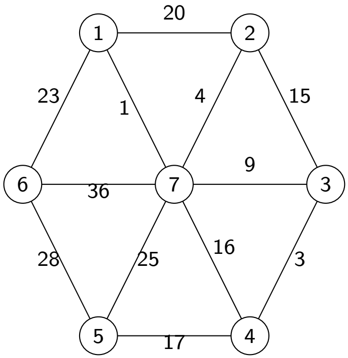
CMP_SC 3050: More greedy algorithms in graphs: minimum spanning trees

Minimum Spanning Tree

Input Connected undirected graph $G = (V, E)$ with edge costs/weights $w(i, j)$

Goal Find $T \subseteq E$ such that (V, T) is connected and total cost/weight of all edges in T is smallest



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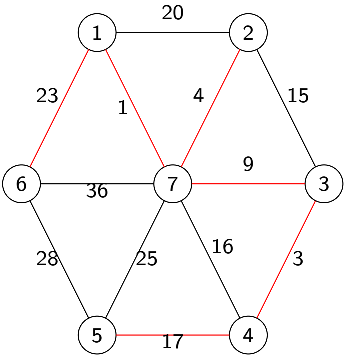

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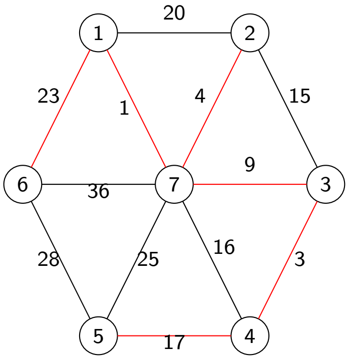
- T is the minimum spanning tree (MST) of G



Difference between MSTs and Shortest Paths

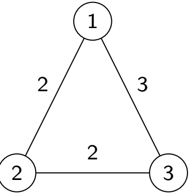


Figure: Graph G

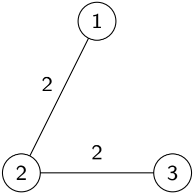


Figure: MST of G

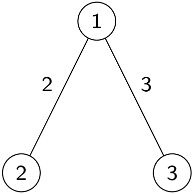


Figure: Shortest Paths (from 1)

Applications

- Network Design
  - Designing networks (roads, computer, electrical) with minimum cost but maximum connectivity
- Approximation algorithms for computationally hard problems
  - Can be used to bound the optimality of algorithms to approximate Travelling Salesman Problem, Steiner Trees, etc.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network

Greedy Template

```
Initially E is the set of all edges in G
T is empty (* T will store edges of a MST *)
while E is not empty
    choose i ∈ E
    if (i satisfies condition)
        add i to T
return the set T
```

Main Task: In what order should edges be processed? When should we add edge to spanning tree?

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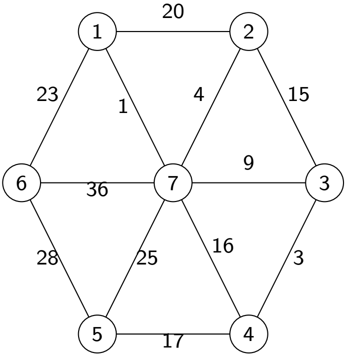
Kruskal's Algorithm

Process edges in the order of their costs (starting from the least) and add edges to T as long as they don't form a cycle.



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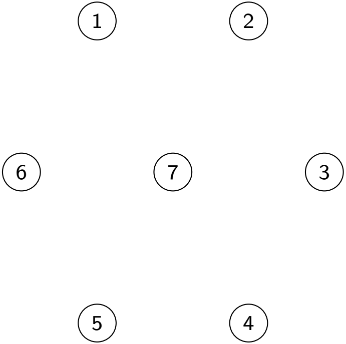

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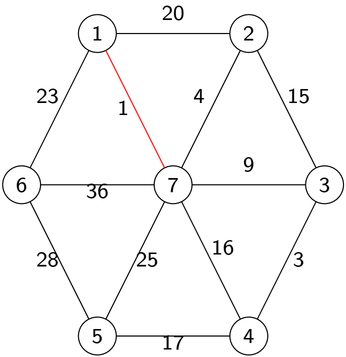

