THEORY AND METHODS FOR THE ANALYSIS OF SOCIAL NETWORKS

LAB6

Woo Min Kim

Department of Statistical Science, Duke University

LATENT SPACE MODEL

Model representation:

$$logit\ P(Y_{ij} = 1) = \beta^T X_{ij} - d(Z_i, Z_j)$$

X is a covariate array.

β is a coefficient.

Z_i, Z_j are latent location variable for nodes *i* and *j*.

$d(Z_i, Z_j)$ is a distance function.

Captures homophily among nodes.

LATENT EIGENMODEL

Model representation:

$$logit\ P(Y_{ij} = 1) = \beta^T X_{ij} - d(Z_i, Z_j)$$

The model can be generalized by setting

$d(u_i, v_j) = \mathrm{u}_\mathrm{i}^\mathrm{T} \Lambda v_j$ where Λ is a diagonal matrix

This model is called latent eigenmodel.

It captures both homophily of nodes and stochastic equivalence among nodes in a symmetric relational data.

LATENT EIGENMODEL

An interpretation of the latent eigenmodel is that each node i has a vector of unobserved characteristics.

Let $\vec{u}_i = \{u_{i1}, u_{i2}, \cdots, u_{iK}\}$. The similar values of u_{ik} and u_{jk} will contribute positively or negatively to the relationship between i and j; it is dependent on $\lambda_k > 0$ or $\lambda_k < 0$.

The model can represent both positive or negative homophily in varying degrees, and stochastically equivalent nodes (nodes with the same or similar latent vectors) may or may not have strong relationships with one another. (Hoff, 2013)

LATENT EIGENMODEL

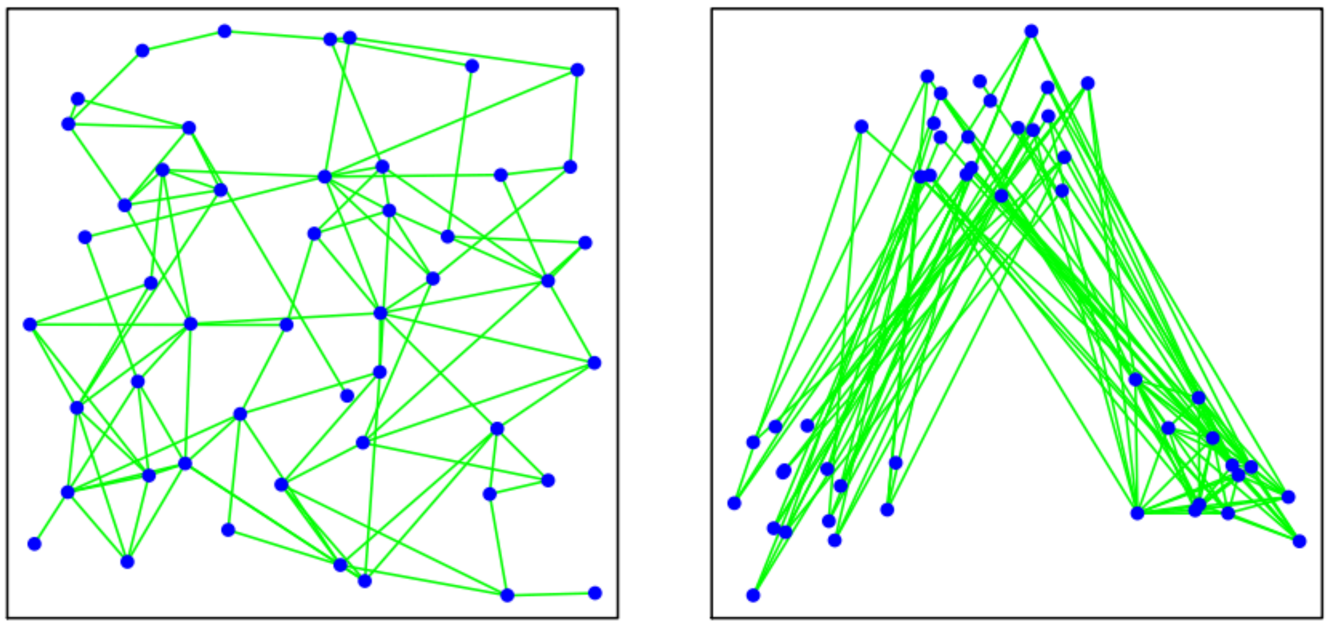

Figure 1: Networks exhibiting homophily (left panel) and stochastic equivalence (right panel).

(Hoff, 2013, p. 2)

LATENT SVD MODEL

Eigenmodel can also be generalized by setting $d(u_i, v_j) = \mathrm{u_i^T v_j}$ and this model is called latent SVD model.

We can also add more terms such as sender and receiver effects, a_i, $\mathrm{b_j}$, as in $\mathrm{P1}$ model.

