

Mechanism Design for Ad Auctions with Display Prices

Bin Li^{1*}, Yahui Lei²

¹School of Computer Science & Engineering, Nanjing University of Science and Technology

² Meituan Inc.

cs.libin@njust.edu.cn, leiyahui@meituan.com

Abstract

In many applications, ads are displayed together with the prices, so as to provide a direct comparison among similar products or services. The price-displaying feature not only influences the consumers' decisions, but also affects the advertisers' bidding behaviors. In this paper, we study ad auctions with display prices from the perspective of mechanism design, in which advertisers are asked to submit both the costs and prices of their products. We provide a characterization for all incentive compatible auctions with display prices, and use it to design auctions under two scenarios. In the former scenario, the display prices are assumed to be exogenously determined. For this setting, we derive the welfare-maximizing and revenue-maximizing auctions for any realization of the price profile. In the latter, advertisers are allowed to strategize display prices in their own interests. We investigate two families of allocation policies within the scenario and identify the equilibrium prices accordingly. Our results reveal that the display prices do affect the design of ad auctions and the platform can leverage such information to optimize the performance of ad delivery.

1 Introduction

Online advertising has been an indispensable part of modern advertising market. According to the newly released report of IAB [IAB, 2022], full-year online advertising revenue has reached \$189.3 billion in 2021. An important reason for wide-spread adoption of online advertisement comes from its high return on investment for advertisers, compared to other traditional marketing methods [Moran and Hunt, 2005]. As an efficient tool for deriving revenue, auctions are commonly used to allocate the display opportunities. Every day, tens of billions of ad auctions are conducted in real time to decide which advertisers' ads are shown, how these ads are arranged, and what the advertisers are charged. To date, online advertising platforms [Google Ads Help, 2022; Microsoft Advertising Editor, 2022;

Facebook Business Help Center, 2022] have developed various types of products for different types of advertisers, such as pay-per-mille or pay-per-impression (PPM), pay-per-click (PPC), and pay-per-action (PPA). In the classic ad auction setting, such as sponsored search, ads are presented in the form of hyper-links together with relevant keywords or well-designed creatives, serving as portals of advertisers' websites or products. Which ads are displayed depends on advertisers' bids and the relevance of their ads to the context [Varian, 2007]. However, in many real applications, like Temu or Ctrip, ads (or products) are displayed also together with the prices, e.g., it can be the per-night price of a room or the group purchase price of a commodity. The price-displaying feature brings two significant changes for the advertising system. On the one hand, the prices provide a direct comparison among similar products or services, which can easily influence the consumers' decisions. One the other hand, the price information also affects the advertisers' bidding behaviors and the efficiency of the underlying ad auctions [Castiglioni *et al.*, 2022].

As the display prices has changed the bidding language and the way advertisers participate in the ad auction, fundamental investigation into mechanism design for auctions with display prices should be made. In this paper, we study how the presence of display prices affects ad auction design. In our model, advertisers are asked to submit both the costs and prices of their products and the advertising platform allocates the display opportunities and decides the charges based on the submitted information. Our model differs from the classic one in two ways. Firstly, rather than submitting a single bid for the display opportunities, we ask advertisers to submit both the costs and prices of their products. Secondly, in classic ad auctions, a convention, like a purchase, of an ad is exogenously determined and is independent of the advertisers' submission, while in our model conventions are essentially determined by the submitted display prices as the price information can dramatically affect the consumers' behaviors. Based on the framework of mechanism design, we carry out a systemic investigation on ad auctions with display prices. Specifically, we characterize all ad auctions that can incentivize truthful cost reports, and use the characterization to design auctions under two scenarios. In the former scenario, the display prices are assumed to be exogenously determined. For this scenario, we derive the welfare-maximizing

* Corresponding Author.

and revenue-maximizing auctions for any realization of the price profile. In the latter scenario, the advertisers are allowed to strategize their display prices. For this setting, we investigate two families of allocation policies and compute the equilibrium price report accordingly. Our results show that the display prices do affect the design of ad auctions, and the advertising platform can leverage such information to further optimize the performance of ad delivery.

In addition to the great industrial success, ad auctions have attracted a lot of attention from the research community. Since Overture, for the first time, adopted the generalized first price auction mechanism in its sponsored search system in 1997 [Edelman and Ostrovsky, 2007; Jansen and Mullen, 2008], many researchers from economics, computer sciences, management science, etc. have been working on different aspects of ad auctions for decades. Some of them focus on studying the theoretical properties of the deployed ad auctions [Edelman *et al.*, 2007; Varian, 2007; Wilkens *et al.*, 2017], while some others are devoted to design new auction mechanisms towards different scenarios and objectives [Laffont and Robert, 1996; Li *et al.*, 2019; Golrezaei *et al.*, 2021]. With the development of AI, it is even possible to design ad auctions automatically [Aggarwal *et al.*, 2019; Shen *et al.*, 2017; Yang *et al.*, 2019], based on advanced machine learning techniques and massive transaction data. To the best of our knowledge, Castiglioni *et al.* [Castiglioni *et al.*, 2022] is the very first to study ad auctions with display prices. The authors study the allocation efficiency of two widely used auctions, namely VCG and GSP, in the presence of display prices, and analyze the Price of Anarchy (PoA) and the Price of Stability (PoS) in the direct and indirect realizations of these two auctions, respectively. In contrast, we focus on the counterpart and study new ad auctions from the perspective of mechanism design.

The reminder of this paper is organized as follows. Section 2 presents the basic model of auction with display prices and defines some general concepts of an auction mechanism. Section 3 characterizes all truthful auctions with display prices. Following the characterization, Section 4 investigates the welfare-maximizing and revenue-maximizing auctions, under the assumption of non-strategic display prices. Section 5 studies two families of auction mechanisms in the general settings and Section 6 summarizes this work.

