1 Reconstruction of algebraic shapes

We consider methods for the sampling and recovery of a class of binary images of the form

$$I = \chi_D$$

where $D \subset \mathbb{R}^2$ is a bounded open region whose boundary ∂D is an *algebraic curve* of a fixed degree n, i.e., the zero locus of a bivariate polynomial p of degree n:

$$\partial D = \left\{ (x_1, x_2) \in \mathbb{R}^2 \colon\, p(x_1, x_2) = \sum_{0 \le i,j,i+j \le n} a_{i,j}\, x_1^i\, x_2^j = 0 \right\}. \tag{1}$$

We refer to such region D as an *algebraic domain* or *algebraic shape* and, without loss of generality, we may assume that it is contained inside the rectangular region $\Omega = [0, L] \times [0, L] \subset \mathbb{R}^2$.

In many practical situations, we may assume that we have access only to a discrete set of uniform samples of the binary image $I = \chi_D$, that is, input data are of the form

$$d_k = I * \phi_T(k) = \iint_\Omega I(x)\, \tfrac{1}{T} \phi(\tfrac{x}{T} - k)\, dx, \quad k \in \mathbb{Z}^2, \tag{2}$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an appropriate sampling kernel and $T \ge 1$ is a parameter. More generally, the samples may be corrupted by additive noise. That is, we observe the samples

$$y_k = d_k + n = A(I; k) + n,$$

where A is a sampling (linear) operator

[There should be a way to represent the vector $y = (y_k)$ as $y = AI + n$, where A and I are matrices]

The inverse problem to solve is the recovery of the image I.

Remark. The assumption that D is an algebraic shape implies that the image I can be parametrized with a small number of parameters, namely, $\binom{n+2}{2} - 1$ parameters, where $\binom{n+2}{2}$ is the number of coefficients of a bivariate polynomial of degree n and we have subtracted one unit since we can normalize the polynomial by fixing a_0 (say $a_0 = 1$). Hence the class of images considered by this problems lives a low dimensional space as compared the class of discrete or continuous images supported in the rectangular region $\Omega = [0, L] \times [0, L] \subset \mathbb{R}^2$. Because of this observation, in the literature this class of data is associated with the class of signals with *finite rate of innovation* (FRI) [2,15,25].

[The image I is not sparse but the derivative of I is a sparse image since it only contains the boundary od D which is an algebraic curve. How can we express the recovery problem with a sparsity constraint? In the literature of compressed sensing there should be problems that could be related to this one.]

[As an alternative to algebraic domains, we could also considered curves associated with trigonometric curves. IN that case, the formulation of the probelms in terms of annihilating equations is probably easier.]

1.1 Background: Analytic solution

Below, I have a summary of an analytic approach used to solve the reconstruction problem stated above.

By classical results [10], an algebraic curve of degree n can be uniquely determined from its set of two-dimensional moments

$$M_{i,j} = \iint_{\Omega} x_1^i x_2^j \, I(x_1, x_2) \, dx_1 \, dx_2 \tag{3}$$

of order less than or equal to n. An algorithmic approach for the reconstruction of bounded algebraic domains from their moments was first presented in [8,17] but this approach is very sensitive to noise. Even though it was shown that one can improve the stability of the reconstruction by increasing the number of moments [13], the problem of recovering an algebraic domain remains unstable in the sense that small errors in the computation of the image moments may have a significant impact on the recovery algorithm.

Fatemi et al. [7] have shown that one can express the image moments (3) as appropriate linear combinations of the image samples (2), where the coefficients of the linear combinations depend on the sampling kernel ϕ. Hence, they derive a linear system for the recovery of the algebraic shape of the form

$$\mathbf{M}\,\mathbf{a} = 0 \tag{4}$$

where $\mathbf{a}$ is the vector of the unknown polynomial coefficients $\{a_{i,j}\}$ in (1) and the matrix $\mathbf{M}$ contains the computed moments (3). Their approach assumes that ϕ is *polynomial generating* up to a certain degree m, that is, there exist coefficients $c_k^{(\alpha)}$ such that

$$\sum_{k \in \mathbb{Z}^2} c_k^{(\alpha)} \phi(x - k) = x^{\alpha}, \text{ for } |\alpha| = 0, \ldots, m. \tag{5}$$

It turns out that the direct numerical solution of (4) recovers the polynomial coefficients if the image moments are computed from noiseless image samples. However, if the image samples are corrupted by even a small additive noise, then the numerical reconstruction fails in general. To remedy the instability of the recovery, Fatemi et al. [7] introduced a modified formulation based on 'generalized moments' leading to a constrained optimization problem that is solved using an iterative regularized reconstruction algorithm.

