Problem Statement

- Investigate the physics of a damped, driven pendulum by accurate integration of its equation of motion

Core Task 1

The equation of motion for the angular displacement of a damped, driven pendulum was given in the manual as:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\sin\theta - q\frac{\mathrm{d}\theta}{\mathrm{d}t} + F\sin(2t/3) \tag{1}$$

This second-order differential equation was re-written as a pair of linked first-order equations:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \tag{2}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\sin\theta - q\omega + F\sin(2t/3) \tag{3}$$

This form of the equations is much easier to numerically evaluate through algorithms - for this exercise I chose to use the 4th-order Runge-Kutta algorithm from `scipy.integrate.odeint`. After implementing the algorithm, I used the initial conditions $(\theta_0, \omega_0) = (0.01, 0, 0)$ as well as set $q = F = 0$ (i.e. no damping or driving force) to test the accuracy of my solution with the analytical one (for small angles, $\theta = \cos(t)$). This was done by directly plotting the angular displacement and angular velocities of my solution with the analytical one on the same graph in the function `Test_ODE`, where excellent agreement is found up to 500 oscillations. The energy conservation of my code was also tested (in `Energy_Conservation`) plotting the energy ($\frac{1}{2}\theta^2 + \frac{1}{2}\dot{\theta}^2$) as a fraction of its initial value against time - the energy of the system is found to decrease linearly with time, losing 10% after ~ 6400 oscillations.

Now that the code is checked to have high accuracy as long as the number of oscillations is not too great, the initial amplitude of the pendulum is increased beyond the small-angle regime up to $\theta_0 = \pi$. We wish to investigate how the initial amplitude affects the period of the pendulum, so an amplitude-period graph is produced (in `Amplitude_Period)` - the total time taken T to cross the origin 100 times was recorded, with $2\frac{T}{100}$ taken as the period of the system. This period was calculated efficiently using some very convenient commands from `numpy`, namely `np.diff`,`np.sign`,and `np.where` which allowed me to reduce the array of angular velocities with time to an array of 1s and 0s, where 1 indicates a change in sign of the angular velocity (i.e. the pendulum crosses the origin). It can be seen that the period varies quite slowly with amplitude for small θ_0 but it rapidly increases as θ_0 approaches π. The value of the period for $\theta_0 = \frac{\pi}{2}$ was found to be 7.41626128715.

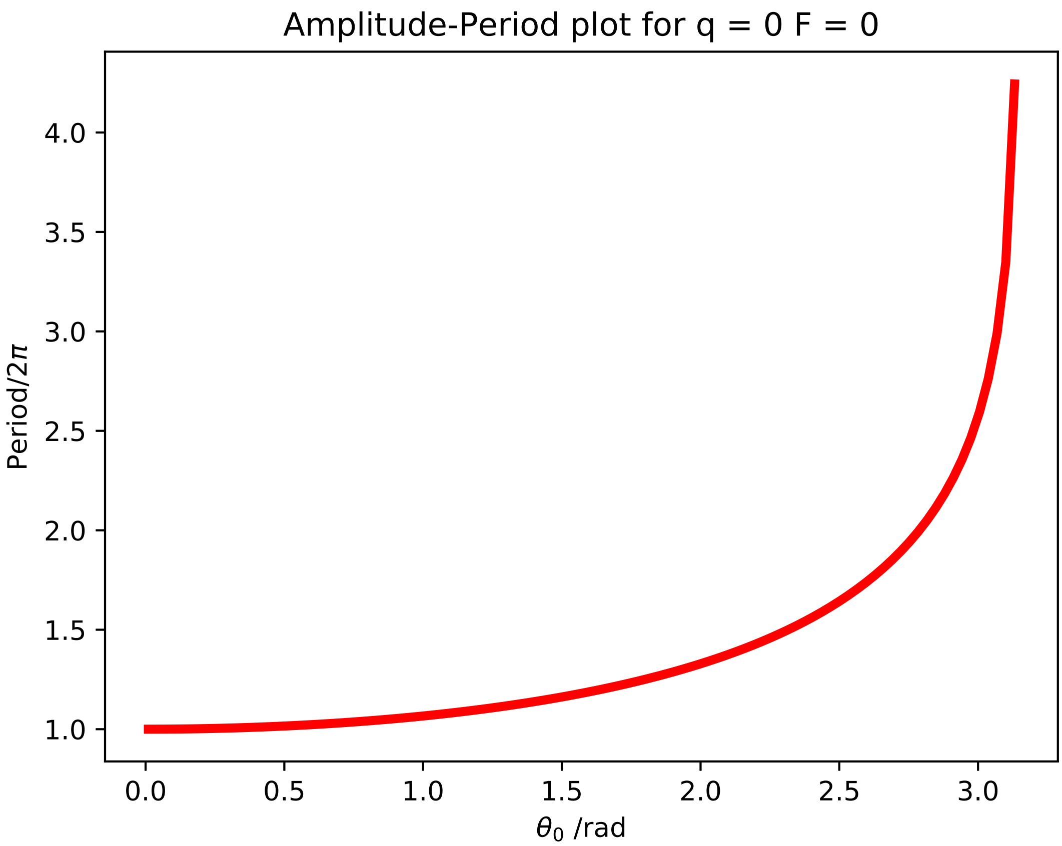

Figure 1: The period of the pendulum plotted as a function of initial amplitude from $\theta_0 = 0 \to \pi$ for $q = 0, F = 0$.

The energy conservation for large θ_0 was also investigated, where it was found that the energy is highly non-conserved, exhibiting oscillations in time with unpredictable behavior as θ_0 is increased. A part of this irregularity no doubt comes from the inadequate definition of energy given above, which really only applies in the small-angle regime.

Core Task 2

Now we turn on the damping and plot the angular displacements and velocities for $\theta_0 = 0.01, q = 0.5, 1, 10$, checking that they produce sensible results - as expected, we obtain solutions that correspond to the lightly damped, critically damped, and over-damped regimes of the damped harmonic oscillator.

Setting $q = 0.5$, now the driving force is switched on with the angular displacements and velocities plotted for $F = 0.5, 1.2, 1.44,$ and 1.465. For $F = 0.5$, the plot was as expected with the pendulum amplitudes stabilising at a value much larger than that for the non-damped pendulum. For larger values of F, it is observed that the pendulum begins to exhibit complicated, aperiodic motion. The amplitude-period plots are highly irregular for large values of F, indicating a high sensitivity of the system to initial conditions.

1 Supplementary Task 1

To further investigate this sensitivity, the angular displacements and velocities for $\theta_0 = 0.2$and $\theta_0 = 0.20001$ were plotted on top of each other for $\sim 10,000$ oscillation with $F = 1.2$. It can be seen that the solutions diverge with time and evolve very differently, again illustrating the strong sensitivity of the system to initial conditions (the "Butterfly effect").

Supplementary Task 2

A more substantial attempt at investigating the chaotic behavior of the system was done by plotting the angular displacement against angular velocity (i.e. phase space) with $\theta_0 = 0.1$and $\theta_0 = 0.3$ for $q = 0, 0.1, 0.24, 0.5$and $F = 0, 0.5, 1.2, 1.44$. All of the plots are shown in a 4x4 grid of subplots. The column with $F = 0$ is simply understood, with a clear $\frac{\pi}{2}$ phase shift between the angular displacement and velocity when no damping/driving is present, while both displacement and velocity spiral in towards the origin as damping is switched on. With $F \neq 0$, the plots show highly irregular behavior in phase space, indicating an onset of chaos.