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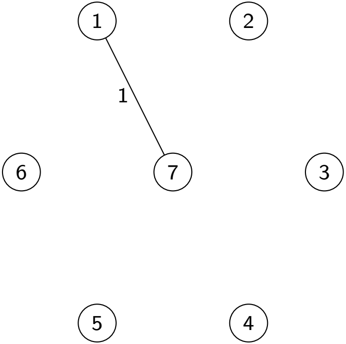

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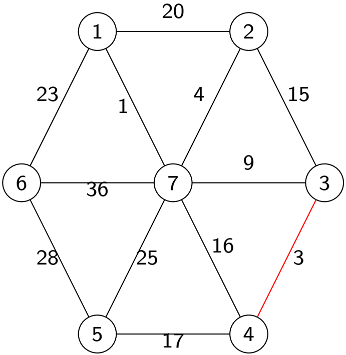

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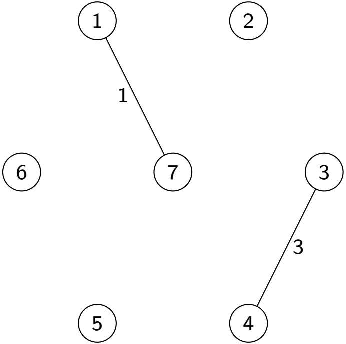

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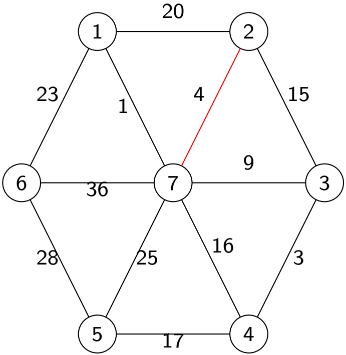

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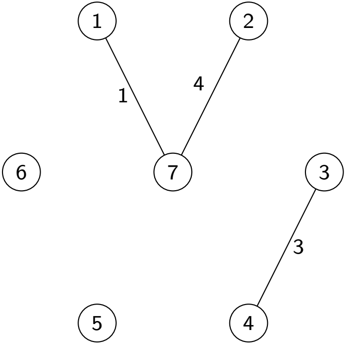

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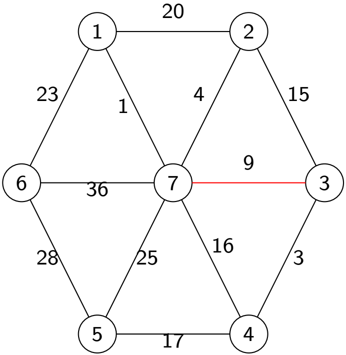

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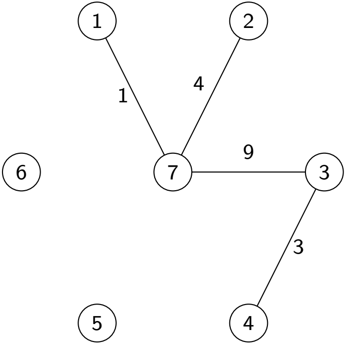

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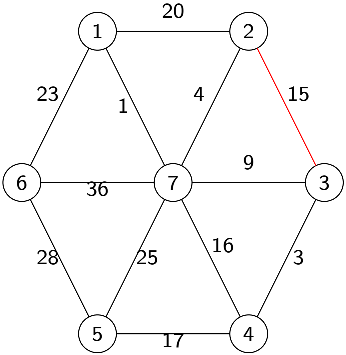

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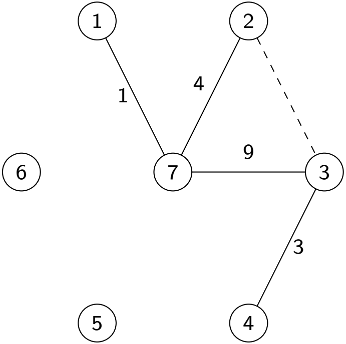

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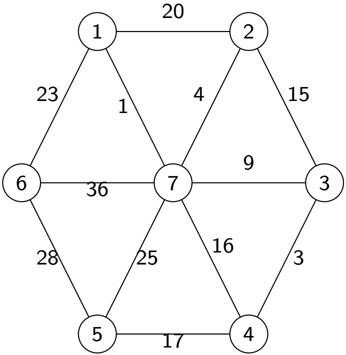

