Linear Algebra.

§1-3 Exercises.

1. (a). F, it should be checked that the $\overset{F?}{\text{fields}}$ and the $\overset{(+,\times)?}{\text{operations}}$ used for V and W are the same.

(b). F, $\phi := \{\}$ is not a subspace of any vector space ∵ $\vec{0} \notin \phi$.

(c). T, Choose $W = \{0\}$, the zero subspace.

(d). F, e.g. $V := \mathbb{R}^2$ is a $\underset{\text{with } (\mathbb{R},+,\times)}{\text{vector space}}$. But $\{?\} \cap \{2,3\} = \{2\}$ is not a su

(e). T, trivial.

(f). F. by def $tr(A)_{n\times n} := \sum_{i=1}^{n} a_{ii}$.

(g). F, $W = \{(a_1,a_2,0) \mid a_1,a_2 \in \mathbb{R}\}$. but $W \neq \mathbb{R}^2$.

2. Trivial.

3. pf. $M := aA+bB$. $N := aA^t+bB^t$. Then $M_{ij} = aA_{ij}+bB_{ij} = a(A^t)_{ji} + b(B^t)$
$= N_{ji}$. $\Rightarrow$ $M^t = N$ ▨

4. pf. For $A \in M_{n\times n}(F)$, $((A_{ij})^t)^t = (A_{ji})^t = A_{ij}\ \forall (i,j)$, $\Rightarrow$ $(A^t)^t = A$ ▨

5. pf. 令 $M := A+A^t$. $M_{ij} = A_{ij}+(A_{ij})^t = A_{ij}+A_{ji}$; $M_{ji} = A_{ji}+(A_{ji})^t = A_{ji}+$
Hence, $M_{ij} = M_{ji}\ \forall i,j$. $\Rightarrow$ M is symmetric ▨

6. pf. For $A,B \in M_{n\times n}(F)$, 令 $M := aA+bB$. Then $tr(M) = \sum_{i=1}^{n}(aA_{ii}+bB_{ii})$
$= \sum_{i=1}^{n} aA_{ii} + \sum_{i=1}^{n} bB_{ii} = a\sum A_{ii} + b\sum B_{ii} = a(tr(A)) + b(tr(B))$. ▨

7. pf. If D is a diagonal matrix, then $D_{ij} = D_{ji} = 0\ \forall i \neq j$ ▨

8. (a). $W_1 = \{(a_1, \frac{1}{3}a_1, -\frac{1}{3}a_1) \mid \forall a_1 \in \mathbb{R}\}$. ∴ ① $\vec{0} \in W_1$. ② $(a_1, \frac{1}{3}a_1, -\frac{1}{3}a_1) \overset{\in W_1}{+} (a_2, \frac{1}{3}a_2, \frac{-1}{3}a$
$= (\boxed{a_1+a_2}, \frac{1}{3}\boxed{a_1+a_2}, -\frac{1}{3}\boxed{a_1+a_2}) \Rightarrow \in W_1$ ③ For any $c \in F$, $c\underbrace{(a_1, \frac{1}{3}a_1, -\frac{1}{3}a_1)}_{\in W_1}$.
$= (\boxed{ca_1}, \frac{1}{3}\boxed{ca_1}, -\frac{1}{3}\boxed{ca_1}) \Rightarrow \in W_1$. 故 W_1 is a subspace of $\mathbb{R}^3$ ▨

(b) $W_2 = \{(a_3+2, a_2, a_3) \mid a_2, a_3 \in \mathbb{R}\}$. is not a subspace of $\mathbb{R}^3$ since $\vec{0} \notin W_2$.

(c). $W_3 := \{(a_1, a_2, a_3) \mid 2a_1 - 7a_2 + a_3 = 0\}$. ① $0 \in W_3$ ② 加法封閉? $\vec{a}, \vec{b} \in W_3$. Then
$2(a_1+b_1) - 7(a_2+b_2) + (a_3+b_3) = \ldots = 0$ ③ 乘法封閉? $\vec{a} \in W_3$. Then $2(ca_1) - 7(ca_2) + ($
$= \ldots = 0$. 故 W_3 is a subspace of $\mathbb{R}^3$ ▨

(d). $W_4 := \{(a_1, a_2, a_3) \mid a_1 - 4a_2 - a_3 = 0\}$. is also a subspace of $\mathbb{R}^3$ (similar to (c)). ▨

(e). $W_5 := \{(a_1, a_2, a_3) \mid a_1 + 2a_2 - 3a_3 = 1\}$. is not a subspace of $\mathbb{R}^3$ since $\vec{0} \notin W_5$.

(f). $W_6 := \{(a_1, a_2, a_3) \mid 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$. ∴ ① $\vec{0} \in W_6$. ② $5(a_1+b_1)^2 - 3(a_2+b_2)^2 + 6(a_3+b_3)^2$

$= \cancel{5a_1^2} + \cancel{5b_1^2} + 10a_1b_1 - \cancel{3a_2^2} - \cancel{3b_2^2} - 6a_2b_2 + \cancel{6a_3^2} + \cancel{6b_3^2} + 12a_3b_3.$

$= 10a_1b_1 - 6a_2b_2 + 12a_3b_3$. 不一定 $=0$. 故 W_6 not a subspace of $\mathbb{R}^3$ ▨

9. $W_1 \cap W_3 = \{(a_1,a_2,a_3) \mid a_1 = 3a_2,\ a_3 = -a_2,\ 2a_1 - 7a_2 + a_3 = 0\} = \{(a_1,a_2,a_3) \mid a_2 = 0,\ a_1 = 3a_2,\ a_3$

$= \{0\}$, a subspace of $\mathbb{R}^3$ ☆

$W_1 \cap W_4 = W_1$, a subspace of $\mathbb{R}^3$ ☆ (by Ex.8).

$W_3 \cap W_4 = \{(a_1,a_2,a_3) \mid 2a_1 - 7a_2 + a_3 = 0 \ \& \ a_1 - 4a_2 - a_3 = 0\} = \{(a_1,a_2,a_3) \mid \left[\begin{matrix} a_1 - 3a_2 + 2a_3 = 0 \\ -a_2 - 3a_3 = 0 \end{matrix}\right.\}$

$= \{(a_1,a_2,a_3) \mid a_2 = -3a_3,\ a_1 = -11a_3\} = \{(-11, -3, -1)a_3 \mid a_3 \in \mathbb{R}\}$.

, clearly, a subspace of $\mathbb{R}^3$ ▨

10. pf. • ① $\vec{0} \in W_1$. ② trivial 的加法封閉 ③ trivial 乘法封閉。 $\Rightarrow W$ subspace of F^n

• $\vec{0} \notin W_2$.

11. For $n > 1$, ① $\vec{0} \in W$, ② 兩個 deg $= n$ 的相加可能導致新的 deg $< n$. , fail!..

For $n = 1$, ① $\vec{0} \in W$., ② $k_1 + k_2 \in W\ \forall k_1, k_2 \in F$. ③ If $k \in W$, $ck \in W\ \forall c \in F$

$\therefore$ $n=1$ case, W a subspace of $P(F)$., but in general case, no! ☆

12. pf. Define the set of all upper triangular matrices to be S.

① $O \in S$. ② $A + B \in S$. ③ $\forall c \in F$, $A \in S$, we have $cA \in S$ ▨

13. ① zero function $\in S$. ② $(f+g)(s_0) = 0$ ③ $(cf)(s_0) = 0$ ▨

14. ① zero function $\in S$. ② finite nonzero points + finite nonzero pts = finite.

③ 原 finite nonzero pts. $\times c$ 後, still the same # of nonzero pts..

15. Ans. Yes. Define the set of all diff. real-valued functions defined on $\mathbb{R}$ is $C^2(\mathbb{R})$.

Then. ① zero function $\in C^2(\mathbb{R})$ ② If $f \,\&\, g \in C^2(\mathbb{R})$, so is $f+g$. ③ $af \in C^2(\mathbb{R})$, $a \in \mathbb{R}$. ▨

16. ① zero function $\in C^n(\mathbb{R})$. ② $f \,\&\, g \in C^n(\mathbb{R}) \Rightarrow f+g \in C^n(\mathbb{R})$. ③ $af \in C^n(\mathbb{R})$., $a \in \mathbb{R}$. ▨

17. Claim: A subset W of vector space V is a subspace of $V \Leftrightarrow \left\{ \begin{matrix} W \neq \emptyset \text{ and } \forall a \in F, x \\ \text{we have } ax \in W,\ x+y \end{matrix} \right.$

pf. ($\Rightarrow$). W, $\overset{\nearrow \text{must contain } \vec{0}}{\text{subspace}}$. but $\emptyset =: \{\}$ is not a subspace of $V \Rightarrow W \neq \emptyset$ ☆

($\Leftarrow$) Now suppose ③ $ax \in W$, ② $x+y \in W$ whenever $a \in F$, $x, y \in W$.; and $W \neq \emptyset$

Then. W 至少有一個元素, say z. 則 ① $0z = 0 \in W$ (by ③). ▨

18. Claim: A subset W of V is a subspace of $V \Leftrightarrow$ ① $0 \in W$ & $ax + y \in W$ whenever $a \in F$, x, y

pf. ($\Rightarrow$). W, subspace of $V \Rightarrow \vec{0} \in W$. If $a \in F$, $x, y \in W$, then. $ax \in W$, and then $(ax) + y$

($\Leftarrow$). If $x, y \in W$, $a \in F$, then $ax + y \in W$. Take $a = 1$ ($a \in F \Rightarrow \frac{1}{a} \in F \Rightarrow 1 \in F$). $\Rightarrow$ ② $x + y \in W$

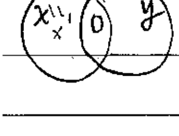
Now, $ax + y \in W$, $y \in W \Rightarrow$ ③ $ax \in W$ using ② ▨

No. Ch1.
Date : :

Linear Algebra.

19. Claim: Let W_1, W_2 be subspaces of V Then $W_1 \cup W_2$ is a subspaces $\Leftrightarrow$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

pf. ($\Rightarrow$) Since W_1, W_2 are subspaces, $\vec{0} \in W_1 \cap W_2 (\neq \emptyset)$. If $\exists x \in W_1 \cap W_2$, then $x \in \left\{\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right.$



and $ax \in \left\{\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right. \Rightarrow ax \in W_1 \cap W_2$. If $\exists x, y \in W_1 \cap W_2$, then $x, y \in \left\{\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right. \Rightarrow x+y \in W_1 \cap W_2$.

So, $W_1 \cap W_2$ is also a subspace of V.

Now, suppose $x \in W_1$ but $x \notin W_2$. If $\exists y \in W_2$ but $y \notin W_1$; then since $x, y \in W_1 \cup$

we have $x+y \in W_1 \cup W_2$. Note that $x+y$ ~~ϕ~~ $\notin W_1 \cap W_2$, (otherwise, $x+y \in W_1$ & $\in$

$\Rightarrow y \in W_1$ and $x \in W_2$ ※) $\therefore x+y \in W_1$ or W_2.

If $x+y \in W_1$, then $y \in W_1$ since $x \in W_1$. ※ If $x+y \in W_2$, $x \in W_2$. ※

故原假設不成立！$\Rightarrow$ If $y \in W_2$, then y mut $\in W_1$ ※

($\Leftarrow$) $W_1 \cup W_2 = \begin{cases} W_1, & \text{if } W_2 \subseteq W_1 \\ W_2; & \text{if } W_1 \subseteq W_2 \end{cases}$, clearly a subspace of V. ▨ W_2

20 pf. Given $w_1, w_2, \cdots, w_n \in W$, $a_1, \dots, a_n \in F$, $\Rightarrow a_i w_i \in W \ \forall i = 1, \dots, n. \Rightarrow \sum_{i=1}^{n} a_i w_i \in W$ ▨

21. Define $S :=$ the set of conv. seq.(s) $\{a_n\}$. (i.e. $\lim_{n\to\infty} a_n$ exists).

V is a vector space consisting of all the real-valued sequences $\{a_n\}$.

① $\{a_n\} = \{0\}$ is a conv. seq. ② If two conv. seq. $\{a_n\}, \{b_n\}$, then $\{a_n\} + \{b_n\} =: \{a_n +$

also converges ($\lim_{n\to\infty}(a_n+b_n)$ exists) ③ $c\{a_n\} := \{ca_n\}$ also conv. ($\lim_{n\to\infty} ca_n$ exists!). ▨

22. even (pf): ① zero function. ② If g_1 & g_2 even, so is (g_1+g_2). ③ ag also even. ※

similar case for odd. ▨

23. (a). $W_1 + W_2 := \{x+y \mid x \in W_1, y \in W_2\}$. Since both W_1 & W_2 are subspace of V, $0 \in W_1, W_2$.

Take $y=0$, then. $W_1 + W_2 \supseteq W_1$; take $x=0$, $W_1 + W_2 \supseteq W_2$. ① $0 \in W_1 + W_2$. ※

② If $z, w \in W_1 + W_2$, $\exists x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$ s.t. $x_1 + x_2 = z$, $y_1 + y_2 = w$. $\Rightarrow z + w = \underset{\in W_1 + W_2}{\overset{\in W_1}{(x_1+y_1)} + \overset{\in W}{(x_2+y}}$ ※

③ Given $a \in F$, $az = a(x_1+x_2) = \overset{\in W_1}{ax_1} + \overset{\in W_2}{ax_2} \in W_1 + W_2$. ▨

(b). 令S為任意V的subspace，且contains W_1, W_2,

If $z \in W_1 + W_2$, $\exists x_1 \in W_1$, $x_2 \in W_2$ s.t. $z = x_1 + x_2$, But $x_1 \in W_1 \subset S$, $x_2 \in W_2 \subset S$ 且 S 有 subspace 性質，故 $z \in S$ ▨

24. $W_1 := \{(a_1, \cdots, a_{n-1}, 0)\}$; $W_2 := \{(0, 0, \cdots, 0, a_n)\}$ are subspaces of F^n.

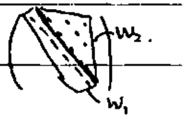
① $W_1 \cap W_2 = \{\vec{0}\}$. ② $W_1 + W_2 =: \{x+y \mid x \in W_1, y \in W_2\} = F^n$. $\therefore F^n = W_1 \oplus W_2$. ▨

25. pf. $W_1 = \{f \mid f = a_{2n+1}x^{2n+1} + \cdots + a_1 x\}$, $W_2 := \{f \mid f = b_{2m}x^{2m} + \cdots + a_2 x^2 + a_0\}$ are subspaces $P($

① $W_1 \cap W_2 = \{0\}$ ② $W_1 + W_2 = P(F)$ ▨

26. ① W_1 & W_2 are subspaces of $M_{m\times n}(F)$. ② $W_1 \cap W_2 = 0$ ③ $W_1 + W_2 = M_{m\times n}(F)$ ▨

27. $V := \{M \in M_{n\times n}(F) \mid M_{ij} = 0 \text{ if } i > j\}$. ① W_1 & W_2 are subspaces of $M_{n\times n}(F)$.



② $W_1 \cap W_2 = 0$ ③ $W_1 + W_2 = V$ ▨ (Binary field).

28 F no characteristic 2 ⇒ F 不是 $1+1=0$ 的 field. — (*)

- $W_1 := \{M \mid M^t = -M, M \in M_{n\times n}(F)\}$. ① $0 \in W_1$. ② If $M_1, M_2 \in W_1$, then $(M_1+M_2)^t = M_1^t + M_2^t = -(M_1+M_2)$

  ③ $\forall a \in F$, $(aM)^t = aM^t = a(-M) = -aM$. Hence, W_1 is a subspace of $M_{n\times n}(F)$ ※

- ~~現在沒了(*), 在証 $M_{n\times n}(F) = W_1 \oplus W_2$ 之前, 前述③要重驗: $-(aM) = -(a(M))$~~

  ~~$= -(a(-M^t)) = (-aM^t) = aM^t = (aM)^t$ 故③仍成立。~~

- $W_2 := \{M \in M_{n\times n}(F) \mid F \text{ no}(*) , M^t = M\}$, still a subspace of $M_{n\times n}(F)$.

  ① $W_1 \cap W_2 = \{A \in M_{n\times n}(F) \mid A^t = -A \text{ \& } A^t = A\} = \{A \mid -A = A\} = \{0\}$.

  ② Given $M \in M_{n\times n}(F)$, $M = \overset{=:A}{\boxed{\frac{M+M^t}{2}}} + \overset{=:B}{\boxed{\frac{M-M^t}{2}}}$, A satisfies $A^t = A$. & B satisfies $B^t = -B$.

  $\therefore \forall M \in M_{n\times n}(F)$, $\exists A \in W_2$, $B \in W_1$ s.t. $M = A + B \Rightarrow M_{n\times n}(F) \subseteq W_1 + W_2$

  Clearly, $M_{n\times n}(F)$ contains all n×n matrix $\Rightarrow M_{n\times n}(F) \supseteq W_1 + W_2$ ▨

29. ① they're all subspaces of $M_{n\times n}(F)$. ② $W_1 \cap W_2 = 0$ ③ Given $M \in M_{n\times n}(F)$,

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} = \begin{pmatrix} M_{11} & & \boxed{M_{1n}} \\ & \ddots & \vdots \\ \boxed{M_{1n}} & & M_{nn} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ \underset{M_{11}-M_{1n}}{\blacksquare} & & 0 \end{pmatrix} = \overset{\in W_2}{\text{symmetric matrix}} + \begin{pmatrix} 0 & 0 \\ \blacktriangle & 0 \end{pmatrix} \overset{\in W_1}{\text{matrix}}.$$

$\therefore \forall M \in M_{n\times n}(F)$, $\exists \bar{M} \in W_2$, $B \in W_1$ s.t. $M = \bar{M} + B$. ▨

30. $V = W_1 \oplus W_2 \Leftrightarrow \forall v \in V, \exists! (\overset{\in W_1}{x_1}, \overset{\in W_2}{x_2})$ s.t. $v = x_1 + x_2$. 已知 W_1, W_2 are subspaces of V.

pf (⇒). $V = W_1 \oplus W_2 \Rightarrow \begin{cases} W_1 + W_2 = V \\ W_1 \cap W_2 = \{0\} \end{cases}$ $0 \in V$, $0 = \overset{\in W_1}{0} + \overset{\in W_2}{0}$ (存在性)

If ∃ 另一对 $(\overset{\neq 0, \in W_1}{x_1}, \overset{\neq 0, \in W_2}{x_2})$ s.t. $0 = \overset{\neq 0}{x_1} + \overset{\neq 0}{x_2}$, then $\overset{\in W_1}{x_1} = -\overset{\in W_2}{x_2}$, but this relation implies

that $x_2 \in W_1$ (additive inverse) or $x_1 \in W_2$. $\rightarrow\!\!\!\!\times$ ($W_1 \cap W_2 = \{0\}$).

Given $\overset{\neq 0}{v} \in V$, $\exists x_1 \in W_1, x_2 \in W_2$ s.t. $v = x_1 + x_2$. (by $W_1 + W_2 = V$). If $\exists x_1' \in W_1, x_2' \in W_2$

s.t. $v = x_1' + x_2'$, then $x_1' + x_2' = x_1 + x_2 \Rightarrow \overset{\in W_1}{(x_1' - x_1)} = \overset{\in W_2}{\boxed{x_2' - x_2}} \xrightarrow[(W_1 \cap W_2 = \{0\})]{} x_1 = x_1'$ & $x_2 = x_2'$ ※

(⇐). Given $v \in V$, $\exists! \overset{\in W_1}{x_1}, \overset{\in W_2}{x_2}$ s.t. $v = x_1 + x_2 \Rightarrow V \subseteq W_1 + W_2$. Take $v = 0 = x_1 + x_2$,

已知 $0 \in W_1, W_2$ 故 $0 = 0 + 0$ hold., by 唯一性, $\nexists\ x_1 \neq v, x_2 \neq v$ s.t. $0 = x_1 + x_2$, or $x_1 = -$

If $\overset{\neq 0}{y} \in W_1 \cap W_2$, then $y \in W_1, y \in W_2 \Rightarrow -y \in W_1, y \in W_2 \Rightarrow 0 = \overset{\neq 0}{-y} + \overset{\neq 0}{y}$ $\rightarrow\!\!\!\!\times$. $\therefore W_1 \cap W_2 = \{0$

Since W_1, W_2 are subspaces of V, $W_1 + W_2 \subseteq V$. ▨

31.(a). " $v+W$ is a subspace of $V \Leftrightarrow v\in W$ ". 已知 W, subspace of V.

pf. ($\Rightarrow$). $v+W := \{v+w \mid w\in W\}$. $0\in v+W \Rightarrow \exists_{w=\overset{\in W}{-v}}$ s.t. $0=v+w$. $\Rightarrow v\in W$ #

($\Leftarrow$). $v\in W \Rightarrow -v\in W \Rightarrow$ ① $0\in v+W$ ② $\forall x,y\in v+W$, say $x=v+w_x$, $y=v+w_y$, th

$x+y = 2v+w_x+w_y = v+\boxed{w_x+w_y+(v)}^{\in W} = v+\bar{w}$ for some $\bar{w}\in W$.

③ $\forall x\in v+W$, $a\in F$, $ax = av+aw_x = v+\boxed{aw_x+(a-1)v}^{\in W}$ ▨

(b). " $v_1+W = v_2+W \Leftrightarrow v_1-v_2\in W$.

pf. ($\Rightarrow$). Given $x\in W$, $\exists y\in W$ s.t. $v_1+x=v_2+y$. $\Rightarrow v_1-v_2=\boxed{y-x}\in W$. (subspace性質

($\Leftarrow$). $v_1-v_2\in W$. Given $x\in W$, $v_1+x = v_2+(\overset{\in W}{v_1-v_2}+\overset{\in W}{x}) = v_2+y$ for some $y\in W$

$\Rightarrow v_1+W\subseteq v_2+W$ Similarly, $v_1+W \supseteq v_2+W$ ▨

(c). • 令 $v_1+W=v_1'+W$, $v_2+W=v_2'+W$. $\Rightarrow v_1-v_1'\in W$ & $v_2-v_2'\in W$.

Then $(v_1+W)+(v_2+W) \overset{\text{def}}{=} (v_1+v_2)+W \overset{(b)}{=} (v_2+v_1')+W \overset{(b)}{=} (v_1'+v_2')+W$

& $(v_1'+W)+(v_2'+W) \overset{\text{def}}{=} (v_1'+v_2')+W$.

• $a(v_1+W) \overset{\text{def}}{=} av_1+W \overset{(b)}{=} av_1'+W$. since $av_1-av_1' = a(v_1-v_1')\in W$.

& $a(v_1'+W) \overset{\text{def}}{=} av_1'+W$ ▨

~~subspace of V.~~

(d). $S := \{v+W \mid v\in V\} := V/W$ is a vector space. 自己check.

§1.4.

1. (a). T. (b). F, $\text{span}(\emptyset) = \{0\}$. (c). T. (Thm 1.5.) (d) F, 不可乘零.

(e) T. (f). F. 反例 p.28. Example 2.

2. (a). $(x_1,x_2,x_3,x_4) = \{(1,1,0,0)r+(1,-1,2,-1)s+(0,1,0,2),\ r,s\in\mathbb{R}\}$

(b). $(x_1,x_2,x_3) = (-2,-4,-3)$.

(c) no solution.

(d) ~ (f) 略.

3. (a). $\begin{cases} x+2y=-2 \\ 3x+4y=0 \\ -y=-1 \end{cases} \Rightarrow$ no solution. $\Rightarrow$ $(-2,0,3)$ 不是 v_2,v_3 之線性組合. #

(b). $(1,2,-3) = 5\cdot(-3,2,1) + 8\times(2,-1,-4)$ 是! #

(c). not!

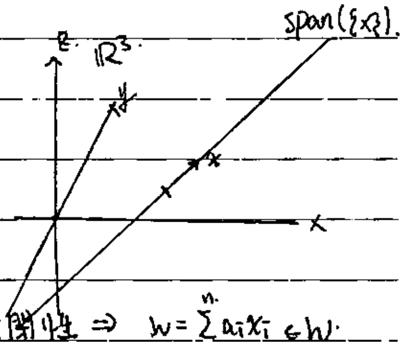
(d)~(f) 略.

4. trivial.

5 trivial.

6. $\forall\ (x,y,z) \in F^3$. $\begin{cases} x = r+s \\ y = r+t \\ z = s+t \end{cases} \Rightarrow x+y+z = 2(r+s+t)$. $\therefore r = \frac{1}{2}(x+y-z)$, $s = \frac{1}{2}(x-y+z)$, $t = (\frac{1}{2})(-x+y+z)$.

$\therefore (x,y,z)^t = \begin{pmatrix}1\\1\\0\end{pmatrix} \times r + \begin{pmatrix}1\\0\\1\end{pmatrix} \times s + \begin{pmatrix}0\\1\\1\end{pmatrix} \times t$ ▨



10. $(S_m) = \mathrm{span}(\{M_1, M_2, M_3\}) =$, $S_m := \{$Symmetric matrices$\}$.

$\begin{pmatrix} r & t \\ t & s \end{pmatrix} = M_1 \times r + M_2 \times s + M_3 \times t$, symmetric matrix.

11. $\mathrm{span}(\{x\}) \overset{def}{=} \{ ax \mid a \in F \}$ ※

12. $(\Rightarrow)$. $\forall\ w \in \mathrm{span}(W)$, $\exists\ x_1, \ldots, x_n \in W$, $a_1, \ldots, a_n \in F$ s.t.

$w = \sum a_i x_i$. since W is a subspace $\Rightarrow$ 加/乘法封閉性 $\Rightarrow w = \sum^{n} a_i x_i \in W$.

$\therefore \mathrm{span}(W) \subseteq W$. By def, $\mathrm{Span}(W) \supseteq W$ ※

$(\Leftarrow)$. 已知 $\mathrm{Span}(W) = W$. Then by thm 1.5, $\mathrm{Span}(W)$ is a subspace of V. ▨

13. S_1, S_2 are subspaces of V. & $S_1 \subseteq S_2$. If $x \in \mathrm{span}(S_1)$, $\exists\ s_1, \ldots, s_n \in S_1^{\subseteq S_2}$, $a_1, \ldots, a_n \in F$

s.t. $x = a_1 s_1 + \cdots + a_n s_n$. $\Rightarrow x \in \mathrm{span}(S_2)$. $\Rightarrow \mathrm{span}(S_1) \subseteq \mathrm{Span}(S_2)$ ※.

