Solutions of Atiyah's Introduction to Commutative Algebra

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Contents

Introduction

These exercises are from M.F. Atiyah and I.G. MacDonald's Introduction to Commutative Algebra [AM19].

I started reading this book and doing the exercises on 2024/01/13, and finished them on 2024/03/13. Well, anyway, this is my first completely-finished book!

There are many helpful online resources for this book, such as "Errata for Atiyah–Macdonald" (`https://mathoverflow.net/q/42241`) on MathOverflow.

Chapter 1

Rings and Ideals

Problem 1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Suppose $x^n = 0$, then $(1+x)(1-x+\cdots+(-1)^{n-1}x^{n-1}) = 1+(-1)^n x^n = 1$. ■

Problem 2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent [If $b_0 + b_1 x + \cdots + b_m x^m$ is the inverse of f, prove by induction on r that $a_n^{r+1} b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]

ii) f is nilpotent $\Leftrightarrow a_0, \ldots, a_n$ are nilpotent.

iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Then $a_n b_m = 0$, hence $a_n g = 0$ (because $a_n g$ annihilates f and has degree $< m$). Now show by induction that $a_{n-r} g = 0$ $(0 \leq r \leq n)$.]

iv) f is said to be *primitive* if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Proof.

i) $(\Rightarrow)$ Let $g(x) = b_0 + b_2 x + \cdots + b_m x^m$ be such that $fg = 1$. Then $a_0 b_0 = 1$, $a_n b_m = 0$. We focus on the $nm-1$ term of fg, that is $a_{n-1} b_m + a_n b_{m-1} = 0$, since $a_n b_m = 0$ multiply by a_n we get $a_n^2 b_{m-1} = 0$. By induction we have $a_n^{m+1} b_0 = 0$, i.e. a_n is nilpotent. Then by Ex. 1, $f - a_n x^n$ is still a unit, by induction $a_1, \ldots, a_n$ are all nilpotent.
$(\Leftarrow)$ Let $k \in \mathbb{N}$ such that $a_1^k, \ldots, a_n^k = 0$. Then $f^{nk} = a_0^{nk} \in A^\times$.

ii) ($\Rightarrow$) Suppose $f^m = 0$, then $a_n^m = 0$, and $f - a_n x^n$ is still a nilpotent element (nilpotent elements form a ring). By induction all a_i are nilpotent.
($\Leftarrow$) Clear.

iii) ($\Rightarrow$) We choose $g(x) = b_0 + b_1 x + \cdots + b_m x^m$ of lowest degree, such that $fg = 0$. Then $a_n b_m = 0$, and by the minimality of g, $a_n g = 0$. By induction, $a_r g = 0$. So $b_s f = 0$.
($\Leftarrow$) Trivial.

iv) ($\Rightarrow$) The ideal I generated by all coefficients of fg is contained in $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_m)$. So if $I = (1)$, then so do $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_m)$.
($\Leftarrow$) If fg is not primitive, then all of its coefficients are contained in some prime ideal $\mathfrak{p}$ (since the maximal one is definitely prime). If f and g are primitive, let $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{i=0}^m b_i x^i \in A[x] \setminus \mathfrak{p}[x]$. Let r, s be the smallest number such that $a_r, b_s \notin \mathfrak{p}$ respectively. Since $fg \in \mathfrak{p}[x]$, then the $(r+s)$-th term of $f(x)g(x)$ is

$$\sum_{\substack{i+j=r+s \\ i \leq r}} a_i b_j \in \mathfrak{p} \implies a_r \in \mathfrak{p} \text{ or } b_s \in \mathfrak{p},$$

contradiction. So f or g is not primitive. [*Remark.* c.f. Exercise 7 of Chapter 2, the proofs are the same.] ■

Problem 3. Generalize the result of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Proof. Let $f(x) = f(x_1, \ldots, x_r) = \sum_{\nu \in \mathbb{N}^r} a_\nu x^\nu \in A[x_1, \ldots, x_r]$.

i) f is a unit in $A[x_1, \ldots, x_r] \iff a_0$ is a unit in A and a_ν is nilpotent for all $\nu \neq 0$. [Use the fact that $A[x_1, \ldots, x_r] = A[x_1, \ldots, x_{r-1}][x_r]$.]

ii) f is nilpotent $\iff a_\nu$ is nilpotent for all $\nu \in \mathbb{N}^r$.

iii) f is a zero-divisor $\iff$ there exists $a \in A$, $a \neq 0$, such that $af = 0$. [Use induction on r. First work in $A[x_r][x_1, \ldots, x_{r-1}]$, use induction hypothesis to find a nonzero $g(x_r) \in A[x_r]$ of smallest degree such that $f(x)g(x_r) = 0$, then work in $A[x_1, \ldots, x_{r-1}][x_r]$, and the remaining stuffs are the same as those in Exercise 2.]

iv) Let $f, g \in A[x_1, \ldots, x_r]$, then fg is primitive $\iff f, g$ are primitive. ■

Problem 4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Proof. Let $\mathfrak{R}_J$ be the Jacobson radical. Let $f(x) = a_0 + \cdots + a_n x^n \in \mathfrak{R}_J$, then $1 - f(x)g(x)$ is a unit for all $g(x) \in A[x]$, in particular, $1 - xf(x)$ is a unit. Then by Exercise 2 i), $a_0, \ldots, a_n$ are nilpotent. So by Exercise 2 ii), $f(x)$ is nilpotent. ■

Problem 5. Let A be a ring and let $A[\![x]\!]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

i) f is a unit in $A[\![x]\!] \Leftrightarrow a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true? (See Chapter 7, Exercise 2.)

iii) f belongs to the Jacobson radical of $A[\![x]\!] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[\![x]\!]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[\![x]\!]$.

Proof.

i) ($\Rightarrow$) Let $g(x) = \sum_{m=0}^{\infty} b_m x^m$, $fg = 1$, then $a_0 b_0 = 1$.
($\Leftarrow$) We can construct the inverse of $f(x)$ by induction. The base case is clear. If we have constructed $g_N(x) = \sum_{m=0}^{N} b_m x^m$ such that each coefficient of 1–Nth degree fg_N is 0, let $b_{N+1} = -a_0^{-1} \sum_{i=1}^{N} a_i b_{N-i+1}$, and let $g_{N+1}(x) = \sum_{m=0}^{N+1} b_m x^m$.

ii) Let $f^N = 0$, then $a_0^N = 0$, and $f(x) - a_0$ is nilpotent. By induction a_n is nilpotent for every $n \geq 0$.

iii) ($\Rightarrow$) Clear.
($\Leftarrow$) Follows from i).

iv) We have a natural injection $\varphi : A/\mathfrak{m}^c \hookrightarrow A[\![x]\!]/\mathfrak{m}$. For every $a \in A \setminus \mathfrak{m}^c$, $ag(x) = 1 + f(x)$ for some $f \in \mathfrak{m}$ and $g \in A[\![x]\!]$. Taking out the constant terms of the above equation we see that $ab = 1 + c$ for some $b \in A$ and $c \in \mathfrak{m}^c$. Hence $A/\mathfrak{m}^c$ is also a field and $\mathfrak{m}^c$ is maximal in A. Since $A[\![x]\!]/\mathfrak{m}^c[\![x]\!] \cong (A/\mathfrak{m}^c)[\![x]\!]$ and $\mathfrak{m}^c[\![x]\!] \subseteq \mathfrak{m}$, we see that $\mathfrak{m} = \mathfrak{m}^c[\![x]\!]$.

v) Let $\mathfrak{p}$ be a prime ideal in A, $\mathfrak{p}[x]$ is an ideal in $A[\![x]\!]$ and $A[\![x]\!]/\mathfrak{p}[\![x]\!] \cong (A/\mathfrak{p})[\![x]\!]$ is clearly integral. Hence $\mathfrak{p}[\![x]\!]$ is prime in $A[\![x]\!]$. Since $\mathfrak{p} = (\mathfrak{p}[\![x]\!])^c$, we see that $\mathfrak{p}$ is the contraction of a prime ideal of $A[\![x]\!]$. ■

Problem 6. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Proof. Let $\mathfrak{N}$ and $\mathfrak{N}'$ be the nilradical and the Jacobson radical, respectively. By definition $\mathfrak{N}' \subseteq \mathfrak{N}$. For every nonzero $x \in \mathfrak{N}'$, if $(x) \not\subseteq \mathfrak{N}$, then there is $a \in A$, $(ax)^2 = ax$ with $ax \neq 0$, that is, $ax(1 - ax) = 0$. Since $1 - ax \in A^\times$ for all $a \in A$, we must have $ax = 0$, a contradiction. So $x \in (x) \subseteq \mathfrak{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$, and $\mathfrak{N} = \mathfrak{N}'$. ■

Problem 7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Proof. For prime ideal $\mathfrak{p}$ and $x \notin \mathfrak{p}$, $0 = x(x^{n-1} - 1) \in \mathfrak{p} \implies x^{n-1} - 1 \in \mathfrak{p}$. That is, every non-zero element $x + \mathfrak{p}$ in $A/\mathfrak{p}$ has an inverse $x^{n-2} + \mathfrak{p}$. ■

Problem 8. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has a minimal elements with respect to inclusion.

Proof. The set of prime ideals is not empty (there is at least a maximal ideal). This set satisfies the requirement of Zorn's lemma, namely, let $\{\mathfrak{p}_i\}_{i\in I}$ be a chain of prime ideals, and let $\mathfrak{p} = \bigcap_{i\in I} \mathfrak{p}_i$. For $x, y \in A$, suppose $x \notin \mathfrak{p}$, then there is i_0 such that $x \notin \mathfrak{p}_i$ for all $i \geq i_0$. Hence for $i \geq i_0$, $y \in \mathfrak{p}_i$, and, $y \in \mathfrak{p}$, we can see that $\mathfrak{p}$ is a prime ideal. Applying Zorn's lemma we get a minimal prime ideal. ■

Problem 9. Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Proof. Let $\phi : A \to A/\mathfrak{a}$ be the canonical homomorphism. We have

$$\mathfrak{R}_{A/\mathfrak{a}} = \bigcap_{\mathfrak{p} \text{ prime in } A/\mathfrak{a}} \mathfrak{p}$$

and $r(\mathfrak{a}) = \phi^{-1}(\mathfrak{R}_{A/\mathfrak{a}})$. We also know that ϕ induces a correspondence of prime ideals

$$\{\text{prime ideals } \mathfrak{p} \supseteq \mathfrak{a} \text{ in } A\} \xrightarrow{\phi} \{\text{prime ideals } \mathfrak{p} \text{ in } A/\mathfrak{a}\}.$$

So

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}.$$ ■

Problem 10. Let A be a ring, $\mathfrak{R}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal;

ii) every element of A is either a unit or nilpotent;

iii) $A/\mathfrak{R}$ is a field.

Proof. i) $\Rightarrow$ ii) Let $\mathfrak{R}$ be the only prime ideal. We know that $\mathfrak{R}$ is also the nilradical of A, and is also maximal. So $A/\mathfrak{R}$ is a field. For $x \notin \mathfrak{R}$, there is y s.t. $xy \in 1 + \mathfrak{R} \subseteq A^\times$ (by Ex. 1). So there are only units and nilpotent element in A.

ii) $\Rightarrow$ iii) Let $\mathfrak{R}$ be the nilradical, it is obvious maximal, so $A/\mathfrak{R}$ is a field.

iii) $\Rightarrow$ i) Since

$$\mathfrak{N} = \bigcap_{\mathfrak{p}\text{ prime}} \mathfrak{p} \subseteq \text{ some maximal ideal.}$$

The condition $A/\mathfrak{N}$ is a field implies that $\mathfrak{N}$ is maximal. So there is only one prime ideal. ∎

Problem 11. A ring A is *Boolean* if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$;

ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;

iii) every finitely generated ideal in A is principal.

Proof.

i) $x + 1 = (x + 1)^2 = x^2 + 2x + 1 = x + 2x + 1 \implies 2x = 0.$

ii) By Ex. 7, prime ideals in this ring are maximal.Also by Ex. 7, we know $x + \mathfrak{p} = 1 + \mathfrak{p}$ for $x \neq 0$, so $A/\mathfrak{p}$ has only zero and identity.

iii) $(x, y) = (x + y)$. This is because $(x - y)(x + y) = x^2 - y^2 = x - y$, this element and $x + y$ generate (x, y). ∎

Problem 12. A local ring contains no idempotent $\neq 0, 1$.

Proof. Let $\mathfrak{m}$ be the maximal ideal. For if $x^2 = x$, i.e. $x(x - 1) = 0$, we have $x \in \mathfrak{m}$ or $x - 1 \in \mathfrak{m}$. If $x \in \mathfrak{m}$, then $x - 1 \notin \mathfrak{m}$ (otherwise they generate the whole ring), so $x - 1 \in A^\times$ (otherwise it is contained in another maximal ideal), let $x = 1 + a$. Then $x^2 = x \implies a(a + 1) = 0$, since $a \in A^\times$, $a + 1 = 0$. That is , $x = 0$. If $x - 1 \in \mathfrak{m}$, similarly we can get $x = 1$. ∎

Construction of an algebraic closure of a field (E. Artin).

Problem 13. Let K be a field and let Σ be the set of all irreducible monic polynomials in f in one determinate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomial $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \bigcup_{i=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\bar{K}$ be the set of all elements of L which are algebraic over K. Then $\bar{K}$ is an algebraic closure of K.

Proof. If $\mathfrak{a} = (1)$, then there is a combination $\sum_{i=1}^{n} g_i(X) f_i(x_{f_i}) = 1$. Since this is a finite sum, only finitely many variables involved. We can extend K for n times, such that every polynomial f_i has a root a_i. Then we substitute x_{f_i} by a_i to evaluate the combination to get $0 = 1$, a contradiction. ■

Problem 14. In a ring A, let Σ be the set of all ideals in which every elements is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof. First, $(0) \in \Sigma$, so it is not empty. And apparently we can use Zorn's lemma on Σ, and get a maximal element $\mathfrak{p}$. If there are some $x, y \notin \mathfrak{p}$ with $xy \in \mathfrak{p}$. Then by maximality of $\mathfrak{p}$, there are $u, v \in A$, $p_1, p_2 \in \mathfrak{p}$, not zero-divisors $a, b \in A$, s.t.

$$uy + p_1 = a, \quad vx + p_2 = b,$$

multiply by x, a respectively, we know that $ax, ab \in \mathfrak{p}$. But a, b are not zero-divisors, and so is ab, which indicates that $ab \notin \mathfrak{p}$. So $\mathfrak{p}$ is prime. ■

The prime spectrum of a ring

Problem 15. Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V\big(r(\mathfrak{a})\big)$.

ii) $V(0) = X$, $V(1) = \varnothing$.

iii) if $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V\Big(\bigcup_{i\in I} E_i\Big) = \bigcap_{i \in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cup V(\mathfrak{b}))$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology.* The topological space X is called the *prime spectrum* of A, and is written $\mathrm{Spec}(A)$.

Proof.

i) The first equation is trivial, the second one follows from Ex. 9, since $r(\mathfrak{a}) = r(r(\mathfrak{a})) \implies r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$.

ii) Obvious.

iii) ($\subseteq$) $\mathfrak{p} \in V\big(\bigcup_{i\in I} E_i\big) \implies \mathfrak{p} \in V(E_i)$ for all $i \in I$.
($\supseteq$) $\mathfrak{p} \in V(E_i)$ for all $i \in I \implies \bigcup_{i\in I} E_i \subseteq \mathfrak{p}$.

iv) Obviously $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{ab})$. From $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, we know $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{ab})$. For $\mathfrak{p} \in V(\mathfrak{ab})$, if there is $y \in \mathfrak{b}$, $y \notin \mathfrak{p}$, then for every $x \in \mathfrak{a}$, since $xy \in \mathfrak{p}$, we must have $x \in \mathfrak{p}$, i.e. $\mathfrak{a} \subseteq \mathfrak{p}$. So $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b}), V(\mathfrak{a}) \cup V(\mathfrak{b})$. ■

Problem 16. Draw pictures of $\operatorname{Spec}(\mathbb{Z})$, $\operatorname{Spec}(\mathbb{R})$, $\operatorname{Spec}(\mathbb{C}[x])$, $\operatorname{Spec}(\mathbb{R}[x])$, $\operatorname{Spec}(\mathbb{Z}[x])$.

Proof.

$$\begin{aligned}
\operatorname{Spec}(\mathbb{Z}) =& \big\{(p) \mid p \in \mathbb{N}, p \text{ prime}\big\} \cup \big\{(0)\big\},\\
\operatorname{Spec}(\mathbb{R}) =& \big\{(0)\big\},\\
\operatorname{Spec}(\mathbb{C}[x]) =& \big\{(x-c) \mid c \in \mathbb{C}\big\} \cup \big\{(0)\big\},\\
\operatorname{Spec}(\mathbb{R}[x]) =& \big\{(x-c) \mid c \in \mathbb{R}\big\} \cup \big\{(x^2+bx+c) \mid b, c \in \mathbb{R}, b^2-4c<0\big\} \cup \big\{(0)\big\},\\
\operatorname{Spec}(\mathbb{Z}[x]) =& \big\{(p) \mid p \in \mathbb{N}, p \text{ prime}\big\} \cup \big\{\big(f(x)\big) \mid f \text{ irreducible in } \mathbb{Z}[x]\big\}\\
& \cup \big\{\big(p, f(x)\big) \mid p \text{ prime}, f \text{ irreducible in } \mathbb{F}_p[x]\big\} \cup \big\{(0)\big\}.
\end{aligned}$$
■

Problem 17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$;

ii) $X_f = \varnothing \Leftrightarrow f$ is nilpotent;

iii) $X_f = X \Leftrightarrow f$ is a unit;

iv) $X_f = X_g \Leftrightarrow f\big((f)\big) = r\big((g)\big)$;

v) X is quasi-compact (that is, every open covering of X has a finite subcovering).

vi) More generally, each X_f is quasi-compact.

vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

The sets X_f are called *basic open sets* of $X = \operatorname{Spec}(A)$.
[To prove (v), remark that it is enough to consider a covering of X by basic open sets X_{f_i} $(i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i\in J} g_i f_i \quad (g_i \in A)$$

where J is some *finite* subset of I. Then the X_{f_i} $(i \in J)$ cover X.]

Proof. By Ex. 15 iii), $\operatorname{Spec}(A) \setminus V(E) = \cup_{f \in E} X_f$.

i) Follows from Ex. 15.

ii) Since nilradical $= \cap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$.

iii) Otherwise x is contained in a maximal ideal, which is also prime.

iv) Follows from Ex. 9.

v) Given an open covering $\{X_f\}_{f \in E}$ of X (w.l.o.g. consider the basic open covering), we know that $V(E) = \varnothing$, that is, no prime ideal in X contains E. So ideal $(f)_{f \in E} = (1)$ (otherwise it is contained in a maximal ideal, which is prime), that is, there is a combination $\sum_{i=1}^{n} g_i f_i = 1$. Let $E' = \{f_1, f_2, \ldots, f_n\}$, and $V(E') = \varnothing$. So $\{X_{f_i}\}_{1 \leq i \leq n}$ is an open covering of X.

vi) As in v), given an open covering $\{X_g\}_{g \in E}$ of X_f, consisting of basic open sets, we have $V(f) \supseteq V(E)$. Let $I = \langle E \rangle$. If $f^n \in I$ for some $n \in \mathbb{N}$, then we are done, since $V(f) = V(f^n)$ and we can proceed as in v). If $f^n \notin I$ for all $n \in \mathbb{N}$, we take the localization of X at f, i.e. let $S = \{f^n \mid n \in \mathbb{N}\}$ and mapping of ideals $\phi : X \to X_f$ by $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$. Then $\phi(I)$ is not maximal in X_f, and there is a maximal ideal $\mathfrak{m}$ containing it, then $f \notin \phi^{-1}(\mathfrak{m}) =: \mathfrak{p}$ is prime, and $I \subseteq \mathfrak{p}$, a contradiction.

vii) Both directions are obvious. The finite subcovering corresponds to a family of generating basic open sets of the given open subset, and basic open sets are quasi-compact by vi). ■

Problem 18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) the set $\{x\}$ is closed (we say that x is a "closed point") in $\operatorname{Spec}(A) \Leftrightarrow \mathfrak{p}_x$ is maximal;

ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;

iii) $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$;

iv) X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

Proof.

i) Closed sets are of the form $V(E)$, so this is just the definition.

ii) $\overline{\{x\}} = \bigcap\{\text{all closed sets containing } x\} = V(\mathfrak{p}_x)$.

iii) $y \in \overline{\{x\}} \implies \overline{\{y\}} \subseteq \overline{\{x\}} \implies V(\mathfrak{p}_y) \subseteq V(\mathfrak{p}_x) \implies \mathfrak{p}_y \subseteq \mathfrak{p}_x$.

iv) For distinct $x, y \in \mathrm{Spec}(A)$, suppose $\mathfrak{p}_x \not\subseteq \mathfrak{p}_y$, let $f \in \mathfrak{p}_x \setminus \mathfrak{p}_y$. Then $\mathrm{Spec}(A) \setminus V(f)$ is a neighborhood of y, which does not contain x. ■

Problem 19. A topological space X is said to be *irreducible* if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proof. Let $\mathfrak{R}$ be the radical of A.

($\Rightarrow$) If $\mathfrak{R}$ is not prime, i.e., there are $x, y \notin \mathfrak{R}$, but $xy \in \mathfrak{R}$. Then since X is irreducible, there is a prime ideal $x, y \notin \mathfrak{p}$. But we know that $\mathfrak{R} \subseteq \mathfrak{p}$, then $xy \in \mathfrak{p}$, this will implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, contradiction. So $\mathfrak{R}$ is prime.

($\Leftarrow$) Since $X_f = \varnothing \iff f \in \mathfrak{R}$, if $\mathfrak{R}$ is a prime ideal, for every $X_x, X_y \neq \varnothing$, $x, y \notin \mathfrak{R}$. So the space X is irreducible. ■

Problem 20. Let X be a topological space.

i) If Y is an irreducible (Exercise 19) subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.

ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.

iii) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

iv) If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A (Exercise 8).

Proof.

i) If U_1, U_2 are two nonempty open sets in $\overline{Y}$, s.t. they don't intersect, then we must have $Y \subseteq U_1$ or $Y \subseteq U_2$ (otherwise $U_1 \cap Y, U_2 \cap Y$ are two nonempty open sets of Y that don't intersect), w.l.o.g. let $Y \subseteq U_2$. Since $\overline{Y} \setminus Y$ is closed, that is, every point in it is a limit point of Y, this implies that $U_2 = \varnothing$, a contradiction. So $\overline{Y}$ is irreducible.

ii) Let $Y \subseteq X$ be irreducible, and let $S = \{Z \subseteq X \mid Y \subseteq Z \text{ irreducible}\}$, and let $Y_0 = \bigcup_{Z \in S} Z$. We show that Y_0 is irreducible Suppose it is not, then there are two nonempty open sets U_1, U_2 in Y_0 that don't intersect. For every $Z \in S$, we have $Z \subseteq U_1$ or $Z \subseteq U_2$, so $Y \subseteq U_1$ or $Y \subseteq U_2$. Since $U_1 \cap U_2 = \varnothing$, we may assume that $Y \cap U_2 = \varnothing$, so for every $Z \in S$, $Z \subseteq U_1$. That is, $U_1 = Y_0$ and $U_2 = \varnothing$, a contradiction. So Y_0 is irreducible, and maximal. [*Note.* There is a proof using Zorn's lemma, but this is totally unnecessary.]

iii) Single points (singleton).

iv) Given any nonempty irreducible subset $S \subseteq X$, by Ex. 8 we know that there is a minimal element $\mathfrak{p} \in X$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$ for every $\mathfrak{p}' \in S$. It is irreducible, since for two closed sets $V(I_1), V(I_2)$, $V(\mathfrak{p}) \subseteq V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$, then $I_1 \cap I_2 \subseteq \mathfrak{p}$, so $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$, this implies $V(\mathfrak{p}) \subseteq V(I_1)$ or $V(\mathfrak{p}) \subseteq V(I_2)$. $V(\mathfrak{p})$ is also maximal as an irreducible subset, for if there is irreducible $S' \supseteq V(\mathfrak{p})$ (then $\mathfrak{p} \in S'$), we have a prime ideal $\mathfrak{p}'$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is minimal, $\mathfrak{p}' = \mathfrak{p}$, so $V(\mathfrak{p})$ is a maximal irreducible subset. ∎

Problem 21. Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that

i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.

ii) If $\mathfrak{a}$ is an ideal of A, then $\phi^{*-1}\big(V(\mathfrak{a})\big) = V(\mathfrak{a}^{\mathrm{e}})$.

iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*\big(V(\mathfrak{b})\big)} = V(\mathfrak{b}^{\mathrm{c}})$.

iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V\big(\ker\phi\big)$ of X. (In particular, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\mathfrak{R})$ (where $\mathfrak{R}$ is the nilradical of A) are naturally homeomorphic.)

v) If ϕ is injective, the $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in X $\Leftrightarrow \ker\phi \subseteq \mathfrak{R}$.

vi) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

Proof. We have the following diagram of Spec functor.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"'] & X = \operatorname{Spec}(A) \\ B \arrow[r] & Y = \operatorname{Spec}(B) \arrow[u, "\phi^*=\operatorname{Spec}(\phi)"'] \end{tikzcd}$$

i) For every $\mathfrak{p} \in Y_{\phi(f)}$, $f \notin \phi^{-1}(\mathfrak{p})$, so $(\phi^*)^{-1} \supseteq Y_{\phi(f)}$. For $\mathfrak{p} \in Y$, s.t. $\phi^*(\mathfrak{p}) \in X_f$, we must have $\phi(f) \notin \mathfrak{p}$, so $(\phi^*)^{-1}(X_f) \subseteq Y_{\phi(f)}$. So the equality holds.

ii) $(\phi^*)^{-1}(V(\mathfrak{a})) = V(\phi(\mathfrak{a})) = V(\mathfrak{a}^{\mathrm{e}})$.

iii) For every $\mathfrak{p} \in V(\mathfrak{b})$, $\phi^*(\mathfrak{p}) \supseteq \mathfrak{b}^c$, so "$\subseteq$" holds (notice that $V(\mathfrak{b}^c)$ is closed). For the other direction, let $V(I) \supseteq \phi^*(V(\mathfrak{b}))$ be a closed set in X, with radical ideal I (that is, $I = r(I) = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$). Since $\mathfrak{b}^c \subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$, we have $\mathfrak{b}^c \subseteq I$, and $V(\mathfrak{b}^c) \subseteq V(I)$. So "$\supseteq$" holds, and the equality holds.

iv) If ϕ is surjective, then ϕ^* is a correspondence

$$\{\text{ideals containing } \ker\phi \text{ in } X\} \xleftarrow{\phi^*} \{\text{ideals in } Y\}.$$

And $\phi^*(V(g)) = V(\phi^{-1}(g))$.

v) By iii), $\overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = X$, as $\ker\phi \subseteq \mathfrak{R}$.

vi) Let $Z = \operatorname{Spec}(C)$. For every $\mathfrak{p} \in Z$, $(\psi\phi)^*(\mathfrak{p}) = (\psi\phi)^{-1}(\mathfrak{p}) = \phi^{-1}\psi^{-1}(\mathfrak{p}) = \phi^*\psi^*(\mathfrak{p})$.

vii) $X = \operatorname{Spec}(A) = \{(0), \mathfrak{p}\}$, $Y = \operatorname{Spec}(B) = \{(A/\mathfrak{p}) \times \{0\}, \{0\} \times K\}$, and

$$\phi^*((A/\mathfrak{p}) \times \{0\}) = (0), \quad \phi^*(\{0\} \times K) = \mathfrak{p}.$$

So ϕ^* is indeed bijective. However, $\{0\} \times K$ is open in Y, while $\mathfrak{p}$ is not open in X, so ϕ^* is not an homeomorphism. ■

Problem 22. Let $A = \prod_{i=1}^n A_i$ be the direct product of rings A_i. Show that $\operatorname{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_i, where X_i is canonically homeomorphic with $\operatorname{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following statements are equivalent:

i) $X = \operatorname{Spec}(A)$ is disconnected.

ii) $A \cong A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.

iii) A contains an idempotent $\neq 0, 1$.

In particular, the spectrum of a local ring is always connected (Exercise 12).

Proof. If $\mathfrak{p}$ is a prime ideal in A, then it must be prime on each of its coordinates. If $\mathfrak{p}$ is not of the form

$$A_1 \times \cdots \times A_{i-1} \times \mathfrak{p}_i \times A_{i+1} \times \cdots \times A_n$$

for some prime ideal $\mathfrak{p}_i$, then let's say

$$\mathfrak{p} = \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_i \times A_{i+1} \times \cdots \times A_n.$$

Then for $x = (1, 0, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$, $xy \in \mathfrak{p}$, but $x, y \notin \mathfrak{p}$. Then everything is clear.

i) $\Rightarrow$ ii) Then we have two nonempty closed sets $V(I_1), V(I_2)$, such that $V(I_1) \cap V(I_2) = \varnothing$ and $V(I_1) \cup V(I_2) = X$. So $I_1 + I_2 = (1)$ (otherwise it is contained in a maximal ideal, contradicting the selection of I_1, I_2), let $i_1 \in I_1$, $i_2 \in I_2$, and $i_1 + i_2 = 1$. Then easy to see that $V(I_1) = V(i_1)$, $V(I_2) = V(i_2)$. In particular, for every prime ideal $\mathfrak{p}$, $i_1 i_2 \in \mathfrak{p}$, so $i_1 i_2 \in \mathfrak{R}$ (the nilradical of A), i.e. there is $n \in \mathbb{N}_{>0}$, $i_1^n i_2^n = 0$. Besides, it is also easy to see that $V(I_1) = V(i_1^n)$, $V(I_2) = V(i_2^n)$, and $(i_1^n) + (i_2^n) = (1)$. Let $a, b \in A$ be such that $a i_1^n + b i_2^n = 1$, let $j_1 = a i_1^n$, $j_2 = a i_2^n$, then again $V(I_1) = V(j_1)$, $V(I_2) = V(j_2)$, and $j_1 j_2 = 0$, $j_1 + j_2 = 1$. Since for every $j \in (j_1)$, $j = a j_1$, $a j_1 = a j_1 (j_1 + j_2) = a j_1 j_1$, we see that j_1 is an idempotent, or an 'identity' in (j_1), we also have the same result for j_2. So for $j \in (j_1) \cap (j_2)$, $j = j j_1 = j j_2 = j j_1 j_2 = 0$, in another way of saying this, $(j_1) \cap (j_2) = 0$. Now by let $\phi : A \to (A/(j_1)) \times (A/(j_2))$ be the canonical homomorphism, since $(j_1) + (j_2) = (1)$ and $(j_1) \cap (j_2) = (0)$, by Chinese Remainder Theorem, it is an isomorphism. Since neither $(j_1) = (1)$, nor $(j_2) = (1)$, $(A/(j_1)), (A/(j_2))$ are two nonzero rings. [*Remark.* This is so hard that I had been thinking it for one day!]

ii) $\Rightarrow$ iii) $(1, 0)^2 = (1, 0) \neq 0, 1$ and $(0, 1)^2 = (0, 1) \neq 0, 1$.

iii) $\Rightarrow$ i) Let $x \neq 0, 1$ be an idempotent, then so is $1 - x$. Let $\mathfrak{p}$ be a prime ideal, since $x, 1 - x \in \mathfrak{p} \implies 1 \in \mathfrak{p}$ which is impossible, and since $x(1 - x) = 0 \in \mathfrak{p}$, $x \in \mathfrak{p}$ or $1 - x \in \mathfrak{p}$. So $V(x) \cap V(1 - x) = \varnothing$ and $V(x) \cup V(1 - x) = X$, and X is disconnected. ■

Problem 23. Let A be a Boolean ring (Exercise 11), and let $X = \operatorname{Spec}(A)$.

i) For each $f \in A$, the set X_f (Exercise 17) is both open and closed in X.

ii) Let $f_1, \ldots, f_n \in A$. Show that $X_{f_1} \cup \cdots X_{f_n} = X_f$ for some $f \in A$.

iii) The sets X_f are the only subsets of X which are both open and closed. [Let $Y \subseteq X$ be both open and closed. Since Y is open, it is a union of basic open sets X_f. Since Y is closed and X is quasi-compact (Exercise), Y is quasi-compact. Hence Y is a finite union of basic open sets; now use (ii) above.]

iv) X is a compact Hausdorff space.

Proof.

i) f is an idempotent, from Ex. 22 and the construction of factor rings, we can let $A \cong A_1 \times A_2$, such that f corresponds to the identity in A_1. Then the open set $X_f = V(\{1\} \times A_2)$ is also closed.

ii) $X_{f_1} \cup \cdots \cup X_{f_n} = V((f_1, \ldots, f_n))$. From Ex. 11 iii), finitely generated ideal $(f_1, \ldots, f_n) = (f)$ is principal, so $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$.

iii) Since we have an open cover for the open set $X \setminus V(E) = \bigcap_{f \in E} X_f$. If $X \setminus V(E)$ is also closed, then $V(E)$ is open, and $\big\{X_f \mid f \in E\big\} \cup \big\{V(E)\big\}$ is an open cover for X. By Exercise 17 v), we can find a finite subcover for X, this induces a finite subcover for $X \setminus V(E)$. So $X \setminus V(E)$ is a finite union of X_f where $f \in E$, by ii), $X \setminus V(E) = X_g$ for some $g \in A$.

iv) By Ex. 17 v), X is quasi-compact. By Exercise 11 ii), every prime ideal in A is maximal, so for distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$, $\mathfrak{p}_1 + \mathfrak{p}_2 = (1)$ (otherwise there is a maximal ideal containing both of them). Let $p_1 \in \mathfrak{p}_1$, $p_2 \in \mathfrak{p}_2$, such that $p_1 + p_2 = 1$. Again, from the construction in the proof of Ex. 22, we can let $A \cong A_1 \times A_2$, such that $p_1 \in A_1$ and $p_2 \in A_2$ (as a consequence $\mathfrak{p}_1 \subseteq A_1$ and $\mathfrak{p}_2 \subseteq A_2$, of course all inclusions here are understood up to isomorphism). It is then obvious that two point $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have disjoint neighborhoods. ■

Problem 24. Let L be a lattice, in which the sup and inf of two elements a, b are denoted by $a \vee b$ and $a \wedge b$ respectively. L is a *Boolean lattice* (or *Boolean algebra*) if

i) L has a least element and a greatest element (denoted by $0, 1$ respectively).

ii) Each of $\vee, \wedge$ is distributive over the other.

iii) Each $a \in L$ has a unique "complement" $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

(For example, the set of all subsets of a set, ordered by inclusion, is a Boolean lattice.)

Let L be a Boolean lattice. Define addition and multiplication in L by the rules

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad ab = a \wedge b.$$

Verify that in this way L becomes a Boolean ring, say $A(L)$.

Conversely, starting from a Boolean ring A, define an ordering on A as follows: $a \leq b$ means that $a = ab$. Show that, with respect to this ordering, A is a Boolean lattice. [The sup and inf are given by $a \vee b = a + b + ab$ and $a \wedge b = ab$, and the complement by $a' = 1 - a$.] In this way we obtain a one-to-one correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

Proof. ($\Rightarrow$) There are some basic facts:

$$(a \wedge b)' = (a' \vee b'), \quad (a \vee b)' = (a' \wedge b'), \quad 0' = 1$$
$$(a \wedge b') \vee (a' \wedge b)' = (a' \vee b) \wedge (a \vee b') = (a \wedge b) \vee (a' \wedge b').$$

For addition, we can calculate by above formulae,

$$\begin{aligned}(a + b) + c &= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c) \\ &= a + (b + c).\end{aligned}$$

and obviously $a+b=b+a$, $a+a=0$, $a+0=a$. For multiplication, $(ab)c=a(bc)$, $ab=ba$, $a1=a$. For distributivity,

$$\begin{aligned} ca+cb &= \big(c\wedge a\wedge(c'\vee b')\big)\vee\big((c'\vee a')\wedge c\wedge b\big)\\ &= \big((c\wedge a\wedge b')\vee(c\wedge a'\wedge b)\big)\\ &= c\wedge\big((a\wedge b')\vee(a'\wedge b)\big)\\ &= c(a+b). \end{aligned}$$

For Booleanness, obviously $aa=a$. So $A(L)$ is indeed a Boolean ring.

($\Leftarrow$) Let $a\vee b=a+b+ab$, $a\wedge b=ab$ and $a'=1-a$, we verify that this is a Boolean lattice. Recall that in a Boolean ring, $2x=0$. We see that $a(a\vee b)=a^2+ab+a^2b=a \implies a\leq a\vee b$ (the same for$b\leq a\vee b$), and $(a\wedge b)a=a^2b=a\wedge b \implies a\wedge b\leq a$ (the same for $a\wedge b\leq b$). ... (MISSING PROOF FOR SUP AND INF) ... . 0 and 1 are the smallest and largest elements respectively. And for distributivity,

$$\begin{aligned} &a\wedge(b\vee c)=a(b+c+bc)=ab+ac+a^2bc=(a\wedge b)\vee(a\wedge c),\\ &a\vee(b\wedge c)=a+bc+abc=(a+b+ab)(a+c+ac)=(a\vee b)\wedge(a\vee c). \end{aligned}$$

So what we defined is a indeed Boolean lattice. ■

Problem 25. From the last two exercises deduce Stone's theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.

Proof. Since $(a,b)=(a+b+ab)$, $X_a\cup X_b=X_{a+b+ab}=X_{a\vee b}$, and $(a)(b)=(ab)$, $X_a\cap X_b=X_{ab}=X_{a\wedge b}$, the sup and inf still preserve well in X. By Ex. 23 iv), $X=\operatorname{Spec}(A)$ is compact Hausdorff. By Ex. 24, Boolean lattices are in bijection with Boolean rings. So we get the correspondence between Boolean lattice and the lattice of open-and-closed subsets of some compact Hausdorff topological space. ■

Problem 26. Let A be a ring. The subspaces of $\operatorname{Spec}(A)$ consisting of the *maximal* ideals of A, with the induced topology, is called the *maximal spectrum* of A and is denoted by $\operatorname{Max}(A)$. For arbitrary commutative rings it does not have the nice functorial properties of $\operatorname{Spec}(A)$ (see Exercise 21), because the inverse image of a maximal under a ring homomorphism need not be maximal.

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X (add and multiply functions by adding and multiplying their values). For each $x\in X$, let $\mathfrak{m}_x$ be the set of all $f\in C(X)$ such that $f(x)=0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the (surjective) homomorphism $C(X)\to\mathbb{R}$ which takes f to $f(x)$. If $\tilde{X}$ denotes $\operatorname{Max}\big(C(X)\big)$, we have therefore defined a mapping $\mu: X\to\tilde{X}$, namely $x\mapsto\mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto $\tilde{X}$.

i) Let $\mathfrak{m}$ be a maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$, that is,

$$V = \{x \in X : f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say $U_{x_1}, \dots, U_{x_n}$, cover X. Let

$$f = f_{x_1}^2 + \cdots + f_{x_n}^2.$$

Then f does not vanish at any point of X, hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty.

Let x be a point of V. Then $\mathfrak{m} \subseteq \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is surjective.

ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X. Hence $x \neq y \Rightarrow \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.

iii) Let $f \in C(X)$; let

$$U_f = \{x \in X : f(x \neq 0)\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$$

Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. $\tilde{U}_f$) form a basis of the topology of X (resp. $\tilde{X}$) and therefore μ is a homeomorphism.

Thus X can be constructed from the ring of functions $C(X)$.

Proof. Since all maximal ideals are of the form $\mathfrak{m}_x$, we have $\tilde{U}_f = \{\mathfrak{m}_x \in \tilde{X} \mid f \notin \mathfrak{m}_x\} = \{\mathfrak{m}_x \mid x \in X, x \notin U_f\} = \mu(U_f)$.

For topology in X, we see that $U_f \cap U_g = U_{fg}$, $\bigcup_{f \in E} U_f = \{x \in X \mid \exists f \in E, f(x) \neq 0\}$. [*Note.* We didn't prove that this topology is the original topology of X.]

For topology in $\tilde{X}$, obviously $\tilde{U}_f \cap \tilde{U}_g = \tilde{U}_{fg}$, $\bigcup_{f \in E} \tilde{U}_f = \{\mathfrak{m} \in \tilde{X} \mid \exists f \in E, f \notin \mathfrak{m}\}$. ■

Affine algebraic varieties

Problem 27. Let k be an algebraically closed field and let

$$f_\alpha(t_1, \dots, t_n) = 0$$

be a set of polynomial equations in n variables with coefficients in k. The set X of all points $x = (x_1, \dots, x_n) \in k^n$ which satisfy these equation is an *affine algebraic variety.*

Consider the set of all polynomials $g \in k[t_1, \dots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring, and is called the *ideal of the variety* X. The quotient ring

$$P(X) = k[t_1, \dots, t_n]/I(X)$$

is the ring of polynomial functions on X, because two polynomials g, h define the same polynomial function on X if and only if $g - h$ vanished at every point of X, that is, if and only if $g - h \in I(X)$.

Let ξ_i be the image of t_i in $P(X)$. The ξ_i $(1 \le i \le n)$ are the *coordinate functions* on X: if $x \in X$, then $\xi_i(x)$ is the ith coordinate of x. $P(X)$ is generated as a k-algebra by the coordinate functions, and is called *coordinate ring* (or affine algebra) of X.

As in Exercise 26, for each $x \in X$ let $\mathfrak{m}_x$ be the ideal of all $f \in P(X)$ such that $f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $\tilde{X} = \operatorname{Max}\big(P(X)\big)$, we have defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i $(1 \le i \le n)$, and hence $\xi_i - x_i$ is in $\mathfrak{m}_x$ but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is *surjective.* This is one form of Hilbert's Nullstellensatz (see Chapter 7).

Proof. He is so right! ■

Problem 28. Let $f_1, \dots, f_m$ be elements of $k[t_1, \dots, t_n]$. They determine a *polynomial mapping* $\phi : k^n \to k^m$: if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \dots, f_m(x)$.

Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \to Y$ is said to be *regular* is ϕ is the restriction to X of a polynomial mapping from k^n to k^m.

If η is a polynomial function on Y, then $\eta \circ \phi$ is a polynomial function on X. Hence ϕ induces a k-algebra homomorphism $P(Y) \to P(X)$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a one-to-one correspondence between the regular mappings $X \to Y$ and the k-algebra homomorphisms $P(Y) \to P(X)$.

Proof. Let $F : \{X \to Y\} \to \{P(Y) \to P(X)\}$ by $\phi \mapsto (\eta \mapsto \eta \circ \phi)$, we show that this is a bijection.

To prove the injectivity, let $\phi_1, \phi_2 : X \to Y$ be regular, such that $F(\phi_1) = F(\phi_2)$. Let $\pi_i : (y_1, \dots, y_m) \mapsto y_i$. Then $\pi_i\phi_1 = \pi_i\phi_2$ for all $1 \le i \le m$, so $\phi_1 = \phi_2$.

For surjectivity, let $\{\xi_i\}_{i=1}^m$ be the coordinate functions on Y. Given a k-algebra homomorphism $\Phi : P(Y) \to P(X)$, let $\phi_i = \Phi(\xi_i)$, $\phi = (\phi_1, \dots, \phi_m)$ is regular. Since $P(Y)$ is generated by $\{\xi_i\}_{i=1}^m$ as a k-algebra, for every $\eta \in P(Y)$, $\Phi(\eta) = \eta\phi$, so $\Phi = F(\phi)$, and F is surjective. ■

Chapter 2

Modules

Exercise 2.15. Let A, B be rings, let M be an A-module, P a B-module and N an (A, B)-bimodule (that is, N is simultaneously an A-module and a B-module and the two structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A$, $b \in B$, $x \in N$). Then $M \otimes_A N$ is naturally a B-module, $N \otimes_B P$ an A-module, and we have

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Proof. We define $(m \otimes n)b = m \otimes (nb)$, $a(n \otimes p) = (an) \otimes p$. Given $p \in P$, let $f_p : (m, n) \mapsto m \otimes (n \otimes p)$, this map is obvious A-bilinear, so we get an induced A-linear map $f_p : m \otimes n \mapsto m \otimes (n \otimes p)$, it is also B-linear. Let $f : (m \otimes n, p) \mapsto f_p(m \otimes n) = m \otimes (n \otimes p)$, this is B-bilinear, so again we get a (A, B)-linear map $f : (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$. In the same way we can see that it has an inverse, and is also (A, B)-linear. So $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$. ∎

Exercise 2.20. If $f : A \to B$ is a ring homomorphism and M is flat A-module, then $M_B = B \otimes_A M$ is a flat B-module. (Use the canonical isomorphisms (2.14), (2.15).)

Proof. Let $E_\bullet$ be an exact sequence of B-modules, then $E_\bullet \otimes_B (B \otimes_A M) \cong (E_\bullet \otimes_B B) \otimes_A M \cong E_\bullet \otimes_A M$, also which is exact. So $B \otimes_A M$ is a flat B-module. ∎

Problem 1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ is m, n are coprime.

Proof. There there exists $x, y \in \mathbb{Z}$, $xm + yn = 1$, $(a + m\mathbb{Z}) \otimes (b + n\mathbb{Z}) = (xm + yn)(a + m\mathbb{Z}) \otimes (b + n\mathbb{Z}) = xm(a + m\mathbb{Z}) \otimes (b + n\mathbb{Z}) + yn(a + m\mathbb{Z}) \otimes (b + n\mathbb{Z}) = 0$. ∎

Problem 2. Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. [Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M.]

Proof. We have a natural exact sequence

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0.$$

Tensor this sequence by M, we have

$$\mathfrak{a} \otimes_A M \to A \otimes_A M \to (A/\mathfrak{a}) \otimes_A M \to 0.$$

Since $A \otimes_A M \cong M$, $\mathfrak{a} \otimes_A M \cong \mathfrak{a}M$, we get $(A/\mathfrak{a})\otimes_A \cong M/\mathfrak{a}M$. ■

Problem 3. Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$. [Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}$ by Exercise 2. By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$. But $M \otimes_A N = 0 \Rightarrow (M \otimes_A N)_k = 0 \Rightarrow M_k \otimes_k N_k = 0 \Rightarrow M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Proof. Let $\mathfrak{m}$ be the maximal ideal. If $M, N \neq 0$, then $\mathfrak{m}M \neq M$, $\mathfrak{m}N \neq N$ (since A is a local ring, M, N are finitely generated, otherwise by Corollary 2.7 $M = 0$ or $N = 0$), so $M/\mathfrak{m}M, N/\mathfrak{m}N \neq 0$ are vector spaces over $A/\mathfrak{m}$. Let $(m, n) \mapsto (m + \mathfrak{m}M) \otimes (n + \mathfrak{m}N)$, it is an A-bilinear map, so it induces

$$M \otimes_A N \twoheadrightarrow (M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N).$$

Since $\mathfrak{m}$ annihilates $(M/\mathfrak{m}M)\otimes_A(N/\mathfrak{m}N)$, we may regard it (which is $(M/\mathfrak{m}M)\otimes_{A/\mathfrak{m}} (N/\mathfrak{m}N)$ in this case) as an $A/\mathfrak{m}$-module. But $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (N/\mathfrak{m}N) \neq 0$ is a vector space over $A/\mathfrak{m}$, we must have $M \otimes_A N \neq 0$, contradiction. ■

Problem 4. Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Proof. Since for any A-module N,

$$N \otimes_A M \cong \bigoplus_{i\in I}(N \otimes_A M_i).$$

And since for every A-linear map $f : A \to M$, $f \otimes \mathrm{id}_M = \bigoplus_{i\in I} f \otimes \mathrm{id}_{M_i}$, the equivalence is clear. ■

Problem 5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. [Use Exercise 4.]

Proof. $A[x] \cong \bigoplus_{n\in\mathbb{N}} A$ and A is flat as a A-module $\implies$ $A[x]$ is flat. It is also an A-algebra, with $f : A \to A[x]$, $a \mapsto a$. ■

Problem 6. For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r \quad (m_i \in M).$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.

Show that $M[x] \cong A[x] \otimes_A M$.

Proof. The obvious product of $a_0+a_1x+\cdots+a_sx^s$ and $m_0+m_1x+\cdots+m_rx^r$ should be $n_0+n_1x+\cdots n_{r+s}x^{r+s} \in M[x]$, where $n_k = \sum_{i+j=k,i\leq s} a_i m_j$, and obviously this define an $A[x]$-module structure on $M[x]$.

For the second part, viewed as A-modules, $A[x] \cong \bigoplus_{n\in\mathbb{N}} A$, and

$$A[x] \otimes_A M \cong \bigoplus_{n\in\mathbb{N}} (A \otimes_A M) \cong \bigoplus_{n\in\mathbb{N}} M$$

gives the isomorphism. ■

[*Remark.* Can we prove this in a naive way: Since the map $A[x] \times M \to M[x]$ by $(\sum_i a_i x^n, m) \mapsto \sum_i a_i m x^n$ is A-bilinear, we have an induced A-linear map $f : A[x] \otimes_A M \to M[x]$. f is obviously surjective. But how to prove the injectivity? Or asking, what role does direct sum play here?]

