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# The multivariate Variance Gamma model: basket option pricing and calibration

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In this paper, we propose a methodology for pricing basket options in the multivariate Variance Gamma model introduced in Luciano and Schoutens [*Quant. Finance* 6(5), 385–402]. The stock prices composing the basket are modelled by time-changed geometric Brownian motions with a common Gamma subordinator. Using the additivity property of comonotonic stop-loss premiums together with Gauss-Laguerre polynomials, we express the basket option price as a linear combination of Black & Scholes prices. Furthermore, our new basket option pricing formula enables us to calibrate the multivariate VG model in a fast way. As an illustration, we show that even in the constrained situation where the pairwise correlations between the Brownian motions are assumed to be equal, the multivariate VG model can closely match the observed Dow Jones index options.

**Keywords:** Basket option; Comonotonicity; Multivariate Variance Gamma; Multivariate calibration

**JEL Classification:** G13

## 1. Introduction

Nowadays, different methods to price single name options, also called vanilla options, are available, and for a variety of models the price of a vanilla call option can be calculated in a fast and efficient way. The industry standard for modelling stock price behaviour is proposed in Black and Scholes (1973), where the asset price process is described by a geometric Brownian motion. This approach results in closed-form expressions for vanilla option prices, which, to a large extent, explains the popularity of this model. It is well known that assuming a geometric Brownian motion is a too simplistic approach, in that it cannot account for different stylized facts of asset log returns, such as skewness and kurtosis. Furthermore, market crashes like the one on Black Monday, or the default of Lehman Brothers are observed more frequently than the Black & Scholes model prescribes.

The above-mentioned shortcomings of the Black & Scholes model have led to the search for and the development of more realistic stochastic models to describe the behaviour of stock prices. A flexible class of stochastic models is the class of the Lévy models; see for example Sato (1999) and Schoutens (2003) for an overview of Lévy processes and their applicability in finance. These advanced stock price models allow for jumps, excess kurtosis and skewness and, as a result, are more suitable for modelling stock price behaviour than the

Black & Scholes model. In Madan and Seneta (1990), the Variance Gamma (VG) process was introduced for modelling the behaviour of a single stock. In Carr and Madan (1999), it was shown that in this model setting vanilla option prices can be approximated using the Fast Fourier Transform (FFT) in an accurate and computationally efficient way. Furthermore, this approach can be extended to other Lévy processes, as long as the characteristic function of the Lévy process is given in closed form.

The increased volume of multi-asset derivatives has shifted research to the question how to model the joint dynamics of a number of dependent stock prices. The most straightforward choice is to extend the univariate Black & Scholes model to the multivariate case, by using correlated Brownian motions (see e.g. Björk 1998, Dhaene *et al.* 2013). This multivariate stock price model uses a lognormal distribution for the single stock prices and a Gaussian dependence structure. However, a Gaussian dependence structure is not realistic; for example, it does not allow for tail dependence. In Luciano and Schoutens (2006), a multivariate version of the VG process was introduced. The individual stock prices are still modelled using a VG process, but they are dependent through a common time change. This new multivariate model can benefit from the strengths of the univariate VG process and also introduces a non-Gaussian dependence structure. Extensions of this multivariate Lévy process can be found in Moosbrucker (2006), Semeraro (2008), Leoni and Schoutens (2008), Luciano and Semeraro (2010) and Guillaume (2013).

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Throughout this paper, we assume that a basket always consists of a number of stocks. The distribution of the basket is in general unknown and, as a result, basket option pricing relies on approximations or numerical methods like Monte Carlo simulation. In this paper, we search for an approximate basket option pricing formula when the stock prices composing the basket are modelled by a multivariate VG model. Conditional on the time change, the basket is a weighted sum of correlated geometric Brownian motions. An extensive bibliography is dedicated to the problem of finding accurate approximations for basket options when dealing with correlated Brownian motions. Here, we use convex upper and lower bounds to derive approximate basket option prices (see e.g. Kaas *et al.* 2000, Vanduffel *et al.* 2005). Having an accurate approximation for the conditional basket option price, the (unconditional) basket option price is determined in closed form using a Gauss-Laguerre quadrature formula. It turns out that the approximate basket option price is given by a weighted sum of Black & Scholes prices, where the weights and values of the Gamma subordinator are based on Gauss-Laguerre polynomials. Note also that our methodology for pricing basket options can be extended to the situation where the joint stock price dynamics can be described by time-changed Brownian motions with common time change. Simulation results show that our methodology is able to closely approximate the basket option price. Furthermore, we can derive the approximate distribution function of the basket.

Calibration of a multivariate stock price model is not straightforward, because a large amount of data is required. In Leoni and Schoutens (2008), the authors propose two calibration procedures of the multivariate VG model. A first approach assumes that only multi-asset derivatives are traded and the marginal parameters are estimated using historical data. The second procedure uses only vanilla option data, but assumes that the pairwise correlations between the Brownian motions are exogenously given. We propose a methodology for calibrating the multivariate VG model with single correlation parameter in a fully market implied way (see also Linders and Schoutens 2014a, 2014b). Our approximate basket option pricing formula is an essential tool in order to have an efficient and fast calibration of the multivariate VG model. We assume that multi-asset derivatives are traded and their prices can be observed. This situation occurs when we consider a traded stock market index like the Dow Jones, EUROSTOXX50, S&P 500, ... A basket option is then often called an index option. The calibration is carried out in two steps. Firstly, the marginal parameters and the distribution of the common time change can be calibrated to market quotes of the vanilla options by using the Carr-Madan formula. Secondly, the availability of traded index options together with an approximate basket option pricing formula enables us to calibrate the remaining correlation parameter, where we make the simplifying assumption that all pairwise correlations are the same.

We illustrate this calibration procedure using the Dow Jones index, a price weighted stock market index consisting of 30 American stocks. Using the available vanilla and index option prices, we are able to calibrate the multivariate VG model. To test if the calibrated model is suitable for multi-asset derivative pricing, we investigate if the model index option prices are comparable to their quoted counterparts. Remarkably, we

can conclude that the quoted DJ option prices can be closely matched by the calibrated multivariate VG model.

To the best of our knowledge, only a few papers have investigated the performance of different multivariate stock price models for reproducing observed vanilla and index option curves (see e.g. Jourdain and Sbair 2012, Cont and Deguest 2013). Multivariate stock price models based on correlated VG processes and their calibration are discussed in Ballotta *et al.* (2014), Ballotta and Bonfiglioli (2014) and Linders and Schoutens (2014a). However, these models will, in general, fail to provide an accurate fit and produce an *implied correlation smile* when calibrated to index option data (see also Fonseca *et al.* 2007, Garcia *et al.* 2009, Tavin 2013, Austing 2014, Linders and Schoutens 2014b).

## 2. Convex order and comonotonicity

In this section, we summarize some definitions and results concerning convex order, inverse distributions and comonotonicity needed afterwards.

Given two random variables  $X$  and  $Y$ ,  $X$  is said to precede  $Y$  in *convex order sense*, notation  $X \preceq_{cx} Y$ , if

$$\begin{cases} \mathbb{E}[(X - K)_+] \leq \mathbb{E}[(Y - K)_+] \\ \mathbb{E}[(K - X)_+] \leq \mathbb{E}[(K - Y)_+] \end{cases}, \quad \text{for all } K \in \mathbb{R}. \quad (1)$$

Consider the random vector  $(X_1, X_2, \dots, X_n)$  and the weighted sum  $S = \sum_{i=1}^n w_i X_i$ , where  $w_i > 0$ . Assume that the marginal stop-loss premiums  $\mathbb{E}[(X_i - K)_+]$  can be determined for any  $K$ . Even if we have full information about the marginal distributions, calculating the stop-loss premium  $\mathbb{E}[(S - K)_+]$  is, in general, not straightforward as it requires information about the dependence among the marginals. Specifying this dependence structure can be done by choosing an appropriate copula. In most situations, the distribution of  $S$  is unknown or will be too cumbersome to work with. However, in case the components of a random vector possess a perfect positive dependence structure, the stop-loss premium of the sum  $S$  can be calculated in closed form.

The random vector  $(X_1, \dots, X_n)$  is said to be *comonotonic* if

$$(X_1, \dots, X_n) \stackrel{d}{=} (F_{X_1}^{-1}(U), \dots, F_{X_n}^{-1}(U)), \quad (2)$$

where  $U$  is in the sequel a uniform  $(0, 1)$  r.v. and ' $\stackrel{d}{=}$ ' denotes 'equality in distribution'. If  $S$  is a weighted sum of comonotonic random variables, the stop-loss premium  $\mathbb{E}[(S - K)_+]$  can be decomposed in stop-loss premiums of the marginals with appropriately chosen strikes. We state this result in Theorem 1. For a proof of this theorem, we refer to Kaas *et al.* (2000).

**THEOREM 1 (Decomposition formula)** *Consider a comonotonic random vector  $(X_1, X_2, \dots, X_n)$  and denote the weighted sum by  $S$ . Assume that  $F_S$  is continuous and strictly increasing on  $[0, +\infty)$ . For  $K \geq 0$ , the stop-loss premium  $\mathbb{E}[(S - K)_+]$  can be decomposed into a linear combination of stop-loss premiums of the marginals involved:*

$$\mathbb{E}[(S - K)_+] = \sum_{i=1}^n w_i \mathbb{E}[(X_i - K)_+], \quad (3)$$

where

$$K_i = F_{X_i}^{-1}(F_S(K)), \quad \text{for } i = 1, \dots, n, \quad (4)$$

and  $F_S(K)$  satisfies the following relation:

$$\sum_{i=1}^n w_i K_i = K. \quad (5)$$

In case the cdf  $F_S$  is not continuous and strictly increasing, a similar decomposition formula (3) can be proven for the stop-loss premium  $\mathbb{E}[(S - K)_+]$  of a comonotonic sum  $S$ , but then the expression for the strike price  $K_i$  will be slightly different. A sufficient condition for the cdf  $F_S$  of the comonotonic sum  $S$  to be strictly increasing and continuous is that the marginal cdf's  $F_{X_i}$  are strictly increasing and continuous. Furthermore, for appropriately chosen strikes  $K_i$ , the decomposition formula (3) remains valid for  $K < 0$  (see e.g. [Dhaene et al. 2002a](#), [Chen et al. 2015](#)).

For an extensive overview of the theory of comonotonicity, including proofs of the results mentioned in this subsection, we refer to [Dhaene et al. \(2002a\)](#). Financial and actuarial applications of the concept of comonotonicity are described in [Dhaene et al. \(2002b\)](#). An updated overview of applications of comonotonicity can be found in [Deelstra et al. \(2011\)](#).

### 3. The multivariate VG model

Consider a finite time horizon of  $T$  years and suppose that we are currently at time 0. We introduce the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ , where  $\mathbb{P}$  is the ‘real-world’ probability measure. The filtered probability space is assumed to satisfy the ‘usual conditions’ of completeness and right-continuity. Interest rates, stock and bond prices, as well as prices of other financial assets are described by stochastic processes (which are assumed to be semi-martingales) on this filtered measurable space. For simplicity in notation and terminology, we assume a deterministic interest rate  $r$ .