Now, $\mathrm{Span}(S_1) = V$ & $S_1 \subseteq S_2 \Rightarrow V \subseteq \mathrm{Span}(S_2)$. , but $V \overset{def\,(\text{or thm 1-5})}{\supseteq} \mathrm{Span}(S_2)$. ▨

14. Given $x \in \mathrm{Span}(S_1 \cup S_2)$, $\exists\ s_1, \ldots, s_n \in S_1 \cup S_2$, $a_1, \ldots, a_n \in F$ s.t. $x = a_1 s_1 + \cdots + a_n s_n$. Actually,

"$\subseteq$" some of $\{s_i\}_{i=1}^{n}$ belong to S_1, and some S_2. $\therefore x = (\sum_{s_i \in S_1} s_i a_i) + (\sum_{s_j \in S_2} a_j s_j) \in (\mathrm{span}(S_1) + \mathrm{span}(S_2))$.

"$\supseteq$" $\therefore$ Given $y \in (\mathrm{Span}(S_1) + \mathrm{Span}(S_2))$, ... similar to the above case ▨

15. Given $x \in \mathrm{span}(S_1 \cap S_2)$ $\exists\ z_1, \ldots, z_n \in S_1 \cap S_n$, $a_1, \ldots, a_n \in F$ s.t. $x = \sum a_i z_i$. but both $\{z_i\}_{i=1}^{n} \in S_1$, $\in S_2$.

$\therefore x \in \mathrm{Span}(S_1)$ & $x \in \mathrm{Span}(S_2) \Rightarrow x \in \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. Hence, $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$ ※

"=" Example: $S_1 := \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\} \subset \mathbb{R}^3$. $S_2 := \{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\} \subset \mathbb{R}^3$. = x-axis. ※

Then $\mathrm{Span}(S_1 \cap S_2) =$ x-axis in $\mathbb{R}^3$; $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2) =$ x,y-plane $\cap$ x,z-plane

"$\subsetneq$" Example: $S_1 := \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\} \subseteq \mathbb{R}^3$ $S_2 := \{\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}-1\\1\\0\end{pmatrix}\} \subseteq \mathbb{R}^3$.

Then $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(\emptyset) = \{0\}$; $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2) =$ x,y-plane $\cap$ x,y-plane = x,y-plane ▨

16. Given $u \in \mathrm{Span}(S)$. $\exists\ v_1, \ldots, v_n \in S$, $a_1, \ldots, a_n \in F$, s.t. $u = \sum a_i v_i$. If $\exists\ b_1, \ldots, b_n \in F$ s.t. $u = \sum b_i v_i$

then $\sum a_i v_i = \sum b_i v_i \Rightarrow 0 = \sum \overbrace{(a_i - b_i)}^{c_i} v_i = \sum c_i v_i \xRightarrow{S\ \text{property}} c_i = 0\ \forall i \Rightarrow a_i = b_i\ \forall i$ ▨

17. S 之間 linearly indep. $\dim(W) < \infty$.

§1-5.

1. (a). F, not "every", just "exist" (b). T, eg. $1 \cdot 0 = 0$, nontrivial representation.

(c) F (text) (d). F (text) (e). T (thm 1.6) (f). T (def).

Linear Algebra.

P.7 ∈F.

3. $\begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix}a+\begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix}b+\begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix}c+\begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix}d+\begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix}e=0.$ $\therefore \begin{cases}a+d=0\\a+e=0\\b+d=0\\b+e=0\\c+d=0\\c+e=0\end{cases}$ $\left.\begin{matrix}a+d=0\\a+e=0\end{matrix}\right\} d=e.$ choose $d=e=1.$, then $a=-1=b=$

$\Rightarrow$ linearly dependent ▨

7. $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is a linearly indep set, generating $\begin{pmatrix}a&0\\0&b\end{pmatrix}$ (diag. matrices set). ▨

8. (a). $F=\mathbb{R}$. for $a,b,c\in\mathbb{R}$, $a\begin{pmatrix}1\\1\\0\end{pmatrix}+b\begin{pmatrix}1\\0\\1\end{pmatrix}+c\begin{pmatrix}0\\1\\1\end{pmatrix}=0 \Rightarrow \begin{matrix}a+b=0\\a+c=0,\ b+c=0\end{matrix}$ $\therefore a=-b=-c$ $\nRightarrow b=-c$ $\Rightarrow a=b=c=0$

(b). F has 特徵(ii) (Appendix C), $\therefore a+b=0=b+c=c+a$

choose $a=b=c=1$. $(1+1=0)$. $\therefore$ linearly dependent ▨

9. $\overset{\in V}{u,\neq v}$. $\{u,v\}$ is linearly dependent $\Leftrightarrow$ u or v is a multiple of the other.

pf ($\Rightarrow$). linearly dep. $\Rightarrow$ $au+bv=0$ for a,b 至少有一個不是零. ... trivial ▨

($\Leftarrow$). Say $u=\overset{\neq 0}{c}v$ $\therefore 1u-cv=0$ ▨

10. $\left\{\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}2\\3\\4\end{pmatrix},\begin{pmatrix}3\\5\\7\end{pmatrix}\right\}$. Each is a linear combination of the others. ▨

11. S, L.I. over $\mathbb{Z}_2$. $\therefore \#\operatorname{span}(S)=2^n$ (by $x=\sum_{i=1}^{n}a_iu_i$, $a_i\in\mathbb{Z}_2$).

12. "Prove Thm1.6." and its corollary: Let V be a vector space. Let $S_1\subseteq S_2\subseteq V$. If S_1

linearly dependent (LD), then, so is S_2.

pf. Suppose not, then $\forall v_1,\dots,v_n\in S_2$, if $\sum_{i=1}^{n}a_iv_i=0$, a_i must be zero $\forall i$.

$\Rightarrow$ if $\sum_{v_i\in S_1}a_iv_i=0$, a_i must be zero $\forall i$. ✗ ▨

Corollary: $S_1\subseteq S_2\subseteq V$. If S_2 is linearly independent, so is S_1.

pf. $\forall v_1,\dots,v_m\in S_1$, they also belong to S_2. $\therefore$ If $\sum^{m}a_iv_i=0$, then $a_i=0\ \forall i\Rightarrow$ LI.

13. (a) "$\{u,v\}$ is LI $\Leftrightarrow$ $\{u+v,u-v\}$ is LI.

pf. ($\Rightarrow$) If $\overset{\in F}{a_1}(u+v)+\overset{\in F}{a_2}(u-v)=0$, then $(a_1+a_2)u+(a_1-a_2)v=0 \xRightarrow{\{u,v\}\text{ is LI}} \begin{cases}a_1+a_2=0\\a_1-a_2=0\end{cases} \Rightarrow a$ F 非 cha

($\Leftarrow$). If $a_1u+a_2v=0$, then (令 $u+v=p$, $u-v=q$) $0=a_1\left(\frac{p+q}{2}\right)+a_2\left(\frac{p-q}{2}\right)$.

$=\frac{1}{2}(a_1+a_2)p+\frac{1}{2}(a_1-a_2)q \underset{\{p,q\}\text{ is L.I.}}{\Rightarrow} \begin{cases}a_1+a_2=0\\a_1-a_2=0\end{cases} \Rightarrow a_1=a_2=0$ ▨

(b). "$\{u,v,w\}$ is L.I $\Leftrightarrow$ $\{u+v,v+w,w+u\}$ is L.I.

pf. ($\Rightarrow$). If $a_1(u+v)+a_2(v+w)+a_3(w+u)=0$, $\Rightarrow$. $(a_1+a_3)u+(a_1+a_2)v+(a_2+a_3)w=0 \Rightarrow \begin{cases}a_1+a_3\\a_2+a\\a_3+a\end{cases}$ $\Rightarrow a_i=$

($\Leftarrow$) If $a_1u+a_2v+a_3w=0$, then (令 $u+v=p$, $v+w=q$, $w+u=r$).

$0=a_1\left(\frac{p-q+r}{2}\right)+a_2\left(\frac{p+q-r}{2}\right)+a_3\left(\frac{-p+q+r}{2}\right) \Rightarrow \begin{cases}a_1+a_2-a_3=0\\-a_1+a_2+a_3=0\\a_1-a_2+a_3=0\end{cases} \Rightarrow a_i=0\ \forall i$ ▨

14. pf ($\Rightarrow$) S is L.D. $\Rightarrow \sum^{n}a_iv_i=0$ for some $v_i\in S$ and some $a_i\in F$, not all zero, say $a_1\neq 0$

$\Rightarrow v_1=-\frac{a_2}{a_1}v_2-\frac{a_3}{a_1}v_3-\cdots-\frac{a_n}{a_1}v_n$. $\therefore\ \exists v$ being a linear combination of some ot

vectors. If this is not the case, $S=\{0\}$, trivially a L.D set ▨

($\Leftarrow$). Trivial ▨

-15 trivial.

16. trivial by corollary of Thm1.6. $\quad // \begin{pmatrix} M_{1i} \\ \vdots \\ M_{i-1,i} \\ 0 \end{pmatrix} \neq 0$.

17. $M \in M_{n\times n}(F)$. & $M_{ii} \neq 0\ \forall i$., being upper triangular matrix. 令 $v_i = \begin{pmatrix} M_{1i} \\ M_{2i} \\ \vdots \\ M_{ni} \end{pmatrix}$, $i=1,\dots,n$.

If $\sum a_i v_i = 0$, then $a_i = 0\ \forall i$ (由. $\sum a_i v_i = 0 = \begin{pmatrix} \sum a_i M_{1i} \\ \sum a_i M_{2i} \\ \vdots \\ \sum a_i M_{ni} \end{pmatrix} = \begin{pmatrix} \sum_1 a_i M_{1i} \\ \sum_2 a_i M_{2i} \\ \vdots \\ a_n M_{nn} \end{pmatrix} \Rightarrow a_n = 0$

$\underset{代回}{\Rightarrow} 0 = \begin{pmatrix} \sum_1 a_i M_{1i} \\ \vdots \\ a_{n-1}M_{n-1,n-1} + \boxed{a_n} M_{n-1,n} \\ \boxed{a_n} M_{nn} \end{pmatrix} \Rightarrow a_{n-1} = 0 \underset{代回}{\Rightarrow} \dots$, Finally, $a_i = 0\ \forall i$). ▨.

18 trivial.

19. If $\sum a_i A_i^t = 0$, then $0 = \sum a_i A_i^t = \sum (a_i A_i)^t = (\sum a_i A_i)^t \Rightarrow 0 = \sum a_i A_i \Rightarrow a_i = 0\ \forall i$ ▨

20. If $af + bg = 0$, $a,b \in \mathbb{R}$, then $ae^{rt} + be^{st} = 0$ $(r \neq s)$. $(\forall t)$. $\Rightarrow a = b = 0$ ▨

§1.6.

1. (a). F, basis $= \varnothing \overset{p.31}{\leftarrow}$ L.I. & generates $\{0\}$. (b). T. (c) F, 反例: $P(F)$. (d) F.
(e) T. (f). F, dim $= n+1$. (g) F, $m \times n$. (h) T. (i) F, S not necessary a L.I. set.
(j) T. (k). T. (l). T.

2. Ans. (a)(c)(d).

4. Absolutely not! $\dim(P_3(R)) = 4$. $\Rightarrow$ basis should contain exactly 4 vectors. ☆

5. Absolutely not! $\dim(R^3) = 3$. $\Rightarrow$ A L.I. set cannot contain more than 3 vectors ☆

6. F^2: $\beta_1 = \{(1,0), (0,1)\}$; $\beta_2 = \{(1,-1), (1,1)\}$; $\beta_3 = \{(1,2), (2,1)\}$.

$M_{2\times 2}(F)$: $\beta_1 = \{E^{ij}, i=1,2. j=1,2\}$; $\beta_2 = \{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\}$; $\beta_3 = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$

8. $W = \{(a_1,a_2,a_3,a_4,a_5) \in \mathbb{R}^5 \mid a_1+a_2+a_3+a_4+a_5 = 0\}$. $\beta = \{u_1, u_3, u_5, u_7, u_8\}$. ☆

9. $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} a + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} b + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} c + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} d \Rightarrow a = a_1$; $b = a_2 - a_1$; $c = a_3 - a_2$; $d = a_4 - a_3$ ▨

10. (a). $g(x) = (-6)\frac{(x+1)(x-1)}{(-2+1)(-2-1)} + (5) \times \frac{(x+2)(x-1)}{(-1+2)(-1-1)} + 3 \times \frac{(x+2)(x+1)}{(1+2)(1+1)} = (-2)(x^2-1) - \frac{5}{2}(x^2+x-2) + \frac{1}{2}(x^2+3x+2)$
$= -4x^2 - x + 8$.

(b)~(d).
Similar method for other case.

11. Just prove that they are all L.I. & by replacement thm, $\Rightarrow$ generating sets $\overset{+L.I.}{\Rightarrow}$ bases ▨

13. $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$. $\Rightarrow$ solution form $= (a,a,a)$. $\Rightarrow \beta = \{(1,1,1)\}$. ☆

14. For W_1: $\beta = \{(1,0,1,0,0), (0,0,1,-1,0), (0,1,0,0,0), (0,0,0,0,1)\}$. $\dim(W_1) = 4$.

For W_2: $\beta = \{(0,1,1,1,0), (1,0,0,0,-1)\}$ $\dim(W_2) = 2$.

17. $\beta = \{A_{ij} \in M_{n\times n}(F) \mid A_{ij} = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & -1 & \cdots & & 0 \end{pmatrix}_{n\times n} = \begin{cases} 1, & \text{position at } (i,j) \\ -1, & \text{at } (j,i) \\ 0, & \text{o.w.} \end{cases}\}$. dimension $= \frac{(n-1)+1}{2} \times (n-1)$
$= \frac{n^2-n}{2}$ ▨

⑱. $\beta = \{\{e_i\} \mid \{e_i\} :=$ a sequence having $a_i = 1$ & $a_j = 0 \; \forall j \neq i$, $i \in \mathbb{N}\}$.

1°. β is L.I. $_*$ 2°. Given a sequence $\{a_n\}$ in V, only finite nonzero entries. then $\{a_n\} = \sum_{i=1} a_i * \{e_i\} \Rightarrow$

19. Given $v \in V$, $\exists!$ $a_1, \ldots, a_n \in F$ s.t. $v = \sum_{i=1}^{n} a_i \cdot u_i$, where $\beta = \{u_1, \cdots, u_n\}$.

Hence, $\text{span}(\beta) \supseteq V$, clearly, $\text{span}(\beta) \subseteq V$. So, $\text{span}(\beta) = V$.

If $\sum a_i u_i = 0$, then since the uniqe expression for $0 \in V$ is $0 = 0$, so we

have $a_i = 0 \; \forall i = 1, \ldots, n$. $\Rightarrow \beta$ is L.I. ▨

⑳. (a). If $S = \phi$ or $\{0\}$, then $V = \{0\}$. Otherwise, we can choose a nonzero vector $u_1 \in S$ & con

pick u_{k+1} s.t. $u_{k+1} \notin \text{span}(\{u_1, \cdots, u_k\})$. Note that $k \leq n$; o.w, we can find linearly in

subset with size $> n$. $\times$. Claim: $k = n$. pf. If $k < n$, then it means that we

find any vectors in S to be the vector u_{k+1}. $\Rightarrow \forall v \in S$, $v \in \text{span}(\underbrace{\{u_1, \ldots, u_k\}}_{W}) \Rightarrow$ spa

$\because V \subseteq \text{span}(S)$ $\therefore V \subseteq \text{span}(W)$ $\underset{\text{Replacement Thm.}}{\Rightarrow}$ $n = \dim(V) \leq k$. $\times$ ▨

(b). The basis has exactly n vectors. By part (a), $\exists$ basis $\beta \subseteq S$. Hence, size

21. ($\Rightarrow$) V is infinite dimensional vector space. Suppose it has no infinite LI subsets

That is, all LI subsets are finite. Then none of these LI subsets can genera

But V has a LI subset that generates V (i.e. basis). (偷偷用到 p.61 Corollary

- <Another>. Let S be the infinite-dimensional generating set of V. Use the proces

in #20 (§1-6) part(a). Claim: $k > N \; \forall N \in \mathbb{N}$. pf: If k terminate at

integer N_0, we find a L.I. set generating S, and thus V. So we find a

basis $\times$ ✲.

($\Leftarrow$). V contains an infinite L.I. subset β. Suppose V is finte dimensional, say dim

Then $\exists \beta' \subseteq \beta$ s.t. β' 的 size $= n+1$. Pick a basis α with size n. $\Rightarrow \alpha$

V. By Replacement Thm, $\underset{\text{generate}}{\textcircled{n}} \geq \overset{\text{L.I. set.}}{\textcircled{n+1}}$ $\times$ ▨

22. Claim: The desired answer is $W_1 \subseteq W_2$ (i.e. $\dim(W_1 \cap W_2) = \dim(W_1) \Leftrightarrow W_1$

pf. ($\Rightarrow$). Let α be the basis of $W_1 \cap W_2$ & β the basis of W_1.

α & β are finite since W_1 & W_2 are finite dimensional. And then α and β

same size. If $W_1 \nsubseteq W_2$ (i.e. $W_1 \supsetneq W_2$), then $\exists$ vector $v \in W_1 - W_2 \Rightarrow v \notin$ spa

and $v \in \text{span}(\beta)$. Also, $W_1 \cap W_2 = W_2 \doteq \text{span}(\beta)$. Then $W_1 \supsetneq W_2 \Rightarrow \text{span}(\alpha) \supsetneq$ sp

$\Rightarrow \text{size}(\alpha) > \text{size}(\beta)$. $\times$ ✲

ch

($\Leftarrow$). Trivial ▨

23 (a). Ans: v is a linear combination of $v_1, \cdots, v_k$. "$\dim(W_1) = \dim(W_2) \Leftrightarrow v \in W_1$."

proof: ($\Rightarrow$) If $v \notin W_1 := \text{span}(\alpha) = \text{span}(\{v_1, \cdots, v_k\})$, then $\alpha \cup \{v\}$ would be a L.I. set that generates W_2 ($= \text{span}(\{v_1, \cdots, v_k, v\})$). By Replacement thm,

$\dim(W_1) < \dim(W_2)$. $\rightarrow\!\!\times$

($\Leftarrow$) Trivial. ∎

(b). Now $\dim(W_1) \neq \dim(W_2)$. —① We know that $W_1 \subseteq W_2$. In fact, $W_1 \subsetneq W_2$ by ①.

Then $\dim(W_1) < \dim(W_2)$.

24. Note that $\{f, f', \cdots, f^{(n)}\} =: \beta$ is a linearly indep. sets (since they all have the different degree). $f(x) \in P_n(R)$. Since $\text{size}(\beta) = n+1 = \dim(P_n(R))$, we conclude that β is a basis of $P_n(R)$ $\therefore \forall g \in P_n(R)$, $\exists$ scalars $c_0, \cdots, c_n \in F$ s.t.

$g(x) = c_0 f(x) + c_1 f'(x) + \cdots + c_n f^{(n)}(x)$. ∎

25. $Z := \{(v, w) \mid v \in V \ \& \ w \in W\}$; a vector space. $\dim(W) = n$ $\dim(V) = m$.

$(\alpha, 0) \cup (0, \beta)$ would be a basis of Z if α, β are the basis of V and W respectively.

$\therefore \dim(Z) = m + n$. ∎

26. Fix $a \in R$ 令 $S =: \{f \in P_n(\mathbb{R}) \mid f(a) = 0\}$.

The basis is $\{(x-a), (x^2-a^2), \cdots, (x^n-a^n)\}$. $\therefore \dim(S) = n$ #

27. $W_1 := \{f \in P(F) \mid f(x) = a_1 x + a_3 x^3 + \cdots\}$.

$W_2 := \{f \in P(F) \mid f(x) = a_0 + a_2 x^2 + \cdots\}$.

$S_1 := W_1 \cap P_n(F)$: basis $= \{x, x^3, \cdots, x^{odd}\}$ $\dim(S_1) = \lfloor \frac{n+1}{2} \rfloor$

$S_2 := W_2 \cap P_n(F)$. basis $= \{1, x^2, \cdots, x^{even}\}$ $\dim(S_2) = \lceil \frac{n+1}{2} \rceil$.

28 V over $\mathbb{C}$ with $\dim = n$, say the basis is α.

Now V over $\mathbb{R}$, then the basis becomes $\alpha \cup \boxed{i\alpha}$ $\therefore \dim = 2n$. ∎ $\{iv \in V \mid v \in \alpha\}$.

29. (a). following the hint, let $\{u_1, \cdots, u_k\} =: \gamma$ be the basis of $W_1 \cap W_2$, and

$\{u_1, \cdots, u_k, v_1, \cdots, v_m\} =: \beta_1$ for W_1, $\{u_1, \cdots, u_k, w_1, \cdots w_p\} =: \beta_2$ for W_2:

$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. for any element $s \in W_1 + W_2$,

$s = (a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_m v_m) + (c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_p w_p)$ for some scalars $\in F$

$= \tilde{a}_1 u_1 + \cdots + \tilde{a}_k u_k + b_1 v_1 + \cdots + b_m v_m + d_1 w_1 + \cdots + d_p w_p$.

$\therefore \text{span}(\{u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_p\}) \supseteq W_1 + W_2$. $\Rightarrow \dim(W_1 + W_2)$ is finite #

續下頁。

Clearly, $\mathrm{span}(\{\overbrace{u_1,\dots,u_k,v_1,\dots v_m,w_1,\dots,w_p}^{\text{Indep.}}\}) = W_1+W_2$

$\overset{\in W_2}{w_i}$ is L.I. of $u_1,\dots,u_k,v_1,\dots,v_m$, o.w. $w_i \in W_1 \Rightarrow w_i \in W_1\cap W_2 \Rightarrow w_i \in \mathrm{span}(\{u_1,\dots,u_k\})$,

contradicting to the fact that $\{u_1,\dots,u_k,w_1,\dots,w_p\}$ is a basis for W_2.

☆Hence, $\{u_1,\dots,u_k,v_1,\dots,v_m,w_1,\dots,w_p\}$ is linearly indep., whence a basis of W_1+W_2.

$\Rightarrow \dim(W_1+W_2) = k+m+p = \dim(W_1)+\dim(W_2)-\dim(W_1\cap W_2)$ ■

(b) " $V = W_1\oplus W_2 \Leftrightarrow \dim(V) = \dim(W_1)+\dim(W_2)$ "

proof. $(\Rightarrow)$ 已知 $W_1\cap W_2=\{0\} \Rightarrow \dim(W_1\cap W_2)=0$ ☆

$(\Leftarrow)$ By part(a), $\Rightarrow \dim(W_1\cap W_2)=0 \Leftrightarrow W_1\cap W_2=\{0\} \Leftrightarrow$ basis $=\phi$. ■

30. • W_1 is a subspace of $V := M_{2\times 2}(\mathbb{F})$ ∵ $O\in W_1$. $\forall M\in W_1, \forall c\in\mathbb{F}, cM\in W_1$.

$\forall M_1,M_2\in W_1$, $M_1+M_2\in W_1$ ☆ $\left(W_1=\left\{\begin{pmatrix} a & b\\ c & a\end{pmatrix} \middle| a,b,c\in\mathbb{F}\right\}\right)$.

• W_2 is a subspace of V : trivial ☆

$W_1\cap W_2 = \left\{\begin{pmatrix} 0 & a\\ -a & 0\end{pmatrix} \middle| a\in\mathbb{F}\right\}$

~~$\dim(W_1\cap W_2) = 1$~~

• $\dim(W_1) = 3$ $\left(\text{basis} = \left\{\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}\right\}\right)$.

• $\dim(W_2) = 2$ $\left(\text{basis} = \left\{\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right\}\right)$.

• $\dim(W_1+W_2) = 3+2-\dim(W_1\cap W_2) = 3+2-1 = 4$.

31. $\dim(W_1)=m$; $\dim(W_2)=n$, $m\geq n$.

(a). $(W_1\cap W_2)$ is a subspace of W_1 & W_2. $\overset{\text{p.50 Thm 1.11.}}{\Rightarrow} \dim(W_1\cap W_2)\leq m$ & $\leq n$.

$\Rightarrow \dim(W_1\cap W_2) \leq \min(m,n) = n$ ■

(b) $\dim(W_1+W_2) = m+n-\underbrace{\dim(W_1\cap W_2)}_{\geq 0} \leq m+n$ ■

32.(a) $W_1 :=$ xy-plane in $\mathbb{R}^3$. $W_2 :=$ a straight line in $\mathbb{R}^3$ that lies in xy-plane & pass th origin.

Then both are subspaces of $\mathbb{R}^3$ & $\dim(W_1\cap W_2) = 1$. ☆ $(\dim(W_1)=2, \dim(W_2)=1)$

(b). $W_1 :=$ xy-plane in $\mathbb{R}^3$ $W_2 :=$ z-axis line in $\mathbb{R}^3$.

Then both are subspaces of $\mathbb{R}^3$ & $W_1\cap W_2=\{0\}$. $\Rightarrow \dim(W_1+W_2) = \underset{2}{\dim(W_1)}+\underset{1}{\dim(W_2)} =$

(c). $W_1 :=$ xy-plane in $\mathbb{R}^3$ $W_2 :=$ xz-plane in $\mathbb{R}^3$

Then both are subspaces of $\mathbb{R}^3$ & $\dim(W_1)=m=2$. $\dim(W_2)=2$.

$\dim(W_1\cap W_2)=1 \overset{<2}{.}$ $\dim(W_1+W_2) = 2+2-1 = 3 < m+n = 4$. ■

33.(a). $V = W_1\oplus W_2 \Rightarrow W_1\cap W_2=\{0\} \Rightarrow \beta_1\cap\beta_2=\phi$ ☆ (Say $\beta_1=\{u_1,\dots,u_k\}$, $\beta_2=\{v_1,\dots,v_m\}$

$\forall i$, v_i cannot be generated by β_1, or $\overset{\neq 0}{v_i}\in W_1\cap W_2$.) Also, v_i is indep. of v_j $\forall j\neq i$.

Then $\beta_1\cup\beta_2$ is linearly indep. $\forall v\in V = W_1\oplus W_2$, $v\in\mathrm{span}(\beta_1\cup\beta_2)$.