PARAMETER ESTIMATION (BAYESIAN APPROACH)

Model specification by probit link

- $Y_{ij} = I_{\{[Z_{ij}>0]\}}$
- $Z_{ij} = \beta^T X_{ij} + a_i + b_j + u_i^T v_j + \epsilon_{ij}$
- $\beta \sim N(\vec{0}, \Sigma_\beta)$, where β is a vector of length p.
- X_{ij} is also a vector of length p. (dyadic covariate)
- $(a_i, b_i) \sim iid\ N(\ (0,0), \Sigma_{ab}\ )$
- $(\vec{u}_i, \vec{v}_i) \sim iid\ N((0,0,0,0), \Sigma_{uv}\ )$
- $\epsilon_{ij} \sim iid\ N(0,1)$
- No dyadic dependency $(\epsilon_{ij}, \epsilon_{ji})$ are independent.

UPDATE Z

- $Z_{ij} \mid \beta, a, b, u, v \sim N\left(\beta^T X_{ij} + a_i + b_j + u_i^T v_j, 1\right)$
- Want to update Z_{ij} given Y_{ij}
- $Y_{ij} = \begin{cases} 1, & if\ Z_{ij} > 0 \\ 0, & otherwise \end{cases}$
- $\Pr(Z_{ij} \mid Y_{ij} = 0, \beta, a, \cdots) \propto \Pr(Y_{ij} = 0 \big| Z_{ij}, \cdots \big) \Pr\big(Z_{ij} \big| \cdots)$

$$= 1_{[Z_{ij} \leq 0]} \cdot dN(Z_{ij}; \beta^T X_{ij} + a_i + b_j + u_i^T v_j, 1)$$

$$\sim \mathrm{TN}_{-}(\beta^T X_{ij} + a_i + b_j + u_i^T v_j, 1)$$

- $\Pr(Z_{ij} \mid Y_{ij} = 1, \beta, a, \cdots) \propto 1_{[Z_{ij} > 0]} \cdot dN(Z_{ij}; \beta^T X_{ij} + a_i + b_j + u_i^T v_j, 1)$

$$\sim \mathrm{TN}_{+}(\beta^T X_{ij} + a_i + b_j + u_i^T v_j, 1)$$

UPDATE β

Let $\hat{Z}_{ij} = Z_{ij} - a_i - b_j - u_i^T v_j$.

$$\Pr(\beta \mid \hat{Z})$$
$$\propto \prod_{i \neq j} \exp\left\{-\frac{1}{2}\left(\hat{Z}_{ij} - X_{ij}^T\beta\right)^T\left(\hat{Z}_{ij} - X_{ij}^T\beta\right)\right\} \exp\left\{-\frac{1}{2}\beta^T\Sigma_\beta^{-1}\beta\right\}$$
$$= \exp\left\{-\frac{1}{2}\left(\Theta_\beta\beta - vec\left(\hat{Z}\right)\right)^T\left(\Theta_\beta\,\beta - vec\left(\hat{Z}\right)\right)\right\} \exp\left\{-\frac{1}{2}\beta^T\Sigma_\beta^{-1}\beta\right\}$$

Where $\Theta_\beta = \begin{pmatrix} | & \cdots & | \\ vec(X_1) & \dots & vec(X_p) \\ | & \cdots & | \end{pmatrix}$

UPDATE β

$\Pr(\beta \,|\, \hat{Z}) \propto$

$$= \exp\left\{-\frac{1}{2}\left(\Theta\beta - vec(\hat{Z})\right)^T (\Theta\beta - vec(\hat{Z}))\right\} \exp\left\{-\frac{1}{2}\beta^T \Sigma_\beta^{-1}\beta\right\}$$

$$\propto \exp\left\{-\frac{1}{2}(\beta^T \Theta^T \Theta\beta - 2\beta^T \Theta^T vec(\hat{Z}))\right\} \exp\left\{-\frac{1}{2}\beta^T \Sigma_\beta^{-1}\beta\right\}$$

Where $vec(\hat{Z})$ is vectorization of $\hat{Z}$ except when $i = j$.

By Normal-Normal Conjugacy,

$\sim N(A_n^{-1} m_n, A_n^{-1})$, where $A_n = A_0 + A_1$, $m_n = m_0 + m_1$

$$A_0 = \Sigma_\beta^{-1}, A_1 = \Theta^T \Theta$$

$$\mathrm{m}_0 = 0, \mathrm{m}_1 = \Theta^T vec(\hat{Z})$$

UPDATE a_i, b_i

Let $\bar{Z}_{ij} = Z_{ij} - X_{ij}^T\beta - u_i^T v_j$.

$$\Pr(a_i \mid \hat{Z}, b_i) = \Pr(\bar{Z}|a_i, b_i)\Pr(a_i|b_i)$$

$$\propto \prod_{j\neq i} \exp\left\{-\frac{1}{2}(\bar{Z}_{ij} - a_i - b_j)^T(\bar{Z}_{ij} - a_i - b_j)\right\} \exp\left\{-\frac{1}{2}(a_i - \mu_{a|b})' \Sigma_{a|b}^{-1}(a_i - \mu_{a|b})\right\}$$

Where $\mu_{a|b} = \Sigma_{ab[1,2]}\Sigma_{ab[2,2]}^{-1}b_i$, $\Sigma_{a|b} = \Sigma_{ab[1,1]} - \Sigma_{ab[1,2]}\Sigma_{ab[2,2]}^{-1}\Sigma_{ab[2,1]}$

$$= \exp\left\{-\frac{1}{2}\sum_{j\neq i}(a_i - \theta_j)^2\right\} \exp\left\{-\frac{1}{2}(a_i - \mu_{a|b})' \Sigma_{a|b}^{-1}(a_i - \mu_{a|b})\right\}$$

Where $\theta_j = \bar{Z}_{ij} - b_j$, with fixed i.

