t for a particular multiplier $\mu_{k,t}$, where the expectation is taken over the distribution G_t . More formally, for a given auction defined by an allocation rule \mathbf{x} and a payment rule \mathbf{p} (see Section 2) and a given distribution G_t we have

$$Z_t(\mu_{k,t}) = \mathbb{E} \left[p_k(b_{k,t}, \mathbf{b}_{-k,t}) \right] \text{ and } V_t(\mu_{k,t}) = \mathbb{E} \left[v_{k,t} \cdot x_k(b_{k,t}, \mathbf{b}_{-k,t}) \right], \quad (4.1)$$

where the bid is $b_{k,t} = v_{k,t}/(1 + \mu_{k,t})$ and the expectations are over $(v_{k,t}, \mathbf{b}_{-k,t}) \sim G_t$. Under the same conventions, agent's expected (quasilinear) utility $U_t(\cdot)$ is then

$$U_t(\mu_{k,t}) = \mathbb{E} \left[v_{k,t} \cdot x_k(b_{k,t}, \mathbf{b}_{-k,t}) - p_k(b_{k,t}, \mathbf{b}_{-k,t}) \right] = V_t(\mu_{k,t}) - Z_t(\mu_{k,t}).$$

Note that, like G_t , the functions $V_t(\cdot)$, $Z_t(\cdot)$ and $U_t(\cdot)$ can depend on the observed history up to round t . Monotonicity of the auction allocation and payment rules implies that, for any fixed history, Z_t and V_t are monotonically non-increasing.

In what follows, we drop the dependence on the agent's index k from our notation. For each round t , we write $v_t = v_{k,t}$, $\mu_t = \mu_{k,t}$ for the agent's value and multiplier, and $b_t = v_t/(1 + \mu_t)$ for its bid. Also, we write $\rho = \rho_k$ and $\epsilon = \epsilon_k$.

4.2 Unified Objective and Pacing Regret

For a unified presentation of both utility- and value-optimization, we employ a unified objective: an arbitrary convex combination of the two. Specifically, the unified objective is defined as

$$\Phi^{\text{uni}}(\mu_1, \dots, \mu_T) := \sum_{t \in [\tau_k]} \gamma \cdot U_t(\mu_t) + (1 - \gamma) \cdot V_t(\mu_t), \quad (4.2)$$

where $\gamma \in [0, 1]$ is a fixed parameter and τ_k is the stopping time (i.e., the first time when the budget is not strictly positive). The t -th summand on the right-hand side of (4.2) is denoted $\Phi_t^{\text{uni}}(\mu_t)$.

Our regret bounds are relative to a non-standard benchmark: essentially, a sequence of pacing multipliers $\mu_1^*, \dots, \mu_T^* \in [0, \bar{\mu}]$ with expected spend $Z_t(\mu_t^*) = \rho$ in each round t . The formal definition needs to account for the possibility that $Z_t(0) < \rho$:

Definition 4.1. For each round t and any auction history up to round t , a *perfect pacing multiplier* is any $\mu_t^* \in [0, \bar{\mu}]$ such that $Z_t(\mu_t^*) = \rho$, or $\mu_t^* = 0$ if $Z_t(0) < \rho$. A *perfect pacing sequence* (for a given auction history up to round T) is a sequence $\mu^* = (\mu_1^*, \dots, \mu_T^*)$, where each μ_t^* is a perfect pacing multiplier.

Our assumptions (stated below) ensure that such μ^* exists, although in general it need not be unique. However, all perfect pacing sequences μ^* have the same unified objective $\Phi^{\text{uni}}(\mu^*)$.¹⁵

Thus, we treat $\Phi^{\text{uni}}(\mu^*)$ as a benchmark, and define *pacing regret* relative to this benchmark:

$$R_{\text{pace}}(T) := \mathbb{E} \left[\Phi^{\text{uni}}(\mu^*) - \Phi^{\text{uni}}(\mu_1, \dots, \mu_T) \right], \quad (4.3)$$

where μ^* is (any) perfect pacing sequence for a given auction history up to round T .

Remark 4.2. We emphasize that the benchmark $\Phi^{\text{uni}}(\mu^*)$ is defined with respect to the realized history of bids. As with most regret guarantees in the literature, the benchmark does not include the counterfactual impact of a change of bid by agent k in round t on future bids by other agents. Rather, our pacing regret guarantees apply to the benchmark of perfect pacing in hindsight assuming that other bidders do not change their bidding behavior in response.

Remark 4.3. A perfect pacing sequence μ^* is not necessarily best-in-hindsight. That is, fixing the entire history, μ^* does not necessarily optimize $\Phi^{\text{uni}}(\cdot)$ among all sequences $\nu \in [0, \bar{\mu}]^T$. In fact, μ^* may even be worse in terms of $\Phi^{\text{uni}}(\cdot)$ than the best fixed pacing multiplier. This is by design. Recall from Section 1.2 that one cannot obtain sublinear regret in an adversarial environment, even against the best fixed pacing multiplier, because of the spend-or-save dilemma [42, 13]. While inevitable, this situation may feel unsatisfying. One interpretation is that the standard benchmark of the best-in-hindsight multiplier is “too hard” in the adversarial environment, and a more “fair” alternative benchmark is needed to make progress and properly express the benefits of algorithms such as ours. The reason our benchmark admits vanishing regret is precisely that it gives up on solving the spend-or-save dilemma, and instead optimizes for each round separately.

4.3 Smoothness Assumptions

We make the following smoothness assumptions on the expenditure function Z_t :

Assumption 4.4. *There exists $\lambda \geq 0$ such that Z_t is λ -Lipschitz for each round t .*

Assumption 4.5. *There exists $\delta > 0$ such that $Z_t(0) \geq \delta$ for each round t . As our bounds will depend inversely on δ , we assume without loss of generality that $\delta \leq \rho$.¹⁶*

Note that these assumptions hold for any history up to round t .

Remark 4.6. We argue that these assumptions are mild in the context of advertising auctions. Indeed, the variation in impression types, click rate estimates, etc., and possibly also the randomness in the auction and/or the bidding algorithms would likely introduce some smoothness into the expected payment $Z_t(\cdot)$, and ensure that the maximum allowable bid would result in a non-trivial payment. In fact, one could eliminate the need for these assumptions by adding a small amount of noise to the auction allocation rule, such as by perturbing bids slightly, and/or a small but positive reserve price, at the cost of a similarly small loss of welfare. Alternatively, the assumptions hold when all agents use bounded pacing multipliers, the joint value profile distribution is sufficiently smooth (for Assumption 4.4),¹⁷ and with probability at least $\epsilon' > 0$, agent k has the highest value while some other agent bids at least $\delta' > 0$ (for Assumption 4.5).

¹⁵This follows by MBB. Fix the history up to some round t . Taking expectations on both sides of Eq. (2.2), one obtains $Z_t(\nu'_t) - Z_t(\nu_t) \geq \frac{1}{1+\nu_t} (V_t(\nu'_t) - V_t(\nu_t))$ for any two pacing multipliers $\nu_t > \nu'_t$. If they are *perfect* pacing multipliers for round t , then $Z_t(\nu_t) = Z_t(\nu'_t) = \rho$ and consequently $V_t(\nu_t) = V_t(\nu'_t)$. It follows that $\Phi_t^{\text{uni}}(\nu_t) = \Phi_t^{\text{uni}}(\nu'_t)$.

¹⁶We only require the weaker condition: $Z_t(0) \geq \delta \times \frac{V_t(0)}{\bar{v}}$ for all t . In particular, this allows $Z_t(0) = 0$ if $V_t(0) = 0$.

¹⁷If the pricing rule is nondecreasing in the agent's bid given the other bids (which is the typical case), Assumption 4.4 holds (for some $\lambda > 0$) the agent's conditional valuation given the other bids almost surely admits a density that is bounded pointwise by some absolute constant $\sigma > 0$. More generally, Assumption 4.4 holds for any bounded pricing rule if the joint valuation distribution has a Lipschitz density, since then the induced density on joint bids is nicely behaved as a function of the agent's multiplier.

We also assume that $\bar{\mu} \geq \bar{v}/\rho$, where $\bar{\mu}, \bar{v}$ are, resp., upper bounds on the largest feasible pacing multipliers and values. Given these assumptions, we can now prove (as promised above) that a perfect pacing sequence is well-defined.

Claim 4.7. *A perfect pacing multiplier μ_t^* exists, for any given round t and history up to this round.*

Proof. Note that $Z_t(\bar{\mu}) \leq \bar{v}/(1 + \bar{\mu}) \leq \rho$. The Lipschitzness and monotonicity of Z_t therefore imply that if there does not exist any μ such that $Z_t(\mu) = \rho$, then it must be that $Z_t(0) < \rho$ (and hence $Z_t(\mu) < \rho$ for all $\mu \in [0, \bar{\mu}]$). This latter case occurs if even bidding the true value v_t generates expected spend less than ρ . In this case we take $\mu_t^* = 0$, which corresponds to the most that the agent can bid without violating the mandate to not bid more than the value v_t . \square

4.4 Main Guarantees: Adversarial Environment

With all definitions in place, we are ready to state our main result of this section: an upper bound on pacing regret, which holds against an arbitrary auction environment and an arbitrary perfect pacing sequence. The guarantee applies uniformly to any mixture of value- and utility-maximization objective, without any dependence on the mixing parameter γ .

Theorem 4.8. *Consider a repeated auction that is individually rational (IR) and satisfies monotone bang-per-buck (MBB) property. Posit Assumptions 4.4 and 4.5. Suppose the path-length $P^* = \sum_{t=1}^{T-1} |\mu_{t+1}^* - \mu_t^*|$ is upper-bounded by some number P with probability 1. Fix parameter $\gamma \in [0, 1]$ in the unified objective. Consider Algorithm 1 with step-size $\epsilon \in (0, \bar{v}]$. Its pacing regret is*

$$R_{\text{pace}}(T) < O\left(\frac{\bar{v}}{\delta} \sqrt{2\lambda T \cdot \text{REG}_1} + \frac{\bar{v}\bar{\mu}}{\epsilon\rho}\right), \quad \text{where } \text{REG}_1 \triangleq \frac{P+1}{\epsilon} \cdot \bar{\mu}^2 + \epsilon T \cdot (\rho + \bar{v})^2. \quad (4.4)$$

Corollary 4.9. *In the setting of Theorem 4.8, assume that parameters $(\bar{v}, \bar{\mu}, \lambda, \delta)$ are absolute constants. The regret bounds simplify as follows:*

- (a) $R_{\text{pace}}(T) < O\left(\sqrt{T\left(\frac{P+1}{\epsilon} + \epsilon T\right)}\right).$
- (b) Taking step-size $\epsilon = 1/\sqrt{T}$ (not knowing P), we obtain $R_{\text{pace}}(T) < \sqrt{P+1} \cdot O\left(T^{3/4}\right).$
- (c) If P is known, taking $\epsilon = \sqrt{\frac{P+1}{T}}$ yields an improved bound, $R_{\text{pace}}(T) < (P+1)^{1/4} \cdot O\left(T^{3/4}\right).$

We emphasize that the environment, as expressed by distributions G_1, \dots, G_T , can change over time. The dependence on this change is summarized with a uniform upper bound P on path-length $P^* = \sum_{t=1}^{T-1} |\mu_{t+1}^* - \mu_t^*|$. We obtain a non-trivial guarantee for an arbitrary non-stationary environment with $P = o(T)$ if P is known, and $P = o(\sqrt{T})$ otherwise.

Path-length P^* is a rather *weak* notion of change. Indeed, distribution G_t can change a lot from one round to another while μ_t^* stays the same. Moreover, if distribution G_t changes arbitrarily from one round to another and stays the same afterwards, the path-length only increases once, by at most $\bar{\mu}$.¹⁸ Therefore, our guarantees go far beyond slight perturbations of the same stochastic environment.

Remark 4.10. This is the first non-trivial regret bound for budget-constrained online bidding in the adversarial environment.

¹⁸For concreteness: if the perfect pacing sequence has at most N changes from one round to another (e.g., because do does the sequence of distributions G_t), then Corollary 4.9(bc) holds with $P = \bar{\mu} N$.

Our guarantees extend beyond the original model: the marginal distribution of the agent's value and the auction itself can change over time, as long as the assumptions hold.

Extension 4.11. *Theorem 4.8 and Corollary 4.9 hold as written if the distribution from which the value profile \mathbf{v}_t is sampled and the auction itself can arbitrarily change from one round to another, possibly depending on the previous rounds.*

4.5 Regret Bounds for Stochastic Environment

Now let us consider the stochastic environment, *i.e.*, assume that the distribution $G_t = G$ does not change over time almost surely. Then neither do the functions $V_t, U_t, Z_t, \Phi_t^{\text{uni}}$, and there exists some $\nu^* \in [0, \bar{\mu}]$ (determined by G) that is a perfect pacing multiplier for all rounds t .

We immediately obtain a regret bound against ν^* (we state it formally for completeness). Let $\Phi_{\text{fix}}^{\text{uni}}(\mu) := \Phi^{\text{uni}}(\mu, \dots, \mu)$ be the unified objective when a fixed pacing multiplier μ is used in all rounds.

Corollary 4.12. *In the setting of Theorem 4.8, suppose distributions G_t do not change over time almost surely and assume that parameters $(\bar{v}, \bar{\mu}, \lambda, \delta)$ are absolute constants. Letting $\epsilon = 1/\sqrt{T}$ be the step-size,*

$$\Phi_{\text{fix}}^{\text{uni}}(\nu^*) - \mathbb{E} \left[\Phi^{\text{uni}}(\mu_1, \dots, \mu_T) \right] \leq O(T^{3/4}). \quad (4.5)$$

It is desirable to have a guarantee against the “best” pacing multiplier — one that maximizes $\Phi_{\text{fix}}^{\text{uni}}(\cdot)$ — and we prove that ν^* comes close for value-maximization. It is crucial for this result that we disallow overbidding (as discussed in Section 2), *i.e.*, only consider $\mu \geq 0$.

Corollary 4.13. *Consider the setting of Corollary 4.12 under value-maximization ($\Phi_t^{\text{uni}} = V_t$). Then*

$$\sup_{\mu \in [0, \bar{\mu}]} \Phi_{\text{fix}}^{\text{uni}}(\mu) - \mathbb{E} \left[\Phi^{\text{uni}}(\mu_1, \dots, \mu_T) \right] \leq O(T^{3/4}). \quad (4.6)$$

Remark 4.14. This is the first non-trivial regret bound for value-maximization in online bidding under budget, in any auction format. While the $T^{3/4}$ regret rate may be suboptimal, we emphasize that the goal of this paper is not (necessarily) to obtain optimal regret rates for a specific environment, but rather to provide a combination of non-trivial aggregate and individual guarantees.

Moreover, ν^* (nearly) maximizes $\Phi_{\text{fix}}^{\text{uni}}(\cdot)$ for utility-maximization in repeated second-price auctions [13], and Algorithm 1 is known to achieve \sqrt{T} regret rate in this setting [13, 17].