Problem 7. Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{i=0}^m b_i x^i \in A[x] \setminus \mathfrak{p}[x]$. If $fg \in \mathfrak{p}[x]$, let r, s be the smallest number such that $a_r, b_s \notin \mathfrak{p}$ respectively. Then the $(r+s)$-th term of $f(x)g(x)$ is

$$\sum_{\substack{i+j=r+s \\ i\leq r}} a_i b_j \in \mathfrak{p} \implies a_r \in \mathfrak{p} \text{ or } b_s \in \mathfrak{p},$$

contradiction. So $f(x)g(x) \notin \mathfrak{p}[x]$, and $\mathfrak{p}[x]$ is a prime ideal.

This is certainly not the case, for example, let $A = k$ be a field. Then $\mathfrak{m} = (0)$, but clearly $(0)[x] = (0)$ is not maximal in $k[x]$, since x is not a unit, and there must a maximal ideal containing both (x) and (0). What we can say for sure is that $\mathfrak{m}[x]$ is a prime ideal. ■

Problem 8.

i) If M and N are flat A-modules, then so is $M \otimes_A N$.

ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Proof.

i) For any exact sequence $E_\bullet$ of A-modules, $E_\bullet \otimes_A (M \otimes_A N) \cong (E_\bullet \otimes_A M) \otimes_A N$ is obviously exact.

ii) Since B is an A-algebra, it is a (A, B)-bimodule. For any exact sequence $E_\bullet$, by (2.15), $(E_\bullet \otimes_A B) \otimes_B N \cong E_\bullet \otimes_A (B \otimes_B N) \cong E_\bullet \otimes_A N$ is exact, so N is a flat A-module. ■

Problem 9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence os A-modules. If M' and M'' are finitely generated, then so is M.

Proof. We denote the A-linear maps of $M' \to M$ and $M \to M''$ to be f and g respectively. Then $M'' \cong M/\ker g \cong M/\operatorname{im} f$ is finitely generated, since $\operatorname{im} f \cong M'$, M is finitely generated. ■

Problem 10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Proof. $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective implies $u(M) + \mathfrak{a}N = N$. Since $u(M) \subseteq N$ are both finitely generated, $\mathfrak{a}$ is contained in the Jacobson radical, by Corollary 2.7, $u(M) = N$, which means $u : M \to N$ is surjective ■

Problem 11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.
[Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $A/\mathfrak{m}$. Hence $m = n$.] (Cf. Chapter 3, Exercise 15.)

If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.

If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Proof. All morphisms here are A-linear maps, otherwise as a ring-isomorphism, $(A^\infty)^2 \cong A^\infty$ (where $A^\infty := \prod_{n=0}^\infty A$), but in this case $2 \neq 1$.[1] Besides, the map $f : A^\infty \times \oplus A^\infty \xrightarrow{\sim}$ by sending the first to even coordinates and sending the second to odd coordinates is not A-linear, because $f\Big((\dots, b_i, \dots)\big((\dots, a_i, \dots), (\dots, a_i', \dots)\big)\Big) \neq (\dots, b_i, \dots) f(a, a')$.

Let $\mathfrak{m}$ be a maximal ideal in A, we may view $A/\mathfrak{m}$ as an A-module, and let $1 \otimes \phi : (A/\mathfrak{m}) \otimes_A A^m \to (A/\mathfrak{m}) \otimes_A A^n$. Since we have an exact sequence $0 \to A^m \xrightarrow{\phi} A^n \to 0$, so the sequence

$$0 \to (A/\mathfrak{m}) \otimes_A A^m \xrightarrow{1 \otimes \phi} (A/\mathfrak{m}) \otimes_A A^n \to 0$$

[1] See also: Question. "Is it true that $R^n \simeq R^m$ as rings implies $m = n$?" *Mathematics Stack Exchange*, `https://math.stackexchange.com/questions/1261177/`, 2015.

is exact, so $1\otimes\phi$ is an isomorphism. Besides, $(A/\mathfrak{m})\otimes_A A^m \cong (A/\mathfrak{m})^m$ and $(A/\mathfrak{m})\otimes_A A^n \cong (A/\mathfrak{m})^n$, so $1\otimes\phi$ induces an isomorphism between two vector spaces over $k/\mathfrak{m}$ of dimensions m, n respectively. So $m = n$.

Using the same technique, when ϕ is surjective, $1\otimes\phi$ is still surjective, so $m \geq n$. But $1\otimes\phi$ may not be injective if we only know that ϕ is injective. ∎

Problem 12. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\ker\phi$ is finitely generated
[Let $e_1, \dots, e_n$ be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$ $(1 \leq i \leq n)$. Show that M is the direct sum of $\ker\phi$ and the submodule generated by $u_1, \dots, u_n$.]

Proof. $M/\ker\phi \cong A^n$. Let $e_1, \dots, e_n \in M$ be the representatives of the basis elements. Let $x_1, \dots, x_m$ be the generators of M. Then every x_i can be uniquely written as a a linear combination of $e_1, \dots, e_n$ and an element $r_i \in \ker\phi$ over A. Then for every element $r \in \ker\phi$, we can write r in terms of a linear combination of x_i's, and hence can be written as a linear combination of e_i's and r_i's. But since $e_1, \dots, e_n$ form a basis, in the above equation their sum lies in $\ker\phi$, hence their terms must all be 0, and r can be written as a linear combination of r_i's. So $\ker\phi$ can be generated by $r_1, \dots, r_m$. ∎

[*Remark.* We have a map $\varphi : (0, \dots, a_i, \dots, 0) \mapsto e_i$, and $m \mapsto \big(\varphi\phi(m), m - \varphi\phi(m)\big)$ gives an isomorphism $M \to A^n \oplus \ker\phi$.]

Problem 13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.
[Define $p : N_B \to N$ by $p(b \otimes y) = by$, and show that $N_B = \operatorname{im} g \otimes \ker p$.]

Proof. Regarded as A-modules, the map $B \times N \to N$ given by $(b, y) \mapsto by$ is bilinear and surjective, so it induces a surjective A-linear map $p : B \otimes_A N \to N$ by $b \otimes y \mapsto by$. So for $y_1, y_2 \in N$, $1 \otimes y_1 = 1 \otimes y_2$ in N_B, we have $1 \otimes (y_1 - y_2) = 0$, hence $y_1 - y_2 = 0$, g is injective.

Besides, we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "g"] & N_B \arrow[r, "p"] & N \arrow[r] \arrow[l, bend left, "g"] & 0 \end{tikzcd}$$

and $pg = \mathrm{id}_N$, so $N_B \cong \operatorname{im} g \oplus \ker p$. ∎

Direct limits

Problem 14. A partially ordered set I is said to be a *directed set* if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let A be a ring, let I be a directed set and let $(M_i)_{i\in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$ let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

(1) μ_{ii} is the identity mapping of M_i, for all $i \in I$;

(2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μij are said to form a *direct system* $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the *direct limit* of the direct system $\mathbf{M}$. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu ij(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the *direct limit* of the direct system $\mathbf{M}$, and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j\mu_{ij}$ whenever $i \leq j$.

Proof. Yes yes yes! ∎

Problem 15. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Proof. Every element of M can be written as a finite sum $\sum_{s=1}^N \mu_{j_s}(x_{j_s})$, for some $x_{j_s} \in M_{j_s}$. Since the poset I is directed, by induction there is $i \in I$, such that $j_s \leq i$ for all $1 \leq s \leq N$, and easy to see that $\mu_{j_s}(x_{j_s}) = \mu_i\mu_{j_si}(x_{j_s})$ and $\sum_{s=1}^N \mu_{j_s}(x_{j_s}) = \mu_i\big(\sum_{s=1}^N \mu_{j_si}(x_{j_s})\big)$.

Without loss of generality, suppose $x_i \neq 0$. If $\mu_i(x_i) = 0$, there is a finite sum $x_i = \sum_{s=1}^N (x_{i_s} - \mu_{i_sj_s}(x_{i_s}))$, where $i_s < j_s$. Let $j_0 \geq j_1, \dots, j_N$. Since for every $1 \leq s \leq N$,

$$x_{i_s} - \mu_{i_sj_s}(x_{i_s}) = \big(x_{i_s} - \mu_{i_sj_0}(x_{i_s})\big) - \big(\mu_{i_sj_s}(x_{i_s}) - \mu_{j_sj_0}(\mu_{i_sj_s}(x_{i_s}))\big),$$

we may assume $j_1 = \cdots = j_N = j_0$.[2] For $k \in I$, let $S_k = \{1 \leq s \leq N \mid i_s = k\}$, then for all $k \neq i$, $\sum_{s\in S_k} x_{i_s} = 0$, hence $\mu_{kj_0}\big(\sum_{s\in S_i} x_{i_s}\big) = 0$. Therefore, $\mu_{ij_0}\big(\sum_{s\in S_i} x_{i_s}\big) = 0$, but $\sum_{s\in S_i} x_{i_s} = x_i$, $\mu_{ij_0}(x_i) = 0$. ∎

[*Remark.* This shows that the definition in Exercise 14 is the same as the definition by disjoint union and quotient, which says that $\varinjlim M_i = \bigsqcup M_i/\sim$, $x_i \sim x_j$ iff for some $i, j \leq k$, $\mu_{ik}(x_i) = \mu_{jk}(x_j)$.]

[2] Reference: Arturo Magidin (`https://math.stackexchange.com/users/742/arturo-magidin`). "Zero image of an element in the direct limit of modules." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/81484`, 2011.

Problem 16. Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Proof.

$$\begin{tikzcd}
 & N & \\
 & M \arrow[u, dashed, "\exists!\alpha"] & \\
M_i \arrow[ur, "\mu_i"] \arrow[rr, "\mu_{ij}"] \arrow[uur, bend left, "\alpha_i"] & & M_j \arrow[ul, "\mu_j"'] \arrow[uul, bend right, "\alpha_j"']
\end{tikzcd}$$

Let $\alpha : \mu_i(x_i) \mapsto \alpha_i(x_i)$, we show that this map is well-defined. For $x_i, y_i \in M_i$, $\mu(x_i - y_i) = 0$, then by Ex. 15 there is $j \in I$, $\mu_{ij}(x_i - y_j) = 0$, and $\alpha_i(x_i - y_i) = \alpha_j \mu_{ij}(x_i - y_i) = 0$. It is easy to see that α is the only way to define the map, and is indeed an A-linear map. ∎

Problem 17. Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_i$ be the embedding of M_i in M_j. Show that

$$\varinjlim M_i \cong \sum M_i = \bigcup M_i.$$

In particular, any A-module is the direct limit of its finitely generated submodules.

Proof. We define a map $\beta : M \to \varinjlim M_i$. For $x \in M$, $x \in M_i$ for some $i \in I$, let $\beta : x \mapsto \mu_i(x)$. We verify that this is well-defined, for if $x \in M_i, M_j$, we know that there is $k \in I$, $x \in M_i, M_j \subseteq M_k$, so $\mu_{ik}(x) = \mu_{jk}(x)$ and $\mu_i(x) = \mu_j(x)$. So β is an A-linear map. But from the universal property of direct limit (Ex. 16), there is a map $\alpha : \varinjlim M_i \to M$, which is also an inverse of β, so $\varinjlim M_i \cong \sum M_i = \bigcup M_i$. ∎

Problem 18. Let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$ be the directed systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ the associated homomorphisms.

A *homomorphism* $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that $\mathbf{\Phi}$ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Proof.

$$\begin{tikzcd}
 & & M \arrow[ddd, dashed, "\exists!\phi"] \\
M_i \arrow[r, "\mu_{ij}"] \arrow[d, "\phi_i"'] \arrow[urr, bend left, "\mu_i"] & M_j \arrow[d, "\phi_j"] \arrow[ur, "\mu_j"'] & \\
N_i \arrow[r, "\nu_{ij}"'] \arrow[drr, bend right, "\nu_i"'] & N_j \arrow[dr, "\nu_j"] & \\
 & & N
\end{tikzcd}$$

We let $\phi : \mu_i(x_i) \mapsto \nu_i\phi_i(x_i)$, we show that it is well-defined. This is because if $\mu_i(x_i) = \mu_i(y_i)$, by Ex. 15 there is $j \in I$, such that $\mu_{ij}(x_i - y_i) = 0$, so $\nu_i\phi_i(x_i - y_i) = \nu_j\phi_j\mu_{ij}(x_i - y_i) = 0$. Besides, ϕ is an A-linear map. ■

Problem 19. A sequence of direct systems and homomorphisms

$$\mathbf{M} \to \mathbf{N} \to \mathbf{P}$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact. [Use Exercise 15.]

Proof.

$$\begin{tikzcd}
M_j \arrow[d, "\phi_j"] & M_i \arrow[l, "\mu_{ij}"'] \arrow[r, "\mu_i"] \arrow[d, "\phi_i"] & M \arrow[d, "\phi"] \\
N_j \arrow[d, "\psi_j"] & N_i \arrow[l, "\nu_{ij}"'] \arrow[r, "\nu_i"] \arrow[d, "\psi_i"] & N \arrow[d, "\psi"] \\
P_j & P_i \arrow[l, "\pi_{ij}"'] \arrow[r, "\pi_i"] & P
\end{tikzcd}$$

Since every element in M is of the form $\mu_i(x_i)$ for some $i \in I$,

$$\operatorname{im}\phi = \bigcup_{i\in I} \operatorname{im}\phi\mu_i = \bigcup_{i\in I} \operatorname{im}\nu_i\phi_i = \bigcup_{i\in I} \nu_i \ker\psi_i.$$

Let $y_i \in N_i$ be such that $\psi\nu_i(y_i) = 0$, then $\pi_i\psi_i(y_i) = 0$ by Ex. 15 there is $j \geq i$, $\pi_{ij}\psi_i(y_i) = 0$, so $\psi_j\nu_{ij}(y_i) = 0$, $\nu_{ij}(y_i) \in \ker\psi_j$. Since $\nu_j\nu_{ij} = \nu_i$,

$$\ker\psi = \bigcup_{j\in I} \nu_j \ker\psi_j,$$

and the sequence $M \to N \to P$ is exact. ■

Tensor products commute with direct limits

Problem 20. Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \varinjlim(M_i \otimes N)$ be its direct limit. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N.$$

[For each $i \in I$, let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear mapping. Passing to the limit we obtain a mapping $g : M \times N \to P$. Show that g is A-bilinear and hence define a homomorphism $\phi : M \otimes N \to P$. Verify that $\phi \circ \psi$ and $\psi \circ \phi$ are identity mappings.]

Proof. Let $y \in N$, $\phi_i^y : x_i \mapsto x_i \otimes y$. Then $\{\phi_i^y\}$ is a homomorphism between the two direct systems (M_i, μ_{ij}) and $(M_i \otimes_A N, \mu_{ij} \otimes 1)$, by Exercise 18 there is a unique A-linear map $\phi^y : \varinjlim M_i \to \varinjlim(M_i \otimes_A N)$, such that the following diagram commutes

$$\begin{tikzcd} M_i \arrow[r, "\mu_i"] \arrow[d, "\phi_i^y"] & \varinjlim M_i \arrow[d, "\phi^y"] \\ M_i \otimes_A N \arrow[r, "\mu_i^{\otimes}"'] & \varinjlim(M_i \otimes_A N) \end{tikzcd}$$

We want to check that given $x \in \varinjlim M_i$, $\phi^y(x)$ is A-linear on N. But this is obvious, since from the explicit (and is also unique) construction in Exercise 18, we see that $\phi^y : \mu_i(x_i) \mapsto \mu_i^{\otimes}(x_i \otimes y)$. So we have a bilinear map $\left(\varinjlim M_i\right) \times N \to \varinjlim(M_i \otimes_A N)$ by $(\mu_i(x_i), y) \mapsto \phi^y(x)$. This induces an A-linear map

$$\left(\varinjlim M_i\right) \otimes_A N \to \varinjlim(M_i \otimes_A N), \quad \mu_i(x_i) \otimes y \mapsto \mu_i^{\otimes}(x_i \otimes y).$$

Since the map ψ is given by $\mu_i^{\otimes}(x_i \otimes y) \mapsto \mu_i(x_i) \otimes y$, we see that ψ has an inverse, hence it is an isomorphism, and $\varinjlim(M_i \otimes_A N) \cong \left(\varinjlim M_i\right) \otimes_A N$. ∎

Problem 21. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I le t$\alpha_{ij} : A_i \to A_j$ be a ring homomorphism, satisfying conditions (1) and (2) of Exercise 14. Regarding each A_i as a $\mathbb{Z}$-module we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \to A$ are ring homomorphisms. The ring A is the *direct limit* of the system (A_i, α_{ij}).

If $A = 0$ prove that $A_i = 0$ for some $i \in I$. [Remember that all rings have identity elements!]

Proof. By Exercise 15, every finite subset of I has a upper bound in I, every element in $\varinjlim A_i$ is of the form $\alpha_i(x_i)$, and for $i \leq j$, $\alpha_j \alpha_{ij}(x_i) = \alpha_i(x_i)$, so we define the operation of $\varinjlim A_i$ as follow. For $x_i \in A_i$, $x_j \in A_j$, $x_k \in A_k$, let $i, j, k \le \ell$, we can lift x_i, x_j, x_k to A_ℓ by $\alpha_{i\ell}, \alpha_{j\ell}, \alpha_{k\ell}$ without changing their values in $\varinjlim A_i$, and define their operation all in A_ℓ. i.e.

$$\begin{aligned}\alpha_i(x_i) + \alpha_j(x_j) &= \alpha_\ell(\alpha_{i\ell}(x_i) + \alpha_{j\ell}(x_j)),\\ \alpha_i(x_i) \cdot \alpha_j(x_j) &= \alpha_\ell(\alpha_{i\ell}(x_i)\alpha_{j\ell}(x_j)).\end{aligned}$$

The distributivity is clear, i.e.

$$\begin{aligned}\alpha_k(x_k) \cdot (\alpha_i(x_i) + \alpha_j(x_j)) &= \alpha_\ell(\alpha_{k\ell}(x_k)(\alpha_{i\ell}(x_i) + \alpha_{j\ell}(x_j)))\\ &= \alpha_\ell(\alpha_{k\ell}(x_k)\alpha_{i\ell}(x_i) + \alpha_{k\ell}(x_k)\alpha_{j\ell}(x_j))\\ &= \alpha_k(x_k) \cdot \alpha_i(x_i) + \alpha_k(x_k) \cdot \alpha_j(x_j).\end{aligned}$$

The identity is $\alpha_i(1_{A_i})$ for any $i \in I$. Everything here is clearly well-defined.

Suppose $A = 0$, if none of A_i is 0, then for any $i \in I$, $\alpha_i(1_{A_i}) = 0$, by Exercise 15, there is $j \ge i$, $1_{A_j} = \alpha_{ij}(1_{A_i}) = 0$, contradiction. [The converse is not true for modules in general, but is true for rings. If $A_i = 0$, then for every $j \ge i$, since α_{ij} is a ring homomorphisms, $1_{A_i} = 0$, we have $1_{A_j} = \alpha_{ij}(1_{A_i}) = 0$, hence $A_j = 0$. Then for every element $\alpha_i(x_i) \in A$, let $k \ge i, j$, $\alpha_i(x_i) = \alpha_k\alpha_{ik}(x_i) = 0$, and $A = 0$.] ■

[*Remark.* We cannot define the multiplication to be componentwise multiplication by representatives, since otherwise we need to check $D^2 \subseteq D$, in particular we may possible have three indices such that $i < j < k$, then $(1_{A_i} - \alpha_{ij}(1_{A_i}))(1_{A_i} - \alpha_{ik}(1_{A_i})) = 1_{A_i} \in D$, which may almost lead to an contradiction.]

Problem 22. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i. Show that $\varinjlim \mathfrak{N}_i$ is the nilradical of $\varinjlim A_i$.

If each A_i is an integral domain, then $\varinjlim A_i$ is an integral domain,

Proof. By Exercise 18, the is an inclusion $\varinjlim \mathfrak{N}_i \hookrightarrow \varinjlim A_i$ (injectivity is easy to check). The direction "$\subseteq$" is obvious. For $\alpha_i(x_i) \in \mathfrak{N}$ the nilradical of $\varinjlim A_i$, there is $j \ge i$, $\alpha_{ij}(x_i^n) = (\alpha_{ij}(x_i))^n = 0$, so $\alpha_{ij}(x_i) \in \mathfrak{N}_j$.

If all A_i are integral, let $x_i, y_i \in A_i$, $\alpha_i(x_iy_i) = 0$, then there is $j \ge i$, $\alpha_{ij}(x_iy_i) = 0$, this implies one of $\alpha_{ij}(x_i)$ and $\alpha_{ij}(y_i)$ is zero, let's say $\alpha_{ij}(x_i) = 0$. Then $\alpha_i(x_i) = \alpha_j\alpha_{ij}(x_i) = 0$, so $\varinjlim A_i$ is integral. ■

Problem 23. Let $(B_\lambda)_{\lambda \in \Lambda}$ be a family of A-algebras. For each finite subset of Λ let B_J denote the tensor product (over A) of the B_λ for $\lambda \in J$. If J' is another finite subset of Λ and $J \subseteq J'$, there is a canonical A-algebra homomorphism $B_J \to B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ. The ring B has a natural A-algebra structure for which the homomorphisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra B is the *tensor product* of the family $(B_\lambda)_{\lambda \in \Lambda}$.

Proof. Of course you're right! ■

Flatness and Tor
In these Exercises it will assumed that the reader is familiar with the definition and basic properties of the Tor functor.[3]

Problem 24. If M is an A-module, the following are equivalent:

i) M is flat;

ii) $\mathrm{Tor}_n^A(M,N)=0$ for all $n>0$ and all A-modules N;

iii) $\mathrm{Tor}_1^A(M,N)=0$ for all A-modules N.

[To show that (i) $\Rightarrow$ (ii), take a free resolution of N and tensor it with M. Since M is flat, the resulting sequence is exact and therefore its homology groups, which are the $\mathrm{Tor}_n^A(M,N)$, are zero for $n>0$. To show that (iii) $\Rightarrow$(i), let $0\to N'\to N\to N''\to 0$ be an exact sequence. Then, from the Tor exact sequence,

$$\mathrm{Tor}_1(M,N'')\to M\otimes N'\to M\otimes N\to M\otimes N''\to 0$$

is exact. Since $\mathrm{Tor}_1(M,N'')=0$ it follows that M is flat.]

Proof. For every exact sequence $0\to N'\to N\to N''\to 0$, we have a long exact sequence

$$\begin{aligned}\cdots\to \mathrm{Tor}_2^A(M,N')\to \mathrm{Tor}_2^A(M,N)\to \mathrm{Tor}_2^A(M,N'')&\\ \to \mathrm{Tor}_1^A(M,N')\to \mathrm{Tor}_1^A(M,N)\to \mathrm{Tor}_1^A(M,N'')&\\ \to M\otimes_A N'\to M\otimes_A N\to M\otimes_A N''\to 0.&\end{aligned}$$

Since N', N or N'' in the short exact sequence can be chosen arbitrarily, the equivalence relation of them is clear. ■

Problem 25. Let $0\to N'\to N\to N''\to 0$ be an exact sequence, with N'' flat. Then N' is flat $\Leftrightarrow$ N is flat. [Use Exercise 24 and the Tor exact sequence.]

Proof. Let $0\to N'\xrightarrow{f} N\xrightarrow{g} N''\to 0$ be short exact, with N'' flat. Let $0\to M'\xrightarrow{d} M\to 0$ be any injection. Since $\mathrm{Tor}_1^A(N'',E)=0$ for every A-module E, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
0 \arrow[r] & M'\otimes N' \arrow[r, "1_{M'}\otimes f"] \arrow[d, "d\otimes 1_{N'}"] & M'\otimes N \arrow[r, "1_{M'}\otimes g"] \arrow[d, "d\otimes 1_N"] & M'\otimes N'' \arrow[r] \arrow[d, "d\otimes 1_{N''}"] & 0\\
0 \arrow[r] & M\otimes N' \arrow[r, "1_M\otimes f"] & M\otimes N \arrow[r, "1_M\otimes g"] & M\otimes N'' \arrow[r] & 0
\end{tikzcd}$$

[3] Advanced Modern Algebra Part 2 by Rotman and Algebra by Lang are good references.

If N is flat, we have an induced injection $\ker(d \otimes 1_{N'}) \hookrightarrow \ker(d \otimes 1_N) = 0$, so N' is flat. If N' is flat, by snake lemma, we have an exact sequence $0 = \ker(d \otimes 1_{N'}) \to \ker(d \otimes 1_N) \to \ker(d \otimes 1_{N''}) = 0$, so $\ker(d \otimes 1_N) = 0$, N is flat.[4] ■

Problem 26. Let N be an A-module. Then N is flat $\Leftrightarrow$ $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A.
[Show first that N is flat if $\mathrm{Tor}_1(M, N) = 0$ for all *finitely generated* A-modules M, by using (2.19). If M is finitely generated, let $x_1, \dots, x_n$ be a set of generators of M, and let M_i be the submodule generated by $x_1, \dots, x_i$. By considering the successive quotients M_i/M_{i-1} and using Exercise 25, deduce that N is flat if $\mathrm{Tor}_1(M, N) = 0$ for all *cyclic* A-modules M, i.e., all M generated by a single element, and therefore of the form $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Finally use (2.19) again to reduce to the case where $\mathfrak{a}$ is a finitely generated ideal.]

Proof. By Proposition 2.19 iv), N is flat if and only if $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated A-module M (coker f is finitely generated in that case). The direction "$\Rightarrow$" is clear, suppose $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for every finitely generated ideal $\mathfrak{a}$ in A. Then for every ideal $\mathfrak{a}$ in A, let $0 \to \mathfrak{a} \to A$ be the inclusion, we want to check that it is still exact after tensoring by N. By Proposition 2.19 iv) again, we only need to check on finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$, then we have an exact sequence $0 \to \mathfrak{a}' \to A \to A/\mathfrak{a}' \to 0$. Since $\mathrm{Tor}_1(A/\mathfrak{a}', N) = 0$, we have an injection $0 \to \mathfrak{a}' \otimes N \to A \otimes N$, so $0 \to \mathfrak{a} \otimes N \to A \otimes N$ is still exact. This means $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all ideals $\mathfrak{a}$ in A.

Let $M = \langle x_1, \dots, x_n \rangle$, and let $M_1 = \langle x_1, \dots, x_i \rangle$. Then we have an exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

for each $1 \leq i \leq n$. Since $M_0 = 0$, $M_n = M$, M_i/M_{i-1} is cyclic ($\cong A/\mathfrak{a}$), we have $\mathrm{Tor}_1(M_0, N) = 0$ and $\mathrm{Tor}_1(M_i/M_{i-1}, N) = 0$. We use induction on i, at each step, by induction hypothesis, $\mathrm{Tor}_1(M_{i-1}, N) = 0$, by Exercise 25 we get $\mathrm{Tor}_1(M_i, N) = 0$, so finally we have $\mathrm{Tor}_1(M, N) = 0$. This implies N is flat. ■

Problem 27. A ring A is *absolutely flat* if every A-module is flat. Prove that the following are equivalent:

i) A is absolutely flat.

ii) Every principal ideal is idempotent.

iii) Every finitely generated ideal is a direct summand of A.

[4] Reference: Algebra, Lang, p. 616.

[i) $\Rightarrow$ ii). Let $x \in A$. Then $A/(x)$ is a flat A-module, hence in the diagram

$$\begin{tikzcd} (x) \otimes A \arrow[r, "\beta"] \arrow[d] & (x) \otimes A/(x) \arrow[d, "\alpha"] \\ A \arrow[r] & A/(x) \end{tikzcd}$$

the mapping α is injective. Hence $\operatorname{im}\beta = 0$, hence $(x) = (x^2)$. ii) $\Rightarrow$ iii). Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and we have $(e) = (x)$. Now if e, f are idempotents, then $(e, f) = (e + f - ef)$. Hence every finitely generated ideal is principal, and generated by an idempotent e, hence is a direct summand because $A = (e) \oplus (1 - e)$. iii) $\Rightarrow$ i). Use the criterion of Exercise 26.]

Proof. i) $\Rightarrow$ ii) Let $a \in A$, then $A/(a)$ and (a) are flat A-modules, and we have an injection $0 \to (a) \to A$, tensoring this by $A/(a)$ we have $0 \to (a)/(a)^2 \to A/(a)$. We also have a surjection $A \to A/(a) \to 0$, tensoring this by (a) we have $(a) \to (a)/(a)^2 \to 0$. Combining them (they are compatible) we have

$$\begin{tikzcd} (a) \arrow[dr] & & A/(a) \\ & (a)/(a)^2 \arrow[ur] \arrow[dr] & \\ 0 \arrow[ur] & & 0 \end{tikzcd}$$

so $(a)/(a)^2 = 0$, $(a) = (a)^2$.

ii) $\Rightarrow$ iii) $(a)^2 = (a^2) = (a)$ implies $ca^2 = a$. Since $ca = (ca)^2$, and $(ca) = (a)$, we may therefore suppose $a^2 = a$ is an idempotent element. Then $a(1 - a) = 0$, for $b \in (a) \cap (1-a)$, $b = ba = b(1-a) = ba(1-a) = 0$. So $(a) \cap (1-a) = (0)$, $(a) + (1-a) = (1)$, $A = (a) \oplus (1 - a)$. Let $\mathfrak{a} = (a_1, \dots, a_n)$ be finitely generated, $(a_i) \not\subseteq (a_j)$ for $i \neq j$, then we can use induction to show that $A = ((a_1, \dots, a_i)) \oplus (1 - (a_1, \dots, a_i))$, $1 \in A^\times$. Hence finally $A = ((a_1, \dots, a_n)) \oplus (1 - (a_1, \dots, a_n)) = \mathfrak{a} \oplus B$.

iii) $\Rightarrow$ i) Let $\mathfrak{a}$ be any finitely generated ideal in A, then we may write $A \cong \mathfrak{a} \oplus \mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq A$. Since for any A-module N, $(\mathfrak{a} \oplus \mathfrak{b}) \otimes_A N = (\mathfrak{a} \otimes_A N) \oplus (\mathfrak{b} \otimes_A N)$, we have an injection

$$0 \to \mathfrak{a} \otimes_A N \to (\mathfrak{a} \otimes_A N) \oplus (\mathfrak{b} \otimes_A N) = A \otimes_A N.$$

Hence $\operatorname{Tor}_1^A(A/\mathfrak{a}, N) = 0$ for any finitely generated ideal $\mathfrak{a} \subseteq A$, and N is a flat A-module. Hence A is absolutely flat. ■

Problem 28. A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 is absolutely flat. Every homomorphic image of an absolutely flat ring is absolutely flat. If a local ring is absolutely flat, then it is a field.

If A is absolutely flat, every non-unit in A is a zero divisor.

Proof.

1. Boolean rings are absolutely absolutely flat.
2. The ring of Chapter 1, Exercise 7 is that for every x there is $n > 1$ (depending on x), $x^n = x$. Then $(x) = (x^n) \subseteq (x^2) \subseteq (x)$. So it is absolutely flat.
3. Homomorphic image of a ring A is $\cong A/\mathfrak{a}$, every principal ideal is still idempotent, hence this ring is absolutely flat.
4. If the maximal ideal of a local ring is not (0), then this ideal contains a nonzero principal ideal, and contains an idempotent element $\neq 0$. But this is impossible due to Exercise 12 of Chapter 1, so the local ring must be a field.
5. Let $x \notin A^\times$, then $(x^2) = (x)^2 = (x)$, and $x^2 = cx$ for some $c \in A^\times$. But since x is not a unit, $c - x \neq 0$, $x(c - x) = 0$ implies x is a zero-divisor. ■

Chapter 3

Rings and Modules of Fractions

Exercise. (page 37) Verify that these definitions are independent of the choices of representatives (a, s) and (b, t), and that $S^{-1}A$ satisfies the axioms of a commutative ring with identity.

Proof. None ■

Problem 1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Proof. Let $M = \langle x_1, \ldots, x_n \rangle$. Then $S^{-1}M = 0$ implies for ever $1 \leq i \leq n$, $x_i/s_i = 0$ for some $s_i \in S$. So there are $t_i \in S$ such that $t_i a_i = 0$. Let $s = \prod_i t_i$, then $tM = 0$. The reverse is obvious. ■

Problem 2. Let $\mathfrak{a}$ be an ideal of a ring A, and le t$S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants. [If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence by Nakayama's lemma we have $S^{-1}M = 0$. Now use Exercise 1.]

Proof. We only need to prove that $1-(S^{-1}\mathfrak{a})(S^{-1}A) \subseteq (S^{-1}A)^{\times}$. Since $(S^{-1}\mathfrak{a})(S^{-1}A) = S^{-1}\mathfrak{a}$, and element of $S^{-1}\mathfrak{a}$ is of the form $a/(1+a')$, $1-a/(1+a') = (1-a+a')/(1+a') \in S^{-1}S$ is invertible.

Let M be a finitely generated A-module, and let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = M$. Then $S^{-1}M$ is finitely generated, and $(S^{-1}\mathfrak{a})(S^{-1}M) = S^{-1}(\mathfrak{a}M) = S^{-1}M$. Since $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, by Nakayama's lemma, $S^{-1}M = 0$. Then by Exercise 1, there exists $x \in S$, $xM = 0$. Besides, $S = 1 + \mathfrak{a}$ implies $s \equiv 1 \pmod{\mathfrak{a}}$. ■

Problem 3. Let A be a ring, let S and t be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Proof. Let

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f"'] & S^{-1}A \arrow[r, "g_2"] & U^{-1}(S^{-1}A) \\ (ST)^{-1}A \arrow[urr, dashed, "\exists! h"'] & & \end{tikzcd}$$

be the homomorphisms, $U = g_1(T) = S^{-1}T$. Let $s \in S$, $t \in T$, then $g_2g_1(st)$ is a unit. If $g_2g_1(a) = 0$, then there is $t/s \in S^{-1}U$, $(ta)/s = (t/s)a = 0$ in $S^{-1}A$, and again there is $s' \in S$ such that $s'ta = 0$ in A. Finally, every element of $U^{-1}(S^{-1}A)$ is of the form $g_2(a/s)g_2(t/s')^{-1} = g_1(a)g_1(s)^{-1}g_1(s')g_1(t)^{-1} = g_1(s'a)g_1(st)^{-1}$ for $a \in A$, $s, s' \in S$ and $t \in T$. So by Corollary 3.2 (the characterization of localization), there exists a unique isomorphism $h : (ST)^{-1}A \to U^{-1}(S^{-1}A)$. ■

Problem 4. Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Proof. We have a ring-homomorphism $g : S^{-1}A \to T^{-1}B$ induced from the universal property, given by $a/s \mapsto f(a)/f(s)$, and a $S^{-1}A$-linear map $h : S^{-1}B \to T^{-1}B$ given by $b/s \mapsto b/f(s)$. $S^{-1}f$ is clearly surjective. Let $b/s \in S^{-1}B$ be such that $b/f(s) = S^{-1}f(b/s) = 0$, then there is $f(s') = t \in T$, $f(s')b = tb = 0$, that is, $b/s = 0$. So $S^{-1}f$ is injective, hence it is an isomorphism. ■

Problem 5. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Proof. Let $x \in A$ be nilpotent in A, suppose $x \neq 0$. Let $\mathfrak{a} = \mathrm{Ann}(x) \neq (1)$. Then there is a maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$, and $x/1$ is nilpotent in $A_\mathfrak{m}$, so $x/1 = 0$ and there is $s \in A \setminus \mathfrak{m}$ such that $sx = 0$. But then $s \in \mathrm{Ann}(x) = \mathfrak{m}$, contradiction. So $x = 0$, A has no nonzero nilpotent element.

The statement for integrality is not necessarily true. For example, let $A = \mathbb{Z}/6\mathbb{Z}$, then the primes ideal are $\mathfrak{p}_1 = (2 + 6\mathbb{Z})$, $\mathfrak{p}_2 = (3 + 6\mathbb{Z})$, and $A \setminus \mathfrak{p}_1 = \{1 + 6\mathbb{Z}, 3 + 6\mathbb{Z}, 5+6\mathbb{Z}\}$, $A \setminus \mathfrak{p}_2 = \{1+6\mathbb{Z}, 2+6\mathbb{Z}, 4+6\mathbb{Z}, 5+6\mathbb{Z}\}$. If $aa'/ss' = 0$ in $A_{\mathfrak{p}_1}$, then there is $t \in A \setminus \mathfrak{p}_1$, $taa' = 0$ in A, easy to check that $ta = 0$ or $taa' = 0$. Similar argument for $A_{\mathfrak{p}_1}$ we see that both of them are integral. But A is clearly not integral. ■

Problem 6. Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A \setminus S$ is a minimal prime ideal of A.

Proof. Obviously we can apply Zorn's lemma on Σ. Let $S \in \Sigma$, then $S^{-1}A \neq 0$ (since $0 \notin S$), so there is a maximal $\mathfrak{m}$ in $S^{-1}A$, its contraction in A is prime, does not intersect S (i.e. $\mathfrak{m}^c \subseteq A \setminus S$). If $A \setminus S$ is not a minimal prime ideal, then there will be a smaller prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}^c$, but then $S \subsetneq A \setminus \mathfrak{p} \in \Sigma$ is strictly larger, contradiction. Conversely, let $\mathfrak{p}$ be a minimal prime ideal, then $A \setminus \mathfrak{p} \in \Sigma$. If $A \setminus \mathfrak{p}$ is not maximal, we can go through the above argument again to find a strictly smaller prime ideal $\subsetneq \mathfrak{p}$, so $A \setminus \mathfrak{p}$ must be maximal. ■

Problem 7. A multiplicatively closed subset S of a ring A is said to be *saturated* if

$$xy \in S \Leftrightarrow x \in S \text{ and } y \in S.$$

Prove that

i) S is saturated $\Leftrightarrow$ $A \setminus S$ is a union of prime ideals.

ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing S, and that $\overline{S}$ is the complement in A of the union of the prime ideals which do not meet S. ($\overline{S}$ is called the *saturation* of S.)
If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\overline{S}$.

Proof.

i) ($\Rightarrow$) Let $f : A \to S^{-1}A$ be the natural homomorphism. Then for all maximal ideal $\mathfrak{m}$ in $S^{-1}A$, $\bigcup_{\mathfrak{m}} \mathfrak{m} = S^{-1}A \setminus (S^{-1}A)^\times$. For $x \in A \setminus S$, $f(x) \notin (S^{-1}A)^\times$ (otherwise $t(aa' - ss') = 0$ for some $t \in S$, $taa' \notin S$ while $tss' \in S$), so $x \in f^{-1}(\mathfrak{m})$ for some $\mathfrak{m}$. Therefore, $A \setminus S = \bigcup_{\mathfrak{m}} f^{-1}(\mathfrak{m})$ is a union of prime ideals.
($\Leftarrow$) Obvious.

ii) Let $\mathfrak{p} \subseteq A \setminus S$, then $\mathfrak{p} + \mathfrak{a} \neq (1)$, there is a maximal ideal $\mathfrak{m} \supseteq \mathfrak{p} + \mathfrak{a}$. Besides, if $\mathfrak{m} \supseteq \mathfrak{a}$ is a maximal ideal, then $1 + \mathfrak{a} \subseteq 1 + \mathfrak{m}$ does not meet $\mathfrak{m}$. So $\overline{S} = A \setminus \bigcup_{\mathfrak{m} \supseteq \mathfrak{a}} \mathfrak{m}$. ■

[*Remark.* As an equivalent proof of i), we consider the following. For every $a \in A \setminus S$, $(a) \cap S = \varnothing$, so the set $\Sigma_a = \{(a) \subseteq \mathfrak{a} \subseteq A \setminus S\}$ (set of ideals) is not empty. We can apply Zorn's lemma on Σ_a to get a maximal element $\mathfrak{p}_a$. If $x, y \notin \mathfrak{p}_a$, $xy \in \mathfrak{p}_a$, by maximality of $\mathfrak{p}_a$, there are $ax + p_1 = s_1$, $by + p_2 = s_2$, but $(ax + p_1)(by + p_2) = s_1 s_2 \in \mathfrak{p}_a, S$, contradiction. So $\mathfrak{p}_a$ is a prime ideal, and $A \setminus S = \bigcup_{a \in A \setminus S} \Sigma_a$ is a union of prime ideals.]

Problem 8. Let S, T be multiplicatively closed subsets of A, such that $S \subseteq T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following are equivalent:

i) ϕ is bijective.

ii) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.

iii) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.

iv) T is contained in the saturation of S (Exercise 7).

v) Every prime ideal which meets T also meets S.

Proof. i) $\Rightarrow$ ii) For each $t \in T$, there is $a/s \in S^{-1}A$, such that $a/s = 1/t$, i.e. $at/1 = s/1$. Then there is $t' \in T$, $t'(at - s) = 0'$. From the injectivity of ϕ, there is $s' \in S$, $s'(at - s) = 0$. So in $S^{-1}A$, $(a/s)(t/1) = 1/1$, $t/1$ is a unit.

ii) $\Rightarrow$ iii) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$. So there is $a/s \in S^{-1}A$, $at/s = 1/1$, i.e. there is $s' \in S$, $s'(at - s) = 0$, and $(s'a)t = ss' \in S$.

iii) $\Rightarrow$ iv) Let $\mathfrak{p} \subseteq A \setminus S$ be a prime ideal. If $\mathfrak{p} \cap T \neq \varnothing$, let $t \in \mathfrak{p} \cap T$. Then there is $x \in A$, $xt \in S$. But $xt \in \mathfrak{p}$, contradiction.

iv) $\Rightarrow$ v) Obvious.

v) $\Rightarrow$ ii) $\Rightarrow$ i) Let $f : A \to S^{-1}A$ be the natural homomorphism, then for each $t \in T$, $f(t) = t/1$ is a unit in $S^{-1}A$ (otherwise it is contained in a maximal ideal, whose inverse image is prime in A, meets T but doesn't meet S. Cf. Remark of Exercise 7), that is, $at/s = 1/1$, $s'(at - s) = 0$, the surjectivity of ϕ follows. Then $s'(at - s) = (s'a)t - ss' = 0$ implies that for if $b/1 = 0$ in $T^{-1}A$ (that is, $tb = 0$ for some $t \in T$ and $b \in A$), then $(s'at)b = (ss')b$, $ss' \in S$, so $b/1 = 0$ in $S^{-1}A$, this show that ϕ is injective. ∎

Problem 9. The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A. Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14). Show that every minimal prime ideal of A is contained in D. [Use Exercise 6.]

The ring $S_0^{-1}A$ is called the *total ring of fractions* of A. Prove that

i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.

ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.

iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \to S_0^{-1}A$ is bijective).

Proof. If $xy \in S_0$, then $x, y \in S_0$, otherwise if $x \notin S_0$, there is $a \neq 0$, $ax = 0$, and $axy = 0$ implies $xy \notin S_0$. If $S \in \Sigma$ (notation in Exercise 6), then $0 \notin SS_0 \in \Sigma$ is a larger multiplicative subset. So every maximal element of Σ contained S_0, and every minimal prime ideal is contained in $A \setminus S_0 = D$.

i) Let S be a multiplicative subset such that the natural homomorphism $\phi : A \to S^{-1}A$ is injective. Let $x \in A$ be such that $x/1 = 0$ in $S^{-1}A$, then there is $s \in S$, $sa = 0$. So we see that $s \in S_0$ (otherwise s will annihilate some $a \neq 0$). Besides, if $S = S_0$, then ϕ is indeed injective, so S_0 is the largest multiplicative subset satisfying the condition.

ii) Let $x/s \in S_0^{-1}A$, then if $x \in S_0$, then x/s is a unit. If $x \in D$, let $ax = 0$ for some $a \neq 0$, then $(a/1)(x/s) = 0$. If $a/1 = 0$, there will be $s' \in S_0$ such that $s'a = 0$, but since $s' \in S_0$, we can only have $a = 0$, contradiction.

iii) Let A be such a ring, then $S_0 = A^\times$. By Exercise 8 iii) $\Rightarrow$ i) (take $S = \{1\}$, $T = S_0$), we see that $\phi : A \to S^{-1}A$ is bijective. ■

Problem 10. Let A be a ring.

i) If A is absolutely flat (Chapter 2, Exercise 27) and S is any multiplicatively closed subset of A, then $S^{-1}A$ is absolutely flat.

ii) A is absolutely flat $\Leftrightarrow$ $A_\mathfrak{m}$ is a field for each maximal ideal $\mathfrak{m}$.

Proof.

i) By Exercise 27 of Chapter 2, A is absolutely flat iff every principal in A is idempotent. Let $\mathfrak{a}' \subseteq S^{-1}A$, then $\mathfrak{a}' = S^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq A$ (e.g. $\mathfrak{a} = \mathfrak{a}'^c$). So $(S^{-1}\mathfrak{a})^2 = S^{-1}\mathfrak{a}$, and $S^{-1}A$ is absolutely flat.

ii) ($\Rightarrow$) For maximal ideal $\mathfrak{m}$, $A_\mathfrak{m}$ is a local ring. Since a local ring cannot have an idempotent element $\neq 0, 1$ (Exercise 12 of Chapter 1), and by i), every principal ideal of $A_\mathfrak{m}$ is generated by an idempotent element (since $(a)^2 = (a) \implies ca^2 = a \implies a = aca \in (ca) \subseteq (a)$), so $A_\mathfrak{m}$ has only two ideals, and hence it is a field. [The statement is true for localization $A_\mathfrak{p}$ at a prime ideal.]

($\Leftarrow$) For every $0 \neq x \in A$, if $\mathrm{Ann}(x) + (x) \neq (1)$, there will be a maximal ideal $\mathfrak{m} \supseteq \mathrm{Ann}(x) + (x)$. Then in $A_\mathfrak{m}$, $(x/1) \neq (0), (1)$ (verify by definition, namely, $x/1 = 0 \implies sx = 0 \implies s \in \mathrm{Ann}(x) \subseteq \mathfrak{m}$ a contradiction, and $x/1 = 0 \implies s'(ax - s) = 0 \implies x \notin \mathfrak{m}$ a contradiction), contradicts the fact that $A_\mathfrak{m}$ is a field. So $\mathrm{Ann}(x) + (x) = (1)$, then there are $a \in \mathrm{Ann}(x)$, $b \in A$, $a + bx = 1$, multiply by x we have $bx^2 = x$, therefore $(bx)^2 = (bx) = (x) = (x)^2$ is idempotent. ■

Problem 11. Let A be a ring. Prove that the following are equivalent:

i) $A/\mathfrak{N}$ is absolutely flat ($\mathfrak{N}$ being the nilradical of A).

ii) Every prime ideal of A is maximal.

iii) $\mathrm{Spec}(A)$ is a T_1-space (i.e., every subset consisting of a single point is closed).

iv) $\operatorname{Spec}(A)$ is Hausdorff.

If these conditions are satisfied, show that $\operatorname{Spec}(A)$ is compact and totally disconnected (i.e. the only connected subsets of $\operatorname{Spec}(A)$ are those consisting of a single point).

Proof. i) $\Rightarrow$ ii) Denote $A/\mathfrak{R}$ by B, then there is a naturally correspondence between prime ideals in A and prime ideals in B. Let $\mathfrak{p} \subseteq \mathfrak{m}$ be a prime ideal contained in a maximal ideal in B. Then we have a natural $\phi : B_\mathfrak{m} \hookrightarrow B_\mathfrak{p}$ (since they are fields by Exercise 10 ii), and homomorphisms between two fields are injective). We then prove the surjectivity of ϕ. Let $s \in B \setminus \mathfrak{p}$, if $s \in B \setminus \mathfrak{m}$, then $B_\mathfrak{m} \ni 1/s \mapsto 1/s \in B_\mathfrak{p}$. If $s \in \mathfrak{m}$, then $s/1 \neq 0$ in $B_\mathfrak{m}$ (otherwise there is $t \in B \setminus \mathfrak{m}$, $ts = 0$, this implies $t \in \mathfrak{p}$ or $s \in \mathfrak{p}$, but none of them is true), so s is a unit in $B_\mathfrak{m}$ (because non-zero elements are invertible in a field), and $s^{-1} \mapsto s^{-1}/1 = 1/s$. Hence ϕ is a bijection, by Exercise 8, $B \setminus \mathfrak{p}$ is contained in the the saturation of $B \setminus \mathfrak{m}$, but the saturation of $B \setminus \mathfrak{m}$ is itself, we must have $\mathfrak{m} \subseteq \mathfrak{p}$, and $\mathfrak{p} = \mathfrak{m}$. So every prime ideal in B is also maximal, so are prime ideals in A (by the natural correspondence).

ii) $\Rightarrow$ i) Without loss of generality, we may assume that the nilradical $\mathfrak{R}$ of A is zero.