$\underset{(\beta_1\cap\beta_2=\phi)}{\Rightarrow}$ $\beta_1\cup\beta_2$ is a basis of V ■

(b). $\overset{W_1}{\beta_1} \cap \overset{W_2}{\beta_2} = \emptyset$, Suppose $\beta_1 \cup \beta_2$ is a basis for V. $\Rightarrow$ $\mathrm{span}(\beta_1 \cup \beta_2) = \overset{?}{V}$ $\Rightarrow$ $W_1 + W_2 = V$.

$\beta_1 = \{u_1, \dots, u_k\}$; $\beta_2 = \{v_1, \dots, v_m\}$: If $\overset{\neq 0}{v} \in W_1 \cap W_2$, then $\overset{\neq 0}{v} = \sum a_i u_i = \sum b_j v_j$ for some not all zero scalars. $\Rightarrow$ $\sum a_i u_i - \sum b_j v_j = 0$ for some not all zero scalars. $\rightarrow\leftarrow$

(since it cause another nontrivial representation of zero, but $\beta_1 \cup \beta_2$ is a basis).

Hence, $W_1 \cap W_2 = \{0\}$. $\Rightarrow$ $V = W_1 \oplus W_2$. ∎

34 (a). Let β be the basis of V & α be the basis of $W_1 (\subseteq V)$.

By Replacement Thm, $\exists$ subset of β, say H, s.t. ① $H \cup \alpha$ generates V ② $H \cap \alpha = \emptyset$. & ③ $\mathrm{size}(\beta) = \mathrm{size}(H) + \mathrm{size}(\alpha)$. $\Rightarrow$ $\alpha' =: H \cup \alpha$ is a basis of V.

Define $W_2 := \mathrm{span}(H)$. (note that H is L.I. since β is).

Then by #33 (§1-6) (The above exercise) part (b), $V = W_1 \oplus W_2$ ∎

Note that even though $\overset{\beta_2 \leftarrow \dots \dots W_2}{\beta_1 \leftarrow \text{basis of } W_1}$, $\beta_1 \cap \beta_2$ may not be a basis of $W_1 \cap W_2$

35. (a). $\alpha := \{u_1, \dots, u_k\}$ (basis of W); $\beta := \{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_n\}$ (basis of V)

Note that $S := V/W = \{v + W \mid v \in V\}$. ~~for some nonzero scalars~~

LI. • Set $a_1(u_{k+1} + W) + \dots + a_n(u_{k+n} + W) = 0$; $\Rightarrow$ $a_1 u_{k+1} + W + a_2 u_{k+2} + W + \dots + a_n u_{k+n} + W = 0$

$\Rightarrow$ $(a_1 u_{k+1} + a_2 u_{k+2} + \dots + a_n u_{k+n}) + W = 0$ -①

If $\exists\ \overset{\neq 0}{v} \in W$ s.t. $\sum_{j=1}^{n} a_j u_{k+j} + v = 0$, $\Rightarrow$ $v = -\sum_{j=1}^{n} a_j u_{k+j} \overset{\text{also}}{=} \sum_{i=1}^{k} b_i u_i$

$\Rightarrow$ $\sum b_i u_i + \sum a_i u_{k+i} = 0$ $\rightarrow\leftarrow$ ($\because \beta$ is a basis).

Hence, $\{u_{k+1} + W, \dots, u_{k+n} + W\}$ is a L.I. set.

Generate • Given $v \in V$, $v = a_1 u_1 + \dots + a_k u_k + a_{k+1} u_{k+1} + \dots + a_{k+n} u_{k+n}$ for some scalars.

Then $v + W = a_1 u_1 + \dots + a_k u_k + a_{k+1} u_{k+1} + \dots + a_{k+n} u_{k+n} + W$.

$= a_1 u_1 + W + a_2 u_k + W + \dots + a_k u_k + W + a_{k+1} u_{k+1} + W + \dots + a_{k+n} u_{k+n} + W$

$= \underbrace{W + W + \dots + W}_{=W} + (a_{k+1} u_{k+1} + W + \dots + a_{k+n} u_{k+n} + W)$.

$= (a_{k+1} u_{k+1} + \dots + a_{k+n} u_{k+n}) + W$

$\therefore$ $\mathrm{span}\{u_{k+1} + W, \dots, u_{k+n} + W\} \supseteq V/W$.

Similar method for the converse proof. $\therefore$ $\mathrm{span}\{u_{k+1} + W, \dots, u_{k+n} + W\} = V/W$. ∎

(b). The formula is: $\underset{n}{\dim(V/W)} = \underset{n+k}{\dim(V)} - \underset{k}{\dim(W)}$. ∎

§1-7 exercises. another : $\mathcal{F} := \{(0,n)\}_{n\geq 1}$.

1. (a). No, counter example : §1-7 Eg.3. (b). No. eg. $\mathcal{F} := \{(0,n)\}_{n\geq 1}$. suppose $M \in \mathcal{F}$ is the maximal element, ~~but~~ say $M = (0,n)$ for some n., but then $(0, n+1))$ contains M ※.

(c). ~~Yes~~, No!! 找很多個 Maximal element 的例子即可, e.g. The family over the set $\{1,2,3\}$ is defined by $\mathcal{F} := \{\{1\},\{2\},\{3\}\}$. Then each are maximal element ∎

(d). Yes, if a chain C has 2 maximal elements, say M_1 & M_2., then by the def of maximal element, $M_1 \nsubseteq M_2$ & $M_2 \nsubseteq M_1$ but chain ⟹ $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$

If $M_1 \subseteq M_2$, then by $M_1 \nsubseteq M_2$, we have $M_1 = M_2$

if $M_2 \subseteq M_1$, then by $M_2 \nsubseteq M_1$, we have $M_2 = M_1$. ∎

(e). Yes, p.60 第二段..

(f). Yes., Thm 1.12.

2. Let $\{a_n\}$ be a convergent seq (in W). Let $W :=$ a subset containing all conv. seq(s). of the vector space V containing all seqs of real numbers.

- Then $\{0\}$ is in W.
- $\forall \{a_n\} \in W$, $\forall c \in \mathbb{R}$, $c\{a_n\} \overset{\text{by def.}}{=} \{ca_n\}$. Then the limit still exists since $\lim_{n\to\infty} ca_n = c \lim a_n = ca < \infty$. Hence, $c\{a_n\} \in W$.
- $\forall \{a_n\}, \{b_n\} \in W$, $\{a_n\} + \{b_n\} = \{a_n + b_n\}$: The limit also exists since $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a+b$. Hence, $\{a_n\} + \{b_n\} \in W$.

⟹ W is a subspace of V. & basis size = size of maximal linearly indep. set.
$\{e_1, e_2, \dots\}$ is a L.I subset of V. ⟹ V is ∞-dim ∎
~~The basis is $\{e_1, e_2, e_3, \dots\}$~~, where $e_i = \{a_j\}$, $\begin{cases} a_j = 1, j = i \\ a_j = 0, j \neq i \end{cases}$.

3. $V = \mathbb{R} \overset{\text{over}}{(\mathbb{Q})}$

proof. Following the hint, we know that π is transcendental, i.e, $\forall f \in P(\mathbb{Q})$, $f(\pi) \neq 0$.

- 1° $\{\pi, \pi^2, \dots, \pi^k\}$ is L.I. : proof : set $\pi^k = \sum_{i=1}^{k-1} a_i \pi^i$ for $a_i \in \mathbb{Q}$ (not all zeroes). Then define $\overset{\in P(\mathbb{Q})}{f(x)} = x^k - \sum_{i=1}^{k-1} a_i x^i \Rightarrow f(\pi) = 0$ ※. ∎
- 2° If we stop collecting the vector π^k at $k = N$. Then $\{\pi, \dots, \pi^N\}$ is a basis for V. But $\pi^{N+1} \in V$, so ∃ scalars (not all zeroes) s.t. $\pi^{N+1} = \sum_{i=1}^{N} a_i \pi^i$. Define $f(x) = x^{N+1} - \sum_{i=1}^{N} a_i x^i$ $(\in P(\mathbb{Q})) \Rightarrow f(\pi) = 0$ ※

Hence, ~~the basis is $\{\pi, \pi^2, \dots\}$~~. ∎ even $\{1, \pi, \pi^2, \dots, e, e^2, \dots\}$ we can only say it's a L.I. set. 不一定是 basis.

§1-7. exercises. 先下頁, 後這頁. p13.

4. Let α be a basis for W. β a basis for V (might be infinite dimensional).

By Replacement thm, $\exists H \subset \beta$ s.t. $\alpha' := H \uplus \alpha$. that generates V. Note that
& $size(\alpha') = size(H) + size(\alpha')$.
α' is L.I. since $H \subset \beta$ is L.I & α is a basis

Thus given a basis for W, by Replacement Thm, we can extend α to α' to be a basis for V ▨

< Another > . By Thm 1.13. ▨

不 8. (Proof of infinite-dimensional version of Thm 1.8, p.43):

會?? (⇒) Set $\beta \overset{?}{=} \{u_1, u_2, \dots\}$ to be a basis for V. ⇒ $span(\beta) = V$ ⇒ $\forall v \in V$, $\exists$ scalars (not all
?? △ zeroes) s.t. $v = \sum^{\infty} a_i u_i < \infty$

6. (Generalization of Thm 1.9). $S_1 \subseteq S_2$, are subsets of V S_1 is L.I S_2 generat

We follow the hint. Let $\mathcal{F} :=$ all L.I subsets of S_2 that contain S_1.

Given a chain in $\mathcal{F}$, say C. Define $U = \bigcup_{\tilde{S} \in C} \tilde{S}$.

① clearly, U contains all elements of C

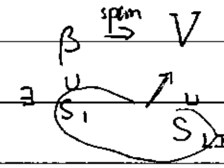
② To prove that $U \in \mathcal{F}$: we first notice that $S_1 \subseteq U \subseteq V$

Second, arbitrary given $u_1, \dots, u_n$ in U. Assum $a_1u_1 + \dots + a_nu_n = 0$ for s
scalars $\in \mathcal{F}$. Because $u_i \in U$ $\forall i = 1, \dots, n$, $\exists A_i \overset{\in C}{}$ s.t. $u_i \in A_i$. But C is a
chain ∴ $\exists A_k \overset{\in C}{}$ s.t. $u_i \in A_k$ $\forall i$. Note that A_k is L.I. (in $C \subset \mathcal{F}$).
Hence, $a_1 = a_2 = \dots = a_n = 0$. Since $u_1, \dots, u_n$ is given arbitrary in U.
We thus have U be L.I. ∴ $U \in \mathcal{F}$. ▨

①+② ⇒ (apply maximal principle). $\mathcal{F}$ contains a maximal member, say β. ($\supseteq S$
By the maximality, β generates S_2, whence V. β is also L.I. ∴ β is a b
for V. ▨

§1-7 P.15.

7. (Generalization of Replacement Thm). Let β be a basis of V. S is a L.I. subset of V. Prove that $\exists\, S_1 \subset \beta$ s.t. $S_1 \cup S$ is a basis for V.



proof: Let $\mathcal{F} := \{\tilde{S} \mid \tilde{S}$ is L.I. subset of β & $\tilde{S} \cup S$ is $\overset{\text{linearly}}{\text{indep.}}\}$.

For each chain C in $\mathcal{F}$, define $U :=$ union of all the members in C.

check that $U \in \mathcal{F}$ and contains all members in C.:

- $U \in \mathcal{F}$: Given $u_1, \dots, u_k$ arbitrarily in U Assume $\sum_{i=1}^{k} a_i u_i = 0$. $\because u_i \in U = \bigcup_{\tilde{S} \in C} \tilde{S}$

$\therefore \exists\, A_i$ s.t. $u_i \in A_i$ for each i. But C is a chain $\Rightarrow \exists\, \bar{A}$ s.t. $u_i \in A_i \subset \bar{A}, \forall i$.

$\bar{A}$ is in $C \subset \mathcal{F} \Rightarrow \bar{A}$ is L.I $\Rightarrow a_1 = a_2 = \cdots = a_k = 0 \Rightarrow U$ is L.I.

Also, $\bar{A} \cup S$ is L.I.. Assume $\underbrace{\sum^{m} b_i v_i}_{S} + \underbrace{\sum^{n} c_j u_j}_{\bar{A}} = 0$ with $v_i \in S\ \forall i$, $\overset{u_j \in U}{u_j \in \bar{A}}\ \forall j$.

Then we have $b_i = c_j = 0\ \forall i, j$ since $\bar{A} \cup S$ is L.I.

Since $u_1, \dots, u_k$ are given arbitrarily in U & $v_1, \dots, v_m$ are given arbitrarily in S, we conclude that $U \cup S$ is L.I.

Thus, $U \in \mathcal{F}$ $*$

Apply Maximal principle. $\Rightarrow \mathcal{F}$ has a maximal member. S_1. So, $\overset{\subset\beta}{S_1}$ is L.I & $S_1 \cup S$ in L.I. It remains to prove that $S_1 \cup S$ generates V.

- By the maximality of S_1, (maximal linearly indep. subset of β). S_1 generates β, and thus V. ∎

(If $\exists\, v \in \beta$ s.t. $v \notin \operatorname{span}(S_1)$, then $S_1 \cup \{v\}$ is L.I $\times$ to maximal prop

No.

Date

Chapter 2. §2-1. Ch2.

1. (a) True: T linear $\Rightarrow T(\sum a_i x_i) = \sum a_i T(x_i)$. (b) F, $T(x+y) = T(x) + T(y) \Rightarrow T(0) = 0$ & $T(2x) = 2T(x)$, generally, $T(nx) = nT(x)$, $n \in \mathbb{Z}$. But T may not satisfy $T(cx) = cT(x)$, $c \in F$.

Eg.

(C). F: Additional condition required is that T must be linear.

Counter-e.g: $T: \mathbb{R}^2 \to \mathbb{R}$ is defined by $T(\vec{x}) = \|\vec{x}\|^2$.

Then $T(\vec{x}) = 0 \Leftrightarrow \vec{x} = (0,0)$ (unique vector s.t. $T(\vec{x}) = 0$).

(d). True: we must have $T(0) = 0$.

(e). False: Thm 2.3 (Dimension Thm) tells us that $\boxed{\dim(V)} =$ nullity(T) + rank(T). if $T: V \to W$ is linear.

Counter-e.g: $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T((x,y)) = (x+y, x, y)$.

Then $\forall c \in F$, $(x_1, y_1) := \vec{a}$, $(x_2, y_2) := \vec{b} \in \mathbb{R}^2$, $T(c\vec{a}+\vec{b}) = (c(x_1+y_1)+x_2+y_2, cx_1+x_2, cy_1+y_2)$;

and $cT(\vec{a}) + T(\vec{b}) = (cx_1+cy_1, cx_1, cy_1) + (x_2+y_2, x_2, y_2) = (c(x_1+y_1)+x_2+y_2, cx_1+x_2, cy_1+y_2)$.

So, $T(c\vec{a}+\vec{b}) = cT(\vec{a}) + T(\vec{b})$. $\Rightarrow$ T is linear.

Also, $T((x,y)) = 0 \Leftrightarrow (x,y) = (0,0)$. Hence, $N(T) = \{0\}$.

By dimension thm, we have nullity(T) = 0, rank(T) = 2 - 0 = 2. < dim(W) = 3. ※

(f). False. (Additional condition required is that T must be one-to-one).

Counter-e.g: $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined to be $T(x) = 0$ $\forall x \in V = \mathbb{R}^2$.

Then T is linear! But T carries every L.I subsets of $V = \mathbb{R}^2$ onto zero in $W = \mathbb{R}^3$.

(g). True, by the corollary to thm 2.6. ※

(h). False. the statement is very like thm 2.6, but the condition for basis for V is unstated. We give a counter-e.g: Let $x_1 = 0 \in V$, $x_2 \neq 0 \in V$ & let $y_1, y_2 \in W - \{0\}$

Then we can't find a linear transformation s.t. $T(x_1) = y_1$, and $T(x_2) = y_2$.

($T(x_1) = T(0) = y_1 \neq 0$, but linear implies that $T(0) = 0$).

4. 意思一下 ==. Given $c \in F$, $A, B \in M_{2\times3}(F)$, $T(cA+B) = \begin{pmatrix} 2ca_{11}+2b_{11}-ca_{12}-b_{12} & ca_{13}+b_{13}+2ca_{12}+2b_{12} \\ 0 & 0 \end{pmatrix}$

and $cT(A)+T(B) = \cdots = T(cA+B) \Rightarrow$ T linear ※ $N(T) = \{A \in M_{2\times3}(F) \mid 2a_{11} = a_{12}, a_{13} = -2a_{12}\}$.

$\Rightarrow N(T) = \left\{\begin{pmatrix} \frac{1}{2}t & t & -2t \\ s & u & v \end{pmatrix} \in M_{2\times3}(F)\right\}$ $\therefore \dim(N(T)) = 4$. (自由度) basis of N(T) is

$\beta = \left\{\begin{pmatrix} \frac{1}{2} & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ ※ For R(T), 先看 $\begin{cases} m = 2a_{11} - a_{12} \\ n = a_{13} + 2a_{12} \end{cases}$ $\forall m,n$

是否有解. $\Rightarrow \begin{cases} a_{12} = 2a_{11} - m \\ a_{12} = \frac{1}{2}a_{13} - \frac{1}{2}n \end{cases} \Rightarrow 2a_{11} - m = \frac{1}{2}a_{13} - \frac{1}{2}n$. 故 Given $m, n \in F$, $\exists a_{11}, a_{13} \in F$ s.t.

續下頁。

…s.t. $2a_{11}-m=\frac{1}{2}a_{13}-\frac{1}{2}n$ 成立。故, $\forall\ \bar{A}=\begin{pmatrix}a&b\\0&0\end{pmatrix}\in M_{2\times2}(\mathbb{F})$, $\exists\ A\in M_{2\times3}(\mathbb{F})$ s.t. $T(A)=\bar{A}$

Then $R(T)=\{\begin{pmatrix}a&b\\0&0\end{pmatrix}\in M_{2\times2}(\mathbb{F})\}$. basis for $R(T)$ is $\gamma=\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\}$.

& $\dim(R(T))=2$. Then we verify the dimension thm.: $\underset{6}{\dim(V)}\underset{\checkmark}{\overset{?}{=}}\underset{4}{\text{nullity}(T)}$

故 dimension thm 成立. $\because N(T)\neq\{0\}$ & T linear $\therefore$ T is not 1-1. ☆

T is not onto since $\text{rank}(T)=2<4=\dim(W)$.

7. ①: T linear $\Rightarrow T(cx+y)=cT(x)+T(y)$. choose $c=1, x=0, y=0$, we have $T(0)=2T(0)\Rightarrow$

②: $(\Rightarrow)$ T linear $\Rightarrow T(x+y)=T(x)+T(y)$ — $\boxed{a}$ & $T(cx)=cT(x)\ \forall c,x,y$ — $\boxed{b}$.

Then $T(cx+y)=T(cx)+T(y)\overset{(\text{by }\boxed{a})}{=}cT(x)+T(y)$ by $\boxed{b}$ ☆

$(\Leftarrow)$ 已知 $T(cx+y)=cT(x)+T(y)\ \forall c\in\mathbb{F}, x,y\in V$.

choose $c=1$. $\Rightarrow T(x+y)=T(x)+T(y)$.

choose $y=0$ $\Rightarrow T(cx)=cT(x)$. ▨

③ T linear. $\Rightarrow T(x-y)=T(x)+T(-y)=T(x)+(-1)T(y)=T(x)-T(y)$ ▨

④ $(\Rightarrow)$. $T(\sum a_ix_i)=\sum T(a_ix_i)=\sum a_iT(x_i)$ ☆

$(\Leftarrow)$. 已知 $T(\sum a_ix_i)=\sum a_iT(x_i)\ \forall x_i\in V, a_i\in\mathbb{F}, i=1,\dots,n$.

Then given $x,y\in V$, $c,1\in\mathbb{F}$, we have $T(cx+1\cdot y)=cT(x)+1\cdot T(y)$. ▨

8. (Rotation) $T_\theta:\mathbb{R}^2\to\mathbb{R}^2$ $T_\theta(a_1,a_2)=(a_1\cos\theta-a_2\sin\theta,\ a_1\sin\theta+a_2\cos\theta)$

check that if T_θ is linear or not:

Let $\vec{x},\vec{y}\in\mathbb{R}^2$. $T_\theta(c\vec{x}+\vec{y})=T_\theta((cx_1+y_1,cx_2+y_2))=((cx_1+y_1)\cos\theta-(cx_2+y_2)\sin\theta,\ (cx_1+y_1)\sin\theta+(cx_2+y_2)$

and $cT_\theta(\vec{x})+T_\theta(\vec{y})=c(x_1\cos\theta-x_2\sin\theta,\ x_1\sin\theta+x_2\cos\theta)+(y_1\cos\theta-y_2\sin\theta,\ y_1\sin\theta+y_2\cos\theta)$

Hence, $T_\theta(c\vec{x}+\vec{y})=cT_\theta(\vec{x})+T_\theta(\vec{y})\ \forall c,\vec{x},\vec{y}$. ☆

• (Reflection). $T:\mathbb{R}^2\to\mathbb{R}^2: (a_1,a_2)\mapsto(a_1,-a_2)$.

Let $\vec{x},\vec{y}\in\mathbb{R}^2$. $T(c\vec{x}+\vec{y})=T((cx_1+y_1,cx_2+y_2))=(cx_1+y_1,(cx_2+y_2)(-1))$.

$cT(\vec{x})+T(\vec{y})=c(x_1,-x_2)+(y_1,-y_2)=(cx_1+y_1,-cx_2-y_2)=T(c\vec{x}+\vec{y})$ ☆

9. 略.

10. T is linear. $T(1,0)=(1,4)$. $T(1,1)=(2,5)$.

Then. $T(2,3)=3T(1,1)-T(1,0)=(6,15)-(1,4)=(5,11)$ ☆ $(T(0,1)=(1,+1))$.

Since T is linear, we check that $N(T)=\{0\}$?

$T(a,b)=T(a,0)+T(0,b)=(a,4a)+(b,+b)=(a+b,4a+b)$.

$\begin{cases}a+b=0\\4a+b=0\end{cases}\Rightarrow a=b=0\Rightarrow N(T)=\{0\}\Rightarrow T$ is 1-1. ☆▨

11. $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right\} =: \beta$ is a basis for $\mathbb{R}^2$. By Thm 2.6, $\exists !$ T. linear transformation, s.t. $T(1,1) = (1,0,2)$, $T(2,3) = (1,-1,4)$.

$T(8,11) = 2T(1,1) + 3T(2,3) = (2,0,4) + (3,-3,12) = (5,-3,16)$. ※

12. No! Since if T is linear, then $T(-2,0,-6) = (-2)T(1,0,3)$,

but $(2,1) \neq (-2)(1,1) = (-2,-2)$. ※

13. $T: V \to W$ is linear. $\gamma := \{w_1, \dots, w_k\}$ is a L.I. subset of $R(T) \subset W$.

令 $S = \{v_1, \dots, v_k\}$ s.t. $T(v_i) = w_i$ $\forall i = 1, \dots, k$. Is S L.I?

proof: Let $a_1v_1 + \cdots + a_kv_k = 0$, $a_i \in \mathbb{F}$ $\forall i$:

$\Rightarrow T(a_1v_1 + \cdots + a_kv_k) = 0 \Rightarrow a_1T(v_1) + \cdots + a_kT(v_k) = a_1w_1 + \cdots + a_kw_k = 0$

$\Rightarrow a_1 = a_2 = \cdots = a_k = 0$ since γ is L.I. $\Rightarrow S$ is L.I. ※ ■

(T linear).

14. (a). ($\Rightarrow$) T is 1-1. Suppose $B \subset V$ is a linearly indep. subset ($B = \{v_1, \dots, v_k\}$).

令 $T(v_i) = w_i$ $\forall i$. Set $a_1w_1 + \cdots + a_kw_k = 0$

$\Rightarrow a_1T(v_1) + \cdots + a_kT(v_k) = 0 \Rightarrow T(a_1v_1 + \cdots + a_kv_k) = 0$

By $N(T) = \{0\}$, $\Rightarrow a_1v_1 + \cdots + a_kv_k = 0$, but $\{v_1, \dots, v_k\}$ is L.I.

$\Rightarrow a_1 = \cdots = a_k = 0$ Hence, $\{w_1, \dots, w_k\}$ is a L.I subset of $R(T) \subset W$. ※

($\Leftarrow$) Suppose T carries L.I. subsets of V onto L.I subsets of W.

Say $\{v_1, \dots, v_k\}$ is a L.I subset of V. Then $\{T(v_1), \dots, T(v_k)\} =: \{w_1, \dots, w_k\}$ is a L.I subset of W. $N(T) = \{x \in V \mid T(x) = 0\}$

but $x = \sum_{i=1}^{k} a_i v_i$ for some scalars a_i. $\Rightarrow 0 = T(x) = \sum a_i T(v_i) = \sum a_i w_i$

$\Rightarrow a_i = 0$ $\forall i$ since $\{w_1, \dots, w_k\}$ is a LI subset of W.

$\Rightarrow x = 0$. $\therefore N(T) = \{0\}$ also T is linear $\therefore T$ is 1-1. ■

(b). T is 1-1. $S \subset V$. T is linear.

($\Rightarrow$). Suppose S is LI. $=: \{v_1, \dots, v_k\}$.

$T(S) = \{T(v_1), \dots, T(v_k)\}$. Set $a_1T(v_1) + \cdots + a_kT(v_k) = 0$

$\Rightarrow T(a_1v_1 + \cdots + a_kv_k) = 0 \overset{1-1}{\Rightarrow} a_1v_1 + \cdots + a_kv_k = 0 \overset{S \text{ is LI.}}{\Rightarrow} a_i = 0$ $\forall i$. $\Rightarrow T(S)$ is L.I ※

($\Leftarrow$). $T(S) = \{T(v_1), \dots, T(v_k)\}$ is L.I. Set $a_1v_1 + \cdots + a_kv_k = 0$.

Then $0 = T(a_1v_1 + \cdots + a_kv_k) = a_1T(v_1) + \cdots + a_kT(v_k)$. $\Rightarrow a_i = 0$ $\forall i$.

$\Rightarrow S$ is LI. ■

§2-1.

$:V\to W.$

(C). T is 1-1, onto. $\beta=:\{v_1,\dots,v_n\}$ is a basis for V.