We assume a financial market where  $n$  different (dividend or non-dividend paying) stocks, labelled from 1 to  $n$ , are traded. For each stock  $i$ , its random price at time  $t$ ,  $0 \leq t \leq T$ , is denoted by  $X_i(t)$ . We denote the stochastic price process of stock  $i$  by  $\{X_i(t) \mid 0 \leq t \leq T\}$ . Hereafter, we will always silently assume that  $X_i(t) \geq 0$  and that all expectations are finite.

#### 3.1. The stock price processes

Consider the process  $\{G(t) \mid t \geq 0\}$  defined on the filtered probability space. It is assumed that this process is a subordinator, i.e. it is a non-decreasing process which starts at zero and has stationary and independent increments. The parameters of the process  $\{G(t) \mid t \geq 0\}$  are chosen such that  $\mathbb{E}[G(t)] = t$ . The characteristic function of the process  $G$  is given by  $\phi_G(u) = \mathbb{E}[e^{iuG(1)}]$ . Throughout this paper, we assume that  $\{G(t) \mid t \geq 0\}$  is a Gamma process with parameters  $at$  and  $b$ , i.e.

$$f_{G(t)}(y) = \frac{b^{at}}{\Gamma(at)} y^{at-1} e^{-yb}, \quad y \geq 0. \quad (6)$$

Table 1. Input for the three-stock basket option.

	1	2	3
$\mu_i$	-0.1368	-0.056	-0.1984
$\sigma_i$	0.1099	0.1677	0.0365
$X_i(0)$	100	200	300
$w_i$	1/3	1/6	1/9

The condition  $\mathbb{E}[G(t)] = t$  implies that  $a = b$  and by using the notation  $\nu = \frac{1}{a}$ , we also have that  $\text{Var}[G(t)] = t\nu$ . Note that our methodology remains valid for other choices of the subordinator.

In the sequel,  $\{\underline{B}(t) \mid t \geq 0\}$  denotes a standard  $n$ -dimensional Brownian motion process independent from  $\{G(t) \mid t \geq 0\}$ . Furthermore, for  $i \neq j$ , we write that

$$\rho_{i,j} = \text{Corr}[\sigma_i B_i(t), \sigma_j B_j(t+s)], \quad (7)$$

where

$$\rho_{i,j} \geq 0. \quad (8)$$

We consider the multivariate VG model introduced in [Luciano and Schoutens \(2006\)](#). The joint risk-neutral dynamics of the stock prices are modelled as follows,

$$X_i(t) \stackrel{d}{=} X_i(0) \exp\{(r - q_i + \omega_i)t + \mu_i G(t) + \sigma_i B_i(G(t))\}, \\ \times \text{ for } i = 1, 2, \dots, n, \quad (9)$$

where

$$\omega_i = \frac{1}{\nu} \log \left( 1 - \frac{1}{2} \sigma_i^2 \nu - \mu_i \nu \right)$$

ensures that the corresponding process  $\{e^{-(r-q_i)t} X_i(t) \mid t \geq 0\}$  is a martingale. Furthermore,  $\mu_i \in \mathbb{R}$  and  $\sigma_i > 0$  for each  $i$ . Stock  $i$  may pay dividends and we assume that these dividends are paid continuously at a rate  $q_i$ . In case the joint dynamics of the stocks are described by (9), each stock can be written as the exponential of a VG process.

The market basket is composed of a linear combination of the  $n$  underlying stocks. Denoting the price of the basket at time  $t$  by  $S(t)$ ,  $0 \leq t \leq T$ , we have that

$$S(t) = w_1 X_1(t) + w_2 X_2(t) + \dots + w_n X_n(t), \quad (10)$$

where  $w_i$ ,  $i = 1, 2, \dots, n$ , are positive weights that are fixed up front. The price of a basket call option with maturity  $T$  and strike price  $K$  is denoted by  $C[K, T]$ . The marginals of the basket  $S$  are modelled by a VG process and according to [Carr and Madan \(1999\)](#), the pricing of vanilla options can be done in a fast and accurate way. In this paper, we describe a methodology to find an accurate approximation, denoted by  $\bar{C}[K, T]$ , for the real basket option price.

The notations  $F_{X_i(T)}$  and  $F_{S(T)}$  will be used for the time-0 cumulative distribution functions (cdf's) of  $X_i(T)$  and  $S(T)$ . In order to avoid unnecessary overloading of the notations, we will hereafter omit the fixed time index  $T$  when no confusion is possible.

#### 3.2. Basket options in the multivariate VG model

Consider the time- $T$  price vector  $(X_1, X_2, \dots, X_n)$ . By equation (9), we can write the distribution of the price level  $S$  of

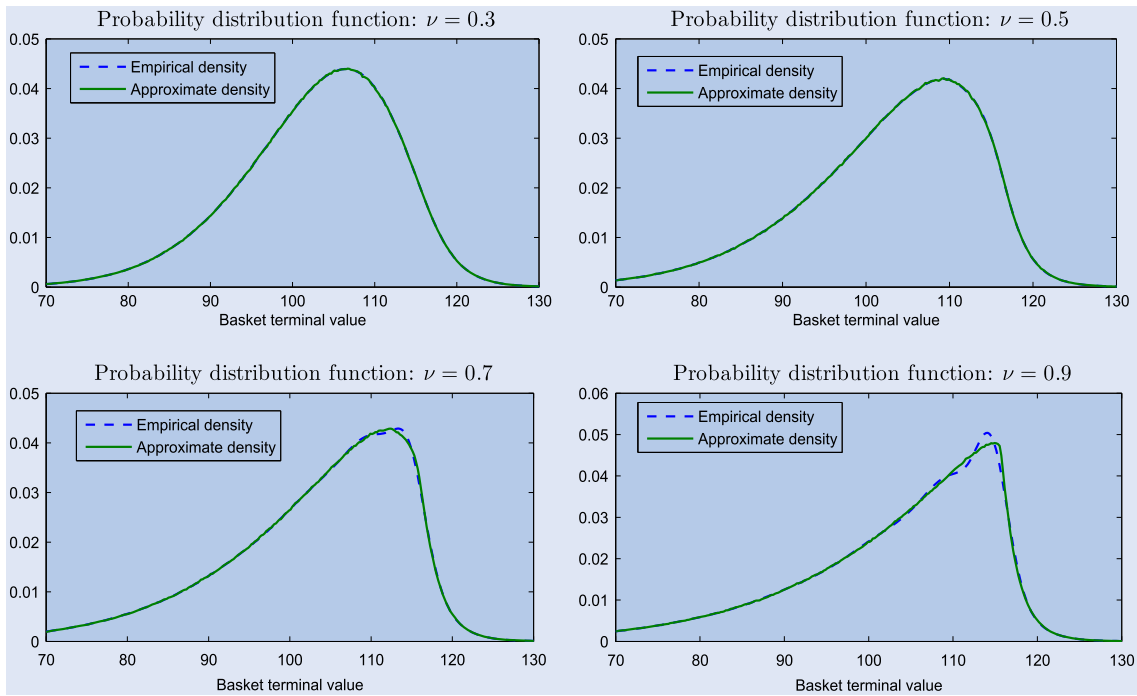


Figure 1. The probability distribution function for the simulated sum  $S^{sim}$  (dashed line) and the approximate sum  $\bar{S}$  (solid line) for different values of  $\nu$ . The other parameters are  $T = 1$ ,  $\sigma_1 = 0.1099$ ,  $\sigma_2 = 0.1677$ ,  $\sigma_3 = 0.0365$  and  $\mu_1 = -0.1368$ ,  $\mu_2 = -0.056$ ,  $\mu_3 = -0.1984$ .

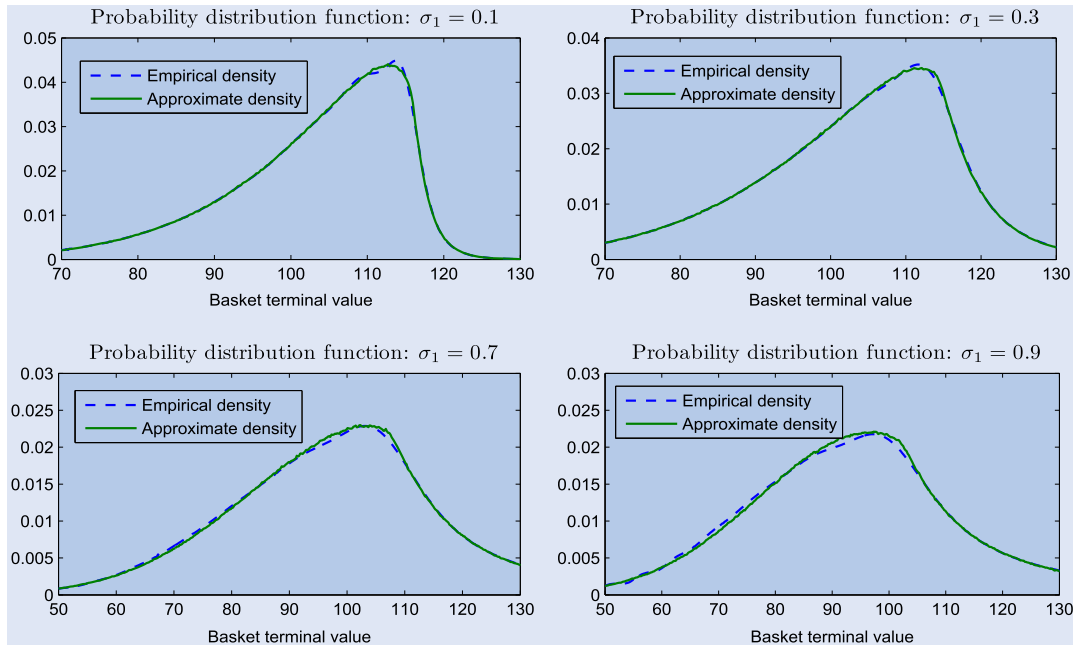


Figure 2. The probability distribution function for the simulated sum  $S^{sim}$  (dashed line) and the approximate sum  $\bar{S}$  (solid line) for different values of  $\sigma_1$ . The other parameters are  $T = 1$ ,  $\nu = 0.7514$ ,  $\sigma_2 = 0.1677$ ,  $\sigma_3 = 0.0365$  and  $\mu_1 = -0.1368$ ,  $\mu_2 = -0.056$ ,  $\mu_3 = -0.1984$ .

the basket at time  $T$  as follows

$$S \stackrel{d}{=} \sum_{i=1}^n w_i X_i(0) \exp \left\{ (r - q_i + \omega_i) T + \mu_i G + \sigma_i \sqrt{G} B_i(1) \right\}, \quad (11)$$

where  $B_i(1) \stackrel{d}{=} N(0, 1)$  and  $G$  is the previously defined subordinator. Conditional on  $G$ , the components in  $S$  are all log-normally distributed:

$$\ln \frac{X_i}{X_i(0)} \stackrel{d}{=} N \left( (r - q_i + \omega_i) T + \mu_i G, \sigma_i^2 G \right). \quad (12)$$

From (11) and (12), it follows that the conditional random variable  $S | G$  is a weighted sum of  $n$  dependent lognormal random variables with pairwise correlations  $\rho_{i,j}$ .