2 Preliminaries

Assume there is a set of advertisers, denoted by $N = \{1, 2, \dots, n\}$, who decide to advertise their products in an online selling platform. For each advertiser $i \in N$, let $c_i \in [\underline{c}_i, \bar{c}_i]$ denote the cost (or type) of her product, which is assumed private information and derived from a distribution \mathcal{C}_i , and let p_i denote the display price that i sets for her product. As the display prices provide a direct comparison among similar products or services, for simplicity we assume the consumers' behaviors are determined by the products' display prices. Formally, let $\lambda_i : \mathcal{R} \rightarrow [0, 1]$ denote the conversion rate function of i 's product. That is, $\lambda_i(p_i)$ represents the probability that a conversion, like a purchase, is acquired under display price p_i . Given an advertiser i , the expected

value when her product is displayed on the selling platform can be formulated as $v_i(c_i, p_i) = (p_i - c_i)\lambda_i(p_i)$. The selling platform runs an auction to allocate the display opportunities. Besides the product cost, each advertiser is asked to report her display price to the auction. Since c_i is private information, advertiser i can cheat the auction to benefit herself. Accordingly, let (c'_i, p_i) denote i 's report, where c'_i is the reported cost and p_i is the reported display price. For convenience, we use \mathbf{c}' and \mathbf{p} to denote the reported type profile and display prices of all advertisers, respectively. In addition, let \mathbf{c}'_{-i} and \mathbf{p}_{-i} be the reported type profile and reported prices of all advertisers except i , i.e., $\mathbf{c}' = (c'_i, \mathbf{c}'_{-i})$ and $\mathbf{p} = (p_i, \mathbf{p}_{-i})$. The formal definition of auction mechanisms with display prices is given below.

Definition 1. An auction mechanism $\mathcal{M} = (\pi, x)$ consists of an allocation policy $\pi = \{\pi_i\}_{i \in N}$ and a payment policy $x = \{x_i\}_{i \in N}$, where $\pi_i : \mathcal{R}_+^{n+1} \rightarrow \{0, 1\}$ and $x_i : \mathcal{R}_+^{n+1} \rightarrow \mathcal{R}$ are the allocation and payment functions for i , respectively.

Given all advertisers' reports $(\mathbf{c}', \mathbf{p})$, $\pi_i(\mathbf{c}', \mathbf{p})$ indicates whether or not advertiser i wins the slot and $x_i(\mathbf{c}', \mathbf{p})$ denotes the amount each advertiser i pays to the platform. For advertiser i with a report (c'_i, p_i) , her utility function under (π, x) is quasi-linear and is defined in the following:

$$u_i(c_i, \mathbf{c}', \mathbf{p}, (\pi, x)) = v_i(c_i, p_i)\pi_i(\mathbf{c}', \mathbf{p}) - x_i(\mathbf{c}', \mathbf{p}). \quad (1)$$

Given an auction mechanism $\mathcal{M} = (\pi, x)$, the social welfare obtained in $(\mathbf{c}', \mathbf{p})$, denoted by $W(\mathbf{c}', \mathbf{p}, \mathcal{M})$, is defined as the total utilities of all agents (including the platform), which can be simplified as $W(\mathbf{c}', \mathbf{p}, \mathcal{M}) = \sum_{i \in N} v_i(c_i, p_i)\pi_i(\mathbf{c}', \mathbf{p})$. We say an auction mechanism is efficient with reported prices (EF-RP) if for all reports $(\mathbf{c}', \mathbf{p})$ it maximizes $W(\mathbf{c}', \mathbf{p}, \mathcal{M})$.

Definition 2. An auction mechanism \mathcal{M} is efficient with reported prices (EF-RP) if for all reports $(\mathbf{c}', \mathbf{p})$

$$\mathcal{M} \in \arg \max_{\mathcal{M}'} W(\mathbf{c}', \mathbf{p}, \mathcal{M}'). \quad (2)$$

Let $\Pi_i(\mathbf{c}') = \arg \max_{\mathbf{p}', \pi'} \sum_{i \in N} v_i(c_i, p'_i)\pi'_i(\mathbf{c}', \mathbf{p}')$. We say an auction mechanism \mathcal{M} is efficient (EF) if for all reports $(\mathbf{c}', \mathbf{p})$ it maximizes $W(\mathbf{c}', \mathbf{p}, \mathcal{M})$ and $\mathbf{p} \in \Pi_i(\mathbf{c}')$.

Definition 3. An auction mechanism \mathcal{M} is efficient (EF) if for all reports $(\mathbf{c}', \mathbf{p})$

$$\mathcal{M} \in \arg \max_{\mathcal{M}'} W(\mathbf{c}', \mathbf{p}, \mathcal{M}'), \quad (3)$$

and the reported prices $\mathbf{p} \in \Pi_i(\mathbf{c}')$.

In other words, the EF-RP property only asks the mechanism to maximize the social welfare with every reports and the reported prices may not belong to $\Pi_i(\mathbf{c}')$, and the EF property requires the mechanism to maximize the social welfare with every reports and the reported prices must lie in $\Pi_i(\mathbf{c}')$. Clearly, if an auction mechanism is EF, it is also EF-RP, but the reverse is not true. We next define several other properties that an auction mechanism should satisfy.

Definition 4. $\mathcal{M} = (\pi, x)$ is incentive compatible (IC) if for all i , all c_i , all \mathbf{p} , and all \mathbf{c}' ,

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}, \mathcal{M}) \geq u_i(c_i, (c'_i, \mathbf{c}'_{-i}), \mathbf{p}, \mathcal{M}). \quad (4)$$

That is, incentive compatibility requires that submitting true product costs is a dominant strategy for all advertisers. Another important concept is called individual rationality. This property guarantees that each advertiser will not receive a negative utility when revealing her product cost truthfully.