The instability of the solution of the system (4) can be explained by observing that the process of converting image samples into image moments can be very sensitive to noise. Such numerical sensitivity is highly dependent on the basis selected for the representation of the algebraic curve, that is, the numerical stability of algebraic curves defined by implicit equations can be enhanced by choosing appropriate polynomial representations. Hence, to derive a more robust method for the recovery of an algebraic curve from the corresponding image samples, here we introduce a novel approach that represents an algebraic curve in terms of non-separable bivariate Bernstein polynomials. Using our representation of algebraic domains in terms of Bernstein polynomials, we derive a new formulation of the *image moment equation* (4) that we can solve directly to recover an algebraic domain from noisy image samples. We show that the numerical reconstruction based on our algorithm is robust to noise and, while its reconstruction performance is comparable with the best regularized algorithm in [7], it is computationally much faster and simpler to implement. This is particularly true in the situation where we use refinable sampling kernels which are *polynomial reproducing* due to the costless computation of the expansion coefficients in (5).

We recall that the problem of accurately recovering image boundaries or edges from image samples has gained renewed interest in recent years with the study of signals with finite rate of innovation (FRI). This area of investigation is concerned with signals that are not band-limited, hence do not satisfy the assumptions of Shannon sampling theory; however, they can be described with a finite number of parameters so that they can still be recovered from their samples using appropriate alternative strategies [2,15,25]. In an effort to apply the FRI framework to images, a number sampling schemes with different sampling kernels were proposed to recover special classes of images with edges [3,18,21,27].

1.2 New result: neural network solution

We have recently designed a simple 3-layer neural network that takes as input the image samples d_k, given by (2), or their noisy version y_k and recovers the image I. The most important ideas in this network is that the output stage consists of 2 layers. The first output layers generates $\binom{n+2}{2}$ coefficients that are identified as the coefficients of a bi-variate polynomial p. The second output layers generates a binary image $I = \chi_D$ where D is the algebraic curve associated with polynomial p. The network performs very competitively even when samples are corrupted by noise.

[Here Wilfredo can expand to make more explicit the prior condition used in the network.]

References

1. L. Baboulaz and P. L. Dragotti, *Exact Feature Extraction Using Finite Rate of Innovation Principles With an Application to Image Super-Resolution*, IEEE Trans. Image Process., 18 n.2 (2009), pp. 281–298.
2. T. Blu, P. L. Dragotti, M. Vetterli, P. Marziliano, and L. Coulot, *Sparse sampling of signal innovations*, IEEE Signal Process. Mag., 25, n. 2 (2008), pp. 31–40.
3. C. Chen, P. Marziliano, and A. C. Kot, *2D finite rate of innovation reconstruction method for step edge and polygon signals in the presence of noise*, IEEE Trans. Signal Process., 60, n. 6 (2012), pp. 2851–2859.
4. C. Conti, C. Deng, K. Hormann, *Symmetric four-directional bivariate pseudo-spline symbols*, Comput. Aided Geom. Design, 60 (2018), pp. 10–17.
5. N. Dyn and D. Levin, *Subdivision schemes in geometric modelling*, Acta Numer., 11 (2002), pp. 73–144.
6. R.T.Farouki, *The Bernstein polynomial basis: A centennial retrospective*, Comput. Aided Geom. Design, 29 (2012), pp. 379–419.
7. M. Fatemi, A. Amini, M. Vetterli, *Sampling and Reconstruction of Shapes With Algebraic Boundaries*, IEEE Trans. Signal Process., 64 n. 22 (2016), pp. 5807–5018.
8. B. Gustafsson, C. He, P. Milanfar, and M. Putinar, *Reconstructing planar domains from their moments*, Inverse Problems, 16 n. 4 (2000), pp. 1053–1070.
9. M.K. Hu, *Visual pattern recognition by moment invariants*, IRE. Trans. Inf. Theory, 8 n. 1 (1962), pp. 179–187.
10. S. Karlin and W. J. Studden, *Tchebycheff Systems, With Applications in Analysis and Statistics*, New York, NY, USA: Interscience, 1966.

