HLSVD

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Hierarchical Latent SVD Model

As an extended idea of latent distance models (LDM), the latent eigenmodel was introduced to capture both homophily and stochastic equivalence among nodes. The main difference between LDMs and eigenmodels is the choice of the function dependent on the latent space. More specifically, eigenmodels use $a(u_i, u_j) = u_i^T \Lambda u_j$ instead of mere physical distance function. Besides, all latent class, distance and eigenmodels only deal with the symmetric function d or a. However; the function also can be asymmetric if there exist sender and receiver effects are different. It is so-called SVD model, and the model specification is as follows:

$$\text{logit}P(Y_{ij} = 1) = \beta^T X_{ij} + a_i + b_j + u_i^T v_j + \epsilon_{ij}$$

where u and v are vectors of latent nodal attributes, and a_i and b_i are terms for nodal heterogeneity.

MCMC Derivation

$$\begin{aligned}
\mathbf{Y}_{ijk} &= \mathbb{1}_{[\mathbf{Z}_{ijk}>0]} \\
\mathbf{Z}_{ijk} &= \sum_{p=0}^{P} \beta_{pk}^T \mathbf{X}_{ijpk} + a_{ik} + b_{jk} + u_{ik}^T v_{jk} + \epsilon_{ijk} \\
(a_{ik}, b_{ik}) &\overset{iid}{\sim} MVN_2((0,0), \mathbf{\Sigma}_{ab}) \\
(u_{ik}, v_{ik}) &\overset{iid}{\sim} MVN_4((0,0,0,0), \mathbf{\Sigma}_{uv}) \\
(\epsilon_{ijk}, \epsilon_{jik}) &\overset{iid}{\sim} MVN_2\Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_k \\ \rho_k & 1 \end{pmatrix} \Big) \\
\beta_p &\sim N(\mu_p, \sigma_p^2) \\
\mu_p &\sim N(\lambda, \tau^2) \\
\sigma_p^2 &\sim IG(\nu_0, S_0) \\
\mathbf{\Sigma}_{ab} &\sim IWishart(\nu_{ab}, S_{ab}^{-1}) \\
\mathbf{\Sigma}_{uv} &\sim IWishart(\nu_{uv}, S_{uv}^{-1})
\end{aligned}$$

Update Z_{ijk}, Z_{jik}:

$$(\mathbf{Z}_{ijk}, \mathbf{Z}_{jik}) \overset{iid}{\sim} MVN_2\Big(\begin{pmatrix} \sum_{p=0}^{P} \mathbf{X}_{ijpk}\beta_{pk} + a_{ik} + b_{jk} + u_{ik}^T v_{jk} \\ \sum_{p=0}^{P} \mathbf{X}_{ijpk}\beta_{pk} + a_{jk} + b_{ik} + u_{jk}^T v_{ik} \end{pmatrix}, \begin{pmatrix} 1 & \rho_k \\ \rho_k & 1 \end{pmatrix} \Big)$$

Let $\eta_{ijk} = \sum_{p=0}^{P} \mathbf{X}_{ijpk}\beta_{pk} + a_{ik} + b_{jk} + u_{ik}^T v_{jk}$

$$\mathbf{Z}_{ijk}|\mathbf{Z}_{jik} \sim N(\eta_{ijk} + \rho_k \cdot (\mathbf{Z}_{jik} - \eta_{jik}), 1 - \rho_k^2)$$

$$\begin{aligned}
\Pr(\mathbf{Z}_{ijk}|\mathbf{Z}_{jik}, \mathbf{Y}_{ijk} = 1, \cdots) &\propto \Pr(\mathbf{Y}_{ijk} = 1|\cdots) \cdot \Pr(\mathbf{Z}_{ijk}|\mathbf{Z}_{jik}) \\
&= \begin{cases} \mathbb{1}_{[\mathbf{Z}_{ijk}>1]} \cdot dN(\mathbf{Z}_{ijk}; \eta_{ijk} + \rho_k \cdot (\mathbf{Z}_{jik} - \eta_{jik}), 1 - \rho_k^2) \text{ if } Y_{ijk} = 1 \\ \mathbb{1}_{[\mathbf{Z}_{ijk}<1]} \cdot dN(\mathbf{Z}_{ijk}; \eta_{ijk} + \rho_k \cdot (\mathbf{Z}_{jik} - \eta_{jik}), 1 - \rho_k^2) \text{ if } Y_{ijk} = 0 \end{cases}
\end{aligned}$$

Update β_k:

Let $\bar{\mathbf{Z}}_{ijk} = \mathbf{Z}_{ijk} - a_{ik} - b_{jk} - u_{ik}^T v_{jk}$

$$\begin{aligned}
\Pr(\bar{\mathbf{Z}}_{ijk}) &\propto \prod_{i \neq j}^{n_k} \exp\left(-\frac{1}{2} \begin{pmatrix} \bar{\mathbf{Z}}_{ijk} - \sum_p \mathbf{X}_{ijpk}\beta_{pk} \\ \bar{\mathbf{Z}}_{jik} - \sum_p \mathbf{X}_{jipk}\beta_{pk} \end{pmatrix}^T \mathbf{\Sigma}_\rho^{-1} \begin{pmatrix} \bar{\mathbf{Z}}_{ijk} - \sum_p \mathbf{X}_{ijpk}\beta_{pk} \\ \bar{\mathbf{Z}}_{jik} - \sum_p \mathbf{X}_{jipk}\beta_{pk} \end{pmatrix} \right) \\
\therefore \Pr(\bar{\mathbf{Z}}_{ijk}|\bar{\mathbf{Z}}_{jik}) &\sim N(\sum_p \mathbf{X}_{ijpk}\beta_{pk} + \rho_k(\bar{\mathbf{Z}}_{jik} - \sum_p \mathbf{X}_{jipk}\beta_{pk}), 1 - \rho_k^2) \\
&\propto \prod_{i \neq j} \exp\left(-\frac{1}{2} \Big(\bar{Z}_{ijk} - \rho_k \sum_p \mathbf{X}_{jipk})\beta_{pk} - \rho_k(\bar{Z}_{jik} - X_{jik}\beta_k) \Big)^2 \cdot (1 - \rho_k^2)^{-1} \right) \\
\therefore \Pr(\beta_k|\mathbf{Z}_k) &\propto \prod_{i \neq j} \exp\left(-\frac{1}{2} \Big((X_{ijk} - \rho_k \sum_p \mathbf{X}_{jipk})\beta_{pk} - \bar{Z}_{ijk} - \rho_k \bar{Z}_{jik} \Big)^2 \cdot (1 - \rho_k^2)^{-1} \right) \cdot \mathrm{dN}(\beta_k; \mu, \mathbf{\Sigma}_\beta) \\
&\propto \exp\left(-\frac{1}{2}(1 - \rho_k^2)^{-1} \Big(\mathbf{\Theta}_\beta \vec{\beta}_k - \mathrm{vec}(\mathbf{Z}_k) + \rho_k \mathrm{vec}(\mathbf{Z}_k^T) \Big)^T \Big(\mathbf{\Theta}_\beta \vec{\beta}_k - \mathrm{vec}(\mathbf{Z}_k) + \rho_k \mathrm{vec}(\mathbf{Z}_k^T) \Big) \right) \\
&\quad \times \exp\left(-\frac{1}{2}(\vec{\beta}_k - \vec{\mu})^T \mathbf{\Sigma}_\beta^{-1} (\vec{\beta}_k - \vec{\mu}) \right)
\end{aligned}$$