Using the tower property, we can write the basket option price  $C[K]$  as follows:

$$C[K] = \int_0^{+\infty} e^{-rT} \mathbb{E}[(S - K)_+ | G = y] f_G(y) dy. \quad (13)$$

Note that for  $y \geq 0$ , the quantity  $e^{-rT} \mathbb{E}[(S - K)_+ | G = y]$  can be considered as the price of a basket call option where the underlying basket  $S | G = y$  is a weighted sum of



correlated lognormal random variables. We introduce the following notation:

$$S_y \equiv S \mid G = y, \quad \text{for } y \geq 0. \quad (14)$$

In order to find an approximation for the basket option price  $C[K]$ , we first approximate the integrand  $e^{-rT} \mathbb{E}[(S - K)_+ \mid G = y]$  using the theory of comonotonicity and afterwards the integral  $\int_0^{+\infty} e^{-rT} \mathbb{E}[(S - K)_+ \mid G = y] f_G(y) dy$  using Gauss-Laguerre polynomials.

For the conditional basket call option price, we introduce the following notation

$$C[K; y] = e^{-rT} \mathbb{E}[(S_y - K)_+].$$

Expression (13) then shows that the price  $C[K]$  for a basket call option is a mixture of the synthetic basket call option prices  $C[K; y]$ . An extensive bibliography is dedicated to the problem of finding accurate approximations for the basket option price  $C[K; y]$ . An analysis of the different approximations for the pricing of basket options in the multivariate Black & Scholes model can be found in Krekel *et al.* (2004). In this paper, we use convex upper and lower bounds to find an approximation for  $C[K; y]$ . This methodology was proposed in Vyncke *et al.* (2004) for Asian options and in Linders (2013) for basket options.

#### 4. The conditional basket option price

We start the search for an approximate basket option price by conditioning on the common time change  $G$ . From section 3.2, the basket  $S_y$  can be written as a weighted sum of correlated lognormal random variables. In this section, we consider convex upper and lower bounds for this conditional sum, which result in an upper and lower bound for the basket option price as well. Under the multivariate VG model, an explicit expression for these bounds is obtained.

##### 4.1. A convex upper bound for the conditional sum

Throughout this section, we fix  $y \geq 0$  and derive an upper bound for  $C[K; y]$  by replacing the real conditional sum  $S_y$  by the random sum  $S_y^c$ , which is defined as follows

$$S_y^c = w_1 F_{X_1|G=y}^{-1}(U) + \dots + w_n F_{X_n|G=y}^{-1}(U). \quad (15)$$

In Kaas *et al.* (2000), it is proven that the comonotonic sum  $S_y^c$  is a convex upper bound for the sum  $S_y$ ,

$$S_y \preceq_{cx} S_y^c. \quad (16)$$

We obtain from the definition in (1) that  $C[K; y]$  can be bounded from above as follows:

$$C[K; y] \leq e^{-rT} \mathbb{E}[(S_y^c - K)_+]. \quad (17)$$

Combining expressions (15) and (12), we find that  $S_y^c$  is a weighted sum of comonotonic lognormal random variables. In this special case, we can determine  $S_y^c$  explicitly in terms of the marginal parameters and the cdf  $\Phi$  of a standard normal distribution.

**THEOREM 2** Consider a market where the assets follow the multivariate VG model (9). Then the conditional random variable  $S_y^c$  is given by the following expression:

$$S_y^c \stackrel{d}{=} \sum_{i=1}^n w_i X_i(0) e^{(r-q_i+\omega_i)T+\mu_i y+\sigma_i \sqrt{y} \Phi^{-1}(U)}. \quad (18)$$

Its variance is given by

$$\begin{aligned} \text{Var}[S_y^c] &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j X_i(0) X_j(0) \\ &\quad e^{2rT+(\omega_i-q_i+\omega_j-q_j)T+(\mu_i+\mu_j)y+\frac{\sigma_i^2+\sigma_j^2}{2}y} (e^{\sigma_i \sigma_j y} - 1). \end{aligned} \quad (19)$$

*Proof* The marginal risk-neutral distributions are given by (12), from which we find that

$$X_i \stackrel{d}{=} X_i(0) \exp \left\{ (r - q_i + \omega_i) T + \mu_i G + \sigma_i \sqrt{G} \Phi^{-1}(U) \right\}.$$

If we combine this expression with Theorem 1 in Dhaene *et al.* (2002a), the inverse cdf  $F_{X_i|G=y}^{-1}$  is given by

$$\begin{aligned} F_{X_i|G=y}^{-1}(p) &= X_i(0) \\ &\quad \times \exp \left\{ (r - q_i + \omega_i) T + \mu_i y + \sigma_i \sqrt{y} \Phi^{-1}(p) \right\}. \end{aligned} \quad (20)$$

Combining this observation with formula (15) proves (18).

The variance  $\text{Var}[S_y^c]$  can be written as

$$\begin{aligned} \text{Var}[S_y^c] &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \text{Cov} \left[ F_{X_i|G=y}^{-1}(U), F_{X_j|G=y}^{-1}(U) \right]. \end{aligned}$$

We have that

$$\begin{aligned} \text{Var}[S_y^c] &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j X_i(0) X_j(0) e^{2rT+(\omega_i-q_i+\omega_j-q_j)T+(\mu_i+\mu_j)y} \\ &\quad \times \text{Cov} \left[ e^{\sigma_i \sqrt{y} \Phi^{-1}(U)}, e^{\sigma_j \sqrt{y} \Phi^{-1}(U)} \right]. \end{aligned} \quad (21)$$

Note that the r.v.  $e^{\sigma_i \sqrt{y} \Phi^{-1}(U)}$  has a lognormal distribution and  $\mathbb{E} \left[ e^{\sigma_i \sqrt{y} \Phi^{-1}(U)} \right] = e^{\frac{\sigma_i^2 y}{2}}$ . The covariance in (21) can now be written as

$$\text{Cov} \left[ e^{\sigma_i \sqrt{y} \Phi^{-1}(U)}, e^{\sigma_j \sqrt{y} \Phi^{-1}(U)} \right] = e^{\frac{(\sigma_i^2 + \sigma_j^2)y}{2}} (e^{\sigma_i \sigma_j y} - 1). \quad (22)$$

Plugging (22) in expression (21) for the variance  $\text{Var}[S_y^c]$  proves (19).  $\square$

We adopt the following notation:

$$V_i = V_i(0) e^{(r-q_i+\omega_i)T+\mu_i y+\sigma_i \sqrt{y} \Phi^{-1}(U)}, \quad \text{for } i = 1, 2, \dots, n, \quad (23)$$

where  $y > 0$  and  $V_i(0) = X_i(0)$ . The vector  $(V_1, V_2, \dots, V_n)$  is a comonotonic vector where each random variable  $V_i$  has a lognormal distribution:

$$\ln \frac{V_i}{V_i(0)} \stackrel{d}{=} N \left( (r - q_i + \omega_i) T + \mu_i y, \sigma_i^2 y \right). \quad (24)$$

Using the lognormality of the random variables  $V_i$ , we can express  $e^{-rT} \mathbb{E} \left[ \left( S_y^c - K \right)_+ \right]$  in a closed form using the Black & Scholes option pricing formula.

**THEOREM 3** Consider a market where the assets follow the multivariate VG process (9). For  $y > 0$ , we have that

$$e^{-rT} \mathbb{E} \left[ \left( S_y^c - K \right)_+ \right] = \sum_{i=1}^n w_i \times \left( X_i(0) e^{(\omega_i - q_i)T + \left( \mu_i + \frac{\sigma_i^2}{2} \right)y} \Phi(d_{i,1}) - K_i e^{-rT} \Phi(d_{i,2}) \right), \quad (25)$$

where

$$d_{i,1} = \frac{\ln \left( \frac{X_i(0)}{K_i} \right) + (r - q_i + \omega_i)T + \mu_i y + \sigma_i^2 y}{\sigma_i \sqrt{y}}, \quad (26)$$

$$d_{i,2} = d_{i,1} - \sigma_i \sqrt{y}. \quad (27)$$

The strikes  $K_i$  are defined by

$$K_i = X_i(0) e^{(r - q_i + \omega_i)T + \mu_i y + \sigma_i \sqrt{y} \Phi^{-1}(F_{S_y^c}(K))} \quad (28)$$

and  $F_{S_y^c}(K)$  is determined such that the following relation holds:

$$\sum_{i=1}^n w_i K_i = K. \quad (29)$$

*Proof* From the definition of  $V_i$  in (23), we find that  $S_y^c \stackrel{d}{=} \sum_{i=1}^n w_i V_i$ . Furthermore, because the marginal cdfs  $F_{V_i}$  are strictly increasing and continuous, the cdf  $F_{S_y^c}$  is also strictly increasing and continuous. By applying Theorem 1, we obtain that

$$\mathbb{E} \left[ \left( S_y^c - K \right)_+ \right] = \sum_{i=1}^n w_i \mathbb{E} \left[ (V_i - K_i)_+ \right], \quad (30)$$

where  $K_i$  follows from expressions (4) and (5). Expression (23) for  $V_i$  results in the following expression for  $F_{V_i}^{-1}(p)$

$$F_{V_i}^{-1}(p) = X_i(0) e^{(r - q_i + \omega_i)T + \mu_i y + \sigma_i \sqrt{y} \Phi^{-1}(p)}.$$

We find that  $K_i$  can be determined by using relations (28) and (29).

The random variable  $V_i$  has a lognormal distribution; see (24). This means that we can modify the Black & Scholes formula to price options on  $V_i$  with maturity  $T$  and strike  $K_i$ :

$$e^{-rT} \mathbb{E} \left[ (V_i - K_i)_+ \right] = e^{-rT} \times \left( X_i(0) e^{(r - q_i + \omega_i)T + \mu_i y + \frac{\sigma_i^2 y}{2}} \Phi(d_{i,1}) - K_i \Phi(d_{i,2}) \right) \quad (31)$$

where  $d_{i,1}$  and  $d_{i,2}$  are given by

$$d_{i,1} = \frac{\ln \left( \frac{X_i(0)}{K_i} \right) + (r - q_i + \omega_i)T + \mu_i y + \sigma_i^2 y}{\sigma_i \sqrt{y}},$$

$$d_{i,2} = d_{i,1} - \sigma_i \sqrt{y}.$$

Plugging these expressions in equation (30) proves the result.  $\square$

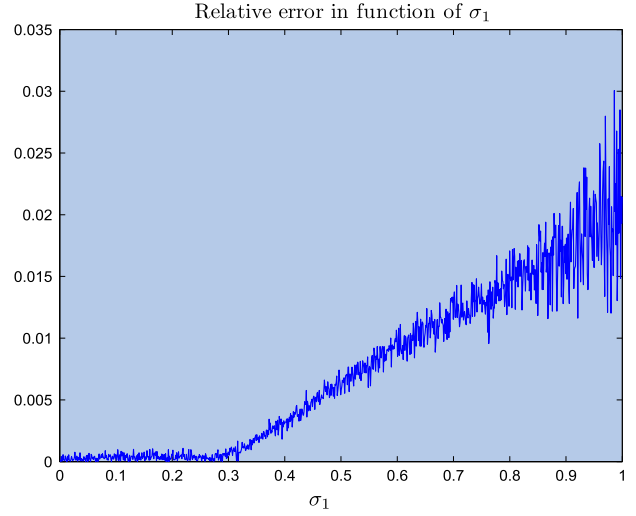


Figure 3. Relative error for different choices of  $\sigma_1$ .