- $T(\beta).=\{T(v_1),\dots,T(v_n)\}$: is L.I.: Set $\sum a_iT(v_i)=0 \Rightarrow T(\sum a_iv_i)=0 \overset{T\text{ is 1-1.}}{\Rightarrow} \sum a_iv_i=$
$\overset{\beta\text{ is L.I.}}{\Rightarrow} a_i=0\ \forall i$. ☆

- T(β) generates W: We know that T(β) is a basis for R(T).
By Dimension Thm, $\dim(V)=\text{nullity}(T)+\text{rank}(T) \overset{1-1}{\Rightarrow} \dim(V)=\dim(R(T))$. ∴ nulli
But onto $\Rightarrow R(T)=W \Rightarrow \dim(V)=\dim(R(T))=\dim(W)$ ∴ T(β) is also a basis
↑ see def at p-551.
for W ∎

15. $T: P(\mathbb{R})\to P(\mathbb{R})$ $f(x)\mapsto\int_0^x f(t)dt$.
- T is linear: $T(cf+g)=\int_0^x cf(t)+g(t)dt=c\int_0^x f(t)dt+\int_0^x g(t)dt=cT(f)+T(g)$ ☆
- T is 1-1: Let $f\in P(\mathbb{R})$, ∃ scalars s.t $f(x)=a_0+a_1x+\dots+a_nx^n+\dots$
$0=\int_0^x f\,dt=\int_0^x a_0+a_1x+\dots+a_nx^n+\dots dx=a_0x+\frac{1}{2}a_1x^2+\dots$
$\Rightarrow a_i=0\ \forall i. \Rightarrow N(T)=\{0\}$.
- Let $f=\sum_{i=0}^n a_ix^i \Rightarrow T(f(x))=\int_0^x \sum a_it^idt=\sum_{i=0}^n \frac{a_i}{i+1}x^{i+1}$ (no const term).
So, there is no function in $P(\mathbb{R})$ s.t $T(f)=k$ for some nonzero const. $\in\mathbb{F}$

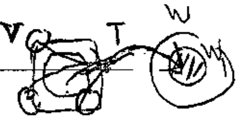
17. (a). If T is onto, $\dim(R(T))=\dim(W)$. (V & W are finite dimensions).
but $\dim(V)<\dim(W) \Rightarrow \dim(V)<\dim(R(T))$. ⟶✱ (T is linear.)
.. why $\dim(V)\geq\dim(R(T))$? Let $\{v_1,\dots,v_n\}=\beta$ be a basis for V.
If T is linear & 1-1. ⇒ T(β) is a basis for R(T).
If T is linear but not 1-1 ⇒ T(β) is a dep. set ∴ the size of the basi
for R(T) would be smaller ☆ ∎

(b). If T is 1-1. ⇒ (Note that T is linear) $N(T)=\{0\}$.
By dimension Thm, $\dim(V)=\dim(R(T))$ but $\dim(V)>\dim(W)$.
$\Rightarrow \dim(R(T))>\dim(W)$. ⟶✱ (since R(T) is a vector subspace of W). ∎

18. $T(\vec{x})=\vec{0}\ \forall\vec{x}\in\mathbb{R}^2$ is the desired linear transformation.

19. $T:\mathbb{R}^2\to\mathbb{R}^2$ $(x_1,x_2)\mapsto(x_1,x_2)$. $U:\mathbb{R}^2\to\mathbb{R}^2$ $(x_1,x_2)\to(x_1+x_2,x_1-x_2)$.
Then $N(T)=\{0\}=N(U)$. & $R(T)=\mathbb{R}^2=R(U)$. ☆

20. • For $T(V_1)$: 1° $0\in V_1\Rightarrow T(0)=0\in T(V_1)$ 2° if $w_1,w_2\in T(V)$ & $c\in\mathbb{F}$,
$\Rightarrow w_1+w_2\overset{(\exists)}{=}T(\overset{\in V}{x_1})+T(\overset{\in V}{x_2})=T(x_1+x_2)\in T(V)$ since $x_1+x_2\in V$. & $cw_1\overset{(\exists)}{=}cT(x_1)=T(cx_1)\in T(V)$
since $cx_1\in V$ ☆



- Define $S = \{x \in V \mid T(x) \in W_1\}$

1° $\because 0 = T(0) \in W_1$ $\therefore 0 \in S$. 2° Given $x_1, x_2 \in S$ & $c \in F$, $\Rightarrow T(x_1), T(x_2) \in W_1$.

Then $T(x_1) + T(x_2) \in W_1$, but $T(x_1) + T(x_2) = T(x_1 + x_2)$, $\therefore x_1 + x_2 \in S$.

Then $cT(x_1) \in W_1$, but $cT(x_1) = T(cx_1)$ $\therefore cx_1 \in S$ ※ ▨

21. (a). trivial. $(a_1, a_2, \dots)$

(b) Given $\bar{a} \in V$. $\exists\ \tilde{\alpha} = (x\ \bar{a}) := (x, a_1, a_2, \dots)$ s.t. $T(\tilde{\alpha}) = \bar{a}$. $\Rightarrow$ onto ※

$N(T) = \{(x, 0, 0, \dots) \mid x \in F\}$. $\Rightarrow$ not 1-1 ※

(c). $N(U) = \{0\}$ $\Rightarrow$ 1-1 ※

But $R(U) = \{(\underline{\underline{0}}, a_1, a_2, \dots)\} \subsetneq V$ ※

22. $\because T: \mathbb{R}^3 \to \mathbb{R}$ is linear. Let $T(1,0,0) = a$, $T(0,1,0) = b$, $T(0,0,1) = c$.

Then we have $T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = ax + by + cz$. $\forall (x,y,z) \in \mathbb{R}^3$.

- $T: F^n \to F$ is linear. Let $T(e_i) = a_i\ \forall i$. Then $T(x_1, x_2, \dots, x_n) = x_1 T(e_1) + \dots + x_n T(e_n)$

$= a_1 x_1 + \dots + a_n x_n$.

- $T: F^n \to F^m$ Let $T(x_1, \dots, x_n) = (\sum_{i=1}^{n} a_i x_i, \sum_{i=2}^{n} a_i x_i, \dots, \sum_{i=m}^{n} a_i x_i)$.

23. A null space of $T: \mathbb{R}^3 \to \mathbb{R}$, being linear, could be a point $= \{0\}$, a line through 0, or a plane through 0. or a whole $\mathbb{R}^3$.

24. $T: \mathbb{R}^2 \to \mathbb{R}^2$.

(a). $T(a,b) = (0,b)$. $(\ (a,b) = \overset{\text{y-axis}}{(0,b)} + \overset{\text{x-axis}}{(a,0)}.\ )$.

(b). V = y-axis ⊕ L-line $\therefore$ If $(a,b) \in \mathbb{R}^2$, $(a,b) = \overset{\text{y-axis}}{(0, b-a)} + \overset{L}{(a,a)}$.

$\therefore T(a,b) = (0, b-a)$.

25. $T: \mathbb{R}^3 \to \mathbb{R}^3$.

(a). $\mathbb{R}^3 =: V = \overset{W_1}{\text{xy-plane}} \oplus \overset{W_2}{\text{z-axis}}$. Given $(a,b,c) \in \mathbb{R}^3$, $(a,b,c) = \overset{W_1}{(a,b,0)} + \overset{W_2}{(0,0,c)}$

(b). $T(a,b,0) = (0,0,c)$.

(c). $\mathbb{R}^3 =$ xy-plane ⊕ $L\ (= \{(a,0,a) \mid a \in \mathbb{R}\})$ Given $(a,b,c) \in \mathbb{R}^3$, $(a,b,c) = \overset{\text{xy-plane}}{(a-c, b, 0)} + \overset{L}{(c,0,c)}$

26. (a). T is the projection on W_1 along W_2. $\Rightarrow$ Given $x \in V = W_1 \oplus W_2$, $x = x_1 + x_2$, where $x_1 \in W_1$ & $x_2 \in W_2$. and then $T(x) = x_1$.

$\because$ Is T linear? Given $x, y \in V$, $c \in F$, $\exists\ x_1, y_1 \in W_1$ & $\exists\ x_2, y_2 \in W_2$, s.t. $\begin{cases} y = y_1 + y_2 \\ x = x_1 + x_2 \end{cases}$.

Then $T(cx + y) = T(\underset{\sim}{cx_1 + y_1} + cx_2 + y_2) = cx_1 + y_1 = cT(x) + T(y)$ ※

To continue on, we next prove that $W_1 = \{x \in V \,|\, T(x) = x)$:

Since. $T(x) = x_1$ if $\overset{\in V}{x} = \overset{\in W_1}{x_1} + \overset{\in W_2}{x_2}$, given $m \in W_1$, we have. $T(m) = m$. $\Rightarrow$. $W_1 \subseteq \{\overset{\in V}{x} | T(x)=$

Now, if $x \in V$ s.t. $T(x) = x$, then since $T(x) \in W_1$, so $x \in W_1$. ▨

(b). • $N(T) = \{x \in V | T(x) = 0\}$ by def.. Since $T(W_2) = 0$, so $W_2 \subseteq N(T)$.

If $x \in N(T)$, then $T(x) = 0$, but x can be written as. $x = \overset{\in W_1}{x_1} + \overset{\in W_2}{x_2}$.

$\therefore T(x) = x_1 = 0 \Rightarrow x = x_2 \in W_2$. Finally, $W_2 = N(T)$ ※

• $T(x) = x \ \forall x \in W_1 \Rightarrow W_1 \subseteq R(T)$. Also, $R(T)$ is naturally contained in W_1.

(c) $\cdot T: W_1 \Rightarrow W_1 . \Rightarrow T(x) = x \ \forall x \in W_1$. ($T$ is a projection on W_1).

(d). $T: V \to V$ where $V = W_1 \oplus W_2 = \{0\} \oplus W_2 \Rightarrow V = W_2$

Note that T is a projection on W_1 along W_2. So, $T: \underset{x \mapsto 0}{W_2 \to W_2}$ ▨

27 (a). Let $\{v_1, \dots, v_k\}$ be a basis for W and by replacement thm, the basis for W can be extended to the basis for V, say $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$.

Define $\text{span}(\{v_{k+1}, \dots, v_n\}) := W'$ Then $W \cap W' = \{0\}$ & $W + W' = V$

So we can define a transformation $T: V \to V$ be the projection on W along

(b). Let $W =$ xy-plane. of the vector space $\mathbb{R}^3 =: V$.

$W_1 =:$ z-axis., $W_2 := \{(s,s,s) | s \in \mathbb{R}\}$ are subspaces of V.

Then $W \oplus W_1 = V$ & $T: V \to V$, $T(a,b,c) = (a,b,0)$ is a projection on W along

$W \oplus W_2 = V$ & $T: V \to V$, $T(a,b,c) = (a-c, b-c, 0)$ is a projection on W alo

28. • $\{0\}$ is T-invariant: $T(0) = 0$ ※

• V is T-invariant since $T: V \to \boxed{V}$ ※

• $R(T)$ is T-invariant: if $x \in R(T) \subseteq V$, then. $x \in V$ $\therefore$ We can take T, i.e. $T(x) \in R$

Hence, $T(R(T)) \subseteq R(T)$ ※

• $N(T)$ is T-invariant: If $x \in N(T)$, then. $T(x) = 0$. Also $T(0) = 0$ $\therefore$ $0 \in N(T)$.

Hence, $T(x) = 0 \in N(T) \ \forall x \in N(T)$. $\Rightarrow T(N(T)) \subseteq N(T)$ ▨

29. • Suppose W is T-invariant. $T_W: W \to W$ is defined by $T_W(x) = T(x) \ \forall x \in W$.

Given $x, y \in W$ & $c \in F$, then $cx + y \in W$ (subspace)., then $T_W(cx+y) = T(cx+y)$.

Also, $cT_W(x) + T_W(y) = cT(x) + T(y)$, but T is linear, we conclude that T_W is line

30. $V = W \oplus W'$. T is the projection on W along some subspace W'

- Given $x \in W$, $T_W(x) = T(x) = \overset{\in W}{x}$ by the def of projection. $\Rightarrow$ $T_W = I_W$ & W is T-invariant ∎

31. $V = R(T) \oplus W$. & W is T-invariant.

(a). W is T-invariant $\Rightarrow$. Given $x \in W$, then $T(x) \in W$.

But. $V = R(T) \oplus W$, that means $R(T) \cap W = \{0\}$.

So, $T(x)$ must be zero since $T(x) \in R(T)$ & $T(x) \in W$, $\forall x \in W$.

$\therefore$ $W \subseteq N(T)$. ∎

(b) V is finite-dimensional. By Dimension Thm, $\dim(V) = \text{nullity}(T) + \text{rank}(T)$.

but $V = R(T) \oplus W \Rightarrow \dim(V) = \dim(R(T)) + \dim(W)$.

Then $\dim(W) = \text{nullity}(T)$. Also, $W \subseteq N(T)$ by part (a)., so $W = N(T)$. ∎

(c). $T: P(\mathbb{R}) \to P(\mathbb{R})$ is defined by $T(f) = f'$ $\forall f \in P(\mathbb{R})$. Then $R(T) = P(\mathbb{R})$.

Then $P(\mathbb{R}) = R(T) \oplus \{0\}$. Let $W = \{0\}$. By part (a), $W \subseteq N(T)$ must be true.

Next, $N(T) = \{ f \in P(\mathbb{R}) \mid T(f) = f' = 0 \} = \{$ all the constants. $\} = \{ f \in P(\mathbb{R}) \mid f = c, c \in \mathbb{R} \}$.

Hence, $N(T) \neq W$. ∎

32. W is T-invariant. $(T_W: W \to W)$

- Given $x \in N(T_W)$., then $T_W(x) = 0 \Rightarrow x \in W$ & $x \in N(T)$. since $T_W(x) = T(x) = 0$.
- Given $x \in N(T) \cap W$, then $T(x) = 0$ & $x \in W$ $\therefore$ $\overset{W \to W}{T_W(x)} = T(x) = 0$ $\Rightarrow N(T_W) = N(T) \cap W$ ∗

- Given $x \in R(T_W)$., then $\exists y \in W$ s.t. $T_W(y) = T(y) = x \in R(T_W) \subseteq W$. but $T(y) = x$ $\therefore$ $x \in T(W)$.
- Given $x \in T(W)$., since W is T-invariant, $x \in T(W) \subseteq W$ $\overset{x \in T(W)}{\Rightarrow}$ $\exists y \in W$ s.t. $T(y) = x \in W$.

Then $T_W(\overset{W}{\overset{\cup}{y}}) = T(y) = x \in W$ $\Rightarrow$. $x \in R(T_W)$. ∎

33. "$R(T) = \text{span}(\{T(v) \mid v \in \beta\})$" under Thm 2.2.

proof. • "$\supseteq$": $T(v_i) \in R(T)$ $\forall i$. & $R(T)$ is a subspace., so $\text{span}(\{T(v) \mid v \in \beta\}) \subseteq R(T)$ ∗

• "$\subseteq$": Given $y \in R(T)$, $\exists x \in V$ s.t. $y = T(x)$. but $x = \sum_{i=1}^{k} a_i v_i$. for some $v_i \in \beta$.

So, $y = T(x) = T(\sum_{i=1}^{K} a_i v_i) = \sum_{i=1}^{k} a_i T(v_i)$. $\in \text{span}(\{T(v) \mid v \in \beta\})$. ∗

34. Given $f: \beta \to W$. Given $x \in V$, $\exists$ finitely many scalars s.t. $x = \sum a_i v_i$, for some $v_i \in \beta$.

For each $v_i \in \beta$., let $w_i = f(v_i)$. Define $T: V \to W$ by $T(x) = \sum^{n} a_i w_i$ if $x = \sum^{n} a_i v_i$ ($\therefore T(x) = f(x)$, $\forall x \in \beta$)

- T is linear: Given $x, y \in V$, $c \in F$, $T(cx + y) = T(c\sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{k} b_j u_j) = c\sum^{n} a_i w_i + \sum^{k} b_j w_j$

$= cT(x) + T(y)$ ∗

§2-1.

· T is unique: Suppose there is another linear transformation. $U: V \to W$ s.t. $U(x) = f(x)\ \forall x \in \beta$.

Given $x \in V$, $x = \sum a_i v_i$ for some $v_i \in \beta$, & for some scalars (unique) $\in \mathbb{F}$.

Then $T(x) = \sum a_i T(v_i) = \sum a_i f(v_i) = \sum a_i U(v_i) \overset{\text{linear}}{=} U(\sum a_i v_i) = U(x)$ ∎

35. V has finite dimension, $T: V \to V$ is linear.

(a) $V = R(T) + N(T)$, $\Rightarrow \dim(V) = \dim(R(T)) + \dim(R(T)) - \dim(R(T) \cap N(T))$.

By dimension thm, $\dim(V) = \text{rank}(T) + \text{nullity}(T) \Rightarrow \dim(R(T) \cap N(T)) = 0 \Rightarrow R(T) \cap N(T) = \{0$

(b) $N(T) \subseteq V$, $R(T) \subseteq V$ Then $N(T) + R(T) \subseteq V$

Note that : $\dim(N(T) + R(T)) = \dim(N(T)) + \dim(R(T)) - \dim(\overset{0}{N(T) \cap R(T)})$.

Dimension thm $\Rightarrow \dim(V) = \dim(N(T)) + \dim(R(T))$ Hence, $\dim(N(T) + R(T)) = \dim(V)$

$(N(T) \cap R(T) = \{0\})$
$\Rightarrow N(T) \oplus R(T) = V$ ∎

36. (a). V: vector space of all seq(s). (infinite-dimensional). $T: V \to V$ is defined by $T(a_1, a_2, \cdots) = (a_2, a_3, \cdots)$.

Then $R(T) = V$, $N(T) = \{(c, 0, 0, 0, \cdots) \in V \mid c \in \mathbb{F}\}$. Clearly, $V = R(T) + N(T)$.

, but $R(T) \cap N(T) \neq \{0\}$ $\therefore V \neq R(T) \oplus N(T)$ ∎

(b). Let $T_1: V \to V$ be defined by $T_1(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots)$, as in the ex 21, sec

$\therefore T_1$ is linear. $R(T) = \{(0, c_2, c_3, \cdots) \mid c_i \in \mathbb{F}\ \forall i\}$, $N(T) = \{(0, 0, 0, \cdots)\} = \{0\}$.

Then $R(T) \cap N(T) = \{0\}$, but $R(T) + N(T) \subsetneq V$ since the seq $(c, 0, 0, \cdots) \in V$ cannot be attained by $R(T) + N(T)$. ∎

37. Let $T: V \to W$ be an additive function, i.e. $T(x+y) = T(x) + T(y)\ \forall x, y \in V$.

Now, V & W are vector spaces over the field $\mathbb{Q}$.

· T is linear: given $x, y \in V$, $c \in \mathbb{Q}$, $T(cx+y) = T(\frac{q}{p}x + y)$, where $p, q \in \mathbb{Z}$.

$= T(\underbrace{\frac{1}{p}x + \frac{1}{p}x + \cdots + \frac{1}{p}x}_{q個} + y) = \underbrace{T(\overset{\in V}{\frac{1}{p}x}) + T(\frac{1}{p}x) + \cdots + T(\frac{1}{p}x)}_{q個} + T(y) = qT(\frac{1}{p}x) + T(y)$.

(Note that $\exists m \in V$ s.t. $m = \frac{1}{p}x\ (\in V) \Rightarrow pm = x \in V$.)

$= qT(m) + T(y) = q(\frac{p \cdot T(m)}{p}) + T(y) = q \cdot (\frac{T(m) + \cdots + T(m)}{p}) + T(y) = \frac{q}{p}(T(pm)) + T(y)$

$= \frac{q}{p}T(x) + T(y)$ ∎

38. $T: \mathbb{C} \to \mathbb{C}$ is defined by $T(z) = \bar{z}$.

· T is additive: Given $x, y \in \mathbb{C}$, say $x = a+bi$, $y = c+di$. Then $T(x+y) = T(a+c+i(b+d))$

$= (a+c) - i(b+d) = a - bi + c - di = T(x) + T(y)$ ※

· T is not linear: 例子: $\underset{\overset{\uparrow}{c}}{i} \cdot \underset{\overset{\uparrow}{x}}{1} \in \mathbb{C}$. $T(i \cdot 1) = T(i) = -i$, but $iT(1) = i$ ∎

39: Following the hint: Let V be the set of real numbers regarded as a vector space over the field of rational numbers. By the corollary to Thm 1.13, V has a basis β. Let $x, y \in \beta$. Define $f: \beta \to V$ by $f(x) = y$. $f(y) = x$. & $f(z) = z$, o.w.

Then by ex34, §2-1, $\exists!$ linear transformation $T: V \to V$ s.t. $T(v) = f(v)\ \forall v \in \beta$.

- Note that T is linear on V ($= \mathbb{R}$ over a field $\mathbb{Q}$). but T is not linear on $\mathbb{R}$, because if we choose $c = \frac{y}{x}$; then $T(cx) = T(y) = x$. & $cT(x) = cy$.
- T is additive on $\mathbb{R}$: Given $x, y \in \mathbb{R}$. $T(x) + T(y) = T(x+y)$. since $c = 1$ & T is linear on V. $(T(1 \cdot x + y) = 1 \cdot T(x) + T(y))$.

- To be more pratical for the problem #39,

let V be the set of $\mathbb{R}$ over $\mathbb{Q}$, as a vector space. let $\beta = \{\pi, \pi^2, \cdots\}$ be a ~~basis~~ L.I. subset for V (see ex3, §1-7) Define $f: \beta \to V$ be $\begin{cases} f(\pi) = \pi^2. \\ f(\pi^2) = \pi \\ f(\pi^n) = \pi^n, n \geq 3. \end{cases}$

Then $\exists!$ linear transformation $T: V \to V$ s.t. $T(v) = f(v)\ \forall v \in \beta$ (Actually, T is defined by $T(x) = \sum a_i f(v_i)$ if $x = \sum a_i v_i$ for some $v_i \in \beta$ & some scalars, the we can prove that such T is linear & $T(v) = f(v)\ \forall v \in \beta$ & unique).

Now, T is additive on $\mathbb{R}$ $(T: \mathbb{R} \to \mathbb{R})$, but T is not linear on $\mathbb{R}$:

e.g. $T(\overset{c}{\pi} \cdot \overset{x}{\pi^2}) = T(\pi^3) = \pi^3$. but $\pi \cdot T(\pi^2) = \pi \cdot \pi = \pi^2$. ▨

40. $\eta: V \to V/W$ by $\eta(v) = v + W$ for $v \in V$. (reference p23).

(a). η is linear: given $x, y \in V$, $c \in F$, $\eta(cx+y) = (cx+y) + W = cx + W + y + W$

$= c(x+W) + y + W = c\eta(x) + \eta(y)$ ※

- $N(\eta) = W$: $N(\eta) = \{x \in V \mid x + W = 0 + W = W\} = W$ ▨

(b). V is finite-dimensional.

Dimension Thm $\Rightarrow$. $\dim(V) = \dim(R(T)) + \dim(N(T)) = \dim(R(T)) + \dim(W)$. $-(*)$

- η is onto: given $x + W \in V/W$ $(x \in V)$, then this x is exactly the desired element in V s.t. $\eta(x) = x + W$. $\Rightarrow$. $V/W \subseteq R(T)$.

$R(T) \subseteq V/W$ is clearly $\Rightarrow$. $V/W = R(T)$ $\Rightarrow$ η is onto ※※

- $(*) \Rightarrow \dim(V) = \dim(V/W) + \dim(W)$. ▨

(C). Under the finite dimensional case, we see that dimension thm is proved by using the replacement thm. Both §1.6#35 & §2.1#40 use almost the same idea from replacement thm.

§2.2. Ch2. p.25.

1. (a). True, by Thm2.7 * (b). True, by the corollary to Thm 2.6 (p.73)

(c). False, $[T]_\beta^\gamma$ is a n×m matrix. (d). True, by thm2.7.

(e). True. (f). False, a transformation of $\mathcal{L}(V,W)$ cannot map element in W i

general*

2. 略. 3略.

4. $T: M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ is defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$.

β = std. ordered basis of $M_{2\times2}(\mathbb{R})$, i.e. $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

$\gamma = \{1, x, x^2\}$. $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1$. $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1+x^2$ $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$ $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2x$.

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{3\times4}$

5. α = std. ordered basis of $M_{2\times2}(\mathbb{R})$. : $\beta = \{1, x, x^2\}$. $\gamma = \{1\}$.

(a) $T: M_{2\times2}(\mathbb{F}) \to M_{2\times2}(\mathbb{F})$: $A \mapsto A^t$. $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4}$ *

(b) $T: P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$: $f(x) \mapsto \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. $[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{4\times3}$. *

(c). $T: M_{2\times2}(\mathbb{F}) \to \mathbb{F}$: $T(A) = tr(A)$. $[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}_{1\times4}$ *

(d). $T: P_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. $[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4. \end{pmatrix}_{1\times3}$ *

(e). $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$. $[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$.

(f). $f(x) = 3 - 6x + x^2$ $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$ *

(g). $[a]_\gamma = (a)_{1\times1}$ $\forall a \in \mathbb{F}$.

6. proof of part (b) of thm2.7, p.82: (i.e. $L(V,W)$ is a vector space).

(VS1) $(T+U)(x) = T(x)+U(x) = U(x)+T(x) = (U+T)(x)$. $\forall$ $T, U \in L(V,W)$.

(VS2). $((T+U)+W)(x) = (T+U)(x)+W(x) = T(x)+U(x)+W(x) = T(x)+(U+W)(x) = (T+(U+W))(x)$

(VS3). zero transformation, 0, is a linear transformation, $0 \in \mathcal{L}(V,W)$.

Then $(0+T)(x) = 0(x) + T(x) = T(x)$. $\forall$ $T \in \mathcal{L}(V,W)$.

(VS4). Given $T \in L(V,W)$, then $-T := U \in \mathcal{L}(V,W)$. s.t. $U+V = 0$

(VS5). Given $T \in \mathcal{L}(V,W)$. chose $c = 1 \in \mathbb{F}$, $(cT)(x) = cT(x) = T(x)$.

(VS6). Given $a, b \in \mathbb{F}$, $T \in \mathcal{L}(V,W)$. $((ab)T)(x) = (ab)T(x) = \sum_{i=1}^{ab} T(x) = a(bT(x)) = (a(bT))(x$

(VS7). Given $a \in \mathbb{F}$, $T, U \in \mathcal{L}(V,W)$, $(a(T+U))(x) = a(T+U)(x) = a(T(x)+U(x)) = aT(x)+aU($
$= (aT)(x)+(aU$
$= (aT+aU)($

(VS8). Given $a, b \in \mathbb{F}$, $T \in \mathcal{L}(V,W)$, $((a+b)T)(x) = (a+b)T(x)$.