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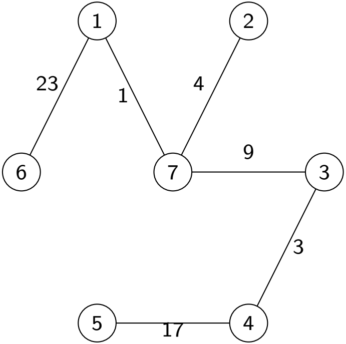

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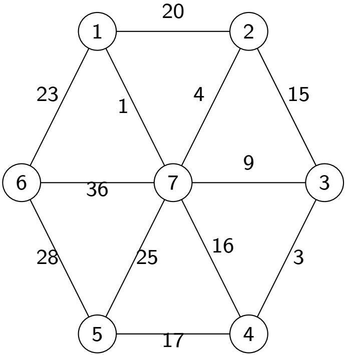
T maintained by algorithm will be a tree. Can start with any vertex. In each iteration, pick edge with least attachment cost to T.
In other words, consider all the edges that have exactly one endpoint in T and choose the one with the least weight amongst these edges.



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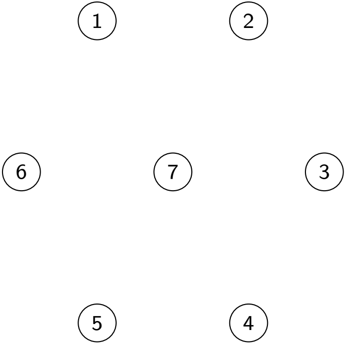

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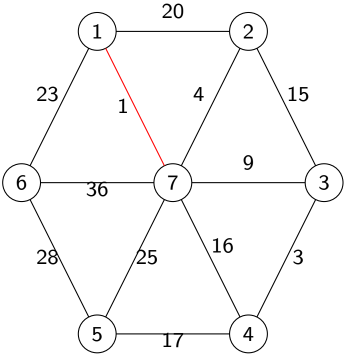

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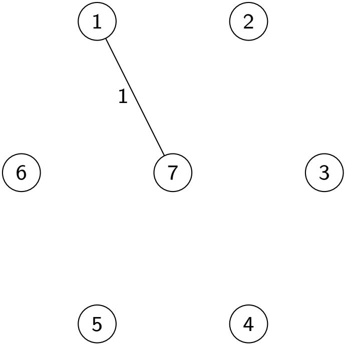

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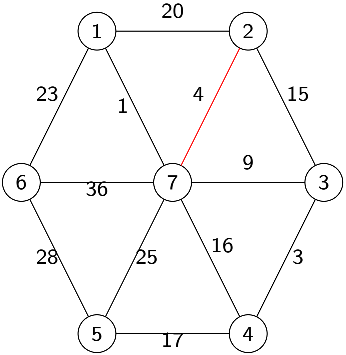

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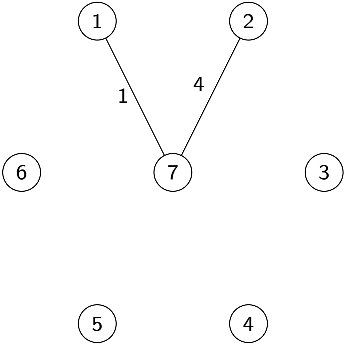

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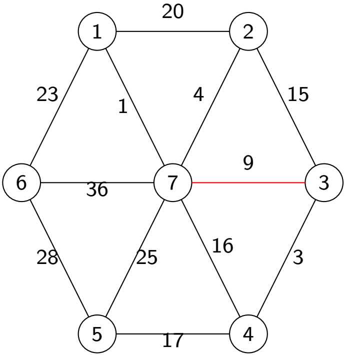

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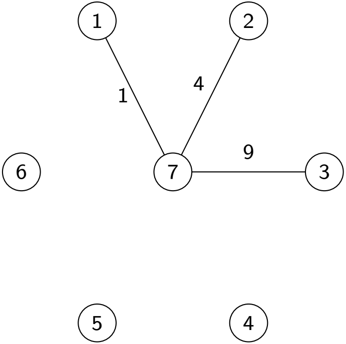

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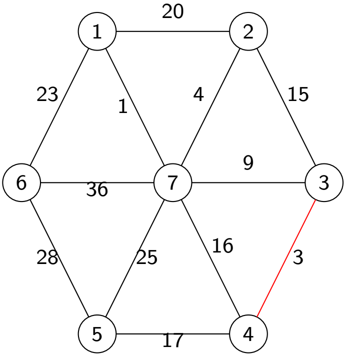