Corollary 4.15 ([13, 17]). *Consider the setting of Corollary 4.12 for utility-maximization ($\Phi_t^{\text{uni}} = U_t$) in repeated second-price auctions. Then*

$$\sup_{\mu \in [0, \bar{\mu}]} \Phi_{\text{fix}}^{\text{uni}}(\mu) - \mathbb{E} \left[\Phi^{\text{uni}}(\mu_1, \dots, \mu_T) \right] \leq O(\sqrt{T}). \quad (4.7)$$

Remark 4.16. Balseiro and Gur [13] prove this regret bound under additional convexity assumptions. Balseiro et al. [17] prove it without convexity assumptions, for a class of algorithms which (implicitly) includes Algorithm 1 as a special case with Euclidean regularizer. The benchmark $\Phi_{\text{fix}}^{\text{uni}}(\nu^*)$ is even stronger in this setting: it matches the best bidding policy (*i.e.*, mapping from value to bid), which may not be linear [13].

Staying on repeated second-price auctions, we saw that ν^* maximizes $\Phi_{\text{fix}}^{\text{uni}}(\cdot)$ for both utility- and value-maximization. Consequently, it does so for any mixture in (4.2). So, we again obtain Eq. (4.6).

Corollary 4.17. *In the setting of Corollary 4.12 for repeated second-price auctions, Eq. (4.6) holds.*

4.6 Discussion: Desiderata for Individual Guarantees

Let's take a step back and discuss what types of results are desirable for individual guarantees. The stochastic environment is commonly understood as a "minimal" desiderata, as a relatively simple special case for which vanishing regret is feasible. A common motivation is that a "small" bidder enters a "large" market in which the bidding dynamics has already converged to a (possibly randomized) stationary state. Of course, the downside is that realistic environments are neither stationary nor guaranteed to converge.

The other extreme is to analyze an adversarial environment, like we do in Theorem 4.8, possibly with improved guarantees for some "nice" special cases. In particular, path-length P^* can quantify convergence to a "well-behaved" environment in which the perfect pacing multiplier does not change over time, so that our guarantee is strong when the convergence is sufficiently fast. However, such "black-box" guarantees for budget-constrained bidders tend to be weak or vacuous in the worst case.

A *third* approach would be to explicitly take advantage of the fact that all agents are controlled by bidding algorithms with particular properties, *e.g.*, by instances of the same bidding algorithm. In particular, if all *other* agents use our algorithm with step size ϵ' (and parameters $(\bar{v}, \bar{\mu}, \lambda, \delta)$ are absolute constants), one can show that the perfect pacing multiplier changes by at most $O(\epsilon')$ in each round, so that the path-length P^* can be upper-bounded as $O(\epsilon'T)$. Plugging this back into Corollary 4.9(b), we obtain vanishing regret with step-size $\epsilon' = o(1/\sqrt{T})$.

Corollary 4.18. *Consider a repeated auction that is individually rational (IR) and satisfies monotone bang-per-buck (MBB) property. Assume that parameters $\bar{v}, \bar{\mu}$ are absolute constants. Suppose all agents use Algorithm 1, so that agent k has step-size $\epsilon_k = 1/\sqrt{T}$ and all other agents have step-size $\epsilon' = f(T)/\sqrt{T}$, where $f(T) \rightarrow 0$. Posit that Assumptions 4.4 and 4.5 are satisfied (see Theorem 4.6) for some absolute-constant λ and δ . Then for agent k , the path-length P^* is at most $O(\epsilon'T)$ with probability 1, so the right-hand side of (4.4) is at most $O\left(T \cdot \sqrt{f(T)}\right)$.*

Note that this analysis yields vanishing regret only for a particular agent k , whereas for all other agents the guarantee that comes out of our analysis is vacuous. And if all agents use the same parameter $\epsilon = 1/\sqrt{T}$, then we can only guarantee path-length $P^* = O(\sqrt{T})$, which does not suffice to guarantee vanishing regret. We leave open the question of whether it is possible to obtain vanishing regret simultaneously for all agents (with any bidding algorithms).

4.7 Stochastic Gradient Descent (SGD) interpretation

To prove Theorem 4.8 it will be helpful to interpret Algorithm 1 as using *stochastic gradient descent* (SGD), a standard algorithm in online convex optimization (see Appendix D). Here, SGD uses pacing multipliers μ as actions, and optimizes an appropriately-defined artificial objective $H_t(\cdot)$ in each round t , where

$$H_t(\mu) = \rho \cdot \mu - \int_0^\mu Z_t(x) dx. \quad (4.8)$$

The per-round spend z_t provides a stochastic signal that in expectation equals the gradient of H_t :

$$H'_t(\mu_t) = \rho - Z_t(\mu_t) = \rho - \mathbb{E}_{G_t}[z_t(\mu_t)]. \quad (4.9)$$

The SGD machinery applies because the function H_t is convex and $(\bar{v} + \rho)$ -Lipschitz (see Appendix E.2).

This interpretation provides a concrete intuition for what Algorithm 1 actually does: for better or worse, it optimizes the aggregate artificial objective $\sum_{t \in [T]} H_t(\mu_t)$, whose optimum is precisely $\sum_t H_t(\mu_t^*)$. The analysis simplifies accordingly, as we can directly invoke the known guarantees for SGD and conclude that the artificial objective comes close to this target optimum. However, outside the well-studied special case of second-price auctions, this objective does not appear to have *a priori* meaning. Much of the proof of

Theorem 4.8 thus involves relating this artificial objective to the objectives of interest, *namely utility and value maximization*, in rather general settings.

Remark 4.19. The fact that Algorithm 1 optimizes an artificial objective whose optimum is attained by the perfect pacing sequence suggests the latter as an appropriate benchmark for analyzing this algorithm.

A similar (but technically different) interpretation appeared in [13] in the context of second-price auctions. Their interpretation relies on Lagrangian duality and does not appear to extend beyond second-price auctions.

Remark 4.20. Our analysis only uses two properties of the dynamics. First, that the dynamics obtains low regret with respect to an adversarially generated sequence of convex functions H_t (we inherit this from SGD). Second, since the regret analysis does not account for early stopping from exhausting the budget, we require that the dynamics do not terminate too early (as given by Lemma 2.2) so as to bound any possible loss on such periods. The proof of Theorem 4.8 generalizes almost immediately to any algorithm satisfying these properties.

4.8 Proof of Theorem 4.8

Our analysis proceeds in three steps. First, we invoke the SGD machinery to show that the sequence of chosen multipliers (μ_t) approximates the target optimum $\sum_t H_t(\mu_t^*)$. Second, we relate the H_t functions to the per-round payments to show that the expected payment is not too far from ρ in each round. Finally, we use the MBB property of the auctions to bound the total regret in either objective.

The following technical lemma will enable us to meaningfully relate the Z_t and H_t functions whose proof is deferred to the Appendix:

Lemma 4.21. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing, λ -Lipschitz function such that $f(0) = 0$. Let $R = \int_0^x f(y)dy$ for some $x \in \mathbb{R}$. Then $|f(x)| \leq \sqrt{2\lambda R}$.*

To begin, observe that Algorithm 1 is equivalent to running (projected) stochastic gradient descent on the sequence of H_t functions from (4.8), where the maximum norm of the gradient is at most \bar{v} using the assumption that the payment is at most the bid, which in turn is at most the maximum possible value. Note that these functions are indeed convex and Lipschitz (see Appendix E.2 for a proof) and indeed, $H'_t(\mu) = \rho - Z_t(\mu) = \rho - \mathbb{E}_{G_t}[z_t(\mu)]$ by construction of H_t . We now wish to bound the sequence regret, $\sum_t \mathbb{E}[H_t(\mu_t) - H_t(\mu^*)]$. While the dynamics may end before time T , we may upper bound the regret by continuing to time T , as each term is nonnegative. Standard analysis of SGD (see Theorem D.2 and Appendix E.2 for details) then implies

$$\text{REG}_1 \triangleq \sum_{t=1}^T \mathbb{E} [H_t(\mu_t) - H_t(\mu^*)] \leq O \left(\frac{P+1}{\epsilon} \cdot \bar{\mu}^2 + \epsilon T \cdot (\rho + \bar{v})^2 \right). \quad (4.10)$$

In the remainder of the proof, we will translate (4.10) into a bound on pacing regret. It suffices to show the regret bound in (4.4) for pacing regret with respect to values and utility individually; the regret guarantee then translates to the unified objective for any choice of $\gamma \in [0, 1]$. It will be technically convenient to perform the analysis for values first and then derive the same bound for utility as a simple variation.

From Lemma 2.2, we may assume that the dynamics continue until time T without stopping at the cost of $O(\bar{v}\bar{\mu}\sqrt{T}/\rho)$ loss in value which we will account for at the end. Define the function $W_t(\mu)$ by

$$W_t(\mu) = \begin{cases} V_t(\mu) & \text{if } Z_t(\mu) < \rho \\ V_t(\mu) \times \frac{\rho}{Z_t(\mu)} & \text{if } Z_t(\mu) \geq \rho. \end{cases}$$

Observe that $W_t(\mu) \leq V_t(\mu)$ by construction; it can be viewed as (approximately) the per-round value obtained by bidding with pacing multiplier μ — if the environment at time t persisted for all rounds —

until the budget is expected to terminate due to overspending. We further observe that $V_t(\mu_t) - V(\mu_t^*) \geq W_t(\mu_t) - W_t(\mu_t^*)$ for all t . To see this, note that $V(\mu_t^*) = W_t(\mu_t^*)$ since $Z_t(\mu_t^*) \leq \rho$, and moreover, the first term on the right side equals the first term of the left side, which is non-negative, with some scaling factor at most 1. Therefore,

$$\sum_{t=1}^T (V_t(\mu_t^*) - V(\mu_t)) \leq \sum_{t=1}^T (W_t(\mu_t^*) - W_t(\mu_t)), \quad (4.11)$$

so it will suffice to show that the right side is not too large.

For any round t , we now consider two cases depending on the value of μ_t and derive a lower bound on the difference in W_t values:

Case 1: $\mu_t < \mu_t^*$. In this case we must have $\mu_t^* > 0$ and hence $Z_t(\mu_t) \geq Z_t(\mu_t^*) = \rho$, and therefore

$$W_t(\mu_t) = V_t(\mu_t) \times \frac{\rho}{Z_t(\mu_t)} \geq W_t(\mu_t^*) \times \frac{\rho}{Z_t(\mu_t)}.$$

By rearranging and adding $W_t(\mu_t^*)$ to each side of the last inequality, it follows that

$$W_t(\mu_t^*) - W_t(\mu_t) \leq \frac{Z_t(\mu_t) - \rho}{Z_t(\mu_t)} \cdot W_t(\mu_t^*) \leq (Z_t(\mu_t) - \rho) \cdot \frac{\bar{v}}{\rho}, \quad (4.12)$$

where we use the fact that $Z_t(\mu_t) \geq \rho$ as well as $W_t(\mu_t^*) \leq \bar{v}$.

Case 2: $\mu_t \geq \mu_t^*$. In this case, we have $Z_t(\mu_t) \leq Z_t(\mu_t^*) \leq \rho$ by assumption, and hence $W_t(\mu_t^*) = V_t(\mu_t^*)$ and $W_t(\mu_t) = V_t(\mu_t)$. We now claim that the MBB property of the auction implies

$$\frac{W_t(\mu_t^*)}{W_t(\mu_t)} = \frac{V_t(\mu_t^*)}{V_t(\mu_t)} \leq \frac{Z_t(\mu_t^*)}{Z_t(\mu_t)}. \quad (4.13)$$

To prove (4.13), define $\gamma = Z_t(\mu_t)/V_t(\mu_t)$ and note that since the auction is individually rational, it must be that an agent bidding $v_t/(1 + \mu_t)$ pays at most $v_t/(1 + \mu_t)$ per unit allocated, and hence $\gamma \leq \frac{1}{1 + \mu_t}$. If we write $Z_t(\mu_t | v_t)$ for the expected value of $Z_t(\mu_t)$ conditional on the realization of v_t (recalling that v_t can be correlated with the competing bids), and similarly for $V_t(\mu_t | v_t)$, then by MBB and linearity of expectation

$$Z_t(\mu_t^* | v_t) - Z_t(\mu_t | v_t) \geq \frac{1}{1 + \mu_t} (V_t(\mu_t^* | v_t) - V_t(\mu_t | v_t))$$

where we used the fact that the bid given v_t and μ_t is $v_t/(1 + \mu_t)$, and $V_t(\cdot | v_t)$ is precisely v_t times the agent's expected allocation. Taking expectations over v_t then implies $Z_t(\mu_t^*) - Z_t(\mu_t) \geq \frac{1}{1 + \mu_t} (V_t(\mu_t^*) - V_t(\mu_t))$. Rearranging and recalling that $Z_t(\mu_t) = \gamma V_t(\mu_t)$, we conclude

$$Z_t(\mu_t^*) \geq \gamma V_t(\mu_t) + \frac{1}{1 + \mu_t} (V_t(\mu_t^*) - V_t(\mu_t)) \geq \gamma V_t(\mu_t^*)$$

which implies (4.13). A rearrangement of (4.13) then implies

$$W_t(\mu_t^*) - W_t(\mu_t) \leq \frac{V_t(\mu_t^*)}{Z_t(\mu_t^*)} \times (Z_t(\mu_t^*) - Z_t(\mu_t)) \leq \frac{\bar{v}}{\min\{\rho, \delta\}} (Z_t(\mu_t^*) - Z_t(\mu_t)). \quad (4.14)$$

The second inequality comes from considering the cases that $\mu_t^* = 0$ (and applying Assumption 4.5) and $\mu_t^* > 0$.

In either case, (4.12) and (4.14) implies that (recalling the assumption $\delta \leq \rho$)

$$W_t(\mu_t^*) - W_t(\mu_t) \leq \frac{\bar{v}}{\delta} |Z_t(\mu_t^*) - Z_t(\mu_t)|. \quad (4.15)$$

We now relate this bound to the H_t functions. Observe that

$$H_t(\mu_t) - H_t(\mu_t^*) = \int_{\mu_t^*}^{\mu_t} [\rho - Z_t(x)] dx = \int_0^{\mu_t - \mu_t^*} [\rho - Z_t(\mu_t^* + x)] dx,$$

and moreover that the integrand $f(x) = \rho - Z_t(\mu_t^* + x)$ is increasing and λ -Lipschitz by the assumption on Z_t . In general, $f(0) \geq 0$ (with equality if $Z_t(\mu_t^*) = \rho$). If we consider $g(x) = f(x) - f(0)$ and apply Lemma 4.21, we obtain

$$|Z_t(\mu_t) - Z_t(\mu_t^*)| = |f(\mu_t - \mu_t^*) - f(0)| \leq \sqrt{2\lambda(H_t(\mu_t) - H_t(\mu_t^*) - (\mu_t - \mu_t^*)(\rho - Z_t(\mu_t^*)))}.$$

Note that if $\mu_t < \mu_t^*$, then $\rho = Z_t(\mu_t^*)$ so the last subtracted term is zero, while if $\mu_t \geq \mu_t^*$, then the last subtracted term is nonnegative. In either case, we deduce that

$$|Z_t(\mu_t) - Z_t(\mu_t^*)| \leq \sqrt{2\lambda(H_t(\mu_t) - H_t(\mu_t^*))} \quad (4.16)$$

By combining these inequalities, we obtain:

$$\begin{aligned} \mathbb{E} \left[\sum_{t=1}^T V_t(\mu_t^*) - V(\mu_t) \right] &\leq \mathbb{E} \left[\sum_{t=1}^T W_t(\mu_t^*) - W_t(\mu_t) \right] && \text{(by (4.11))} \\ &\leq \frac{\bar{v}}{\delta} \mathbb{E} \left[\sum_{t=1}^T |Z_t(\mu_t^*) - Z_t(\mu_t)| \right] && \text{(by (4.15))} \\ &\leq \sqrt{2\lambda} \frac{\bar{v}}{\delta} \mathbb{E} \left[\sum_{t=1}^T \sqrt{H_t(\mu_t) - H_t(\mu_t^*)} \right] && \text{(by (4.16))} \\ &\leq \sqrt{2\lambda T} \frac{\bar{v}}{\delta} \mathbb{E} \left[\sqrt{\sum_{t=1}^T H_t(\mu_t) - H_t(\mu_t^*)} \right] && \text{(by Cauchy-Schwarz)} \\ &\leq \sqrt{2\lambda T} \frac{\bar{v}}{\delta} \sqrt{\mathbb{E} \left[\sum_{t=1}^T H_t(\mu_t) - H_t(\mu_t^*) \right]} && \text{(by Jensen's inequality)} \\ &\leq \frac{\bar{v}}{\delta} \sqrt{2\lambda T \cdot \text{REG}_1} && \text{(by (4.10)).} \end{aligned}$$

Accounting for the $O(\bar{\mu}\bar{v}/\epsilon\rho)$ loss from possibly terminating early using Lemma 2.2 establishes the claimed regret bound of (4.4) for regret with respect to value maximization.