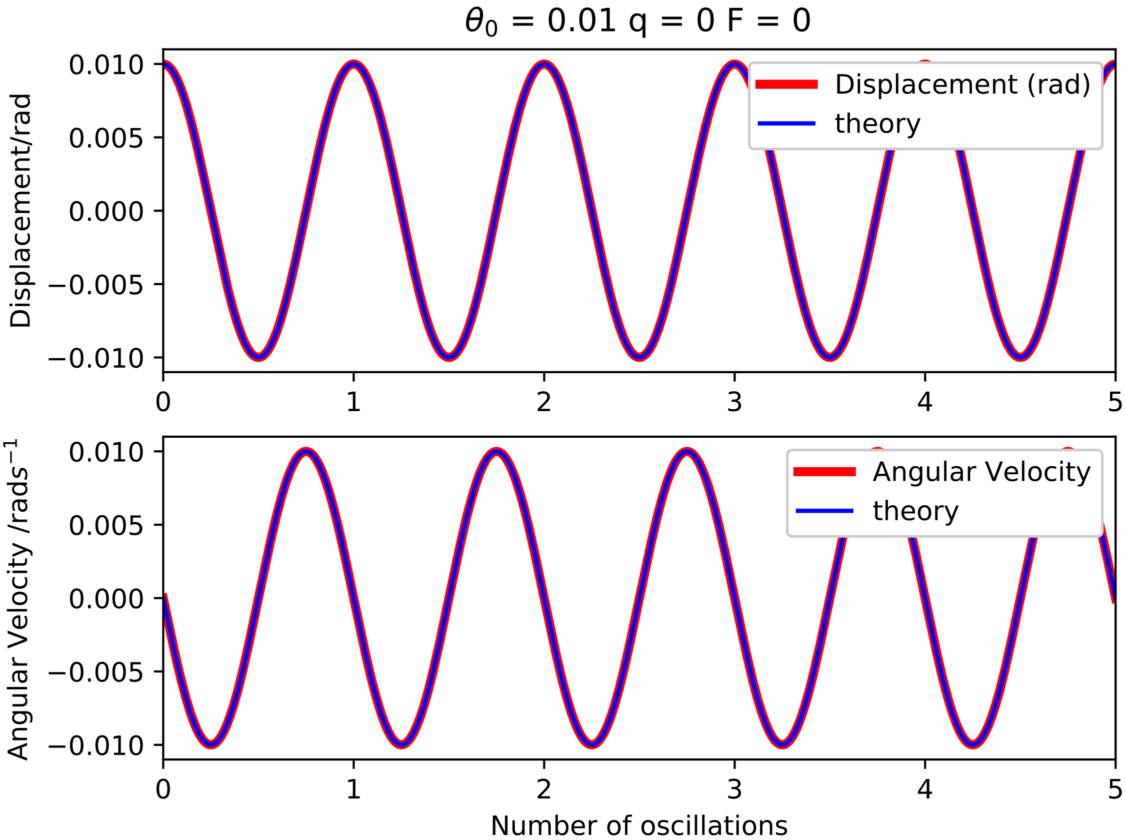

Figure 2: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 0, F = 0$ are plotted alongside the analytical solution. There is excellent agreement.

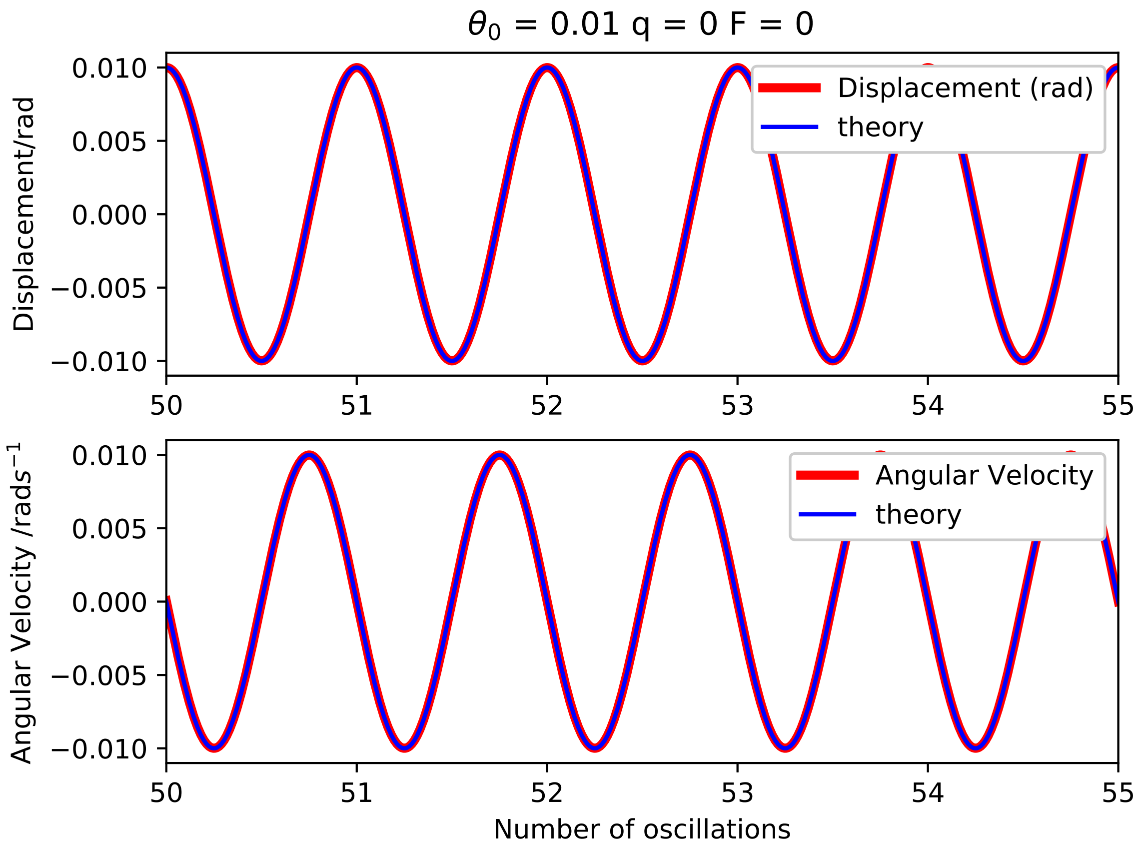

Figure 3: The angular displacement and velocities of the 50-55th oscillations for $\theta_0 = 0.01, q = 0, F = 0$ are plotted alongside the analytical solution. There is excellent agreement.

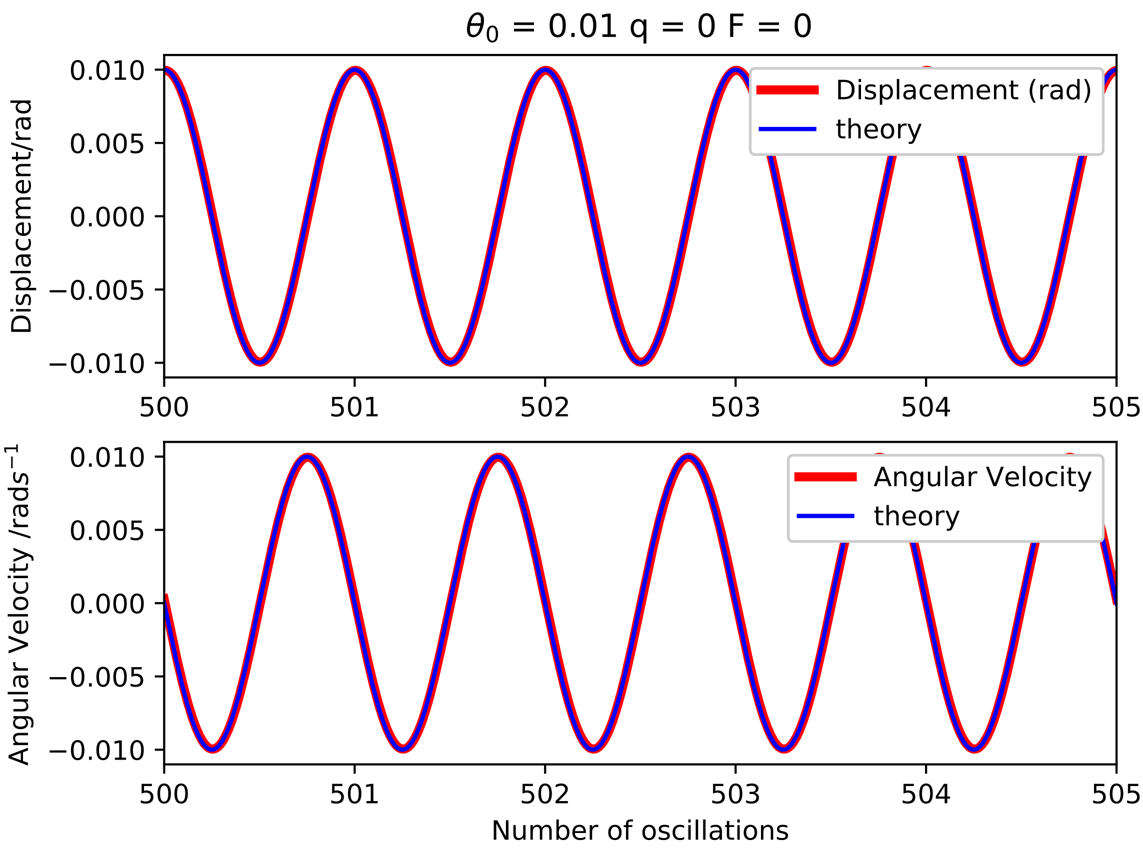

Figure 4: The angular displacement and velocities of the 500-500th oscillations for $\theta_0 = 0.01, q = 0, F = 0$ are plotted alongside the analytical solution. A slight difference between the two can be seen.