Definition 5. $\mathcal{M} = (\pi, x)$ is individually rational (IR) if for all i , all c_i , all \mathbf{p} , and all \mathbf{c}'_{-i} ,

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}, \mathcal{M}) \geq 0. \quad (5)$$

Note that if an auction mechanism violates the IR property, some advertisers may obtain negative utilities when revealing their true types, in which case quitting the auction is the best response. Therefore, individual rationality is also known as the participation constraint. Given advertisers' reports $(\mathbf{c}', \mathbf{p})$, the platform's revenue generated by an auction mechanism \mathcal{M} is the sum of all advertisers' payments, denoted by $R(\mathbf{c}', \mathbf{p}, \mathcal{M}) = \sum_{i \in N} x_i(\mathbf{c}', \mathbf{p})$. A major reason of adopting an auction to sell the display opportunities is to improve the performance of the advertising system, especially to increase the platform's revenue, so an auction with an external subsidy is not compelling.

Definition 6. $\mathcal{M} = (\pi, x)$ is weakly budget balanced (WBB) if for all reports $(\mathbf{c}', \mathbf{p})$,

$$R(\mathbf{c}', \mathbf{p}, \mathcal{M}) \geq 0. \quad (6)$$

In the following contents, we study auctions that satisfy IC, IR and other desired properties in the presence of display prices. We first provide a characterization for all IC and IR auction mechanisms, then using this characterization to design auctions towards different objectives.

3 Characterizations of IC and IR Auctions with Display Prices

In this section, we characterize all IC and IR auctions in the presence of display prices. We first present two necessary properties that an IC mechanism (π, x) should hold, then show the two properties are also sufficient conditions for an auction mechanism to be IC.

Lemma 1. If an auction mechanism (π, x) is IC, then π_i is non-increasing in c'_i for all i , all \mathbf{p} and all \mathbf{c}'_{-i} .

Proof. Consider two reported types c_i^1 and c_i^2 of advertiser i with $c_i^1 > c_i^2$. Incentive compatibility implies that for all \mathbf{p} and \mathbf{c}'_{-i} ,

$$\begin{aligned} v_i(c_i^1, p_i) \pi(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) &\geq \\ v_i(c_i^1, p_i) \pi(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) & \end{aligned}$$

and

$$\begin{aligned} v_i(c_i^2, p_i) \pi(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) &\geq \\ v_i(c_i^2, p_i) \pi(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) &. \end{aligned}$$

Adding above two inequalities, we obtain

$$(v_i(c_i^1, p_i) - v_i(c_i^2, p_i))(\pi_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) - \pi_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p})) \geq 0.$$

Recall that $v_i(c_i, p_i)$ is decreasing in c_i , therefore the above inequality implies that

$$\pi_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) \leq \pi_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}).$$

That is, π_i is non-increasing with c'_i in any IC auction. \square

Lemma 2 unfolds the interconnections of the payment policy and the allocation policy. It shows that in any IC auction, the allocation policy determines the payment policy.

Lemma 2. If an auction mechanism (π, x) is IC, then for all i , all \mathbf{p} and all \mathbf{c}'_{-i} , $x_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ can be formulated as

$$\begin{aligned} v_i(c_i, p_i) \pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz \\ - U_i(\mathbf{c}'_{-i}, \mathbf{p}), \end{aligned} \quad (7)$$

where $U_i(\mathbf{c}'_{-i}, \mathbf{p})$ is independent of i 's cost report.

Proof. Given any IC mechanism $\mathcal{M} = (\pi, x)$, based on Def. 4, we have that for all i , all \mathbf{p} and all \mathbf{c}'_{-i} the following equation must hold:

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) = \max_{c'_i} v_i(c'_i, p_i) \pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}).$$

By the envelope theorem, this equation is equivalent to the following condition:

$$\begin{aligned} \frac{\partial u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p})}{\partial c_i} \\ = \frac{\partial (v_i(c'_i, p_i) \pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}))}{\partial c_i} |_{c'_i=c_i} \\ = -\lambda_i(p_i) \pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}). \end{aligned} \quad (8)$$

Integrating both sides of formula (8) over $[c_i, \bar{c}_i]$ on c_i , we have that i 's utility in an IC auction can be denoted by

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) = U_i(\mathbf{c}'_{-i}, \mathbf{p}) + \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz,$$

where $U_i(\mathbf{c}'_{-i}, \mathbf{p}) = u_i(\bar{c}_i, (\bar{c}_i, \mathbf{c}'_{-i}), \mathbf{p})$ is independent of advertiser i 's cost report. Based on formula (1), we can further get that in an IC auction (π, x) advertiser i 's payment $x_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ can be formulated as

$$\begin{aligned} v_i(c_i, p_i) \pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz \\ - U_i(\mathbf{c}'_{-i}, \mathbf{p}). \end{aligned}$$

\square

As $\lambda_i(p_i)$ and $\pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ are non-negative, Lemma 2 also suggests that an advertiser's utility is non-increasing with her product cost. Our next result indicates that the above two conditions are also sufficient for an auction to be IC.

Theorem 1. An auction mechanism (π, x) is IC if and only if for all i , all \mathbf{p} and all \mathbf{c}'_{-i} :

1. π_i is non-increasing in c_i ;
2. x_i can be formulated as

$$\begin{aligned} v_i(c_i, p_i) \pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz \\ - U_i(\mathbf{c}'_{-i}, \mathbf{p}). \end{aligned}$$

Proof. To prove this theorem, it suffices to prove that if an auction mechanism $\mathcal{M} = (\pi, x)$ satisfies Condition 1 and 2,

then it is IC. Given an advertiser i with true cost c_i , to prove IC, we need to show that the inequality

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) \geq u_i(c_i, (c'_i, \mathbf{c}'_{-i}), \mathbf{p}) \quad (9)$$

holds for all $c'_i \neq c_i$, all \mathbf{p} and all \mathbf{c}'_{-i} . Plugging in the formula of $x_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ and making simplification, the inequality (9) can be reformulated as

$$\begin{aligned} \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz &\geq \pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p})(c'_i - c_i) \\ &+ \int_{c'_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz. \end{aligned} \quad (10)$$

Case 1: If $c'_i > c_i$, the inequality (10) is equivalent to

$$\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p})(c'_i - c_i) \leq \int_{c_i}^{c'_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz, \quad (11)$$

which is true under Condition 1.