We now show how to adapt the preceding analysis to bound the pacing regret for utility-maximization as a nearly direct consequence. By inspection of the proof above, it suffices to show that the following direct analogue of (4.15) holds for utilities:

$$U_t(\mu_t^*) - U_t(\mu_t) \leq \frac{\bar{v}}{\delta} |Z_t(\mu_t^*) - Z_t(\mu_t)|. \quad (4.17)$$

Given (4.17), the remainder of the argument is exactly the same. To see that (4.17) holds, we again consider cases based on the value of μ_t :

Case 1: $\mu_t < \mu_t^*$. In this case, we immediately have

$$\begin{aligned} U_t(\mu_t^*) - U_t(\mu_t) &= \underbrace{(V_t(\mu_t^*) - V_t(\mu_t))}_{\leq 0} + (Z_t(\mu_t) - Z_t(\mu_t^*)) \\ &\leq \underbrace{Z_t(\mu_t) - Z_t(\mu_t^*)}_{\geq 0} = |Z_t(\mu_t) - Z_t(\mu_t^*)|. \end{aligned}$$

The first inequality follows from the fact obtained valuations are non-increasing in μ , while the last equality holds because expenditures are also non-increasing in μ .

Case 2: $\mu_t \geq \mu_t^*$. In this case, we can instead bound

$$U_t(\mu_t^*) - U_t(\mu_t) = (V_t(\mu_t^*) - V_t(\mu_t)) + \underbrace{(Z_t(\mu_t) - Z_t(\mu_t^*))}_{\leq 0} \leq V_t(\mu_t^*) - V_t(\mu_t).$$

This holds again because $Z_t(\mu_t) \leq Z_t(\mu_t^*)$ in this case by monotonicity of expenditure. But in this case, we recall that by construction in the proof of Theorem 4.8 that $V_t(\mu_t^*) - V_t(\mu_t) \leq W_t(\mu_t^*) - W_t(\mu_t)$. So, the previous display combined with the bound for W_t in (4.14) immediately implies

$$U_t(\mu_t^*) - U_t(\mu_t) \leq \frac{\bar{v}}{\delta} |Z_t(\mu_t^*) - Z_t(\mu_t)|.$$

Putting these two cases establishes (4.17) (recall the assumption that $\bar{v} \geq 1$) and thus completes the proof for pacing regret for utilities, and therefore the proof of Theorem 4.8.

5 Aggregate and Individual Guarantees for Other Algorithms

It is natural to ask whether other online bidding algorithms achieve similar liquid-welfare guarantees, or in fact a similar *combination* of aggregate and individual guarantees. A natural no-regret condition alone cannot suffice for liquid welfare: we provide a simple example in Appendix B.2. In this section, we extend our analyses and guarantees to several online bidding algorithms that replace the *stochastic gradient descent* (SGD) update step in Algorithm 1 step with an update step from another well-known algorithm for online convex optimization (OCO). Specifically, we invoke *optimistic gradient descent* (OGD) [52, 53, 48, 44], *optimistic mirror descent* (OMD) [25, 52, 53], and *optimistic follow-the-regularized-leader* (OFTRL) [52].

The new algorithms are defined as follows. Throughout, recall that $\epsilon_k > 0$ is the step-size, and let $\tilde{\nabla}_{k,t} = \rho_k - z_{k,t}$ denote the estimated gradient in round t . In PacingOMD and PacingOFTRL, in each round t , an algorithm is given an estimate $M_{k,t}$ of the *next* gradient. Thus:

PacingOGD Replace Eq. (2.4) in Algorithm 1 with an OGD step: for some fixed $\epsilon'_k \in [-\epsilon_k/2, 0]$,

$$\mu_{k,t+1} \leftarrow P_{[0,\bar{\mu}]} \left(\mu_{k,t} - \epsilon_k \cdot \tilde{\nabla}_{k,t} - \epsilon'_k \cdot \tilde{\nabla}_{k,t-1} \right).$$

PacingOMD Replace Eq. (2.4) in Algorithm 1 with an OMD step: initializing $\hat{\mu}_{k,1} = 0$,

$$\begin{aligned} \hat{\mu}_{k,t+1} &\leftarrow P_{[0,\bar{\mu}]} \left(\hat{\mu}_{k,t} - \epsilon_k \cdot \tilde{\nabla}_{k,t} \right), \\ \mu_{k,t+1} &\leftarrow P_{[0,\bar{\mu}]} \left(\hat{\mu}_{k,t+1} - \epsilon_k \cdot M_{k,t} \right). \end{aligned}$$

PacingOFTRL Replace Eq. (2.4) in Algorithm 1 with an OFTRL step with Euclidean regularizer:

$$\mu_{k,t+1} \leftarrow P_{[0,\bar{\mu}]} \left(-\epsilon_k \left(M_{k,t} + \sum_{s \in [t]} \tilde{\nabla}_{k,s} \right) \right).$$

In line with this notation, one can refer to Algorithm 1 as PacingSGD.

Remark 5.1. In PacingOGD, the case $\epsilon' = 0$ corresponds to the original algorithm (Algorithm 1). The case $\epsilon'_k = -\epsilon_k/2$ corresponds to OGD [30, 28], another OCO algorithm that obtains faster convergence in certain learning-in-games settings. Further work [48, 44] extends OGD to an arbitrary $\epsilon'_k \in [-\epsilon/2, 0]$.

Remark 5.2. While the original versions of OMD and OFTRL with Euclidean regularizer are defined for $\mathcal{K} \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$ and use Bregman divergences, we invoke a simpler equivalent formulation for $d = 1$.

Remark 5.3. OMD and OFTRL achieve the same regret rates as (we invoke for) SGD for the default choice of the next-gradient predictors, $M_{k,t} = \tilde{\nabla}_{k,t}$. They achieve *better* regret rates when the gradient sequences are predictable, in some formal sense, and this is the key advantage of these algorithms [25, 52].

Our analysis is consistent with this intuition. Our liquid-welfare guarantees for PacingOMD and PacingOFTRL apply to arbitrary $M_{k,t}$'s, whereas our regret bounds for these algorithms are for $M_{k,t} = \tilde{\nabla}_{k,t}$. Moreover, our regret analysis extends to any other choice of $M_{k,t}$'s that yields same or better regret bound for the underlying OCO algorithms. Better OCO regret bounds would translate into improved regret bounds for online bidding (but we do not spell this out explicitly).

Regret bounds. All results in Section 4 carry over to PacingOGD and PacingOMD (with $M_{k,t} = \tilde{\nabla}_{k,t}$). This is because OGD and OMD with $M_{k,t} = \tilde{\nabla}_{k,t}$ satisfy the same OCO regret bound as SGD (Theorem D.2), and this regret bound plugs into Section 4 same way as it does for Algorithm 1. Moreover, we obtain an analog of Lemma 2.2, showing that the algorithms do not run out of budget too early (see Lemma F.1). These are the only facts about an algorithm invoked by our analysis in Section 4.

For PacingOFTRL with $M_{k,t} = \tilde{\nabla}_{k,t}$, we only obtain regret bounds for the stochastic environment (Corollaries 4.12 and 4.13 in Section 4.5). We obtain an analog of Lemma 2.2 (see Lemma F.1). Then we invoke the regret bound for OFTRL (Theorem D.2, in the special case when all f_t 's are the same), and observe that it plugs into the analysis in Section 4 for the stochastic environment and implies Corollary 4.12.

Since the results from Section 4 carry over as stated, we do not restate them here. The only change is a slightly stronger condition on the problem parameters: $\bar{\mu} \geq 2\bar{v}/\rho_k + 1$, which stems from Lemma F.1. The relevant background on OCO algorithms is summarized in Appendix D.

Liquid welfare. We obtain essentially the same liquid-welfare guarantee as in Section 3, under a mild additional assumption that $\max_k(\epsilon_k/\rho_k) = o(1)$.

Theorem 5.4. *Fix any core auction and any distribution F over agent value profiles. Suppose that each agent k uses PacingOGD, PacingOMD, or PacingOFTRL. (Different agents can use different algorithms and/or different step-sizes ϵ_k .) The next-gradient predictors $M_{k,t}$ can be arbitrary, as long as $|M_{k,t}| \leq \bar{v}$. Write \mathbf{x} for the corresponding allocation sequence rule. Then for any allocation rule $\mathbf{y}: [0, \bar{v}]^n \rightarrow X$ and $c = 4\bar{v} \cdot \max_k \sqrt{\epsilon_k/\rho_k}$ we have*

$$W(\mathbf{x}, F) \geq (1 - c) \cdot \frac{\bar{W}(\mathbf{y}, F)}{2} - O\left(n\bar{v}\sqrt{T \log(\bar{v}nT)}\right). \quad (5.1)$$

5.1 The General Algorithm Property: Proof Sketch of Theorem 5.4

We obtain Theorem 5.4 by extending the analysis in Section 3. In this analysis, the dependence on the algorithm's details is essentially captured by the following property: the amount spent over a sequence of rounds with non-zero pacing is lower-bounded by a quantity that depends only on the starting and ending bids, but otherwise not the learning path in between (Lemma 3.8, which is itself a corollary of Eq. (3.3)). In what follows, we define a more general property, c -event-feasibility, that captures all three algorithms in Theorem 5.4, and prove Eq. (5.1) when all agents use algorithms that are c -event-feasible.

Definition 5.5 (c -event-feasible). Consider a variant of Algorithm 1 in which the multiplier update in Eq. (2.4) is replaced with some other update rule which computes $\mu_{k,t} \in [0, \bar{\mu}]$. The algorithm is called c -event-feasible, $c \in (0, 1)$ if for each round t before it runs out of budget, there exists an event $\mathcal{E}_{k,t}$ (deter-

mined by the auction history up to this round) such that the following two properties hold:

$$\sum_{t \in [T]} x_{k,t} v_{k,t} \geq \sum_{t \in [T]} (x_{k,t} v_{k,t} - z_{k,t}) \cdot \mathbf{1}(\mathcal{E}_{k,t}) + (1 - c) \sum_{t \in [T]} \rho_k \cdot \mathbf{1}(\mathcal{E}_{t,k}^c). \quad (5.2)$$

$$\text{Event } \mathcal{E}_{k,t} \text{ implies that } b_{k,t} = \frac{v_{k,t}}{1 + \mu_{k,t}} \geq (1 - c) \cdot v_{k,t}. \quad (5.3)$$

Algorithm 1 is event-feasible by Lemma 3.8, with $\mathcal{E}_{k,t} = \{\mu_{k,t} = 0\}$ and $c = 0$. The definition instantiates to the other algorithms as follows:

Lemma 5.6. *PacingOGD, PacingOMD, and PacingOFTRL are c -event feasible with $c = 4\bar{v}\sqrt{\epsilon_k/\rho_k}$. The events $\mathcal{E}_{k,t}$ are $\{\mu_{k,t} \leq c\}$ for PacingOGD and PacingOFTRL, and $\{\hat{\mu}_{k,t} \leq c\}$ for PacingOMD.*

When each agents uses an algorithm from Definition 5.5, liquid welfare is as follows.

Theorem 5.7. *Fix any core auction and any distribution F over agent value profiles. Suppose each agent uses a c -event-feasible algorithm, for some $c \in (0, 1)$ (possibly a different algorithm for different agents). Let \mathbf{x} be the corresponding allocation sequence rule. Then for any allocation rule $\mathbf{y}: [0, \bar{v}]^n \rightarrow X$,*

$$W(\mathbf{x}, F) \geq (1 - c) \cdot \frac{\bar{W}(\mathbf{y}, F)}{2} - O\left(n\bar{v}\sqrt{T \log(\bar{v}nT)}\right). \quad (5.4)$$

The proof follows that of Theorem 3.4 with relatively minor modifications. We lower bound the liquid welfare using a similar charging argument, but tailored to these new events. The agents satisfying $\mathcal{E}_{k,t}$ at any time t are approximately bidding their value, and so the core auction properties again ensure that any valuation loss can be charged to prices. Lemma 5.6 and Theorem 5.7 are proved in Appendix F.

6 Numerical Evaluation

In this Section we complement our theoretical findings with a numerical simulation study of Algorithm 1. We consider the *multi-player* environment with simultaneous learning by competing bidders. Our simulations are semi-synthetic, based on campaign and bidding data collected from the Bing Advertising platform.

We focus on regret relative to the standard benchmark: the best fixed pacing multiplier in hindsight. Recall that, relative to this benchmark, vanishing regret is achievable for a stochastic environment but is provably impossible for an adversarial environment. The achievability of vanishing regret in a multi-player environment remains unknown and is currently an open question.

This gap motivates our simulation study, which focuses on this multi-player environment with simultaneous learning. Our simulations suggest that simultaneous execution of Algorithm 1 yields vanishing regret, with regret rate $O(T^\alpha)$ for $\alpha \leq 3/5$. We also compare the performance of Algorithm 1 with some other approaches to online bid optimization in the literature, analyzing both empirical regret rate and liquid welfare outcomes on our semi-synthetic dataset.

6.1 Data and Simulation Description

Our data consists of campaign and auction data collected over a 7-day period in April 2022. The dataset contains daily budget targets and other campaign parameters (such as maximum bid, when specified) for campaigns from a North American advertising segment. It also contains, for a random subsample of $N \approx 2.4M$ auction instances over this period, the list of participating campaigns, click probability predictions for each participant, realized bids from each participant, and auction outcomes (including auction winner and payment). All monetary measurements are expressed in normalized units.

Given this dataset, we simulate a joint execution of Algorithm 1 as follows. For any campaign that participates in fewer than $\theta = 1000$ auction instances, we maintain the bid that they originally placed in the dataset. For all other campaigns, we generate new bids by simulating the execution of an online bidding algorithm for each participant of each auction instance. The target spend rate for campaign k is calculated by taking T_k to be the average number of per-day auction instances for each campaign, and per-impression values are taken to be proportional to platform estimated click rates.¹⁹

Given the bids for all auction participants, we simulate the auction outcome using estimated click rates and a simplified auction rule. We take there to be only a single winner in each auction instance, corresponding to the highest total bid. Payments are calculated according to a specified payment rule; we consider both first-price and second-price payment rules in our experiments.