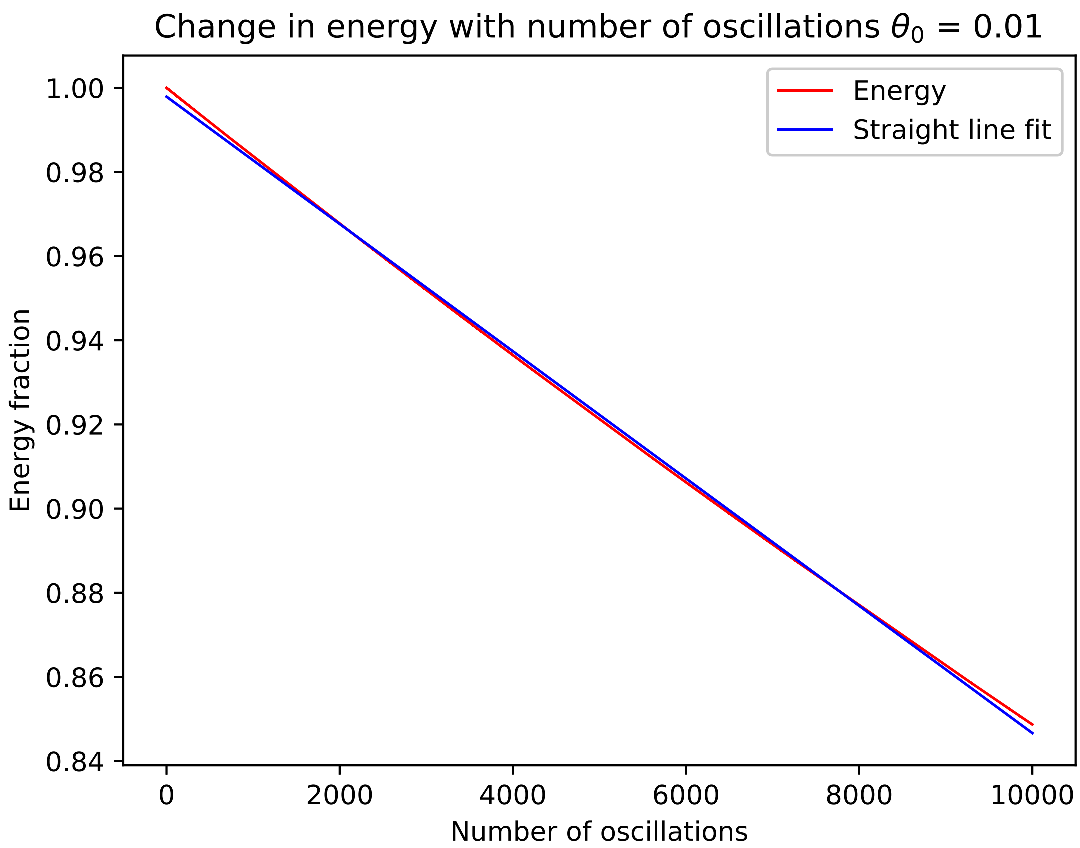

Figure 5: The energy as a fraction of the initial value against the number of oscillations for $\theta_0 = 0.01$, with a straight line fit plotted as well. The energy decreases linearly with time.

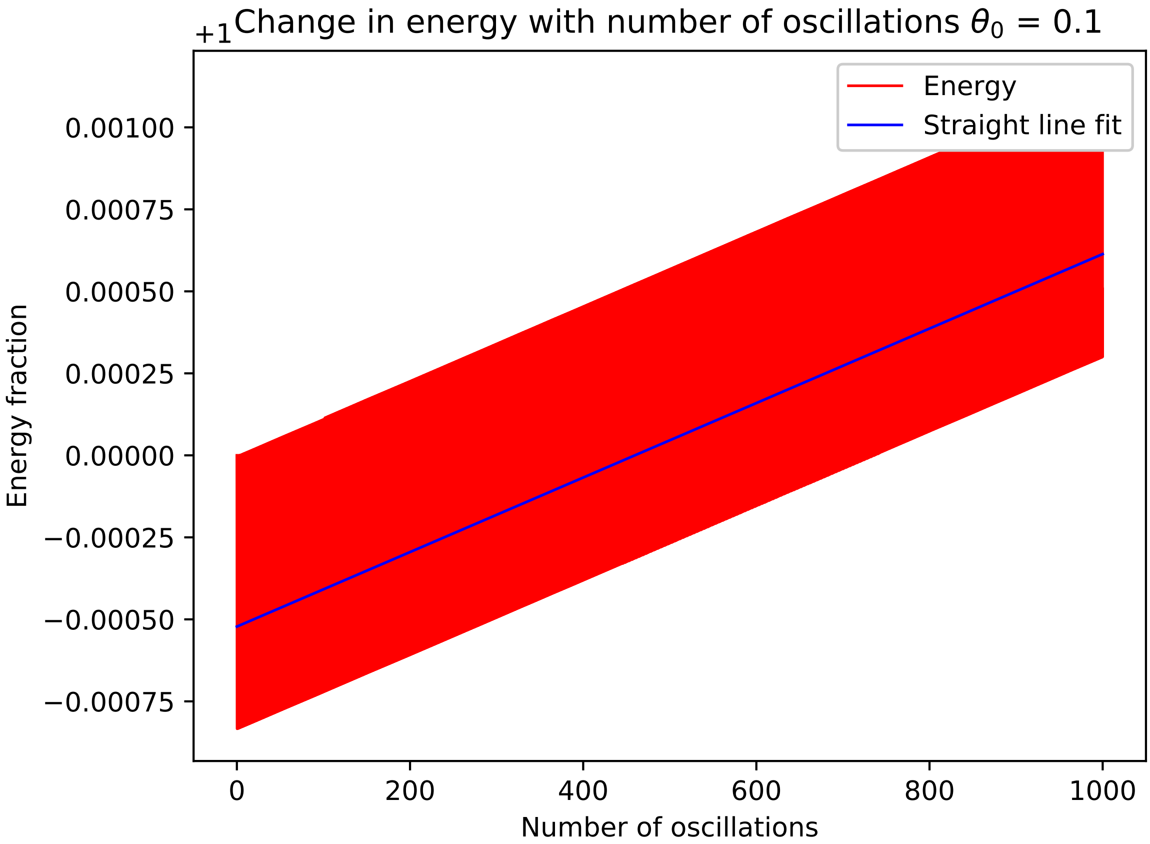

Figure 6: The energy as a fraction of the initial value against the number of oscillations for $\theta_0 = 0.1$, with a straight line fit plotted as well. The energy oscillates and increases with time.

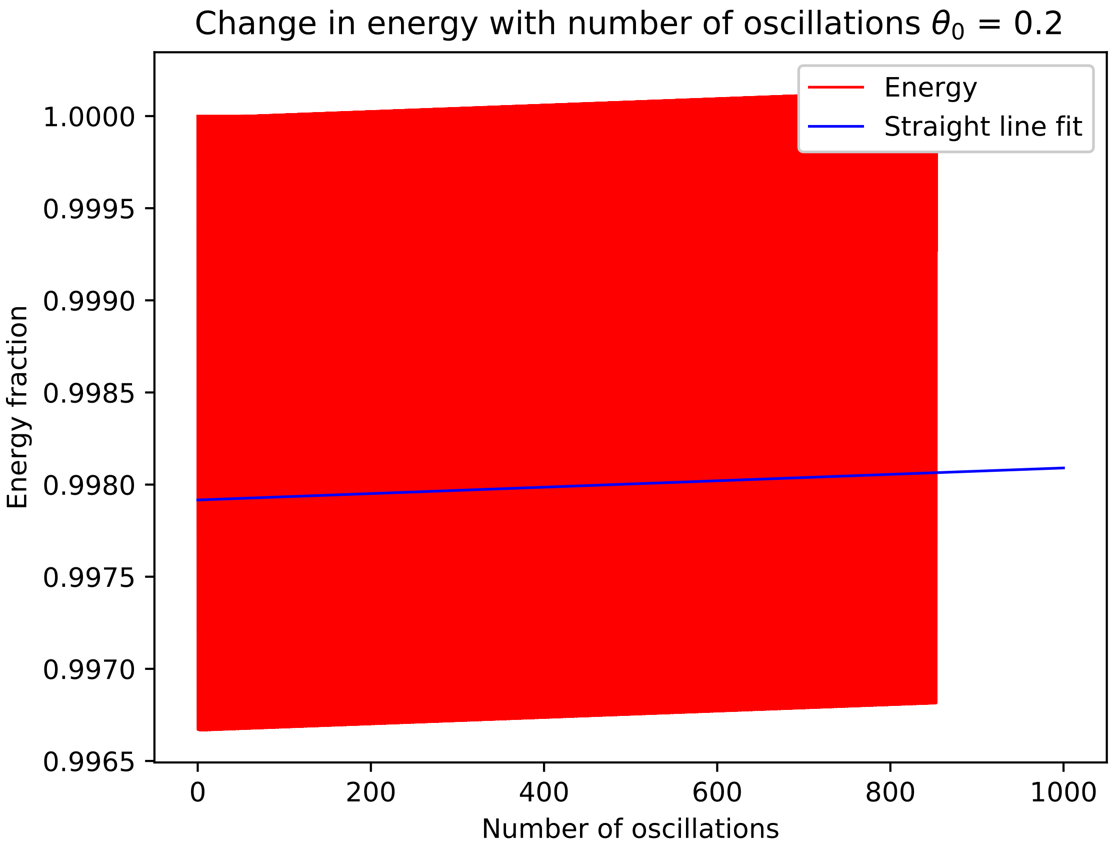

Figure 7: The energy as a fraction of the initial value against the number of oscillations for $\theta_0 = 0.2$, with a straight line fit plotted as well. The energy oscillates very rapidly about a barely increasing value.