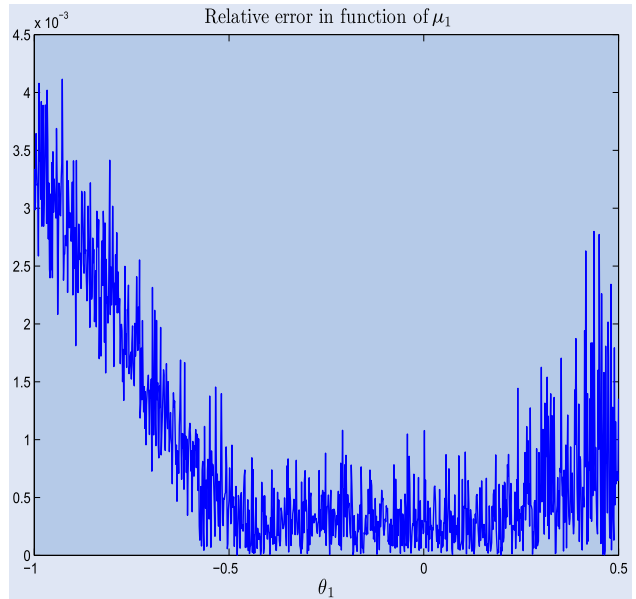


Figure 4. Relative error for different choices of  $\mu_1$ .

#### 4.2. A convex lower bound for the conditional sum

In this section, we search for a lower bound for  $C[K; y]$ . We replace the sum  $S_y$  by  $S_y^l$ , given by

$$S_y^l = \mathbb{E} [S_y | \Lambda_y],$$

where  $\Lambda_y$  is defined as follows

$$\Lambda_y \equiv \Lambda | G = y$$

$$= \sum_{j=1}^n \lambda_j \ln \left( \frac{X_j}{X_j(0)} \right) | G = y,$$

where the  $\lambda_j$  are real numbers. Note that the weights  $\lambda_j$  may depend on  $y$ . However, if no confusion is possible we will omit the dependence on  $y$  in the notation of the weights in order to keep the notation simple. We will illustrate with some numerical examples that this choice for  $\Lambda$  gives tight approximations. However, other choices for  $\Lambda_y$  exist (see Deelstra et al. 2004). In Kaas et al. (2000), it is proven that the sum  $S_y^l$  is a convex

lower bound for the sum  $S_y$ :

$$S_y^l \leq_{cx} S_y. \quad (32)$$

As a result, we find from the definition in (1) that  $C[K; y]$  can be bounded from below as follows:

$$e^{-rT} \mathbb{E} \left[ \left( S_y^l - K \right)_+ \right] \leq C[K; y].$$

In the following theorem, we formulate an explicit expression for the sum  $S_y^l$ .

**THEOREM 4** Consider a market where the assets follow the multivariate VG process (9). For  $y \geq 0$ , the conditional basket  $S_y^l$  is given by the following expression:

$$S_y^l \stackrel{d}{=} \sum_{i=1}^n w_i X_i(0) \exp \left\{ (r - q_i + \omega_i) T + \mu_i y + \frac{\sigma_i^2 y (1 - r_i^2)}{2} + r_i \sigma_i \sqrt{y} \Phi^{-1}(U) \right\}, \quad (33)$$

with  $r_i = \text{Corr} \left[ \Lambda_y, \ln \left( \frac{X_i}{X_i(0)} \right) \mid G = y \right]$  and

$$\Lambda_y \stackrel{d}{=} \sum_{j=1}^n \lambda_j \left( (r - q_j + \omega_j) T + \mu_j y + \sigma_j \sqrt{y} B_j(1) \right). \quad (34)$$

The conditional variance  $\text{Var}[S_y^l]$  is given by

$$\text{Var}[S_y^l] = \sum_{i=1}^n \sum_{j=1}^n w_i w_j X_i(0) X_j(0) e^{2rT + (\omega_i - q_i + \omega_j - q_j)T + (\mu_i + \mu_j + \frac{1}{2}(\sigma_i^2 + \sigma_j^2))y} (e^{r_i r_j \sigma_i \sigma_j y} - 1).$$

*Proof* By using expression (12), we find that expression (34) holds for the random variable  $\Lambda_y$ . Furthermore,  $\Lambda_y$  has a normal distribution. Remark that for a bivariate normal distribution  $(X, Y)$  with  $\rho = \text{Corr}[X, Y]$ , the r.v.  $X \mid Y$  has again a normal distribution with mean:

$$\mathbb{E}[X \mid Y] = \mathbb{E}[X] + \rho \sqrt{\frac{\text{Var}[X]}{\text{Var}[Y]}} (Y - \mathbb{E}[Y]), \quad (35)$$

and variance  $\text{Var}[X] (1 - \rho^2)$ . Using expression (35), we find that  $\ln \left( \frac{X_i}{X_i(0)} \right) \mid G, \Lambda$  has a normal distribution with mean

$$\mathbb{E} \left[ \ln \left( \frac{X_i}{X_i(0)} \right) \mid G = y, \Lambda_y \right] = (r - q_i + \omega_i) T + \mu_i y + r_i \sigma_i \sqrt{y} \left( \frac{\Lambda_y - \mathbb{E}[\Lambda_y]}{\sqrt{\text{Var}[\Lambda_y]}} \right),$$

and variance

$$\text{Var} \left[ \ln \left( \frac{X_i}{X_i(0)} \right) \mid G = y, \Lambda_y \right] = \sigma_i^2 y (1 - r_i^2),$$

where  $r_i = \text{Corr} \left[ \Lambda_y, \ln \left( \frac{X_i}{X_i(0)} \right) \mid G = y \right]$ . We then find that

$$\mathbb{E}[X_i \mid G = y, \Lambda_y] = X_i(0) \exp \left\{ (r - q_i + \omega_i) T + \mu_i y + \frac{\sigma_i^2 y (1 - r_i^2)}{2} + r_i \sigma_i \sqrt{y} \Phi^{-1}(U) \right\},$$

which proves (33).

We can write the variance as follows

$$\begin{aligned} \text{Var}[S_y^l] &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j X_i(0) X_j(0) \\ &\times e^{2rT + (\omega_i - q_i + \omega_j - q_j)T + (\mu_i + \mu_j + \frac{1}{2}(\sigma_i^2(1 - r_i^2) + \sigma_j^2(1 - r_j^2)))y} \\ &\times \text{Cov} \left[ e^{r_i \sigma_i \sqrt{y} \Phi^{-1}(U)}, e^{r_j \sigma_j \sqrt{y} \Phi^{-1}(U)} \right]. \end{aligned}$$

If we use that for any  $\sigma$ ,  $\mathbb{E}[e^{\sigma \Phi^{-1}(U)}] = e^{\frac{\sigma^2}{2}}$ , we find that

$$\begin{aligned} \text{Cov} \left[ e^{r_i \sigma_i \sqrt{y} \Phi^{-1}(U)}, e^{r_j \sigma_j \sqrt{y} \Phi^{-1}(U)} \right] \\ = e^{\frac{1}{2}(r_i^2 \sigma_i^2 + r_j^2 \sigma_j^2)y} (e^{r_i r_j \sigma_i \sigma_j y} - 1), \end{aligned}$$

from which we find the desired result.  $\square$

In this paper, we choose the weights  $\lambda_j$  of the conditioning r.v.  $\Lambda_y$  by the ‘maximal variance’ method, proposed in Vanduffel *et al.* (2005). The convex order relation (32) implies that  $\text{Var}[S_y^l] \leq \text{Var}[S_y]$ . In Dhaene *et al.* (2002a), it was proven that  $\text{Var}[S_y^l] = \text{Var}[S_y]$  implies that  $S_y^l \stackrel{d}{=} S_y$ ; see Cheung *et al.* (2015) for a generalization of this result. Using these observations, we choose  $\lambda_j$  such that  $\text{Var}[S_y^l]$  is as close as possible to  $\text{Var}[S_y]$ . One can prove that the following choice is optimal:

$$\lambda_j = w_j X_j(0) \exp \left\{ (r - q_j + \omega_j) T + \mu_j y + \frac{\sigma_j^2 y}{2} \right\}, \quad \text{for } j = 1, 2, \dots, n.$$

For other choices of the conditioning random variable  $\Lambda$ , we refer to Deelstra *et al.* (2004) and Deelstra and Hainaut (2014).

**Remark 5** (Calculation of  $r_i$ ) Take  $y > 0$  and consider the conditional random variable  $S_y^l$ . If we denote the variance of  $\Lambda_y$  by  $y \sigma_{\Lambda_y}^2$ , we find from relations (7) and (34) that

$$\sigma_{\Lambda_y}^2 = \sum_{i=1}^n \lambda_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^n \lambda_i \lambda_j \sigma_i \sigma_j \rho_{i,j}.$$

The correlation coefficient  $r_i$  is then given by

$$r_i = \frac{\sum_{j=1}^n \lambda_j \sigma_j \rho_{i,j}}{\sigma_{\Lambda_y}}.$$

A sufficient condition for  $r_i$  to be positive is that  $\rho_{i,j} \geq 0$ , for  $i, j = 1, 2, \dots, n$  and  $i \neq j$ .  $\nabla$

Define the random variable  $W_i$  as follows:

$$W_i = W_i(0) \exp \left\{ (r - q_i + \omega_i) T + \mu_i y + \frac{\sigma_i^2 y (1 - r_i^2)}{2} + r_i \sigma_i \sqrt{y} \Phi^{-1}(U) \right\},$$



Table 2. Call option prices for the three-stock basket option outlined in table 1 and different choices of  $\nu$ .

$T$	$\nu$	$K$	$\bar{C}[K]$	$C^{sim}[K]$	$\epsilon[K](\%)$	Length CI
1 year	0.5	75	27.3230	27.3250	0.009	1.30E − 02
		90	13.5860	13.5890	0.016	1.09E − 02
		100	5.9625	5.9640	0.026	7.65E − 03
		110	1.2033	1.2046	0.105	3.20E − 03
		125	0.0059	0.0061	2.458	2.51E − 04
	0.75	75	27.4430	27.4480	0.018	1.40E − 02
		90	13.9640	13.9670	0.019	1.14E − 02
		100	6.4267	6.4277	0.016	7.87E − 03
		110	1.3973	1.3980	0.051	3.29E − 03
		125	0.0050	0.0051	2.183	2.40E − 04
	0.9	75	27.5260	27.5310	0.017	1.44E − 02
		90	14.1660	14.1690	0.022	1.15E − 02
		100	6.6545	6.6574	0.044	7.95E − 03
		110	1.4923	1.4925	0.015	3.31E − 03
		125	0.0045	0.0047	3.385	2.35E − 04
2 years	0.5	75	29.6150	29.6150	0.002	1.88E − 02
		90	16.6940	16.6940	0.002	1.63E − 02
		100	9.4983	9.4990	0.007	1.30E − 02
		110	4.2349	4.2360	0.027	8.68E − 03
		125	0.5521	0.5540	0.349	2.82E − 03
	0.75	75	29.8430	29.8450	0.008	2.03E − 02
		90	17.2470	17.2490	0.009	1.72E − 02
		100	10.1970	10.1990	0.017	1.37E − 02
		110	4.8496	4.8517	0.043	9.30E − 03
		125	0.7020	0.7033	0.185	3.06E − 03
	0.9	75	29.9840	29.9850	0.001	2.10E − 02
		90	17.5400	17.5400	0.004	1.76E − 02
		100	10.5490	10.5480	0.008	1.40E − 02
		110	5.1581	5.1579	0.004	9.58E − 03
		125	0.7826	0.7829	0.041	3.17E − 03