Let $\mathfrak{m}$ be a maximal ideal, we want to show that for every $x \in \mathfrak{m}$, $0 \in x(1+\mathfrak{m})$. If this is not the case, let $x \in \mathfrak{m}$ be such that $0 \notin x(1+\mathfrak{m})$, then for any $n \in \mathbb{N}$, $0 \notin x^n(1+\mathfrak{m})$,[1] contradiction ($n = 0$ can be discussed separately). Let $T = \{x^n(1+y) \mid x, y \in \mathfrak{m}, n \in \mathbb{N}\}$ be a multiplicative subset, and $\overline{T}$ be its saturation. By Exercise 8 i) $\Leftrightarrow$ iv), the natural homomorphism $T^{-1}A \to \overline{T}^{-1}A$ is an isomorphism. However, for any prime ideal $\mathfrak{p} \subseteq A \setminus T \subseteq A \setminus (1+\mathfrak{m})$, we must have $\mathfrak{p} \subseteq \mathfrak{m}$,[2] but $x \in T$ implies $x \notin \mathfrak{p}$, therefore $\mathfrak{p} \subsetneq \mathfrak{m}$, and such prime ideal does not exist (we have assumed all prime ideals are maximal). So $\overline{T} = A \setminus \bigcup_{\mathfrak{p} \subseteq A \setminus T} \mathfrak{p} = A \setminus \varnothing = A$, and the natural homomorphism becomes an injection $T^{-1}A \hookrightarrow 0$, then $T^{-1}A = 0$. But this can happen iff $0 \in T$, contradiction. So for every $x \in \mathfrak{m}$, there is $y \in \mathfrak{m}$, $x(1+y) = 0$.

So for every $x \in \mathfrak{m}$, let $y \in \mathfrak{m}$ be such that $x(1+y) = 0$. Since $1 + \mathfrak{m} \subseteq A \setminus \mathfrak{m}$, we see that $x/1 = 0/1$ in $A_\mathfrak{m}$. Hence the image of $\mathfrak{m}$ in $A_\mathfrak{m}$ is (0), the only ideals in $A_\mathfrak{m}$ is (0) and (1), therefore $A_\mathfrak{m}$ is a field (because $A \neq 0$, $0 \notin A \setminus \mathfrak{m}$, we have $0/1 \neq 1/1$). By Exercise 10 ii), A is absolutely flat.

Remember we have supposed the nilradical of A is 0, but we can apply the above argument to $A/\mathfrak{R}$, since we have a natural correspondence between the prime or maximal ideals in A and those in $A/\mathfrak{R}$. [I am really a genius ha! How hard this direction is!!!]

i), ii) $\Rightarrow$ iv) Without loss of generality we may assume that $\mathfrak{R} = 0$, and we will work in A throughout this part. For two distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$, they are

[1] Otherwise if $x^n(1+y) = 0$, we have $(x(1+y))^n = 0 \implies x(1+y) = 0$ (since we assume that A has no nilpotent element $\neq 0$).

[2] Otherwise $p + m = 1$, i.e. $p \in 1 + \mathfrak{m}$, for some $p \in \mathfrak{p}$, $m \in \mathfrak{m}$.

maximal, so there is $\mathfrak{p}_1 + \mathfrak{p}_2 = (1)$, $p_1 + p_2 = 1$ for some $p_1 \in \mathfrak{p}_1$, $p_2 \in \mathfrak{p}_2$.[3] Since A is absolutely flat, by Exercise 27 ii) of Chapter 2, $(p_1^2) = (p_1)^2 = (p_1)$, i.e. there is $c \in A$, $cp_1^2 = p_1$, and cp_1 is then idempotent. So by Exercise 22 of Chapter 1, $A = A_1 \times A_2$, where we may assume that $A_1 = (cp_1) = (p_1)$. Recall that every prime ideal in $A_1 \times A_2$ is of the form $\mathfrak{q}_1 \times A_2$ or $A_1 \times \mathfrak{q}_2$ for some prime ideal $\mathfrak{q}_1 \subseteq A_1$ and $\mathfrak{q}_2 \subseteq A_2$. So if a prime ideal in $A_1 \times A_2$ doesn't contain p_1,[4] then it must be of the form $\mathfrak{q}_1 \times A_2$. Let $p_2' = p_2 - cp_1p_2 = (1-p_1)(1-cp_1) \in \mathfrak{p}_2$, then $p_2' \in \{0\} \times A_2$ (the motivation is to 'shift' p_2 slightly so that it is contained in A_2), this is because $p_1p_2' = p_1(1-p_1)(1-cp_1) = p_1(1-cp_1) = 0$ (cp_1 can be thought as the identity in A_1 and $1 - p_1$ a unit in A_2). Hence p_2' is 0 on the first coordinate, and the prime ideal $\mathfrak{q} \times A_2 \ni p_2'$. Therefore, we have two basic open set $\mathfrak{p}_1 \in \operatorname{Spec}(A) \setminus V(p_2')$ and $\mathfrak{p}_2 \in \operatorname{Spec}(A) \setminus V(p_1)$, they do not intersect each other, hence the space $\operatorname{Spec}(A)$ is Hausdorff. Further more, since $p_1 + p_2 = 1$, every prime ideal cannot both contain p_1 and p_2' (if some prime ideal does, then it contains $p_1 + p_2' + cp_1p_2 = 1$), so the two basic open sets actually also give a partition of the whole space $\operatorname{Spec}(A)$, and thus $\mathfrak{p}_1$ and $\mathfrak{p}_2$ lie in different connected components of $\operatorname{Spec}(A)$. [The basic ideal for this proof is all what we have done in Exercise 22 of Chapter 1.]

ii) $\Leftrightarrow$ iii) Let $\mathfrak{p}$ be any prime ideal, then it is also maximal. $V(\mathfrak{p}) = \{\mathfrak{p}\}$ is closed. For the opposite direction, let $\mathfrak{p} \in \operatorname{Spec}(A)$, then $\{\mathfrak{p}\}$ is closed, by definition, $\{\mathfrak{p}\} = V(I)$ where we must have $I \subseteq \mathfrak{p}$. But if $\mathfrak{p}$ is not maximal, it will be contained in a strictly larger prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$, then $\mathfrak{m} \in V(I)$, contradiction.

iii) $\Rightarrow$ iv) Have no idea, why don't we follow iii) $\Rightarrow$ i) $\Rightarrow$ i), ii) $\Rightarrow$ iv)?

iv) $\Rightarrow$ iii) A basic exercise in topology.

For the last part, if these conditions are satisfied, then $\operatorname{Spec}(A)$ is Hausdorff, but it is already quasi-compact, so it is compact. We have proved that it will be totally disconnected at the end of i), ii) $\Rightarrow$ iv). ∎

Problem 12. Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if $\operatorname{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

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[3] Of course this implies $p_1, p_2 \notin \mathfrak{R}$.

[4] Here p_1 is considered to be an element of $A_1 \times A_2$ by inclusion, the same for following.

[For iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 2, Exercise 15 and Exercise 20, show that if $1 \times x = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x = 0$.]

Proof. Let $x, y \in T(M)$, and let $a, b \in A \setminus \{0\}$ be such that $ax, by = 0$. Since A is integral, $ab \neq 0$, and $ab(x + y) = 0$. So $T(M)$ is indeed a submodule.

i) Let $x \in M$, such that x is a torsion element in $M/T(M)$. Then there is a nonzero $a \in A$, $ax \in T(M)$. Let $b \in A$, $b \neq 0$, $b(ax) = 0$. Then $(ba)x = 0$, that is, $x \in T(M)$, and $x = 0$ in $M/T(M)$.

ii) If $x \in T(M)$, let $a \in A$ nonzero, $ax = 0$. Then $f(ax) = af(x) = 0$, so $f(x) \in T(N)$.

iii) By ii) $0 \to T(M') \to T(M) \to T(M'')$ is still a sequence. The middle arrow is obvious, since it is always injective. Denote the right arrow by $f : M \to M''$. Let $x \in M$, $f(x) = 0$, then there is $x' \in M'$, $x' \mapsto x$. Since x is torsion, let $a \in A$, $a \neq 0$, then $ax' \mapsto ax = 0$, i.e. $ax' = 0$ (this is an injection). So $x' \in T(M')$, the sequence is exact.

iv) Follow the hint. We let the direct system to be $(A\xi_i, \mu_{ij})$, where the index set is $I = K^\times$, $\xi_i = i$. We define $i \leq j$ if and only if $\xi_i\xi_j^{-1} \in A$, and $\mu_{ij} : \xi_i \mapsto \xi_i$ for $i \leq j$. Let $\mu_i : \xi_i \mapsto \varinjlim A\xi_i$ be the inclusion, then we show $K = \varinjlim A\xi_i$. We let $\nu_i : \xi_i \mapsto \xi_i$ be the inclusion of $A\xi_i$ in K, obviously ν_i is compatible with the direct system, hence by universal property there is a A-linear map $h : \varinjlim A\xi_i \to K$. Obviously h is surjective, for if $x \in \varinjlim A\xi_i$, $h(x) = 0$, we may let $x = \mu_i(a\xi_i)$ for some i, then $\nu_i(a\xi_i) = 0$, hence $a\xi_i = 0$, and $x = 0$. So h is also injective.

Then $K \otimes_A M = \varinjlim(A\xi_i \otimes_A M)$. If $1 \otimes x = 0$ in $K \otimes_A M$, then $1 \otimes x = 0$ in some $A\xi_i \otimes M$. But $A\xi_i \otimes_A M \cong M$, and thus through the canonical isomorphism we see that $\xi_i^{-1}x = 0$, therefore $x \in T(M)$. ■

Problem 13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

i) M is torsion-free.

ii) $M_\mathfrak{p}$ is torsion-free for all prime ideals $\mathfrak{p}$.

iii) $M_\mathfrak{m}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Proof. The direction "$\supseteq$" is clear. Let $x \in M$, $a \in A \setminus \{0\}$, $s \in S$, such that $ax/s = 0$, i.e. $ax = 0$ (A is integral), $x \in T(M)$, $x/s \in S^{-1}(TM)$.

i) $\Rightarrow$ ii) $\Rightarrow$ iii) Clear.

iii) $\Rightarrow$ i) If $T(M) \neq 0$, let $x \in T(M)$, $x \neq 0$, then $\mathrm{Ann}(x) \neq (1)$. Let $\mathfrak{m} \supseteq \mathrm{Ann}(x)$ be a maximal ideal, and let $S = A \setminus \mathfrak{m}$. Then $0 = T(S^{-1}M) = S^{-1}(TM)$, we have an inclusion $T(M) \hookrightarrow S^{-1}(TM) = 0$. This shows that there is $s \in S$, $sx = 0$, but this can happen only if $s \in \mathrm{Ann}(x) \subseteq \mathfrak{m}$, contradiction. So $T(M) = 0$. ∎

Problem 14. Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Prove that $M = \mathfrak{a}M$. [Pass to the $A/\mathfrak{a}$-module $M/\mathfrak{a}M$ and use (3.8).]

Proof. $M/\mathfrak{a}M$ is an $A/\mathfrak{a}$-module. For every maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$, $(M/\mathfrak{a}M)_{\mathfrak{m}/\mathfrak{a}} = 0$ (because for every $x + \mathfrak{a}M \in M/\mathfrak{a}M$, there is $s + \mathfrak{a} \notin \mathfrak{m}/\mathfrak{a}$, $sm = 0$). By Proposition 3.8, $M/\mathfrak{a}M = 0$, i.e. $M = \mathfrak{a}M$. ∎

Problem 15. Let A be a ring, and let F be the A-module A^n. Show that every set of n generators of F is a basis of F. [Let $x_1, \dots, x_n$ be a set of generators and $e_1, \dots, e_n$ the canonical basis of F. Define $\phi : x_i \mapsto e_i$. Then ϕ is surjective and we have to prove that it is an isomorphism. By (3.9) we may assume that A is a local ring. Let N be the kernel of ϕ and let $k = A/\mathfrak{m}$ be the residue field of A. Since F is a flat A-module, the exact sequence $0 \to N \to F \to F \to 0$ gives an exact sequence $0 \to k \otimes N \to k \otimes F \stackrel{1\otimes\phi}{\to} k \otimes F \to 0$. Now $k \otimes F = k^n$ is an n-dimensional vector space over k; $1 \otimes \phi$ is surjective, hence bijective, hence $k \otimes N = 0$.

Also N is finitely generated, by Chapter 2, Exercise 12, hence $N = 0$ by Nakayama's lemma. Hence ϕ is an isomorphism.]

Deduce that every set of generators of F has at least n elements.

Proof. Follow the hints. Let $x_1, \dots, x_n$ be the generators, and let $e_1, \dots, e_n$ be the identities in each coordinate. Let $\phi : F \to F$ by $e_i \mapsto x_i$, and N be its kernel. We may assume that A is a local ring by Proposition 3.9, let $k = A/\mathfrak{m}$, where $\mathfrak{m}$ is the only maximal ideal. Then we have an exact sequence $0 \to N \to F \stackrel{\phi}{\to} F \to 0$, since F is flat, by Exercise 25 of Chapter 2, it induces an exact sequence $0 \to k \otimes_A N \to k \otimes_A F \stackrel{1\otimes\phi}{\to} k \otimes_A F \to 0$. But as A-modules $k \otimes_A F \cong k^n$, we may viewed it as a n-dimensional vector space over k (since $\mathfrak{m} \subseteq \mathrm{Ann}(k^n)$). From the sequence we see that $1 \otimes \phi$ is surjective, it is an k-linear map between two vector spaces, hence it is bijective, and then $0 = k \otimes N \cong N/\mathfrak{m}N$. By Exercise 12 of Chapter 2, N is finitely generated, $\mathfrak{m}$ is contained in the Jacobson radical (the ring A is local by assumption), by Nakayama's lemma, $N = 0$. So ϕ is an isomorphism.

If $x_1, \dots, x_m$ with $m < n$, we can add some arbitrary elements to the generates so that it has n elements. Then these n generators form a basis of F, we have an isomorphism of F to itself, given by $x_i \mapsto e_i$. This shows that if we pick out any of these n generators, they won't generate F any more, so $m \geq n$, contradiction. ∎

Problem 16. Let B be a flat A-algebra. Then the following conditions are equivalent:

i) $\mathfrak{a}^{\mathrm{ec}} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A.

ii) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

iii) For every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^{\mathrm{e}} \neq (1)$.

iv) If M is any non-zero A-module, then $M_B \neq 0$.

v) For every A-module M, the mapping $x \mapsto 1 \otimes x$ is injective.

[For i) $\Rightarrow$ ii), use (3.16). ii) $\Rightarrow$ iii) is clear.
iii) $\Rightarrow$ iv): Let x be a non-zero element of M an let $M' = Ax$. Since B is flat over A it is enough to show that $M'_B \neq 0$. We have $M' \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a} \neq (1)$, hence $M'_B \cong B/\mathfrak{a}^{\mathrm{e}}$. Now $\mathfrak{a} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, hence $\mathfrak{a}^{\mathrm{e}} \subseteq \mathfrak{m}^{\mathrm{e}} \neq (1)$. Hence $M'_B \neq 0$.
iv) $\Rightarrow$ v): Let M' be the kernel of $M \to M_B$. Since B is flat over A, the sequence $0 \to M'_B \to M_B \to (M_B)_B$ is exact. But (Chapter 2, Exercise 13, with $N = M_B$) the mapping $M_B \to (M_B)_B$ is injective, hence $M'_B = 0$ and therefore $M' = 0$.
v) $\Rightarrow$ i): Take $M = A/\mathfrak{a}$.]

B is said to be *faithfully flat* over A.

Proof. Let $f : A \to B$ be the ring homomorphism that the A-algebra B equipped with.

i) $\Rightarrow$ ii) By Proposition 3.16, every prime ideal $\mathfrak{p}$ of A is the contraction of a prime of B. So the map $\mathfrak{q} \mapsto \mathfrak{q}^{\mathrm{c}}$ of prime ideals of B is surjective, and the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced by f is surjective.

ii) $\Rightarrow$ iii) Every prime ideal of A is contracted from a prime ideal of B, and since $\mathfrak{b}^{\mathrm{ce}} \subseteq \mathfrak{b}$, everything is clear.

iii) $\Rightarrow$ iv) Let $x \in M$ be a nonzero element, and let $N = Ax$. Since B is a flat A-module, we have an injection $B \otimes_A N \hookrightarrow B \otimes_A M$. Since $N \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A, $B \otimes_A (A/\mathfrak{a}) \cong B/\mathfrak{a}B$. Besides, $f(\mathfrak{a}) \subseteq \mathfrak{a}^{\mathrm{e}} \neq (1)$, so $B/\mathfrak{a}B \neq 0$ as an A-module, and hence $M_B \neq 0$.

iv) $\Rightarrow$ v) If for some A-module M, the map $x \mapsto 1 \otimes x$ is not injective, let N be its kernel. Since B is flat, the inclusion $N \hookrightarrow M$ gives rise to the inclusion $N_B \hookrightarrow M_B$. So we have a sequence $N \to N_B \hookrightarrow M_B$, which sends $x \mapsto 1 \otimes x \mapsto 1 \otimes x = 0$ in M_B. Hence $N_B = 0$. But $N \neq 0$, this contradicts iv).

v) $\Rightarrow$ i) Let $\mathfrak{a}$ be an ideal of A, and let $M = A/\mathfrak{a}$. Then we have an injection $g : A/\mathfrak{a} \hookrightarrow B \otimes_A (A/\mathfrak{a}) \cong B/\mathfrak{a}^{\mathrm{e}}$. But since $\mathfrak{a}^{\mathrm{ece}} = \mathfrak{a}^{\mathrm{e}}$, $g(\mathfrak{a}^{\mathrm{ec}}/\mathfrak{a}) = 0$, hence $\mathfrak{a}^{\mathrm{ec}} = \mathfrak{a}$. ■

Problem 17. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

Proof. Let $0 \to M' \to M$ be an injection, then we have an injection $0 \to B \otimes_A M \to C \otimes_B (B \otimes_A M)$, similar for M'. So we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C \otimes_B (B \otimes_A M') \arrow[r] & C \otimes_B (B \otimes_A M) \\ & B \otimes_A M' \arrow[u, hook] \arrow[r] & B \otimes_A M \arrow[u, hook] \end{tikzcd}$$

The exactness of the top row is provided by $C \otimes_B (B \otimes_A M) = (C \otimes_B B) \otimes_A M \cong C \otimes_A M$ and the one for M'. It is then easy to see that the bottom row is also an injection, so f is flat. ■

Problem 18. Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective. [For $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by (3.10), and $B_\mathfrak{q}$ is a local ring of $B_\mathfrak{p}$, hence is flat over $B_\mathfrak{p}$. Hence $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ and satisfies condition (3) of Exercise 16.]

Proof. Let $S = A \setminus \mathfrak{p}$, $T = B \setminus \mathfrak{q}$. f induces a ring homomorphism $f' : A_\mathfrak{p} \to B_\mathfrak{q}$ (this is well defined, since $f(S) \subseteq T$). Let M be an $A_\mathfrak{p}$-module, since $S^{-1}(M \otimes_A B) \cong M_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} = M \otimes_{A_\mathfrak{p}} B_\mathfrak{p}$, by Proposition 3.3, $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Since $B_\mathfrak{q} = (B_\mathfrak{p})_\mathfrak{q}$, $B_\mathfrak{q}$ is a flat $B_\mathfrak{p}$-module, hence is a flat $A_\mathfrak{p}$-module, equipped with f'. By Proposition 3.11 iv), for every maximal ideal in $A_\mathfrak{p}$, it is of the form $S^{-1}\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal in A and doesn't meet S. Since $f(\mathfrak{m}) \subseteq \mathfrak{q}$, $f'(S^{-1}\mathfrak{m})^e = T^{-1}B\mathfrak{m} \subseteq \mathfrak{q}$, it doesn't meet T, hence $(f(\mathfrak{m}))^e \neq (1)$. By Exercise 16 iii) $\Rightarrow$ ii), the $f^* : \operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective. ■

Problem 19. Let A be a ring, M an A-module. The *support* of M is defined to be the set of $\operatorname{Supp}(M)$ of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. Prove the following results:

i) $M \neq 0 \Leftrightarrow \operatorname{Supp}(M) \neq \varnothing$.

ii) $V(\mathfrak{a}) = \operatorname{Supp}(A/\mathfrak{a})$.

iii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$.

iv) If $M = \sum M_i$, then $\operatorname{Supp}(M) = \bigcup \operatorname{Supp}(M_i)$.

v) If M is finitely generated, then $\operatorname{Supp}(M) = V\big(\operatorname{Ann}(M)\big)$ (and is therefore a closed subset of $\operatorname{Spec}(A)$).

vi) If M, N are finitely generated, then $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$. [Use Chapter 2, Exercise 3.]

vii) If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\operatorname{Supp}(M/\mathfrak{a}M) = V\big(\mathfrak{a} + \operatorname{Ann}(M)\big)$.

viii) If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $\operatorname{Supp}(B \otimes_A M) = f^{*-1}\big(\operatorname{Supp}(M)\big)$.

Proof.

i) Since $M_\mathfrak{p} = 0 \iff A \setminus \mathfrak{p} \cap \operatorname{Ann}(x) \neq \varnothing$ for every $x \in M$. If $M \neq 0$, let $x \in M$ and $x \neq 0$, then $\operatorname{Ann}(x) \neq (1)$, let $\mathfrak{m} \supseteq \operatorname{Ann}(x)$ be a maximal ideal. Then for if $a \in A \setminus \mathfrak{m}$, $ax = 0$ implies $a \in \operatorname{Ann}(x)$, contradiction. Hence $M_\mathfrak{m} \neq 0$, and $\operatorname{Supp}(M) \neq \varnothing$. The reverse is trivial.

ii) Let $\mathfrak{p} \subseteq A$ be a prime ideal. If $\mathfrak{a} \subseteq \mathfrak{p}$, let $(A/\mathfrak{a})_\mathfrak{p} \neq 0$, otherwise there is $b \in A \setminus \mathfrak{p}$, $b(1 + \mathfrak{a}) = 0$, this implies $b \in \mathfrak{a} \subseteq \mathfrak{p}$, contradiction. If $\mathfrak{a} \not\subseteq \mathfrak{p}$, let $b \in \mathfrak{a} \setminus \mathfrak{p}$, then for every $a \in A$, $b(a + \mathfrak{a}) = 0 + \mathfrak{a}$, hence $(A/\mathfrak{a})_\mathfrak{p} = 0$. So $V(\mathfrak{a}) = \operatorname{Supp}(A/\mathfrak{a})$.

iii) For every prime ideal $\mathfrak{p} \subseteq A$, we have an exact sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$. From this we see that $M_\mathfrak{p} = 0$ iff $M'_\mathfrak{p} = M''_\mathfrak{p} = 0$. Hence $\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$.

iv) By the remark in i), $\operatorname{Supp}(M) = \operatorname{Supp}\big(\bigoplus_i M_i\big) = \bigcup_i \operatorname{Supp}(M_i)$.

v) By ii), if $M = A/\mathfrak{a}$ is generated by one element, $\operatorname{Supp}(A/\mathfrak{a}) = V(\mathfrak{a})$. By iv), let $M = \sum_i M_i$, where $M_i \cong A/\mathfrak{a}_i$ is generated by an element, then $\operatorname{Supp}(M) = \bigcup_i \operatorname{Supp}(A/\mathfrak{a}_i) = \bigcup_i V(\mathfrak{a}_i) = V\big(\prod_i \mathfrak{a}_i\big) = V\big(\bigcap_i \mathfrak{a}_i\big) = V(\operatorname{Ann}(M))$.

vi) By Exercise 3 of Chapter 2, if for prime ideal $\mathfrak{p}$, $(M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} = 0$, then either $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$. Hence $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(N) \cap \operatorname{Supp}(N)$.

vii) By v), $\operatorname{Supp}(M/\mathfrak{a}M) = V(\operatorname{Ann}(M/\mathfrak{a}M)) = V(\mathfrak{a} + \operatorname{Ann}(M))$.

viii) ($\subseteq$) Let $\mathfrak{p} \subseteq A$ be a prime ideal, such that $M_\mathfrak{p} = 0$, and let $\mathfrak{q} \in f^{*-1}(\mathfrak{p})$. Then $(B \otimes_A M)_\mathfrak{q} = B_\mathfrak{q} \otimes_B B \otimes A = B_\mathfrak{q} \otimes_A M$. Since $(B_\mathfrak{q})_\mathfrak{p} = B_\mathfrak{q}$, $B_\mathfrak{q} \otimes_A M = (B_\mathfrak{q})_\mathfrak{p} \otimes_A M = (B_\mathfrak{q} \otimes_A M)_\mathfrak{q} = B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0$. [Particular, we know that M is finitely generated, as another proof, this result is immediately from v), since $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(B \otimes_A M)$.]

($\supseteq$) Let $\mathfrak{q} \subseteq B$ be a prime ideal and $\mathfrak{p} = f^*(\mathfrak{q})$, such that $(B \otimes_A M)_\mathfrak{q} = B_\mathfrak{q} \otimes_A M = 0$. Let $k(\mathfrak{q}) = B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ and $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be the residue field at $\mathfrak{q}$ and $\mathfrak{p}$ respectively. Then we have $k(\mathfrak{q}) \otimes_A M = (B/\mathfrak{q}) \otimes_B B_\mathfrak{q} \otimes_A M = 0$. Notice that the map $A \to B$ induces a map $A/\mathfrak{p} \to B/\mathfrak{q}$, and this map induces $k(\mathfrak{p}) \to k(\mathfrak{q})$, all of them are well-defined, and the last one endows $k(\mathfrak{q})$ with a $k(\mathfrak{p})$-algebra.
Since $k(\mathfrak{p}) \otimes_A M$ is a finite dimensional $k(\mathfrak{p})$-vector space, $k(\mathfrak{p}) \otimes_A M \cong \big(k(\mathfrak{p})\big)^n$,

and then $0 = k(\mathfrak{q}) \otimes_A M = k(\mathfrak{q}) \otimes_{k(\mathfrak{p})} \big(k(\mathfrak{p}) \otimes_A M\big) = \big(k(\mathfrak{q}) \otimes_{k(\mathfrak{p})} k(\mathfrak{p})\big)^n$. But $k(\mathfrak{q}) \otimes_{k(\mathfrak{p})} k(\mathfrak{p}) = k(\mathfrak{q}) \neq 0$, hence $k(\mathfrak{p}) \otimes_A M = 0$ and $(A_\mathfrak{p} \otimes_A M)/\mathfrak{p}(A_\mathfrak{p} \otimes_A M) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A M = k(\mathfrak{p}) \otimes_A M = 0$. Since $\mathfrak{p}A_\mathfrak{p}$ is the only maximal ideal in the local ring $A_\mathfrak{p}$, by Nakayama's lemma, $M_\mathfrak{p} = A_\mathfrak{p} \otimes_A M = 0$.[5] ■

Problem 20. Let $f : A \to B$ be a ring homomorphism, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated mapping. Show that

i) Every prime ideal of A is a contracted ideal $\Leftrightarrow$ f^* is surjective.

ii) Every prime ideal of B is an extended ideal $\Rightarrow$ f^* is injective.

Is the converse of ii) true?

Proof.

i) Both ways are just very very trivial, they follow from how we define f^*.

ii) Let $\mathfrak{q}_1, \mathfrak{q}_2 \subseteq B$ be two prime ideals such that $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2) = \mathfrak{p}$. Since both $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are extended, $ff^*(\mathfrak{q}_1) = \mathfrak{q}_1$, $ff^*(\mathfrak{q}_2) = \mathfrak{q}_2$, hence $\mathfrak{q}_1 = \mathfrak{q}_2$.

For the converse of ii), it may not be true. For example, let $A = B = \mathbb{F}_p[t]$, $f : t \mapsto t^p$. Then every extended ideal is not prime (except the zero one). But given an irreducible polynomial $g(t)$, we have $g(t^p) = (g(t))^p \in (g(t))$, and $g(t) \in f^*((g(t)))$, since $\mathbb{F}_p[t]$ is a PID, we must have $f^*((g(t))) = (g(t))$. This shows that f^* is injective, but not every prime ideal of B is an extended ideal. ■

Problem 21.

i) Let A be a ring, S a multiplicatively closed subset of A, and $\phi : A \to S^{-1}A$ the canonical homomorphism. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto its image in $X = \mathrm{Spec}(A)$. Let this image be denoted by $S^{-1}X$.
In particular, if $f \in A$, the image of $\mathrm{Spec}(X_f)$ in X is the basic open set X_f (Chapter 1, Exercise 17).

ii) Let $f : A \to B$ be a ring homomorphism. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, and let $f^* : Y \to X$ be the mapping associated with f. Identifying $\mathrm{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in$X$, and $\mathrm{Spec}(S^{-1}B)$ ($= \mathrm{Spec}(f(S)^{-1}B)$) with its canonical image in $S^{-1}Y$, show that $S^{-1}f^* : \mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = f^{*-1}(S^{-1}X)$.

[5] Reference: The Stacks project "Section 080S(10.40): Supports and annihilators." *The Stacks project*, `https://stacks.math.columbia.edu/tag/080S`, 2024. Lemma 0BUR(10.40.6).

iii) Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^{\mathrm{e}}$ be its extension in B. Let $\bar{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the homomorphism induced by f. If $\mathrm{Spec}(A/mfka)$ is identified with its canonical image $V(\mathfrak{a})$ in X, and $\mathrm{Spec}(B/\mathfrak{b})$ with its image $B(\mathfrak{b})$ in Y, show that $\bar{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$.

iv) Let $\mathfrak{p}$ be a prime ideal of A. Take $S = A \setminus \mathfrak{p}$ in ii) and then reduce $\mathrm{mod} S^{-1}\mathfrak{p}$ as in iii). Deduce that the subspace $f^{*-1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\mathrm{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_{\mathfrak{p}}$.
$\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$ is called the *fiber* of f^* over $\mathfrak{p}$.

Proof.

i) Every point (prime ideal) in $S^{-1}A$ is an extended ideal of a prime ideal $\mathfrak{p} \subseteq A$, so ϕ^* is injective. By Exercise 21 i) of Chapter 1, ϕ^* is continuous, so we only need to verify that ϕ^* maps basic open sets to basic open sets. Let $f/s \in S^{-1}A$, then $\phi^*\big((\mathrm{Spec}(S^{-1}A))_{f/s}\big) = \phi^*\big((\mathrm{Spec}(S^{-1}A))_{f/1}\big) = \{\mathfrak{p} \in X \mid \mathfrak{p} \subseteq A \setminus S, f \notin \mathfrak{p}\} = X_f \cap S^{-1}X$.

ii) For every prime ideal in $S^{-1}B$, we can lift it into a prime ideal in B, then pass through f^* to get a prime ideal in $S^{-1}A$.

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r, "f^*"] & \mathrm{Spec}(A) \\ \mathrm{Spec}(S^{-1}B) \arrow[u, hook] \arrow[r, "S^{-1}f^*"'] & \mathrm{Spec}(S^{-1}A) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "f^*"] & X \\ S^{-1}Y \arrow[u, hook] \arrow[r, "S^{-1}f^*"'] & S^{-1}X \arrow[u, hook] \end{tikzcd}$$

If a prime ideal in Y meets $f(S)$, then its image under f^* meets S, hence $S^{-1}Y = f^{*-1}(S^{-1}X)$.

iii) The same as ii).

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r, "f^*"] & \mathrm{Spec}(A) \\ \mathrm{Spec}(B/\mathfrak{b}) \arrow[u, hook] \arrow[r, "\bar{f}^*"'] & \mathrm{Spec}(A/\mathfrak{a}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "f^*"] & X \\ V(\mathfrak{b}) \arrow[u, hook] \arrow[r, "\bar{f}^*"'] & V(\mathfrak{a}) \arrow[u, hook] \end{tikzcd}$$

But we don't have the corresponding result of ii).

iv) Since $\mathfrak{p}^e = \mathfrak{p}B$, $S^{-1}\mathfrak{p}^e = S^{-1}\mathfrak{p}B = \mathfrak{p}B_\mathfrak{p}$, we have the diagram

$$\begin{array}{ccc}
\operatorname{Spec}(B) & \xrightarrow{f^*} & \operatorname{Spec}(A) \\
\uparrow & & \uparrow \\
\operatorname{Spec}(B_\mathfrak{p}) & \xrightarrow{S^{-1}f^*} & \operatorname{Spec}(A_\mathfrak{p}) \\
\uparrow & & \uparrow \\
\operatorname{Spec}(\frac{B_\mathfrak{p}}{\mathfrak{p}B_\mathfrak{p}}) & \xrightarrow{\overline{S^{-1}f}^*} & \operatorname{Spec}(\frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}})
\end{array}$$

Let $\mathfrak{q} \subseteq B$ be a prime ideal, such that $f^*(\mathfrak{q}) = \mathfrak{p}$. Then $f(\mathfrak{p}) \subseteq \mathfrak{q}$, and $\mathfrak{p}B \subseteq \mathfrak{q}$. Regarded as an element in $\operatorname{Spec}(B_\mathfrak{p})$, $\mathfrak{p}B_\mathfrak{p} \subseteq \mathfrak{q}B_\mathfrak{p}$. Conversely, given a prime ideal $\mathfrak{q} \subseteq B$ such that $\mathfrak{p}B_\mathfrak{p} \subseteq \mathfrak{q}B_\mathfrak{p}$ and is also prime (proper), then $f(\mathfrak{p}) \subseteq \mathfrak{q}$, $f^*(\mathfrak{q}) \supseteq \mathfrak{p}$, hence it has to be equal (otherwise $f^*(\mathfrak{q})$ is not a prime ideal in A, contradicts what we have proved in ii) on restriction). So we gave a bijection between subspace of $S^{-1}Y$ of which each point contains $\mathfrak{p}B_\mathfrak{p}$, and the subspace of Y of which each point contains $f(\mathfrak{p})$ and does not meet $f(A\backslash\mathfrak{p})$. From ii) we know that $\operatorname{Spec}(B_\mathfrak{p})$ is homeomorphic to $S^{-1}Y$, so this gives a homeomorphism from $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ to the subspace $f^{*-1}(\mathfrak{p})$ of Y. ■

Problem 22. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the canonical image of $\operatorname{Spec}(A_\mathfrak{p})$ in $\operatorname{Spec}(A)$ is equal to the intersection of all the open neighborhoods of $\mathfrak{p}$ in $\operatorname{Spec}(A)$.

Proof. The canonical image of $\operatorname{Spec}(A_\mathfrak{p})$ in $\operatorname{Spec}(A)$ is all prime ideals contained in $\mathfrak{p}$, and is precisely $\bigcap_{U\ni\mathfrak{p}} U = \bigcap_{f\notin\mathfrak{p}} X_f$. ■

Problem 23. Let A be a ring, let $X = \operatorname{Spec}(A)$ and let U be a basic open set in X (i.e., $U = X_f$ for some $f \in A$: Chapter 1, Exercise 17).

i) If $U = X_f$, show that the ring $A(U) = A_f$ depends only on U and not on f.

ii) Let $U' = X_g$ be another basic open set such that $U' \subseteq U$. Show that there is an equation of the form $g^n = uf$ for some integer $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho : A(U) \to A(U')$ (i.e. $A_f \to A_g$) by mapping a/f^m to ay^m/g^{mn}. Show that ρ depends only on U and U'. This homomorphism is called the *restriction* homomorphism.

iii) If $U = U'$, then ρ is the identity map.

iv) If $U \supseteq U' \supseteq U''$ are basic open sets in X, show that the diagram

$$\begin{tikzcd}
A(U) \arrow[rr] \arrow[dr] & & A(U'') \\
& A(U') \arrow[ur] &
\end{tikzcd}$$

(in which the arrows are restriction homomorphisms) is commutative.

v) Let x $(= \mathfrak{p})$ be a point of X. Show that

$$\varinjlim_{U \ni x} A(U) \cong A_{\mathfrak{p}}.$$

The assignment of the ring $A(U)$ to each basic open set U of X, and the restriction homomorphisms ρ, satisfying the conditions iii) and iv) above, constitutes a *presheaf of rings* on the basis of open sets $(X_f)_{f \in A}$. v) says that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

Proof.

i) Let $X_f = X_g$, this implies $r(f) = r(g)$, so there is $m, n \in \mathbb{N}$, $a, b \in A$, such that $ag = f^m$, $bf = g^n$. We define $\phi : A_f \to A_g$ by $1/f \mapsto b/g^n$, and $\psi : A_g \to A_f$ by $1/g \mapsto a/f^m$. Then $\psi\phi : 1/f \mapsto b/g^n \mapsto ba^n/f^{mn} = 1/f$, $\psi\phi = \mathrm{id}_{A_f}$, and similarly $\phi\psi = \mathrm{id}_{A_g}$. Hence $A_f \cong A_g$. [We can also use Exercise 8 iv) to convince ourselves, in this case the saturation of f and g the same.]

ii) $X_g = U' \subseteq U = X_f$ implies $g \in r(f)$, so $g^n = uf$ for some $n \in \mathbb{N}$ and $u \in A$. By i) the map ρ does not depend on the choices of f, g, so it only depends on U, U'.

iii) By the remark of ii), we may choose $g = f$, then ρ is clearly the identity map.

iv) Let $U = X_f$, $U' = X_g$, $U'' = X_h$. Then $h^m = ag$, $g^n = bf$, and $h^{mn} = a^n bf$. $A(U) \to A(U') \to A(U'')$ maps $1/f \mapsto b/g^n \mapsto ba^n/h^{nm}$, $A(U) \to A(U'')$ maps $1/f \mapsto a^n b/h^{mn}$. Since they don't depend on the choice of f, g, h, they commute.

v) For basic open sets $U' \subseteq U$, we denote the restriction map by $\rho_{UU'}$. Let $\mathfrak{p} \in U$ be a basic open set, let $\rho_{X_f} : A(X_f) \to \varinjlim_{U \ni x} A(U)$ and $\sigma_{X_f} : A(X_f) \to A_{\mathfrak{p}}$ be the natural maps. By universal property we have an induced homomorphism $h : \varinjlim_{U \ni x} A(U) \to A_{\mathfrak{p}}$, it is clearly surjective. Since every element in $\varinjlim_{u \ni x} A(U)$ is of the form $\rho_{X_f}(x/f^n)$. If $\sigma_{X_f}(x/1) = 0$ for some $x \in A$, there is $g \in A \backslash \mathfrak{p}$, $gx = 0$. Then $\rho_{X_f X_{fg}}(x/1) = 0$, and $\rho_{X_f}(x/1) = \rho_{X_{fg}} \rho_{X_f X_{fg}}(x/1) = 0$, so h is also injective, it is an isomorphism.

[Recall that here U is a basic open set, but the result is still true if we let U to be any open set, the associated ring will be constructed by inverse limits. The idea is very similar to the proof of Exercise 12 iv).] ∎

Problem 24. Show that the presheaf of Exercise 23 has the following property. Let $(U_i)_{i \in I}$ be a covering of X by basic open sets. For each $i \in I$ let $s_i \in A(U_i)$ be such that, for each pair of indices i, j, the images of s_i and s_j in $A(U_i \cap U_j)$ are equal.

Then there exists a unique $s \in A$ $(= A(X))$ whose image in $A(U_i)$ is s_i, for all $i \in I$. (This essentially implies that the presheaf is a *sheaf*.)

Proof. Since the space X is quasi-compact, we may assume that the open covering $(U_i)_{i=1}^n$ is finite.[6] Let $U_i = X_{f_i}$, since $U_i \cap U_j = X_{f_i f_j}$, $s_i|_{X_{f_i f_j}} = s_j|_{X_{f_i f_j}}$, there exists N_{ij} such that $(f_i f_j)^{N_{ij}} s_i = (f_i f_j)^{N_{ij}} s_j$ in A.[7] Let $N = \max_{i,j} N_{ij}$, from $\sum_{i=1}^n (f_i^N) = 1$ we see that there are $a_i \in A$, $\sum_{i=1}^n a_i f_i^N = 1$. Let $s = \sum_{i=1}^n a_i f_i^N s_i$, then for every X_{f_k},

$$f_k^N s|_{X_{f_k}} = \sum_{i=1}^n a_i (f_i f_k)^N s_i = \sum_{i=1}^n a_i (f_i f_k)^N s_k = f_k^N s_k \sum_{i=1}^n a_i f_i^N = f_k^N s_k,$$

hence $s|_{X_{f_i}} = s_i|_{X_{f_i}}$.

It is unique, let $s \in A$ such that $s|_{X_{f_i}} = 0$ for all i, this means there is N_i, $f_i^{N_i} s = 0$. Since $\sum_{i=1}^n (f^{N_i}) = (1)$, there are $b_i \in A$, $\sum_{i=1}^n b_i f^{N_i} = 1$, then $s = \sum_{i=1}^n b_i f^{N_i} s = 0$. ■

Problem 25. Let $f : A \to B$, $g : A \to C$ be ring homomorphisms and let $h : A \to B \otimes_A C$ be defined by $h(x) = f(x) \otimes g(x)$. Let X, Y, Z, T be the prime spectra of $A, B, C, B \otimes_A C$ respectively. Then $h^*(T) = f^*(Y) \cap g^*(Z)$.
[Let $\mathfrak{p} \in X$, and let $k = k(\mathfrak{p})$ be the residue field at $\mathfrak{p}$. By Exercise 21, the fiber $f^{*-1}(\mathfrak{p})$ is the spectrum of $(B \otimes_A C) \otimes_A k \cong (B \otimes_A k) \otimes_k (C \otimes_A k)$. Hence $\mathfrak{p} \in h^*(T)$ $\Leftrightarrow (B \otimes_A k) \otimes_k (C \otimes_A k) \neq 0 \Leftrightarrow B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0 \Leftrightarrow \mathfrak{p} \in f^*(Y) \cap g^*(Z)$.]

Proof. Follow the hints.

$$\begin{tikzcd} B \arrow[r] & B \otimes_A C \\ A \arrow[u, "f"] \arrow[ur, "h"] \arrow[r, "g"'] & C \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d, "f^*"'] & T \arrow[l] \arrow[dl, "h^*"'] \arrow[d] \\ X & Z \arrow[l, "g^*"] \end{tikzcd}$$

Let $\mathfrak{p} \in X$, and let $k = k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be the residue field at $\mathfrak{p}$. Then Exercise 21 iv) implies that $h^{*-1}(\mathfrak{p}) \neq \varnothing \iff \operatorname{Spec}((B \otimes_A C) \otimes_A k) \neq \varnothing \iff (B \otimes_A C) \otimes_A k \neq 0$. Since $(B \otimes_A C) \otimes_A k \cong (B \otimes_A k) \otimes_k (C \otimes_A k)$, $B \otimes_A k$ and $C \otimes_A k$ can be viewed as vector spaces over k, $h^{*-1}(\mathfrak{p}) \neq \varnothing \iff B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0 \iff f^{*-1}(\mathfrak{p}) \neq \varnothing$ and $g^{*-1}(\mathfrak{p}) \neq \varnothing$. ■

Problem 26. Let $(B_\alpha, g_{\alpha\beta})$ be a direct system of rings and B the direct limit. For each α, let $f_\alpha : A \to B_\alpha$ be a ring homomorphism such that $g_{\alpha\beta} f_\alpha = f_\beta$ whenever

[6] And we can do this, the unique s of this finite subcovering also agrees with every other basic open sets that we threw away, since every such s is unique with respect to each finite subcovering.

[7] It is ambiguous on the notation, since they are not elements in A, but they actually are when canceling out the numerators, this follows from the fact that they agree on their intersection. We will use this ambiguous notation throughout the proof.

$\alpha \leq \beta$ (i.e. the B_α form a direct system of A-algebras). The f_α induces $f : A \to B$. Show that

$$f^*(\mathrm{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha)).$$

[Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then $f^{*-1}(\mathfrak{p})$ is the spectrum of

$$B \otimes_A k(\mathfrak{p}) \cong \varinjlim(B_\alpha \otimes_A k(\mathfrak{a}))$$

(since tensor products commute with direct limits: Chapter 2, Exercise 20). By Exercise 12 of Chapter 2 it follows that $f^{*-1}(\mathfrak{p}) = \varnothing$ if and only if $B_\alpha \otimes_A k(\mathfrak{p}) = 0$ for some α, i.e., if and only if $f_\alpha^{*-1}(\mathfrak{p}) = \varnothing$.]

Proof. The same proof as Exercise 25. Let $g_\alpha : B_\alpha \to B$ be the canonical map.

$$\begin{tikzcd}
 & \varinjlim_\alpha B_\alpha & \\
B_\alpha \arrow[ur, "g_\alpha"] \arrow[r, "g_{\alpha\beta}"] & B_\beta \arrow[u, "g_\beta"'] & \\
 & A \arrow[ul, "f_\alpha"] \arrow[u, "f_\beta"'] \arrow[uu, bend right=60, "f"'] &
\end{tikzcd}$$

Let $\mathfrak{p} \in \mathrm{Spec}(A)$, and let $k = k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be the residue field at $\mathfrak{p}$. Then $f^{*-1}(\mathfrak{p}) \neq \varnothing \iff \mathrm{Spec}(B \otimes_A k) \neq \varnothing \iff B \otimes_A k = \varinjlim_\alpha (B_\alpha \otimes_A k) \neq 0$ (Exercise 20 of Chapter 2, tensor products commute with direct limits) $\iff B_\alpha \otimes_A k \neq 0$ for every α (Exercise 21 of Chapter 2, since $B_\alpha \otimes_A k$ are rings and $g_{\alpha\beta} \otimes 1_k$ are ring homomorphisms) $\iff \mathfrak{p} \in \bigcap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$. ■

Problem 27.

i) Let $f_\alpha : A \to B_\alpha$ be any family of A-algebras and let $f : A \to B$ be their tensor product over A (Chapter 2, Exercise 23). Then

$$f^*\big(\mathrm{Spec}(B)\big) = \bigcap_\alpha f_\alpha^*\big(\mathrm{Spec}(B_\alpha)\big).$$

[Use Examples 25 and 26.]

ii) Let $f_\alpha : A \to B_\alpha$ be any finite family of A-algebras and let $B = \prod_\alpha B_\alpha$. Define $f : A \to B$ by $f(x) = \big(f_\alpha(x)\big)$. Then $f^*(\mathrm{Spec}(B)) = \bigcup_\alpha f_\alpha^*\big(\mathrm{Spec}(B_\alpha)\big)$.

iii) Hence the subsets of $X = \mathrm{Spec}(A)$ of the form $f^*\big(\mathrm{Spec}(B)\big)$, where $f : A \to B$ is a ring homomorphism, satisfy the axioms for closed sets in a topological space. The associated topology is the *constructible topology* on X. It is finer than the Zariski topology (i.e., there are more open sets, or equivalently mo closed sets).

iv) Let X_C denote the set X endowed with the constructible topology. Show that X_C is quasi-compact.

Proof.

i) Follows from Exercise 25 and 26.

ii) When the indices is finite, direct products are equivalent to direct sums, and since tensor products commute with direct sums, in this case they also commute with direct products. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, $k = k(\mathfrak{p})$, then $f^{*-1}(\mathfrak{p}) \neq \varnothing \iff B \otimes_A k \cong \prod_\alpha (B_\alpha \otimes_A k) \neq 0 \iff B_\alpha \otimes_A k \neq 0$ for some $\alpha \iff f_\alpha^{*-1}(\mathfrak{p}) \neq \varnothing$ for some α.

iii) Yeah, I agree!

iv) Given a family of A-algebras $f_\alpha : A \to B_\alpha$, $\alpha \in \Lambda$, such that

$$f^*(\operatorname{Spec}(B)) = \bigcap_{\alpha \in \Lambda} f_\alpha^*(\operatorname{Spec}(B_\alpha)) = \varnothing,$$

(this gives an open cover of X_C) where

$$B = \bigotimes_\alpha B_\alpha = \varinjlim_{\text{finite } J \subseteq \Lambda} \bigotimes_{\beta \in J} B_\beta.$$

$f^*(\operatorname{Spec}(B)) = \varnothing$ implies $B = 0$, hence by Exercise 21 of Chapter 2, there is a finite subset $j \subseteq \Lambda$, $\bigotimes_{\beta \in J} B_\beta = 0$. Therefore

$$\bigcap_{\beta \in J} f_\beta^*(\operatorname{Spec}(B_\beta)) = \varnothing$$

gives a finite subcover of X_C. ∎

Problem 28. (Continuation of Exercise 27.)

i) For each $g \in A$, the set X_g (Chapter 1, Exercise 17) is both open and closed in the constructible topology.

ii) Let C' denote the smallest topology on X for which the sets X_g are both open and closed, and let $X_{C'}$ denote the set X endowed with this topology. Show that $X_{C'}$ is Hausdorff.

iii) Deduce that the identity mapping $X_C \to X_{C'}$ is a homeomorphism. Hence a subset E of X is of the form $f^*\big(\operatorname{Spec}(B)\big)$ for some $f : A \to B$ if and only if it is closed in the topology C'.

iv) The topological space X_C is compact, Hausdorff and totally disconnected.

Proof.

i) As a refinement of Zariski topology, X_g is still open (let $f : A \to A_g$). Let $f : A \to A/\langle g \rangle$ be the canonical map, then $f^*(\mathrm{Spec}(A/\langle g \rangle)) = V(g)$. Since $X_g = \mathrm{Spec}(A) \setminus V(g)$, we see that X_g is both open and closed.

ii) For every distinct points $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(A)$, suppose $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$, let $f \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. Then $\mathfrak{p}_2 \in X_f$, $\mathfrak{p}_1 \in V(f)$ are two disjoint open sets.

iii) By i) we see that the topology of $X_{C'}$ is contained in the topology of X_C, hence the identity map $X_C \to X_{C'}$ is continuous.
Since X_C is quasi-compact and $X_{C'}$ is Hausdorff, the identity map $X_C \to X_{C'}$ is closed. Hence the inverse map is also continuous, hence the identity map $X_C \to X_{C'}$ is a homeomorphism.[8]

iv) All right! ∎

Problem 29. Let $f : A \to B$ be a ring homomorphism. Show that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a continuous *closed* mapping (i.e., maps closed sets to closed sets) for the constructible topology.