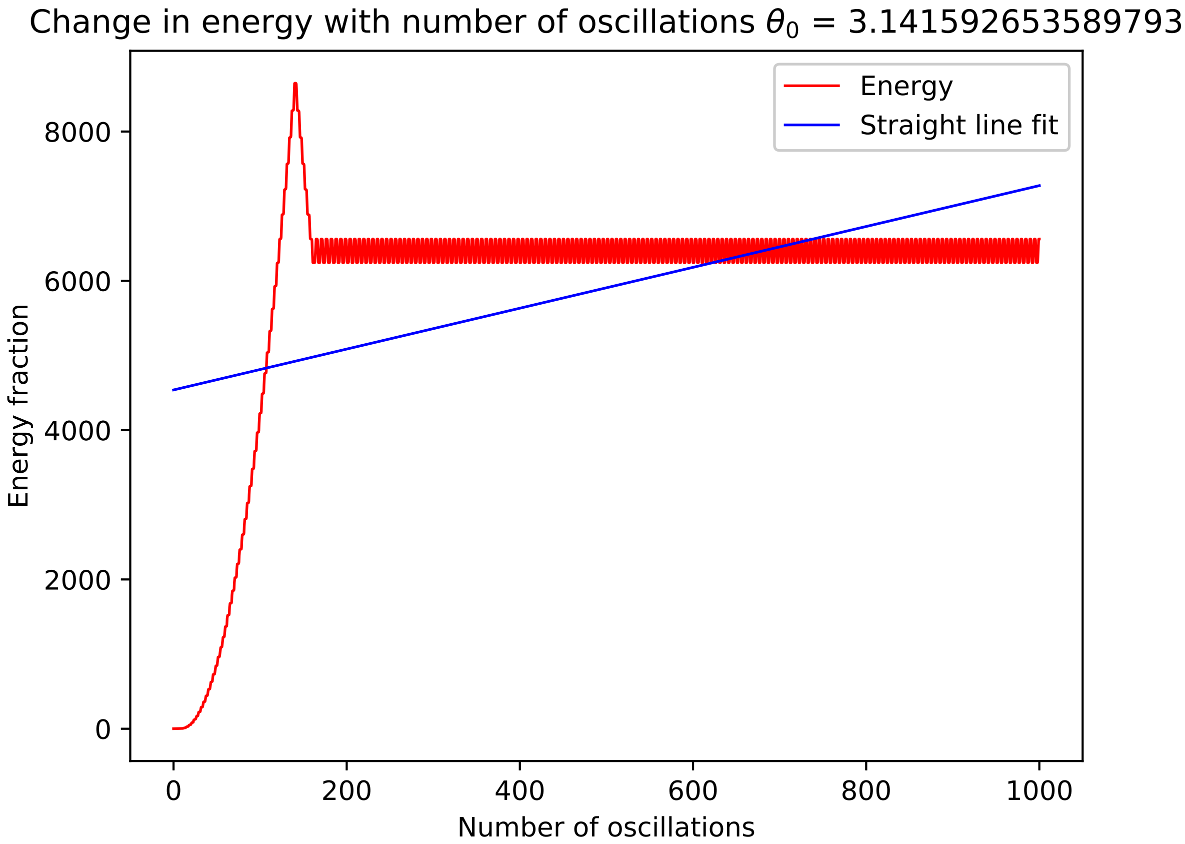

Figure 8: The energy as a fraction of the initial value against the number of oscillations for $\theta_0 = \pi$. The energy jumps to a very high value before stabilising about a fixed value.

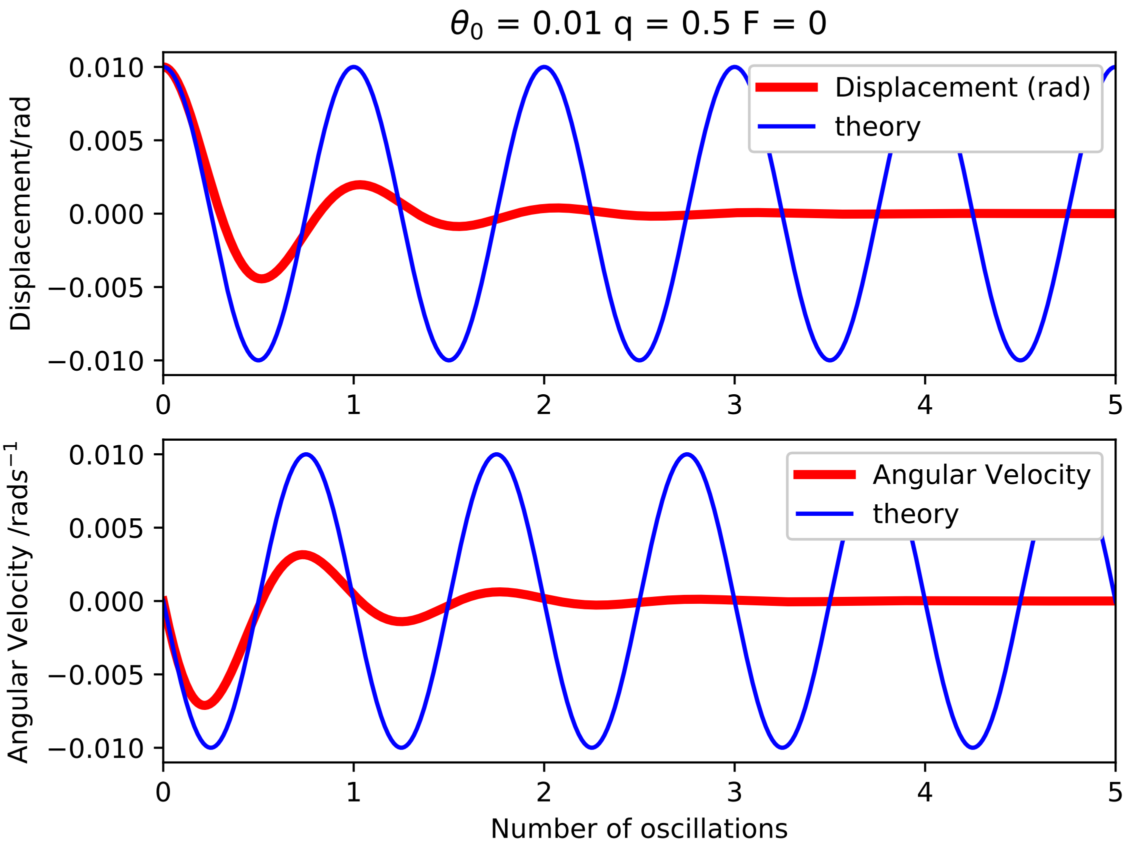

Figure 9: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 0.5, F = 0$ are plotted alongside the analytical solution. The form of the solution follows that of a lightly damped harmonic oscillator.

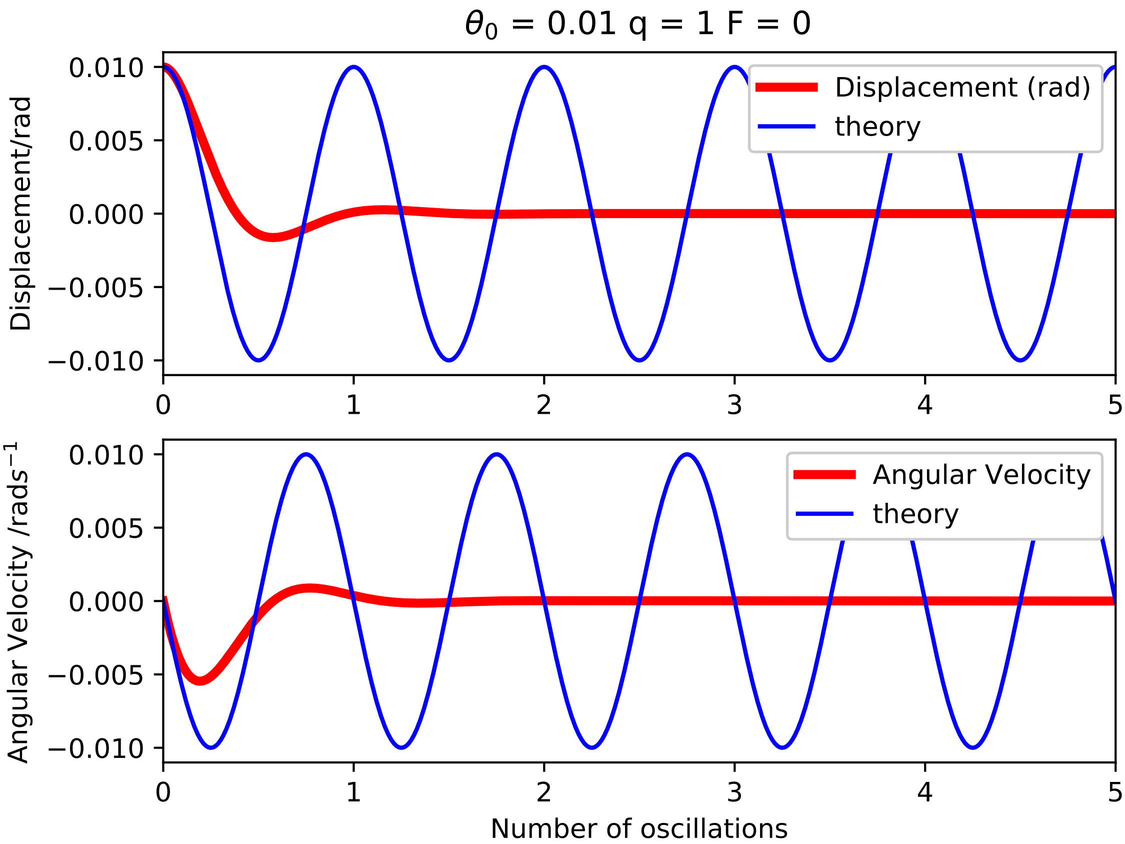

Figure 10: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 1.0, F = 0$ are plotted alongside the analytical solution. The form of the solution follows that of a critically damped harmonic oscillator.