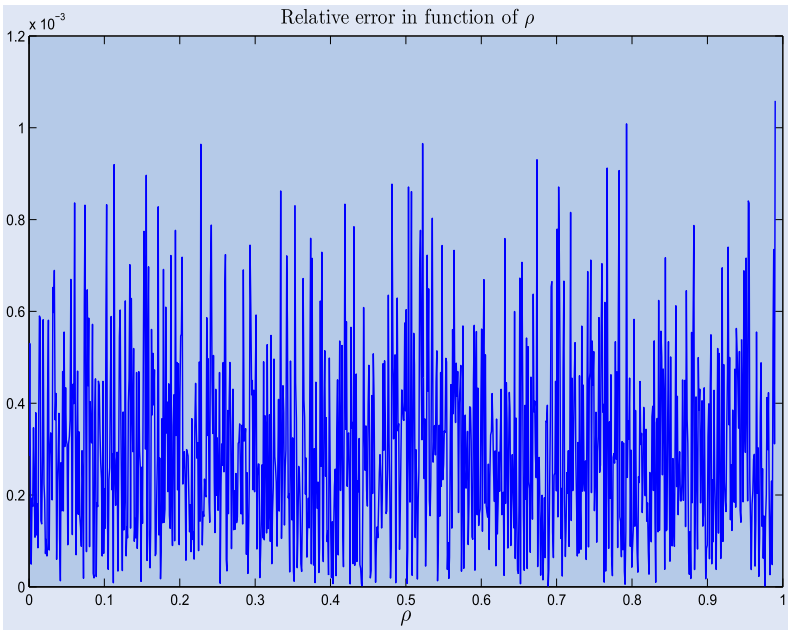


Figure 5. Relative error for different choices of  $\rho$ .

and  $W_i(0) = X_i(0)$ . Then we have that

$$\ln\left(\frac{W_i}{W_i(0)}\right) \stackrel{d}{=} N\left((r - q_i + \omega_i)T + \mu_i y + \frac{\sigma_i^2 y(1 - r_i^2)}{2}, r_i^2 \sigma_i^2 y\right).$$

The sum  $S_y^l$  can be written as

$$S_y^l = \sum_{i=1}^n w_i W_i,$$

Table 3. Call option prices for the three-stock basket option outlined in table 1 for  $\nu = 0.7514$  and different choices of  $\sigma_1$ .

$T$	$\sigma_1$	$K$	$\bar{C}[K]$	$C^{sim}[K]$	$\epsilon[K] (\%)$	Length CI
1 year	0.05	75	27.4220	27.4290	0.024	1.37E – 02
		90	13.8980	13.9020	0.031	1.11E – 02
		100	6.3213	6.3254	0.064	7.64E – 03
		110	1.2914	1.2945	0.245	3.02E – 03
		125	0.0023	0.0025	5.803	1.72E – 04
	0.25	75	27.5440	27.5450	0.003	1.58E – 02
		90	14.3010	14.3010	0.003	1.29E – 02
		100	6.9825	6.9827	0.003	9.44E – 03
		110	2.0228	2.0243	0.075	5.22E – 03
		125	0.1110	0.1127	1.468	1.53E – 03
	0.75	75	28.2450	28.2270	0.067	5.36E – 02
		90	16.7250	16.6180	0.649	5.17E – 02
		100	11.1240	10.9790	1.315	5.00E – 02
		110	7.5784	7.4377	1.892	4.81E – 02
		125	4.9288	4.8252	2.147	4.57E – 02
2 years	0.05	75	29.8010	29.8060	0.018	1.99E – 02
		90	17.1420	17.1460	0.024	1.68E – 02
		100	10.0480	10.0490	0.005	1.33E – 02
		110	4.6822	4.6817	0.010	8.94E – 03
		125	0.6028	0.6032	0.065	2.70E – 03
	0.25	75	30.0430	30.0410	0.005	2.31E – 02
		90	17.7700	17.7670	0.016	1.97E – 02
		100	10.9780	10.9750	0.029	1.61E – 02
		110	5.7923	5.7892	0.053	1.19E – 02
		125	1.4395	1.4420	0.168	6.05E – 03
	0.75	75	31.4240	31.3010	0.394	1.03E – 01
		90	21.4920	21.1820	1.461	1.01E – 01
		100	16.7010	16.3020	2.442	9.93E – 02
		110	13.2680	12.8570	3.194	9.78E – 02
		125	9.9779	9.6303	3.609	9.57E – 02

and given that (8) holds,  $S_y^l$  is a sum of  $n$  comonotonic log-normal random variables. Theorem 2 in Deelstra *et al.* (2004) proves that if assumption (8) does not hold, one can always find  $\lambda_i, i = 1, 2, \dots, n$  such that the corresponding r.v.  $S_y^l$  is a comonotonic sum.

**THEOREM 6** Consider a market where the assets follow the multivariate VG process (9), where all pairwise correlations  $\rho_{i,j}$  are non-negative. For  $y \geq 0$ , we have that

$$\begin{aligned}
& e^{-rT} \mathbb{E} \left[ \left( S_y^l - K \right)_+ \right] \\
&= \sum_{i=1}^n w_i \left( X_i(0) e^{(\omega_i - q_i)T + \left( \mu_i + \frac{\sigma_i^2}{2} \right)y} \Phi(d_{i,1}) \right. \\
&\quad \left. - K_i e^{-rT} \Phi(d_{i,2}) \right), \tag{36}
\end{aligned}$$

where

$$d_{i,1} = \frac{\ln \frac{X_i(0)}{K_i} + (r - q_i + \omega_i)T + \mu_i y + \frac{\sigma_i^2 y(1+r_i^2)}{2}}{r_i \sigma_i \sqrt{y}}, \tag{37}$$

$$d_{i,2} = d_{i,1} - \sigma_i r_i \sqrt{y}. \tag{38}$$

The strikes  $K_i$  are defined by

$$\begin{aligned}
K_i = X_i(0) \exp \left\{ (r - q_i + \omega_i)T + \mu_i y + \frac{\sigma_i^2 y(1 - r_i^2)}{2} \right. \\
\left. + r_i \sigma_i \sqrt{y} \Phi^{-1} \left( F_{S_y^l}(K) \right) \right\}, \tag{39}
\end{aligned}$$

and  $F_{S_y^l}(K)$  is determined such that the following relation holds:

$$\sum_{i=1}^n w_i K_i = K. \tag{40}$$

*Proof* Because all pairwise correlations  $\rho_{i,j}$  are non-negative, the sum  $S_y^l$  is a weighted sum of the comonotonic random variables  $W_1, W_2, \dots, W_n$ . Each  $W_i$  has a lognormal distribution. The remainder of the proof is now similar to the proof of Theorem 3.  $\square$

## 5. An approximate basket option price

In this section, we first derive the approximation  $\bar{C}[K]$  for the basket option price  $C[K]$ . Afterwards, the unconditional basket option price is found by approximating the integral by a Gauss-Laguerre quadrature formula.

Table 4. Call option prices for the three-stock basket option outlined in table 1 for  $\nu = 0.7514$  and different choices of  $\mu_1$ .

$T$	$\mu_1$	$K$	$\bar{C}[K]$	$C^{sim}[K]$	$\epsilon[K](\%)$	Length CI
1 year	-1.5	75	29.9740	29.9850	0.035	3.48E - 02
		90	19.9120	19.8610	0.255	2.93E - 02
		100	14.4570	14.4520	0.036	2.49E - 02
		110	9.8867	10.0430	1.560	2.03E - 02
		125	5.3129	5.1342	3.482	1.34E - 02
	-1	75	29.1870	29.1840	0.012	2.81E - 02
		90	18.0440	18.1050	0.342	2.30E - 02
		100	12.1620	12.1200	0.344	1.87E - 02
		110	7.2898	7.3586	0.936	1.40E - 02
		125	2.5436	2.5331	0.415	7.11E - 03
	-0.5	75	27.3200	27.3210	0.004	1.23E - 02
		90	13.5280	13.5280	0.002	1.02E - 02
		100	5.7517	5.7531	0.024	7.00E - 03
		110	0.8960	0.8974	0.158	2.64E - 03
		125	0.0065	0.0066	1.032	2.98E - 04
2 years	-1.5	75	33.6750	33.6860	0.034	5.75E - 02
		90	25.1820	25.1600	0.088	5.24E - 02
		100	20.5350	20.6910	0.751	4.85E - 02
		110	16.7920	16.9980	1.209	4.45E - 02
		125	12.8450	12.6000	1.950	3.86E - 02
	-1	75	32.6240	32.6270	0.009	4.41E - 02
		90	23.0410	22.9940	0.203	3.90E - 02
		100	17.7900	17.8200	0.168	3.49E - 02
		110	13.4890	13.5670	0.574	3.07E - 02
		125	8.7983	8.6806	1.356	2.43E - 02
	-0.5	75	29.6000	29.6010	0.004	1.78E - 02
		90	16.5860	16.5850	0.003	1.53E - 02
		100	9.2557	9.2551	0.007	1.21E - 02
		110	3.8684	3.8690	0.014	7.79E - 03
		125	0.3415	0.3431	0.459	2.16E - 03

### 5.1. A moment-matching approximation for the basket option price

In this subsection, we derive the approximate basket option price  $\bar{C}[K]$  for  $K \geq 0$  which is based on the conditional upper and lower bounds derived in section 4. First, the conditional approximation  $\bar{C}[K; y]$  is defined as

$$\bar{C}[K; y] = e^{-rT} \left( z_y \mathbb{E} \left[ \left( S_y^l - K \right)_+ \right] + (1 - z_y) \mathbb{E} \left[ \left( S_y^c - K \right)_+ \right] \right), \quad (41)$$

where  $z_y \in [0, 1]$ . Then,  $\bar{C}[K; y]$  can be interpreted as the price of a call option with strike  $K$ , written on a synthetic conditional stock market basket, denoted by  $\bar{S}_y$ , in the sense that the following relation holds

$$\bar{C}[K; y] = e^{-rT} \mathbb{E} \left[ \left( \bar{S}_y - K \right)_+ \right], \text{ for all } K \geq 0.$$

The weights  $z_y$  are chosen such that the approximation  $\bar{S}_y$  is as close as possible to  $S_y$ , defined in (14). We follow the approach of Linders (2013) and take  $z_y$  as follows

$$z_y = \frac{\text{Var} \left[ S_y^c \right] - \text{Var} \left[ S_y \right]}{\text{Var} \left[ S_y^c \right] - \text{Var} \left[ S_y^l \right]}, \quad (42)$$

where  $\bar{C}[K; y]$  is given by (41) and where  $z_y$  is chosen as in (42). Remark that the synthetic random variable  $\bar{S}_y$  always

exists and is unique because of the one-to-one relationship between a convex call option curve and its stop-loss premium.

Second, we define the approximate unconditional basket option price  $\bar{C}[K]$  as follows

$$\bar{C}[K] = \int_0^{+\infty} \bar{C}[K; y] f_G(y) dy, \quad (43)$$

We can then state the following theorem.