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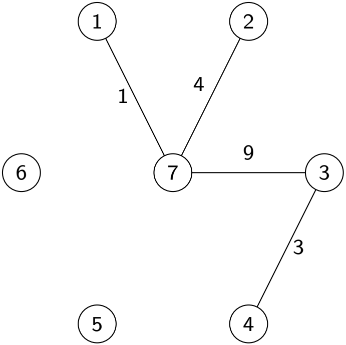

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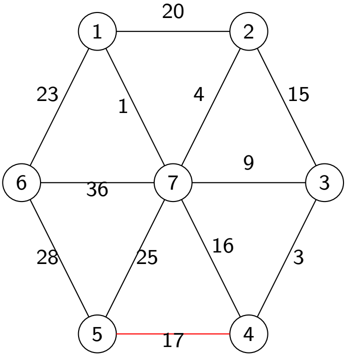

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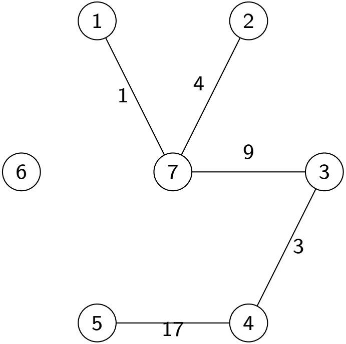

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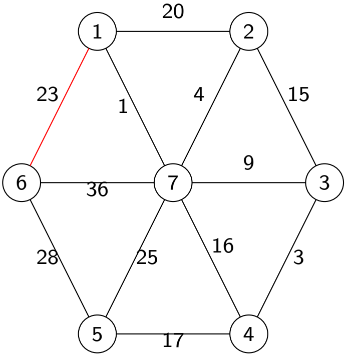

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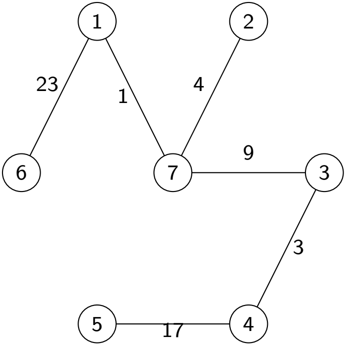

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And for now . . .

No two edge costs are equal.

Key Observation: Cut Property

Let S be a proper non-empty subset of V. Let $e = (v, w)$ be the minimum cost edge with one end in S and the other end in $V \setminus S$. Then every minimum spanning tree contains e

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  - T' may not be a spanning tree!!

Error in the argument: Example

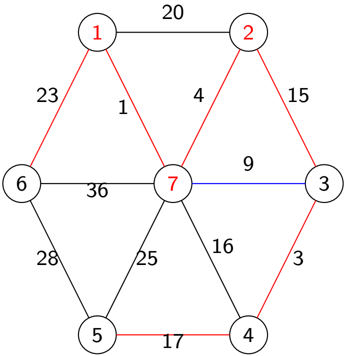

Figure: Problematic example: The Spanning Tree is shown in red. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$.

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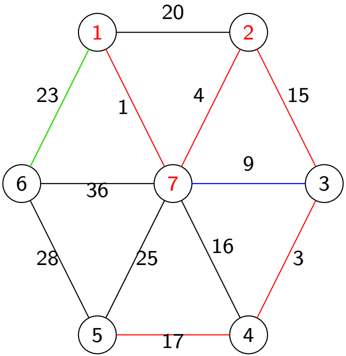
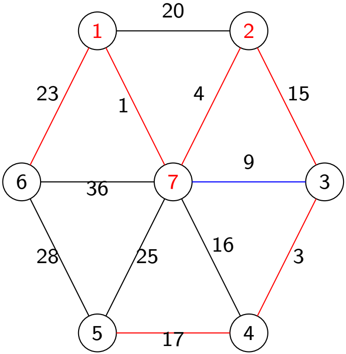


Figure: Problematic example: The Spanning Tree is shown in red. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$. Replacing f by e yields a cycle and does not cover 6.

Why the Cut Property holds?