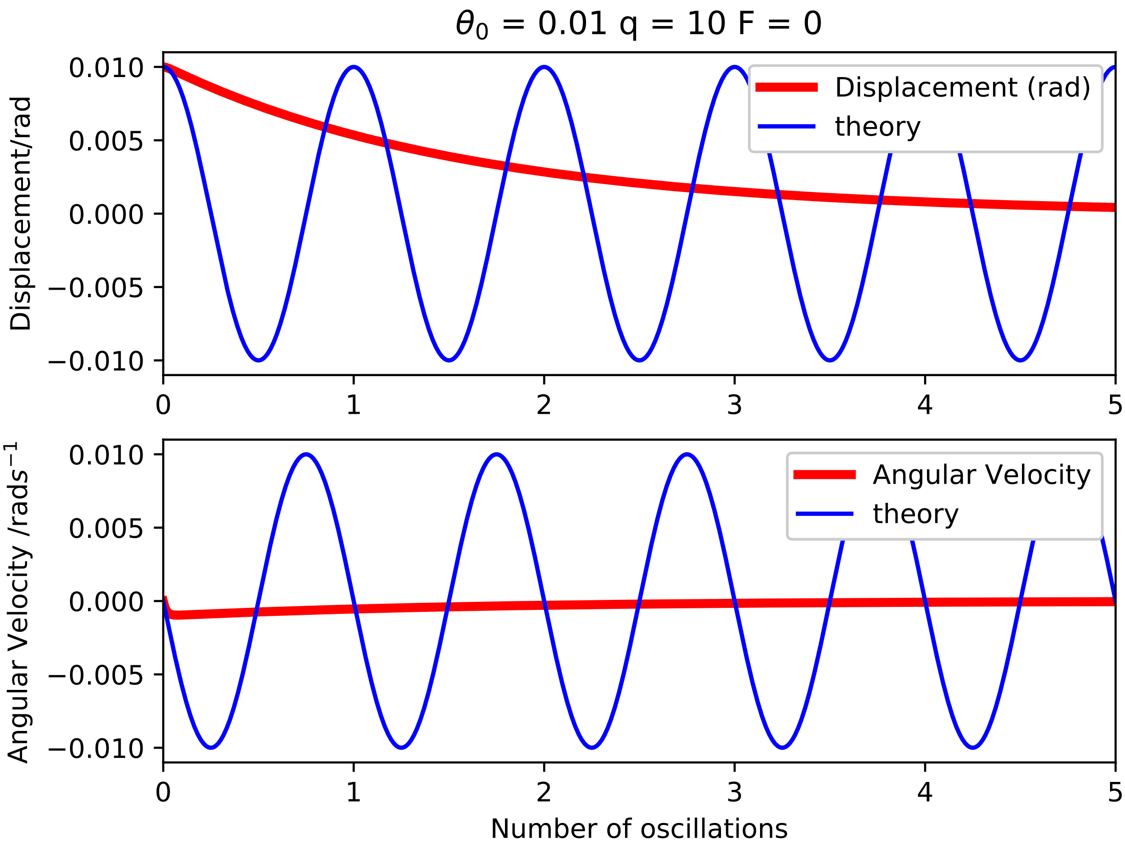

Figure 11: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 10, F = 0$ are plotted alongside the analytical solution. The form of the solution follows that of a over-damped harmonic oscillator.

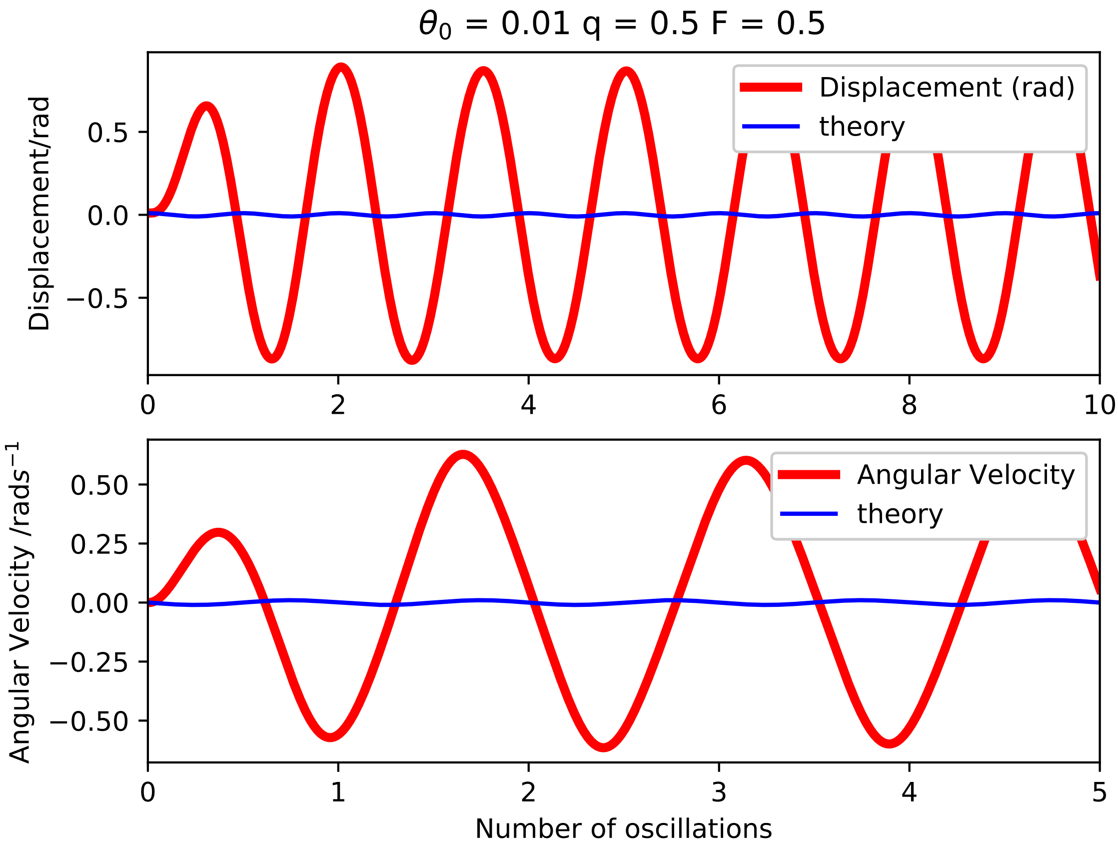

Figure 12: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 0.5, F = 0.5$ are plotted alongside the analytical solution. The form of the solution follows that of a driven, damped harmonic oscillator.

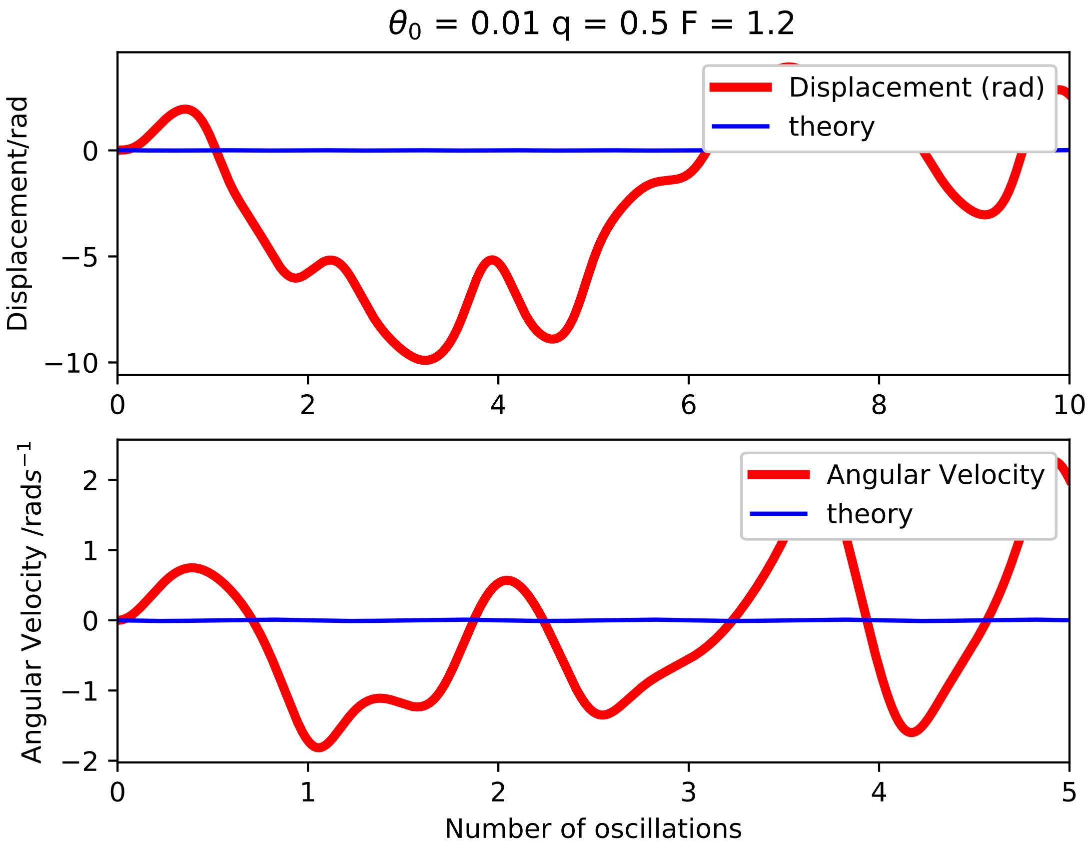

Figure 13: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 0.5, F = 1.2$ are plotted alongside the analytical solution. The form of the solution is aperiodic and complicated.