6.2 Bidding Algorithms

Here we describe the online bidding algorithms evaluated in our simulations.²⁰ All algorithms adjust a bid multiplier that is constrained to lie in $[0, \bar{\mu}]$. All algorithms invoke some gradient-based technique from the literature, treating $G_{k,t} = \rho_k - z_{k,t}$ as a gradient, for each agent k and round t . (For simplicity, we use the technique’s name to label the algorithm.) All meta-parameters are tuned via grid search. We provide a warm start by initializing each campaign’s multiplier to be proportional to $1/\rho_k$.

- **Stochastic Gradient Descent (SGD):** This is Algorithm 1, the online bidding method described in Section 2 and analyzed theoretically in this paper. The multiplier is updated according to the rule $\mu_{k,t+1} = \mu_t - \epsilon_k G_{k,t}$. The learning rate ϵ_k for each agent k is chosen to be proportional to $1/\sqrt{T_k}$ with a tuneable coefficient as meta-parameter.
- **Optimistic Gradient Descent (OGD):** an instantiation of PacingOGD where the multiplier update is $\mu_{k,t+1} = \mu_t - \epsilon_k (2G_{k,t} - G_{k,t-1})$. The learning rate is again chosen to be proportional to $1/\sqrt{T_k}$, with coefficient treated as a tunable meta-parameter.
- **Adaptive Moment Estimation (Adam):** a self-tuning gradient descent method in which the update in each round is a weighted average of prior gradients, scaled by a weighted ℓ_2 -norm of prior gradients. Learning rate is treated as a tunable meta-parameter.
- **Multiplicative Updating (MU):** a bid update method proposed in [19], in which $\mu_{k,t}$ is scaled up or down by a fixed multiplicative factor $(1 + \epsilon)$ depending on whether the previous-round’s spend was above or below the target spend ρ_k . The factor ϵ is a tunable meta-parameter.

6.3 Regret Analysis

Following each execution of the simulation, we calculate the total regret for each bidding agent by determining the single best linear policy multiplier in hindsight. This is done via binary search, up to an error of 10^{-9} . We run $n = 100$ simulations for each scenario, each using a random subsample of 95% of auction instances. This provides an estimate of the regret of each agent k .

For second-price auctions, we consider both utility maximization and value maximization as objectives for the purpose of calculating regret, recalling that our benchmark of the best fixed pacing multiplier in

¹⁹In all visualizations and reported welfare metrics, exact values and counts are renormalized to prevent data leakage.

²⁰In addition to Algorithm 1, we have chosen as comparators two “advanced” versions of SGD (incorporating, resp., the “optimistic correction” $G_{k,t} - G_{k,t-1}$ and weighted gradient-averaging) and a multiplicative approach from an important early work on simultaneous learning in auctions [19]. A more detailed empirical investigation of other variations is left for future studies.

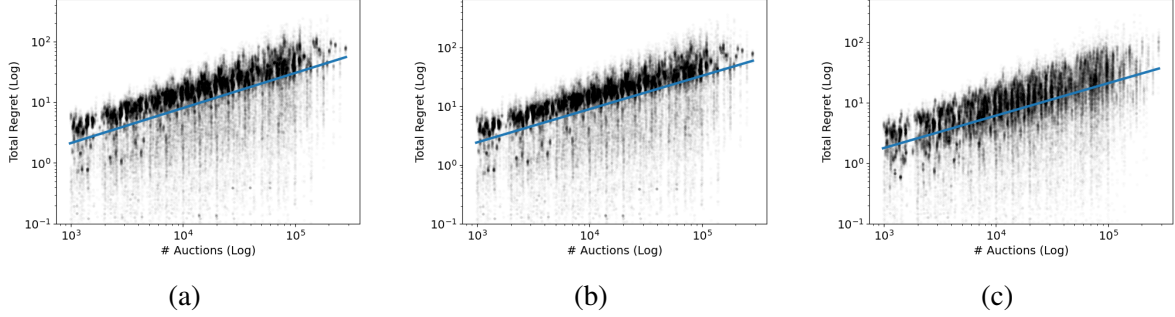


Figure 1: Regressing on observed regret in log-log scale to estimate regret rate for simultaneous multi-agent use of Algorithm 1. Each dot plots the total regret of a simulated campaign against the number of auction instances in which it participates. Standard errors in the linear regression are included but not visible. Plots are for (a) the utility objective in second-price auctions, (b) the value objective in second-price auctions, and (c) the value objective in first-price auctions.

hindsight is optimal for both objectives. For first-price auctions we consider the value maximization objective, since the best fixed pacing multiplier in hindsight is optimal for value maximization but not for utility maximization.

To estimate the rate of regret growth for each agent over time, we simulate longer time periods by iterating over the collection of auction instances in our dataset multiple times. We simulated K iterations over the dataset, where $K \in \{1, 2, 5, 10, 20, 50, 100\}$. To reduce the potential impact of cyclic data patterns, each of the K iterations includes only a subsample of the impression auctions drawn independently at random, where each impression is included in each iteration with probability $\beta = 1/2$. We then calculate regret for each agent, as described above, as the number of auction instances grows.

6.4 Results

Figure 1 illustrates the total regret obtained in our simulation of Algorithm 1 over all simulated campaigns. Each point on the plot corresponds to a combination of campaign (i.e., bidding agent representing an advertiser) and choice of K , and plots normalized auction count against total regret metric. Further visualizations of per-campaign regret evolution are provided in Appendix G. Hypothesizing a regret rate of the form $O(T^\alpha)$, we estimate α using log-log regression. Figure 1 illustrates the outcome of log-log regression for all campaigns and simulation treatments, separately for first-price and second-price auction formats. We obtain an estimated slope of $\alpha = 0.573$ for second-price auctions (standard error 0.0026) and $\alpha = 0.540$ for first-price auctions (standard error 0.0028).

We repeat this estimation exercise for all algorithms in our comparison set. Figure 2 plots the resulting regression for first-price and second-price auctions, with parameter estimates listed in Table 1. Table 1 also lists the total liquid welfare generated over the simulated campaigns for each algorithm and auction format. For liquid welfare comparisons, we ran at $n = 1000$ simulations to test for statistical significance in comparisons between algorithms. The top entries in each category, at a statistical significance of at least $p = 0.05$, are displayed in bold.

We find that SGD and OGD have comparable performance with no statistically significant differences in terms of regret or liquid welfare. Relative to them, Adam has an improved regret rate for second-price auctions and a slightly worse regret rate for first-price auctions, though with higher absolute regret and lower liquid welfare for the auction counts covered in our dataset. This finding aligns with the intuition that advanced self-tuning methods like Adam are most effective when the number of iterations is very large, whereas the generally slower convergence time that comes with self-tuning may come at a loss for campaigns

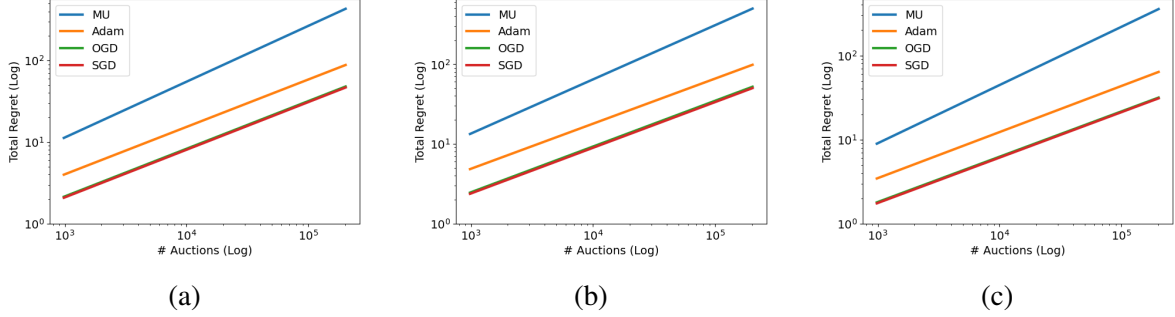


Figure 2: Visual comparison of regressed regret, in log-log scale, as a function of the number of auctions, for different learning algorithms. Plots are for (a) the utility objective in second-price auctions, (b) the value objective in second-price auctions, and (c) the value objective in first-price auctions. Note that OGD is partially obscured due to proximity to SGD.

that activate less frequently. Multiplicative updating noticeably under-performs the other methods in both aggregate liquid welfare and regret. We emphasize that, in all cases, hyperparameters for each algorithm were tuned individually and separately for both first-price and second-price auctions.

Algorithm	SPA			FPA	
	Regret Rate (Utility)	Regret Rate (Value)	Liquid Welfare	Regret Rate (Value)	Liquid Welfare
SGD	0.584 (0.003)	0.573 (0.003)	22.84 (2.25)	0.540 (0.003)	23.42 (2.24)
OGD	0.584 (0.003)	0.574 (0.003)	22.84 (2.25)	0.538 (0.003)	23.41 (2.24)
Adam	0.581 (0.003)	0.565 (0.003)	22.65 (2.22)	0.547 (0.004)	23.15 (2.19)
MU	0.684 (0.003)	0.679 (0.003)	22.60 (2.22)	0.692 (0.003)	23.14 (2.22)

Table 1: Comparison of algorithms under first- and second-price auction formats (resp., FPA and SPA). Estimated (regressed) regret rates are provided (i.e., the estimated α in regret rate of the form $O(T^\alpha)$) as well as the average liquid welfare (in normalized units) over all simulated campaigns. Standard errors provided in parentheses. Top algorithms in each category (up to statistical significance at $p = 0.05$) are represented in bold.

7 Conclusions and Open Questions

The purpose of this paper is to simultaneously guarantee high aggregate welfare and low individual regret for budget-constrained online bidding in a repeated auction, without relying on convergence to an equilibrium. We establish these guarantees for a wide class of auction rules, arbitrarily correlated private values, bandit feedback, and several natural budget-pacing algorithms. Our individual guarantees hold for both utility- and value-maximization.

On a more technical level, the main result is a guarantee on the expected total liquid welfare achieved over multiple auction rounds. This approximation guarantee matches the best possible for a pure Nash equilibrium in a static truthful auction, but is the first to hold without requiring convergence to equilibrium. Our individual guarantees are of independent interest: we obtain the first non-trivial regret bounds for budget-constrained online bidding for (i) the adversarial environment and (ii) value-maximization in the stochastic environment. The result for the adversarial environment holds against non-standard benchmark (perfect pacing sequence) which side-steps impossibility results from prior work and has been fruitfully applied to general bandits-with-knapsacks problems in subsequent work [59, 20]. The modularity of our techniques enables extending both aggregate and individual guarantees from Algorithm 1 to several other algorithms.²¹

Let us emphasize several open questions:

- (1). Improving the approximation factor against our liquid-welfare benchmark appears unlikely (given the negative result for static truthful auctions). Could one identify a different liquid-welfare benchmark — possibly weaker but hopefully more fair — which would allow for a better approximation factor or, ideally, a regret bound with no approximation factor at all?
- (2). The key goal for individual guarantees is vanishing regret for all agents at once, particularly under *self-play* (when all agents use the same algorithm). Previously this goal seemed hopeless for budget-constrained bidders, given the strong impossibility results known for adversarial environments. We provide a plausible framework in which this goal might be achievable: regret bounds relative to the perfect pacing sequence. Recall that our specific guarantee falls just short: Corollary 4.18 requires path-length $P^* = o(\sqrt{T})$, but we can only guarantee $P^* = O(\sqrt{T})$ when all agents run our algorithm. Bridging this gap may be within reach, possibly by leveraging the “optimism” in PacingOMD (see Section 5). Vanishing regret under self-play may also be achievable against the *standard* benchmark: best pacing multiplier in hindsight. Indeed, our semi-synthetic numerical simulations exhibit this for Algorithm 1 and several other bidding algorithms.
- (3). While optimizing the guarantees on individual regret is not our goal per se, it would be desirable to improve upon the $T^{3/4}$ regret rates in our theoretical results and/or compete against non-linear bidding policies for value-maximization. Improved results for the stochastic environment are especially interesting when the same algorithm enjoys non-trivial guarantees for the adversarial environment.

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²¹This addresses an open question from the preliminary conference version of this paper [35].

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A Examples of Auctions

A.1 Single-slot and Multiple-slot Ad Auctions

- *Single-Slot Ad Auctions:* A round corresponds to an ad impression, with a single ad slot available. An impression has a type $\theta_t \in \Theta$; this type might describe, for example, a keyword being searched for, user demographics, intent prediction, etc. In each round t the impression type θ_t is drawn independently from a distribution over types. Each agent k has a fixed value function $v_k: \Theta \rightarrow \mathbb{R}_{\geq 0}$ that maps each impression type to a value for being displayed. The value profile in round t is then $\mathbf{v}_t = (v_1(\theta_t), \dots, v_n(\theta_t))$. A single ad can be displayed each round. We then interpret $x_{k,t} \in [0, 1]$ as the probability that advertiser k is allocated the ad slot, and an allocation profile \mathbf{x}_t is feasible if and only if $\sum_k x_{k,t} \leq 1$.

- *Multiple-Slot Pay-per-click Ad Auctions:* We can generalize the previous example to allow multiple ad slots, using a polymatroid formulation due to [37]. Impression types and agent values are as before, but we now think of there as being $m \geq 1$ slots available in each round, with click rates $1 \geq \alpha_1 \geq \dots \geq \alpha_m \geq 0$. If ad k is placed in slot i , the value to agent k is $v_k(\theta_t) \times \alpha_i$. That is, we think of $v_k(\theta_t)$ as representing both the advertiser-specific click rate (which can depend on the impression type) as well as the advertiser's value for a click. In this case we would take $v_{k,t} = v_k(\theta_t)$ and $x_{k,t} = \alpha_i$. The set of feasible allocation profiles $X \subseteq [0, 1]^n$ is then a polymatroid: $\mathbf{x}_t \in X$ if and only if, for each $\ell \leq m$ and each $S \subseteq [n]$ with $|S| = \ell$, $\sum_{k \in S} x_{k,t} \leq \sum_{i=1}^{\ell} \alpha_i$.

A.2 Core and MBB Auctions

Let us list three notable examples of core auctions. They also satisfy the monotone bang-per-buck (MBB) property, as defined in Eq. (2.2).

- *First-Price Auction* chooses a welfare-maximizing allocation $\mathbf{x}(\mathbf{b}) \in \arg \max_{\mathbf{x}} \{\sum_k b_k x_k\}$. Each agent k then pays her bid for the allocation obtained: $p_k(\mathbf{b}) = b_k x_k(\mathbf{b})$.

To see this auction is MBB, first note that $x_k(\mathbf{b})$ is nondecreasing in b_k . Then observe that if \mathbf{b}_{-k} is fixed, and $b_k < b'_k$, then setting $\mathbf{b}' = (b'_k, \mathbf{b}_{-k})$, we clearly have

$$p_k(\mathbf{b}') - p_k(\mathbf{b}) = b'_k x_k(\mathbf{b}') - b_k x_k(\mathbf{b}) \geq b_k (x_k(\mathbf{b}') - x_k(\mathbf{b})),$$

as needed by the definition of MBB.

- *Second-Price Auction for Single-Slot Ad Auctions.* The second-price auction chooses a welfare-maximizing allocation $\mathbf{x}(\mathbf{b}) \in \arg \max_{\mathbf{x}} \{\sum_k b_k x_k\}$, where feasible allocations \mathbf{x} satisfy $\sum_k x_k \leq 1$. Then each agent k pays x_k times the second-highest bid.