where $\mathbf{\Theta}_\beta$ is a vector length of $n(n-1)$, and,

$$\mathbf{\Theta}_\beta = \Big(\begin{pmatrix} | & \cdots & | \\ \mathrm{vec}(\mathbf{X}_{1k}) & \cdots & \mathrm{vec}(\mathbf{X}_{Pk}) \\ | & \cdots & | \end{pmatrix} - \rho_k \begin{pmatrix} | & \cdots & | \\ \mathrm{vec}(\mathbf{X}_{1k}^T) & \cdots & \mathrm{vec}(\mathbf{X}_{Pk}^T) \\ | & \cdots & | \end{pmatrix} \Big)$$

Thus, we have $\mathbf{V}_n = \mathbf{A}_n^{-1}$ where $\mathbf{A}_n = \mathbf{A}_0 + \mathbf{A}_1$, and $\mathbf{A}_0 = \mathbf{\Sigma}_\beta^{-1}$, $\mathbf{A}_1 = (1 - \rho_k^2)^{-1} \mathbf{\Theta}_\beta^T \mathbf{\Theta}_\beta$

$\mu_n = \mathbf{V}_n m_n$ where $m_n = m_0 + m_1$, and $m_0 = \mathbf{\Sigma}_\beta^{-1}\mu$ and $m_1 = (1 - \rho_k^2)^{-1} \mathbf{\Theta}_\beta^T \Big(\mathrm{vec}(\mathbf{Z}_k) - \rho_k \mathrm{vec}(\mathbf{Z}_k^T) \Big)$

Update a_i, b_i:

Let $\hat{\mathbf{Z}}_{ijk} = \mathbf{Z}_{ijk} - \sum_p \mathbf{X}_{ijpk}\beta_{pk} - u_{ik}^T v_{jk}$

$$\Pr\Bigg(\begin{pmatrix} a_{ik} \\ b_{ik} \end{pmatrix} \mid \begin{pmatrix} \hat{Z}_{ijk} \\ \hat{Z}_{jik} \end{pmatrix}, \dots \Bigg) \propto$$

$$\prod_{i \neq j}^{n_k} \exp\left(-\frac{1}{2} \begin{pmatrix} \hat{Z}_{ijk} - a_{ik} - b_{jk} \\ \hat{Z}_{jik} - a_{jk} - b_{ik} \end{pmatrix}^T \Sigma_\rho^{-1} \begin{pmatrix} \hat{Z}_{ijk} - a_{ik} - b_{jk} \\ \hat{Z}_{jik} - a_{jk} - b_{ik} \end{pmatrix} \right) \cdot \mathrm{dMVN}_2\Bigg(\begin{pmatrix} a_{ik} \\ b_{ik} \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_{ab} \Bigg)$$

$$\therefore \Pr(\hat{\mathbf{Z}}_{ijk} | \hat{\mathbf{Z}}_{jik}) \sim N(a_{ik} - (-b_{jk} - \rho_k(\hat{\mathbf{Z}}_{jik} - a_{jk} - b_{ik})), 1 - \rho_k^2)$$

$$\therefore \Pr(a_{ik} | \mathbf{Z}_k) \propto \prod_{j \neq i} \exp\left(-\frac{1}{2}(1-\rho_k^2)^{-1}(\hat{\mathbf{Z}}_{ijk} - a_{ik} - b_{jk} - \rho_k(\hat{\mathbf{Z}}_{jik} - a_{jk} - b_{ik}))^2 \right)$$

$$\cdot\, \mathrm{dN}(a_{ik} | b_{ik} \;;\; \mu_{a|b} = \mathbf{\Sigma}_{ab[1,2]} \mathbf{\Sigma}_{ab[2,2]}^{-1} b_{ik},\; \sigma_{a|b}^2 = \mathbf{\Sigma}_{ab[1,1]} - \mathbf{\Sigma}_{ab[1,2]} \mathbf{\Sigma}_{ab[2,2]}^{-1} \mathbf{\Sigma}_{ab[2,1]})$$

$$\propto \exp\left(-\frac{1}{2}(1-\rho_k)^{-1} \sum_{j \neq i} (a_{ik} - \theta_j)^T (a_{ik} - \theta_j) \right) \cdot \exp\left(-\frac{1}{2}(a_{ik} - \mu_{a|b})^2 \sigma_{a|b}^{-2} \right)$$

where $\theta_j = -b_{jk} - \rho_k(\hat{\mathbf{Z}}_{jik} - a_{jk} - b_{ik}) + \hat{\mathbf{Z}}_{ijk}$ and i is fixed. Then, we can easily figure out its posterior mean and variance.

$\mathbf{V}_n = \mathbf{A}_n^{-1}$ where $\mathbf{A}_n = \mathbf{A}_0 + \mathbf{A}_1,$ and $\mathbf{A}_0 = \sigma_{a|b}^{-2}, \mathbf{A}_1 = (n_k - 1)(1-\rho_k^2)^{-1}$

$\mu_n = \mathbf{V}_n m_n$ where $m_n = m_0 + m_1$, and $m_0 = \sigma_{a|b}^{-2} \mu_{a|b}$ and $m_1 = (1-\rho_k^2)^{-1} \sum_{j \neq i} \theta_j$

Likewise, we can update b_{ik}.

Update u_i, v_i:

Let $\tilde{Z}_{ijk} = Z_{ijk} - X_{ijk}\beta_k - a_{ik} - b_{jk}$

$$p\Bigg(\begin{pmatrix} u_{ik} \\ v_{ik} \end{pmatrix} \mid \begin{pmatrix} \tilde{Z}_{ijk} \\ \tilde{Z}_{jik} \end{pmatrix}, \dots \Bigg) \propto$$

$$\prod_{i \neq j}^{n_k} \exp\left(-\frac{1}{2} \begin{pmatrix} \tilde{Z}_{ijk} - u_{ik}^T v_{jk} \\ \tilde{Z}_{jik} - u_{ik}^T v_{jk} \end{pmatrix}^T \Sigma_\rho^{-1} \begin{pmatrix} \tilde{Z}_{ijk} - u_{ik}^T v_{jk} \\ \tilde{Z}_{jik} - u_{ik}^T v_{jk} \end{pmatrix} \right) \cdot \mathrm{dMVN}_4\Bigg(\begin{pmatrix} u_{ik} \\ v_{ik} \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma_{uvk} \Bigg)$$