Case 2: If $c'_i < c_i$, the inequality (10) is equivalent to

$$\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p})(c_i - c'_i) \geq \int_{c'_i}^{c_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz, \quad (12)$$

which is also true under Condition 1. \square

Besides the IC property, another desired property is individual rationality, which requires all advertiser's utility to be non-negative when acting truthfully.

Theorem 2. An IC auction mechanism (π, x) is IR if and only if for all i , all \mathbf{p} and all \mathbf{c}'_{-i} ,

$$U_i(\mathbf{c}'_{-i}, \mathbf{p}) \geq 0. \quad (13)$$

Proof. (“ \Rightarrow ”) For any IC auction mechanism \mathcal{M} , we have that advertiser i 's utility $u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p})$ is

$$U_i(\mathbf{c}'_{-i}, \mathbf{p}) + \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz.$$

Since $\lambda_i(p_i) \geq 0$ and $\pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})$ is non-negative, we know that if $U_i(\mathbf{c}'_{-i}, \mathbf{p}) \geq 0$, then $u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) \geq 0$.

("“ \Leftarrow ”) If $U_i(\mathbf{c}'_{-i}, \mathbf{p}) < 0$ for some \mathbf{p} and \mathbf{c}'_{-i} , then an advertiser with a report (\bar{c}_i, p_i) will obtain a negative utility, which violates the IR property. \square

Theorem 1 and 2 show the importance of the price information in auction design. In order to incentivize advertisers to report their product costs truthfully, the selling platform need take the display prices into consideration when designing ad auctions. As the platform acts as a profit-maximizing agent, for convenience's sake the term $U_i(\mathbf{c}_{-i}, \mathbf{p})$ will be treated as zero hereafter for all i , all \mathbf{c}_{-i} and all \mathbf{p} , without violating the IC and IR properties.

To evaluate the performance of a given auction, we need to figure out how each advertiser submits her display price. If the display prices are exogenously determined, the techniques for traditional auctions can apply here. Otherwise, we need to compute the display prices in equilibrium at first. In the following, we first study auctions with non-strategic display prices, then turn to the general scenario where the advertisers are allowed to strategize their display prices.

4 Auction Design with Non-Strategic Display Prices

To keep and improve advertising performance, in some commerce activities advertisers seldom adjust their display prices frequently, especially when they also have offline brick-and-mortar shops. In this section, we investigate the auction design problem under the assumption of non-strategic display prices, i.e., the reported prices are exogenously determined and not affected by the underlying ad auctions.

4.1 Welfare-Maximizing Auctions with Reported Prices

Social welfare reflects the efficiency of a given allocation, which is defined as the summation of all agents' utilities. Based on Theorem 1 and Theorem 2, we next propose an auction mechanism, called welfare maximizer with reported prices (short for WM-RP), to maximize the social welfare. As the reported display prices are assumed to be independent of the auctions, we adopt the concept of EF-RP (see Def. 2) to characterize social welfare maximization.

Welfare Maximizer with Reported Prices (WM-RP)

- **Allocation policy:** Given reports $(\mathbf{c}', \mathbf{p})$, allocate the ad slot to maximize $\sum_{i=1}^N v_i(c'_i, p_i) \pi_i(\mathbf{c}', \mathbf{p})$, break tie arbitrarily.
- **Payment policy:** For all advertiser $i \in N$, her payment $x_i(\mathbf{c}', \mathbf{p})$ is defined below:
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 0$, then $x_i(\mathbf{c}', \mathbf{p}) = 0$;
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 1$, then $x_i(\mathbf{c}', \mathbf{p})$ is defined as

$$v_i(v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i), p_i),$$

where v_i^{-1} is the inverse function of v_i w.r.t. c_i and $v^{(2)}(\mathbf{c}', \mathbf{p})$ denotes the second highest value.

In the WM-RP, the slot is allocated to the advertiser with the highest reported value, and only the winning advertiser pays to the platform. Next, we prove that the WM-RP maximizes the social welfare for any reported display price profile.

Proposition 1. The WM-RP is IC, IR and EF-RP.

Proof. To prove this proposition it is sufficient to show that the WM-RP is IC and IR. Firstly, it is straightforward that the allocation policy is non-increasing in c_i for all i , all \mathbf{p} and all \mathbf{c}_{-i} , so the first condition of Theorem 1 is satisfied. Secondly, we show that the payment policy is identical to (7) with $U_i(\mathbf{c}'_{-i}, \mathbf{p}) = 0$. According to the allocation policy, the slot will be allocated to the one with the highest value $v_i(c'_i, p_i)$. Given reports $(\mathbf{c}', \mathbf{p})$, let $v^{(2)}(\mathbf{c}', \mathbf{p})$ be the second highest value under $(\mathbf{c}', \mathbf{p})$. For all losers i , $\pi_i(\mathbf{c}', \mathbf{p}) = 0$ for all $c''_i \geq c'_i$ and therefore i 's payment $x_i(\mathbf{c}', \mathbf{p})$ is zero according to (7). For the winner i , we know that her allocation $\pi_i(\mathbf{c}', \mathbf{p}) = 1$ as long as