$= aT(x) + bT(x) = (aT)(x) + (bT)(x) = (aT+bT)(x)$. ∎

7. proof of part (b) of thm 2.8, p.83. ($T, U: V \overset{\beta}{\to} \overset{\gamma}{W}$ are linear) γ: basis.

Given $a \in \mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$; $\gamma = \{w_1, \dots, w_m\}$. $\exists$ unique scalars a_{ij} s.t. $1 \le j \le n$.

$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ Then $([aT]_\beta^\gamma)_{i_0 j_0} = a \cdot a_{i_0 j_0}$ & $(a[T]_\beta^\gamma)_{i_0 j_0} = a([T]_\beta^\gamma)_{i_0 j_0}$

$= a(a_{i_0 j_0}) = a \cdot a_{i_0 j_0}$ $\forall i_0, j_0$ ※ ▨

8. $T: V \to \mathbb{F}^n: x \to [x]_\beta$.

- T is linear: given $x, y \in V$, $c \in \mathbb{F}$, By thm 2.8, $T(cx+y) = [cx+y]_\beta = [cx]_\beta + [y]_\beta = c[x]_\beta + [y]_\beta$ ▨

9. $T: V \to V: z \to \bar{z}$, where $V =: \mathbb{C}$ over the field $\mathbb{R}$. $\beta = \{1, i\}$.

- T is linear: given $z_1, z_2 \in V$, $c \in \mathbb{R}$, $T(cz_1 + z_2) = \overline{cz_1 + z_2} = \overline{cz_1} + \overline{z_2} = \bar{c}\,\bar{z_1} + \bar{z_2} = c\bar{z_1} + \bar{z_2}$ ($c \in \mathbb{R}$)

$= cT(z_1) + T(z_2)$ ※

- $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{2\times 2}$ Recall by §2.1 #38 that T is not linear if V is regarded as a vector space over the field $\mathbb{C}$.

10. $$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}.$$

11. $\dim(W) = k$, T-invariant. $\dim(V) = n$.

Given γ, a basis for W, say $\gamma = \{v_1, \dots, v_k\}$. Since $T(W) \subseteq W$, $T(v_j) \in W$ $\forall 1 \le j \le k$.

i.e. $T(v_j) = \sum a_{ij} v_i$ for the unique scalars a_{ij}. $-(*)$

By replacement thm, we can extend the basis γ for W to the basis β for V.

Let $\underset{(k\times k)}{A} =: [T]_\gamma$ Then $$[T]_\beta \overset{\text{by}(*)}{=} \begin{matrix} k\{ \\ n-k\{ \end{matrix} \begin{pmatrix} A & B \\ \underbrace{O}_{k} & C \end{pmatrix}.$$ ▨

12. $\dim(V) < \infty$. $V = \underset{(T\text{-invariant})}{W \oplus W'}$ T is the projection on W along W' $\Rightarrow T(x) = x_1$ if $x = \overset{W}{x_1} + \overset{W'}{x_2}$

Note that $T(x) = x$ $\forall x \in W$. Let γ be a basis for W, say $\dim(W) = k \le n$, and

$\gamma = \{v_1, \dots, v_k\}$. By replacement thm, $\beta =: \gamma \cup \{v_{k+1}, \dots, v_n\}$ is a basis for V, where $v_{k+1}, \dots, v_n$ belongs to W'.

Then $T(v_j) = v_j$ $\forall 1 \le j \le k$. & $T(v_j) = 0$ $\forall k+1 \le j \le n$.

Thus, $[T]_\beta = \begin{bmatrix} I_{k\times k} & O \\ O_{n-k\times k} & O \end{bmatrix}$ ※ ▨

13. $\overset{\ne 0}{T}, \overset{\ne 0}{U}: V \to W$, linear. & $R(T) \cap R(U) = \{0\}$.

Assume $aT + bU = 0$ for some not all zeros $\Rightarrow$ Given $x \in V$, $aT(x) + bU(x) = 0$

$T \not\equiv 0 \Rightarrow \exists \overset{\ne 0}{y} \in R(T) \Rightarrow \exists \overset{\in V}{x_0}$ s.t. $T(x_0) = y$. [illegible] $\Rightarrow aT(x_0) + bU(x_0) = 0 = ay + bU(x_0)$

$\underset{(\text{linear})}{\Rightarrow} U(\frac{-bx_0}{a}) = y \in R(U)$. So we found $y \ne 0 \in R(T) \cap R(U)$. $\rightarrow\leftarrow$ (※ ▨)

§2-2 Ch2. p.27.

14. $V = P(\mathbb{R})$. $T_{j\geq 1}(f(x)) = f^{(j)}(x)$.

Given $n \in \mathbb{N}$. Assume $a_1T_1 + \cdots + a_nT_n = 0$ for some scalars. $\in \mathbb{F}$., $\forall f \in P(\mathbb{R})$.

取 $f(x) = x^n$, 則 $T_1(f) = nx^{n-1}$, $T_2(f) = n(n-1)x^{n-2}$, $\cdots$, $T_n(f) = n!x^0 = n!$

Hence, we have $0 = a_1 \cdot nx^{n-1} + a_2 \cdot n(n-1)x^{n-2} + \cdots + n!$

$\Rightarrow 0 = \tilde{a}_1x^{n-1} + \tilde{a}_2x^{n-2} + \cdots + \tilde{a}_n \cdot 1 \Rightarrow \tilde{a}_i = 0\ \forall i$ since $\{1, x, \cdots, x^{n-1}\}$ is L.I.

$\Rightarrow a_i = 0\ \forall i \Rightarrow \{T_1, \cdots, T_n\}$ is L.I. ▨

15. $S^0 := \{T \in \mathcal{L}(V,W) \mid T(x) = 0\ \forall x \in S\}$.

(a) S^0 subspace: ① $0 \in S_0$ ② given $T, U \in S^0$, then $(T+U)(x) = T(x) + U(x) = 0\ \forall x \in S$.

$\Rightarrow T+U \in S_0$ (加法封閉). ③ given $c \in \mathbb{F}$, $T \in S^0$, then $(cT)(x) = cT(x) = c \cdot 0 = 0\ \forall x \in S$

$\Rightarrow cT \in S_0$. ▨

(b). $S_1 \subseteq S_2 \Rightarrow S_2^0 \subseteq S_1^0$: Given $T \in S_2^0$, then $T(x) = 0\ \forall x \in S_2 \Rightarrow T(x) = 0\ \forall x \in S_1$

since $S_1 \subseteq S_2$. $\Rightarrow S_2^0 \subseteq S_1^0$ ▨

(c). Given $T \in (V_1+V_2)^0$, then $T(x) = 0\ \forall x \in V_1 + V_2$, i.e. $x = x_1 + x_2$ for some $x_1 \in V_1$, $x_2 \in$

Now, if $y \in V_1$, then $y \in V_1 + V_2 \Rightarrow T(y) = 0$. Hence, $T \in V_1^0$ Similarly, $T \in V_2^0$

Hence, $(V_1+V_2)^0 \subseteq V_1^0 \cap V_2^0$

- For the converse, given $T \in V_1^0 \cap V_2^0 \Rightarrow T(x) = 0\ \forall x \in V_1$ or $x \in V_2$.

$\Rightarrow T = 0\ \forall x \in V_1 + V_2$. (since if $x \in V_1 + V_2$, $x = \overset{\in V_1}{x_1} + \overset{\in V_2}{x_2} \Rightarrow T(x) = T(x_1 + x_2) = T(x_1) + T(x_2) =$

☆16. $\underset{③\beta\text{ basis}}{\dim(V)} = \dim(W)$, $T: V \to W$ is linear.

proof: By dimension thm, $\dim(V) = \text{nullity}(T) + \text{rank}(T)$. Let $\bar{\beta} = \{v_1, \ldots, v_k\}$, a basis fo

By replacement thm, $\bar{\beta}$ can be extended to $\boxed{\beta} = \{v_1, \ldots, v_k\} \cup \{v_{k+1}, \ldots, v_n\}$, a basis for V.

Claim: $S := \{T(v_{k+1}), T(v_{k+2}), \cdots, T(v_n)\}$ is a basis for $R(T)$.

proof (p.70).

- generates: $R(T) \overset{\text{Thm2.2.}}{=} \text{span}(\{T(v_1), \ldots, T(v_n)\}) \overset{T(v_i)=0 \text{ since } v_i \in N(T)\ \forall i=1,\ldots,k.}{=} \text{span}(\{T(v_{k+1}), \ldots, T(v_n)\}) = \text{span}(S$
- L.I.: Assume $\sum_{k+1}^{n} b_iT(v_i) = 0 \overset{\text{linear}}{\Rightarrow} T(\sum b_iv_i) = 0 \Rightarrow \sum b_iv_i \in N(T)$.

$\Rightarrow \sum_{k+1}^{n} b_iv_i = \sum_{i=1}^{k} a_iv_i \Rightarrow a_1v_1 + a_2v_2 + \cdots - b_{k+1}v_{k+1} - b_{k+2}v_{k+2} - \cdots - b_nv_n = 0$

$\Rightarrow a_i = b_j = 0\ \forall i = 1, \ldots k,\ j = k+1, \ldots n$. since $\beta = \{v_1, \ldots, v_n\}$ is L.I. ※ (claim finished

Note that $T(v_i) = 0\ \forall i = 1, \ldots, k$ ($\because v_i \in N(T)$). —①

☆ Next we can extend S (basis for $R(T)$) to $\boxed{\gamma} = S \cup \{u_1, \ldots, u_k\}$ since $\dim(V) = \dim(W)$.

Then $T(v_j) = 0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_k + \sum_{i=k+1}^{n} \underset{1\text{ or }0.}{\boxed{\delta_{ij}}} T(v_i)$. —②

By ① & ②, we get $[T]_\beta^\gamma = \begin{bmatrix} O_{k\times k} & O \\ O & I_{n-k} \end{bmatrix}$ ▨

No.

Date §2.3 Ch2.

1.(a). F, $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$ if U, T are linear.

(b). T. (C). F, $[U(w)]_\gamma = [U]_\beta^\gamma [w]_\beta$. if U linear. (d). T.

(e). F, $T^2_{(x)} = T(T(x))$. 沒這種東西, 因 $T: V \to W$. (除非 $W \subseteq V$).

When it it true? If $\alpha = \beta$ (basis for V = basis for W), then

$T^2(v_j) = T(T(v_j)) = T(\sum_i a_{ij} v_i) = \sum_i a_{ij} (\sum_k a_{ki} v_k) \Rightarrow A \times [T(v_j)] = [T]_\alpha^\beta [T]_\alpha^\beta [v_j]_\alpha$.

$[T^2]_\alpha [v_j]_\alpha$. $\underset{\text{Thm2.6.}}{\overset{\text{unique.}}{\Rightarrow}}$ $[T^2]_\beta = ([T]_\beta)^2$.

Note that T^2 is linear : $T^2(cx+y) = T(T(cx+y)) = T(cT(x)+T(y)) = cT^2(x) + T^2(y)$ ※

(f). F, 令 $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x,y) = (y,x)$. Then $[T]_\alpha^\alpha =: A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then $T^2(x,y) = I \Rightarrow A^2 = I$, but $A \neq I$ or $-I$. ※

(g). ~~T~~. F. note that $T: V \to W$. but $L_A: F^n \to F^m$. 是 isomorphism 但 不見得相同.

E.g. Let $T: M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a+b) + (2d)x + bx^2$. (see p.84 #4).

Hence, $[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{3\times4} := A$.

Use this A to define $L_A: F^4 \to F^3$ by $L_A(x) = Ax$, $(x \in F^4)$.

Then $\underbrace{[T(\overset{\in M_{2\times2}(\mathbb{R})}{M})]_\gamma}_{3\times1} = \underbrace{[T]_\beta^\gamma}_{3\times4} \underbrace{[M]_\beta}_{4\times1} := A \cdot x = L_A(x)$. ※

(h). 令 $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$ by $T(ax+b) = a$. (i.e. differentiation).

$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{2\times2} := A$. Then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = O$ but $A \neq O$.

(i). True, $\forall x$, $L_{A+B}(x) = (A+B)(x) = Ax + Bx = L_A(x) + L_B(x)$ ▨

(j). True.

9. $U: F^2 \to F^2$ is defined by $(a,b) \mapsto (b,b)$. $[U]_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} := A$.

$T: F^2 \to F^2$ is defined by $(a,b) \mapsto (a+b, 0)$. $[T]_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} := B$.

Then given $x \in F^2$, $UT(x) = UT(a,b) = U(a+b, 0) = (0,0)$ $\forall a, b \in F \Rightarrow UT = T_0$.

& $TU(x) = T(b,b) = (2b, 0) \neq T_0(a,b)$. Also, $AB = O$, but $BA = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

10. ($\Rightarrow$) A is diagonal $\Rightarrow A_{ij} = \delta_{ij} A_{ij}$.

($\Leftarrow$). $A_{ij} = \delta_{ij} A_{ij}$. $\Rightarrow$ $A_{ij} = 0$ if $i \neq j$. $\Rightarrow$ A is diagonal ▨

11. ($\Rightarrow$). $T^2 = T_0$ & T is linear: Given $y \in R(T)$, $\exists \overset{\in V}{x}$ s.t. $y = T(x)$. Then $T(y) = T(T(x))$

$= T^2(\overset{\in V}{x}) = T_0(x) = 0 \Rightarrow R(T) \subseteq N(T)$. ※

($\Leftarrow$). $R(T) \subseteq N(T)$ & T is linear. Given $x \in V$, $\exists y$ s.t. $T(x) = y \in R(T)$. Then $T^2(x) = T(T(x))$

$= T(y) \overset{\downarrow}{=} 0 = T_0(x)$ ▨

§2.3

12.(a). $\overset{(\text{linear})}{U^{Z}_{W}T^{W}_{V}}$ is 1-1. • T is 1-1: if $T(x)=T(y)$ for $x, y \in V$, then $U(T(x))=U(T(y))$.
$\Rightarrow UT(x)=UT(y)$, but UT is 1-1 $\Rightarrow x=y$. ▨

• U "may not" be 1-1: (pf) If $U(m)=U(n)$ $m,n \in R(T) \subseteq W$, then $\exists\, x, y \in V$ s.t. $T(x)=m$.
& $T(y)=n \Rightarrow U(T(x))=U(T(y)) \Rightarrow UT(x)=UT(y) \Rightarrow x=y$ since UT is 1-1.
$\Rightarrow T(x)=m=T(y)=n$. (We found that U must be 1-1 on R(T).)

但在 W − R(T) 的地方亂搞的話, U 就不會在整個 W 上 1-1.

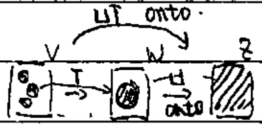
Counter E.g. $W=R(T)\oplus(R(T)^c)$ by replacement thm (i.e. $\beta=\{v_1,\dots,v_k\}$ is a basis for R(T), suppose $\dim(W)=n$. then replacement thm $\Rightarrow \exists\, \alpha=\{v_{k+1},\dots,v_n\}$ s.t. $\alpha\cup\beta$ is a basis for W and thus this α would satisfy $\operatorname{span}(\alpha)\cap\operatorname{span}(\beta)=\{0\}$. ∴ we have a sub-space $\underset{(R(T))^c}{S}$ s.t. α is a basis for S & $R(T)\oplus S=W$) (see p.58 (§1.6) #34).

令 $U: W\to Z$ by $U(x)=x_1$ for $x^{\in W}=x_1^{\in R(T)}+x_2^{\in R(T)^c}$.

∵ U is linear: (p.77 #26) given $x, y \in V$, $c\in F$, $\exists\, x_1, y_1 \in R(T)$ & $x_2, y_2 \in R(T)^c$ s.t. $x=x_1+x_2$, $y=y_1+y_2$.
Then $U(cx+y)=U(cx_1+y_1+cx_2+y_2)=cx_1+y_1=cU(x)+U(y)$ ※

∵ U is not 1-1: Choose $x=x_1\in R(T)$. & $y=x+\overset{\neq 0}{y_2}$, $\overset{\neq 0}{y_2}\in(R(T))^c$, then
$U(x)=x_1=U(y)$ but $x\neq y$. ▨

(b). UT is onto. $\Rightarrow$ given $z\in Z$, $\exists\, v\in V$ s.t. $UT(v)=z=U(T(v))$. That means,
$\exists\, y=T(v)$ s.t. $U(y)=z$ $\forall z$. ▨



∵ T may not be onto:
E.g. 令 $V=\mathbb{R}^2$, $W=\mathbb{R}^2$. $Z=\mathbb{R}$. & define $T:\mathbb{R}^2\to\mathbb{R}^2$ by $T(a,b)=(a,0)$. (linear).
& $U:\mathbb{R}^2\to\mathbb{R}$ by $U(a,b)=a+b$ (linear)
Then UT is onto. & U is onto, but T is not onto. ($R(T)=\{(x,0)\in\mathbb{R}^2\}$).

(C). U & T are 1-1 & onto. $\Rightarrow \dim(V)=\text{nullity}(T)+\text{rank}(T)=0+\dim(W)$. &
$\dim(W)=\text{nullity}(U)+\text{rank}(U)=0+\dim(Z)$ $\Rightarrow \dim(V)=\dim(W)=\dim(Z)$.
Now, let $L=UT$. By dimension thm, $\dim(V)=\text{nullity}(L)+\text{rank}(L)$.
If we have $\text{rank}(L)=\dim(W)$, then $UT=L$ is 1-1 ※.
Claim $\text{rank}(L)=\dim(W)$.
pf: given $z\in Z$, $\exists\, w_0\in W$ s.t. $\overset{(\text{onto})}{U(w_0)=z}$. also, $\exists\, v\in V$ s.t. $\overset{(\text{onto})}{T(v)=w_0}$ for this w_0
Then $UT(v)=U(T(v))=U(w_0)=z$. $\Rightarrow$ UT is onto. ▨

13. $A,B\in M_{n\times n}(\mathbb{F})$. "$tr(AB)=tr(BA)$":

• $tr(AB)=\sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n}A_{ik}B_{ki}\right) \bullet \sum_{k=1}^{n}\left(\sum_{i=1}^{n}B_{ki}A_{ik}\right) = \sum_{k=1}^{n}(BA)_{kk} = tr(BA)$ ▨

14.(a) • $z\in\mathbb{F}^p$: "Bz is a linear combination of the columns of B; in particular, $Bz=\sum a_jv_j$ if $z=\begin{pmatrix}a_1\\ \vdots\\ a_p\end{pmatrix}$".

pf. By Thm2.13(b), we know that $v_j=Be_j$, where v_j denotes the jth column of B. Then $\underset{m\times p\ p\times 1}{Bz}=\begin{pmatrix}\sum_k B_{1k}a_k\\ \sum_k B_{2k}a_k\\ \vdots\\ \sum_k B_{nk}a_k\end{pmatrix}_{n\times 1} = \sum_k a_k\begin{pmatrix}B_{1k}\\ B_{2k}\\ \vdots\\ B_{nk}\end{pmatrix} = \sum_{k=1}^{p}a_kv_k$. ▨

(b). $u_j:=\begin{pmatrix}\overset{2\times n\ n\times p}{(AB)_{1j}}\\ \vdots\\ (AB)_{gj}\end{pmatrix} = \begin{pmatrix}\sum A_{1k}B_{kj}\\ \sum A_{2k}B_{kj}\\ \vdots\\ \sum A_{gk}B_{kj}\end{pmatrix} = \sum B_{kj}\cdot\begin{pmatrix}A_{1k}\\ A_{2k}\\ \vdots\\ A_{gk}\end{pmatrix} =: \sum_{k=1}^{n}B_{kj}\times S_k$ ▨

(c). $w\in\mathbb{F}^m$. Note that $wA=(A^tw^t)^t$. 令 $A^t=B$ & $w^t=z$. By part (a),

$Bz=\sum a_kv_k$, where v_k denotes the jth column of B.

Now, wA is the transpose of Bz, hence, $wA=\sum a_k(v_k)^t$, i.e. wA is the linear combination of the rows of A with coeffi. a_k being the coordinates of w. ▨

(d). $AB=(B^tA^t)^t \Rightarrow (AB)_{row\,i}=(B^tA^t)_i$. 令 $B^t=C$, $A^t=D$

By part (b), $(B^tA^t)_i=(CD)_i=\sum_k D_{ki}S_i$, where S_i denotes the ith column of $\underset{\parallel\,B^t}{C}$.

Then $\overset{(AB)_{row\,i}}{\cancel{(AB)_{row\,i}}} = \sum D_{ki}S_i = \sum_k A_{ik}S_i$, where $\overset{\parallel}{S_i}$ is the ith row of B ▨

15. $\underset{n\times p}{A}:=(v_1\ v_2\ \cdots\ v_p)$, $M\in M_{m\times n}(\mathbb{F})$. 令 $\underset{m\times p}{MA}=(u_1\ \cdots\ u_p)$.

Suppose $v_j=\sum_{k\in S}a_kv_k$ for some subset $S\subset\{1,2,\dots,p\}\setminus\{j\}$.

$u_j=Mv_j$ (by thm 2.13(a)) $= M\left(\sum_{k\in S}a_kv_k\right)=\sum_{k\in S}a_k(Mv_k)=\sum_{k\in S}a_ku_k$. ▨

16. (a). $rank(T)=rank(T^2)$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "T^2\cdot\text{ linear}"'] & \overset{\text{linear}}{V} \arrow[r, "T"] & V \end{tikzcd}$$

Note that T^2 is also linear. By dimension thm, $nullity(T)=nullity(T^2)$. ①

If $x\in N(T)$, $\Rightarrow T(x)=0 \Rightarrow T^2(x)=T(T(x))=T(0)=0 \Rightarrow x\in N(T^2)$. Hence, $\underbrace{N(T)=N(T^2)}_{(*)}$ by ①.

Now, if $y\in N(T)\cap R(T)$, then $\exists\, x\in V$, s.t. $T(x)=y$ & $T(y)=0$ i.e. $T^2(x)=0$

$\Rightarrow x\in N(T^2) \Rightarrow x\in N(T)$ by (*). $\Rightarrow T(x)=0=y \Rightarrow y=0$. $\Rightarrow N(T)\cap R(T)=\{0\}$.

Next, $\dim(R(T)+N(T))=\dim(R(T))+\dim(N(T))-\cancelto{0}{\dim(R(T)\cap N(T))} = \dim(V)$.

$\Rightarrow V=R(T)\oplus N(T)$. ▨

<Another idea> • $R(T)$ is a T-invariant space. $(T(R(T))\subseteq R(T))$.

• $\dim(R(T))=rank(T)\overset{題目}{=}rank(T^2)=\dim(T(T(V)))=\dim(T(R(T)))=rank(T|_{R(T)})$.

$\Rightarrow T|_{R(T)}$ is onto, and thus, 1-1. $(\dim(R(T))=nullity(T|_{R(T)})+rank(T|_{R(T)}))$.

$\Rightarrow N(T|_{R(T)})=\{0\}=N(T)\cap R(T)$. ▨

※16(b). In general $\mathrm{rank}(T^{k+1}) \le \mathrm{rank}(T^k)$. (Since $T^{s+1}(V) = T^s(R(T)) \subseteq T^s(V)$.).

But the integer: $\mathrm{rank}(T^s) \in \{0,1,2,\cdots,\dim(V)\} \Rightarrow \exists k$ s.t. $\mathrm{rank}(T^k) = \mathrm{rank}(T^{k+1})$.

$\Rightarrow T^{k+1}(V) = T^k(V)$. $\overset{T(T^k(V))=(T^k(V))}{\Rightarrow}$. $T^s(V) = T^k(V)\ \forall s \ge k$. Then $T^m(T^k(V)) = T^{m-1}(T^k(V)) = \cdots = T^k(V)$. Choose $s=2k$, $T^{2k}(V) = T^k(V)$.

By part(a), $V = R(T^k) \oplus N(T^k)$ ■

※17. Claim: $V = \{y \mid T(y)=y\} \oplus N(T)$. if $T^2 = T$.

pf. • If $y \in R(T)$. $T(y) = T(T(\overset{\in V}{x}))$ for some x. $= T^2(x) \overset{T^2=T}{=} T(x) = y$; $\therefore R(T) \subseteq \{y \mid T(y)=y\}$

need V finite dim — If $T(y) = y \in V$, then $y \in R(T)$. $\Rightarrow R(T) = \{y \mid T(y)=y\}$

• $\mathrm{rank}(T^2) = \mathrm{rank}(T)$. $\Rightarrow$(#16). $R(T) \cap N(T) = \{0\}$ & $V = R(T) \oplus N(T)$. ※

• $\overset{\in V}{x} = T(x) + (x - T(x))$. Note that $T^2(x) = T(T(x)) = T(x)$. $\therefore T(x) \in \{y \mid T(y)=y\}$.

and $T(x - T(x)) = T(x) - T^2(x) = 0 \Rightarrow$. $x - T(x) \in N(T)$ $\Rightarrow V = \{y \mid T(y)=y\} + N(T)$.

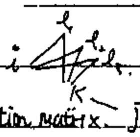
• If $x \in \{y \mid T(y)=y\} \cap N(T)$, then $T(x)=x$ & $T(x)=0 \Rightarrow x=0 \Rightarrow V = \{y \mid T(y)=y\} \oplus N($

Hence, if $T^2=T$, then T must have the property that $R(T) = \{y \mid T(y)=y\}$.

$\therefore$ T must be the projection on W_1 along W_2. for some W_1 & W_2 s.t. $V = W_1 \oplus W_2$. ■

18. 略.

※19. $B^3_{ij} = \sum_k B^2_{ik} B_{kj} = \sum_k \left(\sum_l B_{il} B_{lk} \right) B_{kj}$.

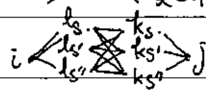


Explaination: $B^2_{ik} = \sum_l B_{il} B_{lk}$ is the two stage connection matrix.

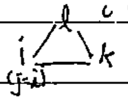
If $B^2_{ik} = C \in$ const., that means there are C people $(l_1, \cdots l_c)$ s.t. i can connect (雙向) to each of the C people, and then connect to k. In figure: $i \leftrightarrow l_1 \leftrightarrow k$. or $i \leftrightarrow l_2 \leftrightarrow k$ ······· $i \leftrightarrow l_c \leftrightarrow k$ ※

共S種.