To see this auction is MBB, fix \mathbf{b}_{-k} and let b^* denote the largest single bid in \mathbf{b}_{-k} . Note that if $b_k \geq b^*$ then the second-highest bid is b^* so $p_k(\mathbf{b}) = b^* x_k(\mathbf{b})$, and if $b_k < b^*$ then $x_k(\mathbf{b}) = p_k(\mathbf{b}) = 0$ so $p_k(\mathbf{b}) = b^* x_k(\mathbf{b})$. Thus, for any $b_k < b'_k$, we have

$$p_k(\mathbf{b}') - p_k(\mathbf{b}) = b^* (x_k(\mathbf{b}') - x_k(\mathbf{b})) \geq b_k (x_k(\mathbf{b}') - x_k(\mathbf{b}))$$

as needed by the definition of MBB, where the inequality follows because if $b_k < b^*$ then the left- and right-hand sides are both zero.

- *Generalized Second-Price (GSP) Auction for Multi-Slot Ad Auctions.* In the GSP auction, slots are allocated greedily by the respective bid, and each agent pays a price per unit equal to the next-highest bid. Formally, given the bids \mathbf{b} , we let π be a permutation of the agents so that $b_{\pi(1)} \geq b_{\pi(2)} \geq \dots \geq$

$b_{\pi(n)}$. That is, $\pi(1)$ is the highest-bidding agent, then $\pi(2)$, etc. Agent $\pi(k)$ is then allocated to slot k for each $k \leq m$. That is, $x_{\pi(k)} = \alpha_k$ for each $k \leq m$ and $x_{\pi(k)} = 0$ for each $k > m$. The payments are set so that $p_{\pi(k)} = x_{\pi(k)} b_{\pi(k+1)}$ for all $k < n$, and $p_{\pi(n)} = 0$.²²

We note that GSP is a core auction; see Appendix A.3 for a proof.

To see that the MBB property holds, let $b_k < b'_k$ and suppose that with bid profiles \mathbf{b} , agent k is assigned the j th slot, while under \mathbf{b}' , the agent is assigned the $\ell \leq j$ th slot (where we allow the value of a slot to be 0 if the agent is not in the top m bids). If $j = \ell$, then it is easy to see that the price and allocation of agent k is unchanged, trivially confirming that the MBB condition holds. If instead $j > \ell$, then it is easy to see that $b_{\pi'(\ell+1)} \geq b_k$, with π' the permutation under \mathbf{b}' , and hence

$$p_k(\mathbf{b}') - p_k(\mathbf{b}) \geq \alpha_\ell b_k - \alpha_\ell b_{\pi(j+1)} \geq b_k(\alpha_\ell - \alpha_j) = b_k(x_k(\mathbf{b}') - x_k(\mathbf{b})).$$

This implies that the MBB property holds for single-slot auctions.

A.3 Proof: Generalized Second Price is a Core Auction

In this section we show that the GSP auction for sponsored search allocation with separable click rates is a core auction. Recall the definition of a GSP auction. There are $m \geq 1$ slots with click rates $1 \geq \alpha_1 \geq \dots \geq \alpha_m \geq 0$. There are n bidders, each bidder placing a bid $b_i \geq 0$. We will reindex agents in order of bid, so that $b_1 \geq b_2 \geq \dots \geq b_n$. Without loss of generality we will assume $m = n$ (by adding extra bidders with bid 0 or extra slots with click rate 0), and we will define $b_{n+1} = \alpha_{m+1} = 0$ for convenience.

In the GSP auction, slots are allocated greedily by bid, and each agent pays a price per unit equal to the next-highest bid. That is, agent i receives slot i for a declared value of $b_i \alpha_i$, and pays $b_{i+1} \alpha_i$.

We claim that the GSP auction is a core auction. First, since $b_{i+1} \leq b_i$ for all i , we have that $b_{i+1} \alpha_i \leq b_i \alpha_i$, and hence each bidder pays at most her declared welfare for the allocation received.

It remains to show the second property of a core auction. Choose any subset of bidders $S \subseteq [n]$. The allocation \mathbf{y} to agents in S that maximizes declared welfare is the one that allocates greedily in index order. More formally, for each $i \in S$, let $\sigma(i)$ be 1 plus the number of elements of S with index less than i . For example, if $S = \{2, 6, 7\}$, then $\sigma(2) = 1$, $\sigma(6) = 2$, and $\sigma(7) = 3$. Then the declared-welfare-maximizing allocation \mathbf{y} to agents in S is such that $y_i = \alpha_{\sigma(i)}$ for each i , for a total declared welfare of $\sum_{i \in S} b_i \alpha_{\sigma(i)}$. The core auction property on subset of bidders S therefore reduces to showing that

$$\sum_{i \notin S} b_{i+1} \alpha_i + \sum_{i \in S} b_i \alpha_i \geq \sum_{i \in S} b_i \alpha_{\sigma(i)}. \quad (\text{A.1})$$

To establish inequality (A.1), we first note that

$$\begin{aligned} \sum_{i \in S} b_i \alpha_{\sigma(i)} - \sum_{i \in S} b_i \alpha_i &= \sum_{i \in S} b_i (\alpha_{\sigma(i)} - \alpha_i) \\ &= \sum_{i \in S} \sum_{j=\sigma(i)}^{i-1} b_i (\alpha_j - \alpha_{j+1}) \\ &\leq \sum_{i \in S} \sum_{j=\sigma(i)}^{i-1} b_{j+1} (\alpha_j - \alpha_{j+1}). \end{aligned}$$

This final double summation contains an instance of the term $b_{j+1}(\alpha_j - \alpha_{j+1})$ for each $i \in S$ such that $\sigma(i) \leq j < i$. But for each j and each i such that $\sigma(i) \leq j < i$ (i.e., such that the “new” allocation to agent

²²Note that $p_{\pi(k)} = 0$ whenever $x_{\pi(k)} = 0$.

i under \mathbf{y} is slot j or better, and the “old” allocation is worse than slot j), there must be some $k \leq j$ such that $k \notin S$. The number of such i is therefore at most the number of agents at index j or less that are not in S . More precisely, observe that $j \geq |\{i \in S : i \leq j\}| + |\{i \in S : \sigma(i) \leq j < i\}|$. This holds as $\sigma(i) \leq i$ for all i and is injective, so the sets on the right hand side are evidently disjoint and σ maps each such element to a unique index in $[j]$. But $j = |\{i \in S : i \leq j\}| + |\{k \notin S : k \leq j\}|$, so cancelling terms shows that $|\{k \notin S : k \leq j\}| \geq |\{i \in S : \sigma(i) \leq j < i\}|$. Thus, by rearranging the order of summation, we have

$$\begin{aligned} \sum_{i \in S} \sum_{j=\sigma(i)}^{i-1} b_{j+1}(\alpha_j - \alpha_{j+1}) &\leq \sum_{j=1}^m b_{j+1}(\alpha_j - \alpha_{j+1}) \times |\{k \leq j : k \notin S\}| \\ &= \sum_{k \notin S} \sum_{j \geq k} b_{j+1}(\alpha_j - \alpha_{j+1}) \\ &\leq \sum_{k \notin S} \sum_{j \geq k} b_{k+1}(\alpha_j - \alpha_{j+1}) \\ &\leq \sum_{k \notin S} b_{k+1} \alpha_k \end{aligned}$$

where the final inequality is a telescoping sum. We therefore conclude that

$$\sum_{i \in S} b_i \alpha_{\sigma(i)} - \sum_{i \in S} b_i \alpha_i \leq \sum_{k \notin S} b_{k+1} \alpha_k$$

and rearranging yields the desired inequality (A.1).

B Additional Discussions

B.1 Interpreting Liquid Welfare as Compensating Variation

Compensating variation (CV) is a measure of welfare change. It refers to an amount of money an agent would need to be given in order to reach their original utility after some change in a market, typically a change in prices. For a textbook discussion, see, for example, Chapter 5 of [51].

For example, suppose that an agent’s utility can be written as $U(w, x)$, where w represents the agent’s wealth (in money), x is some allocation of non-monetary goods, and U is non-decreasing in w . Suppose there is a default outcome (w_0, x_0) for the agent, denoting their wealth and allocation prior to some proposed change. Following some change in the marketplace, the agent’s outcome shifts to (w_1, x_1) . The CV of this change, for this agent, is the minimum monetary transfer p such that

$$U(w_1 + p, x_1) = U(w_0, x_0),$$

if such a p exists. More generally, to allow for discontinuities in the utility function, we define the CV as the supremum of transfers for which the agent would have lower utility:

$$\text{CV} := \sup_p \{U(w_1 + p, x_1) < U(w_0, x_0)\}.$$

We claim that liquid welfare can be interpreted as (the negation of the) compensating variation in our setting of a budget-constrained agent with budget B and valuation function v over a space X of goods that includes a null outcome \emptyset with $v(\emptyset) = 0$.

To see this, suppose the agent's utility function is $U(w, x) = w + v(x)$ if $w \geq 0$, and $U(w, x) = -\infty$ if $w < 0$. We will take the default outcome to be $(w_0, x_0) = (B, \emptyset)$: no allocation and a wealth of B . The alternative outcome is $(w_1, x_1) = (B, x)$, an allocation $x \in X$ and no change in wealth. If $v(x) \leq B$, then

$$U(B - v(x), x) = B - v(x) + v(x) = B = U(B, 0)$$

so the compensating variation is $-v(x)$. If $v(x) > B$ we have

$$U(B + p, x) = -\infty < U(B, 0)$$

for all $p < -B$ and

$$U(B + p, x) = B + p + v(x) > B = U(B, 0)$$

for all $p \geq -B > -v(x)$, and hence $CV = -B$. We conclude that the compensating variation is precisely $-\min\{B, v(x)\}$, the negation of the liquid welfare of allocation x .

B.2 Vanishing Regret Does Not Imply Approximately Optimal Liquid Welfare

In this work we construct bidding algorithms that simultaneously achieve low individual regret and an approximation to the optimal aggregate liquid welfare. This combination is reminiscent of similar results from the analysis of smooth games, which include many auction games. No-regret learning algorithms converge (in distribution) to coarse correlated equilibria (CCE), and for smooth games it is known that the allocations obtained at CCE approximately optimize the aggregate welfare [18, 55, 57]. Thus, for auction games that satisfy smoothness conditions, achieving low regret directly implies (on a per-instance basis) an aggregate welfare approximation.

These results do not directly apply in our case: our class of games includes non-smooth games, and our welfare metric is different. But one might still wonder whether a similar direct implication applies. In this section we present an example showing that this direct implication does not hold in our setting with budgets and liquid welfare, even for single-item second-price auctions. We provide an auction environment and construct a coarse correlated equilibrium for the agents. The fact that agents employ bidding strategies that forms a coarse correlated equilibrium means, in particular, that all agents achieve zero regret. Nevertheless, in our example, the allocation that results from this bidding equilibrium has an unbounded approximation factor with respect to expected liquid welfare.

Proposition B.1. *There exists an instance of a second-price auction for a single good and two agents, described by a distribution over valuations, and a coarse correlated equilibrium of the auction (i.e., a pair of bidding strategies under which each agent has regret zero) such that the resulting expected liquid welfare is arbitrarily small compared to the optimal liquid welfare.*

Proof. The per-round auction in our example is a second-price auction for a single good. There are two agents. The distribution F over value profiles is such that $(v_1, v_2) = (2, 1)$ with probability 1. The target per-round spend rates for the agents are $(\rho_1, \rho_2) = (1/(1 + \bar{\mu}), 1)$, where $\bar{\mu}$ is some arbitrarily large constant independent of T .

Consider the following bidding strategies for the agents. Agent 1 bids value 2 for all periods and agent 2 bids 0 for all periods. Under this strategy profile, agent 1 receives all the items over the T rounds, and both agents pay nothing.

Under these strategies, agent 1 has zero regret as she obtains the maximum possible value. Agent 2 likewise has zero regret, since no choice of bid less than v_2 can cause her to win in any round. Note that the agents would still have zero regret if their objective were changed to maximizing value minus (any scalar multiple $\lambda \in [0, 1]$ times) payments.

The liquid welfare of this equilibrium is $T/(\bar{\mu} + 1)$, the total budget of agent 1. However, allocating all goods to agent 2 achieves a liquid welfare of T . Since $\bar{\mu}$ is an arbitrarily large constant, this approximation factor is unbounded. □

B.3 Linear Pacing and Advertiser-Feasible Bidding Strategies

In our analysis we restricted our attention to bidding strategies that are *linear* in the following sense. In each round t an agent k first chooses a pacing factor $\mu_{k,t} \geq 0$, then $v_{k,t}$ is revealed and the agent bids $b_{k,t} = v_{k,t}/(\mu_{k,t} + 1)$. The important restriction is that $\mu_{k,t}$ is chosen independently of $v_{k,t}$. This choice of $\mu_{k,t}$ can therefore be interpreted as a linear mapping from $v_{k,t}$ to $b_{k,t}$.

When the underlying auction is truthful, this restriction to linear policies is known to be without loss. However, for non-truthful auctions (such as a first-price auction) a value-maximizing or utility-maximizing agent might be able to strictly improve their outcome with a non-linear mapping from value to bid. For example, in a first-price auction where the highest competing bid is known to be exactly 1, an agent with value 2 and an agent with value 3 would both optimize their quasi-linear utility by bidding (slightly more than) 1 even though this is not implementable with a linear bidding strategy.

One motivation for our restriction to linear policies is that they capture bidding strategies that are implementable by an *external* agent (i.e., an advertiser or third-party bid optimizer) who does not have visibility into the precise value estimates of the platform, such as click-rate estimates.

We now make this intuition more precise by adding click rates to our auction model. We will associate each round t with a potential ad impression to be shown to a user. The auction in round t determines which ad will be shown; the user may or may not click on the ad. Each agent k has a value v_k that is obtained only if their advertisement is clicked. The advertising platform has access to an estimated click rate $c_{k,t} \in [0, 1]$ that describes the likelihood that the user will click on agent k 's advertisement if shown. Thus the expected value to agent k of winning the auction in round t is $v_k c_{k,t}$. We will write $v_{k,t} = v_k c_{k,t}$.

In each round, the agent can place a bid $\beta_{k,t}$ that is interpreted as a willingness to pay per click. Once all agents have placed bids, the mechanism multiplies these by the corresponding click rate estimates to determine the effective bids for winning the auction. These are denoted $b_{k,t} = \beta_{k,t} c_{k,t}$. The auction mechanism in round t is then resolved using the effective bids $b_{k,t}$.

We note that if the click rates $c_{k,t}$ are visible to the agents then this formulation is equivalent to our model from Section 2 as it simply expresses the values $v_{k,t}$ and the bids $b_{k,t}$ in a different way. However, for an advertiser that is external to the platform, the bid $\beta_{k,t}$ must be placed without observing the realization of the estimated click rate $c_{k,t}$. Equivalently, the estimated click rate $c_{k,t}$ is realized after the bid $\beta_{k,t}$ is fixed. We can therefore write $\mu_{k,t} = \frac{v_k}{\beta_{k,t}} - 1$ which is independent of $c_{k,t}$. We then have that for every possible realization of $v_{k,t}$,

$$b_{k,t} = \beta_{k,t} c_{k,t} = \frac{1}{\mu_{k,t} + 1} v_k c_{k,t} = \frac{1}{\mu_{k,t} + 1} v_{k,t}$$

and hence the advertiser's effective bidding strategy will be linear.