Proof. Let $g : B \to C$ be any ring homomorphism, then

$$f^*\big(g^*\big(\mathrm{Spec}(C)\big)\big) = (gf)^*\big(\mathrm{Spec}(C)\big)$$

is closed in $\mathrm{Spec}(A)$. ∎

Problem 30. Show that the Zariski topology and the constructible topology on $\mathrm{Spec}(A)$ are the same if and only if $A/\mathfrak{R}$ is absolutely flat (where $\mathfrak{R}$ is the nilradical of A). [Use Exercise 11.]

Proof. ($\Rightarrow$) This implies that $\mathrm{Spec}(A)$ is Hausdorff, by Exercise iv) $\Rightarrow$ i), $A/\mathfrak{R}$ is absolutely flat.

($\Leftarrow$) Let $X = \mathrm{Spec}(A)$. By Exercise 11 i) $\Rightarrow$ iv), X is Hausdorff and compact, for every basic open set $X_f \subseteq X$, by Exercise 17 vi) of Chapter 1, X_f is quasi-compact, hence X_f is closed (the space it lives in is Hausdorff). So X is at least a $X_{C'}$ topology (defined in Exercise 28), by Exercise 28 iii), the topology of X is also the constructible topology. ∎

[8]Reference: Tarizadeh, A. "Flat topology and its dual aspects." *arXiv:1503.04299*, `https://arxiv.org/pdf/1503.04299.pdf`, 2021. Remark 2.5. This is actually an easy exercise in topology once we know that X_C is quasi-compact and $X_{C'}$ is Hausdorff, but why I stuck!

Chapter 4

Primary Decomposition

Problem 1. If an ideal $\mathfrak{a}$ has a primary decomposition, then $\operatorname{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Proof. Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition, and let $\mathfrak{p}_i = r(\mathfrak{q}_i)$. For a prime ideal $\mathfrak{p} \supseteq$ such that it is minimal in $A/\mathfrak{a}$, then by Proposition 1.11 ii) there is $\mathfrak{q}_i \subseteq \mathfrak{p}$, consequently $\mathfrak{p}_i = \mathfrak{p}$ (from the minimality of $\mathfrak{p}$). Hence the set of minimal prime ideals in $A/\mathfrak{a}$ is one-to-one corresponding to $\{\mathfrak{p}_i\}_{i=1}^n$, hence is finite. By Exercise 20 iv) of Chapter 1, $\operatorname{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components. ■

Problem 2. If $\mathfrak{a} = r(\mathfrak{a})$, then $\mathfrak{a}$ has no embedded prime ideals.

Proof. Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition, and $\mathfrak{p}_i = r(\mathfrak{q}_i)$, then $r(\mathfrak{a}) = \bigcap_{i=1}^n \mathfrak{p}_i = \mathfrak{a}$. Hence $\mathfrak{a}$ has a decomposition of prime ideals. From the 1st uniqueness theorem, the set $\{p_i\}_{i=1}^n$ is independent of the choice of decomposition, therefore the decomposition $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{p}_i$ is minimal.[1] Hence for $i \neq j$, $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$, every prime ideal belong to $\mathfrak{a}$ is minimal, and hence $\mathfrak{a}$ has no embedded prime ideals. ■

Problem 3. If A is absolutely flat, every primary ideal is maximal.

Proof. Let $\mathfrak{q} \subseteq A$ be a primary ideal. Since A is absolutely flat, by Exercise 27 ii) of Chapter 2, for every $x \in A$, $(x)^2 = (x)$, there is $c \in A$ such that $cx^2 = x$. Suppose $x \notin \mathfrak{q}$, then we see that $cx \notin \mathfrak{q}$, and $x^n \notin \mathfrak{q}$ for every $n > 0$ (because $(x^n) = (x)^n = (x)$). Hence $x + \mathfrak{q}$ is not in the nilradical of $A/\mathfrak{q}$, this implies that the nilradical of $A/\mathfrak{q}$ is zero, and $A/\mathfrak{q}$ is an integral domain. Then $cx^2 = x$ implies $c + \mathfrak{q}$ is an inverse of $x + \mathfrak{q}$ in $A/\mathfrak{q}$, $A/\mathfrak{q}$ is then a field and $\mathfrak{q}$ is a maximal ideal. ■

Problem 4. In the polynomial ring $\mathbb{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.

[1] Since the set of prime ideals belonging to $\mathfrak{a}$ are fixed, namely they are precisely $\{\mathfrak{p}_i\}_{i=1}^n$, the decomposition $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{p}_i$ must be minimal.

Proof. It is well-known that maximal ideals in $\mathbb{Z}[t]$ is of the form $\big(p, f(t)\big)$, where p is a prime number and $f(t) \in \mathbb{Z}[t]$ is irreducible when reduced in $\mathbb{F}_p[t]$. Hence $\mathfrak{m} = (2, t)$ is maximal. We have $r\big((4, t)\big) = (2, t)$, by Proposition 4.2 $\mathfrak{q}$ is $\mathfrak{m}$-primary. Clearly for $n > 1$, $t \notin \mathfrak{m}^n$, therefore $\mathfrak{m}^n \neq \mathfrak{q}$. ■

Problem 5. In the polynomial ring $K[x, y, z]$ where K is a field and x, y, z are independent indeterminates, let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{m} = (x, y, z)$; $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?

Proof. We have

$$\begin{aligned} \mathfrak{a} &= (x, y)(x, z) = (x^2, xy, xz, yz), \\ \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 &= (x, yz) \cap (x, y, z)^2 = (x^2, xy, xz, yz) = \mathfrak{a}. \end{aligned}$$

$\mathfrak{m}$ is isolated, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are embedded. ■

Problem 6. Let X be an infinite compact Hausdorff space, $C(X)$ the ring of real-valued continuous functions on X (Chapter 1, Exercise 26). Is the zero ideal decomposable in this ring?

Proof. The zero ideal is not decomposable in general.

Since the nilradical ($= r((0))$) of $C(X)$ is zero, if the zero ideal is decomposable, we can write $(0) = \bigcap_{i=1}^n \mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i$. For every ideal $\mathfrak{a} \subseteq C(X)$, set

$$Z(\mathfrak{a}) = \big\{x \in X \mid \forall f \in C(X), f(x) = 0\big\}$$

be the zero set, we see that $Z(\mathfrak{a}) = \varnothing$ iff $\mathfrak{a} = (1)$.

We give an example that the statement fails. Let

$$Y = \big\{1/n \mid n \in \mathbb{Z}_{>0}\big\} \quad \text{and} \quad X = Y \cup \{0\} \subseteq \mathbb{R},$$

we can see that X is infinite compact Hausdorff, and every function in $C(X)$ is completely determined on Y. For every prime ideal $\mathfrak{p} \subseteq C(X)$, from the above remark we have $Z(\mathfrak{p}) \neq \varnothing$. We want to show that $|Z(\mathfrak{p})| = 1$. Suppose it is not the case, choose a point $x_0 \in Z(\mathfrak{p}) \setminus \{0\}$. Define

$$f_1(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0 \end{cases}, \quad f_2(x) = \begin{cases} 0, & x = x_0 \\ 1, & x \neq x_0 \end{cases},$$

then $f_1, f_2 \in C(X) \setminus \mathfrak{p}$, but $f_1 f_2 = 0 \in \mathfrak{p}$, contradiction. Therefore $\mathfrak{p} = \mathfrak{m}_{x_0} = \{f \in C(X) \mid f(x_0) = 0\}$ for $x_0 > 0$, or $Z(\mathfrak{p}) = \{0\}$. For the latter one (i.e. $Z(\mathfrak{p}) = \{0\}$), by some easy argument we can see that for every $m > 0$, the function

$$g_m(x) = \begin{cases} 1, & x = 1/m \\ 0, & x \neq 1/m \end{cases} \in \mathfrak{p},$$

hence the zero set of the intersection of some such prime ideals is still $\{0\}$. So $\bigcap_{i=1}^n \mathfrak{p}_i \supsetneq (0)$, the zero ideal in this case is not decomposable. ■

Problem 7. Let A be a ring and let $A[x]$ denote the ring of polynomials in one indeterminate over A. For each ideal $\mathfrak{a}$ of A, let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$.

i) $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ to $A[x]$.

ii) If $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

iii) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$. [Use Chapter 1, Exercise 2.]

iv) If $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition in A, then $\mathfrak{a}[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.

v) If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

Proof.

i) Obviously $\mathfrak{a}[x] \subseteq \mathfrak{a}^{\mathrm{e}}$. Besides, $\mathfrak{a}[x]$ is an ideal in $A[x]$, hence $\mathfrak{a}[x] = \mathfrak{a}^{\mathrm{e}}$.

ii) This is just Exercise 7 of Chapter 2.

iii) We have $A[x]/\mathfrak{q}[x] \cong (A/\mathfrak{q})[x]$. Since $\mathfrak{q}$ is primary, non-zero elements in $A/\mathfrak{q}$ are nilpotent. By Exercise 2 ii) of Chapter 1, non-zero elements in $(A/\mathfrak{q})[x]$ are nilpotent, hence $\mathfrak{q}[x]$ is primary in $A[x]$. Moreover, by Exercise 2 ii) of Chapter 1 again, the nilradical of $(A/\mathfrak{q})[x]$ (i.e. the radical of $\mathfrak{q}[x]$ in $A[x]$) is the image of $\mathfrak{p}[x]$, hence $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary.

iv) We do have $\mathfrak{a}[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$. Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$. By iii), $r(\mathfrak{q}_i[x]) = \mathfrak{p}_i[x]$ are distinct. Obviously $\bigcap_{j \neq i} \mathfrak{q}_j[x] \not\subseteq \mathfrak{q}_i[x]$. So the decomposition is minimal.

v) Follows from iv) and the 1st uniqueness theorem. ■

Problem 8. Let k be a field. Show that in the polynomial ring $k[x_1, \ldots, x_n]$ the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$ $(1 \leq i \leq n)$ are prime and all their powers are primary. [Use Exercise 7.]

Proof. $k[x_1, \ldots, x_n]/\mathfrak{p}_i \cong k[x_{i+1}, \ldots, x_n]$ is integral, hence $\mathfrak{p}_i$ is prime. To show that $\mathfrak{p}_i^m$ are primary, we use induction on (i, n). The case $(i, n) = (1, n)$ where $n \in \mathbb{Z}_{>0}$ is trivial. For $1 < i \leq n$, suppose the case $(i-1, n-1)$ is proved, by Exercise 7 iii), $\mathfrak{p}_i^m = \mathfrak{p}_{i-1}^m[x_i]$ is $\mathfrak{p}_i$-primary in $k[x_1, \ldots, x_{n-1}][x_n] = k[x_1, \ldots, x_n]$. Hence $\mathfrak{p}_i^m$ are all primary in $k[x_2, \ldots, x_n]$. ■

Problem 9. In a ring A, let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a)$. Show that $x \in A$ is a zero divisor $\Leftrightarrow$ $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.

Let S be a multiplicatively closed subset of A, and identify $\operatorname{Spec}(S^{-1}A)$ with its image in $\operatorname{Spec}(A)$ (Chapter 3, Exercise 21). Show that

$$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A).$$

If the zero ideal has a primary decomposition, show that $D(A)$ is the set of associated prime ideals of 0.

Proof. ($\Rightarrow$) If $x \in A$ is a zero divisor, there is a non-zero $a \in A$ such that $x \in (0 : a)$. Let $\mathfrak{p}$ be a minimal prime ideal containing $(0 : a)$, $\mathfrak{p} \in D(A)$.

($\Leftarrow$) Let $\mathfrak{p} \in D(A)$ be minimal with respect to $(0 : a)$, where $a \in A$. Let $x \in \mathfrak{p}$, suppose x is not a zero-divisor, let $S = \{x^l \mid l \geq 0\}$. Then S does not meet $(0 : a)$ in $A_\mathfrak{p}$, hence the contraction of a minimal prime ideal in $S^{-1}A_\mathfrak{p}$ containing $(0 : a)$ is smaller than $\mathfrak{p}$ in A, contradiction. Hence x is a zero-divisor.

Let $\mathfrak{p} \subseteq A$ be a prime ideal such that $S \cap \mathfrak{p} = \varnothing$. Since for $a \in A$,

$$(0 : a/1) \subseteq S^{-1}\mathfrak{p} \iff (0 : a) \subseteq \mathfrak{p},$$

we see that $S^{-1}\mathfrak{p} \in D(S^{-1}A)$ iff $\mathfrak{p} \in D(A)$. Hence $D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A)$.

Let $(0) = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition, $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Let $\mathfrak{p} \in D(A)$ with respect to $(0 : a)$ for some $a \in A$. Then $r(0 : a) = \bigcap_{i=1}^n r(\mathfrak{q}_\mathfrak{i} : a) = \bigcap_{a \notin \mathfrak{q}_j} \mathfrak{p}_j \subseteq \mathfrak{p}$, hence $(0 : a) \subseteq \mathfrak{p}_j \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is minimal, $\mathfrak{p} = \mathfrak{p}_j$.[2] ∎

Problem 10. For any prime ideal $\mathfrak{p}$ in a ring A, let $S_\mathfrak{p}(0)$ denote the kernel of the homomorphism $A \to A_\mathfrak{p}$. Prove that

i) $S_\mathfrak{p}(0) \subseteq \mathfrak{p}$.

ii) $r\big(S_\mathfrak{p}(0)\big) = \mathfrak{p} \Leftrightarrow \mathfrak{p}$ is a minimal prime ideal of A.

iii) If $\mathfrak{p} \supseteq \mathfrak{p}'$, then $S_\mathfrak{p}(0) \subseteq S_{\mathfrak{p}'}(0)$.

iv) $\bigcap_{\mathfrak{p} \in D(A)} S_\mathfrak{p}(0) = 0$, where $D(A)$ is defined in Exercise 9.

Proof.

i) $a/1 = 0$ iff there is $s \notin \mathfrak{p}$, $sa = 0$, then $a \in \mathfrak{p}$.

[2]Reference: C. Fujinomiya (`https://math.stackexchange.com/users/693946/c-fujinomiya`). "Atiyah-McDonald Exercise 4.9." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/3412337`, 2019.

ii) ($\Rightarrow$) If $\mathfrak{p}$ is not minimal, let $\mathfrak{p}' \subsetneq \mathfrak{p}$ be a smaller prime ideal. By iii) and i), $S_\mathfrak{p}(0) \subseteq S_{mfkp'}(0) \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$, hence $r\big(S_\mathfrak{p}(0)\big) \subsetneq \mathfrak{p}$.
($\Leftarrow$) If $r\big(S_\mathfrak{p}(0)\big) \subsetneq \mathfrak{p}$, let $x \in \mathfrak{p} \setminus r\big(S_\mathfrak{p}(0)\big)$. Then $(A_\mathfrak{p})_{x/1} \neq 0$, it has a maximal ideal, the contraction of this ideal in A is a prime ideal $\mathfrak{p}'$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and $x \notin \mathfrak{p}'$, hence $\mathfrak{p}$ is not minimal.

iii) $a/1 = 0$ in $A_\mathfrak{p}$ iff there is $s \notin \mathfrak{p} \supseteq \mathfrak{p}'$, $sa = 0$. This implies $a/1 = 0$ in $A_{\mathfrak{p}'}$.

iv) For every $x \neq 0$, let $\mathfrak{p} \supseteq (0 : x)$ be the minimal prime ideal, $\mathfrak{p} \in D(A)$. Then $x \notin S_\mathfrak{p}(0)$ ($sx \neq 0$ for $s \in A \setminus \mathfrak{p} \subseteq A \setminus (0 : x)$), hence $\bigcap_{\mathfrak{p} \in D(A)} \mathfrak{p} = 0$. ■

Problem 11. If $\mathfrak{p}$ is a minimal prime ideal of a ring A, show that $S_\mathfrak{p}(0)$ (Exercise 10) is the smallest $\mathfrak{p}$-primary ideal.

Let $\mathfrak{a}$ be the intersection of the ideals $S_\mathfrak{p}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals of A. Show that $\mathfrak{a}$ is contained in the nilradical of A.

Suppose that the zero ideal is decomposable. Prove that $\mathfrak{a} = 0$ if and only if every prime ideal of 0 is isolated.

Proof. If $\mathfrak{p}$ is minimal, $A/S_\mathfrak{p}(0) \hookrightarrow A_\mathfrak{p}$, and by Exercise 10 ii), $S_\mathfrak{p}(0)$ is a $\mathfrak{p}$-primary ideal. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Let $y \in S_\mathfrak{p}(0)$, and $x \notin \mathfrak{p}$ such that $xy = 0 \in \mathfrak{q}$, then $y \in \mathfrak{q}$, $S_\mathfrak{p}(0) \subseteq \mathfrak{q}$.

Since $S_\mathfrak{p}(0) \subseteq \mathfrak{p}$, everything is clear.

Let $\mathfrak{P}$ be the set of minimal prime ideal of A, $\mathfrak{P}$ is exactly the set of isolated prime ideals of 0. By Exercise 10 iv), every prime ideal of 0 is isolated iff $D(A) = \mathfrak{P}$.

($\Rightarrow$) $0 = \mathfrak{a} = \bigcap_{\mathfrak{p} \in \mathfrak{P}} S_\mathfrak{p}(0)$ is a primary decomposition, hence $D(A) \subseteq \mathfrak{P}$ and we must have $D(A) = \mathfrak{P}$.

($\Leftarrow$) Suppose $D(A) = \mathfrak{P}$. Let $0 = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition, and $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Since $S_{\mathfrak{p}_i}(0)$ is the smallest $\mathfrak{p}_i$-primary ideal, $S_{\mathfrak{p}_i}(0) \subseteq \mathfrak{q}_i$, then

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in \mathfrak{P}} S_\mathfrak{p}(0) \subseteq \bigcap_{i=1}^n \mathfrak{q}_i = 0,$$

hence $\mathfrak{a} = 0$. ■

Problem 12. Let A be a ring, S a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}\mathfrak{a}$ in A. The ideal $S(\mathfrak{a})$ is called the *saturation* of $\mathfrak{a}$ with respect to S. Prove that

i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$.

ii) $S\big(r(\mathfrak{a})\big) = r\big(S(\mathfrak{a})\big)$.

iii) $S(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a}$ meets S.

iv) $S_1\big(S_2(\mathfrak{a})\big) = (S_1 S_2)(\mathfrak{a})$.

If $\mathfrak{a}$ has a primary decomposition, prove that the set of ideals $S(\mathfrak{a})$ (where S run through all multiplicatively closed subsets of A) is finite.

Proof. By Proposition 3.11 ii), $S(\mathfrak{a}) = \bigcup_{s\in S}(\mathfrak{a} : s)$.

i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = \bigcup_{s\in S}(\mathfrak{a} : s) \cap (\mathfrak{b} : s) = \bigcup_{s\in S}(\mathfrak{a} \cap \mathfrak{b} : s) = S(\mathfrak{a} \cap \mathfrak{b})$.

ii) $S\big(r(\mathfrak{a})\big) = \bigcup_{s\in S}\big(r(\mathfrak{a}) : s\big) = r\big(\bigcup_{s\in S}(\mathfrak{a} : s)\big) = r\big(S(\mathfrak{a})\big)$.

iii) Obvious.

iv) We have

$$\begin{aligned} S_1\big(S_2(\mathfrak{a})\big) &= \bigcup_{s_1\in S_1} \big(S_2(\mathfrak{a}) : s_1\big) = \bigcup_{s_1\in S}\Big(\bigcup_{s_2\in S_1} (\mathfrak{a} : s_2) : s_1\Big) \\ &= \bigcup_{s_1\in S}\bigcup_{s_2\in S_1} \big((\mathfrak{a} : s_2) : s_1\big) = \bigcup_{s_1\in S}\bigcup_{s_2\in S_1} (\mathfrak{a} : s_2 s_1) \\ &= \bigcup_{s\in S_1 S_2} (\mathfrak{a} : s) = (S_1 S_2)(\mathfrak{a}). \end{aligned}$$

Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition. Then

$$S(\mathfrak{a}) = \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i = \bigcap_{j=1}^{m} \mathfrak{q}_{i_j},$$

where i_j are all indices such that $\mathfrak{q}_{i_j} \cap S = \varnothing$. Hence the set of ideals $S(\mathfrak{a})$ is finite. ∎

Problem 13. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. The *nth symbolic power of* $\mathfrak{p}$ is defined to be the ideal (in the notation of Exercise 12)

$$\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$$

where $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$. Show that

i) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;

ii) if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component;

iii) if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$-primary component;

iv) $\mathfrak{p}^{(n)} = \mathfrak{p}^n \Leftrightarrow \mathfrak{p}^n$ is $\mathfrak{p}$-primary.[3]

[3] There is a typo in the original statement, where the second '$\mathfrak{p}^n$' was '$\mathfrak{p}^{(n)}$', but this is obviously wrong. Double check: Pi314 (`https://mathoverflow.net/users/39203/pi314`). "Errata for Atiyah–Macdonald." *MathOverflow*, `https://mathoverflow.net/q/140533`, 2023.

Proof.

i) $r(\mathfrak{p}^{(n)}) = S_\mathfrak{p}(r(\mathfrak{p}^n)) = S_\mathfrak{p}(\mathfrak{p}) = \mathfrak{p}$. Besides, $\mathfrak{p}^n A_\mathfrak{p}$ is primary in $A_\mathfrak{p}$, hence as the contraction of $\mathfrak{p}^n A_\mathfrak{p}$, $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.

ii) Let $\mathfrak{p}^n = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition, $\mathfrak{p}_i = r(\mathfrak{q}_i)$, and assume that $\mathfrak{p}_1 = \mathfrak{p}$.[4] By Proposition 4.8, $\mathfrak{p}^{(n)} = S_\mathfrak{p}(\mathfrak{p}^n) = \bigcap_{i=1}^n S_\mathfrak{p}(\mathfrak{q}_i) = \mathfrak{q}_1$ is a $\mathfrak{p}$-primary component in the decomposition.

iii) Let $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition, $\mathfrak{p}_i = r(\mathfrak{q}_i)$. And let $r(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$ be a prime ideal, since by Exercise 12 ii),

$$\mathfrak{p} = r(\mathfrak{p}^{(m+n)}) = r(S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})) = S_\mathfrak{p}(r(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})) \subseteq S_\mathfrak{p}(\mathfrak{p}') = \mathfrak{p}',$$

$\mathfrak{p} = \mathfrak{p}'$, $\mathfrak{p}$ is an isolated prime ideal of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$, assume $\mathfrak{p}_1 = \mathfrak{p}$ (easy to see that $\mathfrak{p}$ is an associated prime ideal). Hence all other associated prime ideals except $\mathfrak{p}$ meets $S_\mathfrak{p}$, and $\mathfrak{p}^{(m+n)} = S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = \bigcap_{i=1}^n S_\mathfrak{p}(\mathfrak{q}_i) = \mathfrak{q}_1$ is the $\mathfrak{p}$-primary component.

iv) ($\Rightarrow$) Then $\mathfrak{p}^n$ has a primary decomposition, and has only one associated prime ideal, hence it is $\mathfrak{p}$-primary.
($\Leftarrow$) $\mathfrak{p}$ is then decomposable, has itself to be the only one associated prime ideal. By ii), $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary component of $\mathfrak{p}^n$, it can only be $\mathfrak{p}^{(n)} = \mathfrak{p}^n$. ■

Problem 14. Let $\mathfrak{a}$ be a decomposable ideal in a ring A and let $\mathfrak{p}$ be a maximal element of the set of ideals $(\mathfrak{a} : x)$, where $x \in A$ and $x \notin \mathfrak{a}$. Show that $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$.

Proof. Let $\mathfrak{p} = (\mathfrak{a} : x)$, and let $a, b \in A$ such that $abx = 0$ and $bx \neq 0$, then $(\mathfrak{a} : x) \subseteq (\mathfrak{a} : bx) \ni a$, hence $a \in (\mathfrak{a} : x)$, and $(\mathfrak{a} : x) = \mathfrak{p}$ is a prime ideal. By the 1st uniqueness theorem, $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$. ■

Problem 15. Let $\mathfrak{a}$ be a decomposable ideal in a ring A, let Σ be an isolated set of prime ideals belonging to $\mathfrak{a}$, and let $\mathfrak{q}_\Sigma$ be the intersection of the corresponding primary components. Let f be an element of A such that, for each prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$, we have $f \in \mathfrak{p} \Leftrightarrow \mathfrak{p} \notin \Sigma$, and let S_f be the set of all powers of f. Show that $\mathfrak{q}_\Sigma = S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ for all large n.

Proof. Clearly $\mathfrak{q}_\Sigma = S_f(\mathfrak{a})$. Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition,

$$(\mathfrak{a} : f^n) = \bigcap_{i=1}^n (\mathfrak{q}_i : f^n).$$

[4] This is because $\mathfrak{p}$ is the smallest prime ideal containing $r(\mathfrak{p}^n) = \mathfrak{p}$, by Proposition 4.6 there must be a $\mathfrak{p}$-primary component in the decomposition.

Since for $\mathfrak{q}_i \notin \Sigma$, $f \in r(\mathfrak{q}_i)$, and $f^{n_i} \in \mathfrak{q}_i$ for some $n_i > 0$, and since the set of associated prime ideals is finite, the above equation and Lemma 4.4 i) show that $(\mathfrak{a} : f^n) = \mathfrak{q}_\Sigma$ for all large enough n. ■

Problem 16. If A is a ring in which every ideal has a primary decomposition, show that every ring of fractions $S^{-1}A$ has the same property.

Proof. It follows from Proposition 4.9, since every ideal in $S^{-1}A$ is the extension of an ideal in A. ■

Problem 17. Let A be a ring with the following property.
(L1) For every ideal $\mathfrak{a} \neq (1)$ in A and every prime ideal $\mathfrak{p}$, there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p} = (\mathfrak{a} : x)$, where $S_\mathfrak{p} = A \setminus \mathfrak{p}$.

Then every ideal in A is an intersection of (possible infinitely many) primary ideals.
[Let $\mathfrak{a}$ be an ideal $\neq (1)$ in A, and let $\mathfrak{p}_1$ be a minimal element of the set of prime ideals containing $\mathfrak{a}$. Then $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary (by Exercise 11), and $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap \big(\mathfrak{a} + (x)\big)$.

Now let $\mathfrak{a}_1$ be a minimal element of the set of ideals $\mathfrak{b} \supseteq \mathfrak{a}$ such that $\mathfrak{q}_1 \cap \mathfrak{b} = \mathfrak{a}$, and choose $\mathfrak{a}_1$ so that $x \in \mathfrak{a}_1$, and therefore $\mathfrak{a}_1 \not\subseteq \mathfrak{p}_1$. Repeat the construction starting with $\mathfrak{a}_1$, and so on. At the nth stage we have $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots, \cap \mathfrak{q}_n \cap \mathfrak{a}_n$ where the $\mathfrak{q}_i$ are primary ideals, $\mathfrak{a}_n$ is maximal among the ideals $\mathfrak{b}$ containing $\mathfrak{a}_{n-1} = \mathfrak{a}_n \cap \mathfrak{q}_n$ such that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{b}$, and $\mathfrak{a}_n \not\subseteq \mathfrak{p}_n$. If at any stage we have $\mathfrak{a}_n = (1)$, the process stops, and $\mathfrak{a}$ is a finite intersection of primary ideals. If not, continue by transfinite induction, observing that each $\mathfrak{a}_n$ strictly contains $\mathfrak{a}_{n-1}$.]

Proof. Given an ideal $\mathfrak{a} \neq (1)$, let $\mathfrak{p}_1 \supseteq \mathfrak{a}$ be a minimal prime ideal, by Exercise 11, $\mathfrak{q}_1 := S_{\mathfrak{p}_1}(\mathfrak{a})$ is a $\mathfrak{p}_1$-primary ideal, $S_{\mathfrak{p}_1}(\mathfrak{a}) = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$, and $\mathfrak{a} = \mathfrak{q}_1 \cap \big(\mathfrak{a} + (x)\big)$. Choose $\mathfrak{a}_1$ to be a maximal element in the set $\big\{\mathfrak{b} \mid \mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{b}, \mathfrak{b} \supseteq \mathfrak{a} + (x)\big\}$, then $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{a}_1$ and $\mathfrak{a} \subsetneq \mathfrak{a}_1 \not\subseteq \mathfrak{q}_1$. Denote $\mathfrak{a}_0 = \mathfrak{a}$. We have finished the base case.

Let

$$\Sigma = \Big\{\big(I, \{\mathfrak{q}_i\}_{i\in I}, \mathfrak{a}_I\big) \Big| \mathfrak{a} = \bigcap_{i\in I} \mathfrak{q}_i \cap \mathfrak{a}_I, \mathfrak{q}_i \text{ primary}\Big\},$$

for $\big(I, \{\mathfrak{q}_i\}_{i\in I}, \mathfrak{a}_I\big), \big(I', \{\mathfrak{q}'_{i'}\}_{i'\in I'}, \mathfrak{a}'_{I'}\big) \in \Sigma$, we say

$$\begin{aligned} &\big(I, \{\mathfrak{q}_i\}_{i\in I}, \mathfrak{a}_I\big) \leq \big(I', \{\mathfrak{q}'_{i'}\}_{i'\in I'}, \mathfrak{a}'_{I'}\big) \\ \text{iff} \quad &\{\mathfrak{q}_i\}_{i\in I} \subseteq \{\mathfrak{q}'_{i'}\}'_{i'\in I'} \quad \text{and} \quad \mathfrak{a}_I \subseteq \mathfrak{a}'_{I'}. \end{aligned}$$

It is easy to see that we can apply Zorn's lemma[5] on Σ to obtain a maximal element $\big(I, \{\mathfrak{q}_i\}_{i\in I}, \mathfrak{a}_I\big)$. For this maximal element we must have $\mathfrak{a}_I = (1)$, otherwise we can construct a strictly larger element as in the base case. Hence $\mathfrak{a} = \bigcap_{i\in I} \mathfrak{q}_i$. ■

[5] We need to verify that Σ is an inductively ordered set. If $\Big\{\Big(I^{(j)}, \{\mathfrak{q}_i^{(j)}\}_{i\in I^{(j)}}, \mathfrak{a}^{(j)}_{I^{(j)}}\Big)\Big\}_{j\in J}$ is a chain, we can see that $\Big(\bigcup_{j\in J} I^{(j)}, \bigcup_{j\in J}\bigcup_{i\in I^{(j)}}\{\mathfrak{q}_i^{(j)}\}, \bigcup_{j\in J}\mathfrak{a}^{(j)}_{I^{(j)}}\Big) \in \Sigma$ is an upper bound of it. Besides, $\Sigma \neq \varnothing$ since we have the base case. Hence we can use Zorn's lemma on Σ.

Problem 18. Consider the following condition on a ring A:
(L2) Given an ideal $\mathfrak{a}$ and a descending chain $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$ of multiplicatively closed subsets of A, there exists an integer n such that $S_n(\mathfrak{a}) = S_{n+1}(\mathfrak{a}) = \cdots$.
Prove that the following are equivalent:

i) Every ideal in A ha a primary decomposition;

ii) A satisfies (L1) and (L2).

[For i) $\Rightarrow$ ii), use Exercise 12 and 15. For ii) $\Rightarrow$ i) show, with the notation of the proof of Exercise 17, that if $S_n = S_{\mathfrak{p}_1} \cap \cdots \cap S_{\mathfrak{p}_n}$ then S_n meets $\mathfrak{a}_n$, hence $S_n(\mathfrak{a}) = (1)$, and therefore $S_n(\mathfrak{a}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$. Now use (L2) to show that the construction must terminate after a finite number of steps.]

Proof. i) $\Rightarrow$ ii) Of course we then have (L1). Since the number of associated prime ideals of an ideal $\mathfrak{a}$ is finite, by Proposition 4.9 it is easy to see that $S_n(\mathfrak{a})$ is constant for all large enough n.

ii) $\Rightarrow$ i) Let $\mathfrak{a}$ be an ideal, in the proof of Exercise 17, we may construct $\mathfrak{a}_i$ and $\mathfrak{q}_i = S_{\mathfrak{p}_i}(\mathfrak{a}_{i-1})$ starting from $\mathfrak{a}_0 = \mathfrak{a}$, each step is obtained by repeating the base case, such that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{a}_n$, $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots$ and $\mathfrak{a}_n \not\subseteq \mathfrak{p}_n$. Let $S_n = \bigcap_{i=1}^n (A \setminus \mathfrak{p}_i)$, then S_n meets $\mathfrak{a}_n$, hence $S_n(\mathfrak{a}) = \bigcap_{i=1}^n \mathfrak{q}_i$. By (L2) $S_n(\mathfrak{a})$ is stationary. This means $\mathfrak{a}_n = (1)$ for some n, and $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$. ■

Problem 19. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Show that every $\mathfrak{p}$-primary ideal contains $S_{\mathfrak{p}}(0)$, the kernel of the canonical homomorphism $A \to A_{\mathfrak{p}}$.

Suppose that A satisfies the following condition: for every prime ideal $\mathfrak{p}$, the intersection of all $\mathfrak{p}$-primary ideals of A is equal to $S_{\mathfrak{p}}(0)$. (Noetherian rings satisfy this condition: see Chapter 10.) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals, none of which is a minimal prime ideal of A. Then there exists an ideal $\mathfrak{a}$ in A whose associated prime ideals are $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.
[Proof by induction on n. The case $n = 1$ is trivial (take $\mathfrak{a} = \mathfrak{p}_1$). Suppose $n > 1$ and let $\mathfrak{p}_n$ be maximal in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. By the induction hypothesis there exists an ideal $\mathfrak{b}$ and a minimal primary decomposition $\mathfrak{b} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{n-1}$, where each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. If $\mathfrak{b} \subseteq S_{\mathfrak{p}_n}(0)$, let $\mathfrak{p}$ be a minimal prime ideal of A contained in $\mathfrak{p}_n$. Then $S_{\mathfrak{p}_n}(0) \subseteq S_{\mathfrak{p}}(0)$, hence $\mathfrak{b} \subseteq S_{\mathfrak{p}}(0)$. Taking radicals and using Exercise 10, we have $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{n-1} \subseteq \mathfrak{p}$, hence some $\mathfrak{p}_i \subseteq \mathfrak{p}$, hence $\mathfrak{p}_i = \mathfrak{p}$ since $\mathfrak{p}$ is minimal. This is a contradiction since no $\mathfrak{p}_i$ is minimal. Hence $\mathfrak{b} \not\subseteq S_{\mathfrak{p}_n}(0)$ and therefore there exists a $\mathfrak{p}_n$-primary ideal $\mathfrak{q}_n$ such that $\mathfrak{b} \not\subseteq \mathfrak{q}_n$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ has the required properties.]

Proof. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Since $S_{\mathfrak{p}}(0) = \bigcup_{s \notin \mathfrak{p}} (0 : s)$, let $sa = 0$ for some $s \notin \mathfrak{p}$ and $a \in A$, then $s \notin r(\mathfrak{q})$, hence $a \in \mathfrak{q}$ and $S_{\mathfrak{p}}(0) \subseteq \mathfrak{q}$.

Use induction on n. When $n = 1$, just take $\mathfrak{a} = \mathfrak{p}_1$. Suppose $n > 1$, and the case $n - 1$ is proved. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct non-minimal prime ideal in A, and assume that $\mathfrak{p}_n$ is maximal among them. By inductive hypothesis, there is an ideal with a minimal primary decomposition $\mathfrak{b} = \bigcap_{i=1}^{n-1} \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. If $\mathfrak{b} \subseteq S_{\mathfrak{p}_n}(0)$, let $\mathfrak{p} \subsetneq \mathfrak{p}_n$ be a minimal prime ideal, then $r(\mathfrak{b}) \subseteq r\big(S_{\mathfrak{p}_n}(0)\big) \subseteq r\big(S_{\mathfrak{p}}(0)\big) \subseteq \mathfrak{p}$ (Exercise 10 iii)), hence $\mathfrak{p}_j \subseteq \mathfrak{p}$ for some $1 \leq j \leq n-1$, this implies $\mathfrak{p}_j = \mathfrak{p}$ is minimal, contradiction. Hence $\mathfrak{b} \not\subseteq S_{\mathfrak{p}_n}(0)$, there is a $\mathfrak{p}_n$-primary ideal $\mathfrak{q}_n$ such that $\mathfrak{b} \not\subseteq \mathfrak{q}_n$. Let $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{q}_n$.

Then $\mathfrak{a} = \bigcap_{i=1}^{n}$ is a minimal primary decomposition. To see this, let $S = A \setminus \bigcup_{i=1}^{n-1} \mathfrak{p}_i$. Suppose $\mathfrak{q}_j \supseteq \bigcap_{i \neq j} \mathfrak{q}_i$, where $j < n$, then $\mathfrak{q}_j = S(\mathfrak{q}_j) \supseteq \bigcap_{i \neq j,n} \mathfrak{q}_i$, contradiction. On the other hand, $\mathfrak{b} \not\subseteq \mathfrak{q}_n$ implies the case $\mathfrak{q}_n \supseteq \bigcap_{i<n} \mathfrak{q}_i$ will not happen. Hence the decomposition $\mathfrak{a} = \bigcap_{i=1}^{n}$ is minimal, and the associated prime ideals of $\mathfrak{a}$ are exactly $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. ■

Primary decomposition of modules

Practically the whole of this chapter can be transposed to the context of modules over a ring A. The following exercises indicate how this is done.

Problem 20. Let M be a fixed A-module, N a submodule of M. The *radical* of N in M is defined to be

$$r_M(N) = \{x \in A : x^q M \subseteq N \text{ for some } q > 0\}.$$

Show that $r_M(N) = r(N : M) = r\big(\mathrm{Ann}(M/N)\big)$. In particular, $r_M(N)$ is an *ideal*.

State and prove the formulas for r_M analogous to (1.13).

Proof. Clearly $r_M(N) = r(N : M)$. By Exercise 2.2 ii) of Chapter 2 (on page 20), $(N : M) = \mathrm{Ann}(M/N)$ (actually this is obvious), hence $r(N : M) = r\big(\mathrm{Ann}(M/N)\big)$ is an ideal.

Analog of Exercise 1.13:

i) $r_M(N) \supseteq \mathrm{Ann}(M/N)$. Proved.

ii) $r_M\big(r_M(N)\big) = r_M(N)$. Clear.

iii) $r_M(N_1 \cap N_2) = r_M(N_1) \cap r_M(N_2)$. We have $r_M(N_1 \cap N_2) = r\big(\mathrm{Ann}(M/(N_1 \cap N_2))\big) = r\big(\mathrm{Ann}(M/N_1) \cap \mathrm{Ann}(M/N_2)\big) = r_M(N_1) \cap r_M(N_2)$.

iv) $r_M(N) = (1) \Leftrightarrow N = M$. Clear.

v) $r_M(N_1 + N_2) \supseteq r_M\big(r_M(N_1) + r_M(N_2)\big)$. We have

$$\begin{aligned} r_M(N_1 + N_2) &= r\Big(r\big(\mathrm{Ann}(M/(N_1 + N_2))\big)\Big) \\ &\supseteq r\big(r\big(\mathrm{Ann}(M/N_1)\big) + r\big(\mathrm{Ann}(M/N_2)\big)\big) \\ &= r_M\big(r_M(N_1) + r_M(N_2)\big). \end{aligned}$$

vi) If $\mathfrak{p}$ is prime, $r_M(\mathfrak{p}^n M) \supseteq \mathfrak{p}$. Clear. ■

Problem 21. An element $x \in A$ defines an endomorphism ϕ_x of M, namely $m \mapsto xm$. The element x is said to be a *zero-divisor* (resp. *nilpotent*) in M if ϕ_x is not injective (rest. is nilpotent). A submodule Q of M is *primary in M* if $Q \neq M$ and every zero-divisor in M/Q is nilpotent.

Show that if Q is primary in M, then $Q : M$ is a primary ideal and hence $r_M(Q)$ is a prime ideal $\mathfrak{p}$. We say that Q is $\mathfrak{p}$-primary (in M).

Prove the analogues of (4.3) and (4.4).

Proof. Let $\mathfrak{q} = (Q : M) = \operatorname{Ann}(M/Q)$, and $xy \in \mathfrak{q}$ with $x \notin \mathfrak{q}$. Then $xy(M/Q) = 0$ and $y(M/Q) \neq M/Q$, y is a zero-divisor of M/Q, it is then nilpotent, i.e. $y \in r\big(\operatorname{Ann}(M/Q)\big)$. Hence $\mathfrak{q} = (Q : M)$ is primary.

(4.3) If Q_i $(1 \leq i \leq n)$ are $\mathfrak{p}$-primary, then $Q = \bigcap_{i=1}^n Q_i$ is $\mathfrak{p}$-primary.
Clearly $r_M(Q) = \mathfrak{p}$. Let $x, y \in A$ such that $xy \in Q$ and $y \notin Q$. Then $xy \in Q_i$ and $y \notin Q_i$ for some i, and $x \in r_M(Q_i) = \mathfrak{p} = r_M(Q)$. Hence Q is $\mathfrak{p}$-primary.

(4.4) Let $Q \subseteq M$ be a $\mathfrak{p}$-primary module, $x \in M$. Then

i) If $m \in Q$ then $(Q : m) = (1)$.
Clear.

ii) If $m \notin Q$ then $(Q : m)$ is $\mathfrak{p}$-primary, and therefore $r(Q : m) = \mathfrak{p}$.
Let $x \in (Q : m)$, since $m \notin Q$ and Q is primary, $x \in r(Q : M) = \mathfrak{p}$. Besides, let $x, y \in A$ such that $xy \in (Q : m)$ and $y \notin (Q : m)$. Then $x(ym) \in Q$ and $ym \notin Q$ force $x \in r_M(Q) = \mathfrak{p}$, hence $(Q : m)$ is a $\mathfrak{p}$-primary ideal in A.

iii) If $x \notin \mathfrak{p}$ then $(Q : x) = Q$.
Let $m \in M \setminus Q$, suppose $xm \in Q$, we have $x \in r(Q : M) = \mathfrak{p}$ since Q is primary, contradiction. ■

Problem 22. A *primary decomposition of N in M* is a representation of N as an intersection

$$N = Q_1 \cap \cdots \cap Q_n$$

of primary submodules of M; it is a *minimal primary decomposition* if the ideals $\mathfrak{p}_i = r_M(Q_i)$ are all distinct and if none of the components Q_i can be omitted from the intersection, that is if $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$ $(1 \leq i \leq n)$.

Prove the analogue of (4.5), that the prime ideals $\mathfrak{p}_i$ depend only on N (and M). They are called the *prime ideals belonging to N in M*. Show that they are also the prime ideals belonging to 0 in M/N.

Proof. Let $m \in M$, if $r(N : m) = \bigcap_{i=1}^n r(Q_i : m)$ is a prime ideal, by Proposition 1.11 ii) $r(N : m) = r(Q_i : m)$ for some i, hence by analog of Lemma 4.4 ii) $r(N : m) = r_M(N) = \mathfrak{p}_i$ (it cannot be the case (1) since it has to be a prime ideal) is a prime ideal belonging to N in M.

On the other hand, let $m_k \in \left(\bigcap_{j\neq k} Q_j\right) \setminus Q_k$, then $r(N : m_k) = \bigcap_{i=1}^n r(Q_i : m_k) = \mathfrak{p}_k$. Hence the set $\{\mathfrak{p}_i\}_{i=1}^n$ is exactly the set of prime ideals that occur in $\{(N : m) \mid m \in M\}$, and hence is independent of the decomposition.

Quotient everything by N we see that $\{\mathfrak{p}_i\}_{i=1}^n$ are also those prime ideals belonging to 0 in M/N. ■

Problem 23. State and prove the analogues of (4.6)–(4.11) inclusive. (There is no loss of generality in taking $N = 0$.)

Proof. Throughout the statements and proofs we assume that $N = \bigcap_{i=1}^n Q_i$ is a minimal primary decomposition, and $\mathfrak{p}_i = r_M(Q_i)$. Denote $S(N) = (S^{-1}N)^{\mathrm{c}}$.

(4.6) Any prime ideal $\mathfrak{p} \supseteq r_M(N)$ contains a minimal prime ideal belonging to N in M.
Since $\mathfrak{p} \supseteq r_M(N) = \bigcap_{i=1}^n r_M(Q_i) = \bigcap_{i=1}^n \mathfrak{p}_i$, by Proposition 1.11 ii) $\mathfrak{p}_i \supseteq \mathfrak{p}$ for some i. Hence $\mathfrak{p}$ contains a minimal prime ideal belonging to N in M.

(4.7) $\bigcup_{i=1}^n \mathfrak{p}_i = \{x \in A \mid (N : x) \neq N\}$.
We have

$$\bigcup_{m\in M\setminus N} (N : m) = r\Big(\bigcup_{m\in M\setminus N} (N : m)\Big) = \bigcup_{m\in M\setminus N} r(N : m)$$
$$= \big\{x \in A \mid (N : x) \neq N\big\} \subseteq \big\{x \in A \mid \exists i(x \in \mathfrak{p}_i)\big\} = \bigcup_{i=1}^n \mathfrak{p}_i,$$

hence $\{x \in A \mid (N : x) \neq N\} \subseteq \mathfrak{p}_i = r(N : m_i)$ for some i and $m_i \in M \setminus N$. Therefore $\bigcup_{i=1}^n \mathfrak{p}_i = \{x \in A \mid (N : x) \neq N\}$.

(4.8) Let S be a multiplicatively closed subset of A, and Q be a $\mathfrak{p}$-primary submodule of M.

i) If $S \cap \mathfrak{p} \neq \varnothing$, then $S^{-1}Q = S^{-1}M$.
Let $s \in S \cap \mathfrak{p}$, then $s^n M \subseteq N$ for some n. Hence $S^{-1}N = S^{-1}M$.

ii) If $S \cap \mathfrak{p} = \varnothing$, then $S^{-1}Q$ is $S^{-1}\mathfrak{p}$-primary and its contraction in M is Q.
Since $S \cap \mathfrak{p} = \varnothing$, $s \in S$, $m \in M$ and $sm \in Q$ implies $m \in Q$. We have $r_{S^{-1}M}(S^{-1}Q) = S^{-1}\big(r_M(Q)\big) = S^{-1}\mathfrak{p}$ and $(S^{-1}Q)^{\mathrm{c}} = \bigcup_{s\in S}(Q : s) = Q$. Besides, $S^{-1}M/S^{-1}Q \cong S^{-1}(M/Q)$ implies $S^{-1}Q$ is $S^{-1}\mathfrak{p}$-primary in M.

(4.9) Let S be a multiplicatively closed subset of A, and Q_i are numbered so that S meets $\mathfrak{p}_{m+1}, \dots, \mathfrak{p}_m$ but not $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then

$$S^{-1}N = \bigcap_{i=1}^m S^{-1}Q_i, \quad S(N) = \bigcap_{i=1}^m Q_i,$$

and these are minimal decompositions.
$S^{-1}N = S^{-1}\bigcap_{i=1}^{n} S^{-1}Q_i = \bigcap_{i=1}^{m} S^{-1}Q_i$, contract this equation we get the second one. Since $S^{-1}Q_i$ is $S^{-1}\mathfrak{p}_i$-primary, the first one is a primary decomposition, and easy to see that they are both minimal.

(4.10) Let $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_m}\}$ be an isolated set of prime ideals of N in M. Then $Q_{i_1} \cap \cdots \cap Q_{i_m}$ is independent of the decomposition.
Let $S = A \setminus \bigcup_{j=1}^{m} \mathfrak{p}_{i_j}$, then $S(N) = Q_{i_1} \cap \cdots \cap Q_{i_m}$ is independent of the choice of decomposition.

(4.11) The isolated primary components are uniquely determined by N (in M).
Clear. ■

Chapter 5

Integral Dependence and Valuations

Problem 1. Let $f : A \to B$ be an integral homomorphism of rings. Show that $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a *closed* mapping, i.e. that it maps closed sets to closed sets. (This is a geometrical equivalent of (5.10).)

Proof. Let $\mathfrak{b} \subseteq B$ be an ideal, and let $\mathfrak{a} = f^{-1}(\mathfrak{b})$. Since B is integral over $f(A)$, for every prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$, by Theorem 5.10, there is a prime ideal $\mathfrak{q} \subseteq B$ such that $\mathfrak{q} \cap f(A) = f(\mathfrak{p})$, hence $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ (recall $\operatorname{im} f \cong A/\ker f$, and $\ker f \subseteq \mathfrak{a}$). So the map $f^*|_{V(\mathfrak{b})} : V(\mathfrak{b}) \to V(\mathfrak{a})$ is surjective, and that f^* is a closed mapping. ■

Problem 2. Let A be a subring of a ring B such that B is integral over A, and let $f : A \to \Omega$ be a homomorphism of A into an algebraically field Ω. Show that f can be extended to a homomorphism of B into Ω. [Use (5.10).]