Now, $B^3_{ij} = S$. means that



$\Rightarrow (B^3)_{ii}$



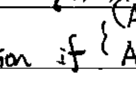
$\therefore$ To see that a person i is belonging to a clique. we found the value $(B^3)_{ii}$. ■

20.(a). $A = \begin{pmatrix} 0&1&0&1 \\ 1&0&0&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 0&1&0&1 \\ 1&0&0&0 \\ 0&0&0&1 \\ 1&0&1&0 \end{pmatrix} \Rightarrow B^2 = \begin{pmatrix} 2&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&2 \end{pmatrix} \Rightarrow B^3 = \begin{pmatrix} 5&0&3&0 \\ 0&3&0&3 \\ 3&0&2&0 \\ 0&3&0&5 \end{pmatrix} \Rightarrow$ Every person belon

to a clique for some clique ※

(b). $A = \begin{pmatrix} 0&0&1&1 \\ 1&0&0&1 \\ 1&0&0&1 \\ 1&0&1&0 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 0&0&1&1 \\ 0&0&0&0 \\ 1&0&0&1 \\ 1&0&1&0 \end{pmatrix} \Rightarrow B^2 = \begin{pmatrix} 2&0&1&1 \\ 0&0&0&0 \\ 1&0&2&1 \\ 1&0&1&2 \end{pmatrix} \Rightarrow B^3 = \begin{pmatrix} 6&0&5&5 \\ 0&0&0&0 \\ 5&0&6&5 \\ 5&0&5&6 \end{pmatrix} \Rightarrow$ Except the peron

every peron belongs to a clique ※



$\therefore$ clique $= \{1,3,4\}$ (only one) ※

21. An incidence matrix A is associated with a dominance relation if $\begin{cases} (A_{ii} = 0\ \forall i). \\ A_{ij} = 1 \Leftrightarrow A_{ji} = 0.\ \forall i \ne j. \end{cases}$

Sol: $(A + A^2)_{ij} > 0$ is equivalent to that i can reach j within two steps.

(看詳解). Claim: Every tournament has a king (i.e. v can reach all other vertices within two steps).

pf. Given arbitrary vertex v_1. If v_1 is a king, then we've done.

If v_1 is not a king, $\exists v_2$ s.t. v_1 cannot reach v_2 within 2 steps.

- $\because v_1 \not\to v_2$ $\therefore v_2 \to v_1$. (property of dominance relation).
- If $v_1 \to W$ for some W, then $v_2 \to W$, o.w. we'll have $v_2 \not\to W \Rightarrow W \to v_2$ $\Rightarrow \cancel{v_1 \to W \to v_2}$.

Hence, $d^+(v_2) > d^+(v_1)$. (dominate 的数量比較多).

Continuing the process, we have $d^+(v_1) < d^+(v_2) < \cdots d^+(v_k)$ for some k. (since there are only finite vertices) $\Rightarrow$ v_k is the king. ▨

22. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ Graph:  $\therefore$ every vertex is a king! ▨

23. #(nonzero entries of A) $= \frac{n^2-n}{2} = \frac{n(n-1)}{2}$ ▨

§ section 2.4.

1. (a). False, 1° T is not necessary invertible 2° If T is invertible, $([T]_\alpha^\beta)^{-1} = [T^{-1}]_{\boxed{\beta}}^{\boxed{\alpha}}$ ☆

(b). True. proof: $(\Rightarrow)$ $\overset{:V\to W}{T}$ invertible $\Rightarrow \exists \overset{:W\to V}{g}$ s.t. $T \circ g = I_W$ & $g \circ T = I_V$.

$\because$ T is 1-1: If $T(x) = T(y)$ for some $x, y \in V$, then $gT(x) = gT(y)$.

$\Rightarrow x = y$ ☆

- T is onto: given $y \in W$. we know that $g(y) = z \in V$ for some z.

Also, $T(g(y)) = T \circ g(y) = I_W(y) = y$. $\Rightarrow$ $T(\overset{\in V}{z}) = y$. ☆
$\quad = T(z)$

Hence, we conclude that given $y \in W$, $\exists z \in V$ s.t. $T(z) = y$. ☆☆

$(\Leftarrow)$. T is 1-1 & onto. $\Rightarrow$ For any $b \in W$, $\underset{\text{onto}}{\exists}\underset{\text{1-1}}{!}$ $a \in V$ s.t. $T(a) = b$.

Now, define $g: W \to V$ s.t. $g(b) = a$, if $T(a) = b$.

Then, $T \circ g = I_W$ & $g \circ T = I_V$ $\Rightarrow$ T is invertible ▨

(c). False, L_A can only map F^n to F^m. but T maps V to W ☆

(d). False, $M_{2\times3}(F) \cong F^6$.

(e). True, by Thm 2.19.

(f). False, 反例: $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{2\times3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}_{3\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. ☆

(g). True. (h). True. $\because$ A invertible $\underset{\text{Ex11}}{\Leftrightarrow}$ $\exists A^{-1}$ $\Leftrightarrow$ $\exists L_{A^{-1}}$ $\Leftrightarrow$ L_A invertible ☆

(i) True, by lemma ☆

§2-4.

4. • AB is invertible: Since A, B are invertible, both A & B are 1-1 & onto, so AB is also 1-1 & onto ⇒ AB is invertible *

• $(AB)^{-1} = B^{-1}A^{-1}$: 令 $C = B^{-1}A^{-1}$. Then $C(AB) = B^{-1}A^{-1}AB = I_n$ & $(AB)C = ABB^{-1}A^{-1} = I_n$. Hence, $(AB)^{-1} = C = B^{-1}A^{-1}$ * ▨

5. • A^t is invertible: A invertible ⇒ $A \in M_{n\times n}(\mathbb{F})$ for some $n \in \mathbb{N}$. & A is 1-1 & onto, then $\text{nullity}(A^t) = \text{nullity}(A) = 0$ & $\text{rank}(A^t) = \text{rank}(A) = n$.

⇒ A^t is also 1-1 ; onto ▨

• $(A^t)^{-1} = (A^{-1})^t$: 令 $C = (A^{-1})^t$. Then $C(A^t) = (A^{-1})^t A^t = (AA^{-1})^t = I_n$

& $(A^t)C = A^t(A^{-1})^t = (A^{-1}A)^t = I_n$ So, $(A^t)^{-1} = (A^{-1})^t$ ▨

6. A is invertible & $AB = O$, then $B \overset{?}{=} O$.

Sol: $\exists A^{-1}$. $A^{-1}(AB) = A^{-1}O = O$ ⇒ $B = O$. ▨

7. A is $\overset{\text{be an}}{n\times n}$ matrix.

(a). $A^2 = O$. Suppose A is invertible, ⇒ $A^{-1}A^2 = A^{-1}O = O$ ⇒ $A = O$ but O is not invertible ⨯ ▨

(b). $AB = O$ for some nonzero $n\times n$ matrix. B. If A is invertible, $A^{-1}AB = A^{-1}O = $
⇒ $B = O$ ⨯. Hence, A couldn't be invertible.

Explanation: 把 A 看作 linear transformation, B 是 domain 裡的一個元素.

則 $AB = O$ ⇒. $B \in N(A)$. 且 $B \neq O$, 故 A 不可能 1-1. ⇒ 不可能 invertible,

8. $\overset{:V\to V}{T}$ linear. V is finite-dimensional. "T invertible ⇔ $[T]_\beta$ invertible"

(⇒) $\exists T^{-1} : V \to V$ s.t. $TT^{-1} = T^{-1}T = I_V$. Let $\dim(V) = n$:

We have $I_n = [I_V]_\beta = [TT^{-1}]_\beta = [T]_\beta[T^{-1}]_\beta$. ;

$I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\beta[T]_\beta$. . .

Hence, $[T]_\beta$ is invertible & $([T]_\beta)^{-1} = [T^{-1}]_\beta$ ☆

(⇐). 令 $\underset{(n\times n)}{A} = [T]_\beta$. $\exists B$ s.t. $AB = BA = I_n$. By thm 2.6., $\exists\, U \in \mathcal{L}(V,V)$.

s.t. $U(v_j) = \sum_{i=1}^{n} B_{ij} v_j$. for $j = 1, 2, \dots, n$. , where $\beta = \{v_1, v_2, \dots, v_n\}$.

⇒ $[U]_\beta = B$.

Now, $[UT]_\beta = [U]_\beta[T]_\beta = BA = I_n$. & $[TU]_\beta = [T]_\beta[U]_\beta = AB = I_n$.

⇒ $U = T^{-1}$ and. T invertible ▨.

Similar proofs for corollary 2. ☆ ▨

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "L_B"] \arrow[rr, bend left, "L_{AB}"] & \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^n \end{tikzcd}$$

9. • $\underset{(n\times n)}{AB}$ is invertible & $A, B \in M_{n\times n}(\mathbb{F})$. ⇒ L_{AB} is invertible $L_{AB}(= L_A L_B)$ is 1-1 & onto. ⇒ L_B is 1-1 & L_A is onto by §2-3 #12.

but $n = \overset{0}{\cancel{\text{nullity}(L_B)}} + \text{rank}(L_B)$; $n = \text{nullity}(L_A) + \overset{n}{\text{rank}(L_A)}$ ⇒ $\text{nullity}(L_A) = 0$.

So, L_B is onto & L_A is 1-1.

Hence, both L_A & L_B are 1-1 & onto. ⇔ L_A, L_B are invertible.

⇔ A, B are invertible ▣

• Give an example to show that arbitrary matrices A, B need not be invertible if AB is invertible.

Sol. (Idea: want AB invertible, so B is 1-1 & A is onto must hold. 再避免 square matrix 即可.)

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{2\times 3}$; $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}_{3\times 2}$

Then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ is invertible ※

10 略.

11. see p.103 e.g 5. $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is defined by $T(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}$.

~~* Lagrange interpolation :~~

~~choose $\beta = \{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$. Then $T(\beta) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 27 & 64 \end{pmatrix} \right\}$.~~

~~$$\underset{(*)\ (\exists!)}{T(f)} := \underbrace{\frac{(f-x)(f-x^2)(f-x^3)}{(1-x)(1-x^2)(1-x^3)}}_{a_1} \times \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{w_1} + \underbrace{\frac{(f-1)(f-x^2)(f-x^3)}{(x-1)(x-x^2)(x-x^3)}}_{a_2} \times \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{w_2}$$~~

~~$$+ \frac{(f-1)(f-x)(f-x^3)}{(x^2-1)(x^2-x)(x^2-x^3)} \times \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix} + \frac{(f-1)(f-x)(f-x^2)}{(x^3-1)(x^3-x)(x^3-x^2)} \times \begin{pmatrix} 1 & 8 \\ 27 & 64 \end{pmatrix}. \quad ※$$~~

~~• Note that $T(\beta)$ is L.I. so if $T(f) = 0$ for some $f \in P_3(\mathbb{R})$, then each coefficent must be zero. and $f(1) = f(2) = f(3) = f(4) = 0$.~~

錯誤用法! In general, 這種造法會使得 a_i 是向量 in $P_3(\mathbb{R})$ 而不是一個純量.

如果 $\beta = \{v_i\}_{i=1}^{n}$ 且 $T(v_i) = w_i$ $\forall i = 1, \ldots, n$. 一般而言, by thm 2.6 的造法,

$\exists!$ T s.t. $T(f) = T(\sum a_i v_i) = \sum a_i T(v_i) = \sum a_i w_i$. —① (唯一!!).

照 (*) 的造法, 但又要滿足 Thm 2.6 的式子①, 則 $a_j = \frac{\prod_{i\neq j}(f-v_i)}{\prod_{i\neq j}(v_j-v_i)}$ 必成立 $\forall j$.

顯而易見的, 此式子不見得存在 (向量內積?), 就算右邊式子存在, 等號也不見得成立!

§2-4.

#續 11. $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, $T(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}$.

① Lagrange method. 令 $f_j(x) = \dfrac{\prod_{i\neq j}^{4} (x-c_i)}{\prod_{i(\neq j)} (c_j-c_i)}$, where $c_1=1$, $c_2=2$, $c_3=3$, $c_4=4$. $(\in \mathbb{R})$.

Then $\beta=\{f_1, f_2, f_3, f_4\}$ is a basis for $P_3(\mathbb{R})$. Let $w_i = T(f_i)\ \forall i$.

By Thm2.6, $\exists!\ T^{(\text{linear})}$ s.t. $T(f) = \sum a_i T(f_i) = \sum a_i w_i$, where $f = \sum a_i f_i$.

Now, if $T(f)=0$ for some $f \in P_3(\mathbb{R})$, then $0 = \sum a_i T(f_i) =$

$a_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $\Rightarrow$ $a_i = 0\ \forall i$ since $\gamma := \{T(f_i) \mid i=1,\dots,4\}$ is a basis for $M_{2\times 2}(\mathbb{R})$. Hence, $f=0$ if $T(f)=0$.

12. (proof of thm 2.21). Say $\dim(V)=n$. β is an ordered basis for V.

- $\phi_\beta : V \to \mathbb{F}^n$ is defined by $x \mapsto [x]_\beta$. By Thm2.8 (p.82), ϕ_β is linear.
- ϕ_β is 1-1 since $\phi_\beta(x) = [x]_\beta = 0_{\mathbb{F}^n} \Leftrightarrow x = 0_V$.
- By dimension thm, $\dim(V) = \dim(\mathbb{F}^n) = \overset{0}{\text{nullity}(\phi_\beta)} + \text{rank}(\phi_\beta)$, $\Rightarrow$ ϕ_β is onto ※

Hence, ϕ_β is invertible, hence an isomorphism. ▨

13. equivalence relation. ① $x \sim x$ ② $y \sim x$ whenever $x \sim y$. ③ If $x \sim y$ & $y \sim z$, then $x \sim z$.

I am lazy ※

14. $V = \left\{\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \middle| a,b,c \in \mathbb{F}\right\}$. Let $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ be the basis (ordered) for

Then $\phi_\beta : V \to \mathbb{F}^3$ is defined by $x^{\in V} \mapsto [x]_\beta$. By Thm2.21, ϕ_β is an isomorphism ▨

$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

15. $\dim(V) = \dim(W) = n$. $T: V \to W$ is linear. β is a basis for V. Prove that "T is an isomorphism $\Leftrightarrow$ $T(\beta)$ is a basis for W."

proof. ($\Rightarrow$). T is an isomorphism $\Rightarrow$ T 1-1 & onto. Let $\beta = \{v_1, \dots, v_n\}$ &

$T(v_i) = w_i\ \forall i = 1, \dots, n$. Assume $0 = \sum_{i=1}^{n} a_i w_i$, then $0 = \sum a_i w_i = T(\sum a_i v_i) \overset{(1-1)}{\Rightarrow} \sum a_i v_i = 0$

$\overset{(\beta \text{ is L.I.})}{\Rightarrow} a_i = 0\ \forall i$. $\Rightarrow$ $T(\beta)$ is a L.I. subset. Now, given $w \in W$, $\overset{(\text{onto})}{\exists}\, x \in V$ s.t. $T(x) = w$

In fact, ∃! not all zero scalars s.t. $x = \sum a_i v_i$. ∴ $w = T(x) = \sum a_i w_i$.

Hence, $T(\beta)$ generates W. ※

($\Leftarrow$) $T(\beta) =: \{w_1, \dots, w_n\}$ is a basis for W.

- If $T(x) = 0$ for some $x \in V$, then $0 = T(x) = T(\sum a_i v_i) = \sum a_i T(v_i) = \sum a_i w_i$.

  $\Rightarrow a_i = 0\ \forall i$ since $T(\beta)$ is L.I. $\Rightarrow N(T) = 0 \Rightarrow T$ is 1-1.
- Given $w \in W$, $w = \sum a_i w_i$ for some scalars. $\Rightarrow$ $w = \sum a_i w_i = \sum a_i T(v_i) = T(\sum a_i v_i)$

  $= T(x)$ for some x. $\Rightarrow$ T is onto. (OR, by dimension thm, $\underset{n}{\dim(V)} = 0 + \text{rank}(T)$) ▨

16. $B \in M_{n\times n}(\mathbb{F})$, & B is invertible. $\Phi: M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ is defined by $A \mapsto B^{-1}AB$.

(proof of Φ being an isomorphism):

- If $\Phi(A) = B^{-1}AB = 0$ for some A, then $B(B^{-1}AB)B^{-1} = B0B^{-1} \Rightarrow A = 0$. ∴ Φ is 1-1.
- ∵ Dimension Thm $\Rightarrow$ $\dim(M_{n\times n}(\mathbb{F})) = \overset{0}{\cancel{\text{nullity}(\Phi)}} + \text{rank}(\Phi) \Rightarrow \Phi$ is onto. ▨

17. (a). $T: V \to W$ is an isomorphism. V_0 is a subspace of V.

Let $\overset{\{v_1,\dots,v_k\}}{\beta}$ be an ordered $\underset{\{v_1,\dots,v_n\}.}{\text{basis}}$ for V_0. By replacement thm, β can be extended to a basis for V, say α. By §2-4 #15 (p.108), $T(\alpha)$ is a basis for W. & $T(\alpha) = T(\beta) \amalg T(\alpha\setminus\beta)$ Now, we claim that $T(\beta)$ is a basis for $T(V_0)$:

- Given $y \in T(V_0)$. $\exists x \in V_0$ s.t. $T(x) = y$, but $x \in V_0 \Rightarrow x = \sum_{i=1}^{k} a_iv_i$ Then $T(x) = T(\sum^{k} a_iv_i) = \sum a_iT(v_i) \in \text{span}(T(\beta))$.
- Assume $0 = \sum^{k} a_iT(v_i)$, $\Rightarrow 0 = T(\sum a_iv_i) \overset{T\ 1\text{-}1}{\Rightarrow} 0 = \sum a_iv_i \overset{\beta\ \text{L.I.}}{\Rightarrow} a_i = 0\ \forall i \Rightarrow T(\beta)$ is L.I.

Hence, $T(\beta)$ is a basis for $T(V_0)$ $\Rightarrow$ $T(V_0)$ is a subspace of W since $T(\beta) \subseteq T(\alpha)$. ▨

(b). $\dim(V_0) = \#\beta = \#T(\beta) = \dim(T(V_0))$. ▨

18. $p(x) = 1+x+2x^2+x^3$. We show that $L_A\phi_\beta(p(x)) = \phi_\gamma T(p(x))$. $(T(f) = f')$.

$$\begin{tikzcd} P_3(\mathbb{R}) \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & P_2(\mathbb{R}) \arrow[d, "\phi_\gamma"] \\ \mathbb{F}^4 \arrow[r, "L_A"'] & \mathbb{F}^3 \end{tikzcd}$$

$\text{LHS} = L_A\begin{pmatrix}1\\1\\2\\1\end{pmatrix} = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}1\\1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\4\\3\end{pmatrix}$.

$\text{RHS} = \phi_\gamma(1+4x+3x^2) = \begin{pmatrix}1\\4\\3\end{pmatrix}$. ▨

19. $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is defined by $M \mapsto M^t$. $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$.

(a). $[T]_\beta = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$

(b). $A := [T]_\beta$. $M := \begin{pmatrix}1&2\\3&4\end{pmatrix}$. Then $L_A\phi_\beta(M) = L_A\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}1\\3\\2\\4\end{pmatrix}$. $\phi_\beta T(A) = \phi_\beta\begin{pmatrix}1&3\\2&4\end{pmatrix} = \begin{pmatrix}1\\3\\2\\4\end{pmatrix}$ ▨

20. - $\text{rank}(T) = \text{rank}(L_A)$: T linear $\Rightarrow$ $R(T)$ is a subspace of W.

$$\begin{tikzcd} \overset{ⓝ}{V} \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & \overset{ⓜ}{W} \arrow[d, "\phi_\gamma"] \\ \mathbb{F}^n \arrow[r, "L_A"', "A = [T]_\beta^\gamma\ m\times n"'] & \mathbb{F}^m \end{tikzcd}$$

ϕ_β & ϕ_γ are iso. $\overset{§2\text{-}4\ \#17}{\Rightarrow}$ $\dim(V) = \dim(\mathbb{F}^n)$ and $\dim(R(T)) = \dim(\phi_\gamma(R(T))) = \dim(\phi_\gamma T(V))$. but we know that

$\phi_\gamma T(x) = [T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta = L_A[x]_\beta = L_A\phi_\beta(x)$, $\forall x \in V$.

Hence, $\dim(\phi_\gamma T(V)) = \dim(L_A\phi_\beta(V)) = \dim(L_A(\mathbb{F}^n)) = \dim(R(L_A))$. ▨

- $\text{nullity}(T) = \text{nullity}(L_A)$: By dimension thm, $n = \text{nullity}(T) + \text{rank}(T) = \text{nullity}(L_A) + \text{rank}(L_A)$.

$\Rightarrow$ $\text{nullity}(T) = \text{nullity}(L_A)$ since $\text{rank}(T) = \text{rank}(L_A)$ ▨

§2-4.

21. $T_{ij}(v_k) = \begin{cases} w_i, & k=j. \\ 0, & o.w. \end{cases}$ where $\beta = \{v_1, \dots, v_n\}$ & $\gamma = \{w_1, \dots, w_m\}$.

"$S := \{T_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis for $\mathcal{L}(V,W)$."

proof. · S is L.I.: given $x \in V$. assume $0 = \sum_{i,j} a_{ij} T_{ij}(x)$.

$\Rightarrow 0 = \sum_{i,j} a_{ij} T_{ij}(\sum_{k=1}^{n} b_k v_k) = \sum_{i,j} a_{ij} \sum_k b_k T_{ij}(v_k) = \sum_{i,j} a_{ij} \cdot b_j w_i = \sum_i^m (\sum_j^n a_{ij} b_j) w_i$.

$\Rightarrow \sum_{j=1}^{n} a_{ij} b_j = 0 \ \forall i = 1, \dots, m$ since γ is L.I.

$\Rightarrow \underset{1 \times n}{(b_1, \dots, b_n)} \underset{n \times m}{\overset{:=A}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \ddots & & \vdots \\ a_{n1} & \cdots & & a_{nm} \end{pmatrix}}} = 0$

But x is arbitrary $\Rightarrow \{b_i\}_{i=1}^{n}$ is arbitrary. $\Rightarrow A = 0$ ※

· S generates $\mathcal{L}(V,W)$: size of $S := \#S = mn$., and $\dim(\mathcal{L}(V,W))$.

$= \dim(M_{m\times n}(\mathbb{F})) = mn$ by Thm 2.20 & 2.19.

$\Rightarrow$ S is the maximal linearly independent subset of $\mathcal{L}(V,W)$.

(By. Thm 1.12, p.60) $\Leftrightarrow$ S is a basis. ▨

Let $(M^{ij})_{kl} = \begin{cases} 1, & \text{if } k=i, l=j. \\ 0, & \text{~~if~~ } o.w. \end{cases}$ $\therefore M^{ij} = \begin{pmatrix} \cdot & \overset{j\text{t}}{\downarrow} & \cdot \\ i \to & ① & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ ※

$\Phi: \mathcal{L}(V,W) \to M_{m\times n}(\mathbb{F})$ is defined by $\Phi(T_{ij}) = M^{ij}$.

"Φ is an isomorphism"

proof. 証法①：說明 Φ 是 1-1 & onto.

証法②：利用 §2-4 #15, p.108 : $\Phi(S) = \{M^{ij} \mid 1 \le i \le m, 1 \le j \le n\}$

is a basis for $M_{m\times n}(\mathbb{F})$ $\Rightarrow$. Φ is an isomorphism ▨

22. $T: P_n(\mathbb{F}) \to \mathbb{F}^{n+1}$ is defined by $T(f) = (f(c_0), f(c_1), \dots, f(c_n))$.

Define $f_j = \dfrac{\prod_{i\neq j}(x - c_i)}{\prod_{i \neq j}(c_j - c_i)}$ $\forall j = 0, \dots, n$. Then $\beta := \{f_0, f_1, \dots, f_n\}$ is a basis. (p.

Given $f \in P_n(\mathbb{F})$, $\exists!$ scalars s.t. $f = \sum a_i f_i$.

Then $T(f) = (f(c_0), \dots, f(c_n)) = (a_0, a_1, \dots, a_n)$.

Since $T(f_j) = e_j^t \ \forall j = 0, \dots, n.$, $T(\beta)$ is a basis for $\mathbb{F}^{n+1}$.

Hence, by §2-4 #15 (p.108), T is an isomorphism ▨

23. Infinte dimension case. (we cannot use §2-4 #15). $T: V \to W$
$\sigma \mapsto \sum_{i=0}^{n} \sigma(i) x^i_n$

· onto: given any polynomial $f \in P(\mathbb{F})$ By the def. of poly., $f = \sum_{i=0}^{n} a_i x^i$ for some $\overset{(\text{finite})}{n} \in \mathbb{N}$. (o.w. $f = \sum_{i=0}^{\infty} a_i x^i$, which is a power series, not a poly.).

$\exists\, \sigma \in V$ s.t. $\sigma(i) = a_i \cdot \forall i = 0, n$. Then $T(\sigma) = f$ ※

· 1-1: T is linear, $N(T) = 0$ ▨

24. $T: V \to Z$. is linear & onto. $\overline{T}: V/N(T) \to Z$. is defined by $\overline{T}(v+N(T)) = T(v)$.

(a). $\overline{T}$ well-defined: If $v+N(T) = v'+N(T)$, then. $T(v) - T(v') = T(v-v')$
$= \overline{T}((v-v')+N(T)) = \overline{T}(0+N(T)) = T(0) = 0$

(b). $\overline{T}$ is linear: given. $v, v' \in V$. given $c \in F$, $\overline{T}(c(v+N(T)) + (v'+N(T)))$
$= \overline{T}(cv+N(T) + v'+N(T)) = T(cv+v') = cT(v) + v' = c\overline{T}(v+N(T)) + \overline{T}(v'+N(T))$ ▨

(c). $\overline{T}$ is an isomorphism: ∵ $N(\overline{T}) = \{v+N(T) \in V/N(T) \mid \overline{T}(v+N(T)) = T(v) = 0\}$.
$= \{v+N(T) \in V/N(T) \mid v \in N(T)\} = \{0+N(T)\}$ ⇒ $\overline{T}$ is 1-1.
$\nwarrow$ an element in $V/N(T)$.