$$\therefore \Pr(\tilde{Z}_{ijk} | \tilde{Z}_{jik}) \sim \mathrm{N}(u_{ik}^T v_{jk} + \rho_k(\tilde{Z}_{jik} - u_{jk}^T v_{ik}), 1 - \rho_k^2)$$

$$\therefore \Pr(u_{ik} | \tilde{Z}_k) \propto \prod_{j \neq i} \exp\left(-\frac{1}{2}(1-\rho_k^2)^{-1}(\tilde{Z}_{ijk} - u_{ik}^T v_{jk} - \rho_k(\tilde{Z}_{jik} - u_{jk}^T v_{ik}))^2 \right)$$

$$\mathrm{dMVN}_2(u_{ik} | v_{ik} \;;\; \mu_{u|v} = \Sigma_{uv[1,2]} \Sigma_{uv[2,2]}^{-1} v_{ik},\; \Sigma_{u|v} = \Sigma_{uv[1,1]} - \Sigma_{uv[1,2]} \Sigma_{uv[2,2]}^{-1} \Sigma_{uv[2,1]})$$

$$\propto \exp\left(-\frac{1}{2}(1-\rho_k^2)^{-1}(\mathbf{\Theta}_V u_{ik} - \vec{\mu}_u)^T (\mathbf{\Theta}_v u_{ik} - \vec{\mu}_u) \right) \cdot \exp\left(-\frac{1}{2}(u_{ik} - \mu_{u|v})^T \Sigma_{u|v} (u_{ik} - \mu_{u|v}) \right)$$

$$\text{Where } \mathbf{\Theta}_V = \begin{pmatrix} | & | \\ \vec{v}_{jk[-i,1]} & \vec{v}_{jk[-i,2]} \\ | & | \end{pmatrix} \text{,and } \vec{\mu}_u = \begin{pmatrix} | \\ \tilde{Z}_{i,\cdot,k} - \rho_k(\tilde{Z}_{\cdot,i,k} - u_\cdot^T v_i) \\ | \end{pmatrix}$$

$$\text{, and dimensions are } (n_k - 1) \times 2 \text{and } (n_k - 1) \times 1, \text{respectively.}$$

Then, we have $\mathbf{V}_n = \mathbf{A}_n^{-1}$ where $\mathbf{A}_n = \mathbf{A}_0 + \mathbf{A}_1,$ and $\mathbf{A}_0 = \Sigma_{u|v}^{-1}, \mathbf{A}_1 = (1-\rho_k^2)^{-1} \mathbf{\Theta}_V^T \mathbf{\Theta}_V$

$\mu_n = \mathbf{V}_n m_n$ where $m_n = m_0 + m_1$, and $m_0 = \Sigma_{u|v}^{-1}\mu_{u|v}$ and $m_1 = (1-\rho_k^2)^{-1}\mathbf{\Theta}_V^T \vec{\mu}_u$.

Update ρ_k: Let $\tilde{\epsilon}_{ijk} = Z_{ijk} - X_{ijk}\beta_k - a_{ik} - b_{jk} - u_{ik}^T v_{jk}$

$$p\left(\rho_k | \begin{pmatrix} \tilde{\epsilon}_{ijk} \\ \tilde{\epsilon}_{jik} \end{pmatrix}, \dots \right) \propto \prod_{i \neq j}^{n_k} \exp\left(-\frac{1}{2} \begin{pmatrix} \tilde{\epsilon}_{ijk} \\ \tilde{\epsilon}_{jik} \end{pmatrix}^T \Sigma_\rho^{-1} \begin{pmatrix} \tilde{\epsilon}_{ijk} \\ \tilde{\epsilon}_{jik} \end{pmatrix} \right) \cdot \mathrm{dUnif}(\rho_k; -1, 1)$$

To update ρ_k, Metropolis-Hastings is needed due to non-conjugacy.

Update μ:

$$\begin{aligned} p(\mu|\vec{Y}, \dots) &\propto \prod_{k=1}^{K} \mathrm{dNorm}(\beta_k; \mu, \sigma) \cdot \mathrm{dNorm}(\mu; \lambda, \tau) \\ &\sim \mathrm{N}(\mu; \frac{\lambda/\tau + \sum_k \beta_k/\sigma^2}{(1\tau + K/\sigma^2)}, (1/\tau + K/\sigma^2)^{-1}) \end{aligned}$$

Update σ^2:

$$\begin{aligned} p(\sigma^2|\vec{Y}, \dots) &\propto \prod_{k=1}^{K} \mathrm{dNorm}(\beta_k; \mu, \sigma) \cdot \mathrm{dIG}(\sigma^2; \nu_0, S_0) \\ &\sim \mathrm{IG}(\sigma^2; \frac{\nu_0 + K}{2}, \frac{\nu_0 \cdot S_0 + \sum_k (\beta_k - \mu)^2}{2}) \end{aligned}$$

Update Σ_{ab}:

$$\begin{aligned} p(\Sigma_{abk}|\vec{Y}, \dots) &\propto \prod_{i=1}^{n_k} \mathrm{dMVN}\left(\begin{pmatrix} a_{ik} \\ b_{ik} \end{pmatrix}; \vec{0}, \Sigma_{ab} \right) \cdot \mathrm{dIWishart}(\Sigma_{ab}; \nu_{ab}, S_{abk}^{-1}) \\ &\sim \mathrm{IWishart}(\Sigma_{abk}; \nu_{ab} + n_k, [S_{ab} + \sum_{i=1}^{n_k} \begin{pmatrix} a_{ik} \\ b_{ik} \end{pmatrix} \begin{pmatrix} a_{ik} \\ b_{ik} \end{pmatrix}^T]^{-1}) \end{aligned}$$

Update Σ_{uv}:

$$\begin{aligned} p(\Sigma_{uvk}|\vec{Y}, \dots) &\propto \prod_{i=1}^{n_k} \mathrm{dMVN}\left(\begin{pmatrix} u_{ik} \\ v_{ik} \end{pmatrix}; \vec{0}, \Sigma_{uv} \right) \cdot \mathrm{dIWishart}(\Sigma_{uvk}; \nu_{uv}, S_{uv}^{-1}) \\ &\sim \mathrm{IWishart}(\Sigma_{uvk}; \nu_{uv} + n_k, [S_{uv} + \sum_{i=1}^{n_k} \begin{pmatrix} u_{ik} \\ v_{ik} \end{pmatrix} \begin{pmatrix} u_{ik} \\ v_{ik} \end{pmatrix}^T]^{-1}) \end{aligned}$$

Coding (Temporary)