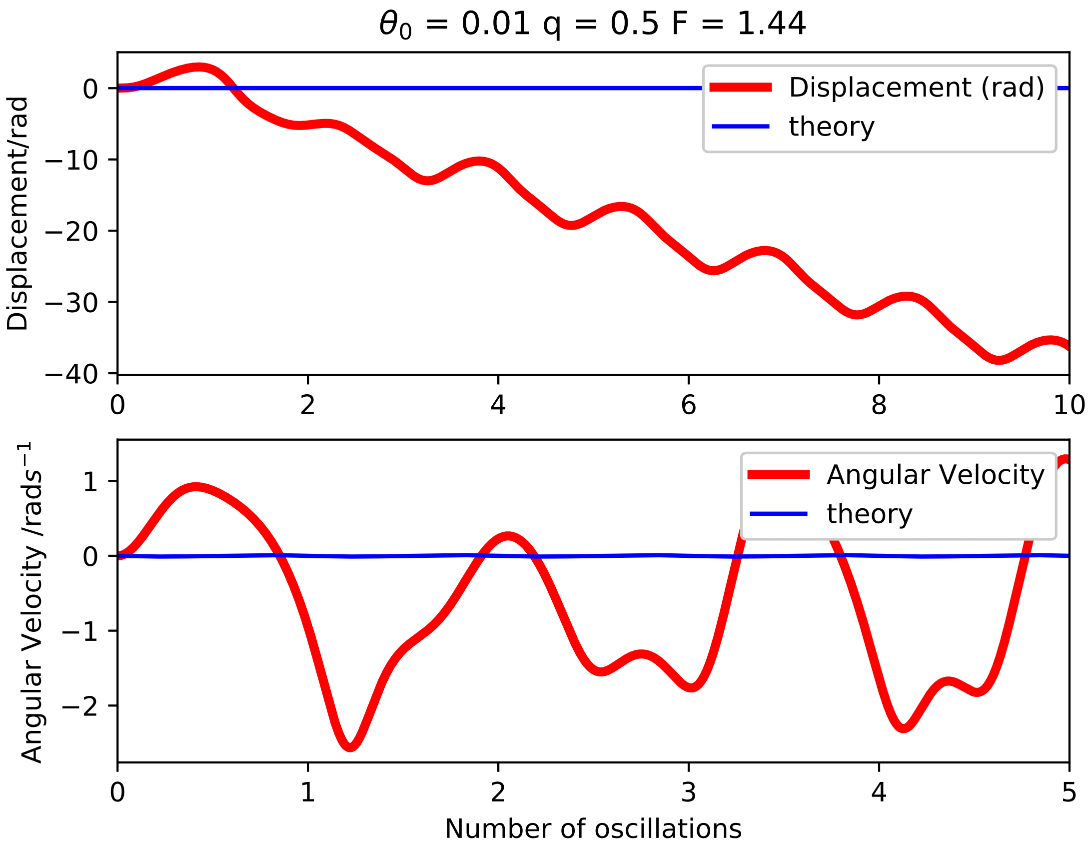

Figure 14: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 0.5, F = 1.44$ are plotted alongside the analytical solution. The form of the solution is aperiodic and complicated.

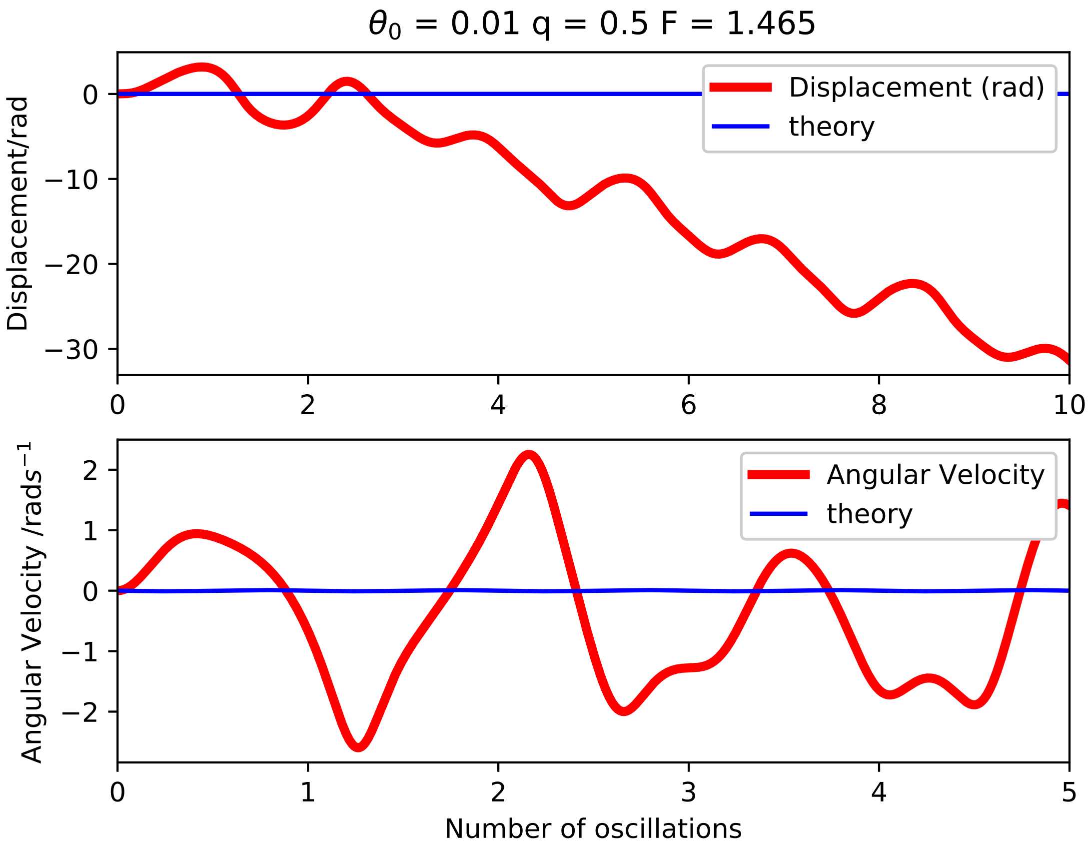

Figure 15: The angular displacement and velocities of the first 5 oscillations for $\theta_0 = 0.01, q = 0.5, F = 1.465$ are plotted alongside the analytical solution. The form of the solution is aperiodic and complicated.