∵ onto: given $z \in Z$, since. T is onto, $\exists v \in V$ s.t. $T(v) = z$. but $\overline{T}(v+N(T))$
$= T(v) = z$. ▨

(d). given $x \in V$. $\overline{T}\eta(x) = \overline{T}(x+N(T)) = T(x)$. ▨

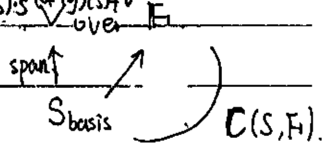
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\eta"'] & Z \\ V/N(T) \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

25. $\Psi: C(S,F) \to V$ is defined by $\begin{cases} \Psi(f) = 0 \text{ if } f \text{ is the zero function.} \\ \Psi(f) = \sum_{s \in S, f(s) \neq 0} f(s) \cdot s., \text{ o.w.} \end{cases}$

Prove that Ψ is an isomorphism.

proof. ① Ψ is linear.: $\Psi(cf+g) = \sum_{s \in S, (cf+g)(s) \neq 0} (cf+g)(s) \cdot s = \sum_{(cf+g)(s)=0} (cf+g)(s) \cdot s + \sum_{(cf+g)(s) \neq 0} (cf+g)(s) \cdot s$
$= c\sum_{f(s) \neq 0} f(s) \cdot s + \sum_{g(s) \neq 0} g(s) \cdot s$. ▨

② $N(\Psi) = 0$ ∴ (1-1).



③ onto: given $x \in V$, $x = \sum_{i=1}^{n} a_i s_i$ for some finite scalars. & $s_i \in S$. $\forall i$.
$\exists f \in C(S,F)$ s.t. $f(s) = 0$ for all but a finite number of vector in S.
and $f(s_i) = a_i$, $i = 1, \ldots, n$. Then $\Psi(f) = \sum_{f(s) \neq 0} f(s) \cdot s = \sum_{i=1}^{n} f(s_i) s_i$
$= \sum a_i s_i = x$. ▨

Thus every nonzero vector space can be viewed as a space of functions. ※

Section§2-5

1 (a). False, $Q = [I_V]_{\beta'}^{\beta}$ ∴ $x_j' = \sum Q_{ij} x_i$ ⇒ jth column of Q is $[x_j']_\beta$.

(b). True, by Thm 2.22. (c). True. $[T]_\beta = [I_V]_{\beta'}^{\beta} [T]_{\beta'} [I_V]_{\beta}^{\beta'}$

(d). False, by def A is similar to B ⇔. $\exists$ invertible Q s.t. $A = Q^{-1}BQ$... not $Q^{\textcircled{t}}BQ$.

(e). True. $T: V \to V$. By Thm 2.23, $[T]_\gamma = Q^{-1}[T]_\beta Q$. for some $Q \in M_{n \times n}(F)$.

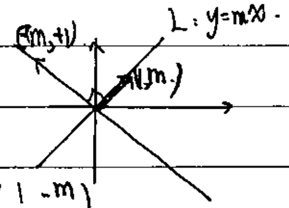
2. (a). $Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}_{\beta'}^{\beta \uparrow}$ (b). $Q = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. (c). $Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$. (d) $Q = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$. ▨

4. $Q =: [I_V]_{\beta'}^{\beta\,\mathrm{std}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$. ▨

§2-5.

6.(a). $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$. , $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\overset{\text{std}}{\uparrow}}_{\beta}$

(C) $[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. ∎



7.(a). $L: y=mx$. T is the reflection of $\mathbb{R}^2$ about L

$\Rightarrow T(1,m) = (1,m)$ & $T(-m,+1) = (+m,-1)$.

Let $\beta' = \left\{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ +1 \end{pmatrix} \right\}$; $\beta = \{e_1, e_2\}$. Then $Q =: [I_V]^{\beta}_{\beta'} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$.

Then $[T]_\beta = Q[T]_{\beta'}Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \cdot \frac{1}{1+m^2}$

$= \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ +m & -1 \end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$.

Thus, $T(x,y) = \frac{1}{1+m^2} \times \left((1-m^2)x + 2my, \ 2mx + (m^2-1)y \right)$.

(b). T is the projection on L along the line perpendicular to L.

$\Rightarrow T(1,m) = (1,m)$ & $T(-m,+1) = (0,0)$. Let $\beta' = \left\{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ +1 \end{pmatrix} \right\}$ $Q = \begin{pmatrix} 1 & -m \\ m & +1 \end{pmatrix}^{\beta}_{\beta'}$

$[T]_\beta = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \cdot \frac{1}{1+m^2} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{1+m^2} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$.

Thus, $T(x,y) = \frac{1}{1+m^2} \cdot (x+my, \ mx+m^2y)$. ∎

<Another>: given $(x,y) \in \mathbb{R}^2$, 令 $\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} 1 \\ m \end{pmatrix} + b\begin{pmatrix} -m \\ 1 \end{pmatrix}$.

$\Rightarrow \begin{cases} x = a - bm \\ y = ma + b \end{cases} \Rightarrow a = \frac{x+my}{1+m^2}$; $b = \frac{-mx+y}{1+m^2}$. Then $\left[T\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} a \\ 0 \end{pmatrix}$.

$\left[T\begin{pmatrix} x \\ y \end{pmatrix}\right]_\beta = Q\left[T\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\beta'} = Q\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ ma \end{pmatrix} = \frac{1}{1+m^2}(x+my, \ mx+m^2y$

8 (proof of the generalization version of Thm 2.23) $T: \underset{\beta,\beta'}{\overset{\text{finite dim.}}{V}} \to \underset{\gamma,\gamma'}{W}$ linear.

"$[T]^{\gamma'}_{\beta'} = P^{-1}[T]^{\gamma}_{\beta}Q$, where $Q = [I_V]^{\beta}_{\beta'}$, $P = [I_W]^{\gamma}_{\gamma'}$."

proof: Note that $I_W T I_V = I_W T = T I_V = T$.

$\overset{\text{Thm 2.11}}{\Rightarrow} P[T]^{\gamma'}_{\beta'} = [I_W]^{\gamma}_{\gamma'}[T]^{\gamma'}_{\beta'} = [I_W T]^{\gamma}_{\beta'} = [T I_V]^{\gamma}_{\beta'} = [T]^{\gamma}_{\beta} \cdot [I_V]^{\beta}_{\beta'} = [T]^{\gamma}_{\beta} \cdot Q$. ∎

9 略.

10. A is similar to $B \Rightarrow \exists$ invertible matrix Q s.t. $A = Q^{-1}BQ$.

$\Rightarrow tr(A) = tr(Q^{-1}BQ) = \sum_{i,j,k} (Q^{-1})_{ij} B_{jk} Q_{ki} = \sum_{i,j,k} Q_{ki}(Q^{-1})_{ij} B_{jk} = \sum_{j,k} (I_n)_{kj} B_{jk}$

$= 0 + \sum_{k=j} (I_n)_{kk} B_{kk} = tr(B)$ ∎

<Another>. $tr(A) = tr(Q^{-1}BQ) = tr(Q^{-1}(BQ)) = tr((BQ)Q^{-1}) = tr(B)$ (by hint) ∎

11. $Q = [I_V]^{\beta}_{\alpha}$; $R = [I_V]^{\gamma}_{\beta}$

(a). $RQ = [I_V]^{\gamma}_{\beta}[I_V]^{\beta}_{\alpha} = [I_V I_V]^{\gamma}_{\alpha} = [I_V]^{\gamma}_{\alpha}$ ※ (b). $Q = [I_V]^{\beta}_{\alpha} \Rightarrow Q^{-1} = ([I_V]^{\beta}_{\alpha})^{-1} = [I_V^{-1}]^{\alpha}_{\beta}$

$= [I_V]^{\alpha}_{\beta}$ ∎

12. (proof of coro in p.115) $Q[L_A]_\gamma = [I_{\mathbb{F}^n}]_\gamma^{std}[L_A]_\gamma = [I\cdot L_A]_\gamma^{std=:\beta} = [L_A]_\gamma^\beta = [L_A\cdot I]_\gamma^\beta$

$= [L_A]_\beta^\beta [I]_\gamma^\beta = AQ$ ■ ($\beta =$ std basis for $\mathbb{F}^n$).

13. Define $x_j' = \sum_{i=1}^n Q_{ij} x_i$ for $1 \le j \le n$, where $\beta = \{x_1, \cdots, x_n\}$ is a basis for V.

- Claim: $\beta' = \{x_1', x_2', \cdots, x_n'\}$ is also a basis for V.

pf: Assume $\sum a_i x_i' = 0 \Rightarrow 0 = \sum_j^n a_j x_j' = \sum_j^n a_j \sum_i^n Q_{ij} x_i = \sum_{i=1}^n \left(\sum_{j=1}^n a_j Q_{ij}\right) x_i$

$\overset{\beta\text{ basis}}{\Rightarrow} \sum_{j=1}^n a_j Q_{ij} = 0, \forall i = 1, \cdots, n. \Rightarrow \begin{pmatrix} & Q & \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

同時左乘 Q^{-1} (Q invertible). $\Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = Q^{-1} \cdot \vec{0} = \vec{0}$ ※(claim finished).

- $Q = [I_V]_{\beta'}^{\beta}$: Since $x_j' = \sum_{i=1}^n Q_{ij} x_i$, i.e. a vector x_j' can be written as a linear combination of $\underset{(\beta\text{, basis.})}{x_i, i=1,\cdots,n}$, with coefficients from j-th column of Q, we obtain that $Q = [I_V]_{\beta'}^{\beta}$ by def. ■

*14. $A, B \in M_{m\times n}(\mathbb{F})$. If $\exists$ invertible matrices $P_{(m\times m)}$ & $Q_{(n\times n)}$ s.t. $B = P^{-1}AQ$, prove that $\exists$ vector space V with n-dim. & W with m-dim & $\exists\ \beta, \beta'$ for V. $\exists\ \gamma, \gamma'$ for W, and $\exists\ T: V \to W$, linear, st. $A = [T]_\beta^\gamma$, $B = [T]_{\beta'}^{\gamma'}$.

proof: Following the hint, let $V = \mathbb{F}^n$, $W = \mathbb{F}^m$, $T = L_A : \mathbb{F}^n \to \mathbb{F}^m$.

Let β and γ be the std ordered basis for $\mathbb{F}^n$, $\mathbb{F}^m$, respectively.

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "T"] & \mathbb{F}^m \\
{}_\beta\mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^m_\gamma \\
{}_{\beta'}\mathbb{F}^n \arrow[r, "L_B"] & \mathbb{F}^m_{\gamma'}
\end{tikzcd}$$

Then $[T]_\beta^\gamma = [L_A]_\beta^\gamma = A$ ※

Next, define $x_j' = \sum_{i=1}^n Q_{ij} x_i$, where $\beta = \{x_1, \dots, x_n\}$.

By §2-5 #13, p.118, $\beta' =: \{x_1', \cdots, x_n'\}$ is a basis for $\mathbb{F}^n$, and $Q = [I_{\mathbb{F}^n}]_{\beta'}^{\beta}$

Also define $y_j' = \sum_{i=1}^m P_{ij} y_i$, where $\gamma = \{y_1, \cdots, y_n\}$.

Similarly, we have $\gamma' =: \{y_1', y_2', \cdots, y_n'\}$ is a basis for $\mathbb{F}^m$, and $P = [I_{\mathbb{F}^m}]_{\gamma'}^{\gamma}$

Now, $B = P^{-1}AQ = ([I_{\mathbb{F}^m}]_{\gamma'}^{\gamma})^{-1}[L_A]_\beta^\gamma [I_{\mathbb{F}^n}]_{\beta'}^{\beta} = [I_{\mathbb{F}^n}^{-1}]_{\gamma}^{\gamma'}[L_A]_\beta^\gamma [I_{\mathbb{F}^n}]_{\beta'}^{\beta}$

$= [I_{\mathbb{F}^m} L_A I_{\mathbb{F}^n}]_{\beta'}^{\gamma'} = [L_A]_{\beta'}^{\gamma'} = [T]_{\beta'}^{\gamma'}$ ■

section.

§ 2-6.

1. (a) False, e.g. $T: V \to \mathbb{R}^2$ can be linear. (In fact, every linear functional is a linear transformation ※

(b). True. $T \overset{\in \mathcal{L}(\mathbb{F},\mathbb{F})}{:} \mathbb{F} \to \mathbb{F}$, could be represented as $[T]_\beta = A_{1\times 1}$.

(c). True. (p119 下部). *(d). True.

(e). False, $T: V \to V^*$ is defined by $T(\overset{\in V}{x}) = T(\sum a_i v_i) = \sum a_i T(v_i) := \sum a_i 2 f_i$, where $f_i(x) = \begin{cases} 1, & x = v \\ 0, & x \neq v \end{cases}$ ($\beta = \{v_1, \cdots, v_n\}$)

Then $T(\beta) = \{2f_1, 2f_2, \cdots, 2f_n\} \neq \beta = \{f_1, f_2, \cdots, f_n\}$ ※ but T is iso. (1-1 & onto)

§2-6.

(f). True. (g). True, $\dim(V^*) \overset{\text{iso}}{=} \dim(V) \overset{\text{by 題目}}{=} \dim(W) \overset{\text{iso}}{=} \dim(W^*)$. 再用 Thm 2.19. ✕ (前提是 finite dimen

(h). False.

2. Check that $T \in \mathcal{L}(V, \mathbb{F})$.

3. (a). $V = \mathbb{R}^3$. $V^* = \mathcal{L}(V, \mathbb{R})$ 令 $\beta^* = \{f_1, f_2, f_3\}$ is the dual basis for β.

Then $f_i(v_j) = \delta_{ij}\ \forall i, j = 1, \dots, 3$.

$$\begin{cases} f_1(1,0,1) = 1 = f_1(e_1) + f_1(e_3) \\ f_1(1,2,1) = 0 = f_1(e_1) + 2f_1(e_2) + f_1(e_3) \\ f_1(0,0,1) = 0 = f_1(e_3) \end{cases} \Rightarrow \begin{array}{l} f_1(e_1) = 1,\ f_1(e_2) = -\frac{1}{2}.\ f_1(e_3) = 0 \\ \therefore f_1(x, y, z) = x - \frac{1}{2}y \end{array}$$

$$\begin{cases} f_2(1,0,1) = 0 = f_2(e_1) + f_2(e_3) \\ f_2(1,2,1) = 1 = f_2(e_1) + 2f_2(e_2) + f_2(e_3). \\ f_2(0,0,1) = 0 = f_2(e_3). \end{cases} \Rightarrow \begin{array}{l} f_2(e_1) = 0\ \ f_2(e_2) = \frac{1}{2}\ \ f_2(e_3) = 0 \\ \therefore f_2(x, y, z) = \frac{1}{2}y. \end{array}$$

Similarly, $f_3(x, y, z) = z - x$. Hence, $\beta^* = \{x - \frac{1}{2}y, \frac{1}{2}y, -x + z\}$

(b). $V = P_2(\mathbb{R})$ $\beta = \{1, x, x^2\}$ 令 $\beta^* = \{f_1, f_2, f_3\}$ is the dual basis of β.

Then $f_1(a + bx + cx^2) = a$, $f_2(a + bx + cx^2) = b$; $f_3(a + bx + cx^2) = c$.

4. • $\dim(V^*) = \dim(V) = 3$. $\therefore$ it suffice to show that β^* is L.I. (or equivently, β^* generates V^*). Assume $0 = \sum c_i f_i(x)$. $\forall x \in V$. $= c_1(x - 2y) + c_2(x + y + z) + c_3(y - 3z) = (c_1 + c_2)x + (-2c_1 + c_2 + c_3)y + (c_2 - 3c_3)z\ \ \forall x.$ $\Rightarrow c_1 = c_2 = c_3 = 0$ ▨

• Let $\beta = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ s.t. $\beta^* = \{f_1, f_2, f_3\}$ is its dual basis.

Then $\begin{cases} x_1 - 2y_1 = 1 \\ x_1 + y_1 + z_1 = 0 \\ y_1 - 3z_1 = 0 \end{cases} \Rightarrow \vec{x}_1 = (0.4, -0.3, -0.1).$; $\begin{cases} x_2 - 2y_2 = 0 \\ x_2 + y_2 + z_2 = 1 \\ y_2 - 3z_2 = 0 \end{cases} \Rightarrow \vec{x}_2 = (0.6, 0.3, 0.1)$;

$\begin{cases} x_3 - 2y_3 = 0 \\ x_3 + y_3 + z_3 = 0 \\ y_3 - 3z_3 = 1. \end{cases} \Rightarrow \vec{x}_3 = (0.2, 0.1, -0.3)$ Hence, $\beta = \left\{ \begin{pmatrix} 0.4 \\ -0.3 \\ -0.1 \end{pmatrix}, \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix}, \begin{pmatrix} 0.2 \\ 0.1 \\ -0.3 \end{pmatrix} \right\}$.

5. • $\dim(V^*) = 2$. Assume $0 = \sum c_i f_i \Rightarrow 0 = c_1 \int_0^1 p(t)dt + c_2 \int_0^2 p(t)dt$.

代入 $p(x) = 1 \Rightarrow 0 = c_1 + 2c_2.$
$ = x \Rightarrow 0 = \frac{1}{2}c_1 + 2c_2.$ $\Rightarrow c_1 = c_2 = 0$ ✕

• $\beta := \{p_1(x), p_2(x)\}$ s.t. $\beta^* = \{f_1, f_2\}$ is its dual basis.

Then $\begin{cases} \int_0^1 p_1(x)dx = 1. \\ \int_0^2 p_1(x)dx = 0 \end{cases}$ (令 $p_1(x) \overset{\in P_1(\mathbb{R})}{=} ax + b$). $\Rightarrow \begin{cases} \frac{1}{2}a + b = 1 \\ 2a + 2b = 0 \end{cases} \Rightarrow p_1(x) = -2x + 2.$

$\begin{cases} \int_0^1 p_2(x)dx = 0 \\ \int_0^2 p_2(x)dx = 1. \end{cases} \Rightarrow p_2(x) = x - \frac{1}{2}$ $\therefore \beta = \{-2x + 2, x - \frac{1}{2}\}$ ▨

6. (a). $T^t(f) \overset{(x,y)}{=} fT \overset{(x,y)}{=} f(3x+2y, x) = (6x+4y)+x = 7x+4y$.

(b). $\beta^* := \{f_1, f_2\}$ is a dual basis of $\beta = \{e_1, e_2\}$. $\therefore \begin{cases} f_1(x,y)=x \\ f_2(x,y)=y \end{cases}$.

$\Rightarrow T^t(f_1) = f_1T = f_1(3x+2y, x) = 3x+2y = 3f_1+2f_2$.

$T^t(f_2) = f_2T = f_2(3x+2y, x) = x = f_1$.

Hence, $[T^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

(c). $[T]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$. $\Rightarrow ([T]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} = [T^t]_{\beta^*}$.

7.

If $p(x)=a+bx$.

(a). $\overset{:W^*\to V^*}{T^t}(f) = fT \overset{(p(x))}{=} f(p(0)-2p(1), p(0)+p'(0)) = -p(0)-2p(1)-2p'(0) \overset{\downarrow}{=} -3a-4b$ *

(b). $\underset{=\{f_1,f_2\}}{\beta^*}$ & $\underset{=\{g_1,g_2\}}{\gamma^*}$ is the dual space of $\underset{=\{1,x\}}{\beta^{std}}$ & $\underset{=\{e_1,e_2\}}{\gamma^{std}}$, respectively.

令 $[T^t]^{\beta^*}_{\gamma^*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\begin{cases} T^t(g_1) = af_1+cf_2 \\ T^t(g_2) = bf_1+df_2 \end{cases}$

$\Rightarrow \begin{cases} g_1T(1) = g_1(-1,1) = g_1(-e_1)+g_1(e_2) = -1 = T^t(g_1)(1) = af_1(1)+cf_2(1)=a. \\ g_1T(x) = g_1(-2,1) = 2g_1(-e_1)+g_1(e_2) = -2 = T^t(g_1)(x) = af_1(x)+cf_2(x)=c \end{cases}$

also, $\begin{cases} g_2T(1) = g_2(-1,1) = +1 = T^t(g_2)(1) = bf_1(1)+df_2(1) = b. \\ g_2T(x) = g_2(-2,1) = +1 = T^t(g_2)(x) = bf_1(x)+df_2(x) = d. \end{cases}$

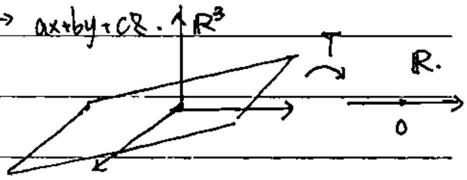
Hence, $[T]^{\beta^*}_{\gamma^*} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$.

(C). $([T]^\gamma_\beta)^t = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^t = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = [T^t]^{\beta^*}_{\gamma^*}$.

8. • Every plane could be written as $P = \{(x,y,z) \mid ax+by+cz=0\}$ (through the origin).

Consider a transformation $T: \mathbb{R}^3 \to \mathbb{R}$ $(x,y,z) \mapsto ax+by+cz$.

So $T \in (\mathbb{R}^3)^*$ & $P = N(T)$.



• For the case in $\mathbb{R}^2$, actually every line through the origin has the form $L = \{(x,y) \mid ax+by=0\}$. and hence is the null space of a vector $T_{(x,y)\mapsto ax+by}$ in $(\mathbb{R}^2)^*$ ▨

9. " $\overset{:\mathbb{F}^n\to\mathbb{F}^m}{T}$ linear $\Leftrightarrow \exists f_1, \dots, f_m \in (\mathbb{F}^n)^*$ s.t. $T(x) = (f_1(x), \dots, f_m(x))\ \forall x \in \mathbb{F}^n$.

proof. ($\Rightarrow$). Following the hint, define $f_i(x) = (g_iT)(x)$ for $x \in \mathbb{F}^n$, $\forall 1 \le i \le m$, where $\gamma =: \{g_1, \dots, g_m\}$ is the dual basis of the std ordered basis for $\mathbb{F}^m$.

Let $[T]^\gamma_\beta := A$, where β is the std ordered basis for $\mathbb{F}^n$. Then $T(v_j) = \sum_i A_{ij} e_i$, where $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{e_1, \dots, e_m\}$ 續下頁...

(續 9). $T(x) = T(\sum_{j=1}^{n} a_j v_j) = \sum_{j=1}^{n} a_j T(v_j) = \sum_{j=1}^{n} a_j \sum_{i=1}^{m} A_{ij} e_i = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_j A_{ij}) e_i$

$= (\sum_{j=1}^{n} a_j A_{1j}, \sum_{j=1}^{n} a_j A_{2j}, \ldots, \sum_{j=1}^{n} a_j A_{mj}) \in \mathbb{F}^m$

Also, $f_k(x) = g_k T(x) = \underline{g_k}(\sum_{i=1}^{m} (\sum_{j=1}^{n} a_j A_{ij}) \underline{e_i}) \underset{\uparrow}{=} \sum_{j=1}^{n} a_j A_{kj} \in \mathbb{R}, \forall k = 1, \ldots, m.$

since $g_k(e_i) = \delta_{ik}$. (coordinate function).

Hence, we obtain that $T(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ ※

(⇐). $\exists f_1, \ldots, f_m \in (\mathbb{F}^n)^*$ s.t. $T(x) = (f_1(x), \ldots, f_m(x))$.

Given $x, y \in \mathbb{F}^n$, $c \in \mathbb{F}$. then $T(cx+y) = (f_1(cx+y), \ldots, f_m(cx+y))$

$= (cf_1(x)+f_1(y), \ldots, cf_m(x)+f_m(y))$ by $f_i \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}) = (\mathbb{F}^n)^*$.

$= c(f_1(x), \ldots, f_m(x)) + (f_1(y), \ldots, f_m(y)) = cT(x) + T(y)$ ▨

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "T"] \arrow[d, dash, squiggly, "\beta"'] & \mathbb{F}^m \arrow[d, dash, squiggly, "\gamma"] \\ (\mathbb{F}^n)^* & (\mathbb{F}^m)^* \arrow[l, "T^t"'] \end{tikzcd}$$

$\gamma^* =: \{g_1, \ldots, g_m\}$

$T(x) = (g_1 T(x), g_2 T(x), \ldots, g_m T(x))$.

make sense very much※

(10). $V = P_n(\mathbb{F})$. $c_0, \ldots, c_n$ is the distinct scalars in $\mathbb{F}$.

(a). For $0 \le i \le n$. define $f_i \in V^*$ by $f_i(p(x)) = p(c_i)$.

Following the hint, assume $0 = \sum_{i=0}^{n} a_i f_i(w)$, $\forall w \in P_n(\mathbb{F})$.

代入 $p(x) = (X-C)\cdots(X-C_n)$. $\Rightarrow 0 = a_0$

再代入 $p(x) = (x-c_0)(x-c_2)\cdots(x-c_n) \in P_n(\mathbb{F}) \Rightarrow 0 = a_1$.

Continue the procee. we find that $a_0 = a_1 = \cdots = a_n = 0$

∴ $\beta^* =: \{f_0, f_1, \ldots, f_n\}$ is L.I.

But $\dim(V) = n+1 = \#\beta$. Hence, β^* is a basis (for V^*). by Coro2(b), p47-48. ※

(b). Given $p_k(x) \in P_n(\mathbb{F})$ s.t. $p_k(c_j) = \delta_{kj}$ (fix k). If $\exists$ another $l(x) \in P_n(\mathbb{F})$ s.t. $l(c_j) = \delta_{kj}$, let $\beta^* = \{f_0, \ldots, f_n\}$ defined in part (a) be the basis for V^*.

By corollary to Thm 2.26, $\exists$ a basis for V, $\beta = \{v_0, \ldots, v_n\}$ s.t. β^* is its dual basis.

So $p_k(x) = \sum_{i=0}^{n} a_i v_i(x)$ for some scalars $a_0, \ldots, a_n$.

$\Rightarrow f_j(p_k(x)) = \delta_{kj} = \sum a_i f_j(v_i(x)) = \sum_{i=0}^{n} a_i \delta_{ji} = a_j \Rightarrow \begin{cases} a_k = 1 \ \& \\ a_j = 0 \quad \forall j \neq k. \end{cases}$

Similarly, $l(x) = \sum_{i=0}^{n} b_i v_i(x)$, $b_i = \begin{cases} 1 & i = k. \\ 0 & i \neq k. \end{cases} \Rightarrow p_k(x) = l(x)$ ▨

(By the fact that every element can be uniquely written as the linear combination of the vectors of the basis) ∴ $p_k(x) = l(x)$, that is, unique ▨

(C). Check $g(x) = \sum_{j=0}^{n} a_i p_i(x)$ has the property that $g(c_i) = a_i$ for $0 \le i \le n$.