$$v_i(c'_i, p_i) \geq v^{(2)}(\mathbf{c}', \mathbf{p}),$$

which is equivalent to the condition that

$$c'_i \leq v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i),$$

where v_i^{-1} is the inverse function of v_i w.r.t c_i (recall that v_i is non-increasing in c_i , so the reverse function v_i^{-1} is existing). Let $U_i(\mathbf{c}'_{-i}, \mathbf{p}) = 0$ for all i , all \mathbf{c}'_{-i} and all \mathbf{p} . Then according to (7), i 's payment $x_i(\mathbf{c}', \mathbf{p})$ is identical to

$$\begin{aligned} & v_i(c'_i, p_i) \pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i) \int_{c'_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz \\ &= v_i(c'_i, p_i) - \lambda_i(p_i) \int_{c'_i}^{v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i)} 1 dz \\ &\quad - \lambda_i(p_i) \int_{v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i)}^{\bar{c}_i} 0 dz \\ &= v_i(c'_i, p_i) - \lambda_i(p_i)(v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i) - c'_i) \\ &= v_i(v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i), p_i). \end{aligned}$$

□

As v_i^{-1} is the inverse function of v_i , the winner's payment $v_i(v_i^{-1}(v^{(2)}(\mathbf{c}', \mathbf{p}), p_i), p_i)$ is exactly $v^{(2)}(\mathbf{c}', \mathbf{p})$ —the second highest reported value. Recall that given any monotonic allocation policy, the only freedom of designing an IC auction is the choices of $U_i(\mathbf{c}'_{-i}, \mathbf{p})$. Since IR requires $U_i(\mathbf{c}'_{-i}, \mathbf{p}) \geq 0$ and we set $U_i(\mathbf{c}'_{-i}, \mathbf{p})$ to be zero in the WM-RP, the following result is straightforward.

Corollary 1. *Among all IR, IC and EF-RP auctions, the WM-RP maximizes the platform's revenue.*

Besides the allocation efficiency, another desiderata of the platform is revenue. We next investigate how to design auction mechanisms that maximize the platform's revenue.

4.2 Revenue-Maximizing Auctions with Reported Prices

The following lemma gives a succinct description of advertiser's expected payment, which plays a key role in characterizing revenue-maximizing auctions.

Lemma 3. *Given an IC and IR mechanism \mathcal{M} , reported prices \mathbf{p} and a type profile of others \mathbf{c}_{-i} , the expected payment $E_{\mathbf{c} \sim \mathcal{C}}[x_i(\mathbf{c}, \mathbf{p})]$ of advertiser i is equal to:*

$$E_{\mathbf{c} \sim \mathcal{C}}[\pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)], \quad (14)$$

where $\phi_i(c_i, p_i) = v_i(c_i, p_i) - \lambda_i(p_i) \frac{C_i(c_i)}{C'_i(c_i)}$ is called the virtual value of advertiser i .

Proof. Given an IC and IR mechanism \mathcal{M} , reported prices \mathbf{p} and \mathbf{c}_{-i} , according to the proof of Theorem 1 advertiser i 's expected payment $E_{\mathbf{c} \sim \mathcal{C}}[x_i(\mathbf{c}, \mathbf{p})]$ is equal to

$$\begin{aligned} & \int_{\underline{c}_i}^{\bar{c}_i} [v_i(y, p_i) \pi_i(y, \mathbf{c}_{-i}, \mathbf{p})] C'_i(y) dy \\ &\quad - \int_{\underline{c}_i}^{\bar{c}_i} C'_i(y) \int_y^{\bar{c}_i} \lambda_i(p_i) \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz dy. \end{aligned}$$

Since p_i is independent of advertiser i 's type c_i , we can change the order of integration and get that

$$\begin{aligned} & \int_{\underline{c}_i}^{\bar{c}_i} C'_i(y) \int_y^{\bar{c}_i} \lambda_i(p_i) \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz dy \\ &= \int_{\underline{c}_i}^{\bar{c}_i} \lambda_i(p_i) \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) \int_0^z C'_i(y) dy dz \\ &= \int_{\underline{c}_i}^{\bar{c}_i} \lambda_i(p_i) \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) C_i(z) dz. \end{aligned}$$

Therefore, $E_{\mathbf{c} \sim \mathcal{C}}[x_i(\mathbf{c}, \mathbf{p})]$ can be formulated as

$$\begin{aligned} & \int_{\underline{c}_i}^{\bar{c}_i} [v_i(y, p_i) \pi_i(y, \mathbf{c}_{-i}, \mathbf{p}) \\ &\quad - \lambda_i(p_i) \pi_i(y, \mathbf{c}_{-i}, \mathbf{p}) \frac{C_i(y)}{C'_i(y)}] C'_i(y) dy \\ &= E_{\mathbf{c} \sim \mathcal{C}}[\pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)], \end{aligned}$$

where $\phi_i(y, p_i) = v_i(y, p_i) - \lambda_i(p_i) \frac{C_i(y)}{C'_i(y)}$. □

Based on Lemma 3, we can characterize the seller's expected revenue for any given display price profile.

