1 Introduction

Definition of NW-estimator is generally given by the following equation 1 and it can be divided into two parts, namely 2 and 3.

$$NW = \frac{\hat{\alpha(x)}}{\hat{p(x)}} \tag{1}$$

$$\hat{\alpha(x)} = \frac{1}{nh_n}\Sigma y_i \cdot K(\frac{x-x_i}{h_n}) \tag{2}$$

$$\hat{p(x)} = \frac{1}{nh_n}\Sigma K(\frac{x-x_i}{h_n}) \tag{3}$$

2 Variance and Expectation of $\hat{\alpha(x)}$ and $\hat{p(x)}$

$$\begin{aligned}
E(\hat{\alpha(x)}) &= \int \frac{1}{h_n} m(u)K(\frac{x-u}{h_n})p(u)du \\
&= \int m(x+h_n z)k(z)p(x+h_n z)dz \\
&= \int [m(x) + h_n m(x)z + \frac{h_n^2 z^2}{2} m'(x)] \\
&\cdot [p(x) + h_n p(x)z + \frac{h_n^2 z^2}{2} p''(x)]k(z)dz + O(h_n^4) \\
&= p(x)m(x) + \frac{h_n^2}{2} m(x)p''(x)\mu_2(k) + h^2 m'(x)p'(x)\mu_2(k) \\
&+ \frac{h_n^2}{2} m''(x)p(x)\mu_2(k) + O(h_n^4)
\end{aligned} \tag{4}$$

For variance, we can write it in this way $Var(\hat{\alpha(x)}) = E(\hat{\alpha}(x)^2) - E^2(\hat{\alpha(x)})$ Since we already know $E(\hat{\alpha(x)})$, let's focus on the former item now. If we ignore the error terms then we have the following

$$
\begin{aligned}
E(\hat{\alpha}(x)^2) &= E([\frac{1}{nh_n}\Sigma y_i \cdot K(\frac{x-x_i}{h_n})]^2) \\
&= n\int \frac{1}{(nh_n)^2} m^2(u)K^2(\frac{x-u}{h_n})p(u)du + \frac{n(n-1)}{n^2}E^2(\hat{\alpha(x)}) \\
&= \frac{1}{nh_n}\int m(x+h_nz)^2k^2(z)p(x+h_n)h_ndz + \frac{n(n-1)}{n^2}E^2(\hat{\alpha(x)}) \\
&= \frac{1}{nh_n}\int [m^2(x) + 2m(x)m'(x)h_nz + [m''(x)m(x) + m'(x)^2]h_n^2z^2 + O(h_n^3)] \\
&\cdot [p(x) + h_np'(x)z + \frac{h_n^2z^2}{2}p''(x) + O(h_n^3)]k(z)dz + \frac{n(n-1)}{n^2}E^2(\hat{\alpha(x)}) \\
&= \frac{1}{nh_n}[m^2(x)p(x)\mu_2(k) + [2m(x)m'(x)p'(x) + m''(x)m(x)p(x) + m'(x)^2p(x) \\
&+ m^2(x)\frac{p''(x)}{2}]\sigma_k^2h_n^2] + O(h_n^4)] + \frac{n(n-1)}{n^2}E^2(\hat{\alpha(x)})
\end{aligned} \tag{5}
$$

Knowing these we can calculate variance as the following,

$$
\begin{aligned}
Var(\hat{\alpha}(x)) &= \frac{1}{nh_n}[m^2(x)p(x)\mu_2(k) + [2m(x)m^{'}(x)p'(x) + m''(x)m(x)p(x) + m'(x)^2p(x) \\
&+ m^2(x)\frac{p''(x)}{2}]\sigma_k^2h_n^2 + O(h_n^4)] - \frac{1}{n}E^2(\hat{\alpha(x)}) \\
&= O((nh_n^{-1}))
\end{aligned} \tag{6}
$$

A better way is to use the independence,

$$Var(\hat{\alpha}(x)) = Var(\frac{1}{nh_n}\Sigma y_i \cdot K(\frac{x-x_i}{h_n}))$$

$$= n \cdot Var(\frac{1}{nh_n}(m(x_i)+\epsilon_i)K(\frac{x-x_i}{h_n}))$$

$$= n(nh_n)^{-2}[E((m(x_i)+\epsilon_i)^2K(\frac{x-x_i}{h_n})^2) - E^2((m(x_i)+\epsilon_i)K(\frac{x-x_i}{h_n}))]$$

$$= n(nh_n)^{-2}\int\int(m(y)+u)^2K(\frac{y-x}{h_n})^2p(y)f(u)dydu - n^{-1}E^2(\alpha\hat{(x)})$$

$$= n(nh_n)^{-2}\int m(y)^2K(\frac{y-x}{h_n})^2p(y)dy + n(nh_n)^{-2}\sigma_\epsilon^2\int K(\frac{y-x}{h_n})^2p(y)dy - n^{-1}E^2(\alpha\hat{(x)})$$

$$= (nh_n)^{-1}\{m^2(x)p(x)r(k) + [2m(x)m^{'}(x)p'(x) + m''(x)m(x)p(x) + m'(x)^2p(x)$$

$$+ m^2(x)\frac{p''(x)}{2}]\sigma_k^2h_n^2 + O(h_n^4)\} + (nh_n)^{-1}\sigma_\epsilon^2\{p(x)r(k) + m^2(x)\frac{p''(x)}{2}\sigma_k^2h_n^2 + O(h_n^4)\} - n^{-1}E^2(\alpha \tag{7}$$