Proof. Let $\mathfrak{p} = \ker f$, then $\mathfrak{p}$ is a prime ideal,[1] f induces a map $\bar{f} : A/\mathfrak{p} \to \Omega$. Since B is integral over A, by Theorem 5.10 there is a prime ideal $\mathfrak{q} \subseteq B$ such that $\mathfrak{p} = \mathfrak{q} \cap A$, then by Proposition 5.6 $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$, $\operatorname{Frac}(B/\mathfrak{q})/\operatorname{Frac}(A/\mathfrak{p})$ is an algebraic extension.

We can then extend $\bar{f}$ to $\bar{f} : \operatorname{Frac}(A/\mathfrak{p}) \hookrightarrow \Omega$, therefore we have an extension $\bar{f} : \operatorname{Frac}(B/\mathfrak{q}) \hookrightarrow \Omega$.[2] The restriction of $\bar{f}$ gives a homomorphism $B/\mathfrak{q} \to \Omega$, and the composite $B \to B/\mathfrak{q} \to \Omega$ is an extension of f. ■

Problem 3. Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A B \to B' \otimes_A C$ is integral. (This includes (5.6) ii) as a special case.

[1] Since the image of A is a subring of a field, hence is integral.
[2] This is a basic result from field theory.

Proof. Let $x \in B'$, then there are $b_i \in B$, $b_n = 1$, $\sum_{i=0}^n b_i x^i = 0$. For every $c \in C$, in $B' \otimes_A C$ we have $\left(\sum_{i=0}^n b_i x^i\right) \otimes c^n = \sum_{i=0}^n (b_i \otimes c^{n-i})(x \otimes c)^i = 0$, hence $x \otimes c$ is integral over $B \otimes_A C$, and the map $f \otimes 1$ is integral. ■

Problem 4. Let A be a subring of a ring B such that B is integral over A. Let $\mathfrak{n}$ be a maximal ideal of B and let $\mathfrak{m} = \mathfrak{n} \cap A$ be the corresponding maximal ideal of A. Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?
[Consider the subring $k[x^2 - 1]$ of $k[x]$, where k is a field, and let $\mathfrak{n} = (x - 1)$. Can the element $1/(x + 1)$ be integral?]

Proof. Follow the hints. Let $\mathbb{Q} \subseteq k$ be a field (infinite field), $A = k[x^2 - 1]$, $B = k[x]$. Let $\mathfrak{n} = (x - 1)$, then $\mathfrak{m} = (x^2 - 1)$. If $1/(x + 1)$ is integral over $A_{\mathfrak{m}}$, let

$$\sum_{i=0}^{n} \frac{a_i}{s_i} \cdot \frac{1}{(x+1)^i} = 0$$

for some $a_i \in A$, $s_i \in A \setminus \mathfrak{m}$, such that $a_n/s_n = 1$. Then we have

$$1 + \sum_{i=0}^{n-1} \frac{a_i}{s_i} \cdot (x+1)^{n-i} = 0,$$

substitute $x = -1$ yields $1 = 0$, contradiction. So this result may not be true. ■

Problem 5. Let $A \subseteq B$ be rings, B integral over A.

i) If $x \in A$ is a unit in B then it is a unit in A.

ii) The Jacobson radical of A is the contraction of the Jacobson radical on B.

Proof.

i) Let $x \in A$, $y \in B$ such that $xy = 1$. Since y is integral over A, there is $\sum_{i=0}^n a_i y^i = 0$ for some $a_i \in A$, and $a_n = 1$. Then

$$\sum_{i=0}^{n} a_i x^n y^i = \sum_{i=0}^{n} (a_i x^{n-i})(xy)^i = \sum_{i=0}^{n} a_i x^{n-i} = 1 + \sum_{i=0}^{n-1} a_i x^{n-i} = 0,$$

hence $x^{-1} = -\sum_{i=0}^{n-1} a_i x^{n-i-1} \in A$, and x is a unit in A.

ii) $x \in A$ is in the Jacobson radical of A iff $1 - xy \in A^\times$ for every $y \in A$, so "$\subseteq$" is clear. Let $x \in A$ be in the Jacobson radical of B, then $1 - xy \in B^\times$ for every $y \in A$. Since B is integral over A, from i) we know that for every $y \in A$, $1 - xy \in A^\times$, hence it is in the Jacobson radical of A. ■

Problem 6. Let $B_1, \ldots, B_n$ be integral A-algebras. Show that $\prod_{i=1}^n B_i$ is an integral A-algebra.

Proof. For each $x_i \in B_i$, let $\sum_{j=0}^{n_i} a_{ij} x_i^j = 0$ for some $a_{ij} \in B_i$ and $a_{in_i} = 1$. Let $b = (b_1, \dots, b_n)$, then

$$\prod_{i=1}^{n} \Big(\sum_{j=0}^{n_i} a_{ij} b^j \Big) = 0.$$

Hence $\prod_{i=1}^{n} B_i$ is an integral A-algebra. ■

Problem 7. Let A be a subring of a ring B, such that the set $B \setminus A$ is closed under multiplication. Show that $A$is integrally closed in B.

Proof. If there are some elements in $B \setminus A$ integral over A, we let $x \in B \setminus A$ be such an element such that its equation $\sum_{i=0}^{n} a_i x^i = 0$ ($a_n = 1$) is of the smallest degree n. Then we must have $n > 1$, and

$$-a_0 = x \Big(\sum_{i=1}^{n} a_i x^{i-1} \Big) \in A.$$

Since $B \setminus A$ is closed under multiplication, we must have $\sum_{i=1}^{n} a_i x^{i-1} \in A$, this gives an equation of smaller degree (still > 0), contradiction. Hence every integral element in B is in A, i.e. A is integrally closed in B. ■

Problem 8.

i) Let A be a ring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then f, g are in $C[x]$. [Take a field containing B in which the polynomials f, g split into linear factors: say $f = \prod (x - \xi_i)$, $g = \prod (x - \eta_j)$. Each ξ_i and each η_j is a root of fg, hence is integral over C. Hence the coefficients of f and g are integral over C.]

ii) Prove the same result without assuming that B (or A) is an integral domain.

Proof.

i) Since B is integral, let $k \supseteq B$ be a splitting field of f, g. Then $f = \prod_i (x - \xi_i)$, $g = \prod_j (x - \eta_j)$, and each ξ_i, η_j is integral over B, hence integral over A. Since every coefficient of f and g is generated by ξ_i and η_j respectively, and since C is the integral closure of A in B, we must have $f, g \in C[x]$.

ii) We first prove a lemma.

> **Lemma 5.0.1.** Let $h \in B[x]$ be a monic polynomial, there is an extension ring D of B such that $h(x)$ splits in $D[x]$.

Proof of Lemma. Let $n = \deg h$, we use induction on n. The case $n = 1$ is trivial, if $n > 1$, there is a natural injection $B \hookrightarrow B[t]/(h(t)) =: D_1$. As a polynomial in $D_1[x]$, $h(x)$ has a root t, hence $h(x) = (x-t)\bar{h}(x)$, $\deg \bar{h} = n-1$. Repeat this process, we get a ring extension $B \subseteq D$, and $D[x]$ in which $h(x)$ completely splits.[3] $\square$

Back to the problem. Let D be an extension ring of B such that $f(x), g(x)$ splits in $D[x]$. Let $f = \prod_i (x - \xi_i)$ and $g = \prod_j (x - \eta_j)$. Then each of ξ_i, η_j is integral over A, hence the coefficients of f and g (as combinations of ξ_i and η_j) are integral over A, are in B, and hence $f, g \in C[x]$. $\blacksquare$

Problem 9. Let A be a subring of a ring B and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$. [If $f \in B[x]$ is integral over $A[x]$, then

$$f^m + g_1 f^{m-1} + \cdots + g_m = 0 \quad (g_i \in A[x]).$$

Let r be an integer larger than m and the degrees of $g_1, \dots, g_m$, and let $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \cdots + g_m = 0$$

or say

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0,$$

where $h_m = (x^r)^m + g_1(x^r)^{m-1} + \cdots + g_m \in A[x]$. Now apply Exercise 8 to the polynomials $-f_1$ and $f_i^{m-1} + h_1 f_1^{m-1} + \cdots + h_{m-1}$.]

Proof. Follow the hints. Let $f \in B[x]$ integral over $A[x]$, write

$$\sum_{i=0}^{n} g_i(x) f^i(x) = 0$$

for some $g_i \in A[x]$, $g_n(x) = 1$. Let $r \geq \max\{n, \deg g_0, \dots, \deg g_{n-1}\}$, and let $f_1 = f - x^r$. Then

$$\sum_{i=0}^{n} g_i(x)\big(f_1(x) + x^r\big)^i = \sum_{i=0}^{n} h_i(x) f_1^i(x) = 0,$$

where $h_0(x) = \sum_{i=0}^{n} g_i(x) x^{ri} \in A[x]$. Hence

$$f_1(x) \sum_{i=1}^{n} h_i(x) f_1^{i-1}(x) \in A[x]$$

[3]Reference: Rankeya (`https://math.stackexchange.com/users/16981/rankeya`). "Two questions about integral 'splitting ring' extensions." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/130540`, 2014.

is a product of two monic polynomials in $B[x]$. By Exercise 8, $f_1 \in C[x]$, and then $f \in C[x]$.

On the other hand, let $f(x) = cx^m \in C[x]$, then $f(x)$ in clearly integral over $A[x]$. Since integral closure is a ring, we see that $C[x] = \sum_{n=0}^{\infty} Cx^n$ is integral over $A[x]$. Hence $C[x]$ is the integral closure of $A[x]$ in $B[x]$. ■

Problem 10. A ring homomorphism $f : A \to B$ is said to have the *going-up property* (resp. the *going-down property*) if the conclusion of the going-up theorem (5.11) (resp. the going-down theorem (5.16)) holds for B and its subring $f(A)$.

Let $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the mapping associated with f.

i) Consider the following three statements:

(a) f^* is a closed mapping.

(b) f has the going-up property.

(c) Let $\mathfrak{q}$ be any prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(A/\mathfrak{p})$ is surjective.

Prove that (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). (See also Chapter 6, Exercise 11.)

ii) Consider the following three statements:

(a') f^* is an open mapping.

(b') f^* has the going-down property.

(c') For any prime ideal $\mathfrak{q}$ of B, if $\mathfrak{p} = \mathfrak{q}^c$, then $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective.

Prove that (a') $\Rightarrow$ (b') $\Leftrightarrow$ (c'). (See also Chapter 7, Exercise 23.)

[To prove that (a') $\Rightarrow$ (c'), observe that $B_\mathfrak{q}$ is the direct limit of the rings B_t where $t \in B \setminus \mathfrak{q}$; hence, by Chapter 3, Exercise 26, we have $f^*\big(\mathrm{Spec}(B_\mathfrak{q})\big) = \bigcap_t f^*\big(\mathrm{Spec}(B_t)\big)$. Since Y_t is an open neighborhood of $\mathfrak{q}$ in Y, and since f^* is open, it follows that $f^*(Y_t)$ is an open neighborhood of $\mathfrak{p}$ in X and therefore contains $\mathrm{Spec}(A_\mathfrak{p})$.]

Proof.

i) (a) $\Rightarrow$ (b) Let $\mathfrak{q}_1 \subseteq B$ be a prime ideal and $\mathfrak{p}_1 = f^{-1}(\mathfrak{q}_1)$, then $f^*\big(V(\mathfrak{q}_1)\big) = V(\mathfrak{a})$ is closed. Since $\mathfrak{p}_1 \in V(\mathfrak{a})$, we must have $f^*\big(V(\mathfrak{q}_1)\big) = V(\mathfrak{p}_1)$. This shows that for every prime ideal $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$, there is a prime ideal $\mathfrak{q}_2 \supseteq \mathfrak{q}_2$, such that $f^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$, this is just the going-up property.

(b) $\Rightarrow$ (c) For every prime ideal $\mathfrak{p}_1 \supseteq \mathfrak{p}$, by the going-up property, there is a prime ideal $\mathfrak{q}_1 \supseteq \mathfrak{q}$ such that $f^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$. Hence f^* is surjective.

(c) $\Rightarrow$ (b) Let $\mathfrak{q}_1$ be a prime ideal in B, let $\mathfrak{p}_1 = f^{-1}(\mathfrak{q}_1)$. Then for any prime ideal $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$, there is a prime ideal $\mathfrak{q}_2 \supseteq \mathfrak{q}_1$, such that $\mathfrak{p}_2 = f^{-1}(\mathfrak{q}_2)$. So f has the going-up property.

ii) (a') $\Rightarrow$ (c') Given a prime ideal $\mathfrak{q} \subseteq B$ and $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Let $f_t : A \to B \to B_t$. Since $B_{\mathfrak{q}} = \varinjlim_{t \notin \mathfrak{q}} B_t$, by Exercise 26 of Chapter 3, $f^*\big(\operatorname{Spec}(B)\big) = \bigcap_{t \notin \mathfrak{p}} f_t^*\big(\operatorname{Spec}(B_t)\big) \supseteq \operatorname{Spec}(A_{\mathfrak{p}})$. Hence f^* is surjective.

(b') $\Rightarrow$ (c') Let $\mathfrak{q}_1 \subseteq B$ be a prime ideal, and let $\mathfrak{p}_1 = f^{-1}(\mathfrak{q})$. Since f has the going-down property, for every prime ideal $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, there is a prime ideal $\mathfrak{q}_2 \subseteq B$ such that $\mathfrak{p}_2 = f^{-1}(\mathfrak{q}_2)$. Since there is a natural correspondence between prime ideals in A and prime ideals in $A_{\mathfrak{p}_1}$, and the same for B and $B_{\mathfrak{q}_1}$, we have $f^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$ as ideals in $B_{\mathfrak{q}_1}$ and $A_{\mathfrak{p}_1}$.

$$\begin{tikzcd} A_{\mathfrak{p}_1} \arrow[r] & B_{\mathfrak{q}_1} \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

(c') $\Rightarrow$ (b') Almost the same. Let $\mathfrak{q}_1 \subseteq B$ be a prime ideal, and let $\mathfrak{p}_1 = f^{-1}(\mathfrak{q}_1)$. Then for every prime ideal $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, extend it in $A_{\mathfrak{p}_1}$, there is a prime ideal $\mathfrak{q}_2 \subseteq B$ such that the inverse image of its extension in $B_{\mathfrak{q}_1}$ is the extension of $\mathfrak{p}_2$. Then $\mathfrak{p}_2 = f^{-1}(\mathfrak{q}_2)$. ∎

Problem 11. Let $f : A \to B$ be a flat homomorphism of rings. Then f has the going-down property. [Chapter 3, Exercise 18.]

Proof. By Exercise 18 of Chapter 3, the map $f^* : \operatorname{Spec}(B_{\mathfrak{q}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is surjective for some $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Hence by Exercise 10 ii) (c') $\Rightarrow$ (b'), f has the going-down property. ∎

Problem 12. Let G be a finite group of automorphisms of a ring A, and let A^G denote the subring of G-invariants, that is of all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$. Prove that A is integral over A^G. [If $x \in A$, observe that x is a root of the polynomial $\prod_{\sigma \in G}\big(t - \sigma(x)\big)$.]

Let S be a multiplicatively closed subset of A such that $\sigma(S) \subseteq S$ for all $\sigma \in G$, and let $S^G = S \cap A^G$. Show that the action of G on A extends to an action on $S^{-1}A$, and that $(S^G)^{-1}A^G \cong (S^{-1}A)^G$.

Proof. For every $a \in A$, let $f(x) = \prod_{\sigma \in G}(x - \sigma a)$. Then $f(x)$ is monic, $f(a) = 0$, for every $\sigma \in G$, $f^\sigma = f$, hence $f \in A^G[x]$. And hence a is integral over A^G, and therefore A is integral over A^G.

For every $\sigma \in G$ and $a/s \in S^{-1}A$, define $\sigma(a/s) = \sigma a/\sigma s$. Obviously this definition is an group action, it is also well-defined, for if $a/s = a'/s'$, then there is $t \in S$, $t(s'a - sa') = 0$, this implies $\sigma(t)\big(\sigma(s')\sigma(a) - \sigma(s)\sigma(a')\big) = 0$.

Then we let $h : (S^G)^{-1}A^G \to (S^{-1}A)^G$ be the natural map. For surjectivity, let $a/1 \in (S^{-1}A)^G$,[4] then for each τ there is $t_\tau \in S$ such that $t_\tau\big(a - \tau(a)\big) = 0$. Let

$$t = \prod_{\sigma \in G} \prod_{\tau \in G} \sigma(t_\tau),$$

then $t\big(a - \tau(a)\big) = 0$ for every $\tau \in G$, hence $a/1 = ta/t$ with $t \in S^G$, $ta \in A^G$. For injectivity, if $a/s = 0$ in $(S^{-1}A)^G$, we may assume $s \in S^G$ and $a \in A^G$, then there is $t \in S$, $ta = 0$. Therefore $\prod_{\sigma \in G} \sigma(t)a = 0$, $a/s = 0$ in $(S^G)^{-1}A^G$. So h is an isomorphism. ■

Problem 13. In the situation of Exercise 12, let $\mathfrak{p}$ be a prime ideal fo A^G, and let P be the set of prime ideals of A whose contraction is $\mathfrak{p}$. Show that G acts transitively on P. In particular, P is *finite*.
[Let $\mathfrak{p}_1, \mathfrak{p}_2 \in P$ and let $x \in \mathfrak{p}_1$. Then $\prod_\sigma \sigma(x) \in \mathfrak{p}_1 \cap A^G = \mathfrak{p} \subseteq \mathfrak{p}_2$, hence $\sigma(x) \in \mathfrak{p}_2$ for some $\sigma \in G$. Deduce that $\mathfrak{p}_1$ is contained in $\bigcup_{\sigma \in G} \sigma(\mathfrak{p}_2)$, and then apply (1.11) and (5.9).]

Proof. Follow the hints. Let $\mathfrak{q}_1, \mathfrak{q}_2 \in P$, for $x \in \mathfrak{q}_1$, $\prod_{\sigma \in G} \sigma(x) \in \mathfrak{q} \cap A^G \subseteq \mathfrak{p} \subseteq \mathfrak{q}_2$. Hence $x \in \tau(\mathfrak{q}_2)$ for some $\tau \in G$, $\mathfrak{q}_1 \subseteq \bigcup_{\sigma \in G} \sigma(\mathfrak{q}_2)$, and hence by Proposition 1.11 i) $\mathfrak{q}_1 \subseteq \sigma(\mathfrak{q}_2)$ for some $\sigma \in G$. By Exercise 12 A is integral over A^G, by Corollary 5.9 $\sigma(\mathfrak{q}_1) = \mathfrak{q}_2$, hence the action of G on P is transitive. ■

Problem 14. Let A be an integrally closed domain, K its field of fractions and L a finite normal separable extension of K. Let G be the Galois group of L over K and let B be the integral closure of A in L. Show that $\sigma(B) = B$ for all $\sigma \in G$, and that $A = B^G$.

Proof. For every $x \in B$, suppose x is a root of a monic polynomial $f(t) \in A[t]$, then for every $\sigma \in G$, $f^\sigma = f$. Since f splits in L, $\sigma(x)$ is integral over A (the same for $\sigma^{-1}(x)$), hence $\sigma(B) = B$ for every $\sigma \in G$. We know that $K = L^G$, so $B^G = B \cap L^G = B \cap K = A$ (the last equality is because A is integrally closed). ■

Problem 15. Let A, K be as in Exercise 14, let L be any finite extension field of K, and let B be the integral closure of A in L. Show that, if $\mathfrak{p}$ is any prime ideal of A, then the set of prime ideals $\mathfrak{q}$ of B which contract to $\mathfrak{p}$ is finite (in other words, that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has finite fibers).
[Reduce to the two cases (a) L separable over K and (b) L purely inseparable over K. In case (a), embed L in a finite normal separable extension of K, and use Exercise 13 and 14. In case (b), if $\mathfrak{q}$ is a prime ideal of B such that $\mathfrak{q} \cap A = \mathfrak{p}$, show that $\mathfrak{q}$ is the set of all $x \in B$ such that $x^{p^m} \in \mathfrak{p}$ for some $m \geq 0$, where p is the characteristic of K, and hence that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is bijective in this case.]

[4] This assumption is to simplify the proof. Anyway, we can reduce a/s to $a/1$ in a similar manner.

Proof. We first discuss two special cases.

(a) L/K is separable, let $L'/L/K$ be a finite Galois extension, B' the integral closure of A in L'. Fix a prime ideal $\mathfrak{p} \subseteq A$, denote by $P \subseteq \operatorname{Spec}(B)$ and $P' \subseteq \operatorname{Spec}(B')$ the set of prime ideals whose contraction in A is $\mathfrak{p}$. Then the Galois group $G = \operatorname{Gal}(L'/K)$ is finite, by Exercise 13 and Exercise 14, P' is finite. Since for every prime $\mathfrak{q} \in P$, by Theorem 5.10 there is a prime ideal $\mathfrak{q}' \subseteq \operatorname{Spec}(B')$ such that $\mathfrak{q}' \cap B = \mathfrak{q}$, this implies $\mathfrak{q}' \cap A = \mathfrak{p}$, hence $\mathfrak{q}' \in P'$. This induces a surjection $P' \to P$, hence P is finite, we are done.

(b) L/K is purely inseparable, and we don't assume that A is integrally closed. Then for every $x \in L$, $x^{p^m} \in K$ for some $m \geq 0$. Let $\mathfrak{p} \subseteq A$, $\mathfrak{q} \subseteq B$ be a prime ideal such that $\mathfrak{q} \cap A = \mathfrak{p}$. Then for every $x \in \mathfrak{q}$, $x^{p^m} \in A \implies x^{p^m} \in \mathfrak{p} \implies x \in \mathfrak{p}$, hence $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is bijective.

Back to the general case, let L_{s} be the set of all separable elements in L over K, that is,

$$L_{\mathrm{s}} = \big\{x \in L \mid [K(x) : K]_{\mathrm{s}} = [K(x) : K]\big\}.$$

L_{s} is a field, since for $x, y \in L$ separable over K, $[K(x,y) : K]_{\mathrm{s}} = [K(x,y) : K(x)]_{\mathrm{s}}[K(x) : K]_{\mathrm{s}} = [K(x,y) : K]$. Then L/L_{s} is purely inseparable, $B \cap L_{\mathrm{s}}$ is the integral closure of A in L_{s}, and B is the integral closure of $A \subseteq B \cap L_{\mathrm{s}}$ in L.

For every prime ideal $\mathfrak{p} \subseteq A$, by (a) we see that there are only finitely many prime ideals $\mathfrak{q} \subseteq B \cap L_{\mathrm{s}}$ such that $\mathfrak{q} \cap A = \mathfrak{p}$, by (b) there is a bijection between such prime ideals in $B \cap L_{\mathrm{s}}$ and such prime ideals in B, hence the prime ideals $\mathfrak{q} \subseteq B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$ are finite, and the map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ has finite fibers. ■

Noether's normalization lemma

Problem 16. Let k be a field and let $A \neq 0$ be a finitely generated k-algebra. Then there exist elements $y_1, \ldots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \ldots, y_r]$.

We shall assume that k is *infinite*. (The result is still true if k is finite, but a different proof is needed.) Let $x_1, \ldots, x_n$ generate A as a k-algebra. We can renumber the x_i so that $x_1, \ldots, x_r$ are algebraically independent over k and each of $x_{r+1}, \ldots, x_n$ is algebraic over $k[x_1, \ldots, x_r]$. Now proceed by induction on n. If $n = r$ there is nothing to do, so suppose $n > r$ and the result is true for $n-1$ generators. The generators x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$, hence there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \ldots, x_{n-1}, x_n) = 0$. Let F be the homogeneous part of highest degree in f. Since k is infinite, there exist $\lambda_1, \ldots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$. Put $x_i' = x_i - \lambda_i x_n$ $(1 \leq i \leq n-1)$. Show that x_n is integral over the ring $A' = k[x_1,' \ldots, x_{n-1}']$, and hence that A is integral over A'. Then apply induction hypothesis to A' to complete the proof.

From the proof it follows that $y_1, \ldots, y_r$ may be chosen to be linear combinations of $x_1, \ldots, x_n$. This has the following geometric interpretation: if k is algebraically closed and X is an affine algebraic variety in k^n with coordinate ring $A \neq 0$, then there exists a linear subspace L of dimension r in k^n and a linear mapping of k^n onto L which maps X *onto* L. [Use Exercise 2.]

Proof. Let $A = k[x_1, \ldots, x_n]$. We can find a maximal subset, by a suitable renumbering say $\{x_1, \ldots, x_r\}$, of $\{x_1, \ldots, x_n\}$ such that $x_1, \ldots, x_r$ are algebraically independent over k, and then for each $r < i \leq n$, x_i is algebraic over $k[x_1, \ldots, x_r]$.

We use induction on the number of generators n.[5] The case $n = r$ (that is, A is free) is trivial. Suppose $n > r$ and the case $n - 1$ is proved. Since x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$, there is a polynomial $f \in k[t_1, \ldots, t_n]$, $f \neq 0$ such that $f(x_1, \ldots, x_{n-1}, x_n) = 0$.

Let $F = f^{(\deg f)}$ be the homogeneous part of f of degree $\deg f$, since k is infinite, there are $\lambda_1, \ldots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$. Let $x'_i = x_i - \lambda_i x_n$ for $1 \leq i \leq n-1$, and let $A' = k[x'_1, \ldots, x'_{n-1}]$, then

$$f\big((x'_1 + \lambda_1 x_n), \ldots, (x'_{n-1} + \lambda_{n-1} x_n), x_n\big) = 0.$$

Expend this express in terms of x_n we see that x_n is a root of a polynomial in $A'[t_n]$ (with leading coefficient $F(\lambda_1, \ldots, \lambda_{n-1}), 1 \neq 0)$, and since k is a field we can make this polynomial monic. Hence x_n is integral over A', and hence A is integral over A'. Since A' is generated by $n-1$ elements, by induction hypothesis, there are algebraically independent $y_1, \ldots, y_{r'} \in A'$ such that A' is integral over $k[y_1, \ldots, y_{r'}]$, and A is integral over $k[y_1, \ldots, y_{r'}]$.

For the geometric interpretation, let $\xi_i = t_i + I(X) \in k[t_1, \ldots, t_n]/I(X) = A$. By Noether's normalization lemma and the remark, there are some algebraically independent elements $y_1, \ldots, y_r \in A$ $(r \leq n)$, such that we can write $y_j = \sum_{i=1}^n a_{ji}\xi_i$. Then the matrix

$$\varphi = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rn} \end{bmatrix} : k^n \to L \cong k^r$$

induces a linear transformation, where L is spanned by $y_1, \ldots, y_r$ over k. Since $[y_1 \cdots y_r] = \varphi[\xi_1 \cdots \xi_n]^{\mathrm{t}}$ are algebraically independent over k, they are then linearly independent over k, φ is of rank r, therefore φ is surjective. ∎

Nullstellensatz (weak form).

Problem 17. Let X be an affine algebraic variety in k^n, where k is an algebraically closed field, and let $I(X)$ be the ideal of X in the polynomial ring $k[t_1, \ldots, t_n]$ (Chapter 1, Exercise 27). If $I(X) \neq (1)$ then X is not empty. [Let $X = k[t_1, \ldots, t_n/I(X)$ be the coordinate ring of X. Then $A \neq 0$, hence by Exercise 16 there exists a linear subspace L of dimension ≥ 0 in k^n and a mapping of X *onto* L. Hence $X \neq \varnothing$.]

Deduce that every maximal ideal in the ring $k[t_1, \ldots, t_n]$ is of the form $(t_1 - a_1, \ldots, t_n - a_n)$ where $a_i \in k$.

[5] We do not fix the algebra A, moreover, r is not fixed, it depends on the underlying ring.

Proof. If $X = \varnothing$ then $I(X) = (1)$ (why we adapt such a complicated geometric proof).

Let $\mathfrak{m} \subseteq k[t_1, \dots, t_n]$ be a maximal ideal, then $k \subseteq k[t_1, \dots, t_n]/\mathfrak{m}$ is a field. By Noether's normalization lemma, there is a subring $R = k[t'_1, \dots, t'_r] \subseteq k[t_1, \dots, t_n]/\mathfrak{m}$ such that this is an integral extension. By Proposition 5.7 R is also a field, hence $R = k$. Since k is algebraically closed and $k[t_1, \dots, t_n]/\mathfrak{m}$ is integral over k, we must have $k = k[t_1, \dots, t_n]/\mathfrak{m}$. Let

$$k[t_1, \dots, t_n] \twoheadrightarrow k[t_1, \dots, t_n]/\mathfrak{m} = k$$

be the canonical map, and let $a_i \in k$ be the image of t_i, we must then have $\mathfrak{m} = (t_1 - a_1, \dots, t_n - a_n)$.[6] ∎

Problem 18. Let k be a field and let B be a finitely generated k-algebra. Suppose that B is a field. Then B is a finite algebraic extension of k. (This is another version of Hilbert's Nullstellensatz. The following proof is due to Zariski. For other proofs, see (5.24), (7.9).)

Let $x_1 \dots, x_n$ generate B as a k-algebra. The proof is by induction on n. If $n = 1$ the result is clearly true, so assume $n > 1$. Let $A = k[x_1]$ and let $K = k(x_1)$ be the field of fractions of A. By the inductive hypothesis, B is a finite algebraic extension of K, hence each of $x_2, \dots, x_n$ satisfies a monic polynomial equation with coefficients in K, i.e. coefficients of the form a/b where a and b are in A. If f is the product of the denominators of all these coefficients, then each of $x_2, \dots, x_n$ is integral over A_f. Hence B and therefore K is integral over A_f.

Suppose x_1 is transcendental over k. Then A is integrally closed, because it is a unique factorization domain. Hence A_f is integrally closed (5.12), and therefore $A_f = K$, which is clearly absurd. Hence x_1 is algebraic over k, hence K (and therefore B) is a finite extension of k.

Proof. Well! ∎

Problem 19. Deduce the result of Exercise 17 from Exercise 18.

Proof. Let $\mathfrak{m} \subseteq k[t_1, \dots, t_n]$ be a maximal ideal, then $k[t_1, \dots, t_n]/\mathfrak{m}$ is a finitely generated k-algebra, a field extension of k, hence by Exercise 18 a finite algebraic extension of k.

Therefore, for each $1 \le i \le n$, there exists a nonzero polynomial $f \in k[t_i]$ such that $f(t_i) \in \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, we see that $k[t_i] \cap \mathfrak{m}$ is prime, of the form $(t_i - a_i)$ (or (0), but since f is in it, this will not happen). Hence $\mathfrak{m} = (t_1 - a_1, \dots, t_n - a_n)$. ∎

[6] Reference: zcn (`https://math.stackexchange.com/users/115654/zcn`). "Weak nullstellansatz in Atiyah-Macdonald 5.17." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/878520`, 2014.

Problem 20. Let A be a subring of an integral domain B such that B is finitely generated over A. Show that there exists $s \neq 0$ in A and elements $y_1, \dots, y_n$ in B, algebraically independent over A and such that B_s is integral over B'_s, where $B' = A[y_1, \dots, y_n]$. [Let $S = A \setminus \{0\}$ and let $K = S^{-1}A$, the field of fractions of A. Then $S^{-1}B$ is a finitely generated K-algebra and therefore by the normalization lemma (Exercise 16) there exists $x_1, \dots, x_n$ in $S^{-1}B$, algebraically independent over K and such that $S^{-1}B$ is integral over $K[x_1, \dots, x_n]$. Let $z_1, \dots, z_m$ generate B as an A-algebra. Then each z_j (regarded as an element in $S^{-1}B$) is integral over $K[x_1, \dots, x_n]$. By writing an equation of integral dependence for each z_j, show that there exists $s \in S$ such that $x_i = y_i/s$ $(1 \leq i \leq n)$ with $y_i \in B$, and such that each sz_j is integral over B'. Deduce that this s satisfies the conditions stated.]

Proof. Let $S = A \setminus \{0\}$, then by Noether's normalization lemma there are some $x_1, \dots, x_n \in S^{-1}B$ algebraically independent over $S^{-1}A$, such that B is integral over $S^{-1}A[x_1, \dots, x_n]$. Let $B = A[z_1, \dots, z_m]$, each $z_i/1$ is integral over $S^{-1}A[x_1, \dots, x_n]$ as an element in $S^{-1}B$, so we can write

$$\sum_{j=0}^{n_i} \frac{b_{ij}}{s_{ij}} z_i^j = 0$$

for some $b_{ij} \in A[x_1, \dots, x_n]$, $s_{ij} \in S$, and $b_{in_i} = s_{in_i} = 1$.

Since $x_i \in S^{-1}B$, we can choose $s' \in S$ such that $s'x_i \in B$ for all i. Let $s = s' \prod_{i,j} s_{ij}$, and $t_{ij} = s/s_{ij}$, then the above equations can be rewritten as

$$\sum_{j=0}^{n_i} \frac{t_{ij} b_{ij}}{s} z_i^j = 0.$$

Let $y_i = sx_i$. Since $b_{ij} \in A[x_1, \dots, x_n]$, for each i, j there is a large enough integer N_{ij} such that $s^{N_{ij}} b_{ij} \in A[y_1, \dots, y_n]$, we see that z_i is integral over $A_s[y_1, \dots, y_n]$. Hence $y_1, \dots, y_n$ are algebraically independent over A, and $B_s = A_s[z_1, \dots, z_m]$ is integral over B'_s. ■

Problem 21. Let A, B be as in Exercise 20. Show that there exists $s \neq 0$ in A such that, if Ω is an algebraically closed field and $f : A \to \Omega$ is a homomorphism for which $f(s) \neq 0$, then f can be extended to a homomorphism $B \to \Omega$. [With the notation of Exercise 20, f can be extended first of all to B', for example by mapping each y_i to 0; then to B'_s (because $f(s) \neq 0$) , and finally to B_s (by Exercise 2, because B_s is integral over B'_s).]

Proof. Use notation of Exercise 20. Since $y_1, \dots, y_n \in B$ algebraically independent over A, we can extend f to $A[y_1, \dots, y_n]$ (e.g. evaluation map), by localize at s it extends to $A_s[y_1, \dots, y_n] = B'_s$ (since $f(s) \neq 0$, we can localize Ω at $f(s)$). By Exercise 20, B_s is integral over B'_s, by Exercise 2, we can extend f further to B_s, it then induces an extension on B. ■

Problem 22. Let A, B be as in Exercise 20. If the Jacobson radical of A is zero, then so is the Jacobson radical of B.
[Let $v \neq 0$ be an element of B. We have to show that there is a maximal ideal of B which does not contain v. By applying Exercise 21 to the ring B_v and its subring A, we obtain an element $s \neq 0$ in A. Let $\mathfrak{m}$ be a maximal ideal of A such that $s \in \mathfrak{m}$, and let $k = A/\mathfrak{m}$. Then the canonical mapping $A \to k$ extends to a homomorphism g of B_v into an algebraic closure Ω of k. Show that $g(v) \neq 0$ and that $\ker g \cap B$ is a maximal ideal of B.]

Proof. For every $v \in B$, $v \neq 0$, we have an inclusion $A \subseteq B \subseteq B_v$, B_v is a finitely generated A-algebra. Let $s \in A$, $s \neq 0$, and let $\mathfrak{m} \subseteq A$ be a maximal ideal, then the canonical map $f : A \to A/\mathfrak{m} =: k$ does not send s to 0. Hence by Exercise 21 there are $s \in A$, $s \neq 0$ and a homomorphism $f : A \to k \to \Omega$ (e.g. the canonical map given above), where Ω is an algebraic closure of k, such that $f(s) \neq 0$ and we can extend f to B_v. then $f(v) \in A^\times$ is a unit, hence $f(v) \neq 0$.

Besides, since $k \subseteq f(B) \subseteq \Omega$, Ω is algebraic over k (hence integral over $f(B)$) and $\ker f|_B = \ker f \cap B$, we see that $f(B) \cong B/(\ker f \cap B)$ is a field (Proposition5.7). Hence $\ker f \cap B$ is a maximal ideal in B not containing v, and hence the Jacobson radical of B is zero. ■

Problem 23. Let A be a ring. Show that the following are equivalent:

i) Every prime ideal in A is an intersection of maximal ideals.

ii) In every homomorphic image of A the nilradical is equal to the Jacobson radical.

iii) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

[The only hard part is iii) $\Rightarrow$ ii). Suppose ii) false, then there is a prime ideal which is not an intersection of maximal ideals. Passing to the quotient ring, we may assume that A is an integral domain whose Jacobson radical $\mathfrak{R}$ is not zero. Let f be a non-zero element of $\mathfrak{R}$. Then $A_f \neq 0$, hence A_f has a maximal ideal, whose contraction in A is a prime ideal $\mathfrak{p}$ such that $f \notin \mathfrak{p}$, and which is maximal with respective to this property. Then $\mathfrak{p}$ is not maximal and is not equal to the intersection of the prime ideals strictly containing $\mathfrak{p}$.]

A ring A with the three equivalent properties above is called a *Jacobson ring.*

Proof. i) $\Leftrightarrow$ ii) We have a natural correspondence of ideals between a ring and its quotient, and homomorphic image is isomorphic to a quotient, so we only deal with the ring itself. For each prime ideal $\mathfrak{p}$ we write $\mathfrak{p} = \bigcap_{\mathfrak{m}_\mathfrak{p}} \mathfrak{m}_\mathfrak{p}$. Then

$$\bigcap_{\mathfrak{p}} \mathfrak{p} = \bigcap_{\mathfrak{p}} \bigcap_{\mathfrak{m}_\mathfrak{p}} \mathfrak{m}_\mathfrak{p} = \bigcap_{\mathfrak{m}} \mathfrak{m}.$$

ii) $\Rightarrow$ iii) If the prime ideal $\mathfrak{p} \subseteq A$ is not maximal, then

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m} \supseteq \bigcap_{\mathfrak{q} \supsetneq \mathfrak{p}} \mathfrak{q} \supseteq \mathfrak{p}.$$

iii) $\Rightarrow$ i) If there is a prime ideal $\mathfrak{p} \subseteq A$ such that $\mathfrak{p} \neq \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}$ ($\mathfrak{m}$ is maximal), then in $A/\mathfrak{p}$ the nilradical is zero while the Jacobson radical is not. So we may assume that A is integral and its Jacobson radical $\mathfrak{R}$ is nonzero.

Let $f \in \mathfrak{R} \setminus \{0\}$, $\varphi : A \to A_f$ the natural map. Then $A_f \neq 0$, it has a maximal ideal $\mathfrak{n}$, and $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$ is nonzero and prime in A, and $f \notin \mathfrak{m}$. But condition iii) says if $\mathfrak{m}$ is not maximal, then we have $\mathfrak{m} = \bigcap_{\mathfrak{q} \supsetneq \mathfrak{m}} \mathfrak{q}$ ($\mathfrak{q}$ is prime), this implies there is a prime ideal $f \notin \mathfrak{q} \supsetneq \mathfrak{m}$. But since $A_f\varphi(\mathfrak{m}) = \mathfrak{n}$ is maximal, $\mathfrak{q} = \varphi^{-1}\big(A_f\varphi(\mathfrak{q})\big) = \varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$, contradiction. ■

Problem 24. Let A be a Jacobson ring (Exercise 23) and B an A-algebra. Show that if B is either (i) integral over A of (ii) finitely generated as an A-algebra, then B is Jacobson. [Use Exercise 22 for (ii).]

In particular, every finitely generated ring, and every finitely generated algebra over a field, is a Jacobson ring.

Proof.

i) Since A is Jacobson, every prime ideal $\mathfrak{p} \subseteq A$ can be written as $\mathfrak{p} = \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}$ for all such maximal ideals $\mathfrak{m}$. By Proposition 5.10 and Corollary 5.8, each maximal ideal in A can be lifted to a maximal ideal in B, then everything is clear.

ii) Let $\mathfrak{q} \subseteq B$ be a prime ideal, and let $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Then the Jacobson radical of $A/\mathfrak{p}$ is zero. Since $B/\mathfrak{q}$ is a finitely generated $A/\mathfrak{p}$-algebra, by Exercise 22 the Jacobson radical of $B/\mathfrak{q}$ is also zero, hence condition i) of Exercise 23 is satisfied, B is Jacobson. ■

Problem 25. Let A be a ring. Show that the following are equivalent:

i) A is a Jacobson ring;

ii) Every finitely generated A-algebra B which is a field is finite over A.

[i) $\Rightarrow$ ii). Reduce to the case where A is a subring of B, and use Exercise 21. If $s \in A$ is as in Exercise 21, then there exists a maximal ideal $\mathfrak{m}$ of A not containing s, and the homomorphism $A \to A/\mathfrak{m} = k$ extends to a homomorphism g of B into the algebraic closure of k. Since B is a field, g is injective, and $g(B)$ is algebraic over k, hence finite algebraic over k.
ii) $\Rightarrow$ i). Use criterion iii) of Exercise 23. Let $\mathfrak{p}$ be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let f be a non-zero element of B. Then B_f is a finitely

generated A-algebra. If it is a field it is finite over B, hence integral over B and therefore B is a field by (5.7). Hence B_f is not a field and therefore has a non-zero prime ideal, whose contraction in B is a non-zero ideal $\mathfrak{p}'$ such that $f \notin \mathfrak{p}'$.]

Proof. i) $\Rightarrow$ ii) Let $f : A \to B$ be an algebra, we may regard $A/\ker\varphi$ as a subring of B, if B is finite over $A/\ker\varphi$ then of course it is finite over A, hence we may assume $A \subseteq B$. Choose a non-zero $s \in A$ by Exercise 21, then there is a maximal ideal $\mathfrak{m}$ of A such that $s \notin \mathfrak{m}$.[7] We get a map $f : A \to A/\mathfrak{m} \to \Omega$ where Ω is an algebraic closure of $A/\mathfrak{m}$. By Exercise 21, f can be extended to $f : B \to \Omega$. Hence B is a finite algebraic extension of $A/\mathfrak{m}$.

ii) $\Rightarrow$ i) Let $\mathfrak{p} \subseteq A$ be a prime but not maximal ideal, and let $B = A/\mathfrak{p}$, it is not a field. For every $f \in B$, $f \neq 0$, then B_f is finitely generated as an A-algebra. If B_f is a field, then it is finite over A, hence is integral over A and integral over B, but then by Proposition 5.7 B is a field, contradiction. Hence B_f is not a field, it has a non-zero maximal ideal $\mathfrak{n}$, and $\mathfrak{n}$ contracts to B is a non-zero prime ideal that does not contain f. Hence

$$\bigcap_{\mathfrak{p} \supsetneq (0)} \mathfrak{p} = (0) \quad (\mathfrak{p} \text{ is prime}),$$

by Exercise 23 iii) A is Jacobson. ∎

Problem 26. Let X be a topological space. A subset of X is *locally closed* if it is the intersection of an open set and a closed set, or equivalently if it is open in its closure.

The following conditions on a subset X_0 of X are equivalent:

(1) Every non-empty locally closed subset of X meets X_0;

(2) For every closed set E in X we have $\overline{E \cap X_0} = E$;

(3) The mapping $U \mapsto U \cap X_0$ of the collection of open sets of X onto the collection of open sets of X_0 is *bijective.*

A subset X_0 satisfying these conditions is said to be *very dense* in X.

If A is a ring, show that the following are equivalent:

i) A is a Jacobson ring;

ii) The set of maximal ideals of A is very dense in $\mathrm{Spec}(A)$;

iii) Every locally closed subset of $\mathrm{Spec}(A)$ consisting of a single point is closed.

[ii) and iii) are geometrical formulations of conditions ii) and iii) of Exercise 23.]

[7] This is because A as a subring of the field B is therefore integral, its nilradical is zero. Plus A is Jacobson, its Jacobson radical is then zero.

Proof. (1) ⇒ (2) For each $x \in E$ and every open neighborhood U of x, $U \cap E$ meets X_0, hence $\overline{E \cap X_0} \supseteq E$. Since E is closed $\overline{E \cap X_0} = E$.

(2) ⇒ (3) If the map is bijective, then so is the map for closed sets of X. Then the relation $\overline{E \cap X_0} = E$ for all closed sets $E \subseteq X$ shows that the map is bijective.

(3) ⇒ (1) Let $U \subseteq X$ be open and $E \subseteq X$ be closed, such that $U \cap E \neq \varnothing$ and $U \cap E \cap X_0 = \varnothing$. We may assume $E = U^{\mathrm{c}} \cup E$, then $Ec \subseteq U$. Then $U \cap X_0 = E^{\mathrm{c}} \cap X_0$ while $U \supsetneq E^{\mathrm{c}}$, contradiction. Hence $U \cap E$ meets X_0.

i) ⇒ ii) For every locally closed set $U = V(\mathfrak{a}) \setminus V(\mathfrak{b})$, since every prime ideal is an intersection of some maximal ideal, if for every maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$, $\mathfrak{m} \supseteq \mathfrak{b}$, then $\mathfrak{b} \subseteq \mathfrak{a}$, and $U = \varnothing$. Hence if $U \neq \varnothing$, there are some maximal ideal $\mathfrak{m} \in U$, hence U meets $\operatorname{Specm}(A)$.

i) ⇔ ii) Exists $V(\mathfrak{a}) \setminus V(\mathfrak{b}) \neq \varnothing$ does not meet $\operatorname{Specm}(A) \iff$ for every maximal $\mathfrak{m} \supseteq \mathfrak{a}$, $\mathfrak{m} \supseteq \mathfrak{b} \iff \bigcap_{\mathfrak{m} \supseteq \mathfrak{b}} \mathfrak{m} \subseteq \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \mathfrak{m} \implies$ condition ii) of Exercise 23 is false.

Condition ii) of Exercise 23 is false $\implies$ exists ideal $\mathfrak{a}$, $r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supsetneq \mathfrak{a}} \mathfrak{p} \neq \bigcap_{\mathfrak{m} \supsetneq \mathfrak{a}} \mathfrak{m} =: \mathfrak{b} \implies V(\mathfrak{a}) \setminus V(\mathfrak{b}) \neq \varnothing$ does not meet $\operatorname{Specm}(A)$.

i) ⇔ iii) $V(\mathfrak{a}) \setminus V(\mathfrak{b}) = \{\mathfrak{m}\}$ is closed (i.e. $\mathfrak{m}$ is maximal) $\iff$ if $\mathfrak{m}$ is not maximal, then $V(\mathfrak{a}) \setminus V(\mathfrak{b}) = \varnothing \iff$ if $\mathfrak{m}$ is not maximal, then for every prime $\mathfrak{p} \supsetneq \mathfrak{m}$, we must have $\mathfrak{p} \supseteq \mathfrak{b}$ (i.e. $\mathfrak{b} \subseteq \mathfrak{m}$ since $\mathfrak{m}$ is prime) $\iff \bigcap_{\mathfrak{p} \supsetneq \mathfrak{m}} \mathfrak{p} = \mathfrak{m} \iff$ condition iii) of Exercise 23. ■

Valuation rings and valuations

Problem 27. Let A, B be two rings. B is said to *dominate* A is A is a subring of B and the maximal ideal $\mathfrak{m}$ of A is contained in the maximal ideal $\mathfrak{n}$ of B (or, equivalently, if $\mathfrak{m} = \mathfrak{n} \cap A$). Let K be a field and let Σ be the set of all local subrings of K. If Σ is ordered by the relation of domination, show that Σ has maximal elements and that $A \in \Sigma$ is maximal if and only if A is the valuation ring of K. [Use (5.21).]

Proof. Let $\{A_i\}_{i \in I}$ be a chain in Σ, $\mathfrak{m}_i$ the maximal ideal of A_i, then $A = \bigcup_{i \in I} A_i$ is a subring of K, $\mathfrak{m} = \bigcup_{i \in I} \mathfrak{m}_i$ the maximal of A. Hence we can apply Zorn's lemma to get a maximal element in Σ (well, actually $K \in \Sigma$ is already maximal).

Let $A \in \Sigma$ be maximal, by Theorem 5.21 A can be extended to a valuation ring $B \subseteq K$. Since the extension of the maximal ideal $\mathfrak{m}$ of A in B is not a unit ideal, and since valuation rings are local, $\mathfrak{m}$ is contained in the maximal ideal of B. Hence $A = B$ is a valuation ring.

Conversely, let $A \subseteq K$ be a valuation ring. If A is not maximal in Σ, let $B \supsetneq A$ be a maximal element, A is then also a valuation ring. Let $x \in B \setminus \mathfrak{m}_B$, then $x^{-1} \in B \setminus \mathfrak{m}_B$, and either $x \in A$ or $x^{-1} \in A$, but since $x, x^{-1} \notin \mathfrak{m}_A$, $x, x^{-1} \in A$, $A = B$. ■

Problem 28. Let A be an integral domain, K its field of fractions. Show that the following are equivalent:

(1) A is a valuation ring of K;

(2) If $\mathfrak{a}, \mathfrak{b}$ are any two ideals of A, then either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$.