C Omitted Proofs from Section 3: Aggregate Guarantees

C.1 Motivating ex ante liquid welfare

Our definition of ex ante liquid welfare assumes that the same allocation rule y is used in every round. We now show that this is without loss of generality. The following lemma shows that given any allocation sequence rule there is a single-round allocation rule with the same ex ante liquid welfare.

Lemma C.1. *Let $\tilde{y} : [0, \bar{v}]^{nT} \rightarrow X^T$ be an allocation sequence rule that takes in the entire sequence $\mathbf{v}_1, \dots, \mathbf{v}_T$ and allocates $\tilde{y}_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T)$ units to agent k at time t . Then there exists a (single-round) allocation rule $y : [0, \bar{v}]^n \rightarrow X$ such that*

$$\begin{aligned} \tilde{W}(\tilde{y}, F) &\triangleq \sum_{k=1}^n \min \left\{ B_k, \mathbb{E}_{\mathbf{v}_1, \dots, \mathbf{v}_T \sim F} \left[\sum_{t=1}^T \tilde{y}_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T) v_{k,t} \right] \right\} \\ &= \sum_{k=1}^n T \cdot \min \left\{ \rho_k, \mathbb{E}_{\mathbf{v} \sim F} [y_k(\mathbf{v}) v_k] \right\} = \overline{W}(\mathbf{y}, F). \end{aligned}$$

Proof. For each t , by slightly abusing notation, we define an allocation rule $\hat{y}_t : [0, \bar{v}]^n \rightarrow [0, 1]^n$ by

$$\hat{y}_{k,t}(\mathbf{v}_t) \triangleq \mathbb{E}_{\mathbf{v}_{-t} \in F^{T-1}} [y_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T) | \mathbf{v}_t],$$

Note that this is a feasible allocation rule as the set of feasible allocations is convex and closed. We have

$$\begin{aligned} \mathbb{E}_{\mathbf{v}_1, \dots, \mathbf{v}_T \sim F} [y_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T) \cdot v_{k,t}] &= \mathbb{E}_{\mathbf{v}_t} \left[\mathbb{E}_{\mathbf{v}_{-t}} [y_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T) \cdot v_{k,t} | \mathbf{v}_t] \right] \\ &= \mathbb{E}_{\mathbf{v}_t} [\hat{y}_{k,t}(\mathbf{v}_t) \cdot v_{k,t}] = \mathbb{E}_{\mathbf{v} \sim F} [\hat{y}_{k,t}(\mathbf{v}) \cdot v_k]. \end{aligned}$$

Now we define the allocation rule \tilde{y} by setting $\tilde{y}_k = \frac{1}{T} \sum_{t=1}^T \hat{y}_{k,t}$ for each $k \in [n]$, which is again feasible because the set of feasible allocations is convex. By the linearity of the expectations operator, we have

$$\begin{aligned} \mathbb{E}_{\mathbf{v}_1, \dots, \mathbf{v}_T \sim F} \left[\sum_{t=1}^T y_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T) \cdot v_{k,t} \right] &= \sum_{t=1}^T \mathbb{E}_{\mathbf{v}_1, \dots, \mathbf{v}_T \sim F} [y_{k,t}(\mathbf{v}_1, \dots, \mathbf{v}_T) \cdot v_{k,t}] \\ &= \sum_{t=1}^T \mathbb{E}_{\mathbf{v} \sim F} [\hat{y}_{k,t}(\mathbf{v}) \cdot v_k] = \mathbb{E}_{\mathbf{v} \sim F} \left[\sum_{t=1}^T \hat{y}_{k,t}(\mathbf{v}) \cdot v_k \right] = T \cdot \mathbb{E}_{\mathbf{v} \sim F} [\tilde{y}_k(\mathbf{v}) \cdot v_k]. \quad \square \end{aligned}$$

C.2 Proof of Lemma 3.8

The assumption that $\mu_{k,t_2} \neq \bar{\mu}$ means that the agent participates in all periods of $[t_1, t_2]$. Moreover, if $t_2 = t_1 + 1$, then (3.4) is trivial as the third term on the right hand of side of inequality (3.4) is zero and $z_{k,t_1} \geq 0$. Therefore, we may assume $t_2 \geq t_1 + 2$.

By the definition of an epoch, there is no negative projection in the dynamics on the epoch until possibly time t_2 . I.e., $\mu_{k,t} > 0$ for all $t_1 \leq t < t_2$. The pacing recurrence condition implies that

$$\begin{aligned} 0 &< \mu_{k,t_2-1} = P_{[0, \bar{\mu}]}(\mu_{k,t_2-2} + \epsilon(z_{k,t_2-2} - \rho_k)) \leq \mu_{k,t_2-2} + \epsilon(z_{k,t_2-2} - \rho_k) \\ &\leq \epsilon \sum_{t=t_1}^{t_2-2} (z_{k,t} - \rho_k) = \epsilon \left(\sum_{t=t_1}^{t_2-2} z_{k,t} \right) - \epsilon(t_2 - t_1 - 1)\rho_k. \end{aligned}$$

The first and second inequalities follow because the multipliers are positive during an epoch. The third inequality follows from applying the second inequality repeatedly. Note that the inequality holds even if there is a positive projection during the epoch (i.e., if $\mu_{k,t} = \bar{\mu}$ for some $t \in [t_1, t_2]$).

Thus, the expenditure of agent k on this epoch is at least $\sum_{t=t_1}^{t_2-2} z_{k,t} \geq (t_2 - t_1 - 1)\rho_k$. Let us now consider the *value* obtained by the agent on this epoch. Because agents never overbid in the pacing algorithm and because payments are always lower than the value in a core auction, the value obtained by the agent on the epoch is at least the expenditure, which we just lower bounded. To get a slight sharpening of this, note that on the first period of the epoch, the agent actually receives $x_{k,t_1}v_{k,t_1}$ value and pays z_{k,t_1} , which we know is at most $x_{k,t_1}v_{k,t_1}$ from the first property of core auctions and the no overbidding condition. Therefore, we can trade z_{k,t_1} expenditure for $x_{k,t_1}v_{k,t_1}$ value. It follows from the above bound and this observation that

$$\sum_{t=t_1}^{t_2-1} x_{k,t}v_{k,t} \geq x_{k,t_1}v_{k,t_1} - z_{k,t_1} + \sum_{t=t_1}^{t_2-2} z_{k,t} \geq x_{k,t_1}v_{k,t_1} - z_{k,t_1} + (t_2 - t_1 - 1)\rho_k.$$

C.3 Proof of Lemma 3.9

Let $\Delta_t = X_t Y_t + (1 - X_t)\rho - (\mathbb{E}[X_t Y_t + (1 - X_t)\rho | \mathcal{F}_{t-1}])$. Clearly, the sequence Δ_t forms a \mathcal{F}_t -martingale difference sequence by construction. Moreover, we observe that

$$\mathbb{E}[X_t Y_t + (1 - X_t)\rho | \mathcal{F}_{t-1}] = X_t \mathbb{E}[Y_t | \mathcal{F}_{t-1}] + (1 - X_t)\rho = X_t \mathbb{E}[Y_t] + (1 - X_t)\rho,$$

where we use the facts that X_t is \mathcal{F}_{t-1} -measurable and Y_t is independent of \mathcal{F}_{t-1} . It follows that $\Delta_t \in [-\mathbb{E}[Y_t], \bar{v} - \mathbb{E}[Y_t]]$. As an immediate consequence of the Azuma-Hoeffding inequality, we obtain

$$\Pr \left(\sum_{t=1}^T \Delta_t \geq \theta \right) \leq \exp \left(\frac{-2\theta^2}{T\bar{v}^2} \right). \quad (\text{C.1})$$

But observe that

$$\begin{aligned} \left\{ \sum_{t=1}^T \Delta_t \geq \theta \right\} &= \left\{ \sum_{t=1}^T [X_t Y_t + (1 - X_t)\rho] \geq \theta + \sum_{t=1}^T [X_t \mathbb{E}[Y_t] + (1 - X_t)\rho] \right\} \\ &\supseteq \left\{ \sum_{t=1}^T [X_t Y_t + (1 - X_t)\rho] \geq \theta + T \cdot \rho \right\}, \end{aligned}$$

using the assumption $\mathbb{E}[Y_t] \leq \rho$. This inclusion together with (C.1) yields (3.5).

D Online Convex Optimization: Regret Bounds from Prior Work

Recall that we establishing individual guarantees through a reduction to *online convex optimization* (OCO). This section summarizes the relevant background and guarantees from prior work on OCO.

The guarantees for Algorithm 1 invoke a reduction to *stochastic gradient descent* (SGD), a very standard OCO algorithm. The additional guarantees in Section 5 invoke several other gradient-based OCO algorithms: *optimistic gradient descent* (OGD) [52, 53, 48, 44], *optimistic mirror descent* (OMD) [52, 53], and *optimistic follow-the-regularized-leader* (OFTRL) [52] with Euclidean regularizer.

Algorithm 2: Online Convex Optimization with gradient estimates

- 1 **Environment Parameters:** convex set $\mathcal{K} \subset \mathbb{R}$, time horizon T .
 - 2 Algorithm chooses a point $x_1 \in \mathcal{K}$.
 - 3 **for** $t = 1, \dots, T$ **do**
 - 4 Nature chooses a convex function $f_t : \mathcal{K} \rightarrow \mathbb{R}$, possibly depending on history \mathcal{H}_t
 // history \mathcal{H}_t comprises $(x_s, f_s, \tilde{\nabla}_s)$ for each round $s < t$
 - 5 Nature chooses a gradient estimate $\tilde{\nabla}_t \in \mathbb{R}$, possibly at random
 /* $\tilde{\nabla}_t$ is an unbiased estimate of $\nabla f_t(x_t)$ given \mathcal{H}_t :
 $\mathbb{E}[\tilde{\nabla}_t \mid \mathcal{H}_t] = \nabla f_t(x_t)$ */
 - 6 Algorithm observes $\tilde{\nabla}_t$ and chooses $x_{t+1} \in \mathcal{K}$.
-

The general framework of OCO with gradient estimates is summarized in Algorithm 2. While we focus on $\mathcal{K} \subset \mathbb{R}$, we note in passing that all algorithms and guarantees extend to $\mathcal{K} \subset \mathbb{R}^d$, $d \in \mathbb{N}$.

Within this framework, the particular algorithms are defines as follows. Throughout, $\epsilon > 0$ denotes the step-size, and $P_{\mathcal{K}}(x)$ denotes the projection of $x \in \mathbb{R}$ into \mathcal{K} . For OMD and OFTRL, in each round t , an algorithm is given an estimate M_t of the *next* gradient $\nabla f_{t+1}(x_{t+1})$. Thus:

SGD $x_{t+1} \leftarrow P_{\mathcal{K}}(x_t - \epsilon \cdot \tilde{\nabla}_t)$.

OGD $x_{t+1} \leftarrow P_{\mathcal{K}}(x_t - \epsilon \cdot \tilde{\nabla}_t - \epsilon' \cdot \tilde{\nabla}_{t-1})$, where $\epsilon' \in [-\epsilon/2, 0]$ is another parameter.

OMD $\hat{x}_{t+1} \leftarrow P_{\mathcal{K}}(\hat{x}_t - \epsilon \cdot \tilde{\nabla}_t)$ and $x_{t+1} \leftarrow P_{\mathcal{K}}(\hat{x}_{t+1} - \epsilon \cdot M_t)$, where $\hat{x}_1 = 0$.

OFTRL $x_{t+1} \leftarrow P_{\mathcal{K}}\left(-\epsilon \left(M_t + \sum_{s \in [t]} \tilde{\nabla}_s\right)\right)$.

Remark D.1. The original versions of OMD and OFTRL with Euclidean regularizer are defined for $\mathcal{K} \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$ and involve explicit regularizers and Bregman divergences. We only provide a simpler equivalent formulation for $d = 1$. The default choice for M_t is $M_t = \tilde{\nabla}_t$, see Remark 5.3.

We use the following guarantees from prior work:

Theorem D.2. *Consider the framework in Algorithm 2. Fix some numbers $D \geq 1$, $G > 0$ and $P \geq 1$. Assume the feasible set \mathcal{K} has diameter at most D , and that $|\tilde{\nabla}_t| \leq G$ almost surely for all rounds t . Fix an arbitrary comparator sequence $u_1, \dots, u_T \in \mathcal{K}$ satisfying $\sum_{t=1}^{T-1} \|u_{t+1} - u_t\|_2 + 1 \leq P$ almost surely. For OMD and OFTRL, posit that $|M_t| \leq G$ for all rounds t . Then SGD, OGD and OMD satisfy the following*

regret bound, for any given step size $\epsilon > 0$:

$$\mathbb{E} \left[\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u_t) \right] \leq O \left(\frac{D^2 P}{\epsilon} + \epsilon \cdot G^2 T \right). \quad (\text{D.1})$$

Moreover, OFTRL satisfies Eq. (D.1) for the stationary comparator (when all u_t 's are the same).

Remark D.3. The result for SGD and OGD is essentially contained in the proof of [41, Theorem 10.1.1]. Since the latter only considers the special case of (deterministic) gradient descent, we spell out the proof below for the sake of completeness. The result for OMD is contained in [43, Lemma 1]. The result for OFTRL follows from [60, Theorem 19 and Lemma 20], extending an earlier analysis of [54].

D.1 Proof of Theorem D.2 for SGD and OGD

Focus on OGD (SGD being a special case with $\epsilon' = 0$). It will be convenient to study the equivalent reparameterization of the dynamics:

$$x_{t+1} = P_{\mathcal{K}} \left(x_t - \epsilon \cdot \tilde{\nabla}_t - \epsilon' \cdot N_t \right), \quad (\text{D.2})$$

where we define $N_t := \tilde{\nabla}_t - \tilde{\nabla}_{t-1}$ and now have the equivalent assumption that $0 \leq \epsilon' \leq \epsilon$. (We've redefined $\epsilon \leftarrow \epsilon + \epsilon'$ and $\epsilon' \leftarrow -\epsilon'$.) Since we changed the original value of ϵ by a factor of at most 2, it suffices to prove the regret bound for (D.2).