Here $r(k) = \int K^2(z)dz$ and $\sigma_k^2 = \int K^2(z)z^2dz$

3 Moment Generating Function

The next step is to derive the asymptotic behavior of the estimators. Besides MGF, we can also use *CLT for Triangular Arrays*. But For generality, we may first discuss MGF. Consider an estimator

$$\beta\hat{(x)} = \frac{1}{nh_n}\Sigma m(x_i) \cdot K(\frac{x-x_i}{h_n}) \tag{8}$$

$$C = (nh_n)^{1/2}[\beta\hat{(x)} - E(\beta\hat{(x)})] \tag{9}$$

Its MGF can be calculated as below

$$
\begin{aligned}
MGF(C) &= MGF((nh_n)^{1/2}[\hat{\beta(x)} - E(\hat{\beta(x)})]) \\
&= E(\exp(t \cdot (nh_n)^{1/2}[\hat{\beta(x)} - E(\hat{\beta(x)})])) \\
&= \frac{1}{\exp(t \cdot (nh_n)^{1/2} \cdot E(\hat{\beta(x)}))} \cdot E[\exp(t \cdot (nh_n)^{1/2} \cdot \frac{1}{nh_n} \Sigma m(x_i) \cdot K(\frac{x - x_i}{h_n}))] \\
&= \frac{1}{\exp^n(t \cdot (nh_n)^{-1/2} \cdot hE(\hat{\beta(x)}))} \cdot E^n[\exp(t \cdot (nh_n)^{-1/2} \cdot m(x_i) \cdot K(\frac{x - x_i}{h_n}))] \\
&= E^n[\frac{\exp(t \cdot (nh_n)^{-1/2} \cdot m(x_i) \cdot K(\frac{x-x_i}{h_n})}{\exp(t \cdot (nh_n)^{-1/2} \cdot h_n E(\hat{\beta(x)}))}]
\end{aligned}
$$

$$
\begin{aligned}
&E[\frac{\exp(t \cdot (nh_n)^{-1/2} \cdot m(x_i) \cdot K(\frac{x-x_i}{h_n})}{\exp(t \cdot (nh_n)^{-1/2} \cdot h_n E(\hat{\beta(x)}))}] = E[exp(t \cdot (nh_n)^{-1/2} \cdot (m(x_i) \cdot K(\frac{x - x_i}{h_n}) - h_n E(\hat{\beta(x)}))] \\
&= \int exp(t \cdot (nh_n)^{-1/2} \cdot (m(y) \cdot K(\frac{y - x}{h_n}) - h_n E(\hat{\beta(x)}))p(y)dy \\
&= \int [1 + t \cdot (nh_n)^{-1/2} \cdot (m(y) \cdot K(\frac{y - x}{h_n}) - h_n E(\hat{\beta(x)}) \\
&+ [t \cdot (nh_n)^{-1/2} \cdot (m(y) \cdot K(\frac{y - x}{h_n}) - h_n E(\hat{\beta(x)}))]^2] \cdot p(y)dy
\end{aligned}
$$

Note that each term is just r-th absolute central moment(which is actually talking about the same thing as CLT for triangular arrays),

$$
\begin{aligned}
&\int t \cdot (nh_n)^{-1/2} \cdot (m(y) \cdot K(\frac{y - x}{h_n}) - h_n E(\hat{\beta(x)})) \cdot p(y)dy \\
&\quad = \int t \cdot (nh_n)^{-1/2} \cdot m(y) \cdot K(\frac{y - x}{h_n}) \cdot p(y)dy - t \cdot (nh_n)^{-1/2} \cdot h_n E(\hat{\beta(x)}) \\
&\quad = t \cdot (nh_n)^{-1/2} \cdot h_n E(\hat{\beta(x)} - t \cdot (nh_n)^{-1/2} \cdot h_n E(\hat{\beta(x)}) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\int t^2 \cdot (nh_n)^{-1} \cdot (m(y) \cdot K(\frac{y - x}{h_n}) - h_n E(\hat{\beta(x)}))^2 \cdot p(y)dy \\
&= t^2 \cdot h_n[Var(\hat{\beta(x)}) + n^{-1}E^2(\hat{\beta(x)})] \\
&= t^2 \cdot [O(\frac{1}{n}) + O(\frac{h_n}{n})] = t^2 \cdot O(\frac{1}{n})
\end{aligned}
$$

Since we did Taylor expansion before to make the integration plausible, we already used the condition $nh_n \to \infty$

Taking into consideration terms of higher orders, we shall find that

$$\begin{aligned}
&O(\int (m(y)\cdot K(\frac{y-x}{h_n}) - h_n E(\hat{\beta(x)}))^r \cdot p(y)dy) \\
&= O(\int m^r(y)\cdot K^r \frac{y-x}{h_n} \cdot p(y)dy) + O(\int h_n^r E(\hat{\beta(x)})^r \cdot p(y)dy) \\
&= O(\int m^r(x+h_n z)\cdot K^r(x)\cdot p(x+h_n z)h_n dz) + O(h_n^r) \\
&= O(h_n) + O(h_n^r) = O(h_n)
\end{aligned}$$

To summarize, each high order term of t can be written as the following $O((nh_n)^{-r/2}h_n)$

Therefore, MGF can be written as,

$$\begin{aligned}
MGF(B) &= MGF((nh_n)^{1/2}[\hat{\beta(x)} - E(\hat{\beta(x)})]) \\
&= (1 + \frac{C^*}{n}t^2 + \Sigma O((nh_n)^{-r/2}h_n)t^r)^n \\
&= \exp(n\frac{C^*}{n}t^2 + nO((nh_n)^{-3/2}h_n)t^3) \\
&= \exp(C^* t^2 + O((nh_n)^{-1/2}))
\end{aligned}$$