Deduce that if A is a valuation ring and $\mathfrak{p}$ is a prime ideal of A, then $A_\mathfrak{p}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

Proof. (1) $\Rightarrow$ (2) Let $x \in \mathfrak{a}$, $y \in \mathfrak{b}$, suppose $\mathfrak{a} \not\subseteq \mathfrak{b}$, that is, there is $x \in \mathfrak{a}$ such that $x \notin \mathfrak{b}$. For every $y \in \mathfrak{b}$, since A is a valuation ring, we must have either $xy^{-1} \in A$ of $x^{-1}y \in A$, but the latter one is impossible (otherwise $x \in (y) \subseteq \mathfrak{a}$), hence $y \in (x) \subseteq \mathfrak{a}$ and $\mathfrak{b} \subseteq \mathfrak{a}$.

(2) $\Rightarrow$ (1) For every $x, y \in A$, either $(x) \subseteq (y)$ or $(x) \supseteq (y)$, suppose $(x) \subseteq (y)$. Then $x = cy$ for some $c \in A$, and $xy^{-1} = c \in A$. Since every element in K is of the form a/b for some $a, b \in A$, we have either $a/b \in A$ or $b/a \in A$, A is a valuation ring. ■

Problem 29. Let A be a valuation ring of a field K. Show that every subring of K which contains A is a local ring of A.

Proof. Let $A \subseteq B \subseteq K$ be an intermediate subring. Let

$$S = \{x \in A \mid x \neq 0, x^{-1} \in B\},$$

then S is a multiplicative set of A, and $S^{-1}A \subseteq B$. Let $x \in B$, $x \neq 0$, if $x \notin A$, there will be $x^{-1} \in A$, and $x \in S^{-1}A$, hence $S^{-1}A = B$. Clearly $S^{-1}A$ is a local ring. ■

Problem 30. Let A be a valuation ring of a field K. The group U of units of A is a subgroup of the multiplicative group $K^\times$ of K.

Let $\Gamma = K^\times/U$. If $\xi, \eta \in \Gamma$ are represented by $x, y \in K$, define $\xi \geq \eta$ to mean $xy^{-1} \in A$. Show that this defines a total ordering on Γ which is compatible with the group structure (i.e., $\xi \geq \eta \Rightarrow \xi\omega \geq \eta\omega$ for all $\omega \in \Gamma$). In other words, Γ is a totally ordered abelian group. It is called the *value group* of A.

Let $v : K^\times \to \Gamma$ be the canonical homomorphism. Show that $v(x + y) \geq \min\big(v(x), v(y)\big)$ for all $x, y \in K^\times$.

Proof. Let $\xi, \eta \in \Gamma$ represented by $x, y \in K$ respectively. Let $u, v \in U = A^\times$. If $xy^{-1} \in A$, then $(ux)(vy)^{-1} = (uv^{-1})(xy^{-1}) \in A$, so the ordering is well-defined. Obviously that this is a total ordering on Γ. Let $\xi, \eta, \omega \in \Gamma$ be represented by $x, y, z \in K$, then $xy^{-1} \in A$ implies $(xz)(yz)^{-1} = xy^{-1} \in A$, hence $\xi \geq \eta \implies \xi\omega \geq \eta\omega$.

Let $x, y \in K^\times$, then either $xy^{-1} \in A$ or $x^{-1}y \in A$, suppose $xy \in A$. Since then $1 + xy^{-1} \in A$, $(x + y) = (1 + xy^{-1})y$ and $v(x + y) \geq v(y)$. ■

Problem 31. Conversely, let Γ be a totally ordered abelian group (written *additively*), and let K be a field. A *valuation of K with values in* Γ is a mapping $v : K^\times \to \Gamma$ such that

(1) $v(xy) \geq v(x) + v(y)$,

(2) $v(x+y) \geq \min\big(v(x), v(y)\big)$,

for all $x, y \in K^\times$. Show that the set of elements $x \in K^\times$ such that $v(x) \geq 0$ is a valuation ring of K. This ring is called the *valuation ring* of v, and the subgroup $v(K^\times)$ of Γ is the *value group* of v.

Proof. Let $A = \{x \in K^\times \mid v(x) \geq 0\}$. For every $x, y \in A$, we have

$$\begin{aligned} v(x+y) &\geq \min\big(v(x), v(y)\big) \geq 0 \\ v(xy) &= v(x) + v(y) \geq 0 + v(y) = v(y) \geq 0, \end{aligned}$$

hence $x + y \in A$ and $xy \in A$. In particular,

$$\begin{aligned} v(1) = v(1 \cdot 1) = v(1) + v(1) &\implies v(1) = 0 \implies 1 \in A \\ v(1) = v((-1) \cdot (-1)) = v(-1) + v(-1) = 0 &\implies v(-1) \geq 0, \end{aligned}$$

therefore $v(x^{-1}) = -v(x)$, $-1 \in A$, and $A \cup \{0\}$ is ring. Finally, for every $x \in K^\times$, either $v(x) \geq 0$ or $v(x) \leq 0$, hence either $x \in A$ or $x^{-1} \in A$, $A \cup \{0\}$ is a valuation ring of K. ■

Problem 32. Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is *isolated* in Γ if, whenever $0 \leq \beta \leq \alpha$ and $\alpha \in \Delta$, we have $\beta \in \Delta$. Let A be a valuation ring of a field K, with value group Γ (Exercise 31). If $\mathfrak{p}$ is a prime ideal of A, show that $v(A \setminus \mathfrak{p})$ is the set of elements ≥ 0 in an isolated subgroup Δ of Γ, and that the mapping so defined of $\operatorname{Spec}(A)$ into the set of isolated subgroups of Γ is bijective.

If $\mathfrak{p}$ is a prime ideal of A, what are the value groups of the valuation rings $A/\mathfrak{p}$, $A_\mathfrak{p}$?

Proof. We have $v(A \setminus \mathfrak{p}) \geq 0$. Let $0 \leq \beta \leq \alpha$ in Γ be represented by $x \in K^\times$ and $y \in A \setminus \mathfrak{p}$. Then $x \in A$ and there is $u \in A$ such that $y = ux$, but then $u, x \in A \setminus \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. Let Δ be generated by $v(A \setminus \mathfrak{p})$, from above it follows that Δ is an isolated subgroup of Γ.

$(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})^\times/(A/\mathfrak{p})^\times$ and $K^\times/(A_\mathfrak{p} \setminus \mathfrak{p}A_\mathfrak{p})$. ■

Problem 33. Let Γ be a totally ordered abelian group. We shall show how to construct a field K and a valuation v of K with Γ as value group. Let k be any field and lt $A = k[\Gamma]$ be the group algebra of Γ over k. By definition, A is freely generated as a k-vector space by elements x_α ($\alpha \in \Gamma$) such that $x_\alpha x_\beta = x_{\alpha+\beta}$. Show that A is an integral domain.

If $u = \lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n}$ is any non-zero element of A, where the λ_i are all $\neq 0$ and $\alpha_1 < \cdots < \alpha_n$, define $v_0(u)$ to be α_1. Show that the mapping $v_0 : A \setminus \{0\} \to \Gamma$ satisfies conditions (1) and (2) of Exercise 31.

Let K be the field of fractions of A. Show that v_0 can be uniquely extended to a valuation v of K, and that the value group of v is precisely Γ.

Proof. Let $\{c_\alpha\}_{\alpha\in\Gamma}, \{d_\beta\}_{\beta\in\Gamma} \subseteq k$, each of which almost zero, such that

$$0 = \Big(\sum_{\alpha\in\Gamma} c_\alpha x_\alpha\Big)\Big(\sum_{\beta\in\Gamma} d_\beta x_\beta\Big) = \sum_{\alpha\in\Gamma}\Big(\sum_{\tau\in\Gamma} c_\tau d_{\alpha-\tau}\Big)x_\alpha,$$

i.e. $\sum_{\tau\in\Gamma} c_\tau d_{\alpha-\tau} = 0$ for every $\alpha \in \Gamma$. Let m, n be the maximal indices in Γ such that $c_m, d_n \neq 0$. Then $\sum_{\tau\in\Gamma} c_\tau d_{m+n-\tau} = 0 \implies c_m = 0$ or $d_n = 0$. Since $\{c_\alpha\}$ and $\{d_\beta\}$ are almost zero, we can use induction to show that $\{c_\alpha\} = \{0\}$ or $\{d_\beta\} = \{0\}$. Hence A is integral.

Let $u = \sum_{i=1}^n \lambda_i x_{\alpha_i} \neq 0$, $v = \sum_{j=1}^m \mu_i x_{\beta_j} \neq 0$ such that α_i and β_j are both increasing. Then $v_0(uv) = \alpha_1 + \beta_1 = v_0(u) + v_0(v)$, and $v_0(u+v) \geq \alpha_1, \beta_1 \geq \min\big(v_0(u), v_0(v)\big)$.

The only extension is $v(a/b) = v_0(a) - v_0(b)$ for $a, b \in A \setminus \{0\}$. Then for $a_1/b_1, a_2/b_2 \in K^\times$,

$$\begin{aligned} v\Big(\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}\Big) &= v\Big(\frac{a_1a_2}{b_1b_2}\Big) = v_0(a_1) + v_0(a_2) - v_0(b_1) - v_0(b_2) = v\Big(\frac{a_1}{b_1}\Big) + v\Big(\frac{a_2}{b_2}\Big),\\ v\Big(\frac{a_1}{b_1} + \frac{a_2}{b_2}\Big) &= v\Big(\frac{a_1b_2 + a_2b_1}{b_1b_2}\Big) \geq \min\big(v_0(a_1b_2), v_0(a_2b_1)\big) - v_0(b_1) - v_0(b_2)\\ &= \min\Big(v\Big(\frac{a_1}{b_1}\Big), v\Big(\frac{a_2}{b_2}\Big)\Big). \end{aligned}$$

Hence v is a valuation of K, and clearly $v(K^\times) = \Gamma$, hence it is the value group of v. ∎

Problem 34. Let A be a valuation ring and K its field of fractions. Let $f : A \to B$ be a ring homomorphism such that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a *closed* mapping. Then if $g : B \to K$ is any A-algebra homomorphism (i.e., it $g \circ f$ is the embedding of A in K) we have $g(B) = A$.
[Let $C = g(B)$; obviously $C \supseteq A$. Le t$\mathfrak{n}$ be a maximal ideal of C. Since f^* is closed, $\mathfrak{m} = \mathfrak{n} \cap A$ is the maximal ideal of A, whence $A_\mathfrak{m} = A$. Also the local ring $C_\mathfrak{n}$ dominates $A_\mathfrak{m}$. Hence by Exercise 27 we have $C_\mathfrak{n} = A$ and therefore $C \subseteq A$.]

Proof. Completely follow the hints. Let $C = g(B)$, then $C \supseteq A$, and let $\mathfrak{n}$ be a maximal ideal of C. Since f^* is closed, by Exercise 10 i) (a) $\Rightarrow$ (b) f has the going-up property, hence $\mathfrak{m} = \mathfrak{n} \cap A$ is a maximal ideal in B. Since A is a valuation ring, it is known that $A \setminus \mathfrak{m} = A^\times$, we have $A_\mathfrak{m} = A$ and $C_\mathfrak{n}$ dominates A. From Exercise 27 A is maximal with respect to domination, we have $C \subseteq C_\mathfrak{n} \subseteq A$, hence $A = C = g(B)$. ■

Problem 35. From Exercise 1 and 3 it follows that, if $f : A \to B$ is integral and C is any A-algebra, then the mapping $(f \otimes 1)^* : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec}(C)$ is a closed map.

Conversely, suppose that $f : A \to B$ has this property and that B is an integral domain. Then f is integral. [Replacing A by its image in B, reduce to the case where $A \subseteq B$ and f is the injection. Let K be the field of fractions of B and let A' be a valuation ring of K containing A. By (5.22) it is enough to show that A' contains B. By hypothesis $\operatorname{Spec}(B \otimes_A A') \to \operatorname{Spec}(A')$ is a closed map. Apply the result of Exercise 34 to the homomorphism $B \otimes_A A' \to K$ defined by $b \otimes a' \mapsto ba'$. It follows that $ba' \in A'$ for all $b \in B$ and all $a' \in A'$; taking $a = 1$, we have what we want.]

Show that the result just proved remains valid if B is a ring with only finitely many prime ideals (e.g. if B is Noetherian). [Let $\mathfrak{p}_i$ be the minimal prime ideals. Then each composite homomorphism $A \to B \to B/\mathfrak{p}_i$ is integral, hence $A \to \prod(B/\mathfrak{p}_i)$ is integral, hence $A \to B/\mathfrak{R}$ is integral (where $\mathfrak{R}$ is the nilradical of B), hence finally $A \to B$ is integral.]

Proof. Completely follow the hints. If the closed map condition only holds for $g : A \to C$ with $\ker g \supseteq \ker f$, we may regard B and C as $A/\ker f$-algebras. Hence we may assume that f is injective, $A \subseteq B$.

Let $K = \operatorname{Frac} B$, A' the valuation ring of K containing A (Exercise 27). Then A' is an A-algebra, we get a map $f \otimes 1 : B \otimes_A A' \to K$ by $b \otimes a' \mapsto ba'$. Then $(f \otimes 1)^*$ is closed, by Exercise 34 $(f \otimes 1)(B \otimes_A A') = A$, hence $b \otimes 1 \mapsto b \in A \implies B \subseteq A'$. By Corollary 5.22, B is contained in the integral closure of A in K, hence is integral over A.

Let $\{\mathfrak{p}_i\}_{i \in I}$ be the set of minimal prime ideals, where I is finite. Then by what we just proved each $A \to B \to B/\mathfrak{p}_i$ is integral, hence $A \to A/\mathfrak{R} \to \prod_{i \in I}(B/\mathfrak{p}_i)$ is integral, where $\mathfrak{R}$ is the nilradical of B. Since $\mathfrak{R} = \bigcap_{i \in I} \mathfrak{p}_i$, by Chinese remainder theorem, $A/\mathfrak{R} \to \prod_{i \in I}(B/\mathfrak{p}_i)$ is injective. Hence $A \to A/\mathfrak{R}$ is integral, and consequently $A \to B$ is integral. ■

Chapter 6

Chain Conditions

Problem 1.

i) Let M be a Noetherian module and $u : M \to M$ a module homomorphism. If u is surjective, then u is an isomorphism.

ii) If M is Artinian and u is injective, then again u is an isomorphism.

[For (i), consider the submodules $\ker u^n$; for (ii), the quotient modules $\operatorname{coker} u^n$.]

Proof.

i) Let $N = \ker u$, then $M \cong M/N$. If $N \neq 0$, it produces a strictly increasing sequence $M \cong M/N \subsetneq M \cong M/N \subsetneq \cdots$, contradicts a.c.c.

ii) Let $N = \operatorname{coker} u$, then $M \cong N$. Again, if $N \neq M$, we have a strictly decreasing sequence $N \cong M \supsetneq N \cong M \supsetneq \cdots$, contradicts d.c.c. ■

Problem 2. Let M be an A-module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Proof. Let $N \subseteq M$ be a submodule, and let Σ be the set of all finitely generated submodules of N, it is not empty, since $0 \in \Sigma$. Then there is a maximal element $N_0 \in \Sigma$. If $N_0 \neq N$, let $x \in N \setminus N_0$, then $N_0 + Ax$ is finitely generated, contained in N, and strictly contains N_0, contradiction. Hence $N = N_0$ is finitely generated. [This is basically the proof of Proposition 6.2, we just replace Proposition 6.1 by such a weaker condition.] ■

Problem 3. Let M be an A-module and let N_1, N_2 be submodules of M. If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

Proof. Since $N_1/(N_1 \cap N_2) \cong (N_1 + N_2)/N_2 \subseteq M/N_2$ is Noetherian, the exact sequence

$$0 \to N_1/(N_1 \cap N_2) \to M/(N_1 \cap N_2) \to M/N_1 \to 0$$

implies $M/(N_1 \cap N_2)$ is Noetherian. The proof is the same for Artinian modules. ∎

Problem 4. Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A. Prove that $A/\mathfrak{a}$ is a Noetherian ring.

If we replace "Noetherian" by "Artinian" in this result, is it still true?

Proof. Since M is Noetherian A-module, it is finitely generated, say $M = \langle x_1, \ldots, x_n \rangle$. Then for every $1 \leq i \leq n$, $\langle x_i \rangle \cong A/\mathfrak{b}_i$ is Noetherian for some $\mathfrak{a} \subseteq \mathfrak{b}_i \subseteq A$. Then $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{b}_i$. Since each $A/\mathfrak{b}_i$ is a Noetherian A-module, by Exercise 3, $A/\bigcap_{i=1}^n \mathfrak{b}_i = A/\mathfrak{a}$ is a Noetherian A-module, hence it is a Noetherian ring.

It is not true in general. For example, let $A = \mathbb{Z}$, which is not Artinian, and let $M = \{a/p^k \in \mathbb{Q} \mid k \in \mathbb{N}, p \text{ is prime}\}/\mathbb{Z}$ as a $\mathbb{Z}$-module, however, it is Artinian (Example 2 and 3 above Proposition 6.2). ∎

Problem 5. A topological space X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition (or, equivalently, the maximal condition). Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition (or, equivalently, the minimal condition). Show that, if X is Noetherian, then every subspaces of X is Noetherian, and that X is quasi-compact.

Proof. Let $V \subseteq X$, and $\{V_i\}_{i\in I} \subseteq V$ be an open covering of V. Then for each $i \in I$, there is an open set $U_i \subseteq X$, such that $V_i = U_i \cap V$. Since X is Noetherian, there is a maximal element in $\{U_i\}_{i\in I}$, and this element apparently corresponds to a maximal element of $\{V_i\}_{i\in I}$.

Given an open covering $\{U_i\}_{i\in I}$ of X, let Σ be the set of all finite unions of X_i. Then Σ has a maximal element U. If $U \neq X$, we can find some $j \in I$, $U \subsetneq U \cup U_j \in \Sigma$, contradicts the maximality of U. Hence X is quasi-compact. ∎

Problem 6. Prove that the following are equivalent:

i) X is Noetherian.

ii) Every open subspace of X is quasi-compact.

iii) Every subspace of X is quasi-compact.

Proof. i) $\Rightarrow$ iii) Exercise 5.

iii) $\Rightarrow$ ii) Trivial.

ii) $\Rightarrow$ i) Let $U_0 \subseteq U_1 \subseteq \cdots$ be a chain of open sets in X. Let $U = \bigcup_{i=0}^\infty U_i$, it is open, hence is quasi-compact. So U can be cover by a finite subset of $\{U_i\}_{i=0}^\infty$, and the chain must be stationary, thus X is Noetherian. ∎

Problem 7. A Noetherian space is a finite union of irreducible closed subspaces. [Consider the set Σ of closed subsets of X which are not finite unions of irreducible closed spaces.] Hence the set of irreducible components of a Noetherian space is finite.

Proof. Let X be a Noetherian space. By Exercise 20 iii) of Chapter 1, there is a covering of X by irreducible closed subsets, say by $\{U_i\}_{i\in I}$. Let $\mathscr{I}$ be the set of all finite subsets of I, then we have a set of closed sets $\mathscr{U} = \left\{\bigcup_{j\in J} U_j\right\}_{J\in\mathscr{I}}$. Since X is Noetherian, there is a maximal element in $\mathscr{U}$, say $U = \bigcup_{j\in J} U_j$ for some $J \in \mathscr{I}$, then we must have $U = X$, otherwise $U \subsetneq U \cup U_i$ for some $U_i \not\subseteq U$, contradicts the maximality of U. ■

Problem 8. If A is a Noetherian ring then $\operatorname{Spec}(A)$ is a Noetherian topological space. Is the converse true?

Proof. Let $V(I_0) \supseteq V(I_1) \supseteq \cdots$ be a chain of closed subsets in $\operatorname{Spec}(A)$, then it induces a chain $r(I_0) \subseteq r(I_1) \subseteq \cdots$ of ideals in A. Since A is Noetherian, the latter chain is stationary, and hence the first chain is stationary, and that $\operatorname{Spec}(A)$ is a Noetherian space.

Adapt Example 3 above Proposition 6.2, let p be a prime number, and let $G = \{a/p^k \in \mathbb{Q} \mid k > 0\}$. Then G is a ring, satisfying d.d.c but not a.c.c. However, G satisfies a.c.c. for its radical ideals (its radical ideals are those generated by $1/p$), hence $\operatorname{Spec}(G)$ is a Noetherian space, this gives a counterexample. ■

Problem 9. Deduce from Exercise 8 that the set of minimal prime ideals in a Noetherian ring is finite.

Proof. If A is a Noetherian ring, then by Exercise 8, $\operatorname{Spec}(A)$ is a Noetherian space, and by Exercise 7, its set of irreducible components is finite. By Exercise 20 iv) of Chapter 1, there is a correspondence between irreducible components in $\operatorname{Spec}(A)$ and minimal prime ideals in A, and hence there is only a finitely many minimal prime ideals in A. ■

Problem 10. If M is a Noetherian module (over an arbitrary ring A) then $\operatorname{Supp}(M)$ is a closed Noetherian subspace of $\operatorname{Spec}(A)$.

Proof. Since M is Noetherian, it is finitely generated, by Exercise 19 v) of Chapter 3, $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$ is closed in $\operatorname{Spec}(A)$. Since M is a Noetherian A-module, by Exercise 4, $A/\operatorname{Ann}(M)$ is a Noetherian ring. Since every closed subset in $\operatorname{Supp}(M)$ is of the form $V(\mathfrak{a})$, where $\operatorname{Ann}(M) \subseteq \mathfrak{a}$, $\operatorname{Supp}(M) \cong \operatorname{Spec}(A/\operatorname{Ann}(M))$, hence is it is Noetherian space. ■

Problem 11. Let $f : A \to B$ be a ring homomorphism and suppose that $\operatorname{Spec}(B)$ is a Noetherian space (Exercise 5). Prove that $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a closed mapping if and only if f has the going-up property (Chapter 5, Exercise 10).

Proof. ($\Rightarrow$) Exercise 10 i) (a) $\Rightarrow$ (b) of Chapter 5.

($\Leftarrow$) Let $V(\mathfrak{b})$ be a closed set in $\mathrm{Spec}(B)$, where $\mathfrak{b} \subseteq B$ is an ideal. Since $\mathrm{Spec}(B)$ is Noetherian, by Exercise 5, $V(\mathfrak{b}) \cong \mathrm{Spec}(B/\mathfrak{b})$ is a Noetherian subspace. By Exercise 7, there are only finitely many irreducible components in $V(\mathfrak{b})$, hence there are only finitely many minimal prime ideals in $A/\mathfrak{b}$, and hence $V(\mathfrak{b}) = \bigcup_{i=1}^{n} V(\mathfrak{q}_i)$ for some minimal prime ideals in B containing $\mathfrak{b}$. We then have

$$f^*\big(V(\mathfrak{b})\big) = f^*\big(\bigcup_{i=1}^{n} V(\mathfrak{q}_i)\big) = \bigcup_{i=1}^{n} f^*\big(V(\mathfrak{q}_\mathrm{i})\big) = \bigcup_{i=1}^{n} V\big(f^*(\mathfrak{q}_\mathrm{i})\big)$$

(the last equality holds because of the going-up property of f) is a finite intersection of closed sets, hence it is closed in $\mathrm{Spec}(A)$, and f^* is a closed mapping.[1] ■

Problem 12. Let A be a ring such that $\mathrm{Spec}(A)$ is a Noetherian space. Show that the set of prime ideals of A satisfies the ascending chain condition. Is the converse true?

Proof. Since the radical of a prime ideal is the prime ideal itself, by the proof of Exercise 8, it is clear that the set of prime ideals satisfies a.c.c.

Let $A = \mathbb{C}[t]$, then every prime ideal in A is of the form $(t - c)$ for some $c \in \mathbb{C}^\times$ or (0), clearly satisfy a.c.c. But the chain

$$\begin{aligned}\big((t-1)\big) \supseteq \big((t-1)(t-2)\big) \supseteq \big((t-1)(t-2)(t-3)\big) \supseteq \cdots \\ \supseteq \big((t-1)(t-2)\cdots(t-n)\big) \supseteq \cdots\end{aligned}$$

is such that their radicals form a non-stationary descending chain. ■

[1]Reference: zcn (`https://math.stackexchange.com/users/115654/zcn`). "Closedness and going up property." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/962795`, 2014.

Chapter 7

Noetherian Rings

Problem 1. Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. Show that Σ has a maximal elements and that the maximal elements of Σ are prime ideals.
[Let $\mathfrak{a}$ be a maximal element in Σ, and suppose that there exists $x, y \in A$ such that $x \notin \mathfrak{a}$ and $y \notin \mathfrak{a}$ and $xy \in \mathfrak{a}$. Show that there exists a finitely generated ideal $\mathfrak{a}_0 \subseteq \mathfrak{a}$ such that $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$, and that $\mathfrak{a} = \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x)$. Sine $(\mathfrak{a} : x)$ strictly contains $\mathfrak{a}$, it is finitely generated and therefore so is $\mathfrak{a}$.]

Hence a ring in which every prime ideal is finitely generated is Noetherian (I. S. Cohen).

Proof. Let $\{\mathfrak{a}_i\}_{i \in I} \subseteq \Sigma$ be a chain. If $\bigcup_{i \in I} \mathfrak{a}_i = (a_1, \ldots, a_n)$ is finitely generated, there exists $i \in I$, $a_1, \ldots, a_n \in \mathfrak{a}_i \subseteq (a_1, \ldots, a_n)$, this implies $\mathfrak{a}_i = (a_1, \ldots, a_n)$ is finitely generated, contradiction. Hence $\bigcup_{i \in I} \mathfrak{a}_i \in \Sigma$, and we can apply Zorn's lemma to deduce that maximal elements exist in Σ with respect to inclusion.

Let $\mathfrak{p} \in \Sigma$ be a maximal element, and let $x, y \in A$ such that $xy \in \mathfrak{p}$. Suppose $x, y \notin \mathfrak{p}$, then $\mathfrak{p} + (x) = (b_1, \ldots, b_n)$ is finitely generated. For every b_i, we have $b_i = a_i + c_i x$ for some $a_i \in \mathfrak{p}$ and $c_i \in A$, hence $\mathfrak{p} + (x) = (a_1, \ldots, a_n, x)$. Let $\mathfrak{p}_0 = (a_1, \ldots, a_n) \subseteq \mathfrak{p}$, then $\mathfrak{p} + (x) = \mathfrak{p}_0 + (x)$, hence $\mathfrak{p} = \mathfrak{p}_0 + (\mathfrak{p} : x)x$. Since $y \in (\mathfrak{p} : x)$, $\mathfrak{p} \subsetneq (\mathfrak{p} : x)$, it is then finitely generated, and this implies $\mathfrak{p}$ is finitely generated, contradiction. So $\mathfrak{p}$ is a prime ideal. ■

Problem 2. Let A be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in A[\![x]\!]$. Prove that f is nilpotent if and only if each a_n is nilpotent.

Proof. ($\Rightarrow$) Easy to see that a_0 is nilpotent, then $f - a_0$ is also nilpotent, proceed inductively we see that a_i are all nilpotent.

($\Leftarrow$) Since A is Noetherian, $\mathfrak{a} = (a_n)_{n=0}^{\infty} = (b_i)_{i=1}^{m}$ for some $b_i \in A$. Since every b_i is an algebraic combination of finitely many a_n, it is nilpotent, hence $\mathfrak{a}^N = \left((b_i)_{i=1}^{m}\right)^N = (0)$ for some $N \in \mathbb{N}$, and $f^N = 0$.[1] ■

[1] See also `https://math.stackexchange.com/questions/187952/`.

Problem 3. Let $\mathfrak{a}$ be an irreducible ideal in a ring A. Then the following are equivalent:

i) $\mathfrak{a}$ is primary.

ii) for every multiplicatively closed subset S of A we have $(S^{-1}\mathfrak{a})^{c} = (\mathfrak{a} : x)$ for some $x \in S$.

iii) the sequence $(\mathfrak{a} : x^n)$ is stationary, for every $x \in A$.

Proof. i) $\Rightarrow$ ii) If $S \cap \mathfrak{a} \neq \varnothing$, let $x \in S \cap \mathfrak{a}$, then $(S^{-1}\mathfrak{a})^{c} = (1) = (\mathfrak{a} : x)$. If $S \cap \mathfrak{a} = \varnothing$, then for every $s \in S$, $b \in A$, $bs \in \mathfrak{a}$ implies $b \in \mathfrak{a}$ (since $s^n \notin \mathfrak{a}$), hence $(\mathfrak{a} : s) = \mathfrak{a}$, and $(S^{-1}\mathfrak{a})^{c} = \bigcup_{s \in S}(\mathfrak{a} : s) = \mathfrak{a} = (\mathfrak{a} : 1)$.

i) $\Rightarrow$ iii) Let $x \in A$. If $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$, then $(\mathfrak{a} : x^n) = (1)$. If $x^n \notin \mathfrak{a}$ for every $n \in \mathbb{N}$, then for $y \in (\mathfrak{a} : x^n)$, $yx^n \in \mathfrak{a}$, this implies $y \in \mathfrak{a}$. Hence the sequence $(\mathfrak{a} : x^n)$ is stationary.

ii) $\Rightarrow$ iii) Let $S = \langle x \rangle$, then $\bigcup_{s \in S}(\mathfrak{a} : s) = (S^{-1}\mathfrak{a})^{c} = (\mathfrak{a} : y)$ for some $y \in S$. Hence the sequence $(\mathfrak{a} : x^n)$ is stationary.

iii) $\Rightarrow$ i) Let $x, y \in A$ such that $y \notin \mathfrak{a}$ and $xy \in \mathfrak{a}$. Then since $(\mathfrak{a} : x^n)$ is stationary, $(\mathfrak{a} : x^{n+1}) = (\mathfrak{a} : x^n)$ for every $n \geq N_x$, where $N_x \in \mathbb{N}$ depends on x. We must have $(x^n) \cap (y) \subseteq \mathfrak{a}$ $(n \geq N_x)$, for if $a \in (x^n) \cap (y)$, then $ax \in \mathfrak{a}$ implies $x \in \mathfrak{a}$. Hence $\mathfrak{a} = ((x^n) + \mathfrak{a}) \cap ((y) + \mathfrak{a})$, and $x^n \in \mathfrak{a}$ (since we have assumed $y \notin \mathfrak{a}$), and $\mathfrak{a}$ is primary. [One can proceed the proof by quotient by $\mathfrak{a}$, as the proof of Lemma 7.12.] ∎

Problem 4. Which of the following rings are Noetherian?

i) The ring of rational functions of z having no pole on the circle $|z| = 1$.

ii) The ring of power series in z with a positive radius of convergence.

iii) The ring of power series in z with an infinite radius of convergence.

iv) The ring of polynomials in z whose first k derivatives vanish at the origin (k being a fixed integer).

v) The ring of polynomials in z, w all of whose partial derivatives with respect to w vanish for $z = 0$.

In all cases the coefficients are complex numbers.

Proof.

i)

ii)

iii)

iv)

v)

Flag. *Finish this.*

■

Problem 5. Let A be a Noetherian ring, B a finitely generated A-algebra, G a finite group of A-automorphisms of B, and B^G the set of all elements of B which are left fixed by every element of G. Show that B^G is a finitely generated A-algebra.

Proof. We may assume that A is a subring of B, let $B = A[y_1, \dots, y_m]$, and let

$$S = \{\sigma y_i \mid \sigma \in G, 1 \leq i \leq m\} = \{x_1, x_2, \dots, x_n\},$$

B is also generated by S over A. Let

$$s_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i} \quad (s_0 = 1)$$

be the elementary symmetric polynomials in $x_1, \dots, x_n$ over A, then s_i are fixed under G, the ring $B_0 := A[s_1, \dots, s_n] \subseteq B^G$. Since $B = A[x_1, \dots, x_n]$ is finitely generated as a B_0-module,[2] it is finitely generated as a B^G-module. By Proposition 7.8, B^G is then finitely generated as an A-algebra. ■

Problem 6. If a finitely generated ring K is a field, it is a finite field.
[If K has characteristic 0, we have $\mathbb{Z} \subset \mathbb{Q} \subseteq K$. Since K is finitely generated over $\mathbb{Z}$ it is finitely generated over $\mathbb{Q}$, hence by (7.9) is a finitely generated $\mathbb{Q}$-module. Now apply (7.8) to obtain a contradiction. Hence K is of characteristic $p > 0$, hence is finitely generated as a $\mathbb{Z}/(p)$-algebra. Use (7.9) to complete the proof.]

Proof. If $\operatorname{char} K = 0$, then $\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow K$. By Proposition 7.9, K is a finite algebraic extension of $\mathbb{Q}$, then by Proposition 7.8, $\mathbb{Q}$ is a finitely generated $\mathbb{Z}$-algebra, this is impossible. Hence $\operatorname{char} K = p > 0$, $\mathbb{F}_p \hookrightarrow K$, K is a finitely generated $\mathbb{F}_p$-algebra, hence finite algebraic over $\mathbb{F}_p$, and hence a finite field. ■

[2] Each x_k is a root of $\prod_{i=1}^n (x - x_i) = \sum_{i=0}^n (-1)^i s_i x^{n-i}$, the extension of modules

$$B_0 \subseteq B_0[x_1] \subseteq B_0[x_1, x_2] \subseteq \cdots \subseteq B_0[x_1, \dots, x_n] = A$$

is finite. In particular, when $n = 2$, $f(x, y) = \frac{xf(y,x) - yf(x,y)}{x-y} + \frac{f(x,y) - f(y,x)}{x-y} x$. See: orangeskid (`https://math.stackexchange.com/users/168051/orangeskid`). “Ring of polynomials as a module over symmetric polynomials.” *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/1004579`, 2014.

Problem 7. Let X be an affine algebraic variety given by a family of equations $f_\alpha(t_1, \dots, t_n) = 0$ ($\alpha \in I$) (Chapter 1, Exercise 27). Show that there exists a finite subset I_0 of I such that X is given by the equations $f_\alpha(t_1, \dots, t_n) = 0$ for $\alpha \in I_0$.

Proof. Let $\mathfrak{a} = (f_\alpha)_{\alpha \in I}$ be the ideal generated by all f_α in $A[t_1, \dots, t_n]$. Since $A[t_1, \dots, t_n]$ is Noetherian, we can write $\mathfrak{a} = (g_1, \dots, g_m)$. Then each g_i can be written as an algebraic combination of finitely many f_α. Since then number of g_i is finite (m), we can generate $\mathfrak{a} = (g_1, \dots, g_m)$ by finitely many f_α. ■

Problem 8. If $A[x]$ is Noetherian, is A necessarily Noetherian?

Proof 1. Yes. For every ideal $\mathfrak{a} \subseteq A$, the ideal $A[x]\mathfrak{a} \subseteq A[x]$ is finitely generated, we may write it as $\big(f_1(x), \dots, f_n(x)\big)$. Since $\mathfrak{a} = A[x]\mathfrak{a} \cap A = \big(f_1(x), \dots, f_n(x)\big) \cap A = (c_1, \dots, c_n)$, where c_i is the constant term of $f_i(x)$, we see that $\mathfrak{a}$ is finitely generated in A. ■

Proof 2. Yes. The map $\phi : A[x] \to A[x]/(x) = A$ is surjective, hence $A \cong A[x]/\ker\phi$. Since $A[x]$ is Noetherian, its quotient is also Noetherian. ■

Problem 9. Let A be a ring such that

(1) for each maximal ideal $\mathfrak{m}$ of A, the local ring $A_\mathfrak{m}$ is Noetherian;

(2) for each $x \neq 0$ in A, the set of maximal ideals of A which contain x is finite.

Show that A is Noetherian.
[Let $\mathfrak{a} \neq 0$ be an deal in A. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals which contain $\mathfrak{a}$. Choose $x_0 \neq 0$ in $\mathfrak{a}$ and let $\mathfrak{m}_1, \dots, \mathfrak{m}_{r+s}$ be the maximal ideals which contain x_0. Since $\mathfrak{m}_{r+1}, \dots, \mathfrak{m}_{r+s}$ do not contain $\mathfrak{a}$ there exist $x_j \in \mathfrak{a}$ such that $x_j \notin \mathfrak{m}_{r+j}$ ($1 \leq j \leq s$). Since each $A_{\mathfrak{m}_i}$ ($1 \leq i \leq r$) is Noetherian, the extension of $\mathfrak{a}$ in $A_{\mathfrak{m}_i}$ is finitely generated. Hence there exist $x_{s+1}, \dots, x_t$ in $\mathfrak{a}$ whose images in $A_{\mathfrak{m}_i}$ generate $A_{\mathfrak{m})_i}\mathfrak{a}$ for $i = 1, \dots, r$. Let $\mathfrak{a}_0 = (x_0, \dots, x_t)$. Show that $\mathfrak{a}_0$ and $\mathfrak{a}$ have the same extension in $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$, and deduce by (3.9) that $\mathfrak{a}_0 = \mathfrak{a}$.]

Proof. Let $\mathfrak{a} \subseteq A$ be a non-zero ideal. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be all maximal ideals $\supseteq \mathfrak{a}$, and let the images of $a_1, \cdots, a_n \in \mathfrak{a}$ generate $\mathfrak{a}A_{\mathfrak{m}_i}$. Choose a non-zero $a_0 \in A$, let $\mathfrak{m}_{r+1}, \dots, \mathfrak{m}_{r+s}$ be all maximal ideals that contains x_0 but not contains $\mathfrak{a}$, and let $a_{n+1}, \dots, a_{n+s} \in \mathfrak{a}$ such that $a_{n+j} \notin \mathfrak{m}_{r+j}$.

Finally we let $\mathfrak{a}_0 = (a_0, a_1, \dots, a_{n+s})$. For every maximal ideal $\mathfrak{m} \subseteq A$, if $a_0 \notin \mathfrak{m}$, we have $\mathfrak{a}_0 A_\mathfrak{m} = \mathfrak{a}A_\mathfrak{m} = (1)$. If $a_0 \in \mathfrak{m}$, but $\mathfrak{a} \not\subseteq \mathfrak{m}$, then $a_{n+j} \notin \mathfrak{m}$ for some $1 \leq j \leq s$, then still $\mathfrak{a}_0 A_\mathfrak{m} = \mathfrak{a}A_\mathfrak{m} = (1)$. If $\mathfrak{a} \subseteq \mathfrak{m}$, obviously $\mathfrak{a}_0 A_\mathfrak{m} = \mathfrak{a}A_\mathfrak{m}$. Hence $\mathfrak{a}_\mathfrak{m}/(\mathfrak{a}_0)_\mathfrak{m} = (\mathfrak{a}/\mathfrak{a}_0)_\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m}$, by Proposition 3.8 $\mathfrak{a}_0 = \mathfrak{a}$. ■

Problem 10. Let M be a Noetherian A-module. Show that $M[x]$ (Chapter 2, Exercise 6) is a Noetherian $A[x]$-module.

Proof. The same proof as the Hilbert's basis theorem. ■

Problem 11. Let A be a ring such that each local ring $A_{\mathfrak{p}}$ is Noetherian. Is A necessarily Noetherian?

Proof. Not necessarily. For example, every infinite Boolean rings fail, in particular, the ring $A = \prod_{n=1}^{\infty}(\mathbb{Z}/2\mathbb{Z})$ fails.[3] This is because every Boolean rings is Noetherian if and only if it is finite, and by Exercise 12 of Chapter 1, a local Boolean ring can only be $\mathbb{Z}/2\mathbb{Z}$. ■

Problem 12. Let A be a ring and B a faithfully flat A-algebra (Chapter 3, Exercise 16). If B is Noetherian, show that A is Noetherian [Use the ascending chain condition.]

Proof. Let $f : A \to B$ be the A-algebra. For every ascending chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ of ideals in A, $Bf(\mathfrak{a}_1) \subseteq Bf(\mathfrak{a}_2) \subseteq \cdots$ stabilizes. Since B is faithfully flat, by Exercise 16 i) of Chapter 3 $\mathfrak{a}^{ec} = \mathfrak{a}$ for every ideal $\mathfrak{a} \subseteq A$, we see that the original chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ stabilizes. ■

Problem 13. Let $f : A \to B$ be a ring homomorphism of finite type and let $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the mapping associated with f. Show that the fibers of f^* are Noetherian subspaces of B.

Proof. By Exercise 21 iv) of Chapter 3, for every prime ideal $\mathfrak{p} \subseteq A$, $f^{*-1}(\mathfrak{p}) \cong \mathrm{Spec}\big(k(\mathfrak{p}) \otimes_A B\big)$, where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is the residue field. f is of finite type implies $B \cong A[t_1, \ldots, t_n]/\mathfrak{a}$ a quotient of the polynomial ring. Hence $k(\mathfrak{p}) \otimes_A B \cong k(\mathfrak{p})[t_1, \ldots, t_n]/(k(\mathfrak{p}) \otimes_A \mathfrak{a})$ is a Noetherian ring (quotient of a polynomial ring over a field, which is Noetherian), and hence $f^{*-1}(\mathfrak{p})$ is a Noetherian space. ■

Nullstellensatz, strong form

Problem 14. Let k be an algebraically closed field, let A denote the polynomial ring $k[t_1, \ldots, t_n]$ and let $\mathfrak{a}$ be an ideal in A. Let V be the variety in k^n defined by the ideal $\mathfrak{a}$, so that V is the set of all $x = (x_1, \ldots, x_n) \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$. Let $I(V)$ be the ideal of V, i.e. the ideal of all polynomials $g \in A$ such that $g(x) = 0$ for all $x \in V$. Then $I(V) = r(\mathfrak{a})$.
[It is clear that $r(\mathfrak{a}) \subseteq I(V)$. Conversely, let $f \notin r(\mathfrak{a})$, then there si a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ such that $f \notin \mathfrak{p}$. Let $\bar{f}$ be the image of f in $B = A/\mathfrak{p}$, let $C - B_f = B[1/\bar{f}]$, and let $\mathfrak{m}$ be a maximal ideal of C. Since C is a finitely generated k-algebra we have $C/\mathfrak{m} \cong k$, by (7.9). The images x_i in $C/\mathfrak{m}$ of the generators t_i of A thus define a point $x = (x_1, \ldots, x_n) \in k^n$, and the construction shows that $x \in V$ and $f(x) \notin 0$.]

[3]This example comes from: Pete L. Clark (`https://math.stackexchange.com/users/299/pete-l-clark`). "A non-noetherian ring with all localizations noetherian." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/73442`, 2011.

Proof. For $f \notin r(\mathfrak{a})$, there exists a prime ideal $f \notin \mathfrak{p} \subseteq A$. Let $B = A/\mathfrak{p}$ and $C = B_f$, and let $\mathfrak{m} \subseteq C$ be a maximal ideal. By Corollary 7.10, $C/\mathfrak{m} \cong k$, hence the images of t_i in $C/\mathfrak{m} \cong k$ correspond to a point $(x_1, \ldots, x_n) \in V \subseteq k^n$ (since $\mathfrak{p}$ contains $\mathfrak{a}$), and $f(x_1, \ldots, x_n) \neq 0$, therefore $I(V) \subseteq r(\mathfrak{a})$. On the other hand, clearly $r(\mathfrak{a}) \subseteq I(V)$, hence we have $I(V) = r(\mathfrak{a})$. ∎

Problem 15. Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field, and let M be a finitely generated A-module. Then the following are equivalent:

i) M is free;

ii) M is flat;

iii) the mapping of $\mathfrak{m} \otimes M$ into $A \otimes M$ is injective;

iv) $\mathrm{Tor}_1^A(k, M) = 0$.

[To show that iv) ⇒ i), let $x_1, \ldots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a k-basis of this vector space. By (2.8), the x_i generate M. Let F be a free A-module with basis $e_1, \ldots, e_n$ and define $\phi : F \to M$ by $\phi(e_i) = x_i$. Let $E = \ker \phi$. Then the exact sequence $0 \to E \to F \to M \to 0$ gives us an exact sequence

$$0 \to k \otimes_A E \to k \otimes_A F \overset{1\otimes\phi}{\to} k \otimes_A M \to 0.$$

Since $k \otimes F$ and $k \otimes M$ are vector spaces of the same dimension over k, it follows that $1 \otimes \phi$ is an isomorphism, hence $k \otimes E = 0$, hence $E = 0$ by Nakayama's Lemma (E is finitely generated because it is a submodule of F, and A is Noetherian).]

Proof. i) ⇒ ii) ⇒ iii) ⇒ iv) Trivial.

iv) ⇒ i) We have a short exact sequence

$$0 \to \mathfrak{m} \otimes_A M \to A \otimes_A \to k \otimes_A M \to 0.$$

Since $k \otimes_A M$ is a vector space over k, it is a free A-module, the quotient $M/\mathfrak{m}M$ (since M is flat, $\mathfrak{m}M \cong \mathfrak{m} \otimes_A M$) is a free A-module. Let $x_1, \ldots, x_n \in M$ such that their images in $M/\mathfrak{m}M$ are a basis over k, by Proposition 2.8, $x_1, \ldots, x_n$ also generate M. Let $F = A^n$ whose basis is $e_1, \ldots, e_n$, and let $\phi : e_i \mapsto x_i$, we get an exact sequence

$$0 \to k \otimes_A E \to k \otimes_A F \to k \otimes_A M \to 0$$

where $E = \ker \phi$. Since $\dim_k k \otimes_A F = \dim_k k \otimes_A M$, we must have $E/\mathfrak{m}E \cong k \otimes_A E = 0$. But E as a submodule of a Noetherian A-module F, it is finitely generated over A, by Nakayama's lemma, $F = 0$, $M \cong F$ is free. ∎

Problem 16. Let A be a Noetherian ring, M a finitely generated A-module. Then the following are equivalent:

i) M is a flat A-module;

ii) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module, for all prime ideals $\mathfrak{p}$;

iii) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module, for all maximal ideals $\mathfrak{m}$.

In other words, flat = locally free. [Use Exercise 15.]

Proof. i) $\Rightarrow$ ii) Since localization is an exact functor, M is flat implies $M_{\mathfrak{p}}$ is flat. By Exercise 15, $M_{\mathfrak{p}}$ is then free.

ii) $\Rightarrow$ iii) Trivial.

iii) $\Rightarrow$ i) Since $M_{\mathfrak{m}}$ is free (then is flat) for every maximal ideal $\mathfrak{m}$, given injection $0 \to N' \to N$, the map $0 \to N' \otimes_A M_{\mathfrak{m}} \to N \otimes_A M_{\mathfrak{m}}$ is injective. From the fact that injectivity of A-linear map is a local property, we have an injection $0 \to N' \otimes_A M \to N \otimes_A M$, hence M is flat. ■

Problem 17. Let A be a ring and M a Noetherian A-module. Show (by imitating the proofs of (7.11) and (7.12)) that every submodule N of M has a primary decomposition (Chapter 4, Exercise 20–23).

Proof. We say that a submodule N is *irreducible* if $N = N_1 \cap N_2 \implies N = N_1$ or $N = N_2$.

Lemma 7.0.1. Every submodule of M is a finite intersection of irreducible submodules.

Proof. Suppose it is not, let Σ be the set of submodules of M which cannot be written as an intersection of finitely many irreducible submodules. By a.c.c. Σ has a maximal element, let it be N. Then N is not irreducible, there exist $N \subsetneq N_1, N_2 \subseteq M$ such that $N = N_1 \cap N_2$, but N_1 and N_2 are strictly larger, and $N_1 \in \Sigma$ or $N_2 \in \Sigma$, this is a contradiction. □

Lemma 7.0.2. Every irreducible submodule in M is primary.

Proof. Taking quotient by N we may assume that $N = 0$. Let $0 \subseteq M$ be irreducible, and let $x \in A$, $m \in M \setminus \{0\}$ such that $xm = 0$. We denote $\operatorname{Ann}_M(a) = \{m \in M \mid am = 0\}$ for $a \in A$, it is a submodule of M. Consider the chain $\operatorname{Ann}_M(x) \subseteq \operatorname{Ann}_M(x^2) \subseteq \cdots$, it chain is stationary by a.c.c., say $\operatorname{Ann}_M(x^n) = \operatorname{Ann}_M(x^{n+1}) = \cdots$ for some n. Then $x^n M \cap \langle m \rangle = 0$; this is because for every $am \in x^n M \cap \langle m \rangle$, $am = x^n m'$, $0 = xam = x^{n+1}m' \implies m' \in \operatorname{Ann}_M(x^{n+1}) = \operatorname{Ann}_M(x^n) \implies am = x^n m' = 0$. But since $m \neq 0$, $0 \subseteq M$ is irreducible, we must have $x^n M = 0$, hence 0 is a primary submodule of M. □

Combining these two lemmas we see that every submodule of M has a primary decomposition. ■

Problem 18. Let A be a Noetherian ring, $\mathfrak{p}$ be a prime ideal of A, and M a finitely generated A-module. Show that the following are equivalent:

i) $\mathfrak{p}$ belongs to 0 in M;

ii) there exists $x \in M$ such that $\mathrm{Ann}(x) = \mathfrak{p}$;

iii) there exists a submodule of M isomorphic to $A/\mathfrak{p}$.

Deduce that there exists a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each quotient M_i/M_{i-1} is of the form $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A.