For the following function, two main arguments are required: data and edge_covariate. Both arguments should follow a list format. For now, the argument, edge_covariate, takes only one edge-specific covariate.

```r
hsvd = function(data, edge_covariate, dyad_dep = T, num_iter = 1000, verbose = T){

  suppressMessages(require(tmvtnorm))
  suppressMessages(require(mvtnorm))
  suppressMessages(require(truncnorm))
  suppressMessages(require(MCMCpack))

  iternum = num_iter
  Y = data
  COV = edge_covariate
  K = length(Y)
  nk = unlist(lapply(Y, nrow))

   # Create a list to collect MCMC samples
    Chain = list("beta" = list(),
                 "rho" = matrix(nrow=iternum, ncol=K),
                 "sigma_ab" = list(), "sigma_uv" = list(),
                 "U" = list(), "V" = list(),
                 "a" = list(), "b" = list(),
                 "z" = list(), "mu" = matrix(nrow=iternum, ncol=2))

    for (i in 1:K){
      Chain$U[[i]] = Chain$V[[i]] = list()
      for (j in 1:iternum){
        Chain$U[[i]][[j]] = Chain$V[[i]][[j]] = list()
      }
      Chain$beta[[i]] = matrix(nrow=iternum,ncol=2)
      Chain$a[[i]] = Chain$b[[i]] = matrix(nrow=iternum, ncol=nk[i])
    }

    # Initial points for parameters
    {
      beta0 = rep(0,K)
      beta1 = rep(0,K)
      beta = cbind(beta0,beta1)
      mu0 = mu = 0
      s02 = s2 = 1
      # sigma_uv
      sigma_uv =  matrix(0.5,4,4)
      diag(sigma_uv) = 1

      # Random Initial points for U and V
      U = V = list()
      for (i in 1:K){
        U[[i]] = as.matrix(rmvnorm(nk[i],c(0,0),sigma_uv[1:2,1:2]),ncol=2)
        V[[i]] = as.matrix(rmvnorm(nk[i],c(0,0),sigma_uv[3:4,3:4]),ncol=2)
      }


      # Create Starting points for z, a, b
      z = a = b = list()

      # sigma_ab
      sigma_ab = matrix(c(1,0.5,0.5,1),2,2)

      # fill in z,a,b with random numbers
      for (k in 1:K){
        z[[k]] = matrix(rnorm(nk[k]*nk[k]), ncol=nk[k],nrow=nk[k])
        diag(z[[k]]) = NA
        a[[k]] = b[[k]] = numeric(nk[k])
      }

      #initial for rho, beta0, beta1
      if (dyad_dep == T){
        rho = rep(0.5,K)
      } else rho = rep(0,K)
    }


  #MCMC

  ptm <- proc.time()
    for (k in 1:K){
      diag(Y[[k]]) = 0
      diag(Cov1[[k]]) = 0
      diag(z[[k]]) = 0
    }

    for (sim in (1):(iternum)){

      for (k in 1:K){
        index = which(diag(nk[k]) == 1)

        ### Update Z ###
        etas = matrix(cbind(1,c(Cov1[[k]])) %*% beta[k,] +
                        rowSums(U[[k]][rep(seq_len(nrow(U[[k]])), nk[k]),] *
                                  V[[k]][rep(seq_len(nrow(V[[k]])), each=nk[k]),]) +
                        rep(a[[k]], nk[k]) +
                        rep(b[[k]], each=nk[k]),
                      nk[k],nk[k])

        diag(etas) = 0

        m = c(etas) + rho[k]*( c(t(z[[k]])) - c(t(etas)))

        z[[k]] = matrix(rtruncnorm(1,
                                   a = ifelse(c(Y[[k]]), 0, -Inf),
                                   b = ifelse(c(Y[[k]]), Inf, 0),
                                   mean = m, sd = sqrt(1-rho[k]^2)),
                        nrow = nk[k], ncol = nk[k])

        ### Update beta0, beta1 ###
        sigma_beta = matrix(c(s02,0,0,s2),2,2)

        zbar = matrix(
          c(z[[k]])-
        rep(a[[k]], nk[k]) -
        rep(b[[k]], each=nk[k]) -
        rowSums(U[[k]][rep(seq_len(nrow(U[[k]])), nk[k]),] *
                  V[[k]][rep(seq_len(nrow(V[[k]])), each=nk[k]),]),
      nk[k],nk[k])

    diag(zbar) = 0
    rp = 1 / (1-rho[k]^2)
    X = (cbind(1,c(Cov1[[k]])) - rho[k]*(cbind(1, c(t(Cov1[[k]])))))[-index,]

    V_beta = solve( solve(sigma_beta) + rp*crossprod(X))

    mu_beta = V_beta %*% ( solve(sigma_beta) %*% c(mu0,mu)
                           + rp * crossprod(X, c(zbar)[-index] - rho[k] * c(t(zbar))[-index]))

    beta[k,] = rmvnorm(1, mu_beta, V_beta)

    ### Update a, b ###
    zhat = matrix( c(z[[k]]) -
                     c(cbind(1, c(Cov1[[k]]))%*%beta[k,]) -
                     rowSums(U[[k]][rep(seq_len(nrow(U[[k]])), nk[k]),] *
                               V[[k]][rep(seq_len(nrow(V[[k]])), each=nk[k]),]),
                   nk[k], nk[k])

    S11 = sigma_ab[1,1]; S22 = sigma_ab[2,2]; S12 = S21 = sigma_ab[1,2]

    sigma2_a = S11 - S12%*% solve(S22) %*% S21
    sigma2_b = S22 - S21%*% solve(S11) %*% S12

    for (i in sample(1:nk[k])){
      theta_ai = sum(zhat[i,-i] - rho[k]*(zhat[-i,i] - a[[k]][-i] - b[[k]][i]) - b[[k]][-i])

      m_ab = S12%*%solve(S22)%*%b[[k]][i]

      A0 = 1/sigma2_a; A1 = (nk[k]-1) * rp
      m0 = 1/sigma2_a*m_ab; m1 = rp * theta_ai

      mu_ai = 1/(A0+A1)*(m0+m1)
      s2_ai = 1/(A0+A1)
      a[[k]][i] = rnorm(1, mu_ai, sqrt(s2_ai))


      theta_bi = sum(zhat[-i,i] - rho[k]*(zhat[i,-i] - b[[k]][-i] - a[[k]][i]) - a[[k]][-i])
      m_ba = S21%*%solve(S11)%*%a[[k]][i]
      A0 = 1/sigma2_b; A1 = (nk[k]-1) * rp
      m0 = 1/sigma2_b*m_ba; m1 = rp * theta_bi

      mu_bi = 1/(A0+A1)*(m0+m1)
      s2_bi = 1/(A0+A1)
      b[[k]][i] = rnorm(1, mu_bi, sqrt(s2_bi))
    }
```

```
### Update U, V ###

ztilde = matrix( c(z[[k]]) -
                   cbind(1, c(Cov1[[k]]))%*%beta[k,] -
                   rep(a[[k]], nk[k]) - rep(b[[k]], each=nk[k]),
                 nk[k], nk[k])

diag(ztilde) = 0

S11 = sigma_uv[1:2,1:2]; S22 = sigma_uv[3:4,3:4]; S12 = sigma_uv[1:2,3:4]; S21 = sigma_uv[3:4,1

sigma2_u = S11 - S12%*% solve(S22) %*% S21
sigma2_v = S22 - S21%*% solve(S11) %*% S12

for (i in sample(1:nk[k])){
  Theta_V = V[[k]][-i,]
  mu_u = ztilde[i,-i] - c(rho[k]*(ztilde[-i,i] - U[[k]][-i,] %*% V[[k]][i,]))

  mu_uv = S12%*%solve(S22)%*%V[[k]][i,]

  A0 = solve(sigma2_u); A1 = rp*crossprod(Theta_V)

  m0 = solve(sigma2_u)%*%mu_uv; m1 = rp*crossprod(Theta_V,mu_u)

  mu_ui = solve(A0+A1)%*%(m0+m1)

  U[[k]][i,] = mvrnorm(1,mu_ui,solve(A0+A1))

  Theta_U = U[[k]][-i,]
  mu_v = ztilde[-i,i] - c(rho[k]*(ztilde[i,-i] - V[[k]][-i,] %*% U[[k]][i,]))

  mu_vu = S21%*%solve(S11)%*%U[[k]][i,]

  A0 = solve(sigma2_v); A1 = rp*crossprod(Theta_U)
  m0 = solve(sigma2_v)%*%mu_vu; m1 = rp*crossprod(Theta_U,mu_v)
  mu_vi = solve(A0+A1)%*%(m0+m1)

  V[[k]][i,] = mvrnorm(1,mu_vi,solve(A0+A1))
}



### Update rho ###
if (dyad_dep == T){
etilde = matrix(c(z[[k]]) -
                  cbind(1, c(Cov1[[k]]))%*%beta[k,] -
                  rep(a[[k]], nk[k]) - rep(b[[k]], each=nk[k]) -
                  rowSums(U[[k]][rep(seq_len(nrow(U[[k]])), nk[k]),] *
                          V[[k]][rep(seq_len(nrow(V[[k]])), each=nk[k]),]),
                nk[k],nk[k])
diag(etilde) = 0
EM<-cbind(etilde[upper.tri(etilde)],t(etilde)[upper.tri(etilde)] )
emcp<-sum(EM[,1]*EM[,2])
```