Theorem 3. *Given reported prices \mathbf{p} and an IC and IR mechanism \mathcal{M} , the expected revenue $E_{\mathbf{c} \sim \mathcal{C}}[\sum_{i=1}^N x_i(\mathbf{c}, \mathbf{p})]$ of the platform is equal to the the expected virtual social welfare*

$$E_{\mathbf{c} \sim \mathcal{C}}[\sum_{i=1}^N \pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)]. \quad (15)$$

Proof. Take the expectation, with respect to \mathbf{c}_{-i} , of both sides of

$$E_{\mathbf{c} \sim \mathcal{C}}[x_i(\mathbf{c}, \mathbf{p})] = E_{\mathbf{c} \sim \mathcal{C}}[\pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)],$$

we obtain that

$$E_{\mathbf{c} \sim \mathcal{C}}[x_i(\mathbf{c}, \mathbf{p})] = E_{\mathbf{c} \sim \mathcal{C}}[\pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)].$$

Apply linearity of expectations, we have that

$$\begin{aligned} E_{\mathbf{c} \sim \mathcal{C}}[\sum_{i=1}^N x_i(\mathbf{c}, \mathbf{p})] &= \sum_{i=1}^N E_{\mathbf{c} \sim \mathcal{C}}[x_i(\mathbf{c}, \mathbf{p})] \\ &= \sum_{i=1}^N E_{\mathbf{c} \sim \mathcal{C}}[\pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)] \\ &= E_{\mathbf{c} \sim \mathcal{C}}[\sum_{i=1}^N \pi_i(\mathbf{c}, \mathbf{p})\phi_i(c_i, p_i)]. \end{aligned}$$

□

Theorem 3 indicates that the platform's expected revenue is identical to the expectation of the virtual social welfare. To maximize the platform's expected revenue, we can maximize the virtual social welfare pointwisely for any report (\mathbf{c}, \mathbf{p}) . Based on this observation, we now propose the virtual welfare maximizer with reported prices (short for VWM-RP).

Virtual Welfare Maximizer with Reported Prices (VWM-RP)

- **Allocation Policy:** Given reports $(\mathbf{c}', \mathbf{p})$, allocate the ad slot to maximize $\sum_{i=1}^N \phi_i(c'_i, p_i) \pi_i(\mathbf{c}', \mathbf{p})$, break tie arbitrarily.
- **Payment policy:** For all advertiser $i \in N$, her payment $x_i(\mathbf{c}', \mathbf{p})$ is defined below:
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 0$, then $x_i(\mathbf{c}', \mathbf{p}) = 0$;
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 1$, then $x_i(\mathbf{c}', \mathbf{p})$ is defined as $v_i(\phi_i^{-1}(\max\{\phi^{(2)}(\mathbf{c}', \mathbf{p}), 0\}, p_i), p_i)$,
 where ϕ_i^{-1} is the inverse function of ϕ_i w.r.t. c_i and $\phi^{(2)}(\mathbf{c}', \mathbf{p})$ denotes the second highest virtual value.

Different from the WM-RP, in the VWM-RP the allocation policy maximizes the virtual social welfare. Note that if all advertisers' virtual values are negative, the platform will not allocate the slot according to the allocation policy, i.e., the VWM-RP is not EF-RP. To implement the allocation policy of the VWM-RP, we need a regular condition on the distributions, which is defined below.

Definition 7. A distribution \mathcal{C}_i is regular if the virtual value function $\phi_i(c_i, p_i)$ is non-increasing in c_i for all p_i .

Recall that $\phi_i(c_i, p_i)$ can be reformulated as $v_i(c_i + \frac{1}{\sigma_i(c_i)}, p_i)$, where $\sigma(c_i) = C'_i(c_i)/C_i(c_i)$ is the reverse hazard rate of \mathcal{C}_i . Since $v_i(\cdot, \cdot)$ is non-increasing with the first variable, a sufficient condition for regularity is that $\sigma(\cdot)$ is non-increasing. We next show if the distributions are regular, the VWM-RP will maximize the seller's revenue.

Theorem 4. Given a set of regular distributions $\mathcal{C}_1, \dots, \mathcal{C}_n$, the VWM-RP is IC and IR, and maximizes the platform's revenue for any given set of display prices.

Proof. If $\mathcal{C}_1, \dots, \mathcal{C}_n$ are regular, then the allocation policy of the VWM-RP is non-increasing in c_i . Following the proof of Proposition 1, we can verify that the payment policy of the VWM-RP is consistent with (7), and therefore the VWM-RP is IC and IR according to Theorem 1. Since the VWM-RP maximizes the virtual welfare $\sum_{i=1}^N \pi_i(\mathbf{c}', \mathbf{p}) \phi_i(c'_i, p_i)$ pointwisely for each report $(\mathbf{c}', \mathbf{p})$, then based on Theorem 3, we know the VWM-RP maximizes the platform's revenue. \square

If \mathcal{C}_i is not regular, we can use the “ironing technique” to obtain a surrogate $\tilde{\mathcal{C}}_i$ [Myerson, 1981], which is regular and replaces \mathcal{C}_i in the allocation policy.

5 Auction Design with Strategic Display Prices

In this section, we consider the general scenario where advertisers can report their display prices strategically. For this scenario, advertiser i will choose a display price that maximizes her expected utility for any auction mechanism. Since product costs are private information for all advertisers, Bayesian Nash Equilibrium (BNE) is a suitable solution concept for this setting, which is formally defined below.

Definition 8. Given an IC and IR auction \mathcal{M} , a strategy profile $\mathbf{p}^{\mathcal{M}} = (p_1^{\mathcal{M}}, \dots, p_n^{\mathcal{M}})$ is a Bayesian Nash Equilibrium if for all i , all p_i' and all c_i ,

$$\begin{aligned} & E_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[u_i(c_i, \mathbf{c}, (p_i^{\mathcal{M}}(c_i), \mathbf{p}_{-i}^{\mathcal{M}}(\mathbf{c}_{-i})))] \\ & \geq E_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[u_i(c_i, \mathbf{c}, (p_i', \mathbf{p}_{-i}^{\mathcal{M}}(\mathbf{c}_{-i})))], \end{aligned}$$

where $\mathbf{p}_{-i}^{\mathcal{M}}(\mathbf{c}_{-i}) = \{p_j^{\mathcal{M}}(c_j)\}_{j \in N \setminus \{i\}}$.