UPDATE a_i, b_i

$$\Pr(a_i | \hat{Z}, b_i) \propto \exp\left\{-\frac{1}{2}\sum_{j\neq i}\left(a_i - \theta_j\right)^2\right\} \exp\left\{-\frac{1}{2}(a_i - \mu_{a|b})' \Sigma_{a|b}^{-1}(a_i - \mu_{a|b})\right\}$$

By Normal-Normal Conjugacy,

$\sim N(A_n^{-1} m_n, A_n^{-1})$, where $A_n = A_0 + A_1, m_n = m_0 + m_1$

$$A_0 = \Sigma_{a|b}^{-1},\ A_1 = n - 1$$

$$\mathrm{m}_0 = \Sigma_{a|b}^{-1}\mu_{a|b},\ \ \mathrm{m}_1 = \sum_{j\neq i} \theta_j$$

Likewise we can update b_i.

UPDATE $\vec{u}_{i,}\ \vec{v}_i$

Let $\tilde{Z}_{ij} = Z_{ij} - X_{ij}^T\beta - a_i - \mathrm{b_j}$.

$$\Pr(\vec{u}_i \big| \tilde{Z}, \vec{v}_i) = \Pr(\tilde{Z}|\vec{u}_i, \vec{v}_i)\Pr(\vec{u}_i|\vec{v}_i)$$

$$\propto \prod_{j\neq i} \exp\left\{-\frac{1}{2}\left(\tilde{Z}_{ij} - \vec{u}_i^T\vec{v}_i\right)^T\left(\tilde{Z}_{ij} - \vec{u}_i^T\vec{v}_i\right)\right\}\ \exp\left\{-\frac{1}{2}(\vec{u}_i - \mu_{u|v})'\,\Sigma_{u|v}^{-1}(\vec{u}_i - \mu_{u|v})\right\}$$

Where $\mu_{u|v} = \Sigma_{uv[1,2]}\Sigma_{uv[2,2]}^{-1}\vec{v}_i$, $\Sigma_{u|v} = \Sigma_{uv[1,1]} - \Sigma_{uv[1,2]}\Sigma_{uv[2,2]}^{-1}\Sigma_{uv[2,1]}$

$$= \exp\left\{-\frac{1}{2}(\Theta_V\vec{u}_i - \vec{\mu}_u)^T(\Theta_V\vec{u}_i - \vec{\mu}_u)\right\}\exp\left\{-\frac{1}{2}(\vec{u}_i - \mu_{u|v})'\,\Sigma_{u|v}^{-1}(\vec{u}_i - \mu_{u|v})\right\}$$

Where $\Theta_V = \begin{pmatrix} | & | \\ V_{[-i,1]} & V_{[-i,2]} \\ | & | \end{pmatrix}$ size of $(\mathrm{n}-1)\times 2$ since $j \neq i$

And $\vec{\mu}_u = \begin{pmatrix} | \\ \tilde{Z}_{[i,-i]} \\ | \end{pmatrix}$ length of $(\mathrm{n}-1)$

UPDATE $\vec{u}_{i,}\ \vec{v}_i$

$$\Pr(\vec{u}_i \big| \hat{Z}, \vec{v}_i)$$

$$\propto \exp\left\{-\frac{1}{2}(\Theta_V\vec{u}_i\ -\vec{\mu}_u)^T(\Theta_V\vec{u}_i\ -\vec{\mu}_u)\right\} \exp\left\{-\frac{1}{2}(\vec{u}_i - \mu_{u|v})'\,\Sigma_{u|v}^{-1}(\vec{u}_i - \mu_{u|v})\right\}$$

By Normal-Normal Conjugacy,

$\sim N(A_n^{-1}m_n, A_n^{-1})$, where $A_n = A_0 + A_1, m_n = m_0 + m_1$

$$A_0 = \Sigma_{u|v}^{-1},\ A_1 = \Theta_V^T\Theta_V$$

$$\mathrm{m}_0 = \Sigma_{u|v}^{-1}\mu_{u|v}\,, \quad \mathrm{m}_1 = \Theta_V^T\vec{\mu}_u$$

Likewise we can update $\vec{\mathrm{v}}_i$.

WITH DYADIC DEPENDENCY

- More complicated but the method is identical with previous example when updating $\beta, a, b, u,$ and v.
- However, when $(\epsilon_{ij}, \epsilon_{ji}) \sim MVN_2(\vec{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$, updating ρ is problematic. Much more work is needed.
- Hierarchical Modeling is also feasible when multiple networks exist.