We have deterministically by convexity that

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u_t) \leq \sum_{t=1}^T \nabla f_t(x_t)^T (x_t - u_t) = \mathbb{E} \left[\sum_{t \in [T]} \tilde{\nabla}_t^T (x_t - u_t) \right],$$

where the expectation is over all random choices in the dynamics using the assumption on the gradient estimates. Observe that

$$\begin{aligned} \|x_{t+1} - u_t\|^2 &\leq \|x_t - \epsilon \tilde{\nabla}_t - \epsilon' N_t - u_t\|^2 \\ &= \|x_t - u_t\|^2 + \|\epsilon \tilde{\nabla}_t + \epsilon' N_t\|^2 - 2\epsilon \tilde{\nabla}_t^T (x_t - u_t) - 2\epsilon' N_t^T (x_t - u_t) \\ &\leq \|x_t - u_t\|^2 + 9\epsilon^2 G^2 - 2\epsilon \tilde{\nabla}_t^T (x_t - u_t) - 2\epsilon' N_t^T (x_t - u_t) \end{aligned}$$

We first show that the last term is small. We surely have

$$\begin{aligned} \sum_{t=1}^T N_t^T (x_t - u_t) &= \sum_{t=1}^T \tilde{\nabla}_t^T (x_{t+1} - x_t) + \sum_{t=1}^T \tilde{\nabla}_t^T (u_{t+1} - u_t) \\ &\leq G \left(\sum_{t=1}^T \|x_{t+1} - x_t\| + \sum_{t=1}^T \|u_{t+1} - u_t\| \right) \\ &\leq G \left(\sum_{t \in [T]} 3G\epsilon + P \right) = O(\epsilon G^2 T + GP). \end{aligned}$$

where we simply use the definition of N_t and rearrange terms. Therefore, we have

$$\begin{aligned} 2 \sum_{t=1}^T \tilde{\nabla}_t^T(x_t - u_t) &\leq \sum_{t=1}^T \frac{\|x_{t+1} - x_t\|^2 - \|x_t - u_t\|^2}{\epsilon} + O\left(\epsilon' G^2 T + \frac{\epsilon' GP}{\epsilon}\right) \\ &\leq \sum_{t=1}^T \frac{\|x_{t+1} - x_t\|^2 - \|x_t - u_t\|^2}{\epsilon} + O\left(\epsilon G^2 T + GP\right), \end{aligned}$$

where the extra contribution is from the last term we just bounded. Here, we use the fact that $\epsilon' \leq \epsilon$. At this point, the sum is bounded exactly as in Theorem 10.1.1 of [41] for an overall bound of

$$\sum_{t=1}^T \nabla f_t(x_t)^T(x_t - u_t) \leq O\left(\frac{D^2}{\epsilon} P + \epsilon G^2 T + GP\right).$$

Moreover, it is easy to see that GP is bounded by the other terms (up to constants) for any choice of ϵ . This is because the minimum is (up to constants) $DG\sqrt{PT} \gtrsim D^{1/2}GP \geq GP$. Here, we use the fact that $P \leq 1 + 2DT$ and the assumption $D \geq 1$.

E Omitted Proofs from Section 4: Individual Guarantees

E.1 Stochastic environment

We need to show that the perfect pacing multiple ν^* approximately optimizes the objective over all multipliers, for (a) value maximization and (b) second-price auctions. (While part (b) is implicit in [13], we provide a proof here for the sake of completeness.)

Lemma E.1. *In the setting of Corollary 4.12, assume either that (a) the objective is value-maximization: $\Phi_t^{\text{uni}} = V_t$, or that (b) the auction rule is second-price. Then*

$$\Phi_{\text{fix}}^{\text{uni}}(\nu^*) \geq \sup_{\mu \in [0, \bar{\mu}]} \Phi_{\text{fix}}^{\text{uni}}(\mu) - O(\sqrt{T}). \quad (\text{E.1})$$

The rest of this subsection proves this Lemma.

Fix the mixing parameter $\gamma \in [0, 1]$ in the unified objective. Denote $U_t = U$, $V_t = V$, and $Z_t = Z$ (this being a stochastic environment, there's no dependence on the round t). Fix some pacing multiplier $\mu \in [0, \bar{\mu}]$. We will repeatedly use the fact that $\lambda U + (1 - \lambda)V = V - \lambda Z$ by definition, and the same identity holds pointwise as well.

Define the random process $M_k = \sum_{t=1}^k [x_t v_t - \lambda z_t - (V(\mu) - \lambda Z(\mu))]$, where x_t and v_t are the (random) allocation and valuations at each time t when bidding with multiplier μ and z_t is the expenditure. Notice that M_k is a martingale by construction and $\mathbb{E}[M_1] = 0$. Define τ_μ to be first time that such a bidding algorithm runs out of money at the end of the period, or T if this does not occur. Because M_k is a martingale, by the Optional Stopping Theorem,

$$\Phi_{\text{fix}}^{\text{uni}}(\mu) \leq \mathbb{E}\left[\sum_{t=1}^{\tau_\mu} (x_t v_t - \lambda z_t)\right] = \mathbb{E}\left[\sum_{t=1}^{\tau_\mu} (V(\mu) - \lambda Z(\mu))\right] = \mathbb{E}[\tau_\mu] \cdot (V(\mu) - \lambda Z(\mu)), \quad (\text{E.2})$$

where the inequality occurs because the stopping rule counts the (non-negative) value or utility obtained on the period where the agent may (strictly) exceed the budget, and then using Wald's identity in the last step as the function $V(\mu)$ is constant over time by assumption so independent of τ_μ .

We now derive an upper bound on the expectation of the stopping time. On the one hand, we certainly have $\mathbb{E}[\tau_\mu] \leq T$ as $\tau_\mu \leq T$ by definition. An analogous martingale argument with $N_k = \sum_{t=1}^k [z_t - Z(\mu)]$ (where z_t is the (random) expenditure at time t) similarly implies that

$$\mathbb{E}[\tau_\mu] \cdot Z(\mu) = \mathbb{E} \left[\sum_{t=1}^{\tau_\mu} z_t \right] \leq B + \bar{v}, \quad (\text{E.3})$$

where the extra term arises in the analysis because the agent may spend \bar{v} on the final round before she exceeds her budget. Combining (E.2) and (E.3) implies that $\Phi_{\text{fix}}^{\text{uni}}(\mu) \leq (\gamma U(\mu) + (1 - \gamma)V(\mu)) \cdot \min \left\{ T, \frac{B + \bar{v}}{Z(\mu)} \right\}$.

Suppose first that $\mu \geq \nu^*$. In this case, it is clear that $V(\mu) \leq V(\nu^*)$ by monotonicity of the auction. This implies that when $\gamma = 0$, we have

$$\Phi_{\text{fix}}^{\text{uni}}(\mu) \leq V(\mu) \cdot T \leq V(\nu^*) \cdot T,$$

proving part (a).

For part (b), assume further that the auction is second-price. Then we have the inequality $U(\mu) \leq U(\nu^*)$ since utilities are non-increasing in the multiplier for any fixed valuation and competing bid so long as the pacing multipliers are nonnegative. It follows that for second-price auctions, for any $\gamma \in [0, 1]$

$$\Phi_{\text{fix}}^{\text{uni}}(\mu) \leq (\lambda U(\mu) + (1 - \lambda)V(\mu)) \cdot T \leq (\lambda U(\nu^*) + (1 - \lambda)V(\nu^*)) \cdot T$$

If instead $\mu < \mu^*$, note that by definition this implies that $Z(\mu^*) = \rho$. For any $\gamma \in [0, 1]$ and any MBB auction (not necessarily second-price), we then have:

$$\begin{aligned} \Phi_{\text{fix}}^{\text{uni}}(\mu) &\leq (B + \bar{v}) \frac{V(\mu) - \gamma Z(\mu)}{Z(\mu)} \\ &= (B + \bar{v}) \left(\frac{V(\mu)}{Z(\mu)} - \gamma \right) \\ &\leq (B + \bar{v}) \left(\frac{V(\nu^*)}{Z(\nu^*)} - \gamma \right) \\ &= \frac{B + \bar{v}}{\rho} (V(\nu^*) - \gamma Z(\mu^*)) \\ &\leq T \cdot (\lambda U(\nu^*) + (1 - \lambda)V(\nu^*)) + \bar{v}^2 / \rho. \end{aligned}$$

where the inequality is the monotone bang-for-buck property (see the proof of Theorem 4.8) and then using the fact that $B/\rho = T$.

In either case, a similar Wald argument then shows that

$$\Phi_{\text{fix}}^{\text{uni}}(\nu^*) \geq (\gamma U(\nu^*) + (1 - \gamma)V(\nu^*)) \cdot \mathbb{E}[\tau^* - 1]$$

where τ^* is the stopping time of the strategy that paces using ν^* . However, it is easy to see that the expected stopping time is at least $T - O(\sqrt{T})$ by considering the fluctuations of the super-martingale $z_t - \rho$ since $\mathbb{E}[z_t] := Z(\nu^*) \leq \rho$ (using e.g. the Azuma-Hoeffding inequality and integrating the tail). The desired inequality follows, noting that we treat the other parameters as absolute constants by assumption.

E.2 Artificial objective H_t

Lemma E.2. *The function H_t is convex and is $(\bar{v} + \rho)$ -Lipschitz.*

Proof. The convexity is immediate from the fundamental theorem of calculus and monotonicity assumption on Z_t , as the expected expenditure is a weakly decreasing function of the multiplier. To show that the Lipschitz condition holds note that

$$|H_t(y) - H_t(x)| \leq \rho|y - x| + \left| \int_x^y Z_t(s) ds \right| \leq (\rho + Z_t(0))|y - x|.$$

The proof of the Lipschitz condition follows from the fact the expected expenditure function is at most the expected valuation. \square

E.3 Proof of Lemma 4.21

The conclusion is trivial if $\lambda = 0$, so we assume $\lambda > 0$. Moreover, note that we always have $R \geq 0$, and by considering the function $-f(-y)$ if necessary, we may assume without loss of generality that $x \geq 0$ and thus the same for $f(x)$.

For a contradiction, suppose that $f(x) > \sqrt{2\lambda R}$. Then by the assumption that f is λ -Lipschitz, it follows that $f(y) > \sqrt{2\lambda R} - \lambda(x - y)$ for all $y \in [x - \sqrt{2R/\lambda}, x]$ (note $f(y) > 0$ on this region so this region is contained in $[0, x]$, where $f(y) \geq 0$). Hence, we have

$$\int_0^x f(y) dy \geq \int_{x-\sqrt{2R/\lambda}}^x f(y) dy > \int_{x-\sqrt{2R/\lambda}}^x [\sqrt{2\lambda R} - \lambda(x - y)] dy.$$

This latter integral gives the area of a right triangle with height $\sqrt{2\lambda R}$ and base $\sqrt{2R/\lambda}$, which is clearly equal to R . This contradicts the assumption that $R = \int_0^x f(y) dy$, proving the lemma.

F Omitted Proofs from Section 5: Other Pacing Algorithms

F.1 The algorithms do not run out of budget too early

We state and prove an analogue of Lemma 2.2 for PacingOGD, PacingOMD, and PacingOFTRL, showing that the algorithms do not run out of budget too early. For this result, PacingOMD and PacingOFTRL can use arbitrary next-gradient predictors $M_{k,t} \leq \bar{v}$. The only change compared to Lemma 2.2 is the slightly stronger condition on the problem parameters: $\bar{\mu} \geq 2\bar{v}/\rho_k + 1$.

Lemma F.1. *Fix agent k in a core auction with some (possibly adaptive, randomized, adversarially generated) set of valuations and competing bids. Suppose the agent uses PacingOGD, PacingOMD, or PacingOFTRL (the latter two algorithms with arbitrary next-gradient predictors satisfying $|M_{k,t}| \leq \bar{v}$). Let τ_k be the algorithm's stopping time. Assume all valuations are at most \bar{v} , and the parameters satisfy $\bar{\mu} \geq 2\bar{v}/\rho_k + 1$ and $\epsilon_k \bar{v} \leq 1$. Then $T - \tau_k \leq \frac{\bar{\mu}}{\epsilon_k \rho_k} + \frac{\bar{v}}{\rho_k}$ almost surely.*

The analysis is modified from Proposition 2 of Balseiro et al. [17].

Proof. Following the proof of [17, Proposition 2], it suffices to show that under the stated conditions on ϵ_k and $\bar{\mu}$, either the pacing multipliers or the auxiliary sequences (depending on the setting) remain strictly below $\bar{\mu}$ at all times. In each case, the remainder of the argument to bound the stopping time then becomes identical to that in Balseiro et al. [17].

PacingOMD. We first show this for PacingOMD with respect to the auxiliary sequence $\hat{\mu}_{k,t}$. Observe that we deterministically have

$$z_{k,t} \leq \frac{\bar{v}}{1 + \mu_{k,t}} \leq \frac{\bar{v}}{1 + \hat{\mu}_{k,t} - \epsilon_k |M_{k,t}|} \leq \frac{\bar{v}}{1 + \hat{\mu}_{k,t} - \epsilon_k \bar{v}} \leq \frac{\bar{v}}{\hat{\mu}_{k,t}}. \quad (\text{F.1})$$

The first inequality holds since expenditure is at most the bid, which in turn can be bounded using $\hat{\mu}_{k,t}$ after accounting for the optimistic step. The second inequality holds by the assumption $\epsilon_k \bar{v} \leq 1$. Note that we also deterministically always have:

$$\hat{\mu}_{k,t+1} \leq \hat{\mu}_{k,t} + \epsilon_k z_{k,t}. \quad (\text{F.2})$$

We will now combine these inequalities to show that $\hat{\mu}_{k,t} < \bar{v}/\rho_k + 1$ for all t .

First, suppose that at some time t , it holds that $\hat{\mu}_{k,t} \leq \bar{v}/\rho_k$. Then (F.2) immediately implies that

$$\hat{\mu}_{k,t+1} \leq \hat{\mu}_{k,t} + \epsilon_k z_{k,t} \leq \hat{\mu}_{k,t} + 1 < \bar{v}/\rho_k + 1,$$

as claimed.

Suppose instead that $\bar{v}/\rho_k \leq \hat{\mu}_{k,t} < \bar{v}/\rho_k + 1$. Then by (F.1), it immediately follows that $z_{k,t} \leq \rho_k$. If $\hat{\mu}_{k,t+1} > 0$, then the recursion satisfies:

$$\hat{\mu}_{k,t+1} \leq \hat{\mu}_{k,t} + \epsilon_k (z_{k,t} - \rho_k) \leq \hat{\mu}_{k,t} < \bar{v}/\rho_k + 1.$$

If $\hat{\mu}_{k,t+1} = 0$, then we are done as well, so unconditionally $\hat{\mu}_{k,t+1}$ remains strictly below this threshold as well in this case.

Therefore, it follows that under our assumption that $\bar{\mu} \geq 2\bar{v}/\rho + 1 > \bar{v}/\rho + 1$, it deterministically holds that $\hat{\mu}_{k,t} < \bar{\mu}$ at all times t , and thus there is never any projection back to this endpoint in this recurrence. As stated above, the bound on the stopping time then follows the exact same analysis as in Proposition 2 of [17], which is based on writing out the full recurrence for $\hat{\mu}_{k,t}$ until the stopping time in terms of the expenditure.

PacingOFTRL. The argument for PacingOFTRL is nearly identical upon defining the auxiliary sequence

$$\hat{\mu}_{k,t+1} = P_{[0, \bar{\mu}]} \left(-\epsilon_k \left(\sum_{s \in [t]} \tilde{\nabla}_{k,s} \right) \right).$$

The casework depending on the value of $\hat{\mu}_{k,t}$ is the same as for PacingOMD and establishes that $\hat{\mu}_{k,t}$ remains strictly below $\bar{\mu}$ under the same assumptions on $\bar{\mu}$ and ϵ_k .

PacingOGD. For PacingOGD, we show that the pacing multiplier again remains strictly below $\bar{\mu}$ under the stated assumptions. We argue as follows: observe that if $\mu_{k,t} \geq \frac{2\bar{v}}{\rho_k} - 1$, then we must have

$$z_{k,t} \leq \frac{\bar{v}}{1 + \mu_{k,t}} \leq \frac{\rho_k}{2}.$$

Suppose that for some t , it holds that $\mu_{k,t} \geq \frac{2\bar{v}}{\rho_k} - 1$. We claim that in this case, $\mu_{k,t+1} \leq \mu_{k,t}$. Indeed, either $\mu_{k,t+1} = 0$ and we are already done, or otherwise there is no negative projection step and hence the PacingOGD recursion implies that

$$\mu_{k,t+1} \leq \mu_{k,t} + \epsilon_k (z_{k,t} - \rho_k) + \epsilon'_k (z_{k,t-1} - \rho_k) \leq \mu_{k,t} - \epsilon_k \frac{\rho_k}{2} + \epsilon_k \frac{\rho_k}{2} \leq \mu_{k,t}.$$

In the penultimate step, we use the assumption in PacingOGD that $\epsilon'_k \in [-\epsilon_k/2, 0]$. Thus, it follows that $\mu_{k,t+1} \leq \mu_{k,t}$ unconditionally when $\mu_{k,t} \geq \frac{2\bar{v}}{\rho_k} - 1$.