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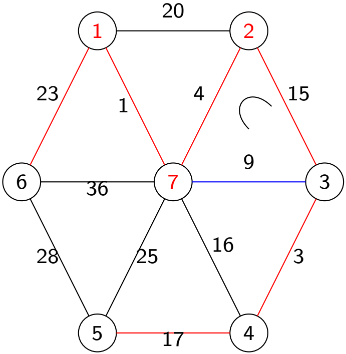

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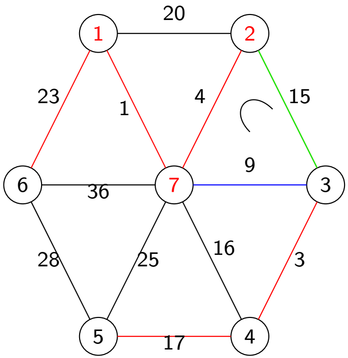

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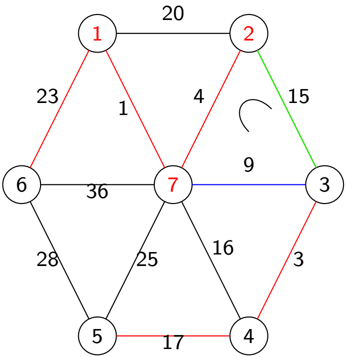

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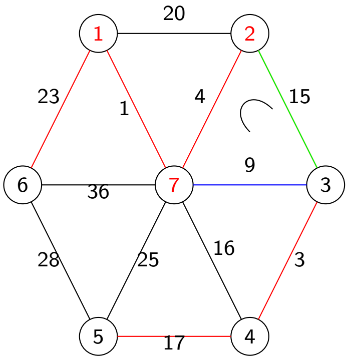

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- In the example on the left, e' is the edge $(2, 3)$
- $T' = (T \setminus \{e'\}) \cup \{e\}$ is a spanning tree of lower cost

Why Cut Property holds (contd)?

Observation

$T' = (T \setminus \{e'\}) \cup \{e\}$ is a spanning tree.

T' is connected.

T' is acyclic

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$T' = (T \setminus \{e'\}) \cup \{e\}$ *is a spanning tree.*

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> If path uses $e' = (v', w')$, then go from v' to v, use edge (v, w) and then go from w to w' in T'

T' is acyclic

> Only one cycle in $T' \cup \{e'\}$, namely, one involving e and e', which is not present in T'

Correctness of Prim's Algorithm

Prim's Algorithm

Pick edge with minimum attachment cost to current tree T, and add to current tree T

- If e is added to tree, then e belongs to every MST.
- Set of edges output is a spanning tree.

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- Set of edges output is a spanning tree.
  - There are no cycles in T because we always add an edge from an "unattached vertex"
  - Algorithm stops when all vertices are connected

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  - Thus, we get a new tree T' by adding this edge to T

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- Start from an arbitrary root vertex r
- Maintain a priority queue Q, where Q is the set of vertices in $V \setminus T$
- Key of a vertex u in Q is the minimum weight of any edge (v, u) where v is in T. We also maintain the predecessor $u.\pi$ which gives this minimum edge
- In each iteration, extract the vertex u from Q having the smallest key. The edge $(u.\pi, u)$ is the edge with minimum cost to T
  - Thus, we get a new tree T' by adding this edge to T
  - We can remove u from Q and update the keys of vertices adjacent to u