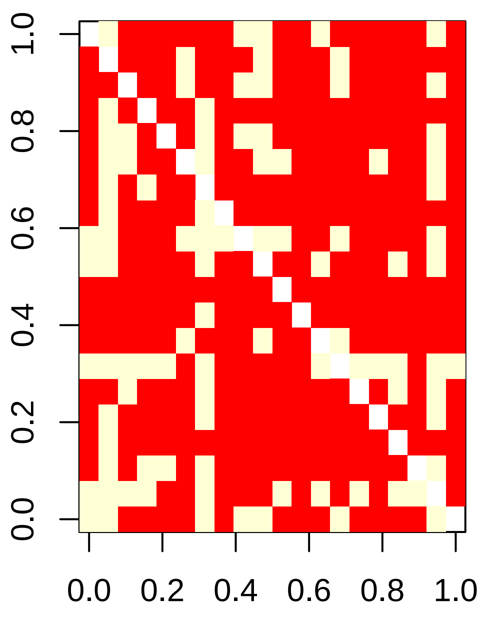
```
    emss<-sum(EM^2)
    m<- nrow(EM)

    sr = 2 * (1-cor(etilde)[1,2]^2)/sqrt(m)

    rho_p = rho[k] + sr * qnorm( runif(1,pnorm( (-1-rho[k])/sr), pnorm( (1-rho[k])/sr)))

    rd <- (-.5*(m*log(1-rho_p^2)+(emss-2*rho_p*emcp)/(1-rho_p^2)))-
      (-.5*(m*log(1-rho[k]^2 )+(emss-2*rho[k]*emcp )/(1-rho[k]^2 )))+
      ( (-.5*log(1-rho_p^2)) - (-.5*log(1-rho[k]^2)) )


    if (log(runif(1)) < rd) rho[k] = rho_p
    }

    Chain$rho[sim,k] = rho[k]
    Chain$beta[[k]][sim,] = beta[k,]
    Chain$U[[k]][[sim]] = U[[k]]
    Chain$V[[k]][[sim]] = V[[k]]
    Chain$a[[k]][sim,] = a[[k]]
    Chain$b[[k]][sim,] = b[[k]]
  }

  ### Update Sigma_ab ###
  Sab0 = matrix(c(1,0,0,1),ncol=2); nuab0 = 2+2
  S_theta_ab = matrix(rowSums(apply(do.call(rbind,Map(cbind,a,b)) ,1,tcrossprod)),2,2)
  sigma_ab = riwish(v = nuab0 + sum(nk) , S = Sab0 + S_theta_ab)

  ### Update Sigma_uv ###
  Suv0 = matrix(c(rep(5,16)),ncol=4); diag(Suv0) = 10; nuuv0 = 4+2
  S_theta_uv = matrix(rowSums(apply(do.call(rbind,Map(cbind,U,V)),1,tcrossprod)),4,4)
  sigma_uv = riwish(v =nuuv0 + sum(nk), S= Suv0 + S_theta_uv)

  ### Update mu0 (mean of beta0) ###
  s00 = 1; mu00 = 0
  s02_n= (1/s00 + (K / s02))^(-1)
  mu0_n= (mu00 / s00 + sum(beta[,1]) / s02) * s02_n
  mu0 = rnorm(1, mu0_n, sqrt(s02_n))

  ### Update mu (mean of beta1) ###
  s10 = 1; mu10 = 0
  s2_n= (1/s10 + (K / s2))^(-1)
  mu_n= (mu10 / s10 + sum(beta[,2]) / s2) * s2_n
  mu = rnorm(1, mu_n, sqrt(s2_n))

  ### Update S0 (var of beta0) ###
  c = 200; d = 1.5  # Hyperparameters

  nu0_n = c + K
  ss0_n = 1/nu0_n * (c * d + (K-1)*var(beta[,1]) + K/(K+1)*(mean(beta[,1])-mu0)^2)
  s02 = 1/ rgamma(1, nu0_n/2, nu0_n*ss0_n/2)

  ### Update S1 (var of beta1) ###
```

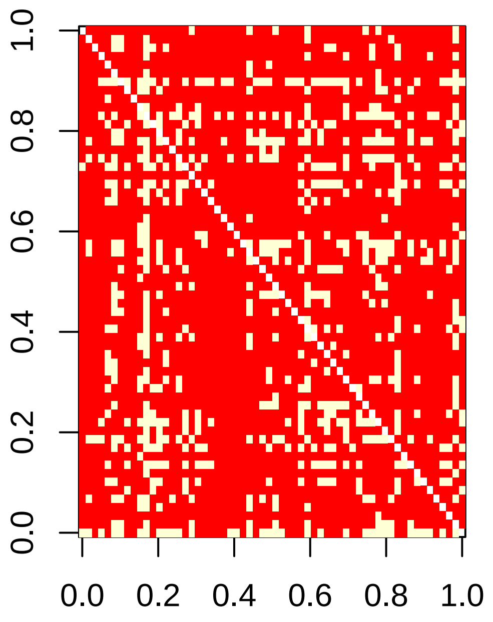
```
      nu_n = c + K
      ss_n = 1/nu_n * (c * d + (K-1)*var(beta[,2]) + K/(K+1)*(mean(beta[,2])-mu)^2)
      s2 = 1/ rgamma(1, nu_n/2, nu_n*ss_n/2)

      Chain$sigma_ab[[sim]] = sigma_ab
      Chain$sigma_uv[[sim]] = sigma_uv
      Chain$mu[sim,] = cbind(mu0,mu)

      if (sim %% 100 == 0 & verbose == T) {
        ptm2 = (proc.time() - ptm)[3]
        cat(sim,"/",iternum,"   (time:", ptm2, ")",
            "     (estimated time left:", ptm2*(iternum-sim)/100, ") \n")
        ptm = proc.time()
      }
    }
    return(Chain)
}
```