In other words, a strategy profile $\mathbf{p}^{\mathcal{M}}$ is a BNE if no one can gain more utilities by unilaterally deviating from $p_i^{\mathcal{M}}(c_i)$. Finding BNE in games is known to be a hard problem both analytically and computationally [Naroditskiy and Greenwald, 2007] and previous works derived the BNE analytically only for the simplest auction settings [Krishna, 2009]. Recall that the price information enters into the allocation policy according to Theorem 1, hence it is impossible to obtain a closed form of the equilibrium prices for all auctions. For tractability, in the following contents we study two special classes of allocation policies, namely the price-independent allocation policy and the affine maximizer allocation policy, in which the equilibrium prices are given analytically.

5.1 Price-Independent Allocation Policy

We first study price-independent allocation policy, where the slot is allocated without considering the reported display prices. It can apply to the circumstances where the display prices are unavailable before the auction or the advertisers tend to adjust their display prices dynamically after the auction. The formal definition of price-independent allocation policy is given below.

Definition 9 (Price-Independent Allocation Policy). We say an allocation policy π is price-independent (PI) if for all $i \in N$, and any two reports $(\mathbf{c}', \mathbf{p}^1)$ and $(\mathbf{c}', \mathbf{p}^2)$,

$$\pi_i(\mathbf{c}', \mathbf{p}^1) = \pi_i(\mathbf{c}', \mathbf{p}^2).$$

Based on Theorem 1, we can easily derive the equilibrium prices for PI allocation policies.

Proposition 2. Given any IC and IR mechanism \mathcal{M} with a PI allocation policy, the price

$$\bar{p}_i^{\mathcal{M}}(c_i) = \arg \max_{p_i'} \{\lambda_i(p_i')\}$$

forms the (dominant-strategy) equilibrium price for all i .

Proof. Given any IC and IR mechanism \mathcal{M} , according to Theorem 1 advertiser i 's utility can be simplified as

$$u_i(c_i, (c_i, \mathbf{c}_{-i}), \mathbf{p}) = \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz.$$

If the allocation policy is PI, then $\int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz$ is independent of p_i . Therefore, the expected utility of advertiser i is identical to

$$\begin{aligned} & E_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[u_i(c_i, \mathbf{c}, \mathbf{p}, \mathcal{M})] \\ & = E_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[\lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz] \\ & = \lambda_i(p_i) E_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[\int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}_{-i}) dz]. \end{aligned}$$

To maximize the expected utility, i can choose the price p_i that maximizes $\lambda_i(p_i)$, no matter what the others report. \square

Proposition 2 indicates that the equilibrium price $\bar{p}_i^M(c_i)$ is a dominate strategy and is independent of the cost reports of advertisers. If the platform considers all PI allocation policies, then the following auction maximizes the revenue.

Virtual Welfare Maximizer with Price-Independent Allocations (VWM-PIA)

Given a set of convention rate functions $\{\lambda_i\}_{i \in N}$, compute the equilibrium price profile $\bar{\mathbf{p}} = \{\bar{p}_i\}_{i \in N}$, where $\bar{p}_i = \arg \max_{p'_i} \lambda_i(p'_i)$.

- **Allocation Policy:** Given reports $(\mathbf{c}', \mathbf{p})$, allocate the ad slot to maximize $\sum_{i=1}^N \phi_i(c'_i, \bar{p}_i) \pi_i(\mathbf{c}', \mathbf{p})$, break tie arbitrarily.
- **Payment policy:** For all advertiser $i \in N$, her payment $x_i(\mathbf{c}', \mathbf{p})$ is defined below:
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 0$, then $x_i(\mathbf{c}', \mathbf{p}) = 0$;
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 1$, then $x_i(\mathbf{c}', \mathbf{p})$ is defined as
$$v_i(\phi_i^{-1}(\max\{\phi^{(2)}(\mathbf{c}', \mathbf{p}), 0\}, p_i), p_i).$$

Theorem 5. Given a set of regular distributions $\mathcal{C}_1, \dots, \mathcal{C}_n$, the VWM-PIA is IC and IR, and maximizes the platform's revenue over all PI allocation policies.

Proof. According to Proposition 2, if the allocation policy is PI, then the equilibrium price is cost-independent. Hence, Lemma 3 and Theorem 3 can extend to the PI allocation policies. Since the allocation policy of the VWM-PIA is PI and the cost distributions are regular, one can check that the payment policy of the VWM-PIA is consistent with (7), i.e., the VWM-PIA is IC and IR. In addition, as the allocation is PI, then each advertiser will submits \bar{p}_i as her display price in equilibrium. Recall that the VWM-PIA maximizes the virtual welfare pointwisely for each report, then according to Theorem 3 the VWM-PIA maximizes the revenue over all PI allocation policies. \square

5.2 Affine Maximizer Allocation Policy

This section investigates another family of allocation policy, called affine maximizer. In this kind of allocation policy, the slot will be allocated to maximize the weighed and boosted social welfare. The platform can utilize such allocation policies to artificially increase the winning chance of advertisers with low values or outright ban certain outcomes in order to boost the revenue [Guo et al., 2017]. The formal definition of an affine maximizer allocation policy is given below.

Definition 10. Given some advertiser weights $\mathbf{w} = (w_1, \dots, w_n) \in \mathcal{R}_+^N$ and boosts $\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{R}^N$, an allocation policy π is called an affine maximizer if for all reports $(\mathbf{c}', \mathbf{p})$, the allocations are

$$\pi(\mathbf{c}', \mathbf{p}) \in \arg \max_{\pi'(\mathbf{c}', \mathbf{p})} \{b_i + \sum_{i \in N} w_i v_i(c'_i, p_i) \pi'_i(\mathbf{c}', \mathbf{p})\}.$$

Let $\psi_i(c'_i, p_i) = b_i + w_i v_i(c'_i, p_i)$ be the weighed and booted value of advertiser i , then an affine maximizer allocation policy will allocate the slot to advertiser i with the maximum $\psi_i(c'_i, p_i)$. Accordingly, ad auctions with an affine maximizer, called affine maximizer auction, is given below.