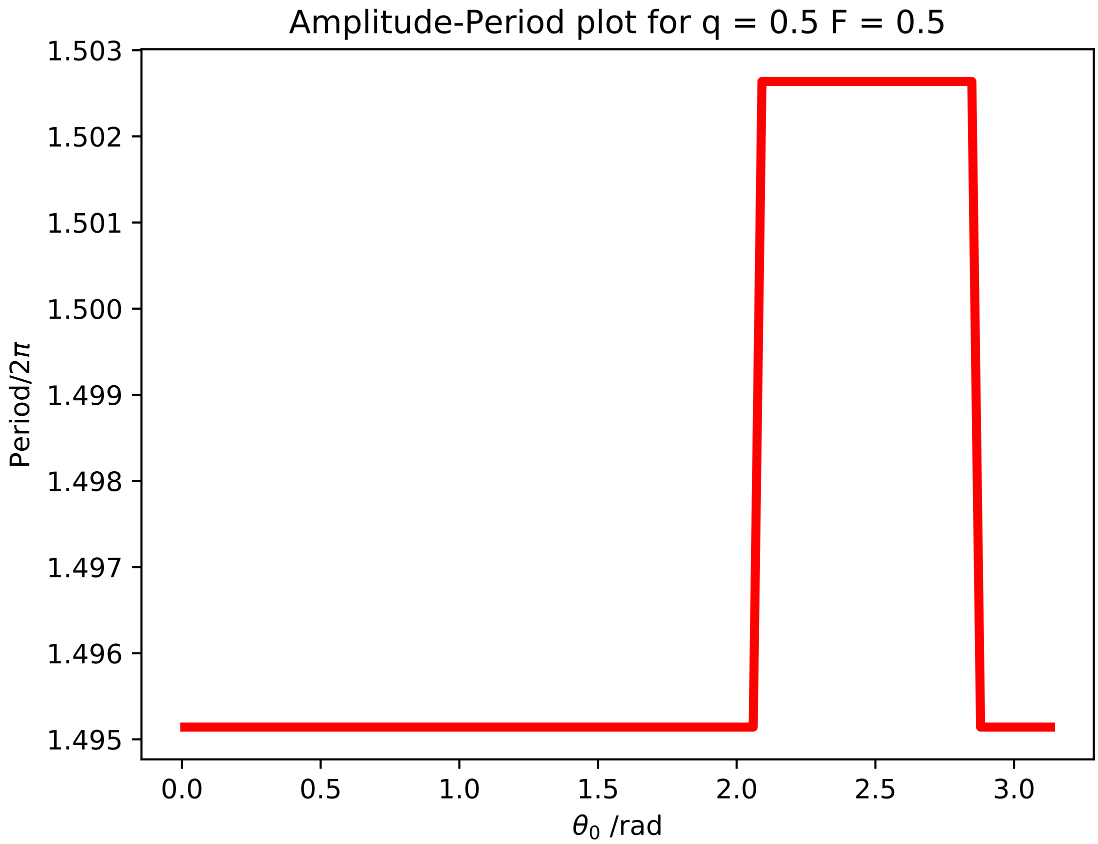

Figure 16: The period of the pendulum plotted as a function of initial amplitude from $\theta_0 = 0 \rightarrow \pi$ for $q = 0.5, F = 0.5$.

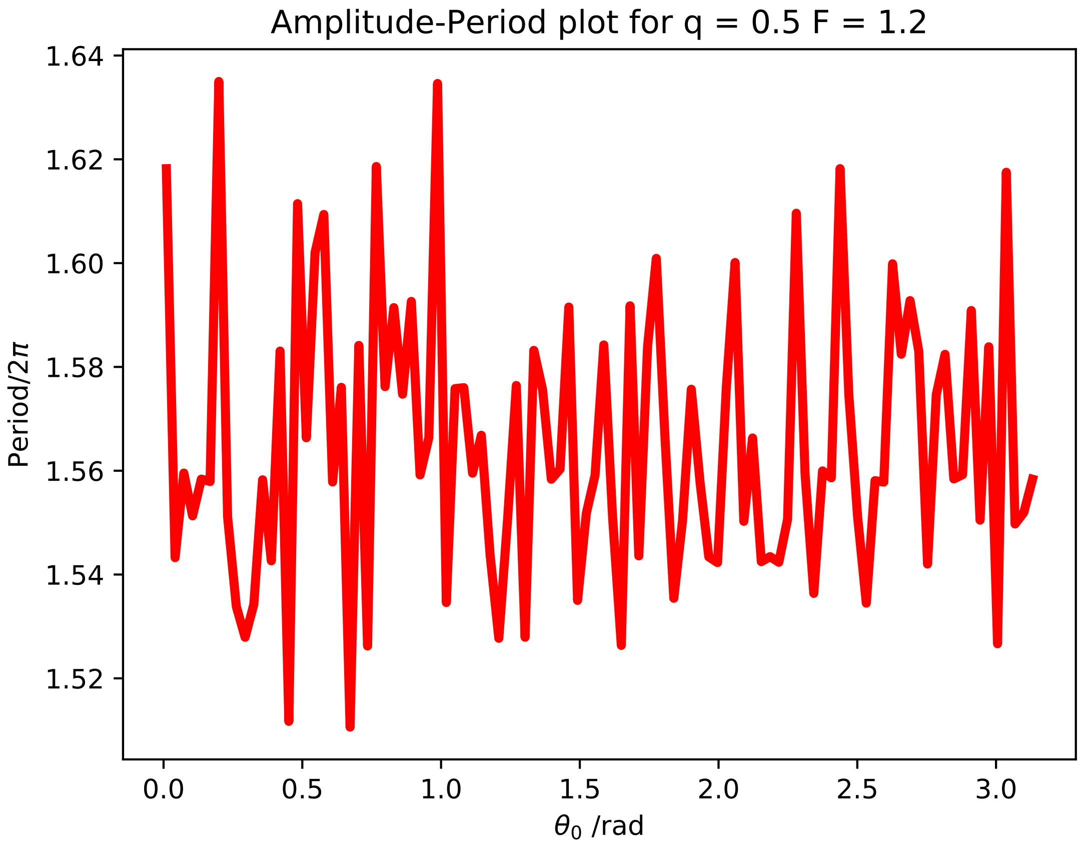

Figure 17: The period of the pendulum plotted as a function of initial amplitude from $\theta_0 = 0 \rightarrow \pi$ for $q = 0.5, F = 1.2$. The form of the plot is highly irregular.

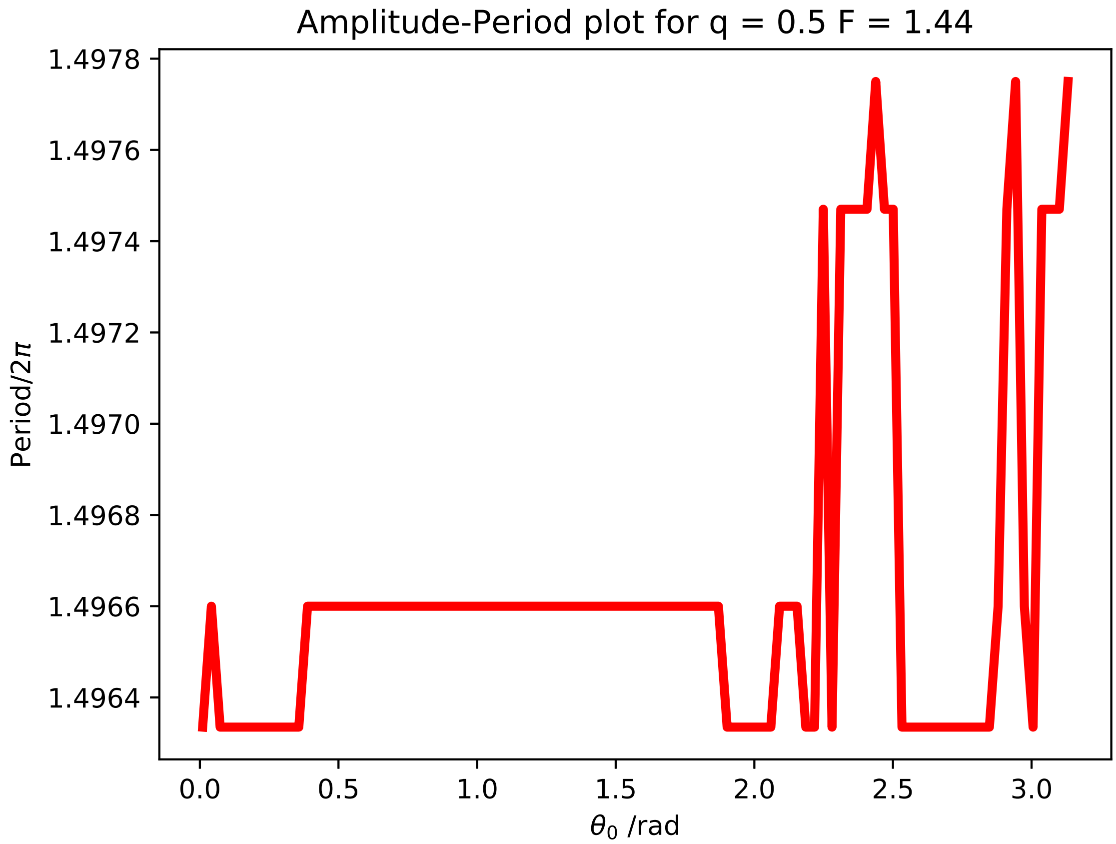

Figure 18: The period of the pendulum plotted as a function of initial amplitude from $\theta_0 = 0 \rightarrow \pi$ for $q = 0.5, F = 1.44$. The form of the plot is highly irregular.