Testing (with simulated data)

```
# 2 networks with sizes of 20 and 40.
suppressMessages(library(mvtnorm))

K = 2
Cov1 = list()
nk = c(20, 60)

set.seed(1)

for (i in 1:K){
  Cov1[[i]] = matrix(rnorm(nk[i]^2),nk[i],nk[i])
  diag(Cov1[[i]]) = NA
}

rho_sim = runif(K,0,0.5)
s2_sim = 0.5^2; s02_sim = 0.5^2
mu0_sim = -2; mu_sim = 1

beta0_sim = rnorm(K,mu0_sim,sqrt(s02_sim))
beta1_sim = rnorm(K,mu_sim,sqrt(s2_sim))

sigma_ab_sim = matrix(c(1,0.3,0.3,1),2,2)
sigma_uv_sim = matrix(c(1,0.7,0.1,0.1,0.7,1,0.1,0.1,0.1,0.1,1,0.45,0.1,0.1,0.45,1),4,4)

a_sim = b_sim = z = list()
for (k in 1:K){
  z[[k]] = matrix(0, ncol=nk[k],nrow=nk[k])
  diag(z[[k]]) = NA
  a_sim[[k]] = b_sim[[k]] = numeric(nk[k])

  for (i in 1:nk[k]){
    ab0 = rmvnorm(1,c(0,0),sigma_ab_sim)
    a_sim[[k]][i] = ab0[1]
    b_sim[[k]][i] = ab0[2]
  }
}

U_sim = V_sim = list()
for (i in 1:K){
  U_sim[[i]] = as.matrix(rmvnorm(nk[i],c(0,0),sigma_uv_sim[1:2,1:2]),ncol=2)
  V_sim[[i]] = as.matrix(rmvnorm(nk[i],c(0,0),sigma_uv_sim[3:4,3:4]),ncol=2)
}

Z = Ysim= list()
for (k in 1:K){
  Z[[k]] = matrix(0, ncol=nk[k], nrow=nk[k])
  for (i in 1:nk[k]){
    for (j in 1:nk[k]){
      if (i == j) next
      Z[[k]][i,j] = beta0_sim[k] + beta1_sim[k] * Cov1[[k]][i,j] + t(U_sim[[k]][i,])%*%V_sim[[k]][j,] +
    }
  }

  w1 = sqrt((1 + sqrt(1-rho_sim^2))/2)
  w2 = sign(rho_sim)*sqrt(1-w1^2)
  EC = matrix(rnorm(length(Z[[k]])),nrow(Z[[k]]),nrow(Z[[k]]))
  EC = ( w1*EC + w2*t(EC) )
  ZS = Z[[k]]+EC
  YS <- 1 * (ZS > 0)
  diag(YS) <- NA
  Ysim[[k]] =YS
}
```

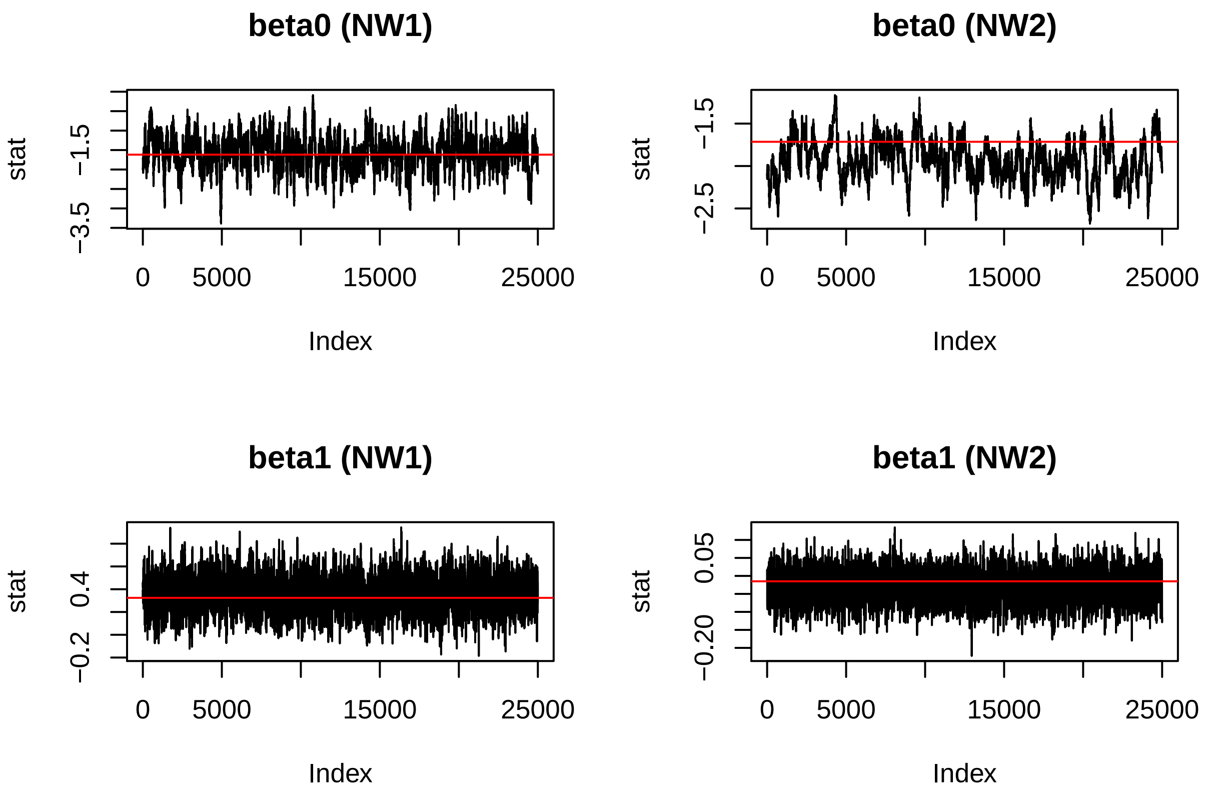
```
# Simulated Adjacency Matrices
# Diagonals are NAs
# red pixels are 0s, yellows are 1s
par(mfrow = c(1,2))
image(t(Ysim[[1]])[ncol(Ysim[[1]]):1,])
image(t(Ysim[[2]])[ncol(Ysim[[2]]):1,])
```