**THEOREM 7** Consider a market where the assets follow the multivariate VG model (9) and let  $\bar{C}[K]$  be defined by (43). Then there exists a r.v.  $\bar{S}$  such that

$$\bar{C}[K] = e^{-rT} \mathbb{E} \left[ \left( \bar{S} - K \right)_+ \right],$$

and moreover,

$$\text{Var}[\bar{S}] = \text{Var}[S].$$

*Proof* We first prove that the optimal value for  $z_y$  is given by (42). Choosing the approximation  $\bar{S}_y$  as close as possible to  $S_y$ , leads to the following criterion,

$$\begin{aligned} \int_{-\infty}^{+\infty} \left( \mathbb{E} \left[ \left( \bar{S}_y - K \right)_+ \right] - \mathbb{E} \left[ \left( S_y - K \right)_+ \right] \right) dK &= 0 \\ \iff z_y &= \frac{\text{Var} \left[ S_y^c \right] - \text{Var} \left[ S_y \right]}{\text{Var} \left[ S_y^c \right] - \text{Var} \left[ S_y^l \right]}, \end{aligned}$$

Table 5. Call option prices for the three-stock basket option outlined in table 1 with correlation matrix  $R$  and for different choices of  $\nu$ .

$T$	$\nu$	$K$	$\bar{C}[K]$	$C^{sim}[K]$	$\epsilon[K] (\%)$	Length CI
1 year	0.5	75	27.4040	27.4020	0.009	1.48E – 02
		90	13.9340	13.9310	0.017	1.24E – 02
		100	6.5583	6.5566	0.027	9.01E – 03
		110	1.7972	1.7966	0.035	4.72E – 03
		125	0.0644	0.0644	0.070	9.66E – 04
	0.75	75	27.5500	27.5550	0.016	1.55E – 02
		90	14.2950	14.2970	0.012	1.26E – 02
		100	6.9386	6.9382	0.005	9.01E – 03
		110	1.9190	1.9168	0.115	4.62E – 03
		125	0.0550	0.0550	0.076	9.03E – 04
	0.9	75	27.6430	27.6430	0.000	1.58E – 02
		90	14.4840	14.4840	0.000	1.26E – 02
		100	7.1273	7.1275	0.002	9.00E – 03
		110	1.9773	1.9767	0.030	4.56E – 03
		125	0.0505	0.0506	0.262	8.73E – 04
2 years	0.5	75	29.7900	29.7880	0.007	2.15E – 02
		90	17.2310	17.2300	0.010	1.85E – 02
		100	10.3170	10.3160	0.017	1.51E – 02
		110	5.1747	5.1735	0.024	1.08E – 02
		125	1.1327	1.1325	0.016	4.92E – 03
	0.75	75	30.0510	30.0490	0.006	2.25E – 02
		90	17.7530	17.7510	0.010	1.91E – 02
		100	10.9120	10.9100	0.018	1.55E – 02
		110	5.6598	5.6575	0.041	1.11E – 02
		125	1.2497	1.2484	0.101	4.98E – 03
	0.9	75	30.2030	30.2120	0.027	2.30E – 02
		90	18.0260	18.0330	0.038	1.93E – 02
		100	11.2150	11.2210	0.056	1.56E – 02
		110	5.9080	5.9130	0.085	1.13E – 02
		125	1.3091	1.3122	0.240	4.99E – 03

where we use  $\mathbb{E}[S_y^c] = \mathbb{E}[S_y] = \mathbb{E}[S_y']$  and the following variance relation,

$$\text{Var}[X] = 2 \int_{-\infty}^{+\infty} (\mathbb{E}[(X - K)_+] - (\mathbb{E}[X] - K)_+) dK.$$

We can then write

$$\begin{aligned} \text{Var}[\bar{S}] - \text{Var}[S] &= \int_0^{+\infty} \left[ 2 \int_{-\infty}^{+\infty} (\mathbb{E}[(\bar{S}_y - K)_+] - \mathbb{E}[(S_y - K)_+]) f_G(y) dy \right. \\ &\quad \left. - \mathbb{E}[(S_y - K)_+] \right] dK \\ &= 0. \end{aligned}$$

□

Note that upper and lower bounds for the basket option price  $C[K]$  can be obtained by choosing  $z_y = 0$  or  $z_y = 1$ , respectively.

## 5.2. Numerical integration

The expression (43) for the approximate basket option price is given in an integral form and has to be evaluated using numerical integration procedures. In this subsection, we show that by using the Gauss-Laguerre quadrature, we arrive at a simple and easy-to-implement approximation for  $\bar{C}[K]$ .

The price  $C[K]$  for a basket option with maturity  $T$  and strike  $K$  is approximated by  $\bar{C}[K]$ , which can be written as

in (43) where  $f_G$  is given by expression (6). Plugging this expression for  $f_G$  into (43) results in

$$\bar{C}[K] = \int_0^{+\infty} \bar{C}[K; y] \frac{\left(\frac{1}{\nu}\right)^{\frac{1}{\nu}T}}{\Gamma\left(\frac{1}{\nu}T\right)} y^{\frac{1}{\nu}T-1} \exp\left\{-y\frac{1}{\nu}\right\} dy.$$

Using the substitution  $x = \frac{y}{\nu}$  and defining  $\beta = \frac{1}{\Gamma\left(\frac{1}{\nu}T\right)}$  and  $\alpha = \frac{1}{\nu}T - 1$ , we can rewrite the approximation  $\bar{C}[K]$  as follows:

$$\bar{C}[K] = \beta \int_0^{+\infty} \bar{C}[K; \nu x] g(x) dx, \quad (44)$$

with

$$g(x) = x^\alpha e^{-x}. \quad (45)$$

Expression (44) shows that the approximation  $\bar{C}[K]$  can be expressed as an integral, where the integrand is the product of the weighting function  $g$  and the smooth function  $\bar{C}$ . Because the weight function is given by (45) with  $\alpha > -1$ , we can use the generalized Gauss-Laguerre polynomial of degree  $d$ , denoted by  $L_d^{(\alpha)}$ , to approximate the integral in expression (44). This approach results in the following approximation for  $\bar{C}[K]$ :

$$\bar{C}[K] \approx \beta \sum_{i=1}^d g_{i;d} \bar{C}[K; \nu y_{i;d}],$$

Table 6. Call option prices for the three-stock basket option outlined in table 1 with correlation matrix  $R$  and for different choices of  $\sigma_1$ .

$T$	$\sigma_1$	$K$	$\bar{C}[K]$	$C^{sim}[K]$	$\epsilon[K](\%)$	Length CI
1 year	0.05	75	27.4920	27.5000	0.031	$1.47\text{E} - 02$
		90	14.1200	14.1260	0.043	$1.19\text{E} - 02$
		100	6.6686	6.6717	0.047	$8.35\text{E} - 03$
		110	1.6315	1.6320	0.032	$3.83\text{E} - 03$
		125	0.0163	0.0165	0.948	$4.52\text{E} - 04$
	0.25	75	27.7810	27.7920	0.038	$1.86\text{E} - 02$
		90	14.9350	14.9430	0.054	$1.53\text{E} - 02$
		100	7.9049	7.9117	0.085	$1.18\text{E} - 02$
		110	2.9873	2.9932	0.195	$7.75\text{E} - 03$
		125	0.4229	0.4247	0.434	$3.44\text{E} - 03$
	0.75	75	28.8780	28.8980	0.068	$6.89\text{E} - 02$
		90	17.8740	17.8590	0.088	$6.70\text{E} - 02$
		100	12.4290	12.4010	0.224	$6.55\text{E} - 02$
		110	8.7823	8.7507	0.361	$6.39\text{E} - 02$
		125	5.7983	5.7795	0.327	$6.18\text{E} - 02$
2 years	0.05	75	29.9350	29.9290	0.020	$2.13\text{E} - 02$
		90	17.4810	17.4750	0.030	$1.80\text{E} - 02$
		100	10.5310	10.5260	0.042	$1.45\text{E} - 02$
		110	5.2244	5.2223	0.041	$1.01\text{E} - 02$
		125	0.9336	0.9340	0.051	$3.86\text{E} - 03$
	0.25	75	30.4890	30.5000	0.034	$2.71\text{E} - 02$
		90	18.7340	18.7420	0.044	$2.33\text{E} - 02$
		100	12.2730	12.2780	0.048	$1.97\text{E} - 02$
		110	7.2454	7.2490	0.050	$1.56\text{E} - 02$
		125	2.6025	2.6042	0.065	$9.75\text{E} - 03$
	0.75	75	32.4340	32.3310	0.317	$1.10\text{E} - 01$
		90	23.0800	22.9000	0.783	$1.08\text{E} - 01$
		100	18.4270	18.2150	1.163	$1.07\text{E} - 01$
		110	14.9330	14.7130	1.500	$1.05\text{E} - 01$
		125	11.3670	11.1640	1.817	$1.03\text{E} - 01$

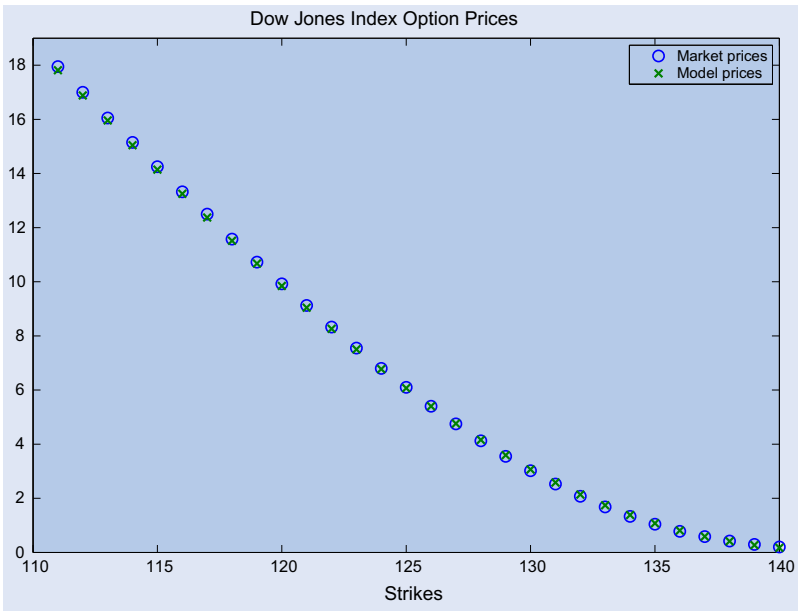


Figure 6. Model (crosses) and market (circles) prices for traded Dow Jones Index options on 18 April 2008 and time to maturity equal to 64 days.

where  $g_{i:d}$  are the quadrature weights and  $y_{i:d}$  is the  $i$ th root of the Gauss-Laguerre polynomial  $L_d^{(\alpha)}$ . A similar approach for determining joint defaults and pricing CDO tranches was followed in [Garcia and Goossens \(2010\)](#).

The function values and the derivative of the Gauss-Laguerre polynomial  $L_d^{(\alpha)}$  are given by the following recurrence relations:



Table 7. Call option prices for the three-stock basket option outlined in table 1 with correlation matrix  $R$  and for different choices of  $\mu_1$ .

