Gluon Density Evolution with DGLAP

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Following the steps in `Dr Sandapen's DGLAP Notes.pdf`, for a gluon density $G(x_{Bj},\ \mu^2) = xg(x_{Bj},\ \mu^2)$, the moment of the gluon density, $G^N(\mu^2)$, is defined as:

$$G^N(\mu^2) = \int_0^1 x_{Bj}^{N-2} G(x_{Bj},\ \mu^2) dx_{Bj}. \tag{1}$$

The evolved moment using DGLAP is given by:

$$G^N(\mu^2) = G^N(\mu_0^2) e^{\tilde{\gamma}_{gg}^{N-1}(\mu^2)}, \tag{2}$$

where $G^N(\mu_0^2)$ is:

$$G^N(\mu_0^2) = \int_0^1 A_g x_{Bj}^{N-2-\lambda_g} (1 - x_{Bj})^{5.6} \mathrm{d}x_{Bj} = A_g \frac{\Gamma(\frac{33}{5})\Gamma(N - 1 - \lambda_g)}{\Gamma(N + \frac{28}{5} - \lambda_g)}, \tag{3}$$

$\tilde{\gamma}_{gg}^{N-1}(\mu^2)$ is:

$$\tilde{\gamma}_{gg}^{N-1}(\mu^2) = \gamma_{gg}^{N-1}(\mu^2) \int_{\mu_0^2}^{\mu^2} \frac{\alpha_s({\mu'}^2)}{{\mu'}^2} \mathrm{d}{\mu'}^2, \tag{4}$$

and $\gamma_{gg}^{N-1}(\mu^2)$ is:

$$\gamma_{gg}^{N-1}(\mu^2) = -\frac{\alpha_s(\mu^2)}{4\pi} \left(\frac{(N-2)(N+1)}{N(N-1)} + \frac{(N-2)(N+5)}{6(N+1)(N+2)} + \sum_{i=2}^{N} \frac{2}{i} + \frac{1}{9} n_f \right). \tag{5}$$

The strong coupling constant $\alpha_s(\mu)$ to leading order is:

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln\left(\frac{\mu}{\Lambda_{QCD}}\right)}, \tag{6}$$

and β_0 is:

$$\beta_0 = 11 - \frac{2n_f}{3}. \tag{7}$$

The gluon density $G(x_{Bj},\ \mu^2)$ can then be recovered from this moment by using an expansion of Jacobi polynomials, $\Theta_n^{(\alpha,\ \beta)}(x_{Bj})$, as is done on page 8 of arXiv:0705.2647v2 [5]. Jacobi polynomials are the general-case solutions to the Jacobi differential equation, and, over a range $[a,\ b]$, their weight function is:

$$w(x) = (a + x)^\beta (b - x)^\alpha. \tag{8}$$

The orthogonal polynomials over this range are:

$$\mathcal{N}^{(n)}(\alpha,\ \beta) \cdot \Theta_n^{(\alpha,\ \beta)} \left(\frac{2(x-a)}{b-a} - 1 \right), \tag{9}$$

where $\mathcal{N}^{(n)}(\alpha,\ \beta)$ is a normalisation constant. For $x_{Bj} \in [0,\ 1]$ (so $a = 0$ and $b = 1$), the orthonormality relationship for Jacobi polynomials is:

$$\int_0^1 x_{Bj}^\beta (1 - x_{Bj})^\alpha \Theta_m^{(\alpha,\ \beta)}(2x_{Bj} - 1) \Theta_n^{(\alpha,\ \beta)}(2x_{Bj} - 1) \mathrm{d}x_{Bj} = \delta_{m,n}, \tag{10}$$

and the normalisation constant is:

$$\mathcal{N}^{(n)}(\alpha,\ \beta) = \frac{1}{\int_0^1 x_{Bj}^{\beta}(1 - x_{Bj})^{\alpha}(\Theta_n^{(\alpha,\ \beta)}(2x_{Bj} - 1))^2 \mathrm{d}x_{Bj}}. \tag{11}$$

The gluon density can be written as:

$$G(x_{Bj},\ \mu^2) = x_{Bj}^{\beta}(1 - x_{Bj})^{\alpha} \sum_{n=0}^{N_{\max}} a_n(\mu^2)\mathcal{N}^{(n)}(\alpha,\ \beta)\Theta_n^{(\alpha,\ \beta)}(2x_{Bj} - 1), \tag{12}$$

where $a_n(\mu^2)$ are the Jacobi moments, given by Equation (4.4) of [5]:

$$a_n(\mu^2) = \sum_{j=0}^{n} c_j^{(n)}(\alpha,\ \beta)G^{j+2}(\mu^2), \tag{13}$$

where $c_j^{(n)}(\alpha,\ \beta)$ are the coefficients of the Jacobi polynomials evaluated at x_{Bj} (not $2x_{Bj} - 1$), and $G^{j+2}(\mu^2)$ is the moment of the gluon density $G^N(\mu^2)$ for $N = j + 2$.

Therefore, the recovered gluon density is:

$$\boxed{G(x_{Bj},\ \mu^2) = x_{Bj}^{\beta}(1 - x_{Bj})^{\alpha} \sum_{n=0}^{N_{\max}} \mathcal{N}^{(n)}(\alpha,\ \beta)\Theta_n^{(\alpha,\ \beta)}(2x_{Bj} - 1) \sum_{j=0}^{n} c_j^{(n)}(\alpha,\ \beta)G^{j+2}(\mu^2)}. \tag{14}$$

[5] uses values of $N_{\max} = 9$, $\alpha = 3.0$, and $\beta = 0.5$.