11. D. Keren, D. Cooper, and J. Subrahmonia, *Describing complicated objects by implicit polynomials*, IEEE Trans. Pattern Anal. Mach. Intell., 16 n.1 (1994), pp. 38–53.
12. M-J. Lai, L. Schumaker, *Spline functions on triangulations.* Cambridge University Press, 2007.
13. J. B. Lasserre and M. Putinar, *Algebraic-exponential Data Recovery from Moments*, Discrete Comput. Geom., 54 (2015), pp. 993–1012.
14. A. Levin, *Polynomial generation and quasi-interpolation in stationary non-uniform subdivision*, Comput. Aided Geom. Design, 20 n. 1 (2003), pp. 41–60.
15. I. Maravic and M. Vetterli, *Sampling and reconstruction of signals with finite rate of innovation in the presence of noise*, IEEE Trans. Signal Process., 53, n. 8 (2005), pp. 2788-–2805.
16. MATLAB, The MathWorks, Inc., Natick, Massachusetts, United States (https://www.mathworks.com/products/matlab.html)
17. P. Milanfar, M. Putinar, J. Varah, B. Gustafsson, and G. Golub, *Shape reconstruction from moments: Theory, algorithms, and applications*, Proc. Int. Symp. Opt. Sci. Technol., SPIE, (2000), pp. 406–416.
18. H. Pan, T. Blu, and P. L. Dragotti, *Sampling curves with finite rate of innovation*, IEEE Trans. Signal Process., 62 n. 2 (2014), pp. 458–471.
19. S. Rad, K. C. Smith, and B. Benhabib, *Application of moments and Fourier descriptors to the accurate estimation of elliptical shape parameters*, Proc. IEEE Int. Conf. Acoust. Speech, Signal Process. (ICASSP), 4 (1991), pp. 2465–2468.
20. T.W. Sederberg, *Planar piecewise algebraic curves*, Comput. Aided Geom. Design, 1 (1984), pp. 241–255.
21. P. Shukla and P. L. Dragotti, *Sampling schemes for multidimensional signals with finite rate of innovation*, IEEE Trans. Signal Process., 55 n. 7 (2007), pp. 3670–3686.
22. I. M. Spiliotis and B. G. Mertzios, *Real-time computation of two-dimensional moments on binary images using image block representation*, IEEE Trans. Image Process., 7 (1998), pp. 1609–1615.
23. G. Taubin, F. Cukierman, S. Sullivan, J. Ponce, and D. J. Krigman, *Parameterized families of polynomials for bounded algebraic curve and surface fitting*, IEEE Trans. Pattern Anal. Mach. Intell., 16 n. 3 (1994), pp. 287–303.
24. K. Tsirikolias and B. G. Mertzios, *Statistical pattern recognition using efficient two-dimensional moments with applications to character recognition*, Pattern Recognit., 26 n. 6 (1993), 877–882.
25. M. Vetterli, P. Marziliano, and T. Blu, *Sampling signals with finite rate of innovation*, IEEE Trans. Signal Process., 50 n. 6 (2002), pp. 1417-–1428.
26. H. Whitney, *Geometric Integration Theory*, Princeton Univ Press, 1957.
27. M. Wischerhoff and G. Plonka, *Reconstruction of polygonal shapes from sparse Fourier samples*, Journal of Computational and Applied Mathematics, 297 (2016), pp. 117–131.
28. L. Yang, F. Albregtsen, *Fast and exact computation of cartesian geometric moments using discrete Green's theorem*, Pattern Recognit., 29 n. 7 (1996), 1061–1073.