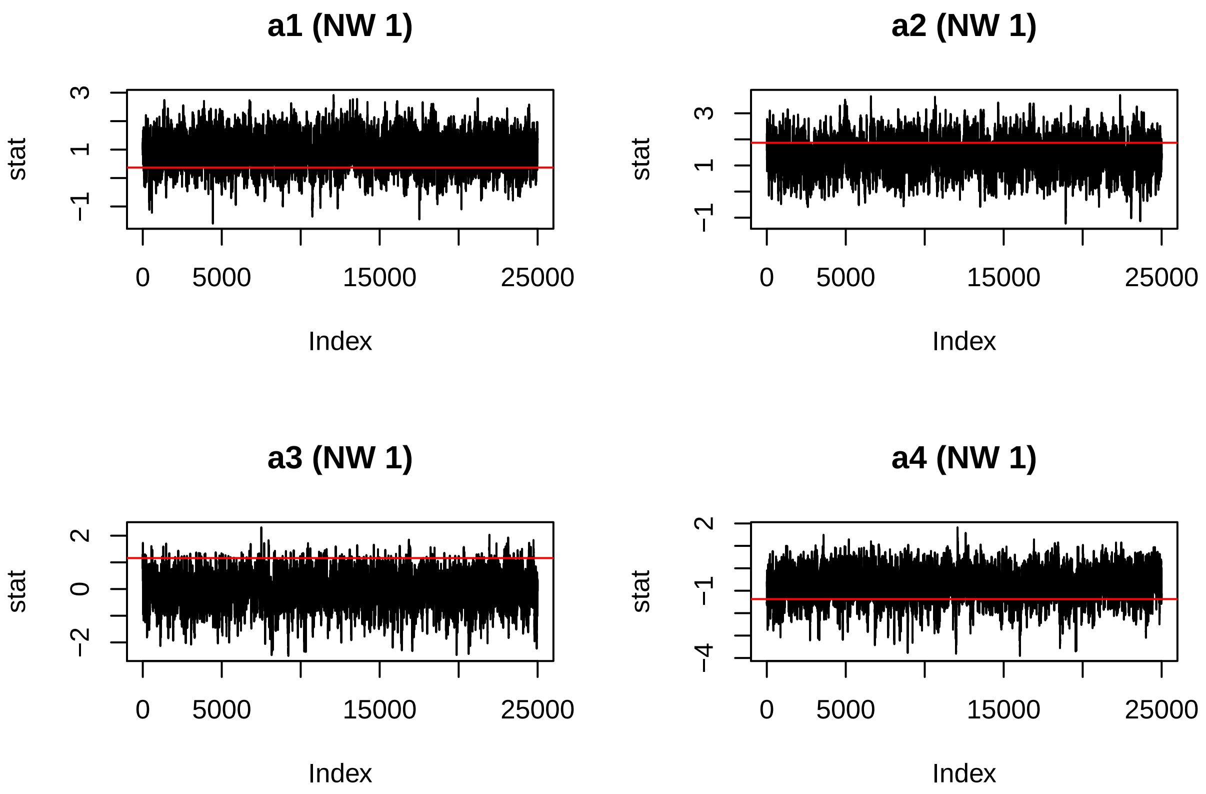


```
# results = hsvd(data = Ysim, edge_covariate = Cov1, dyad_dep = T, num_iter = 3000, verbose = F)
load("hsvd.rdata") # loading 30,000 posterior samples
```

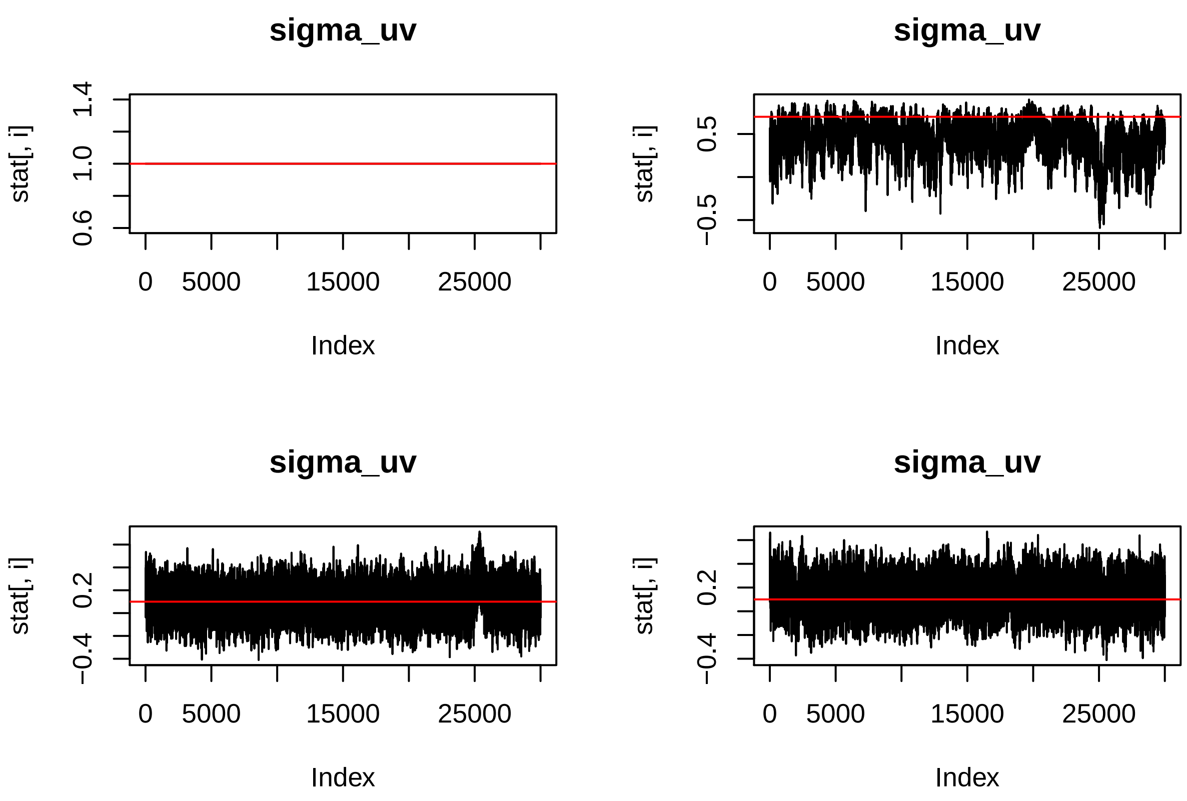
```
burnin = 5000
K=2
{par(mfrow=c(2,2))
  for (i in 1:K){
    stat = results$beta[[i]][-(1:burnin),1]
    lab = "beta0"

    if (NA %in% stat) stat = stat[-which(is.na(stat))]
    plot(stat, type="l", main=paste0(lab," (NW",i,")",sep = ""))
    abline(h = beta0_sim[i], col="red")
  }

  for (i in 1:K){
    stat = results$beta[[i]][-(1:burnin),2]
    lab = "beta1"

    if (NA %in% stat) stat = stat[-which(is.na(stat))]
    plot(stat, type="l", main=paste0(lab," (NW",i,")",sep = ""))
    abline(h = beta1_sim[i], col="red")
  }
  par(mfrow=c(1,1))
}
```



```
k = 1

{par(mfrow=c(2,2))
  for (i in 1:4){
    stat = results$a[[k]][-(1:burnin),i]
    lab = "a"

    if (NA %in% stat) stat = stat[-which(is.na(stat))]

    plot(stat, type="l", main=paste0(lab,i, " (NW ", k, ")", sep = ""))
    abline(h=a_sim[[k]][i], col="red")
  }
  par(mfrow=c(1,1))
}
```



```
{par(mfrow=c(2,2))
  stat = do.call(rbind,lapply(results$sigma_uv, function(x) c(cov2cor(x))))
  lab = "sigma_uv"

  for (i in 1:4){
    if (NA %in% stat) stat = stat[-which(is.na(stat)),]

    plot(stat[,i], type="l", main=paste0(lab,sep = ""))
    abline(h=sigma_uv_sim[i], col="red")
  }
  par(mfrow=c(1,1))
}
```