Pseudocode and analysis of Prim's Algorithm

```
PRIM(G, w, r)
  Q = ∅
  for each u ∈ G.V
      u.key = ∞
      u.π = NIL
      INSERT(Q, u)
  DECREASE-KEY(Q, r, 0)        // r.key = 0
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Initially E is the set of all edges in G
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Implementing Kruskal's Algorithm Efficiently

Maintain the sets of vertices already connected during the running of Kruskal's algorithm

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Need a new data structure to check if two elements belong to same set and to merge two sets.

Disjoint Set Data Structure

A disjoint set data structure $\mathcal{S}$ is used to store a collection of disjoint sets of elements. (Also known as Union-Find data structure.) Supports following operations.

MAKE-SET (x)	:	If x does not belong to a set in $\mathcal{S}$ then create a new set whose only member is x
FIND(x)	:	Find the set in $\mathcal{S}$ containing x
UNION(x,y)	:	Takes the two sets in $\mathcal{S}$ that contain x and y and merges them into one

Implementing Disjoint sets using arrays

Store the name of the sets in an array indexed by the elements of the set.

- For an element $x \in S$, $Set[x]$ will give the name of the set containing x. The name of a set can be the name of a unique element in the set.

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- $\textsc{Find}(x)$ involves reading the entry $Set[x]$: $O(1)$
- $\textsc{Union}(x,y)$ involves updating the entries $Set[z]$ for all elements z in $Set[x]$ and $Set[y]$: $O(n)$

Improving the Array Implementation: weighted union heuristic

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- When performing $\textsc{Union}(x,y)$, append the smaller set to the larger set and update the size of the set. Also now change the entries in the array *Set* of the smaller set to that of the larger set
- $\textsc{Find}(x)$ still takes $O(1)$ time.
- $\textsc{Union}(x,y)$ takes time $O(r)$, where r is the size of the smaller set. Worst case, still $O(n)$.

Running time

Key Observation: UNION takes $O(n)$ time for large sets. But then the union is even larger and so we cannot have too many of these bad unions.

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With weighted union, a sequence of m operations on n elements takes $O(m + n \log n)$ time.

Improving Worst Case Time: Better data structure

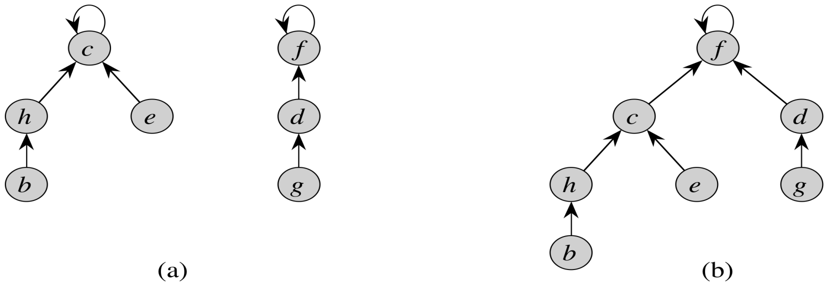

Figure: (a) Two sets $\{c, h, e, b\}$ and $\{f, d, g\}$ (b) Result of $\textsc{Union}(e,g)$

Maintain $\mathcal{S}$ in a forest; each vertex contains a single element and all elements in one tree belong to a set.

