MPHY0030 Coursework Report

Report Q1: Is the polynomial part need and why?

Answer:

The polynomial part is not required as the Gaussian equation is positive definite.

Report Q2: Write down the linear algebra used to represent the spline fitting problem and the solution. Cite the key formula in the report in the relevant part of your code.

Answer:

The transformation function between a set of source points, **p**, and target points, **q**, can be described as:

$$\boldsymbol{u_k}(\boldsymbol{p_i}) = \boldsymbol{q_{i,k}}, \qquad i = 1\text{:}\,\mathrm{n}$$

For each point in the 2 sets, n, and for each dimension of the data, k (for 3 dimensions, k = 1:3).

Viewing this transformation function as a radial basis function, it can be described as:

$$u(\boldsymbol{x}) = \Phi_s(\boldsymbol{x}) + R_s(\boldsymbol{x})$$

Where the sum of polynomials, $\Phi_s(\boldsymbol{x})$, and the sum of RBFs, $R_s(\boldsymbol{x})$, is equal to:

$$\Phi_s(\boldsymbol{x}) = \sum_{j=1}^{M} \beta_j \Phi_j(\boldsymbol{x}), \qquad R_s(\boldsymbol{x}) = \sum_{i=1}^{n} \alpha_i R(||\boldsymbol{x} - \boldsymbol{p}_i||)$$

Combining these 2 equations with the constraints …:

$$\sum_{i=1}^{n} \alpha_i \Phi_j(\boldsymbol{p}_i) = 0, \qquad j = 1\text{:}\,M;$$

… gives the system of linear equations (Including the weighting parameters, λ and σ.):

$$\begin{pmatrix} \boldsymbol{K} + \boldsymbol{\lambda W}^{-1} & \boldsymbol{P} \\ \boldsymbol{P}^T & 0 \end{pmatrix} \times \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{q}_k \\ 0 \end{pmatrix}$$

$$\boldsymbol{W} = diag\left\{\frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_n^2}\right\},$$

Writing the linear equations as:

$$\left((\boldsymbol{K} + \boldsymbol{\lambda W}^{-1}) \times \boldsymbol{\alpha}\right) + (\boldsymbol{P} \times \boldsymbol{\beta}) = \boldsymbol{q}_k$$

$$\left(\boldsymbol{P^T} \times \boldsymbol{\alpha}\right) + (\boldsymbol{0} \times \boldsymbol{\beta}) = \boldsymbol{0}$$

As we are dealing with a Gaussian kernel, the polynomial part, $\Phi_s(\boldsymbol{x})$, can be ignored.

As a result, the transfer function is equal to:

$$u(\boldsymbol{x}) = R_s(\boldsymbol{x})$$

… and, as P is equal to:

$$P_{ij} = \Phi_j(\boldsymbol{p}_i) = 0$$

… the linear equations can be simplified to:

$$(\boldsymbol{K} + \boldsymbol{\lambda W}^{-1}) \times \boldsymbol{\alpha} = \boldsymbol{q}_k$$

Rearranging for the equation coefficient, α, gives us:

$$\boldsymbol{\alpha} = (\boldsymbol{K} + \boldsymbol{\lambda W}^{-1})^{-1} \times \boldsymbol{q}_k$$

For a Gaussian fit, K is equal to:

$$K_{ij} = R\left(\left|\left|\boldsymbol{p}_i - \boldsymbol{p}_j\right|\right|\right) = e^{-\frac{r^2}{2\sigma^2}}$$

Which we insert into the equation for α to give us the solutions of the linear equations in each dimension as:

$$\boldsymbol{\alpha_k} = \left(e^{-\frac{(||\boldsymbol{x_k} - \boldsymbol{p}_k||)^2}{2\sigma^2}} + \boldsymbol{\lambda W}^{-1}\right)^{-1} \times \boldsymbol{q}_k$$

For 3 dimensions, k = 1:3.

Report Q3: What is the best linear algebra algorithm should be implemented to solve this spline fitting problem and why?

Answer:

Singular value decomposition may be the best linear algebra algorithm for this spline fitting problem as

Report Q4: What are the control points here? Can we choose any points as control points at evaluation stage, and why?

Answer:

Report Q5: Do we need the weighting parameter lambda at evaluation stage, and why?

Answer:

The weighting parameter, lambda, is not needed at the evaluation stage as a fit between the initial points, p, and the target points, q, has already been created. The query points in between do not have localization errors thus a weighting parameter is not needed to correct these errors.

Report Q6: Describe the details of your vectorization strategies for kernel computing for large point sets.

Answer:

Looping statements, like the ones used in RBFSpline, are slow to compute with large point sets. Therefore, a matrix can be used as each value can directly correspond to equations which can be solved simultaneously, whereas loops must be read in their entirety, which will increase the processing time.

Report Q7: Discussion of the utility of the Gaussian kernel parameter, sigma.

Answer:

The Gaussian kernel parameter, sigma, describes the variance of the radial basis function kernel. This variance affects the width of the kernel selection, which can affect the accuracy of the transformed image. A small sigma has a narrower kernel which can increase the accuracy of the transformed image, but requires more computing power to solve, increasing the processing time. A larger sigma encompasses more values at the expense of accuracy, decreasing computing time.

Report Q8: What is a reasonable approach to randomly displace the control points, (e.g. what distribution the random transform need to draw from and is there any constraint needed), such that the resulting transformation represented by the moved control points are considered biophysically reasonable.

Report Q9: Does it mean that the interpolated voxel coordinates, driven by the moved control points, represent biophysically plausible deformation?

Report Q10: Describe in details of the steps taken to compute a warped 3D image.

Report Q11: Does it mean that the interpolated voxel coordinates, driven by the moved control points, represent biophysically plausible deformation?

Report Q12: Summarise the effect of the three changes, with example images included in the report.