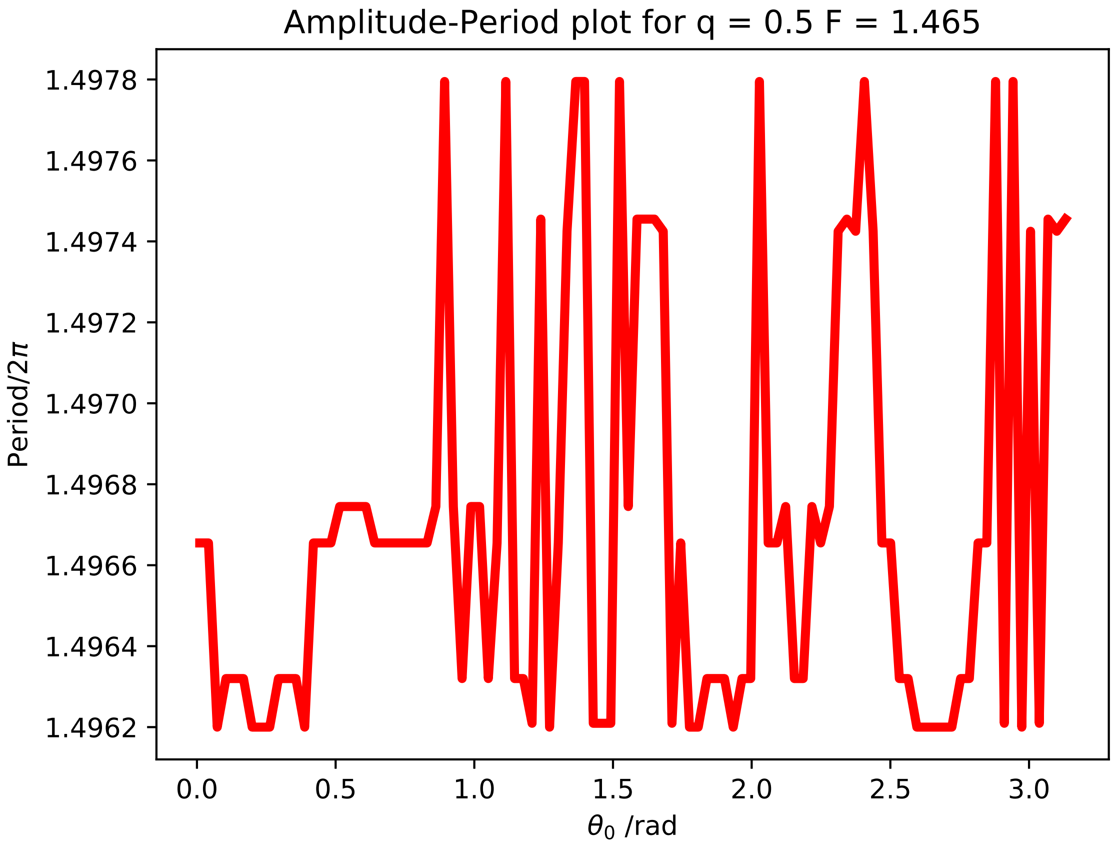

Figure 19: The period of the pendulum plotted as a function of initial amplitude from $\theta_0 = 0 \to \pi$ for $q = 0.5, F = 1.465$. The form of the plot is highly irregular.

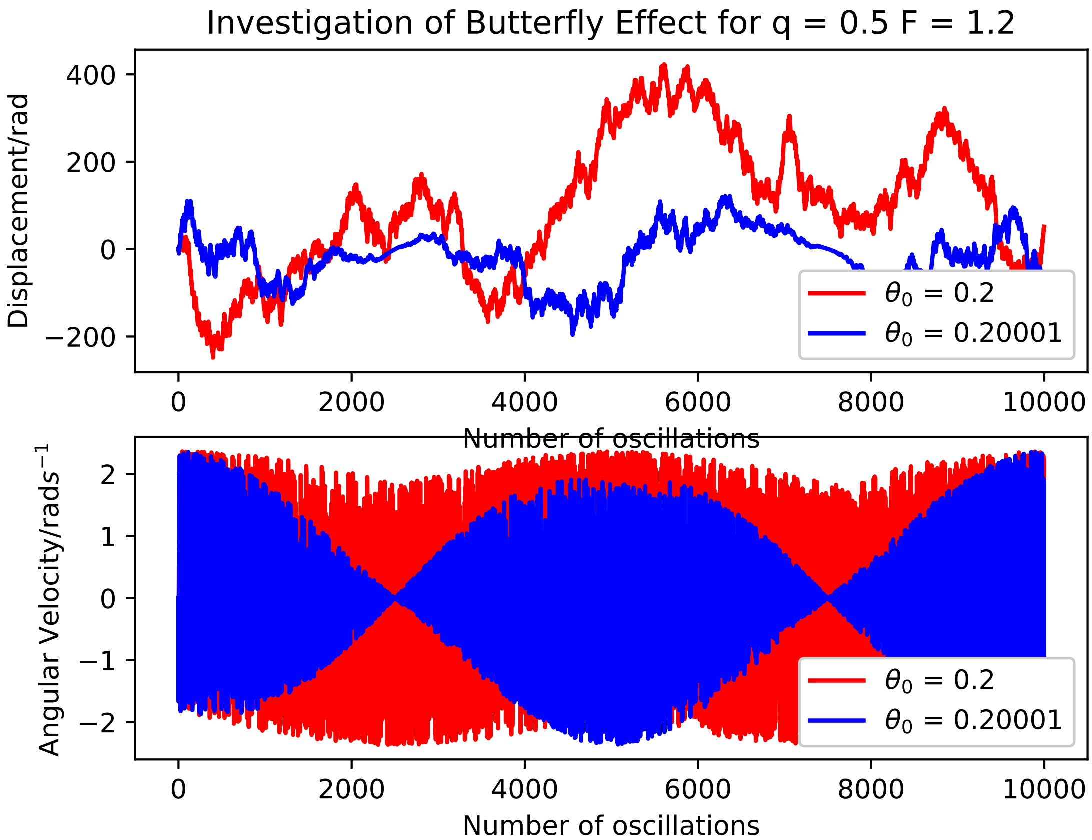

Figure 20: The angular displacements and velocities for $\theta_0 = 0.2$and $\theta_0 = 0.20001, q = 0.5, F = 1.2$, are plotted for $\sim 10,000$ oscillations. The solutions diverge with time despite the close initial conditions.

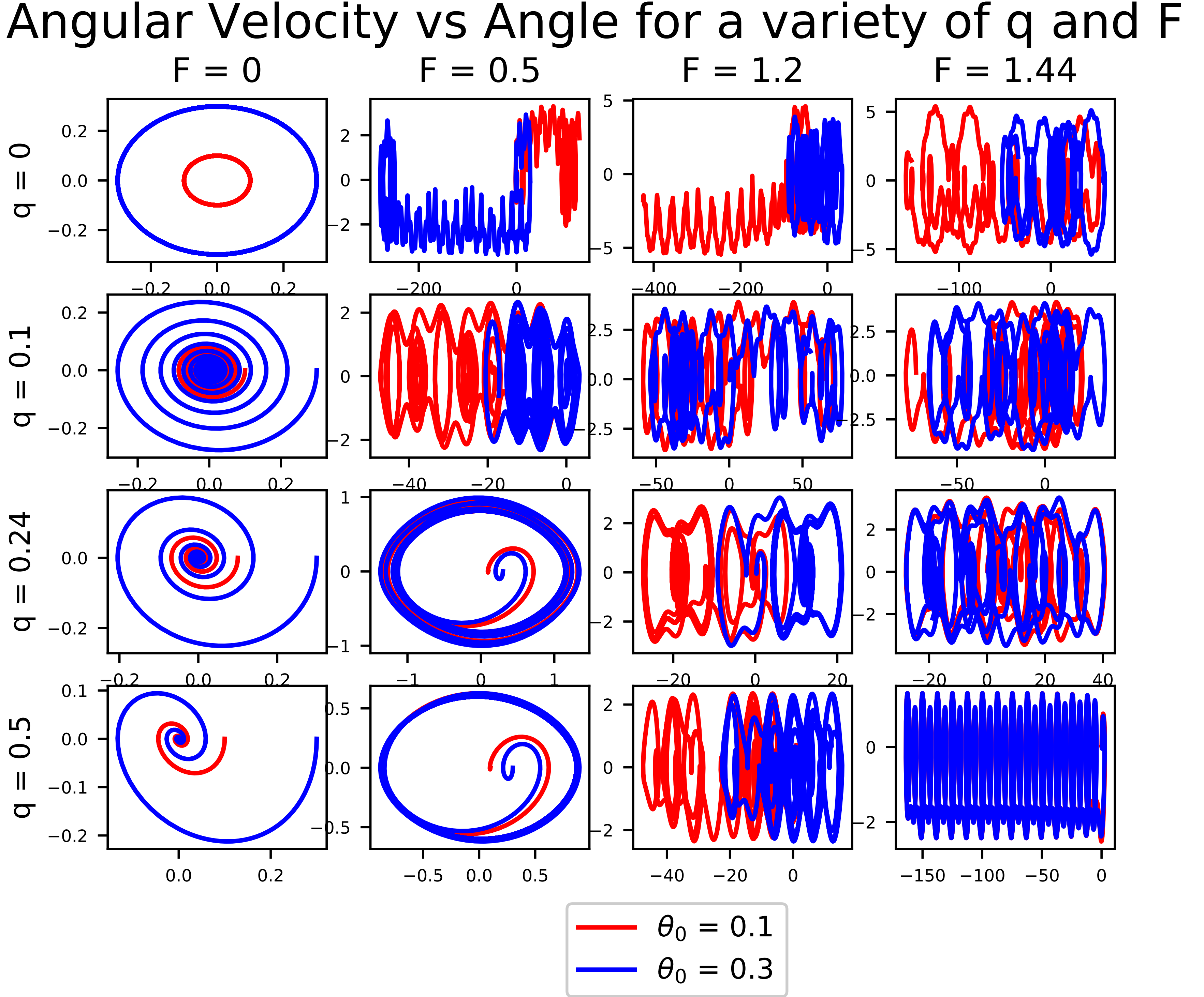

Figure 21: A 4x4 plot of the angular displacements again velocities for $\theta_0 = 0.1$and $\theta_0 = 0.3$ for a variety of q, F values. Highly irregular phase space orbits are seen, indicating chaos behavior in the system.