- In the tree, the child points to its parent. The root points to itself. We can use the element at the root as the name of the set.
- $\textsc{Find}(x)$: Traverse from u to the root
- $\textsc{Union}(x,y)$: Make root of x point to root of y. Takes $O(1)$ time.

Heuristics to improve worst case-behavior: union by rank

Make the root of the smaller tree (fewer nodes) a child of the root of the larger tree

- Don't actually use size

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Make the root of the smaller tree (fewer nodes) a child of the root of the larger tree

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- Use rank, which is an upper bound on height of node
- Make the root with the smaller rank into a child of the root with the larger rank

More Heuristics: Path Compression

Observation: Consecutive calls of $\textsc{Find}(x)$ take $O(\log n)$ time each, but they traverse the same sequence of pointers.

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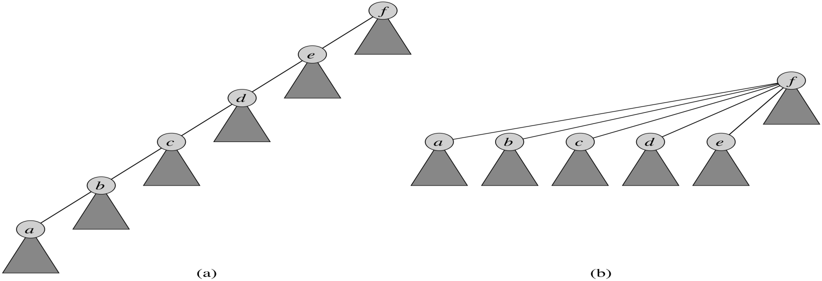


Figure: (a) Tree prior to $\textsc{Find}(a)$. Triangles represent subtrees whose roots are the vertices shown. (b) Result of $\textsc{Find}(a)$

Pseudocode for disjoint-set operations

MAKE-SET(x)
  $x.p = x$
  $x.rank = 0$

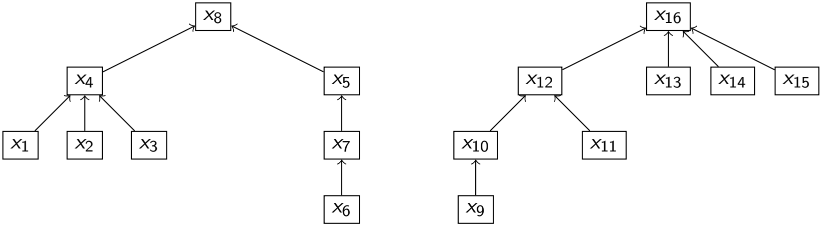
FIND-SET(x)
  if $x \neq x.p$
    $x.p =$ FIND-SET($x.p$)
  return $x.p$

UNION(x, y)
  LINK(FIND-SET(x), FIND-SET(y))

LINK(x, y)
  if $x.rank > y.rank$
    $y.p = x$
  else $x.p = y$
    // If equal ranks, choose y as parent and increment its rank.
    if $x.rank == y.rank$
      $y.rank = y.rank + 1$

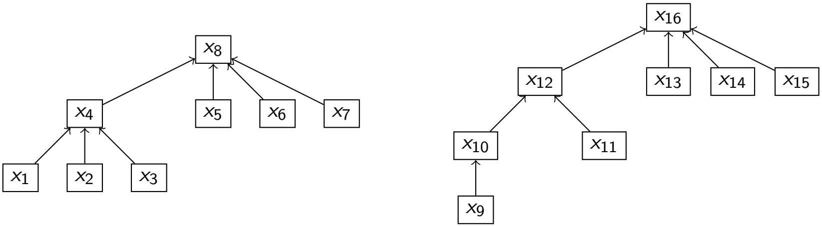
Each vertex has two attributes, p (parent) and rank

Example



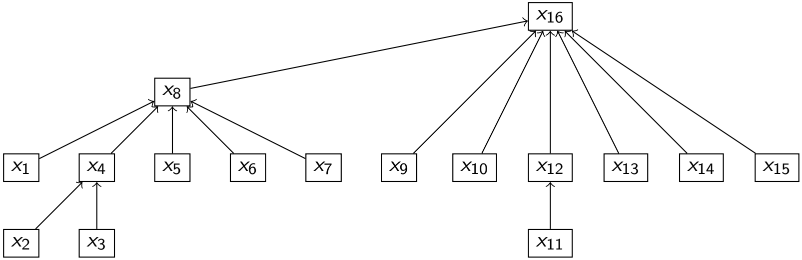
We have two sets each of rank 4.

Example continued: Find(x_6)



We have two sets each of rank 4. Note the rank of the first set does not change.

Example continued: Union(x_1, x_9)



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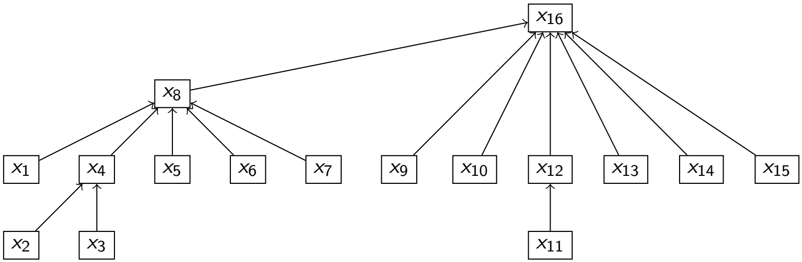

- Notice where x_1, x_9, x_{10} point to.

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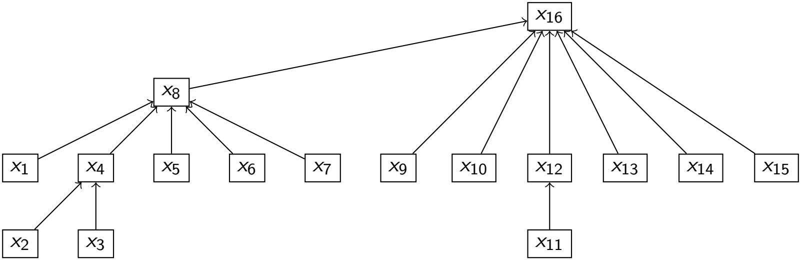

- Notice where x_1, x_9, x_{10} point to.
- The rank of the set is 5.

Running time

If use both union by rank and path compression, a sequence of m operations on n elements takes $O(m\alpha(n))$ time

- α is a very slowly growing function (slower than log) and $m\alpha$ almost linear

n	$\alpha(n)$
$0-2$	0
3	1
$4-7$	2
$8-2047$	3
$2048-A_4(1)$	4

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- $A_4(1)$ is greater than 10^8

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- Therefore, running time $= O(|E|\log|V|)$