Proof. i) $\Rightarrow$ ii) We know that $\mathfrak{p} = r(0 : x)$ for some $x \in M \setminus \{0\}$. Since A is Noetherian, the set $\Sigma = \{(0 : x) \mid r(0 : x) = \mathfrak{p}, x \in M\}$ has a maximal element, say $(0 : x)$. Suppose $(0 : x) \subsetneq \mathfrak{p}$, let $a \in \mathfrak{p} \setminus (0 : x)$. If $(0 : ax) \subseteq \mathfrak{p}$, $(0 : ax) \in \Sigma$, by the maximality of $(0 : x)$ we must have $(0 : ax) = (0 : x)$. But since $a \in \mathfrak{p}$, $a^n \in (0 : x)$ for some n, and

$$(1) = (0 : 0) = (0 : a^n x) = (0 : a^{n-1}x) = \cdots = (0 : ax) = (0 : x),$$

contradiction. Therefore the only possibility is $(0 : ax) \not\subseteq \mathfrak{p}$, let $b \in (0 : ax) \setminus \mathfrak{p}$, we have $(0 : x) \subsetneq (0 : bx) \ni a$. Besides, for each $c \in (0 : bx)$, $bcx = 0 \implies bc \in \mathfrak{p} \implies c \in \mathfrak{p} \implies (0 : bx) \subseteq \mathfrak{p}$ and $(0 : bx) \in \Sigma$, contradicting the maximality of $(0 : x)$. These facts force $\mathfrak{p} = (0 : x) = \mathrm{Ann}(x)$.

ii) $\Rightarrow$ iii) Let $x \in M$ such that $\mathrm{Ann}(x) = \mathfrak{p}$, then $\langle x \rangle \cong A/\mathfrak{p}$.

iii) $\Rightarrow$ i) Let $N \subseteq M$ be the submodule, and let $x \in N$ whose image in $A/\mathfrak{p}$ is non-zero. We then have $(0 : x) = \mathfrak{p}$. Since A is Noetherian and M is finitely generated, M is a Noetherian module, and every submodule is decomposable, in particular, 0 is decomposable. Hence $\mathfrak{p} = (0 : x)$ belongs to 0 in M.

Suppose we have constructed $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i-1} \subsetneq M$, let $\mathfrak{p}_i$ be a prime ideal belonging to 0 in M/M_{i-1}, by iii) there is a submodule $M_{i-1} \subsetneq M_i \subseteq M$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$. By a.c.c. the construction must stop at some step, and the final submodule M_r must be equal to M. ■

Problem 19. Let $\mathfrak{a}$ be an ideal in a Noetherian ring A. Let

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{b}_i = \bigcap_{j=1}^{s} \mathfrak{c}_j$$

be two minimal decompositions of $\mathfrak{a}$ as intersections of *irreducible* ideals. Prove that $r = s$ and that (possible after re-indexing the $\mathfrak{c}_j$) $r(\mathfrak{b}_i) = r(\mathfrak{c}_i)$ for all i.

[Show that for each $i = 1, \dots, r$ there exists j such that

$$\mathfrak{a} = \mathfrak{b}_i \cap \cdots \cap \mathfrak{b}_{i-1} \cap \mathfrak{c}_j \cap \mathfrak{b}_{i+1} \cap \cdots \cap \mathfrak{b}_r.]$$

State and prove an analogous result for modules.

Proof. By taking quotients by $\mathfrak{a}$ we may assume that $\mathfrak{a} = 0$. Fix $1 \leq i \leq r$, and let $I_j = \mathfrak{c}_j \bigcap_{k \neq i} \mathfrak{b}_k$, then $\bigcap_{j=1}^s I_j = 0$. Since we have a natural injection $\varphi : A \hookrightarrow \prod_{k=1}^r A/\mathfrak{b}_k$,

$$\bigcap_{j=1}^{s} (\mathfrak{c}_j + \mathfrak{b}_i)/\mathfrak{b}_i \cong \bigcap_{j=1}^{s} \varphi(I_j) = \varphi\Big(\bigcap_{j=1}^{s} I_j\Big) = 0,$$

and since $\mathfrak{b}_i$ is irreducible, we have $\mathfrak{c}_j \subseteq \mathfrak{b}_i$ and $I_j = 0$ for some j.[4] Let $\lambda : i \mapsto j$. Similarly there is $\mu : j \mapsto i$ (of course the i, j here are re-chosen) such that $\mathfrak{b}_i \subseteq \mathfrak{c}_j$.

From the selection above we see that λ is injective (otherwise contradiction the minimality of $\bigcap_{k=1}^r \mathfrak{b}_k$), $r \leq s$, and conversely $r \geq s$, hence $r = s$.

For the second part, by the 1st uniqueness theorem we see that

$$\{r(\mathfrak{b}_k) \mid k = 1, \dots, r\} = \{r(\mathfrak{c}_k) \mid k = 1, \dots, r\}.$$

We use induction on the maximal length ℓ of chains of associated prime ideals of $\mathfrak{a}$. For $\ell = 1$, there are only isolated prime ideals, $\mathfrak{b}_i \supseteq \mathfrak{c}_{\lambda(i)} \supseteq \mathfrak{b}_{\mu\lambda(i)}$, taking the radical we see that $r(\mathfrak{b}_i) = r(\mathfrak{c}_{\lambda(i)})$, hence the base case is done. Suppose $\ell > 1$ and the case $\ell - 1$ is proved. Since intersection of the components corresponding to a maximal chain of prime ideals is independent of decompositions (2nd uniqueness theorem, Theorem 4.10), considering the first $\ell - 1$ components of both decompositions (by 'first' I mean starting from the smallest one), from the inductive hypothesis, they have the same multi-set of radicals. Since they have the same numbers of components, we see that the two sets of radicals of the components corresponding to the whole chain are the same. By induction we are done.[5]

For modules, let $N = \bigcap_{i=1}^r Q_i = \bigcap_{j=1}^s T_j$ be two minimal irreducible decompositions, then $r = s$ and $(Q_i : M) = (T_i : M)$ after a suitable renumbering. The proof is essentially the same. ■

Remark 7.0.3. I don't think this can be proved by the 1st uniqueness theorem for primary decompositions, since minimal irreducible decompositions may not be minimal primary. An obvious fact is that two irreducible (none of them contains another) is definitely not irreducible, hence such a decomposition is probably reducible as an intersection of primary ideals.

[4]Reference: Emerton (`https://mathoverflow.net/users/2874/emerton`). "Atiyah-MacDonald, exercise 7.19 - 'decomposition using irreducible ideals'." *MathOverflow*, `https://mathoverflow.net/q/15340`, 2010.

[5]Reference: CJD (`https://mathoverflow.net/users/71/cjd`). "Atiyah-MacDonald, exercise 7.19 - 'decomposition using irreducible ideals'." *MathOverflow*, `https://mathoverflow.net/q/15350`, 2010.

Remark 7.0.4. From the selection of λ and μ we see that there is a chain

$$\mathfrak{c}_{i_1} \supseteq \mathfrak{b}_{j_1} \supseteq \mathfrak{c}_{i_2} \supseteq \mathfrak{b}_{j_2} \supseteq \cdots,$$

hence $\mathfrak{c}_i = \mathfrak{b}_j$ for some i, j. If we let $\mathfrak{c}_{i_1}$ to be maximal in the set $\{\mathfrak{c}_1, \ldots, \mathfrak{c}_s\}$, the chain stops at $\mathfrak{c}_{i_2}$. From this we see that the minimal elements in $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_r\}$ and the minimal elements in $\{\mathfrak{c}_1, \ldots, \mathfrak{c}_s\}$ are the same.

Problem 20. Let X be a topological space and let $\mathscr{F}$ be the smallest collection of subsets of X which contains all open subsets of X and is closed with respect to the formation of finite intersections and complements.

i) Show that a subset E of X belongs to $\mathscr{F}$ if and only if E is a finite union of sets of the form $U \cap C$, where U is open and C is closed.

ii) Suppose that X is irreducible and let $E \in \mathscr{F}$. Show that E is dense in X (i.e., that $\overline{E} = X$) if and only if E contains a non-empty open set in X.

Proof.

i) Let $\mathscr{F}' = \left\{\bigcup_{i=1}^n (U_i \cap C_i) \mid U_i \text{ open}, C_i \text{ closed}, n \in \mathbb{Z}_{\geq 0}\right\}$. We must have $U \cap C \in \mathscr{F}$ for U open and C closed, $\mathscr{F}' \subseteq \mathscr{F}$. Besides, let $\left\{V_i^{(1)}\right\}_{i=1}^n$ be open and $\left\{V_i^{(2)}\right\}_{i=1}^n$ be closed,

$$\left(\bigcup_{i=1}^n (V_i^{(1)} \cap V_i^{(2)})\right)^{\mathrm{c}} = \bigcap_{i=1}^n \left((V_i^{(2)})^{\mathrm{c}} \cup (V_i^{(1)})^{\mathrm{c}}\right) = \bigcup_{\substack{\varepsilon_1, \ldots, \varepsilon_n \\ \in \{1,2\}}} \left(\bigcap_{i=1}^n (C_i^{(\varepsilon_i)})^{\mathrm{c}}\right) \in \mathscr{F}',$$

let $\{U_i\}_{i=1}^n, \{U_j'\}_{j=1}^m$ be open and $\{C_i\}_{i=1}^n, \{C_j'\}_{j=1}^m$ be closed,

$$\left(\bigcup_{i=1}^n (U_i \cap C_i)\right) \cap \left(\bigcup_{j=1}^m (U_j' \cap C_j')\right) = \bigcup_{i=1}^n \bigcup_{j=1}^m ((U_i \cap U_j') \cap (C_i \cap C_j')) \in \mathscr{F}'.$$

Hence $\mathscr{F} \subseteq \mathscr{F}'$, and $\mathscr{F} = \mathscr{F}'$.

ii) ($\Rightarrow$) Let $E = \bigcup_{i=1}^n (U_i \cap C_i)$ for some open U_i and closed C_i. Suppose all $U_i \neq \varnothing$. If none of $C_i = X$, since X is irreducible, the open set $E' = \bigcap_{i=1}^n C_r^{\mathrm{c}}$ is non-empty and $E' \cap E = \varnothing$, contradiction. Hence $C_r = X$ for some r, and $U_r = U_r \cap C_r \subseteq E$.
($\Leftarrow$) Clear. ■

Problem 21. Let X be a Noetherian topological space (Chapter 6, Exercise 5) and let $E \subseteq X$. Show that $E \in \mathscr{F}$ if and only if, for each irreducible closed set $X_0 \subseteq X$, either $\overline{E \cap X_0} \neq X_0$ or else $E \cap X_0$ contains a non-empty open subset of X_0. [Suppose $E \notin \mathscr{F}$. Then the collection of closed sets $X' \subseteq X$ such that $E \cap X' \notin \mathscr{F}$ is not empty and therefore has a minimal element X_0. Show that X_0 is irreducible and then that each of the alternatives above leads to the conclusion that $E \cap X_0 \in \mathscr{F}$.] The sets belonging to $\mathscr{F}$ are called the *constructible* subsets of X.

Proof. ($\Rightarrow$) If $\overline{E \cap X_0} = X_0$, the equality also holds in X_0, by Exercise 20 ii) $E \cap X_0$ contains a non-empty open set of X_0.

($\Leftarrow$) Suppose $E \notin \mathscr{F}$, let $\Sigma = \{Y \subseteq X \mid E \cap Y \notin \mathscr{F}, Y \text{ closed}\}$ and $\Sigma \neq \varnothing$. Since X is Noetherian, Σ has a minimal element, say X_0. For if $X_0 = Y_1 \cup Y_2$ for some Y_1, Y_2 closed in X, and $(E \cap Y_1) \cup (E \cap Y_2) \notin \mathscr{F} \implies Y_1 \in \Sigma$ or $Y_2 \in \Sigma$, hence X_0 must be irreducible in X.

By the conditions, if $\overline{E \cap X_0} \neq X_0$ (hence $\subsetneq X_0$), we set $X_1 = \overline{E \cap X_0}$. From the minimality of X_0 we must have $E \cap X_1 \in \mathscr{F}$. But $E \cap X_0 = E \cap X_1 \in \mathscr{F}$, contradiction. On the other hand, if $E \cap X_0 \supseteq V \cap X_0 \neq \varnothing$ for some open set V in X. From the minimality of X_0, we have $E \cap (X_0 \setminus V) \in \mathscr{F}$ and

$$E \cap X_0 = \big(E \cap (X_0 \cap V)\big) \cup \big(E \cap (X_0 \setminus V)\big) = \big(X_0 \cap V\big) \cup \big(E \cap (X_0 \setminus V)\big) \in \mathscr{F},$$

yields a contradiction again. So $E \in \mathscr{F}$. ■

Problem 22. Let X be a Noetherian topological space and let E be a subset of X. Show that E is open in X if and only if, for each irreducible closed subset X_0 of X, either $E \cap X_0 = \varnothing$ or else $E \cap X_0$ contains a non-empty open subset of X_0. [The proof is similar to that of Exercise 21.]

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose E is not open ($E^{\mathrm{c}} = X \setminus E$ is not closed), let τ be the set of all closed sets of X, and let $\Sigma = \{Y \subseteq X \mid E^{\mathrm{c}} \cap Y \notin \tau, Y \in \tau\}$,[6] then $X \in \Sigma \neq \varnothing$. Since X is Noetherian space, Σ has a minimal element, say X_0. If $X_0 = Y_1 \cup Y_2$ for Y_1 and Y_2 closed, we must have $Y_1 \in \Sigma$ or $Y_2 \in \Sigma$ (otherwise $E^{\mathrm{c}} \cap X_0 = (E^{\mathrm{c}} \cap Y_1) \cup (E^{\mathrm{c}} \cap Y_2) \in \tau$), hence X_0 is irreducible in X.

We cannot have $E \cap X_0 = \varnothing$, otherwise $E^{\mathrm{c}} \cap X_0 = X \in \tau$. Therefore $E \cap X_0 \supseteq V \cap X_0 \neq \varnothing$ for some open set V in X. By the minimality of X_0, we have $E^{\mathrm{c}} \cap (X_0 \setminus V) \in \tau$, and

$$E^{\mathrm{c}} \cap X_0 = \big(E^{\mathrm{c}} \cap (X_0 \setminus V)\big) \in \tau,$$

contradiction. Hence E is open in X. ■

Problem 23. Let A be a Noetherian ring, $f : A \to B$ be a ring homomorphism of finite type (so that B is Noetherian). Let $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and let $f^* : Y \to X$ be the mapping associated with f. Then the image under f^* of a constructible subset E of Y is a constructible subset of X.
[By Exercise 20 it is enough to take $E = U \cap C$ where U is open and C is closed in Y; then, replacing B by a homomorphic image, we reduce to the case where E is open in Y. Since Y is Noetherian, E is quasi-compact and therefore a finite union of open

[6] If we denote τ_Z the set of all open sets of a topological space Z, it is the same if we define $\Sigma = \{Y \subseteq X \mid E \cap Y \notin \tau_Y, Y \text{ closed}\}$, but it is a little bit tricky to show that X_0 is irreducible. Besides, I don't think we can use something like $\Sigma = \{Y \subseteq X \mid E^{\mathrm{c}} \cap Y \text{ not open}, Y \text{ closed}\}$.

sets of the form $\operatorname{Spec}(B_g)$. Hence reduce to the case $E = Y$. To show that $f^*(Y)$ is constructible, use the criterion of Exercise 21. Let X_0 be an irreducible closed subset of X such that $f^*(Y) \cap X_0$ is dense in X_0. We have $f^*(Y) \cap X_0 = f^*\big(f^{*-1}(X_0)\big)$, and $f^{*-1}(X_0) = \operatorname{Spec}\big((A/\mathfrak{p}) \otimes_A B\big)$, where $X_0 = \operatorname{Spec}(A/\mathfrak{p})$. Hence reduce to the case where A is an integral domain and f is injective. If $Y_1, \ldots, Y_n$ are the irreducible components of Y, it is enough to show that some $f^*(Y_i)$ contains a non-empty open set in X. So finally we are brought down to the situation in which A, B are integral domains and f is injective (and still of finite type); now use Chapter 5, Exercise 21 to complete the proof.]

Proof. Let $E = U \cap C$ for some open U and closed C in Y. Since $U = V(\mathfrak{b}) = \operatorname{Spec}(B/\mathfrak{b})$ for some ideal $\mathfrak{b}$, we may assume that E is open in Y (by replacing B with $B/\mathfrak{b}$ and f with $A \to B \to B/\mathfrak{b}$).

By Exercise 8 of Chapter 6, Y is Noetherian, hence by Exercise 6 of Chapter 6, E is quasi-compact, and by Exercise 17 of Chapter 1, $E = \bigcup_{i=1}^n Y_{g_i}$ for some $g_i \in B$. Hence we may just prove the special case that $E = Y$.

Let $X_0 \subseteq X$ be an irreducible closed set, such that $\overline{f^*(Y) \cap X_0} = X_0$. By Exercise 20 iv) of Chapter 1, $X_0 = \operatorname{Spec}(A/\mathfrak{p})$ for some minimal prime ideal $\mathfrak{p} \subseteq A$. Hence $\mathfrak{p} \in f^*(Y)$, let $f^*(\mathfrak{q}') = \mathfrak{p}$, and $\mathfrak{q} \subseteq \mathfrak{q}' \subseteq B$ a minimal prime ideal. We may assume that A and B are an integral domains and f is injective (by replacing A with $A/\mathfrak{p}$, B with $B/\mathfrak{q}$ and f with $A/\mathfrak{p} \to B \to B/\mathfrak{q}$). We can see that f is still of finite type. Now $\overline{f^*(Y)} = X$.

We may regard A as a subring of B (f being the inclusion map). Let $0 \neq s \in A$, $s \notin \mathfrak{p}$ a prime ideal in A, and let $\mathfrak{m}$ be the contraction of a maximal ideal of A_s containing $\mathfrak{p}A_s$. Then we get a map

$$h : A \to A/\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to \Omega,$$

where Ω is an algebraically closed field, we have $\ker h = \mathfrak{p}$. By Exercise 21 of Chapter 5, there is a suitable $0 \neq s \in A$ such that h can be extended to a map $h : B \to \Omega$, and $h(s) \neq 0$. Let $\mathfrak{q} = \ker h$, we see that $s \notin \mathfrak{q}$ is a prime ideal of B, and $\ker h \cap A = \mathfrak{p}$. Hence $X_s \subseteq f^*(X)$, by Exercise 21 (or 22), $f^*(Y)$ is constructible.

We now easily see that the general case is true. ■

Problem 24. With the notation and hypotheses of Exercise 23, f^* is an open mapping $\Leftrightarrow$ f has the going-down property (Chapter 5, Exercise 10). [Suppose f has the going-down property. As in Exercise 23, reduce to proving that $E = f^*(Y)$ is open in X. The going-down property asserts that if $\mathfrak{p} \in E$ and $\mathfrak{p}' \subseteq \mathfrak{p}$, then $\mathfrak{p}' \in E$: in other words, that if X_0 is an irreducible closed subset of X and X_0 meets E, then $E \cap X_0$ is dense in X_0. By Exercise 20 and 22, E is open in X.]

Proof. ($\Rightarrow$) Exercise 10 ii) (a') $\Rightarrow$ (b') of Chapter 5.

($\Leftarrow$) For any basic open set Y_g in Y, we want to show that $f^*(Y_g)$ is open in X, it is sufficient to show that $f^*(Y)$ is open in X_h for some $h \in f^{-1}(g)$. We may therefore replace f with $A_h \to B_g$ assume that $A = A_h$, $B = B_g$. Notice that f still has the going-down property.

Let $E = f^*(Y)$, and let $X_0 = V(\mathfrak{p}_0) \subseteq X$ be an irreducible closed set, where $\mathfrak{p}_0$ is a minimal prime ideal. The going-down property of f shows that if for prime ideals $\mathfrak{p} \in E$ and $\mathfrak{p}' \subseteq \mathfrak{p}$, then $\mathfrak{p}' \in E$. Hence if $E \cap X_0 \neq \varnothing$, we have $\mathfrak{p}_0 \in E$ and $\overline{E \cap X_0} = X_0$. By Exercise 20 ii) $E \cap X_0$ contains a non-empty open set of X_0, by Exercise 22, E is open in X.

In the general case, we now see that f^* is an open map on basic open sets, hence is open on every open set of Y. ∎

Problem 25. Let A be Noetherian, $f : A \to B$ of finite type and *flat* (i.e., B is flat as an A-module). Then $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open mapping. [Exercise 24 and Chapter 5, Exercise 11.]

Proof. By Exercise 11 of Chapter 5, f has the going-down property. Then by Exercise 24, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open mapping. ∎

Grothendieck groups

Problem 26. Let A be a Noetherian ring and let $F(A)$ denote the set of all isomorphism classes of finitely generated A-modules. Let C be the free abelian group generated by $F(A)$. With each short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated A-modules we associate the element $(M') - (M) + (M'')$ of C, where (M) is the isomorphism class of M, etc. Let D be the subgroup of C generated by these elements, for all short exact sequences. The quotient group C/D is called the *Grothendieck group* of A, and is denoted by $K(A)$. If M is a finitely generated A-module, let $\gamma(M)$, or $\gamma_A(M)$, denote the image of (M) in $K(A)$.

i) Show that $K(A)$ has the following universal property: for each additive function λ on the class of finitely generated A-modules, with values in an abelian group G, there exists a unique homomorphism $\lambda_0 : K(A) \to G$ such that $\lambda(M) = \lambda_0\big(\gamma(M)\big)$ for all M.

ii) Show that $K(A)$ is generated by the elements $\gamma(A/\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of A. [Use Exercise 18.]

iii) If A is a field, or more generally if A is a principal ideal domain, then $K(A) \cong \mathbb{Z}$.

iv) Let $f : A \to B$ be a *finite* ring homomorphism. Show that the restriction of scalars gives rise to a homomorphism $f_! : K(B) \to K(A)$ such that $f_!\big(\gamma_B(N)\big) = \gamma_A(N)$ for a B-module N. If $g : B \to C$ is another finite ring homomorphism, show that $(g \circ f)_! = f_! \circ g_!$.

Proof.

i) We define $\lambda_0 : \gamma_A(M) \mapsto \lambda(M)$, it can be extended to a map $K(A) \to G$ by $\lambda_0 : \sum_{i=1}^n \gamma_A(M_i) \mapsto \sum_{i=1}^n \lambda(M_i)$. This is well-defined, because we see that for $x \in D$, $\lambda(x) = 0$. Then obviously λ_0 is a group homomorphism, and it is the unique one.

$$\begin{tikzcd} F(A) \arrow[r, "\gamma_A"] \arrow[dr, "\lambda"'] & K(A) \arrow[d, dashed, "\exists!\lambda_0"] \\ & G \end{tikzcd}$$

ii) Let $K'(A)$ be a subgroup of $K(A)$ generated by all $\gamma_A(A/\mathfrak{p})$ modulo D, apparently $\gamma(0) \in K'(A)$. By Exercise 18, for every finitely generated A-module M, there is a chain

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M,$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i$. For each $i > 1$, suppose $M_{i-1} \in K'(A)$, by putting the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0,$$

into γ_A, we have $\gamma_A(M_i) = \gamma_A(M_i) + \gamma_A(A/\mathfrak{p}_i) \in K'(A)$, hence finally we have $M \in K'(A)$, and $K'(A) = K(A)$.

iii) Let $k = \mathrm{Frac}(A)$ be the field of fractions, and let $\lambda : M \mapsto \dim_k(k \otimes_A M)$. By i) there is a map $\lambda_0 : K(A) \to \mathbb{Z}$ such that $\lambda = \lambda_0\gamma_A$, in particular, $\dim_k(k \otimes_A A) = \lambda_0\gamma_A(A) = 1$.

We see that $\gamma_A(M_1) + \gamma_A(M_2) = \gamma_A(M_1 \oplus M_2)$, hence if we want to prove $\ker \lambda_0 = \{\gamma_A(0)\}$, it suffices to prove that for two A-modules M_1 and M_2, $\dim_k(k \otimes_A M_1) = \dim_k(k \otimes_A M)$ implies $\gamma_A(M_1) = \gamma_A(M_2)$.

By ii) $\gamma_A(M)$ is generated by some $A/(p)$, where p is prime. We also see that

$$k \otimes_A (A/(a)) = k/ak = \begin{cases} k, & a = 0 \\ 0, & a \neq 0 \end{cases},$$

therefore

$$\dim_k\Big(k \otimes_A \bigoplus_{i=1}^n \big(A/(p_i)\big)\Big) = \dim_k\Big(k \otimes_A \bigoplus_{j=1}^m \big(A/(q_j)\big)\Big) \quad (p_i,\ q_j \text{ prime})$$

iff $|\{1 \le i \le n \mid p_i = 0\}| = |\{1 \le j \le m \mid q_j = 0\}|$, and this implies $k \otimes_A \bigoplus_{i=1}^n \big(A/(p_i)\big) = k \otimes_A \bigoplus_{j=1}^m \big(A/(q_j)\big)$.

Finally, for $a \neq 0$ (recall that a PID is required to be integral), we have an exact sequence

$$\begin{gathered} 0 \to A \to A \to A/(a) \to 0 \\ x \mapsto ax \end{gathered}$$

therefore $\gamma_A\big(A/(a)\big) = 0$, and it immediately follows that

$$\gamma_A\Big(\bigoplus_{i=1}^{n}(A/(p_i))\Big) = \gamma_A\Big(\bigoplus_{j=1}^{m}(A/(q_j))\Big).$$

Hence $\ker \lambda_0 = \{\gamma_A(0)\}$, and we have $K(A) \cong \mathbb{Z}$.[7]

iv) Every finitely generated B module is finitely generated over A, and if the sequence $0 \to N' \to N \to N'' \to 0$ is exact as B-modules, it is exact as A-modules. So the map $f_! : \gamma_B(N) \mapsto \gamma_A(N)$ is well-defined. The relation $(gf)_! = f_! g_!$ is clear. ■

Problem 27. Let A be a Noetherian ring and let $F_1(A)$ be the set of all isomorphism classes of finitely generated *flat* A-modules. Repeating the construction of Exercise 26 we obtain a group $K_1(A)$. Let $\gamma_1(M)$ denote the image of (M) in $K_1(A)$.

i) Show that the tensor product of modules over A induces a commutative ring structure on $K_1(A)$, such that $\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M \otimes N)$. The identity element of this ring is $\gamma_1(A)$.

ii) Show that the tensor product induces a $K_1(A)$-module structure on the group $K(A)$, such that $\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes N)$.

iii) If A is a (Noetherian) local ring, then $K_1(A) \cong \mathbb{Z}$.

iv) Let $f : A \to B$ be a ring homomorphism, B being Noetherian. Show that extension of scalars gives rise to a ring homomorphism $f_! : K_1(A) \to K_1(B)$ such that $f^!\big(\gamma_1(M)\big) = \gamma_1(B \otimes_A M)$. [If M is flat and finitely generated over$A$, then $B \otimes_A M$ is flat and finitely generated over B.] If $g : B \to C$ is another ring homomorphism (with C Noetherian), then $(f \circ g)^! = f^! \circ g^!$.

v) If $f : A \to B$ is a finite ring homomorphism then

$$f_!\big(f^!(x)y\big) = x f_!(y)$$

for $x \in K_1(A)$, $y \in K(B)$. In other words, regarding $K(B)$ as a $K_1(A)$-module by restriction of scalars, the homomorphism $f^!$ is a $K_1(A)$-module homomorphism.

[7] See also: Wikipedia contributors. "Grothendieck group." *Wikipedia, The Free Encyclopedia*, `https://en.wikipedia.org/w/index.php?title=Grothendieck_group&oldid=1174301713`, 2024. Grothendieck group and extensions/Examples.

Remark. Since $F_1(A)$ is a subset of $F(A)$ we have a group homomorphism $\epsilon : K_1(A) \to K(A)$, given by $\epsilon\big(\gamma_1(M)\big) = \gamma(M)$. If the ring A is finite-dimensional and *regular*, i.e., for all its local rings $A_\mathfrak{p}$ are regular (Chapter 11) it can be show that ϵ is an isomorphism.

Proof. We put a subscript '1' on every set in the construction of $K_1(A)$, in order to distinguish with those for $K(A)$. Hence in this convention, D becomes D_1, etc.

i) We show that $\gamma_1(M) \cdot \gamma_1(N) := \gamma_1(M \otimes_A N)$ can be the multiplication in $K_1(A)$. If there is some $\gamma_1(M) = \gamma_1(M')$, i.e. $(M) - (M') \in D_1$, since N is a flat A-module, 'multiplying' by $\gamma_1(N)$ we see that $(M \otimes_A N) - (M' \otimes_A N) \in D_1$ and $\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M') \cdot \gamma_1(N)$. Hence the multiplication is well-defined.

   The associativity is clear, and since $\gamma_1(M) + \gamma_1(N) = \gamma_1(M \oplus N)$, the distributivity is also clear. Obviously the multiplication is commutative, and $\gamma_1(A)$ is the identity.

ii) If there is $\gamma_1(N) = \gamma_1(N')$, i.e. $(N) - (N') \in D$, since M is a flat A-module, multiplying it on the left by $\gamma_1(M)$ we see that $(M \otimes_A N) - (M \otimes_A N') \in D$. The other side is a little bit tricky, for if $\gamma_1(M) = \gamma_1(M')$, i.e. $(M) - (M') \in D_1$, multiplying it on the right by $\gamma_1(N)$ we still get $(M \otimes_A N) - (M' \otimes_A N) \in D$.[8] The remaining stuffs are trivial.

iii) Notice that we still have the universal property as in Exercise 26 i), the proof is identically the same. Let $\mathfrak{m}$ be the maximal ideal of A, we let $\lambda : M \mapsto \dim_{A/\mathfrak{m}}\big((A/\mathfrak{m}) \otimes_A M\big)$, then by the universal property there is a unique map $\lambda_0 : K_1(A) \to \mathbb{Z}$ such that $\lambda = \lambda_0 \gamma_1$, in particular, $\lambda_0 \gamma_1(A) = 1$.

   Since A is a Noetherian local ring, by Exercise 15 ii) $\Rightarrow$ i), $M \cong A^n$, hence $\gamma_1(A)$ generates $K_1(A)$. And since $\gamma_A(M_1) + \gamma_A(M_2) = \gamma_A(M_1 \oplus M_2)$, every element in $K_1(A)$ is of the form $\gamma_1(A^n) - \gamma_1(A^m)$. Suppose $\lambda_0\gamma_1(A^n) = \lambda_0\gamma_1(A^m)$, that is, $(A/\mathfrak{m})^n = (A/\mathfrak{m})^m$, this implies $n = m$, and then $\gamma_1(A^n) - \gamma_1(A^m) = 0$. Therefore $\ker \lambda_0 = \{0\}$, and $K_1(A) \cong \mathbb{Z}$.

iv) For a finitely generated A-module M, $B \otimes_A M$ is a flat B-module, and is clear finitely generated over B, since B is Noetherian, hence $B \otimes_A M \in F_1(A)$. If $\gamma_1(M) = \gamma_1(M')$, that is, $(M) - (M') \in D_1$, by the tricky part in the proof of ii), $\gamma_1(B \otimes_A M) = \gamma_1(B \otimes_A M')$. Hence the map $f^! : K_1(A) \to K_1(B)$ is well-defined. Since $C \otimes_B B \otimes_A M = C \otimes_A M$, the relation $(fg)^! = f!g!$ is clear.

v) Let $x = \gamma_1(M)$ and $y = \gamma_1(N)$ for some A-module M and B-module N. For

[8] This is because for an exact sequence $0 \to M_1' \to M_1 \to M_1'' \to 0$, since M_1 is flat, $\operatorname{Tor}_1^A(M_1'', N) = 0$, tensoring this sequence by N we still get an exact sequence.

a B-module H, we denote $H_!$ the restriction of scalars by f. Then

$$\begin{aligned} f^!\big(\gamma_1(M)\big)\gamma_1(N) &= \gamma_1(B \otimes_A M)\gamma_1(N) \\ &= \gamma_1(M \otimes_A B \otimes_B N) = \gamma_1(M \otimes_A N), \\ f_!\big(f^!(x)y\big) &= \gamma_1\big((M \otimes_A N)_!\big), \\ \gamma_1(M)f_!\big(\gamma_1(N)\big) &= \gamma_1(M \otimes_A N_!). \end{aligned}$$

Hence the only thing remained is to show that $(M \otimes_A N)_! = M \otimes_A N_!$, but this is obvious, we concluded the proof. ■

Chapter 8

Artin Rings

Problem 1. Let $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = 0$ be a minimal primary decomposition of the zero ideal in a Noetherian ring, and let $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary. Let $\mathfrak{p}_i^{(r)}$ be the rth symbolic power of $\mathfrak{p}_i$ (Chapter 4, Exercise 13). Show that for each $i = 1, \ldots, n$ there exists an integer r_i such that $\mathfrak{p}_i^{(r_i)} \subseteq \mathfrak{q}_i$.

Suppose $\mathfrak{q}_i$ is an isolated primary component. Then $A_{\mathfrak{p}_i}$ is an Artin local ring, hence if $\mathfrak{m}_i$ is its maximal ideal we have $\mathfrak{m}_i^r = 0$ for all sufficiently large r, hence $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r.

If $\mathfrak{q}_i$ is an embedded primary component, then $A_{\mathfrak{p}_i}$ is *not* Artinian, hence the powers $\mathfrak{m}_i^r$ are all distinct, and so the $\mathfrak{p}_i^{(r)}$ are all distinct. Hence in the given primary decomposition we can replace $\mathfrak{q}_i$ by any of the infinite set of $\mathfrak{p}_i$-primary ideals $\mathfrak{p}_i^{(r)}$ where $r \geq r_i$, and so there are infinitely many minimal primary decompositions of 0 which differ only in the $\mathfrak{p}_i$-component.

Proof. Since $\mathfrak{p}_i$ is finitely generated and $\mathfrak{p}_i = r(\mathfrak{q}_i)$, $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{q}_i$ for some r_i. By Lemma 4.4 iii), $\mathfrak{p}_i^{(r_i)} = \bigcup_{s \in A \setminus \mathfrak{p}_i} (\mathfrak{p}_i^{r_i} : s) \subseteq \mathfrak{q}_i$. ∎

Problem 2. Let A be a Noetherian ring. Prove that the following are equivalent:

i) A is Artinian;

ii) $\operatorname{Spec}(A)$ is discrete and finite;

iii) $\operatorname{Spec}(A)$ is discrete.

Proof. i) $\Rightarrow$ ii) Since there are only finitely many prime ideals in A, and every prime ideal is maximal, we see that $\operatorname{Spec}(A)$ is finite, every point is closed, hence the space is discrete.

ii) $\Rightarrow$ iii) Trivial.

iii) $\Rightarrow$ i) $\operatorname{Spec}(A)$ is discrete implies every point is closed, that is, every prime ideal in A is maximal ($\dim A = 0$). Hence, since A is Noetherian, by Theorem 8.5, A is a Artin ring. ∎

Problem 3. Let k be a field and A a finitely generated k-algebra. Prove that the following are equivalent:

i) A is Artinian;

ii) A is a finite k-algebra.

[To prove that i) $\Rightarrow$ ii), use (8.7) to reduce to the case where A is an Artin local ring. By the Nullstellensatz, the residue field of A is a finite extension of k. Now use the fact that A is of finite length as an A-module. To prove ii) $\Rightarrow$ i), observe that the ideals of A are k-vector subspaces and therefore satisfy d.c.c.]

Proof. i) $\Rightarrow$ ii) By Theorem 8.7, $A = \prod_{i=1}^{n} A_i$ for some Artinian local rings A_i, hence we may reduce to the case when A is an Artin local ring, and let $\mathfrak{m}$ be its maximal ideal. Then the field $A/\mathfrak{m}$ is a finitely generate k-algebra, by the weak Nullstellensatz (Corollary 5.24, or Exercise 18 of Chapter 5), $A/\mathfrak{m}$ is a finite algebraic extension of k, hence $A/\mathfrak{m}$ is finitely generated as a k-module. Besides, by Exercise 18 (note that Artinian is Noetherian), there is a chain of submodules of A as an A-module

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M,$$

such that $M_i/M_{i-1} \cong A/\mathfrak{m}$ ($\mathfrak{m}$ is the only prime ideal in A). Hence by induction $M = M_r$ is of finite length as a k-module. Therefore by Proposition 6.10, A as a k-module is of finite dimension, that is, A is a finite k-algebra.

ii) $\Rightarrow$ i) Since every ideal of A is also a k-module, by Proposition 6.10, A satisfies d.c.c. ∎

Problem 4. Let $f : A \to B$ be a ring homomorphism of finite type. Consider the following statements:

i) f is finite;

ii) the fibres of f^* are discrete subspaces of $\operatorname{Spec}(B)$;

iii) for each prime ideal $\mathfrak{p}$ of A, the ring $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra ($k(\mathfrak{p})$ is the residue field of $A_\mathfrak{p}$);

iv) the fibres of f^* are finite.

Prove that i) $\Rightarrow$ ii) $\Leftrightarrow$ iii) $\Rightarrow$ iv). [Use Exercise 2 and 3.]

If f is integral and the fibres of f^* are finite, is f necessarily finite?

Proof. Let $\mathfrak{p}$ be a prime ideal of A. Since f is of finite type, $B \otimes_A k(\mathfrak{p})$ is a finitely generated $k(\mathfrak{p})$-algebra. Since $k(\mathfrak{p})$ is a field, $B \otimes_A k(\mathfrak{p})$ is then a Noetherian ring.

i) $\Rightarrow$ ii) Let $\mathfrak{p}$ be a prime ideal in A, f induces a map $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, and then $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is a finite $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$-algebra. By Exercise 3, $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is Artinian, by Exercise 2, $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is discrete.

ii) $\Rightarrow$ iii) Since $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}\big(B \otimes_A k(\mathfrak{p})\big)$, by Exercise 2, $B \otimes_A k(\mathfrak{p})$ is an Artin ring, by Exercise 3, $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra.

iii) $\Rightarrow$ ii) Let $\mathfrak{p}$ be a prime ideal of A. By Exercise 3, $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra implies $B \otimes_A k(\mathfrak{p})$ is Artinian, by Exercise 2, $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}\big(B \otimes_A k(\mathfrak{p})\big)$ is discrete.

ii) $\Rightarrow$ iv) If $\operatorname{Spec}\big(B \otimes_A k(\mathfrak{p})\big)$ is discrete, by Exercise 2, $\operatorname{Spec}\big(B \otimes_A k(\mathfrak{p})\big)$ is discrete and finite.

The assertion is not true in general. Consider the inclusion $f : \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$, this is obviously integral and has finite fibers (prime ideal of a field is zero). But f is not finite, since the elements $\sqrt{p}$, where p are prime numbers, are linearly independent over $\mathbb{Q}$ (although the proof is not easy).[1] ■

Problem 5. In Chapter 5, Exercise 16, show that X is a finite covering of L (i.e., the number of points of X lying over a given point of L is finite and bounded).

Proof. We use the notation of Exercise 16 of Chapter 5. Let $P(X)$ and $P(L)$ be the coordinate rings of X and L respectively, and let $f : P(L) \to P(X)$ be induced by the surjective linear map $\pi : X \to L$. By Exercise 27 of Chapter 1, there are bijections μ_X and μ_L, and a map f^* induced by f (Corollary 5.8), such that

$$\begin{tikzcd} X \arrow[r, hook, two heads, "\mu_X"] \arrow[d, two heads, "\pi"'] & \operatorname{Specm}\big(P(X)\big) \arrow[d, "f^*"] & \mathfrak{m}_x \arrow[d, mapsto] \\ L \arrow[r, hook, two heads, "\mu_L"] & \operatorname{Specm}\big(P(L)\big) & \mathfrak{n}_{\pi(x)} \end{tikzcd}$$

where $\mathfrak{m}_\bullet \in \operatorname{Specm}\big(P(X)\big)$ and $\mathfrak{n}_\bullet \in \operatorname{Specm}\big(P(L)\big)$. So the problem is equivalent to showing that f^* has finite bounded fibers.

Denote $A = P(X)$ and $B = P(L)$. Let $\mathfrak{n} \in \operatorname{Specm}(B)$, we have

$$f^{*-1}(\mathfrak{n}) = \operatorname{Specm}\big(k(\mathfrak{n}) \otimes_B A\big) \subseteq \operatorname{Spec}\big(k(\mathfrak{n}) \otimes_B A\big),$$

where $k(\mathfrak{n}) = B_\mathfrak{n}/\mathfrak{n}B_\mathfrak{n}$. Since f is integral, by Corollary 5.2, f is finite, by Exercise 4 i) $\Rightarrow$ iv), f^* has finite fibers. Suppose A is generated by m elements as a B-algebra. Fix $\mathfrak{n}$, let $C = k(\mathfrak{n}) \otimes_B A$, and $\mathfrak{m}_1, \dots, \mathfrak{m}_r \in \operatorname{Specm}(C)$, note that they are pairwise coprime. By Chinese remainder theorem, we have a surjection $C \twoheadrightarrow \prod_{i=1}^r (C/\mathfrak{m}_i)$, and

$$m \geq \dim_{k(\mathfrak{n})} C \geq \dim_{k(\mathfrak{n})} \prod_{i=1}^r (C/\mathfrak{m}_i) \geq r.$$

This implies that m is an upper bound for the fibers of f^*. ■

[1] Reference: Alex Youcis (`https://math.stackexchange.com/users/16497/alex-youcis`). "If the ring map $f : A \to B$ is integral, fibres of f^* are finite, then f is finite?" *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/675870`, 2014.

Problem 6. Let A be a Noetherian ring and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal in A. Consider chains of primary ideals from $\mathfrak{q}$ to $\mathfrak{p}$. Show that all such chains are of finite bounded length, and that all maximal chains have the same length.

Proof. Since A is a Noetherian A-module, every chain from $\mathfrak{q}$ to $\mathfrak{p}$ is of length less that the length of A, they are all of bounded lengths.

By Proposition 4.8, we see that the chains of primary ideals from $\mathfrak{q}$ to $\mathfrak{p}$ are in bijection with the chains of primary ideals from 0 to $\mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}$. Hence we may assume that A is an Artin local ring with an unique prime ideal $\mathfrak{p}$, and consider only the chains of primary ideals from 0 to $\mathfrak{p}$.

By Proposition 4.2 ($r(\mathfrak{q})$ maximal $\implies$ $\mathfrak{q}$ primary), every chain of ideals in A is a chain of $\mathfrak{p}$-primary ideal, hence by Proposition 6.7, the maximal chains in A have the same length. ■

Remark 8.0.1. Another approach to the second part is to adapt the proof of Proposition 6.7. By Lemma 4.3, finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary, hence we can do step i) in the proof of Proposition 6.7. Then everything follows.

Chapter 9

Discrete Valuation Rings and Dedekind Domains

Problem 1. Let A be a Dedekind domain, S a multiplicatively closed subset of A. Show that $S^{-1}A$ is either a Dedekind domain or the field of fraction of A.

Suppose that $S \neq A \setminus \{0\}$, and let H, H' be the ideal class groups of A and $S^{-1}A$ respectively. Show that extension of ideals induces a surjective homomorphism $H \to H'$.

Proof. Let K be the field of fractions of A. If S meets every maximal ideal of A, clearly $S^{-1}A = K$. If not, we have $\dim(S^{-1}A) = 1$, and by Proposition 5.12 $S^{-1}A$ is integrally closed, hence it is a Dedekind domain.

Let I, I' be the group of invertible fractional ideals and P, P' the group of principal fractional ideals of A and $S^{-1}A$ respectively. Let $f : A \to S^{-1}A$ be the natural map, then $f(I) = I'$, the induced map $f_* : H \to H'$ is well-defined. For $M' \in P'$, let $M = f_*^{-1}(M')$, then $M \neq 0$, and $M' = f(M) = S^{-1}M$. Hence f_* is surjective. ■

Problem 2. Let A be a Dedekind domain. If $f = a_0 + a_1 + \cdots + a_n x^n$ is a polynomial with coefficients in A, the *content* of f is the ideal $c(f) = (a_0, \ldots, a_n)$ in A. Prove *Gauss's lemma* that $c(f)c(g) = c(fg)$.
[Localize at each maximal ideal.]

Proof 1. If we can proof that $c(fg)_\mathfrak{p} = c(f)_\mathfrak{p} c(g)_\mathfrak{p}$ for each maximal ideal $\mathfrak{p}$, by the local property of localization (Proposition 3.8, apply to $c(f)c(g)/c(fg)$) we are done. Hence we may assume that A is a local ring with maximal ideal $\mathfrak{p} = (x)$.

Let $f(x) = \sum_{i=0}^n a_i x^i \neq 0$ and $g(x) = \sum_{j=0}^m b_j x^j \neq 0$. By Proposition 9.2, we have $c(fg) = (x^t)$, $c(f) = (x^r)$ and $c(g) = (x^s)$. Let $0 \leq i_0 \leq n$ and $0 \leq j_0 \leq m$ be the smallest such that $(a_{i_0}) = (x^r)$ and $(b_{j_0}) = (x^s)$. Then

$$\sum_{i+j=i_0+j_0} a_i b_j = c x^{r+s} \quad (c \in A^\times)$$

since for $i + j = i_0 + j_0$ and $(i, j) \neq (i_0, j_0)$, we have $a_i b_j \in (x^{r+s+1})$, and since $1 + \sum c_i x^{k_i} \in A^\times$ for $c_i \in A$ and $k_i \geq 1$. Besides, we have $c(fg) \subseteq c(f)c(g)$, $t \geq r + s$. Therefore $t = r + s$ and $c(f)c(g) = c(fg)$. ■

Proof 2. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$. Dedekind-Mertens Lemma says

$$c(f)^m c(f)c(g) = c(f)^m c(fg). \tag{9.1}$$

Since $c(f) \neq 0$, $c(f)^k \neq 0$ for $k \geq 0$,[1] $c(f)^m$ is invertible, multiplying its inverse to (9.1) we get $c(f)c(g) = c(fg)$. ■

Problem 3. A valuation (other than a field) is Noetherian if and only if it is a discrete valuation ring.

Proof. Let A be a valuation ring, K its field of fractions, $v : K^\times \to \Gamma$ a valuation of K with valuation ring A and value group Γ. Then $\Gamma \neq 0$, since A is not a field.

($\Rightarrow$) Denote $\Gamma_{>0} = \{x \in \Gamma \mid x > 0\}$. For each $x \in \Gamma_{>0}$, let $\mathfrak{a}_x = \{a \in K^\times \mid v(a) \geq x\}$, it is an ideal of A. Since A is Noetherian, the collection $\{\mathfrak{a}\}_{x \in \Gamma_{>0}}$ has a maximal element, say $\mathfrak{a}_1$ for some $1 \in \Gamma_{>0}$,[2] it is the minimal element in $\Gamma_{>0}$. Then 1 generate Γ, for if there are $k < x < k + 1$ for some $k \in \mathbb{Z}$,[3] we have $0 < x - k < 1$, contradicting the minimality of 1 in $\Gamma_{>0}$. Hence $\Gamma \cong \mathbb{Z}$, v is a discrete valuation.

($\Leftarrow$) Since $v(a) = v(b)$ iff $(a) = (b)$, every ideal in A is of the form $\mathfrak{a}_x$, where $\mathfrak{a}_x = \{a \in A \mid v(a) \geq x\}$. Hence given a chain of ideals in A, it is of the form $\mathfrak{a}_{x_1} \subseteq \mathfrak{a}_{x_2} \subseteq \cdots$, such that $x_1 \geq x_2 \geq \cdots$ in Γ. Since $\Gamma = \mathbb{Z}$, the second chain is stationary, and so is the first one, A is therefore a Noetherian ring. ■

Problem 4. Let A be a local domain which is not a field and in which the maximal ideal $\mathfrak{m}$ is principal and $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0$. Prove that A is a discrete valuation ring.

Proof. Write $\mathfrak{m} = (x)$. For each $a \in \mathfrak{m}$, $a \neq 0$, we must have $a \in \mathfrak{m}^n$ and $a \notin \mathfrak{m}^{n+1}$ for some $n \geq 1$, otherwise we will have $a \in \bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0$, this will lead to $a = 0$. For this n, we have $a = cx^n$ for some $c \notin (x)$, that is, $c \in A^\times$ since A is local. We define a map $\lambda : \mathfrak{m} \setminus \{0\} \to \mathbb{Z}_{>0}$ by $a \mapsto n$, where n is as above.

For every ideal $\mathfrak{a} \subseteq A$, the set $\{\lambda(a) \mid a \in \mathfrak{a}, a \neq 0\}$ has a minimal element, say $n = \lambda(a)$, then we have $\mathfrak{a} = (a) = (x^n)$. Therefore A is a principal ideal domain, clearly it has only one non-zero prime ideal $\mathfrak{m}$ (for if $\mathfrak{p} = (x^k)$ is prime, taking the radical we get $\mathfrak{p} = \mathfrak{m}$), by Proposition 9.2, A is a discrete valuation ring. ■

Problem 5. Let M be a finitely-generated module over a Dedekind domain. Prove that M is flat $\Leftrightarrow$ M is torsion-free.
[Use Chapter 3, Exercise 13 and Chapter 7, Exercise 16.]