$T$	$\mu_1$	$K$	$\bar{C}[K]$	$C^{sim}[K]$	$\epsilon[K](\%)$	Length CI
1 year	0.05	75	30.1190	30.1340	0.052	$3.50E-02$
		90	20.0720	20.0360	0.178	$2.95E-02$
		100	14.5890	14.6210	0.218	$2.51E-02$
		110	10.1190	10.1950	0.753	$2.05E-02$
		125	5.3474	5.2499	1.858	$1.37E-02$
	0.25	75	29.3380	29.3430	0.016	$2.85E-02$
		90	18.3250	18.3280	0.017	$2.34E-02$
		100	12.3520	12.3510	0.003	$1.91E-02$
		110	7.5512	7.5733	0.292	$1.44E-02$
		125	2.6737	2.6853	0.429	$7.44E-03$
	0.75	75	27.4170	27.4240	0.025	$1.43E-02$
		90	13.9050	13.9090	0.026	$1.18E-02$
		100	6.3906	6.3926	0.030	$8.48E-03$
		110	1.5154	1.5163	0.063	$4.32E-03$
		125	0.0620	0.0621	0.260	$1.05E-03$
2 years	0.05	75	33.9380	33.9390	0.002	$5.79E-02$
		90	25.4060	25.4320	0.101	$5.28E-02$
		100	20.8630	20.9450	0.391	$4.89E-02$
		110	17.2410	17.2260	0.089	$4.49E-02$
		125	12.9300	12.7890	1.096	$3.89E-02$
	0.25	75	32.8870	32.8860	0.002	$4.47E-02$
		90	23.3420	23.3210	0.089	$3.95E-02$
		100	18.1430	18.1560	0.068	$3.55E-02$
		110	13.8730	13.8870	0.100	$3.12E-02$
		125	9.0002	8.9539	0.517	$2.49E-02$
	0.75	75	29.7980	29.7960	0.008	$2.07E-02$
		90	17.1730	17.1730	0.002	$1.77E-02$
		100	10.1480	10.1480	0.008	$1.43E-02$
		110	4.8880	4.8896	0.034	$1.01E-02$
		125	0.9028	0.9040	0.133	$4.38E-03$

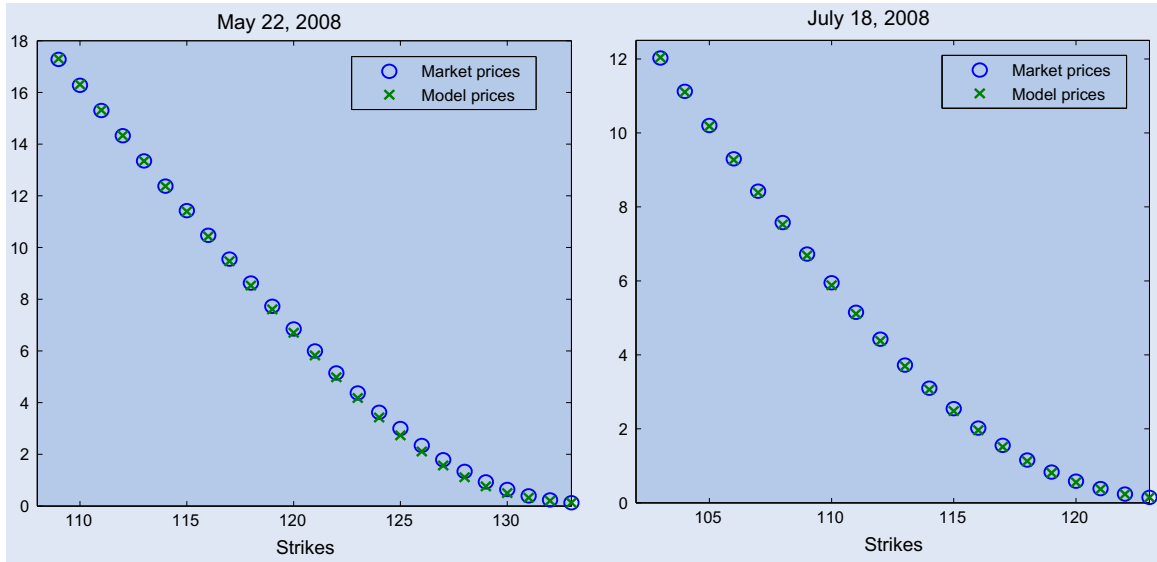


Figure 7. Model (crosses) and market (circles) prices for traded Dow Jones Index options on 18 July 2008 and 22 May 2008. The time to maturity is equal to 29 and 30 days, respectively.

$$(d+1)L_{d+1}^{(\alpha)}(x) = (2d+1+\alpha-x)L_d^{(\alpha)}(x) - (d-\alpha)L_{d-1}^{(\alpha)}(x),$$

$$y \frac{dL_d^{(\alpha)}}{dy}(y) = dL_d^{(\alpha)}(y) - (d+\alpha)L_{d-1}^{(\alpha)}(y),$$

with starting values  $L_0^{(\alpha)}(x) = L_{-1}^{(\alpha)}(x) = 0$ . The roots  $y_{i:d}$  can be determined using a Newton-Raphson iteration. The starting value  $y_{1:d}^{(1)}$  to determine the first root  $y_{1:d}$  is given by

$$y_{1:d}^{(1)} = \frac{(1+\alpha)(3+0.92\alpha)}{1+2.4n+1.8},$$

Table 8. Calibrated marginal parameters for the multivariate VG model on 18 April 2008.

	$X_i(0)$	$\sigma_i$	$\mu_i$	RMSE (%)
Alcoa Incorporated	36.26	0.5374	-0.50720	0.61
American Express Company	45.53	0.3715	-1.18450	1.99
American International group	48.23	0.4076	-1.85920	4.69
Bank of America	38.56	0.4256	-1.30810	2.63
Boeing Corporation	78.66	0.3640	-0.68050	3.70
Caterpillar	85.28	0.3731	-0.71440	2.81
JP Morgan	45.76	0.3490	-0.64090	4.65
Chevron	93.18	0.2168	-0.48380	1.53
Citigroup	25.11	0.4227	-0.65850	6.17
Coca Cola Company	60.11	0.2710	-0.52720	6.55
Walt Disney Company	31.33	0.2962	-0.55880	2.26
DuPont	52.02	0.3222	-0.50080	1.14
Exxon Mobile	94.00	0.2646	-0.59700	8.08
General Electric	32.69	0.2327	-0.28010	4.18
General Motors	20.13	0.6881	-1.33890	4.01
Hewlet-Packard	48.18	0.3927	-0.62160	1.01
Home Depot	28.68	0.4451	-1.08610	0.50
Intel	22.55	0.3652	-0.76170	1.82
IBM	124.40	0.2461	-0.60900	6.43
Johnson & Johnson	66.51	0.1775	-0.29690	2.86
McDonald's	58.30	0.2122	-0.43760	1.79
Merck & Company	39.76	0.4160	-0.91710	8.62
Microsoft	30.00	0.3407	-0.67170	1.59
3M	82.90	0.2608	-0.45860	1.12
Pfizer	20.47	0.2156	0.33030	2.84
Practer & Gamble	67.17	0.1916	-0.44340	1.55
AT&T	37.51	0.3172	-0.71230	0.72
United Technologies	72.51	0.3082	-0.68880	3.06
Verizon	36.03	0.3141	-0.65150	1.04
Wal-Mart Stores	56.31	0.2112	-0.37380	1.17

Table 9. Calibrated parameters for the DJ on three different days. The RMSE and the relative error are determined by comparing the observed DJ index option curve with the corresponding model prices; see also figures 6 and 7.

Day	18 April 2008	22 May 2008	18 July 2008
Time to maturity	64 days	30 days	29 days
$\nu$	0.076312	0.04380	0.03395
$\rho$	0.064745	0.23293	0.21057
RMSE	0.0796	0.1311	0.0367
Relative error	0.0154	0.0574	0.0401

and for the second root  $y_{2:d}$  the starting value is

$$y_{2:d}^{(1)} = y_{1:d} + \frac{15 + 6.25\alpha}{1 + 0.9\alpha + 2.5d}.$$

For the other roots  $y_{i:d}$ ,  $i = 2, 3, \dots, d$ , we use the starting value  $y_{i:d}^{(1)}$ :

$$y_{i:d}^{(1)} = y_{i-1:d} + \left( \frac{1 + 2.55(i-2)}{1.9(i-2)} + \frac{1.26(i-2)\alpha}{1 + 3.5(i-2)} \right) \frac{y_{i-1:d} - y_{i-2:d}}{1 + 0.3\alpha}.$$

Using the starting value  $y_{i:d}^{(1)}$ , the root  $y_{i:d}$  of the function  $L_d^{(\alpha)}$  is determined by using the following iterations

$$y_{i:d}^{(k+1)} = y_{i:d}^{(k)} - \frac{L_d^{(\alpha)}(y_{i:d}^{(k)})}{\frac{dL_d^{(\alpha)}}{dy}(y_{i:d}^{(k)})}.$$

The quadrature weights  $g_{i:d}$  are given by:

$$g_{i:d} = \frac{-\Gamma(\alpha + d)}{d\Gamma(d) L_{d-1}^{(\alpha)}(y_{i:d}) \frac{dL_d^{(\alpha)}}{dy}(y_{i:d})}.$$

Note that given  $\Gamma(\alpha + 1)$ , we can evaluate  $\Gamma(\alpha + d)$  using the relation  $\Gamma(x) = (x-1)\Gamma(x-1)$ , which implies that the Gamma function has to be evaluated only once. Alternatively, one can determine the weights and the roots of  $L_d^{(\alpha)}$  using an eigenvalue problem (see e.g. Press *et al.* 1992).

## 6. Numerical illustration

The bounds derived in this paper can be used to approximate the value of a basket option when the stock price dynamics are described by the multivariate VG process (9). We consider a basket option written on three underlying non-dividend paying

Table 10. The market and model prices for a Dow Jones call option on 18 April 2008 and time to maturity equal to 64 days.

Strikes	Market prices	Model prices
97	31.50	31.40
99	29.50	29.43
100	28.55	28.44
101	27.55	27.46
102	26.58	26.48
103	25.60	25.50
104	24.63	24.52
105	23.65	23.55
106	22.70	22.58
107	21.73	21.62
108	20.78	20.66
109	19.83	19.70
110	18.85	18.76
111	17.95	17.82
112	17.00	16.88
113	16.05	15.96
114	15.15	15.04
115	14.25	14.14
116	13.33	13.25
117	12.50	12.38
118	11.58	11.52
119	10.73	10.67
120	9.93	9.85
121	9.13	9.04
122	8.33	8.26
123	7.55	7.50
124	6.80	6.77
125	6.10	6.07
126	5.40	5.40
127	4.75	4.76
128	4.13	4.16
129	3.55	3.59
130	3.02	3.07
131	2.53	2.58
132	2.08	2.14
133	1.68	1.74
134	1.33	1.38
135	1.04	1.08
136	0.78	0.81
137	0.59	0.59
138	0.42	0.41
139	0.30	0.27
140	0.20	0.17

stocks. The parameter values of our base scenario are listed in table 1. In order to work with realistic parameters, we have taken the calibrated VG parameters reported in section 4.1.1 of Luciano and Schoutens (2006). We start with a multivariate VG model with uncorrelated Brownian motions. Note that this was the original model proposed in Luciano and Schoutens (2006). The stocks have all different spot prices and the weights are chosen such that the spot price of the basket is equal to 100.