Since PacingOGD and our assumption $\epsilon_k \bar{v} \leq 1$ deterministically implies that

$$\mu_{k,t+1} \leq \mu_{k,t} + 2\epsilon_k \bar{v} \leq \mu_{k,t} + 2,$$

it follows that under PacingOGD dynamics, $\mu_{k,t}$ can never exceed $\frac{2\bar{v}}{\rho_k} + 1$. Indeed, we have shown that the sequence must be non-increasing when above $\frac{2\bar{v}}{\rho_k} - 1$ in any iteration, and so can only ever exceed this threshold by 2 from a single step. Therefore, since we assumed that $\bar{\mu} \geq \frac{2\bar{v}}{\rho_k} + 1$, it follows that $\mu_{k,t} < \bar{\mu}$ for all times t . \square

F2 Proof of Theorem 5.7: Liquid Welfare under Event-Feasible Algorithms

Under the assumptions, we may follow the proof of Theorem 3.4 with relatively minor modifications. In this case, we simply again define:

$$R_k(\mathbf{v}) \triangleq \sum_{t=1}^T [\mathbf{1}\{\mathcal{E}_{k,t}\} y_k(\mathbf{v}) v_{k,t} + \mathbf{1}\{\mathcal{E}_{k,t}\} \rho_k].$$

Note that the concentration inequality of Lemma 3.9 applies equally well to this quantity since the events $\mathcal{E}_{k,t}$ are determined by the history through time $t - 1$ by definition.

At this point, we may simply follow the proof, again defining $A \subseteq [n]$ as the set of agents such that $\sum_{t=1}^T x_{k,t} v_{k,t} < B_k$. By our assumption, for each such agent $k \in A$, we have the analogous inequality

$$\sum_{t=1}^T x_{k,t} v_{k,t} \geq \sum_{t=1}^T (x_{k,t} v_{k,t} - z_{k,t}) \cdot \mathbf{1}(\mathcal{E}_{k,t}) + (1 - c) \sum_{t=1}^T \rho_k \cdot \mathbf{1}(\mathcal{E}_{k,t}^c),$$

and again upon summing over $k \in A$, we derive exact analogue of (3.11):

$$\sum_{k \in A} \sum_{t=1}^T x_{k,t} v_{k,t} \geq \sum_{t=1}^T \sum_{k \in A} [\mathbf{1}\{\mathcal{E}_{k,t}\} (x_{k,t} v_{k,t} - z_{k,t})] + (1 - c) \cdot \sum_{k \in A} \sum_{t=1}^T \mathbf{1}\{\mathcal{E}_{k,t}^c\} \cdot \rho_k.$$

We may now again appeal to the core auction assumption in the same way: for any $t \in [1, T]$, if we set $S \subseteq A$ to be the set of agents k satisfying $\mathcal{E}_{k,t}$, we find

$$\begin{aligned} \sum_{k \in A} [\mathbf{1}\{\mu_{k,t} = 0\} (x_{k,t} v_{k,t} - z_{k,t})] &= \sum_{k \in S} (x_{k,t} v_{k,t} - z_{k,t}) \\ &\geq \sum_{k \in S} x_{k,t} b_{k,t} - \sum_{k \in S} z_{k,t} \\ &\geq \sum_{k \in S} y_k(\mathbf{v}_t) b_{k,t} - \sum_{k=1}^n z_{k,t} \\ &\geq (1 - c) \sum_{k \in S} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k=1}^n z_{k,t} \\ &= (1 - c) \sum_{k \in A} \mathbf{1}\{\mathcal{E}_{k,t}\} y_k(\mathbf{v}_t) v_{k,t} - \sum_{k=1}^n z_{k,t}. \end{aligned}$$

Here, we crucially use the fact that $v_{k,t} \geq b_{k,t}$ but that Definition 5.5 imposes a reverse inequality up to a factor $(1 - c)$. We thus obtain the analogous inequality (3.12) on the good event that concentration held:

$$\sum_{k \in [n]} \text{WEL}_{k, \text{GPD}}(\mathbf{v}) \geq (1 - c) \cdot \sum_{k \in [n]} R_k(\mathbf{v}) - \sum_{k \in [n]} \sum_{t \in [T]} z_{k,t} - n\bar{v} \sqrt{T \log(\bar{v}nT)}.$$

The rest of the proof extends verbatim up to this factor of $(1 - c)$ in the final bound.

F.3 Proof of Lemma 5.6: Our Algorithms are Event-Feasible

We need to prove that PacingOGD, PacingOMD, and PacingOFTRL are $O(\bar{v}\sqrt{\epsilon_k/\rho_k})$ -event-feasible. Fix agent k in what follows and define

$$c' := 2\bar{v}\sqrt{\epsilon_k/\rho_k}.$$

PacingOGD. Let $\mathcal{E}_{k,t} = \{\mu_{k,t} \leq c'\}$. As in Lemma 3.8, we will prove the desired conditions hold on each epoch individually, where epochs are defined as before. Let $[t_1, t_2)$ denote a maximal epoch for agent k so that $\mu_{k,t_1} = 0$ and $\mu_{k,t_2-1} > 0$. Because there are no negative projections on an epoch by assumption, writing out the gradient recurrence and rearranging yields:

$$\begin{aligned} \sum_{t=t_1}^{t_2-2} z_{k,t} &\geq (t_2 - t_1 - 1)\rho_k + \frac{\epsilon'_k}{\epsilon_k + \epsilon'_k}(z_{k,t_2-2} - z_{k,t_1-1}) \\ &\geq (t_2 - t_1 - 1)\rho_k - \bar{v}. \end{aligned}$$

In the last step, we use the assumption that $|\epsilon'_k| \leq \epsilon_k/2$ so that the ratio is bounded by 1 in absolute value and drop a negative term.

Say that the epoch is *long* (resp. *short*) iff the length of the interval satisfies:

$$t_2 - t_1 \geq \frac{2\bar{v}}{\rho_k c'}.$$

It is immediate from basic algebra and the assumption $\bar{v} \geq \rho_k$ that on any long epoch:

$$\sum_{t=t_1}^{t_2-2} z_{k,t} \geq (1 - c')(t_2 - t_1)\rho_k. \quad (\text{F.3})$$

If an interval is short, then since each step of PacingOGD can increase the pacer by at most $2\epsilon_k\bar{v}$ by assumption and the number of steps is at most $\frac{2\bar{v}}{\rho_k c'}$, the pacing multiplier can be at most

$$2\epsilon_k\bar{v} \cdot \frac{2\bar{v}}{\rho_k c'} = \frac{4\epsilon_k\bar{v}^2}{\rho_k c'} = 2\bar{v}\sqrt{\epsilon_k/\rho_k} = c'.$$

Therefore, the entire epoch will satisfy $\mathcal{E}_{k,t}$ by construction. This means that for any short epoch,

$$\sum_{t=t_1}^{t_2-1} x_{k,t}v_{k,t} \geq \sum_{t=t_1}^{t_2-1} (x_{k,t}v_{k,t} - z_{k,t}) = \sum_{t=t_1}^{t_2-1} (x_{k,t}v_{k,t} - z_{k,t}) \cdot \mathbf{1}(\mathcal{E}_{k,t}) + \sum_{t=t_1}^{t_2-1} \rho_k \cdot \mathbf{1}(\mathcal{E}_{k,t}^c)$$

If instead the epoch is long, then we can instead bound using (F.3):

$$\begin{aligned} \sum_{t=t_1}^{t_2-1} x_{k,t}v_{k,t} &= \sum_{t=t_1}^{t_2-1} (x_{k,t}v_{k,t} - z_{k,t}) + \sum_{t=t_1}^{t_2-1} z_{k,t} \\ &\geq \sum_{t=t_1}^{t_2-1} (x_{k,t}v_{k,t} - z_{k,t}) \cdot \mathbf{1}(\mathcal{E}_{k,t}) + (1 - c')(t_2 - t_1)\rho_k \\ &\geq \sum_{t=1}^T (x_{k,t}v_{k,t} - z_{k,t}) \cdot \mathbf{1}(\mathcal{E}_{k,t}) + (1 - c') \sum_{t=t_1}^{t_2-1} \rho_k \cdot \mathbf{1}(\mathcal{E}_{k,t}^c). \end{aligned}$$

The first inequality holds from nonnegativity of quasilinear utility under any pacing algorithm in an individually rational auction. Finally, note that on $\mathcal{E}_{k,t}$,

$$\frac{1}{1 + \mu_{k,t}} \geq \frac{1}{1 + c'} \geq 1 - c'.$$

Doing this for each agent k and taking the minimum of ϵ_k/ρ_k yields the desired conditions with $c = c'$.

PacingOMD. We can reduce to the previous analysis. Since the auxiliary sequence $\hat{\mu}_{k,t}$ follows PacingOGD (with $\epsilon'_k = 0$), defining the events

$$\mathcal{E}_{k,t} = \{\hat{\mu}_{k,t} \leq c'\},$$

the same lower bound on valuations holds by an identical analysis. Since $|\mu_{k,t} - \hat{\mu}_{k,t}| \leq \epsilon_k |M_k| \leq \epsilon_k \bar{v}$ by assumption, the event $\mathcal{E}_{k,t}$ implies that

$$\mu_{k,t} \leq 2\bar{v}\sqrt{\epsilon_k/\rho_k} + \epsilon_k \bar{v} \leq 4\bar{v}\sqrt{\epsilon_k/\rho_k},$$

and so we satisfy the desired properties after taking $c = 2c'$.

PacingOFTRL. We use a similar analysis. Again, we prove the desired inequality on each epoch $[t_1, t_2)$ separately defined in terms of the $\mu_{k,t}$. The key observation is that by definition of an epoch:

$$\begin{aligned} \mu_{k,t_1} = 0 &\implies \epsilon_k \sum_{s=1}^{t_1-1} (z_{k,s} - \rho_k) + \epsilon_k (z_{k,t_1-1} - \rho_k) \leq 0 \\ \mu_{k,t_2-1} > 0 &\implies \epsilon_k \sum_{s=1}^{t_2-2} (z_{k,s} - \rho_k) + \epsilon_k (z_{k,t_2-2} - \rho_k) > 0, \end{aligned}$$

so subtracting the first inequality from the latter yields

$$\epsilon_k \sum_{s=t_1}^{t_2-2} (z_{k,s} - \rho_k) + \epsilon_k (z_{k,t_2-2} - z_{k,t_1-1}) > 0.$$

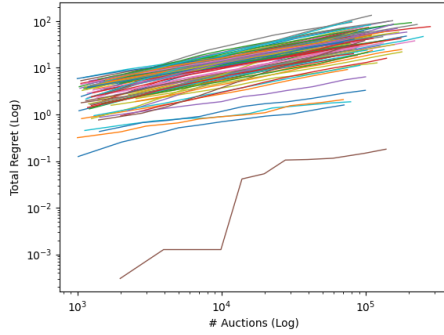
After rearrangement and dropping a negative term, we again find that

$$\sum_{s=t_1}^{t_2-2} z_s > (t_2 - t_1 - 1) \cdot \rho_k - \bar{v}.$$

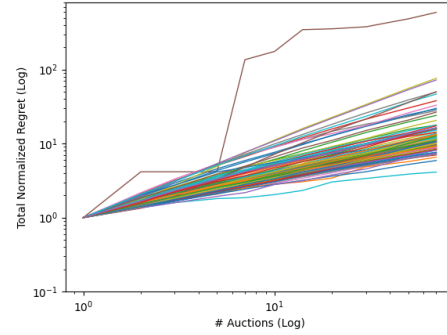
But this is identical to the lower bound we derived for PacingOGD, and so we can follow the same exact analysis by defining the events $\mathcal{E}_{k,t} = \{\mu_{k,t} \leq c'\}$ to obtain the desired conclusion.

G Additional Numerical Simulation Plots

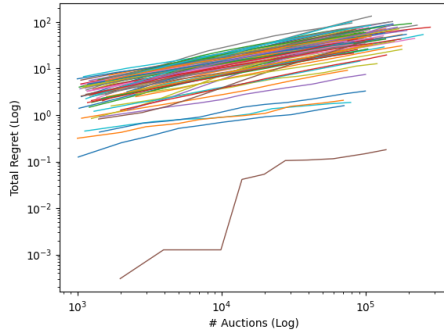
Our empirical estimation of regret rate in Section 6 builds on a hypothesis that regret accumulates at a rate of $\Theta(T^\alpha)$ for some $\alpha \in [0, 1]$. To provide additional evidence for this hypothesis, we illustrate traces of regret evolution for 100 randomly-sampled campaigns from among those that exceed our participation threshold of at least $\theta = 1000$ auction instances. For each campaign, we trace out the accumulation of regret as the time horizon (i.e., the number of auction instances) is amplified. Our hypothesis corresponds to linear evolution of these traces when plotted in log-log scale. See Figure 3.



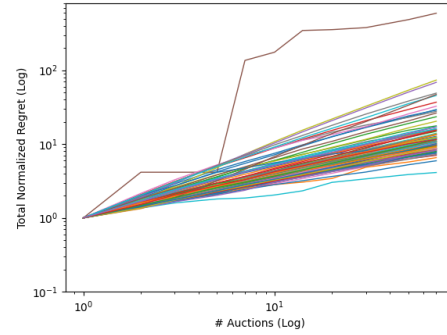
(a) utility-maximization in SPA



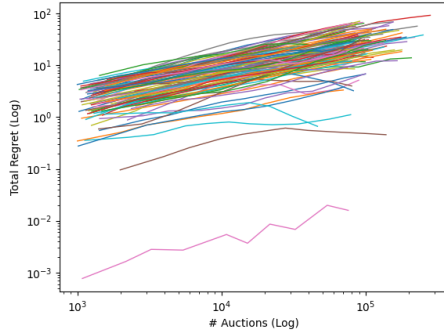
... on normalized data



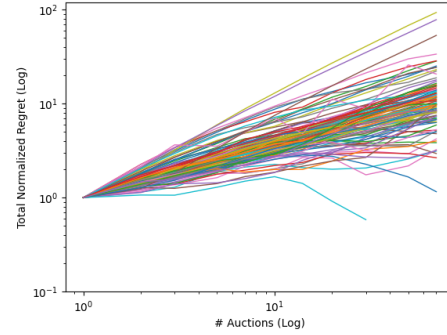
(b) value-maximization in SPA



... on normalized data



(c) value-maximization in FPA



... on normalized data

Figure 3: Illustration of estimated regret in repeated auction simulation. For each of 100 randomly-sampled campaigns, we trace out the evolution of regret as the time horizon (the number of auction instances) is amplified, illustrated in log-log scale. Each line corresponds to a single campaign, plotted for different choices of the amplification parameter K (the number of iterations through the dataset). Results are shown for (a) utility maximization in second-price auctions, (b) value maximization in second-price auctions, and (c) value maximization in first-price auctions. Instances with negative regret are excluded. In the right column we show the same data but normalized so that all traces begin at (1,1).