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[1] This is because A is a Dedekind domain, $c(f) = \prod_\ell \mathfrak{p}_\ell$ for some non-zero prime ideals $\mathfrak{p}_\ell$, and because $\mathfrak{p}_\ell^k \neq 0$ for $k \geq 0$, we have $c(f)^m \neq 0$.

[2] The '1' here is just a notation, since Γ is not a ring, it doesn't have a multiplicative identity.

[3] Here $k \in \mathbb{Z}$ as an element in Γ means the k copies of 1.

Proof. ($\Rightarrow$) Since flatness is a local property, by Exercise 16 of Chapter 7, $M_\mathfrak{p}$ is free for every maximal ideal $\mathfrak{p}$, hence $M_\mathfrak{p}$ is torsion-free. By Exercise 13 of Chapter 3, M is then torsion-free.

($\Leftarrow$) Since A is a Dedekind domain, for every maximal ideal $\mathfrak{p}$, $A_\mathfrak{p}$ is a principal ideal domain, and $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module and is finitely generated and torsion-free, hence by the structure theorem for finitely generated modules over a principal ideal domain[4], $M_\mathfrak{p}$ is free, and is therefore flat. Hence again by the fact that flatness is a local property, M is flat. ■

Problem 6. Let M be a finitely-generated torsion module ($T(m) = M$) over a Dedekind domain A. Prove that M is uniquely representable as a finite direct sum of modules $A/\mathfrak{p}_i^{n_i}$, where $\mathfrak{p}_i$ are non-zero prime ideals of A. [For each $\mathfrak{p} \neq 0$, $M_\mathfrak{p}$ is a torsion $A_\mathfrak{p}$-module; use the structure theorem for modules over a principal ideal domain.]

Proof. By Exercise 13 of Chapter 3, for every maximal ideal $\mathfrak{p}$, $M_\mathfrak{p}$ is a torsion module, it is also a finitely generated $A_\mathfrak{p}$-module. Hence by the structure theorem for finitely generated modules over a principal ideal domain, and by the fact that every non-zero ideal in $A_\mathfrak{p}$ is of the form $\mathfrak{p}^k A_\mathfrak{p}$, we have

$$M_\mathfrak{p} \cong \bigoplus_{j=1}^{N_\mathfrak{p}} (A_\mathfrak{p}/\mathfrak{p}^{n_{\mathfrak{p},j}} A_\mathfrak{p}) = \Big(\bigoplus_{j=1}^{N_\mathfrak{p}} A/\mathfrak{p}^{n_{\mathfrak{p},j}} \Big)_\mathfrak{p} \tag{9.2}$$

for some $n_{\mathfrak{p},j} \geq 1$. Let $M' = \bigoplus_\mathfrak{p} \bigoplus_{j=1}^{N_\mathfrak{p}} A/\mathfrak{p}^{n_{\mathfrak{p},j}}$, from (9.2) we see that $\big((M + M')/M\big)_\mathfrak{p} = \big((M + M')/M'\big)_\mathfrak{p} = 0$ for every maximal ideal $\mathfrak{p}$, hence $M' \subseteq M$ and $M \subseteq M'$, and hence $M = M'$. Since M is finitely generated, there only finitely many factors in the direct sum of $M' = M$, hence we can write

$$M \cong \bigoplus_{i=1}^{N} (A/\mathfrak{p}_i^{n_i})$$

for some $n_i \geq 1$. ■

Problem 7. Let A be a Dedekind domain and $\mathfrak{a} \neq 0$ an ideal in A. Show that every ideal in $A/\mathfrak{a}$ is principal.

Deduce that every ideal in A can be generated by at most 2 elements.

Proof. Since A is a Dedekind domain, we can write $\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{p}_i^{n_i}$ for some maximal ideals $\mathfrak{p}_i$ and $n_i \geq 1$. By Chinese remainder theorem, $A/\mathfrak{a} = \bigoplus_{i=1}^{n} (A/\mathfrak{p}_i^{n_i})$, hence it

[4]Look this up in some textbooks, or see: Wikipedia contributors. "Structure theorem for finitely generated modules over a principal ideal domain." *Wikipedia, The Free Encyclopedia*, `https://en.wikipedia.org/w/index.php?title=Structure_theorem_for_finitely_generated_modules_over_a_principal_ideal_domain&oldid=1208662190`, 2024.

suffices to prove that for every maximal ideal $\mathfrak{p}$, ideals in $A/\mathfrak{p}^n$ are principal. But since $A_\mathfrak{p}$ is local and is a principal ideal domain, $A/\mathfrak{p}^n = (A/\mathfrak{p}^n)_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}$ is a principal ideal ring. Therefore we are done.[5]

For every ideal $\mathfrak{b} \neq (0), (1)$, let $x \in \mathfrak{b}$, $x \neq 0$, and let y generate $\mathfrak{b}$ in $A/(x)$. Then x, y are generators for $\mathfrak{b}$ in A. ∎

Problem 8. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be three ideals in a Dedekind domain. Prove that

$$\begin{aligned} \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) &= (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c}) \\ \mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) &= (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}). \end{aligned}$$

[Localize.]

Proof. We first prove the result when A is a local ring, let $\mathfrak{p} = (x)$ be the maximal ideal. Then A is a local Dedekind domain, and hence is a principal ideal domain. Let $\mathfrak{a} = \mathfrak{p}^a$, $\mathfrak{b} = \mathfrak{p}^b$ and $\mathfrak{c} = \mathfrak{p}^c$, we have

$$\begin{aligned} (x^a) \cap \big((x^b) + (x^c)\big) &= (x^a) \cap \big(x^{\min(b,c)}\big) = \big(x^{\max(a,\min(b,c))}\big) \\ &= \big(x^{\min(\max(a,b),\max(a,c))}\big) = \big((x^a) \cap (x^b)\big) + \big((x^a) \cap (x^c)\big), \\ (x^a) + \big((x^b) \cap (x^c)\big) &= (x^a) \cap \big(x^{\max(b,c)}\big) = \big(x^{\min(a,\max(b,c))}\big) \\ &= \big(x^{\max(\min(a,b),\min(a,c))}\big) = \big((x^a) + (x^b)\big) \cap \big((x^a) + (x^c)\big). \end{aligned}$$

Back to the general case. We see that the result is true when localized at every maximal ideal, by the fact that equality is a local property (just as what we did in the proof of Exercise 6, although this is not proved in the book), the general case is then also true. ∎

Problem 9 (Chinese Remainder Theorem)**.** Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $x_1, \ldots, x_n$ be elements in a Dedekind domain A. Then the system of congruences $x \equiv x_i \pmod{\mathfrak{a}_i}$ $(1 \leq i \leq n)$ has a solution x in $A \Leftrightarrow x_i \equiv x_j \pmod{\mathfrak{a}_i + \mathfrak{a}_j}$ whenever $i \neq j$. [This is equivalent to saying that the sequence of A-modules

$$A \xrightarrow{\phi} \bigoplus_{i=1}^n A/\mathfrak{a}_i \xrightarrow{\psi} \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j)$$

is exact, where ϕ and ψ are defined as follows:
$\phi(x) = (x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$; $\psi(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$ has (i,j)-component $x_1 - x_j + \mathfrak{a}_i + \mathfrak{a}_j$. To show that this sequence is exact it is enough to show that it is exact when localized at any $\mathfrak{p} \neq 0$: in other words we may assume that A is a discrete valuation ring, and then it is easy.]

[5]Reference: jspecter (`https://math.stackexchange.com/users/11844/jspecter`). "If A is a Dedekind domain and $I \subset A$ a non-zero ideal, then every ideal of A/I is principal." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/72548`, 2011.

Proof. As suggested, we show that the sequence

$$A \xrightarrow{\phi} \bigoplus_{i=1}^{n} A/\mathfrak{a}_i \xrightarrow{\psi} \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j) \tag{9.3}$$

is exact, where

$$\begin{aligned}
&\phi : (x_1, \ldots, x_n) \mapsto (x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n),\\
&\psi : (x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) \mapsto \big(x_i - x_j + \mathfrak{a}_i + \mathfrak{a}_j\big)_{i<j}.
\end{aligned}$$

To show that (9.3) is exact, it is sufficient to show that it is exact when localized at every maximal ideal, so we may assume that A is a local Dedekind domain with maximal ideal $\mathfrak{p} = (p)$. Then $\mathfrak{a}_i = (p^{r_i})$, and for simplicity we may assume that $r_1 \geq \cdots \geq r_n$.

Let $(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) \in \ker\psi$, we have

$$x_i - x_j \in \mathfrak{a}_i + \mathfrak{a}_j = \big(p^{\min(r_i, r_j)}\big) = \big(p^{r_j}\big) = \mathfrak{a}_j \quad (i < j),$$

in particular, $x_1 \equiv x_i \pmod{\mathfrak{a}_i}$ for every $i \geq 1$, hence x_1 is a solution of the system, $\phi(x_1) = (x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$, $\ker\psi \subseteq \operatorname{im}\phi$. The converse $\operatorname{im}\phi \subseteq \ker\psi$ follows from $\psi\phi = 0$, therefore $\ker\psi = \operatorname{im}\phi$, the sequence (9.3) is exact. ■

Chapter 10

Completions

Problem 1. Let $\alpha_n : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be the injection of abelian groups given by $\alpha_n(1) = p^{n-1}$, and let $\alpha : A \to B$ be the direct sum of all the α_n (where A is a countable direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, and B is the direct sum of the $\mathbb{Z}/p^n\mathbb{Z}$). Show that the p-adic completion of A is just A but that the completion of A for the topology induced form the p-adic topology on B is the direct *product* of the $\mathbb{Z}/p\mathbb{Z}$. Deduce that p-adic completion is *not* a right-exact functor on the category of all $\mathbb{Z}$-modules.

Proof. We have $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ and $B = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$. Since $pA = 0$, A is discrete with respect to its p-adic topology, hence $\widehat{A} = A$. For the second case, denote

$$A_n := \alpha^{-1}(B_n) = \underbrace{0 \oplus \cdots \oplus 0}_{n \text{ terms}} \oplus (\mathbb{Z}/p\mathbb{Z}) \oplus \cdots$$

where $B_n = p^n B$, then $A/A_n = \bigoplus_{i=1}^{n} \mathbb{Z}/p\mathbb{Z} = \prod_{i=1}^{n} \mathbb{Z}/p\mathbb{Z}$. The completion with respect to this topology is $\widehat{A} = \varprojlim A/A_n = \prod_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$.

Denote $\widehat{A}_1 = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ and $\widehat{A}_2 = \prod_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$. Suppose the p-adic completion is right-exact. Consider the exact sequence $0 \to A \to B \to B/\operatorname{im}\alpha \to 0$, we see that the p-adic topology of $B/\operatorname{im}\alpha$ is the same as the induced topology by the p-adic topology of B. By applying Proposition 10.3 and the hypothesis respectively, we get two exact sequences

$$0 \to \widehat{A}_2 \xrightarrow{f_2} \widehat{B} \xrightarrow{g} \widehat{B/\operatorname{im}\alpha} \to 0, \quad \widehat{A}_1 \xrightarrow{f_1} \widehat{B} \xrightarrow{g} \widehat{B/\operatorname{im}\alpha} \to 0.$$

But then we have

$$\widehat{A}_1 \twoheadrightarrow f_1(\widehat{A}_1) = \ker g = f_2(\widehat{A}_2) \cong \widehat{A}_2,$$

that is, we have a surjection $\bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \prod_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$, but this is impossible, contradiction. ∎

Problem 2. In Exercise 1, let $A_n = \alpha^{-1}(p^n B)$, and consider the exact sequence

$$0 \to A_n \to A \to A/A_n \to 0.$$

Show that $\varprojlim$ is not right exact, and compute $\varprojlim^1 A_n$.

Proof. Taking the inverse limits we get

$$0 \to \varprojlim A_n \to \bigoplus_{n=1}^{n} \mathbb{Z}/p\mathbb{Z} \to \prod_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z},$$

clearly the last arrow cannot be surjective.

Since the systems $\{A\}$ and $\{A/A_n\}$ are surjective, $\varprojlim^1 A = \varprojlim^1 A/A_n = 0$. And since $\bigcap_{n=1}^{\infty} A_n = 0$, $\varprojlim A_n = 0$. By Proposition 10.2 we have an exact sequence

$$0 \to \varprojlim A \to \varprojlim A/A_n \to \varprojlim{}^1 A_n \to 0.$$

This show that $\varprojlim^1 A_n \cong \big(\prod_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}\big)/\big(\bigoplus_{n=1}^{n} \mathbb{Z}/p\mathbb{Z}\big)$. ∎

Problem 3. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal and M a finitely-generated A-module. Using Krull's Theorem and Exercise 14 of Chapter 13, prove that

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \to M_{\mathfrak{m}}),$$

where $\mathfrak{m}$ runs over all maximal ideals containing $\mathfrak{a}$.

Deduce that

$$\widehat{M} = 0 \Leftrightarrow \operatorname{Supp}(M) \cap V(\mathfrak{a}) = \varnothing \quad \big(\text{in } \operatorname{Spec}(A)\big).$$

[The reader should think of $\widehat{M}$ as the "Taylor expansion" of M transversal to the subscheme $V(\mathfrak{a})$: the above result then shows that M is determined in a neighborhood of $V(\mathfrak{a})$ by its Taylor expansion.]

Proof. By Krull's Theorem,

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = \{x \in M \mid \exists \alpha \in \mathfrak{a}, (1-\alpha)x = 0\}.$$

For $x \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n M$, we have $(1-\alpha)x = 0$ for some $\alpha \in \mathfrak{a}$, clearly $x \in \ker(M \to M_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$, hence the "$\subseteq$" holds. For the other direction, let $x \in \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \to M_{\mathfrak{m}})$, we still have $\langle x \rangle_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$, by Exercise 14 of Chapter 3, $\mathfrak{a}\langle x \rangle = \langle x \rangle$, therefore $\alpha x = x$ for some $\alpha \in \mathfrak{a}$, and the "$\supseteq$" is clear.

$\widehat{M} = 0 \iff \bigcap_{n=1}^{\infty} \mathfrak{a}^n M = M \iff$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$, $\ker(M \to M_{\mathfrak{m}}) = 0 \iff$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$, $M_{\mathfrak{m}} = 0$ (since the map is surjective) $\iff$ for all prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$, $M_{\mathfrak{p}} = 0$. ∎

Problem 4. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal in A, and $\widehat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\hat{x}$ be the image of x in $\widehat{A}$. Show that

$$x \text{ not a zero-divisor in } A \Rightarrow \hat{x} \text{ not a zero-divisor in } \widehat{A}.$$

Does this imply that

$$A \text{ is an integral domain} \Rightarrow \widehat{A} \text{ is an integral domain?}$$

[Apply the exactness of completion to the sequence $0 \to A \xrightarrow{x} A$.]

Proof. If x is not a zero-divisor in A, we have an injection $0 \to A \xrightarrow{r} A$ where $r : a \mapsto xa$. By Proposition 10.12, the sequence $0 \to \widehat{A} \xrightarrow{\hat{r}} \widehat{A}$ is still exact, where $\hat{r}$ is the multiplication by $\hat{x}$. Therefore $\hat{x}$ is not a zero-divisor in $\widehat{A}$.

For the second part, it is not always the case, consider the following. Let $A = k[x,y]/(y^2 - x^3 - x^2)$, where k is a field of characteristic $\neq 2$, and let $\mathfrak{a} = (x, y)$. Then we have

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-3)!!}{2^n n!} x^n \in \widehat{A},$$

and $0 = (y - x\sqrt{1+x})(y + x\sqrt{1+x})$ is a product of two non-zero elements in $\widehat{A}$, therefore $\widehat{A}$ is not a integral domain.[1] ■

Remark 10.0.1. This implies Corollary 10.18, which says that in a Noetherian domain, for $\mathfrak{a} \neq (1)$ we have $\bigcap_{n=1}^{\infty} \mathfrak{a}^n = 0$. This is because $\mathfrak{a} \neq (1)$ implies $\widehat{A} \neq 0$, and for $x \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n$, $\hat{x} = 0$ is a zero-divisor, therefore x is a zero-divisor, the only choice is $x = 0$, therefore $x \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n = 0$.

Problem 5. Let A be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in A. If M is any A-module, let $M^{\mathfrak{a}}, M^{\mathfrak{b}}$ denote its $\mathfrak{a}$-adic and $\mathfrak{b}$-adic completions respectively. If M is finitely generated, prove that $(M^{\mathfrak{a}})^{\mathfrak{b}} \cong M^{\mathfrak{a}+\mathfrak{b}}$.
[Take the $\mathfrak{a}$-adic completion of the exact sequence

$$0 \to \mathfrak{b}^m M \to M \to M/\mathfrak{b}^m M \to 0$$

and apply (10.13). Then use the isomorphism

$$\varprojlim_m \left(\varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^m M) \right) \cong \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^n M)$$

and the inclusions $(\mathfrak{a} + \mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a} + \mathfrak{b})^n$.]

Proof. We first prove a lemma.

[1] This example is given by: Qiaochu Yuan (`https://math.stackexchange.com/users/232/qiaochu-yuan`). "Can the completion of a domain be a non-domain?" *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/186549`, 2012.

Lemma 10.0.2. We have

$$\varprojlim_m \big(\varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^m M)\big) \cong \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^n M).$$

Proof. Let $A = \varprojlim_m \big(\varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^m M)\big)$ and $B = \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^n M)$. Clearly we have an injection $\varphi : A \to B$ by $\big((x_{n,m} + \mathfrak{a}^n M + \mathfrak{b}^m M)_{n=1}^\infty\big)_{m=1}^\infty \mapsto (x_{n,n} + \mathfrak{a}^n M + \mathfrak{b}^n M)_{n=1}^\infty$. For every $\mathbf{y} = (x_n + \mathfrak{a}^n M + \mathfrak{b}^n M)_{n=1}^\infty \in B$, let $\mathbf{x} = \big((x_m + \mathfrak{a}^n M + \mathfrak{b}^m M)_{n=1}^\infty\big)_{m=1}^\infty$, clearly $\mathbf{x} \in A$ and $\varphi(\mathbf{x}) = \mathbf{y}$, therefore $A \cong B$. $\square$

We have an exact sequence $0 \to \mathfrak{b}^m M \to M \to M/\mathfrak{b}^m M \to 0$, by Proposition 10.12, take the $\mathfrak{a}$-adic completion we get an exact sequence $0 \to (\mathfrak{b}^m M)^{\mathfrak{a}} \to M^{\mathfrak{a}} \to (M/\mathfrak{b}^m M)^{\mathfrak{a}} \to 0$. Besides, by Proposition 10.13, $(\mathfrak{b}^m M)^{\mathfrak{a}} \cong A^{\mathfrak{a}} \otimes_A (\mathfrak{b}^m M) = \mathfrak{b}^m (A^{\mathfrak{a}} \otimes_A M) \cong \mathfrak{b}^m M^{\mathfrak{a}}$, and we have

$$\varprojlim_n \left(\frac{M}{(\mathfrak{a}^n + \mathfrak{b}^m)M}\right) = \left(\frac{M}{\mathfrak{b}^m M}\right)^{\mathfrak{a}} \cong \frac{M^{\mathfrak{a}}}{(\mathfrak{b}^m M)^{\mathfrak{a}}} \cong \frac{M^{\mathfrak{a}}}{\mathfrak{b}^m M^{\mathfrak{a}}},$$

and then

$$\varprojlim_n \left(\frac{M}{(\mathfrak{a}^n + \mathfrak{b}^n)M}\right) \cong \varprojlim_m \left(\varprojlim_n \left(\frac{M}{(\mathfrak{a}^n + \mathfrak{b}^m)M}\right)\right) \cong \varprojlim_m \frac{M^{\mathfrak{a}}}{\mathfrak{b}^m M^{\mathfrak{a}}} = (M^{\mathfrak{a}})^{\mathfrak{b}}.$$

Finally, notice that we have $(\mathfrak{a} + \mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a} + \mathfrak{b})^n$, the set $\{(\mathfrak{a} + \mathfrak{b})^n\}_{n=1}^\infty$ form the same topology of M as the set $\{\mathfrak{a}^n + \mathfrak{b}^n\}_{n=1}^\infty$, and

$$M^{\mathfrak{a}+\mathfrak{b}} = \varprojlim_n \left(\frac{M}{(\mathfrak{a} + \mathfrak{b})^n M}\right) = \varprojlim_n \left(\frac{M}{(\mathfrak{a}^n + \mathfrak{b}^n)M}\right) \cong (M^{\mathfrak{a}})^{\mathfrak{b}}. \qquad \blacksquare$$

Problem 6. Let A be a Noetherian ring and $\mathfrak{a}$ an ideal in A. Prove that $\mathfrak{a}$ is contained in the Jacobson radical of A if and only if every maximal ideal of A is closed for the $\mathfrak{a}$-topology. (A Noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a *Zariski ring*. Examples are local rings and (by (10.15)(iv)) $\mathfrak{a}$-adic completions.)

Proof. Denote by $\mathfrak{R}$ the Jacobson radical of A.

($\Rightarrow$) $\mathfrak{a}$ is open in A, if $\mathfrak{a} \subseteq \mathfrak{R}$, the cosets of $\mathfrak{a}$ (they are open) give a partition of $A \setminus \mathfrak{m}$, hence $\mathfrak{m}$ is closed (it is also open).

($\Leftarrow$) The basic open sets of the $\mathfrak{a}$-adic topology are of the form $x + \mathfrak{a}^n$. Since for every maximal ideal $\mathfrak{m}$, $1 \notin \mathfrak{m}$, let $1 + \mathfrak{a}^n \not\subseteq A \setminus \mathfrak{m}$ be an basic open neighborhood of 1. Then $\mathfrak{a}^n + \mathfrak{m} \neq (1)$, this implies $\mathfrak{a}^n \subseteq \mathfrak{m}$, taking the radical we have $\mathfrak{a} \subseteq \mathfrak{m}$. Therefore $\mathfrak{a} \subseteq \mathfrak{R}$. $\blacksquare$

Problem 7. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and $\widehat{A}$ the $\mathfrak{a}$-adic completion. Prove that $\widehat{A}$ is faithfully flat over A (Chapter 3, Exercise 16) if and only if A is a Zariski ring (for the $\mathfrak{a}$-topology).
[Since $\widehat{A}$ is flat over A, it is enough th show that

$$M \to \widehat{M} \text{ injective for all finitely generated } M \Leftrightarrow A \text{ is Zariski;}$$

now use (10.19) and Exercise 6.]

Proof. We already know that $\widehat{A}$ is flat over A by Proposition 1.14. Recall that $\widehat{A}$ is faithfully flat over $A \iff$ the map $M \to \widehat{A} \otimes_A M$ is injective for every A-module $M \iff$ the map[2] $M \to \widehat{A} \otimes_A M \cong \widehat{M}$ is injective for every finitely generated A-module $M \iff \bigcap_{n=1}^{\infty} \mathfrak{a}^n M = 0$ for every finitely generated A-module M.

($\Rightarrow$) Now by Exercise 16 iii) of Chapter 3, the extension of a non-unit ideal from A to $\widehat{A}$ is also non-unit. Suppose there is a maximal ideal $\mathfrak{m} \subseteq A$ such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, that is $\mathfrak{a} + \mathfrak{m} = A$. Then $\widehat{A}\mathfrak{a} + \widehat{A}\mathfrak{m} = \widehat{A}$, and there is a maximal $\mathfrak{n} \supseteq \widehat{A}\mathfrak{m}$ in $\widehat{A}$, consequently $\widehat{A}\mathfrak{a} + \mathfrak{n} = \widehat{A}$. But by Proposition 10.15 iv), $\widehat{\mathfrak{a}} = \widehat{A}\mathfrak{a}$ is contained in the Jacobson radical of $\widehat{A}$, which means that $\widehat{A}\mathfrak{a} \subseteq \widehat{\mathfrak{n}}$, therefore $\widehat{A}\mathfrak{a} + \mathfrak{n} = \mathfrak{n} \neq \widehat{A}$, this is a contradiction. So $\mathfrak{a}$ is contained in the Jacobson radical of A.

($\Leftarrow$) If $\mathfrak{a}$ is contained in the Jacobson radical of A, by Corollary 10.19, $\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = 0$ for every finitely generated A-module M, by the above remark $\widehat{A}$ is faithfully flat. ∎

Problem 8. Let A be the local ring of the origin in $\mathbb{C}^n$ (i.e., the ring of all rational functions $f/g \in \mathbb{C}(z_1, \ldots, z_n)$ with $g(0) \neq 0$), let B be the ring of power series in $z_1, \ldots, z_n$ which converge in some neighborhood of the origin, and let C be the ring of formal power series in $z_1, \ldots, z_n$, so that $A \subset B \subset C$. Show that B is a local ring and that its completion for the maximal ideal topology is C. Assuming that B is Noetherian, prove that B is A-flat. [Use Chapter 3, Exercise 17, and Exercise 7 above.]

Proof. [3] Since every proper ideal in B contains some $f \in B$ such that $f(0) = 0$ (otherwise f has an inverse f^{-1} in C, and f^{-1} is also in B, although we didn't show it), we see that the only maximal ideal of B is $\mathfrak{m} = \{f \in B \mid f(0) = 0\}$, hence B is local.

Let $R = \mathbb{C}[z_1, \ldots, z_n]$, and let $\mathfrak{n} = R \cap \mathfrak{m}$, we have an injection $\varphi_m : R/\mathfrak{n}^m \hookrightarrow B/\mathfrak{m}^m$, but it turns out that this is also an isomorphism, and we have an exact sequence $0 \to R/\mathfrak{n}^m \xrightarrow{\sim} B/\mathfrak{m}^m \to 0 \to 0$. By Proposition 10.2, taking the inverse limits of above we get an exact sequence $0 \to \widehat{R} \xrightarrow{\sim} \widehat{B} \to 0$, i.e. $\widehat{R} \cong \widehat{B}$. Besides,

[2] This map is defined by $x \mapsto 1 \otimes x$, and the isomorphism takes $1 \otimes x \mapsto 1 \otimes \hat{x} \mapsto \hat{x}$, where $\hat{x}$ is the image of x in $\widehat{A}$. Hence the composite map $M \to \widehat{M}$, $x \mapsto \hat{x}$ is the natural inclusion.

[3] This is a very ambiguous proof, because I don't know how to deal with power series in full details.

we have $\widehat{R} = k[\![z_1, \ldots, z_n]\!]$ (Example 1 on page 105, and Corollary 10.27), therefore $\widehat{B} \cong \widehat{R} = C$.

Finally, B is then Zariski, by Exercise 7, $C = \widehat{B}$ is faithfully flat over B. Similarly $C = \widehat{A}$ is also faithfully flat over A, since $A = R_{\mathfrak{n}}$ is a local and Noetherian. Therefore by Exercise 17 of Chapter 3, B is flat over A. ■

Problem 9. Let A be a local ring, $\mathfrak{m}$ its maximal ideal. Assume that A is $\mathfrak{m}$-adically complete. For any polynomial $f(x) \in A[x]$, let $\bar{f} \in (A/\mathfrak{m})[x]$ denote its reduction mod $\mathfrak{m}$. Prove Hensel's lemma: if $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\bar{g}(x), \bar{h}(x) \in (A/\mathfrak{m})[x]$ of degrees $r, n-r$ with $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$, then we can lift $\bar{g}(x), \bar{h}(x)$ back to monic polynomials $g(x), h(x) \in A[x]$ such that $f(x) = g(x)h(x)$.
[Assume inductively that we have constructed $g_k(x), h_k(x) \in A[x]$ such that $g_k(x)h_k(x) - f(x) \in \mathfrak{m}^k A[x]$. Then use the fact that since $\bar{g}(x)$ and $\bar{h}(x)$ are coprime we can find $\bar{a}_p(x), \bar{b}_p(x)$, of degrees $\leq n-r, r$ respectively, such that $x^p = \bar{a}_p(x)\bar{g}_k(x) + \bar{b}_p(x)\bar{h}_k(x)$, where p is any integer such that $1 \leq p \leq n$. Finally, use the completeness of A to show that the sequences $g_k(x), h_k(x)$ converge to the required $g(x), h(x)$.]

Proof. Let $g_1 = g$ and $h_1 = h$, we have $f - g_k h_k \in \mathfrak{m}^k[x]$ for $k = 1$. Note that the leading coefficients of g_1, h_1 lies in $A \setminus \mathfrak{m} = A^\times$ since A is local.

Suppose for some $k \geq 1$, we have constructed g_k, h_k such that $f - g_k h_k \in \mathfrak{m}^k[x]$, and $g_k - g_{k-1}, h_k - h_{k-1} \in \mathfrak{m}^{k-1}[x]$. Then $(g_k) + (h_k) = (1)$ in $(A/\mathfrak{m})[x]$, there exist $a, b \in \mathfrak{m}^k[x]$ such that $ag_k + bh_k \equiv 1 \pmod{\mathfrak{m}^{k+1}}$ with $\deg a < \deg h_k$ and $\deg b < \deg g_k$, let $g_{k+1} = g_k + b$, $h_{k+1} = h_k + a$. Therefore

$$g_{k+1}h_{k+1} = (g_k + b)(h_k + a) = g_k h_k + ag_k + bh_k + ab \equiv f \pmod{\mathfrak{m}^{k+1}},$$

and $g_{k+1} - g_k, h_{k+1} - h_k \in \mathfrak{m}^k[x]$.[4]

Now we have constructed two sequences $\{g_k\}_{k=1}^\infty$ and $\{h_k\}_{k=1}^\infty$ of polynomials of bounded degrees, and such that $f \equiv g_k h_k \pmod{\mathfrak{m}^k}$. We see that these two sequences converge in $A[x]$, say converge to g and h respectively. Now $f \equiv gh \pmod{\mathfrak{m}^k}$ for every $k \geq 1$ show that $f = gh$. ■

Problem 10.

i) With the notation of Exercise 9, deduce from Hensel's lemma that if $\bar{f}$ has a simple root $\alpha \in A/\mathfrak{m}$, then $f(x)$ has a simple root $a \in A$ such that $\alpha = a$ mod $\mathfrak{m}$.

ii) Show that 2 is a square in the ring of 7-adic integers.

[4] Reference: Wikipedia contributors. "Hensel's lemma." *Wikipedia, The Free Encyclopedia*, `https://en.wikipedia.org/w/index.php?title=Hensel%27s_lemma&oldid=1193073320`, 2024.

iii) Let $f(x,y) \in k[x,y]$, where k is a field, and assume that $f(0,y)$ has $y = a_0$ as a simple root. Prove that there exists a formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f\big(x, y(x)\big) = 0$.

(This gives the "*analytic branch*" of the curve $f = 0$ through the point $(0, a_0)$.)

Proof.

i) We have $\bar{f}(x) = (x-\alpha)\bar{g}(x)$ and $(x-\alpha) \nmid \bar{g}(x)$, by Hensel's lemma and the explicit construction in Exercise 9, we can lift them back to $(x-a), g(x) \in A[x]$, and $a \equiv \alpha \pmod{\mathfrak{m}}$. Since α is a simple root of $\bar{f}$, $\bar{g}(\alpha) \neq 0$ and $g(a) \neq 0$, a is then a simple root of f.

ii) Let $f(x) = x^2 - 2 \in \mathbb{Z}_7[x]$, then $f(x) \equiv (x-3)(x+3) \pmod 7$. We see that 2 is a simple root of $\bar{f}$, by i), $f(a) = 0$ for some $a \in \mathbb{Z}_7$, that is, $2 = a^2$ in $\mathbb{Z}_7$.

iii) Recall $\widehat{k[x]} = k[\![x]\!]$ with its (x)-topology. Let $\mathfrak{m} = (x) \subseteq k[\![x]\!]$, in $(k[\![x]\!]/\mathfrak{m})[y] = k[y]$, $\bar{f}(x,y) = f(0,y) \in k[y] \subseteq k[\![x]\!][y]$ has a simple root $y = a_0$, by i) there is $y(x) = \sum_{n=0}^{\infty} a_n x^n \in k[\![x]\!]$ such that $f\big(x, y(x)\big) = 0$ in $k[\![x]\!][y]$ after lifting. ■

Problem 11. Show that the converse of (10.26) is false, even if we assume that A is local and that $\widehat{A}$ is a finitely-generated A-module.
[Take A to be the ring of germs of C^∞ functions of x at $x = 0$, and use Borel's Theorem that every power series occurs as the Taylor expansion of some C^∞ function.]

Proof. Let A be the ring of germs of $C^\infty(\mathbb{R};\mathbb{R})$ at $x = 0$, and denote by $[f]$ the image in A of some f. Borel's Theorem says that the natural map $\varphi : A \to \mathbb{R}[\![x]\!]$ is surjective. We have an surjection $A \twoheadrightarrow \mathbb{R}$ by evaluating $x = 0$, let $\mathfrak{m}$ be its kernel, it is then maximal. For $f \in C^\infty(\mathbb{R};\mathbb{R})$ with $f(0) \neq 0$, the power series $\varphi([f])$ has non-zero constant term, by Exercise 5 i) of Chapter 1, it has an inverse $\varphi([g])$. Then $fg = [1]$, this shows that $\mathfrak{m}$ is the only maximal ideal in A, and therefore A is a local ring. And we see that $\mathfrak{m} = ([x])$ (by the same method of lifting to $\mathbb{R}[\![x]\!]$ via φ).

We have $A/\mathfrak{m}^n \cong \mathbb{R}[x]/(x^n)$, then $\widehat{A} \cong \widehat{\mathbb{R}[x]} = \mathbb{R}[\![x]\!]$. Besides, by Corollary 10.27 (or the remark of Theorem 7.5), $\mathbb{R}[\![x]\!]$ is Noetherian, and since the map φ is surjective, $\widehat{A}$ is finitely generated as an A-module. However, since $[e^{-1/x^2}] \in \bigcap_{n=1}^{\infty} \mathfrak{m}^n = \bigcap_{n=1}^{\infty}([x^n])$, by Corollary 10.19, A is not Noetherian. This is a counterexample. ■

Problem 12. If A is Noetherian, then $A[\![x_1, \ldots, x_n]\!]$ is a faithfully flat A-algebra.
[Express $A \to A[\![x_1, \ldots, x_n]\!]$ as a composition of flat extensions, and use Exercise 5(v) of Chapter 1.]

Proof. Let $B \supseteq A$ be a Noetherian ring. By Exercise 5 of Chapter 2, $B[x]$ is flat over B, by Proposition 1.14, $B[\![x]\!] = \widehat{B[x]}$ is flat over $B[x]$, therefore $B[\![x]\!]$ is flat over B. Then by Exercise 5 v) of Chapter 1 and Exercise 16 iii) of Chapter 3, $B[\![x]\!]$ is faithfully flat over B.

Since we can break the map $A \to A[\![x_1, \ldots, x_n]\!]$ into

$$A \to A[\![x_1]\!] \to A[\![x_1, x_2]\!] \to \cdots \to A[\![x_1, \ldots, x_n]\!],$$

it is easy to see that each component is Noetherian, and is also faithfully flat over the previous one by the discussion above. Therefore $A[\![x_1, \ldots, x_n]\!]$ is faithfully flat over A. ■

Chapter 11

Dimension Theory

Problem 1. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial over an algebraically closed field k. A point P on the variety $f(x) = 0$ is *non-singular* $\Leftrightarrow$ not all the partial derivatives $\partial f/\partial x_i$ vanish at P. Let $A = k[x_1, \ldots, x_n]/(f)$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point P. Prove that P is non-singular $\Leftrightarrow$ $A_\mathfrak{m}$ is a regular local ring.
[By (11.18) we have $\dim A_\mathfrak{m} = n - 1$. Now

$$\mathfrak{m}/\mathfrak{m}^2 \cong (x_1, \ldots, x_n)/(x_1, \ldots, x_n)^2 + (f)$$

and has dimension $n - 1$ if and only if $f \notin (x_1, \ldots, x_n)^2$.]

Proof. By Corollary 11.18, $\dim A_\mathfrak{m} = n - 1$. We may assume that $P = (0, \ldots, 0)$, and $\mathfrak{m} = (x_1, \ldots, x_n) + (f)$. Suppose $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n - 1$, then we cannot have $f \in (x_1, \ldots, x_n)^2$, otherwise we have $\mathfrak{m}/\mathfrak{m}^2 \cong k^n$ as k-vector spaces. On the other hand, suppose $f \notin (x_1, \ldots, x_n)^2$, then f the homogeneous part of degree 1 is non-zero, therefore some terms x_i occur in f, and $\dim_k(\mathfrak{m}/\mathfrak{m}^2) < n$, since $\dim A_\mathfrak{m} \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, we must have $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n - 1$.

Besides, since f is a polynomial and $f(0) = 0$, it is clear that

$$f \notin (x_1, \ldots, x_n)^2 \iff \exists i, \left.\frac{\partial f}{\partial x_i}\right|_{x=0} \neq 0.$$

Therefore, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim A_\mathfrak{m} \iff$ not all $\partial f/\partial x_i$ vanish at $x = 0$. ■

Problem 2. In (11.21) assume that A is complete. Prove that the homomorphism $k[t_1, \ldots, t_d] \to A$ given by $t_i \mapsto x_i$ $(1 \leq i \leq d)$ is injective and that A is a finitely-generated module over $k[t_1, \ldots, t_d]$. [Use (10.24).]

Proof. Clearly by Corollary 11.21 the map $\varphi : k[t_1, \ldots, t_d] \to A$ by $t_i \mapsto x_i$ is injective, and maps isomorphically onto $k[x_1, \ldots, x_d]$. Since $\varphi^{-1}(\mathfrak{q}) = (t_1, \ldots, t_d)$, taking their completions and by Corollary 10.3, φ induces an injection $\widehat{\varphi} : k[\![t_1, \dots, t_d]\!] \hookrightarrow \widehat{A} \cong A$, which also maps $t_i \mapsto x_i$.

For the second part, we first prove a lemma.[1]

Lemma 11.0.1. For an Artin local ring A, $\mathfrak{m}$ its maximal ideal. If there is a subfield $k \subseteq A$ isomorphic to $A/\mathfrak{m}$, then $\dim_k A < \infty$.

Proof. Since A is Artinian local, $\mathfrak{m}^{N+1}$ for some N, and each $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an Artin A-module, of finite length. Therefore

$$\dim_k A = \dim_k\left(A/\mathfrak{m}^{N+1}\right) = \sum_{n=0}^{N} \dim_k\left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right) < \infty. \qquad \square$$

Let $B = k[t_1, \dots, t_d]$, $\mathfrak{b} = (t_1, \dots, t_d) \subseteq B$, then $\widehat{B} = k[\![t_1, \dots, t_d]\!]$ with respect to its $\mathfrak{b}$-adic completion. Let $A_n = \mathfrak{q}^n \subseteq A$, this is a $\hat{\mathfrak{b}}$-filtration of A, and $\hat{\mathfrak{b}}\mathfrak{q}^n = \mathfrak{q}^{n+1}$ for $n \geq 0$. Since A is Noetherian local, by Corollary 10.19, $\bigcap_{n=0}^{\infty} \mathfrak{q}^n = 0$, therefore A is Hausdorff. Besides, $G_{\hat{\mathfrak{b}}}(\widehat{B}) \cong G_{\mathfrak{b}}(B) \cong B$, $B/\mathfrak{b} \cong k$, and by the lemma $A/\mathfrak{q}$ is finitely generated as an k-module, so $G_{\mathfrak{a}}(A)$ is a finitely generated $G_{\hat{\mathfrak{b}}}(\widehat{B})$-module. By Proposition 10.24, A is a finitely generated $k[\![t_1, \dots, t_d]\!]$-module. ∎

Problem 3. Extend (11.25) to non-algebraically-closed fields. [If $\bar{k}$ is the algebraic closure of k, then $\bar{k}[x_1, \dots, x_n]$ is integral over $k[x_1, \dots, x_n]$.]

Proof. The proof is basically the one of Theorem 11.25.

By the Normalization Lemma, there is a polynomial ring $B = k[x_1, \dots, x_d] \subseteq A(V)$ such that $d = \dim V$ and $A(V)$ is integral over B. Since B is integrally closed (because B is a UFD), we can apply Lemma 11.26 and reduce to proving that $d = \dim B_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{m} \subseteq A(V)$ and $\mathfrak{n} = \mathfrak{m} \cap B$.

Let $\bar{k}$ be the algebraic closure of k, then $C = \bar{k}[x_1, \dots, x_r]$ is integral over B. Let $\mathfrak{n} = \mathfrak{m} \cap B$, and let $\bar{\mathfrak{n}} \subseteq C$ be such that $\mathfrak{n} = \bar{\mathfrak{n}} \cap B$ (Theorem 5.10). By Corollary 5.8, $\mathfrak{n} \subseteq B$ and $\bar{\mathfrak{n}} \subseteq C$ are both maximal. Therefore

$$\dim A(V)_{\mathfrak{m}} = \dim B_{\mathfrak{n}} = \dim C_{\bar{\mathfrak{n}}} = d,$$

where the last equality is done by the proof of Theorem 11.25 (and is not hard to see this). ∎

Problem 4. An example of a Noetherian domain of infinite dimension (Nagata). Let k be a field and let $A = k[x_1, x_2, \dots, x_n, \dots]$ be a polynomial ring over k in a countably infinite set of indeterminates. Let $m_1, m_2, \dots$ be an increasing sequence

[1] Reference: Emory Sun (`https://math.stackexchange.com/users/477629/emory-sun`). "Question about Exercise 11.2 in Atiyah-MacDonald." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/4494406`, 2022.

of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_i+1}, \ldots, x_{m_{i+1}})$ and let S be the complement in A of the union of the ideals $\mathfrak{p}_i$.

Each $\mathfrak{p}_i$ is a prime ideal and therefore the set S is multiplicatively closed. The ring $S^{-1}A$ is Noetherian by Chapter 7, Exercise 9. Each $S^{-1}\mathfrak{p}_i$ has height equal to $m_{i+1} - m_i$, hence $\dim S^{-1}A = \infty$.

Proof. Absolutely correct! ■

Problem 5. Reformulate (11.1) in terms of the Grothendieck group $K(A_0)$ (Chapter 7, Exercise 26).

Proof. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a Noetherian graded ring, then A is finitely generated by, say, homogeneous elements of degrees $k_1, \ldots, k_s > 0$. Let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded A-module, then each M_n is a finitely generated A_0-module.

Let $F(A_0)$ be the the set of all isomorphism classes of finitely generated A_0-modules, $K(A_0)$ the Grothendieck group of A_0, and $\gamma : F(A_0) \to K(A_0)$ the natural map. Define

$$P(M,t) = \sum_{n=0}^{\infty} \gamma(M_n) t^n \in K(A_0)[\![t]\!].$$

Theorem 11.1 now becomes: $P(M,t) \prod_{i=1}^{s} (1 - t^{k_i}) \in K(A_0)[t]$.[2] [3] ■

Problem 6. Let A be a ring (not necessarily Noetherian). Prove that

$$1 + \dim A \leq \dim A[x] \leq 1 + 2 \dim A.$$

[Let $f : A \to A[x]$ be the embedding and consider the fiber of $f^* : \operatorname{Spec}(A[x]) \to \operatorname{Spec}(A)$ over a prime ideal $\mathfrak{p}$ of A. This fiber can be identified with the spectrum of $k \otimes_A A[x] \cong k[x]$, where k is the residue field of at $\mathfrak{p}$ (Chapter 3, Exercise 21), and $\dim k[x] = 1$. Now use Exercise 7(ii) of Chapter 4.]

Proof. For a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in A, by Exercise 7 ii) of Chapter 4 we have a chain of prime ideals $\mathfrak{p}_0[x] \subsetneq \mathfrak{p}_1[x] \subsetneq \cdots \subsetneq \mathfrak{p}_r[x] \subsetneq \mathfrak{q}$ in $A[x]$, where $\mathfrak{q} = \left\{ \sum_{i=0}^{n} c_i x^i \mid n \geq 0, c_0 \in \mathfrak{p}_r, c_1, \ldots, c_n \in A \right\}$, hence $1 + \dim A \leq \dim A[x]$.

Let $f : A \to A[x]$ be the inclusion, $f^* : \operatorname{Spec}(A[x]) \to \operatorname{Spec}(A)$ the associated map. For a prime ideal $\mathfrak{p} \subseteq A$, denote $k = k(\mathfrak{p})$ the reside field at $\mathfrak{p}$, then $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}(k \otimes_A A[x]) = \operatorname{Spec}(k[x])$ has maximal length 2 of prime ideals, since we have $\dim k[x] = 1$. Now, let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ be a chain of prime ideals in $A[x]$, from the above argument we see that we won't have the case $f(\mathfrak{q}_{i-1}) = f(\mathfrak{q}_i) = f(\mathfrak{q}_{i+1})$.

[2] We don't need $K(A_0)$ to be a ring. If we regard abelian groups as $\mathbb{Z}$-modules, we can define multiplication of $f \in \mathbb{Z}[t]$ and $g \in G[t]$ in a natural way, where G is an abelian group. See Exercise 6 of Chapter 2.

[3] For another perspective of reformulation, see: jdc (`https://math.stackexchange.com/users/7112/jdc`). "Exercise 11.5 from Atiyah-MacDonald: Hilbert-Serre theorem and Grothendieck group." *Mathematics Stack Exchange*, `https://math.stackexchange.com/q/217612`, 2012.

Therefore, after picking out repeating terms, the length of $f(\mathfrak{q}_0) \subseteq f(\mathfrak{q}_1) \subseteq \cdots \subseteq f(\mathfrak{q}_r)$ is at least $\lceil (r+1)/2 \rceil$, and we have $\dim A[x] \leq 1 + 2\dim A$. ∎

Problem 7. Let A be a Noetherian ring. Then

$$\dim A[x] = 1 + \dim A,$$

and hence, by induction on n,

$$\dim A[x_1, \ldots, x_n] = n + \dim A.$$

[Let $\mathfrak{p}$ be a prime ideal of height m in A. Then there exist $a_1, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal belonging to the ideal $\mathfrak{a} = (a_1, \ldots, a_m)$. By Exercise 7 of Chapter 4, $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$ and therefore $\operatorname{height} \mathfrak{p}[x] \leq m$. On the other hand, a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m = \mathfrak{p}$ gives rise to a chain $\mathfrak{p}_0[x] \subset \cdots \subset \mathfrak{p}_m[x] = \mathfrak{p}[x]$, hence $\operatorname{height} \mathfrak{p}[x] = m$. Hence $\operatorname{height} \mathfrak{p}[x] = \operatorname{height} \mathfrak{p}$. Now use the argument of Exercise 6.]

Proof. Let $\mathfrak{p} \subseteq A$ be a prime ideal of height m, then there exist $a_1, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal of the ideal $\mathfrak{a} = (a_1, \ldots, a_m)$ (this is done by applying Dimension Theorem to $A_{\mathfrak{p}}$). By Exercise 7 v) of Chapter 4, $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$, and is therefore of height $\leq m$ (again by Dimension Theorem). Besides, a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m = \mathfrak{p}$ in A gives rise to a chain of prime ideals $\mathfrak{p}_0[x] \subsetneq \mathfrak{p}_1[x] \subsetneq \cdots \subsetneq \mathfrak{p}_m[x] = \mathfrak{p}[x]$ in $A[x]$, hence $\operatorname{height} \mathfrak{p}[x] \geq m$, and hence $\operatorname{height} \mathfrak{p}[x] = m$.

From the proof of Exercise 6 we see that, for every prime ideal $\mathfrak{p} \subseteq A$, there are exactly two distinct prime ideals $\mathfrak{q}_1, \mathfrak{q}_2 \subseteq A[x]$, such that $\mathfrak{p} = \mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$. We can see that after renumbering we have

$$\mathfrak{q}_1 = \mathfrak{p}[x], \quad \mathfrak{q}_2 = \left\{ \sum_{i=0}^{n} c_i x^i \mid n \geq 0, c_0 \in \mathfrak{p}_r, c_1, \ldots, c_n \in A \right\},$$ [4]

and $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$. Let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ be a maximal chain of prime ideals in $A[x]$ (then $m = \dim A[x]$), taking contraction we have a chain of prime ideals $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_m$ in A. If $\mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1}$ for all i, we have $\dim A \geq \dim A[x]$, contradicting Exercise 6. So we can let r be the largest number such that $\mathfrak{p}_r = \mathfrak{p}_{r+1}$. By the above remark we have $\mathfrak{q}_r = \mathfrak{p}_r[x]$, and $\operatorname{height} \mathfrak{q}_r = \operatorname{height} \mathfrak{p}_r = r$, therefore

$$\begin{aligned} \dim A[x] &= \operatorname{height} \mathfrak{q}_r + m - r \\ &= \operatorname{height} \mathfrak{p}_r + m - r \leq \dim A + 1. \end{aligned}$$

We know that $1 + \dim A \leq \dim A[x]$ from Exercise 6, hence $1 + \dim A = \dim A[x]$. ∎

[4] Since for these two choices we have $\mathfrak{p} = \mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$, and $\mathfrak{q}_1, \mathfrak{q}_2$ are also prime ideals.

Bibliography

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