For a number of strikes  $K$ , we compare the approximate option price  $\bar{C}[K]$  with its corresponding Monte Carlo estimate, denoted by  $C^{sim}[K]$ . The approximations  $\bar{C}[K]$  are determined by using Gauss-Laguerre polynomials with degree  $d = 24$ . The quality of the approximation is measured by the relative error  $\varepsilon[K]$ :

$$\varepsilon[K] = \frac{|\bar{C}[K] - C^{sim}[K]|}{C^{sim}[K]}.$$

We first determine option prices for different values of the parameter  $\nu$ . More specifically, we consider the situations where  $\nu = 0.5, 0.75$  and  $0.9$ . The maturities are  $T = 1$  and  $T = 2$ . The results are displayed in table 2. The approximate basket option curve can directly be linked to the probability distribution (pdf)  $f_{\bar{S}}$  of the approximate random variable  $\bar{S}$ . Indeed, we have that

$$f_{\bar{S}}(K) = e^{rT} \frac{\partial \bar{C}^2[K]}{\partial K^2}, \quad (46)$$

provided the second derivative of the basket option curve  $\bar{C}$  exists. In case we replace the approximate basket option curve  $\bar{C}$  in (46) with the corresponding Monte Carlo curve  $C^{sim}$ , we find the empirical pdf  $f_{S^{sim}}$  of the simulated random variable  $S^{sim}$ . A plot of  $f_{\bar{S}}$  and  $f_{S^{sim}}$  is shown in figure 1, where we have taken different values for  $\nu$  and a time to maturity  $T = 1$ . We

conclude that a higher value of  $\nu$  leads to marginal distributions that are more skewed.

Consider the three-stock basket option with parameters given in table 1. In table 3, we compare the approximation  $\bar{C}[K]$  with the Monte Carlo estimate  $C^{sim}[K]$ , for  $\sigma_1 = 0.05, 0.25$  and  $0.75$ . We observe that the error is larger when  $\sigma_1$  is large. A comparison between the probability distribution functions  $f_{\bar{C}}$  and  $f_{C^{sim}}$  for different values of  $\sigma_1$  and a time to maturity equal to  $T = 1$  is given in figure 2. In order to investigate the effect of  $\sigma_1$  on the quality of the approximation, we also determine  $\bar{C}[300]$  and  $C^{sim}[300]$  for an at-the-money basket option where  $\sigma_1$  varies between 0 and 1. Figure 3 depicts the relative error in function of  $\sigma_1$ . In case  $\sigma_1$  is larger than 0.5, the relative error is rising above 1%, whereas the approximation  $\bar{C}[300]$  proves to be accurate when  $\sigma_1$  is below 0.5. For all values of  $\sigma_1$ , the relative error stays below 3% and we conclude that using the approximation  $\bar{C}[K]$  for  $C[K]$  is justified in these situations.

Table 4 reports  $\bar{C}[K]$  and the corresponding Monte Carlo estimates  $C^{sim}[K]$  for the situation where  $\mu_1 = -1.5, -1$  and  $-0.5$ . The other parameters remain unchanged and are listed in table 1. In order to further assess the impact of the parameter  $\mu_1$ , we set  $T = 1$  and determine the approximations  $\bar{C}[300]$  and  $C^{sim}[300]$  for an at-the-money basket option for different choices of  $\mu_1$ . The remaining parameter values are the ones reported in table 1. A graph of the relative error in function of  $\mu_1$  is shown in figure 4. From these results, we conclude that for varying values of  $\mu_1$ , the approximation  $\bar{C}[K]$  always remains accurate.

In the examples described above, the Brownian motions are assumed to be uncorrelated. This is in line with the approach followed in the original paper of Luciano and Schoutens (2006). However, we will now consider the situation where the correlation matrix  $R$  containing the pairwise correlations  $\rho_{i,j}$  between the Brownian motions is given by:

$$R = \begin{pmatrix} 1 & 0.6 & 0.9 \\ 0.6 & 1 & 0.8 \\ 0.9 & 0.8 & 1 \end{pmatrix}.$$

The results for  $\bar{C}[K]$  and the corresponding Monte Carlo estimates  $C^{sim}[K]$  for different values of  $\nu$  and time to maturity equal to  $T = 1$  and  $T = 2$  are listed in table 5. Similarly, we also investigate the impact of changing  $\sigma_1$  or  $\mu_1$ . The results are reported in tables 6 and 7, respectively. One observes that the approximation remains very accurate in the presence of correlated Brownian motions.

In order to investigate the impact of the correlation parameter between the Brownian motions on the relative error, we assume that  $\rho_{i,j} = \rho$  for  $i \neq j$ . For a basket option with parameters given by table 1, we determine the approximations  $\bar{C}[300]$  and  $C^{sim}[300]$  for different values of  $\rho$ . The relative error is shown in figure 5. This figure shows that the relative error is small and relatively stable when we change the single, positive, correlation parameter.

## 7. Calibration and pricing of basket options

The perfect (multivariate) stock price model does not exist and any model has its merits, but also its drawbacks. Therefore, the

goal is to find a simple and intuitive model which is also capable of giving a correct description of the stock price behaviour. In the multivariate VG model (9), the individual stocks are modelled using a VG model and it is well known that this model provides a reasonable fit of the observed vanilla option curves. In order to test if the multivariate VG model is a good model for pricing multi-asset derivatives, we investigate in this section if the model can closely match quoted basket option prices. We illustrate this methodology using the Dow Jones Industrial Average (DJ). This price-weighted stock market index consists of 30 stocks and on each component of the DJ and on the DJ itself, options are traded and their prices are observable. The price of the vanilla option written on stock  $i$  with strike  $K$  and maturity  $T$  is denoted by  $C_i[K, T]$ . Note, however, that vanilla options are of American type, whereas DJ index options are of European type.

We denote by  $X_i, i = 1, 2, \dots, 30$ , the price level of the  $i$ -th component of the DJ at time  $T$ . If the price vector  $(X_1, X_2, \dots, X_{30})$  is described by the multivariate VG model (9), the log returns of stock  $i$  can be described by a VG distribution with parameters  $(\sigma_i, \nu, \mu_i)$ . In Carr and Madan (1999), the authors show how to determine the VG price  $C_i^{VG}[K, T]$  for an option on stock  $i$  by using the FFT. For a series of traded strikes  $K$ , we can observe the call option prices  $C_i[K, T]$  and the parameters  $\mu_1, \mu_2, \dots, \mu_{30}, \sigma_1, \sigma_2, \dots, \sigma_{30}, \nu$  can be determined by simultaneously calibrating the vanilla option curves. For 18 April 2008, the calibrated parameters together with the calibration error are listed in table 8, where we used the available vanilla option prices with a time to maturity of 64 days.

Given the vanilla option curves, all parameters of the multivariate VG model (9) can be calibrated, except the correlation parameters  $\rho_{i,j}$ . Here, we make the simplifying assumption that all pairwise correlations are equal, i.e.

$$\rho_{i,j} = \rho, \text{ if } i \neq j,$$

for some  $\rho \geq 0$ . Index options on the DJ are traded and their observed prices are denoted by  $C[K, T]$  for strike  $K$  and maturity  $T$ . The model price of an index option under the multivariate VG model (9) is denoted by  $C^{VG}[K, T; \underline{\mu}, \underline{\sigma}, \nu, \rho]$ . Note that this price can be closely approximated using the techniques described in section 5. This approximate DJ index option price is denoted by  $\bar{C}^{VG}[K, T; \underline{\mu}, \underline{\sigma}, \nu, \rho]$ . The parameters  $\underline{\mu}, \underline{\sigma}$  and  $\nu$  are calibrated to the vanilla option curves in a first step of the calibration. However, for any choice of  $\rho$ , another index price arises.

The parameter  $\rho$  can be calibrated if multi-asset derivatives are traded. The availability of market quotes for index options together with an approximate basket option pricing formula enable us to determine an implied estimate for  $\rho$  by minimizing the relative error between the market and model index option quotes. Moreover, this second step of the calibration procedure is relatively fast because for a given set of parameters, basket options can be priced in a fast way. The implied average correlation for 18 April 2008 is listed in table 9. Using the calibrated marginal parameters shown in table 8 and the calibrated joint parameters  $\nu$  and  $\rho$  listed in table 9, we can determine the DJ index option price  $\bar{C}^{VG}[K, T; \underline{\mu}, \underline{\sigma}, \nu, \rho]$  and compare this price with the market price  $C[K, T]$  for any traded strike  $K$ ; see figure 6. We find that the multivariate VG model is

capable of closely matching the observed market quotes for DJ options. The numerical values of the Dow Jones market and model prices are listed in table 10. The relative error and the RMSE are shown in table 8.

The implied average correlation is also computed for two other dates, namely 22 May and 18 July 2008, respectively. These values are depicted in table 9. In figure 7, we compare again the DJ index option price with the market price for a series of strikes  $K$ . Both graphs show that by choosing equal pairwise correlations, the multivariate VG model is able to fit the observed market option prices remarkably well. Note that we assume a single correlation parameter  $\rho$  in order to be able to calibrate the model using today's observed index option prices. Indeed, when we allow for a correlation matrix  $R$ ,  $n(n - 1)$  pairwise correlations should be calibrated. For example, in the Dow Jones case, this means that 870 correlation parameters should be calibrated. However, our data set only contains approximately 40 options on the Dow Jones. In order to relax the assumption of equal correlations and allow for different pairwise correlations, one can calibrate the model by using also historical option prices, instead of only using today's option prices.

## 8. Conclusion

The rapid growth of financial markets over the past decades has increased the interest in multi-asset products, such as basket options. In order to price such products, one needs to model the joint dynamics of a number of dependent stock prices. In this paper, we proposed a methodology for pricing basket options where the stock prices composing the basket are modelled by a multivariate VG model. The individual stock prices are then modelled using a VG process, but they are dependent through a common time change. Conditional on the time change, the basket is a weighted sum of correlated geometric Brownian motions. Using the additivity property of comonotonic stop-loss premiums, we derived a closed-form expression for the approximate basket option price as a linear combination of Black & Scholes prices, where the weights and values of the subordinator were based on Gauss-Laguerre polynomials.

In this paper, we searched for a pricing formula that gives us a reasonable compromise between the complexity and the tractability of the underlying model. Complexity here is understood as the ability of the underlying model to capture the characteristics of single-asset and multi-asset option prices, whereas tractability is related to the calibration of the model. Once a pricing formula is available and single-asset as well as multi-asset derivatives are traded, the model can be calibrated. Therefore, the availability of market quotes for options on the index and its components together with an approximate basket option pricing formula enabled us to have an efficient and fast calibration of the multivariate VG model. Two steps were carried out in the calibration procedure. Firstly, the marginal parameters and the distribution of the common time change were calibrated to market quotes of the vanilla options by using the Carr-Madan formula. Secondly, we determined an implied estimate for the average correlation by minimizing the relative error between the market and model index option quotes.

As an illustration, we showed that the multivariate VG model is able to closely match the observed Dow Jones index options. By assuming an equal pairwise correlation between the different stocks, we observed that the model still provides us with an accurate fit as well as a good estimate for the implied correlation.

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No potential conflict of interest was reported by the authors.

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