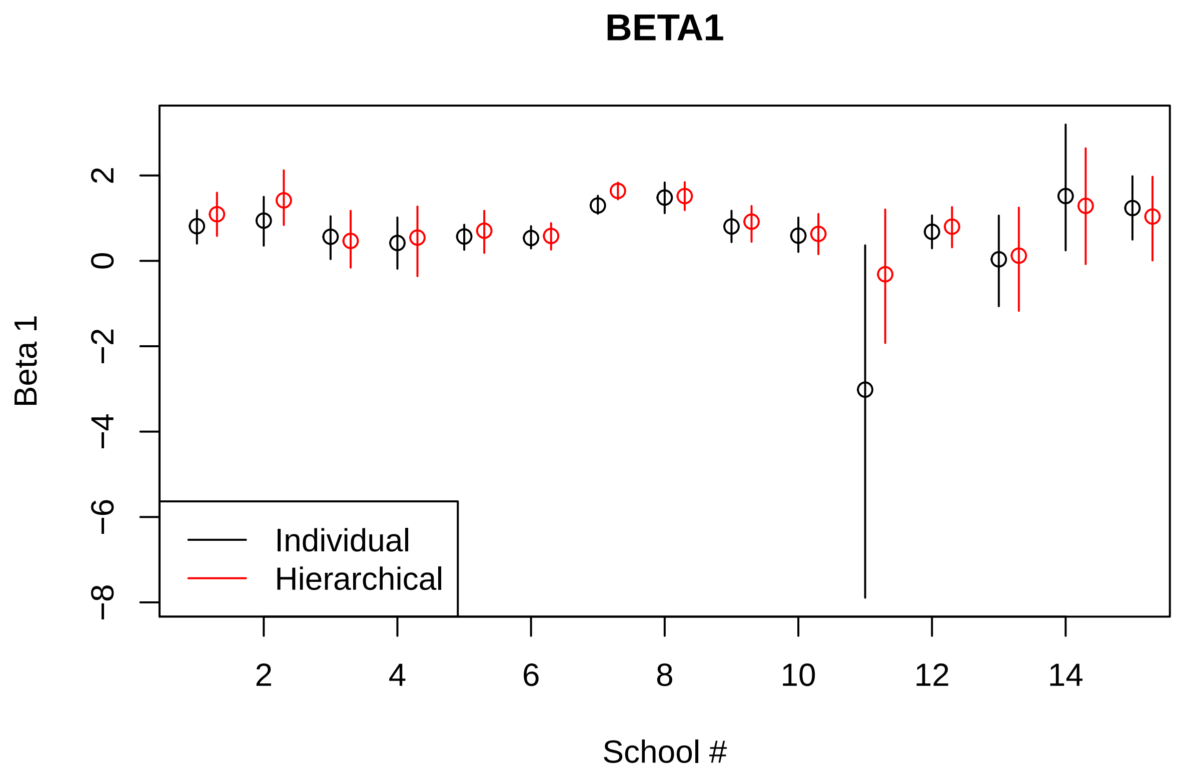
30,000 posterior samples were gathered and trace plots are provided above. As can be seen, the trace plots well captures the true values (red horizontal line). In order to check the shrinkage effects from hierarchical modeling, comparison with AMEN package with real data will follow.

```r
suppressMessages(library(HLSM))

# DATA from HLSM package
# Networks for advice-seeking activities among instructors in 15 different school.
# Edge-specific covariate is 1 if instructor i and j teach the same grade, 0 if otherwise.
K = 15
Cov1 = Y = list()
for (i in 1:K){
  Cov1[[i]] = ps.edge.vars.mat[[i]][,,3]
  diag(Cov1[[i]]) = NA
  Y[[i]] = ps.advice.mat[[i]]
}
nk = unlist(lapply(Y, nrow))
```

```r
Chain = hsvd(data = Y, edge_covariate = Cov1, num_iter = 2000, verbose = F)
```

```r
suppressMessages(library(amen))

AME_FIT = list()
for (k in 1:K){
y.array = array(Y[[k]], dim = c(nk[k],nk[k],1))
x.array = array(Cov1[[k]], dim = c(nk[k],nk[k],1,1))

AME_FIT[[k]] = amen::ame_rep(y.array, x.array,
                         dcor = T, intercept = T,
                         symmetric = F, model="bin",
                                nscan = 5000, plot = F, print = F)
}
```

```
#beta1
par(mfrow=c(1,2))
{
par(mfrow=c(1,1))
chain_b1a = chain_b1b = matrix(nrow=15,ncol=3)
for (k in 1:15){
  cc = Chain$beta[[k]][1000:2000,2]
  chain_b1b[k,1] = mean(cc)
  chain_b1b[k,2:3] = quantile(cc, c(0.025,0.975))

  aa = AME_FIT[[k]]$BETA[,2]
  chain_b1a[k,1] = mean(aa)
  chain_b1a[k,2:3] = quantile(aa, c(0.025,0.975))
}

lb = min(c(chain_b1a,chain_b1b))
ub = max(c(chain_b1a,chain_b1b))
plot(chain_b1a[,1],ylim=c(lb,ub), main="BETA1", ylab = "Beta 1", xlab= "School #")
legend("bottomleft", legend=c("Individual", "Hierarchical"), col=c("black", "red"), lty=c(1,1))
points(seq(1,15)+0.3,chain_b1b[,1],col="red")
segments(seq(1,15),chain_b1a[,2],seq(1,15),chain_b1a[,3])
segments(seq(1,15)+0.3,chain_b1b[,2],seq(1,15)+0.3,chain_b1b[,3], col="red")
}
```



In the data, school 11 has small sample size and its estimation of covariate effect has large variability. However, with shrinkage effect by hierarchical structure, the variabilty of the estimate decreased notably.