Affine Maximizer Auction (AMA)

Predefine a set of weights $\{w_i\}_{i \in N}$ and boosts $\{b_i\}_{i \in N}$.

- **Allocation Policy:** Given reports $(\mathbf{c}', \mathbf{p})$, allocate the ad slot to maximize $\sum_{i=1}^N \psi_i(c'_i, p_i) \pi_i(\mathbf{c}', \mathbf{p})$, break tie arbitrarily.
- **Payment policy:** For all advertiser $i \in N$, her payment $x_i(\mathbf{c}', \mathbf{p})$ is defined below:
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 0$, then $x_i(\mathbf{c}', \mathbf{p}) = 0$;
 - if $\pi_i(\mathbf{c}', \mathbf{p}) = 1$, then $x_i(\mathbf{c}', \mathbf{p})$ is defined as
$$v_i(\psi_i^{-1}(\psi^{(2)}(\mathbf{c}', \mathbf{p}), p_i), p_i),$$

where ψ_i^{-1} is the inverse function of ψ_i w.r.t. c_i and $\psi^{(2)}(\mathbf{c}', \mathbf{p})$ denotes the second highest weighted and boosted value.

Proposition 3. The AMA is IC and IR, and the price

$$\tilde{p}_i^M(c_i) = \arg \max_{p'_i} \{v_i(c_i, p'_i)\}$$

forms the (dominant-strategy) equilibrium price for all i .

Proof. According to the definition of affine maximizer, we know that the allocation policy is non-increasing in c_i . In addition, we can further verify that the payment policy is consistent with (7), and thereby the AMA is IC and IR. Next, we show that $\tilde{p}_i^M(c_i) = \arg \max_{p'_i} \{v_i(c_i, p'_i)\}$ constitutes the equilibrium price for all i . Given any report (\mathbf{c}, \mathbf{p}) , by the payment policy the winner's payment in the AMA equals is

$$\lambda_i(p_i)[\psi_i^{-1}(\psi^{(2)}(\mathbf{c}, \mathbf{p}), p_i) - c_i].$$

As

$$\psi_i^{-1}(y, p_i) = p_i - \frac{y - b_i}{w_i \lambda(p_i)},$$

then winner i 's utility can be reformulated as

$$u_i(c_i, (c_i, \mathbf{c}_{-i}), \mathbf{p}) = v_i(c_i, p_i) - \frac{\psi^{(2)}(\mathbf{c}, \mathbf{p}) - b_i}{w_i}.$$

Notice that $\psi^{(2)}(\mathbf{c}, \mathbf{p})$ is independent of p_i , and thereby choosing p_i that maximizes $v_i(c_i, p_i)$ can maximize the winner's utility. Assume advertiser i submits a display price of $\tilde{p}_i^M(c_i)$, we next show that advertiser i cannot increase her utility by choosing other display prices. If i wins the item with the report $(c_i, \tilde{p}_i^M(c_i))$, then according to our previous analysis, the best submitted price is exactly $\tilde{p}_i^M(c_i)$. Otherwise, if i loses the item with the report $(c_i, \tilde{p}_i^M(c_i))$, then she will still lose by arbitrary p_i according to the definitions of ψ_i and $\tilde{p}_i^M(c_i)$. Therefore, her payment and utility are still zero. Combined with above analysis, we conclude that for all i , all others' reports $(\mathbf{c}_{-i}, \mathbf{p}_{-i})$, submitting $(c_i, \tilde{p}_i^M(c_i))$ can maximize the utility. \square

Recall that the allocation policy of the WM-RP is an instance of the affine maximizer allocation policy, therefore the following result is straightforward.

Corollary 2. *The WM-RP is EF.*

Proof. According to the definition of the WM-RP, its allocation policy is an affine maximizer with $\mathbf{b} = \{0, \dots, 0\}$ and $\mathbf{w} = \{1, \dots, 1\}$. Since the WM-RP is IC and IR for all reported prices, we know that each advertiser will submit a report $(c_i, \tilde{p}_i^M(c_i))$ to maximize her own utility. In other words, in the WM-RP the item will be allocated to the advertiser with the maximum gain $\max_{i \in N} \{v_i(c_i, \tilde{p}_i^M(c_i))\}$, implementing the efficient allocation based on Def. 3 and Proposition 3. \square

Corollary 2 shows that the platform can achieve the maximum social welfare by designing proper auction mechanism, even when the advertisers' strategize on the display prices. To optimize the platform's revenue over all affine maximizer allocation policies, we only need to adjust the parameters (\mathbf{w}, \mathbf{b}), which is a typical optimization problem. Since there is no known short-cut for calculating the expected revenue [Guo *et al.*, 2017], previous studies have developed many techniques to search for the (approximate) optimal parameters, e.g., grid-based gradient descent approach [Likhodedov and Sandholm, 2004; Likhodedov and Sandholm, 2005], linear programming based heuristic [Guo *et al.*, 2017], neural networks [Curry *et al.*, 2022].

6 Conclusion

This paper investigates the problem of ad auction design in the presence of display prices. We provide a characterization for all IC, IR auctions with display prices, and analyze the welfare-maximizing and revenue-maximizing auctions under different scenarios. Our results show that the engagement of the price information does affect the advertisers' bidding behaviors and the platform can leverage the price information to optimize the performance of ad delivery. There are many other works worthy of further investigation. For example, we only analyze two special classes of allocation policies for the strategic display price setting, a more broad class of mechanisms need be further studied. In addition, though the revenue-maximizing auction in non-strategic display price setting is identified, characterizing the revenue-maximizing auction in the general setting is still an open problem.

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