By part (b), $\beta = \{v_0, \ldots, v_n\} = \{\overset{v_0}{\overset{\parallel}{p_0}}, \ldots, \overset{v_n}{\overset{\parallel}{p_n}}\}$ is a basis. ∴ $g(x) \in \overset{\swarrow \text{ degree at most}}{P_n(\mathbb{F})}$.

$\Rightarrow \exists$ scalars s.t. $g(x) = \sum_{i=0}^{n} a_i p_i(x)$. Then $g(c_j) = \sum_{i=0}^{n} a_i p_i(c_j) = \sum a_i \delta_{ij} = a_j$.

The unique property is trivial ▨

(d). Given $p(x)\in P_n(\mathbb{F})$. $\Rightarrow$ $p(x)=\sum_{i=0}^{n} a_i p_i(x)$ for some scalars $a_0,\dots,a_n\in\mathbb{F}$.
Then $p(c_j) = a_j$. $\forall j=0,\dots,n$. $\Rightarrow$ $p(x) = \sum_{i=0}^{n} p(c_i)\,p_i(x)$. ∎

(e). $\forall$ $p(x)\in P_n(\mathbb{F})$, $\int_a^b p(x)\,dx = \int_a^b \sum_{i=0}^{n} p(c_i)\,p_i(x) = \sum_{i=0}^{n}\int_a^b p(c_i)\,p_i(x)$
$= \sum_{i=0}^{n} p(c_i)\int_a^b p_i(x) = \sum_{i=0}^{n} p(c_i)\,d_i$, where $d_i = \int_a^b p_i(x)\,dx$. ∎

- $c_i := a+\frac{i(b-a)}{n}$ for $i=0,1,\dots,n$.

For $n=1$, (trapezoidal rule). $c_i = a+i(b-a), i=0,1.$

$$\therefore \int_a^b p(x)\,dx = \sum_{i=0}^{\overset{n=1}{n}} p(c_i)\,d_i = p(c_0)\cdot\int_a^b \frac{x-c_1}{c_0-c_1}dx + p(c_1)\int_a^b \frac{x-c_0}{c_1-c_0}dx = p(c_0)\frac{b-a}{2}+p(c_1)\frac{b-a}{2}$$

$$\uparrow\ = (p(a)+p(b))\frac{b-a}{2} \text{ ※}$$

For $n=2$. (Simpson's rule). $c_i = a+\frac{i(b-a)}{\cancel{n}2}$ $i=0,1,2$.

$$\int_a^b p(x)\,dx = \sum_{i=0}^{n=2} p(c_i)\,d_i = p(c_0)\int_a^b \frac{(x-c_1)(x-c_2)}{(c_0-c_1)(c_0-c_0)}dx + p(c_1)\int_a^b\frac{(x-c_0)(x-c_2)}{(c_1-c_0)(c_1-c_2)}dx +$$
$$p(c_2)\int_a^b\frac{(x-c_0)(x-c_1)}{(c_2-c_0)(c_2-c_1)}dx = p(a)\frac{b-a}{6} + p\left(\frac{a+b}{2}\right)\cdot 4\,\frac{b-a}{6} + p(b)\cdot\frac{b-a}{6}. \text{ ※ ∎}$$

11

$$\begin{tikzcd} \overset{\text{finite}}{V} \arrow[r, "T", "\text{linear}"'] \arrow[d, "\text{ISO } \psi_1"'] & W^{\text{finite}} \arrow[d, "\psi_2 \text{ ISO}"] \\ V^{**} \arrow[r, "T^{tt}"] & W^{**} \end{tikzcd}$$

$(T^{tt} = (T^t)^t)$.

Given $x\in V$, ① $\psi_2 T(x) = \widehat{T(x)} \in W^{**}$. $V^*\to\mathbb{R}$. $W^*\to V^*$

② $T^{tt}\psi_1(x) = (T^t)^t(\hat{x}) = \hat{x}(T^t) \in W^{**}$

Now, given $g\in W^*$, ① $= \widehat{T(x)}(g) \overset{\text{def}}{=} g(T(x))$.

② $= (\hat{x}T^t)(g) = \hat{x}(T^t(g)) = \hat{x}(\underbrace{gT}_{\in V^*}) = (gT)(x) = g(T(x))$ $\Rightarrow$ ①=② ∎

12. Let $\dim(V)=n$. & $\beta =: \{v_1,\dots,v_n\}$, a basis for V.
$\psi(\beta) = \{\hat{v}_1,\hat{v}_2,\dots,\hat{v}_n\}$. Define f_i to be the coordinate function. ($f_i(v_j)=\delta_{ij}$).
$\therefore$ $\beta^* =: \{f_1,\dots,f_n\}$ is the dual basis of β.
Note that $\hat{v}_j(f_i) = f_i(v_j) = \delta_{ij}$. Hence, $\psi(\beta)$ is the dual basis of β^*.
$\therefore$ That is $\psi(\beta) = \beta^{**}$ ∎

13. Given subset S of the finite-dim vector space V, define the "annihilator" S^o of S to be $S^o := \{f\in V^* \mid f(x)=0\ \forall x\in S\}$

(a). ① $f_0 :=$ zero function, is in S^o. ② $\forall g,h\in S^o$, $(g+h)(x) = g(x)+h(x) = 0+0=0$ $\forall x\in S$.
$\Rightarrow$ $g+h\in S^o$. ③ $\forall c\in\mathbb{F}, g\in S^o$, $(cg)(x) = cg(x) = c\cdot 0 = 0$ $\forall x\in S$. $\Rightarrow$ $cg\in S^o$ ∎

(b). W is a subspace of V & $x\notin W$. $W^0 := \{f\in V^* \mid f(x)=0\ \forall x\in W\}$ Let $\beta := \{v_1,\dots,v_k\}$ be a basis for W. <key> Since $x\notin W$, $\{v_1,\dots,v_k, v_{k+1} := x\}$ is a L.I. subset. Define $\gamma^* := \{f_1,\dots,f_k,f_{k+1}\}$ to be the dual basis of $\{v_1,\dots,\overset{x}{\overset{\shortparallel}{v_{k+1}}}\}$. Now, define $f = f_{k+1}\in V^*$ Then $f(x) = f(\sum_{i=1}^{k} a_i v_i) = \sum_{i=1}^{k} a_i\,\delta_{(k+1)i} = 0$, $\forall x\in W$ $\Rightarrow$ $f\in W^0$ and $f(x) = 1 \neq 0$ ∎

§2-6. Key!

(C). $W := \text{span}(S)$. Claim: $W^0 = S^0$

proof: • $\forall f \in W^0$, $f(x)=0\ \forall x \in W \Rightarrow f(x)=0\ \forall x \in S \subseteq W$. $\therefore W^0 \subseteq S^0$.

• $\forall f \in S^0$, $f(x)=0\ \forall x \in S$. Then $f(y) = f(\sum a_i x_i) = \sum a_i f(x_i) = 0$, $\forall y \in \text{span}(\overset{x_i}{S})$ ※

So, $W^0 = S^0$. $\Rightarrow (W^0)^0 = (S^0)^0$. Also, $\text{span}(\psi(S)) = \psi(W)$ since ψ is an isomorphism.

• It suffices to prove that "$(W^0)^0 = \psi(W)$".

proof: If $\hat{x} \in (W^0)^0$, $\hat{x} \in V^{**}$ s.t. $\hat{x}(f) = f(x) = 0\ \underline{\forall f \in W^0}$.

Now, if $x \notin W$, then by part (b), $\exists f \in W^0$ s.t. $f(x) \neq 0$. $\rightarrow\!\leftarrow$. So $x \in W$.

$\hat{x} \in \psi(W)$. Hence, $(W^0)^0 \subseteq \psi(W)$.

For the converse, given $\hat{x} \in \psi(W)$. ($x \in W$). $\therefore \forall f \in W^0$, $f(x) = 0 = \hat{x}(f)$

i.e. $\hat{x} \in (W^0)^0$ Hence, $\psi(W) \subseteq (W^0)^0$. ▨

(d). ($\Rightarrow$). $W_1 = W_2$. Let $f \in W_1^0 \Leftrightarrow f(x) = 0\ \forall x = W_1 = W_2 \Rightarrow f \in W_2^0$ ※

($\Leftarrow$). $W_1^0 = W_2^0$. Let $x \in W_1$. $\hat{x}(f) = f(x) = 0\ \forall f \in W_1^0 = W_2^0$. If $x \notin W_2$,

by part (b), $\exists f \in W_2^0$ s.t. $f(x) \neq 0$ $\rightarrow\!\leftarrow$. ▨

(e). • If $f \in (W_1+W_2)^0$, $f(x)=0\ \forall x \in W_1+W_2$. $\Rightarrow x = x_1 + x_2$ for $x_i \in W_i$, $i=1,2$.

$\Rightarrow 0 = f(x) = f(x_1+x_2) = f(x_1)+f(x_2) \overset{f \in (W_1+W_2)^0}{=} 0+0$. $\therefore f(y)=0\ \forall y \in W_1$ & $\forall y \in W_2 \Rightarrow \overset{(W_1+W_2)^0}{\subseteq W_1^0 \cap W_2^0}$

• For the converse, $f \in W_1^0 \cap W_2^0 \Rightarrow f(x) = 0\ \forall x \in W_1$ or W_2

$\overset{\text{linear}}{\Rightarrow} f(z) = 0\ \forall z \in W_1+W_2 \Rightarrow f \in (W_1+W_2)^0$ ▨

14. Following the hint, $\{x_1, \cdots, x_k\}$ is a basis for W. Extend it to an ordered basis $\beta := \{x_1, x_2, \cdots, x_n\}$ of V. Let $\beta^* = \{f_1, f_2, \cdots, f_n\}$. Claim: $\{f_{k+1}, \cdots, f_n\}$ is a basis for W^0.

proof. • $f_{k+1}, \cdots, f_n \in W^0$, clearly. If $f \in W^0$, $f(x) = 0\ \forall x \in W$.

Hence, if $x \in V$, $x = \sum^n a_i x_i$ for some scalars. $\Rightarrow f(x) = \sum_{i=1}^n a_i f(x_i) = \sum_{i=k+1}^n \underbrace{a_i f(x_i)}_{\in \mathbb{F}}$.

$= \sum_{i=k+1}^n (a_i \cdot f(x_i)) f_i(x_i)$. $\therefore f \in \text{span}(\{f_{k+1}, \cdots, f_n\})$.

For the converse, $g := \sum_{i=k+1}^n a_i f_i$. Then $g(x) = \sum_{i=k+1}^n a_i f_i(x) = 0\ \forall x \in W \Rightarrow g \in W^0$

$\therefore \text{span}(\{f_{k+1}, \cdots, f_n\}) = W^0$.

• Assume $\sum a_i f_i = 0$. $\forall x \in V$ 代 $x_1 \Rightarrow a_1 = 0$; 代 $x_2 \Rightarrow a_2 = 0$ ....

$\therefore \{f_{k+1}, \cdots, f_n\}$ is L.I. ▨

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "(gT)"'] & W \arrow[d, "g"] \\ & \mathbb{F} \end{tikzcd}$$

15. • If $f \in N(T^t)$, $T^t(f) = 0$. $\Rightarrow fT(x) = 0\ \forall x \in V$. $\Rightarrow f(T(x)) = 0\ \forall x$. $\Rightarrow f \in (R(T))^0$.

• If $f \in (R(T))^0$, $f(T(x)) = 0\ \forall x \in V$. $\Rightarrow T^t(f)(x) = 0\ \forall x \in V$. $\Rightarrow f \in N(T^t)$. ▨

No.

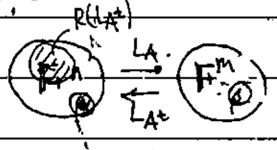
Date

$L_A: \mathbb{F}^n \to \mathbb{F}^m$.

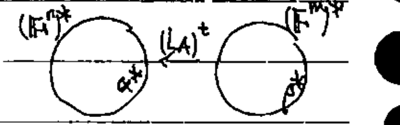
16. Let $A \in M_{m\times n}(\mathbb{F})$. $L_{A^t}: \mathbb{F}^m \to \mathbb{F}^{*n}$ is defined by $x \mapsto A^t x$.
↙subspace. of $W = \mathbb{F}^m$.

~~Define $\{x_1, \dots, x_k\}$ to be an ordered basis of $R(L_{A^t})$. Extend the basis to~~

-- ~~$\beta := \{x_1, \dots, x_m\}$ = basis of $\mathbb{F}^m$. $\beta^* :=$ dual space of β for $(\mathbb{F}^n)^*$~~ $=: \{$~~$f_1, \dots, f_n$~~$\}$.

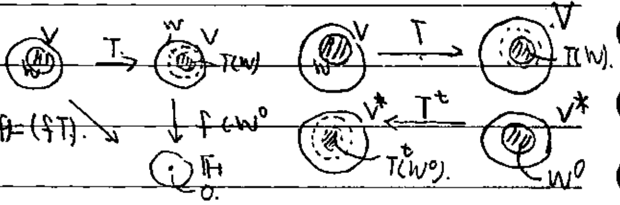


- $\text{rank}(L_A) = \dim(R(L_A)) = m - \dim(R(L_A)^0) = m - \dim(N((L_A)^t))$
  $= \dim(\mathbb{F}^{m*}) - \dim(N((L_A)^t)) = \dim(R((L_A)^t))$
- Next, let α, β be the std. ordered basis for $\mathbb{F}^n$ and $\mathbb{F}^m$. Then $[(L_A)^t]_{\beta^*}^{\alpha^*} = ([L_A]_\alpha^\beta)^t = A^t$
  $= [L_{A^t}]_\beta^\alpha$.
  $\therefore \dim(R((L_A)^t)) = \dim(\phi_{\alpha^*}[R((L_A)^t)]) \overset{*}{=} \dim(R(L_{A^t})) = \text{rank}(L_{A^t})$ ▨
  ↑ $\phi_{\alpha^*}: (\mathbb{F}^n)^* \to \mathbb{F}^n$. $f \mapsto [f]_{\alpha^*}$, iso.



17. (⇒) (W is T-invariant) if $f \in W^0$, $f(x) = 0\ \forall x \in W$. Then $T^t(f)(x) = fT(x) = f(T(x))$
$= 0\ \forall x \in W$ ($\Leftarrow T(x) \in W$). $\therefore T^t(f) \in W^0 \Rightarrow W^0$ is T^t-invariant.

(⇐) (W^0 is T^t-invariant).



Given $x \in W$. If $T(x) \notin W$, then by
#13(p.126), $\exists f \in W^0$ s.t. $f(T(x)) \neq 0$

But, $f(T(x)) = \underbrace{T^t(f)}(x) = 0$ since $T^t(f) \in W^0$, ⨯ ▨

18. S, a basis for V. $\Phi: V^* \to \mathcal{F}(S, \mathbb{F})$. $f \mapsto f_S (= f|_S)$.

- Given $f \in \mathcal{F}(S, \mathbb{F})$, i.e. $f: S \to \mathbb{F}$ is defined by $v \mapsto f(v)$. Then by §2-1 #34,
  $\exists!$ linear transformation $T: V \to \mathbb{F}$ (i.e. $T \in V^*$) s.t. $T(v) = f(v)\ \forall v \in S$.
  ⇒ Φ is onto. (by the existence of such T). & 1-1 (by the uniqueness of T).
- 忘記 check Φ 是 linear 嗎? : $\Phi(f + cg)(s) = (f + cg)_s(s) = f_s(s) + c \cdot g_s(s) = (\Phi(f) + c\Phi(g))(s)$ ▨

19. $W \subsetneq V$. Let S = maximal linearly indep. subset of W, i.e. the basis for W.
By the generalized replacement thm (§1-7 #7), S can be extended to a basis $\tilde{S}$ for V.
Define $g: \tilde{S} \to \mathbb{R}$ by $g(v) = \begin{cases} 0, & \text{if } v \in S \\ 1, & \text{if } v \in (\tilde{S} - S) \neq \phi \end{cases}$. Then by §2-1 #34, $\exists!$ linear transformation $f: V \to \mathbb{R}$ s.t. $f(v) = g(v)\ \forall v \in \tilde{S}$. Thus, $f(x) = 0\ \forall x \in W$. since every element in W can be uniquely written as the linear combination of the vectors in S.
Also, $f \neq$ zero function. since $f(x) = 1$ for $x \in \tilde{S} - S$. ∴ It's the desired function ▨

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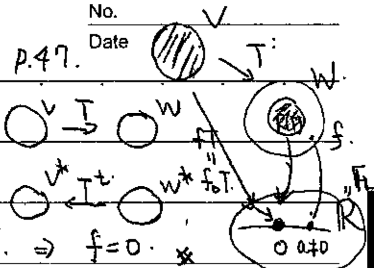
§2-6. p.47.

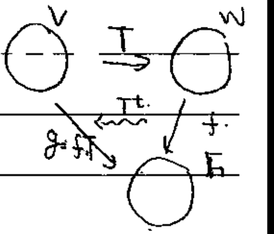


20. (a). " T onto ⇔ T^t 1-1 "

proof. (⇒). check that $N(T^t)=\{0\}$: If $f\in N(T^t)$,
$0=T^t(f)(x)= fT(x)$, $\forall x\in V$. ∵ T is onto, ∴ $f(y)=0$ $\forall y\in W$. ⇒ $f=0$. ※

(⇐). Suppose $R(T)\neq W$ (not onto). By #19 p.127. ∃ nonzero f ∈ W*
s.t. $f(y)=0$ $\forall y\in R(T)$. Define $f_0\in W^*$ is the zero functional. $T^t(f)(x)= f(T(x))=0$ $\forall x\in V$.
and $T^t(f_0)(x)= f_0T(x) = 0$ $\forall x\in V$ $\overset{T^t\ 1\text{-}1}{\Rightarrow}$ $f=f_0$ $\rightarrow\!\!\times$ ▨

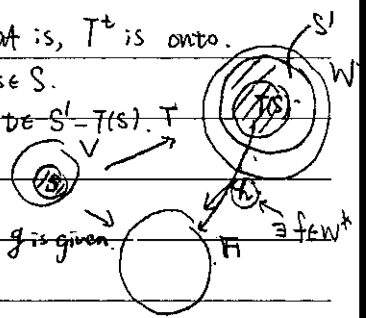
(b). " T^t onto ⇔ T 1-1 "



proof. (⇒). Given $g\in V^*$, ∃ $f\in W^*$ s.t $T^t(f)=fT=g$.
If $T(x)=T(y)$ for some $x,y\in V$., $f(T(x))=f(T(y))$
⇒. $g(x)=g(y)$. ⇒ $\hat{x}(g)=\hat{y}(g)$ $\forall g\in V^*$ $\underset{\hat{g},\,\hat{x}\text{ linear \& 唯一性 (Thm2.6)}}{\overset{\text{(iso) Thm2.26.}}{\Longleftrightarrow}}$ $\hat{x}=\hat{y}$ ⇒ $x=y$ ▨

(⇐). Let S be a basis for V. T 1-1 ⇒. T(S) is L.I. in W. Extend it to a basis for W, say S'.

Claim: $\forall g\in V^*$, ∃ $f\in W^*$ s.t. $T^t(f)=g$. That is, T^t is onto.



pf. Construct $h\in \mathcal{F}(\underline{S},\mathbb{F})$ by $\begin{cases} h(T(s)) = g(s) & \text{for } s\in S.\\ h(t)=0 & \text{for } t\in S'-T(S). \end{cases}$
By §2-6 #18, ∃ $f\in W^*$ s.t. $f_{S'}=h$
∴ $\forall s\in S$, $g(s) = h(T(s)) = f(T(s)) = T^t(f)(s)$.
By §2-1 #34, $g=T^t(f)$ ▨

(⇒) ≪Another proof≫ check that $N(T^{\bullet})=\{0\}$:

Suppose $T(x)=0$ for some $x\in V$. & $x\neq 0$. ∃ $g\in V^*$ s.t. $g(x)\neq 0$.
T^t onto ⇒ ∃ $f\in W^*$ s.t. $T^t(f)=g$. Then $T^t(f)(x)= fT(x)= f(0) \overset{f\text{ is linear}}{=} 0 \neq g(x)$ $\rightarrow\!\!\times$ ▨

Section §2-7.

$ae^{ct}+at^ke^{ct}$ 亦可以是 sol.

1. (a).T. (b).T. (c).F (d). ~~T~~ F, 代數基本定理 ⇒ p(t) splits over C, 且 D−cI has the solution form ae^{ct}; (e).T. (f) F. (g).T.

↑ 少考慮了 $t^ke^{c_it}$, $t^{r_0}e^{c_jt}$ … 的可能性。

2. ⓐ. ~~T~~ False. Let $\{t^2\}$ be a basis of a finite dimensional subspace of C^∞, say W.
- If the homo. linear diff eq. is $y^{(k)}=0$, $k\geq 3\in\mathbb{N}$. then W are solutions, but not its solution space. since $y^{(k)}=0$ has solution space of dim. k−1.
- If the homo. linear diff eq. is $ay'+by=0$. and W are solutions. then $a=b=0$. $\rightarrow\!\!\times$

Hence, there is no homo. linear diff. eq. s.t. its solution space is W ▨

(b) False.
Homogeneous linear diff. eq: $ay' + by = 0$. $a, b \in$ consts.

Suppose its solution space has the basis $\{t, t^2\}$

Then $a + bt = 0$ & $2at + bt^2 = 0$, $\forall t \Rightarrow a = 0$ & $b = 0$. ※

(c) True. If $x \in C^\infty$ is a sol to $ax' + bx = 0$, then $(ax' + bx)' = 0$

$\Rightarrow ax'' + bx' = 0 \Rightarrow x'$ is also a sol. ▨

(d) True. $x \in N(P(D))$, $y \in N(q(D))$ Then $x+y \in N(P(D)q(D))$. since

$$(P(D)q(D))(x+y) \underset{\uparrow}{=} (P(D)q(D))x + (P(D)q(D))y = 0 + 0 = 0 \text{ ※}$$

$D(x+y) = D(x) + D(y)$.

(e) False. 令 $p(t) = t - 1$ $q(t) = t - 2$.

choose $x = e^t$ $y = e^{2t}$

Then $p(t) \cdot q(t) = (t-1)(t-2) = t^2 - 2t + 2$.

but $x \cdot y = e^{3t}$ is not a solution to $P(D)q(D)$.

$$\left((e^{3t})'' - 2(e^{3t})' + 2e^{3t} = (9-6+2)e^{3t} = 5e^{3t} \not\equiv 0 \right). \text{ ▨}$$

3. (a). $p(t) = t^2 + 2t + 1 = (t+1)^2$ $\therefore$ sol basis $= \{e^{-t}, te^{-t}\}$.

(b). $p(t) = t^3 - t = t(t+1)(t-1)$ $\therefore$ sol basis $= \{1, e^t, e^{-t}\}$.

(c). $p(t) = t^4 - 2t^2 + t = t(t^3 - 2t + 1) = t(t-1)(t^2 + t - 1) = t(t-1)(t - \frac{-1+\sqrt5}{2})(t - \frac{-1-\sqrt5}{2})$

$\therefore$ sol basis $= \{1, e^t, e^{\frac{-1+\sqrt5}{2}t}, e^{\frac{-1+\sqrt5}{2}t}\}$

(d) $p(t) = t^2 + 2t + 1 = (t+1)^2$ $\therefore$ sol basis $= \{e^{-t}, te^{-t}\}$.

(e). $p(t) = t^3 - t^2 + 3t + 5 = (t+1)(t^2 - 2t + 5) = (t+1)(t-(1+2i))(t-(1-2i)) \Rightarrow$ sol basis $= \{e^{-t}, e^{+(1+2i)t}, e^{+(1-2i)t}\}$

4. (a). $p(t) = t^2 - t^1 - 1 = (t \div (\frac{1 \pm \sqrt5}{2}))$. $\therefore$ sol basis $= \{e^{+(\frac{1+\sqrt5}{2})t}, e^{+(\frac{1-\sqrt5}{2})t}\}$. ※ ▨

(b). $p(t) = t^3 - 3t^2 + 3t - 1 = (t-1)(t^2 - 4t - 1) = (t-1)(t-(2+\sqrt5))(t-(2-\sqrt5))$ $\therefore$ sol basis $= \{e^t, e^{(2+\sqrt5)t}, e^{(2-\sqrt5)t}\}$ ※

(c) $p(t) = t^3 + 6t^2 + 8t = t(t^2 + 6t + 8) = t(t+4)(t+2)$ $\therefore$ sol basis $= \{1, e^{-4t}, e^{-2t}\}$ ▨

5. 1° zero function $\in C^\infty$ 2° $\forall f, g \in C^\infty$, $f+g \in C^\infty$, trivially.

3° $\forall c \in \mathbb{R}$, $\forall f \in C^\infty$, $(cf) \in C^\infty$, trivially ※ ▨

6. (a). Given $f, g \in C^\infty$, given $c \in \mathbb{F}$, $D(cf+g) = (cf+g)' = cf' + g' = cD(f) + D(g)$ ▨

(b). Given differential operator $p(D)$, $\forall \overset{\in \mathbb{F}}{c}, f, g \in C^\infty$, $p(D)(cf+g)$

$$= \sum_{k=0}^{n} \frac{d^k}{dt^k}(cf+g) \cdot a_k = \sum_{k=0}^{n} a_k (cf^{(k)} + g^{(k)}) = c\sum a_k f^{(k)} + \sum a_k g^{(k)} = cP(D)(f) + P(D)(g) \text{ ▨}$$

§2-7.

p.49

7. · Assume $c_1 \cdot \frac{1}{2}(x+y) + c_2 \cdot \frac{1}{2i}(x-y) = 0$, where $c_1, c_2 \in \mathbb{C}$. 令 $c_1 = a+bi$, $c_2 = c+di$.

$\Rightarrow (a+bi)(x+y) + (c+di)\cdot\frac{1}{i}(x-y) = 0 \Rightarrow ax+ay+ibx+iby+d(x-y)-ic(x-y)=0$

$\Rightarrow (a+d+(b-c)i)x + (a-d+(b+c)i)y = 0 \underset{\{x,y\} \text{ is L.I.}}{\overset{\uparrow}{\Rightarrow}} \begin{cases} (a+d)+i(b-c)=0 \\ (a-d)+i(b+c)=0 \end{cases}$

$\Rightarrow \begin{cases} a-d=0 \\ a+d=0 \\ b-c=0 \\ b+c=0 \end{cases} \Rightarrow a=b=c=d=0 \Rightarrow \{\frac{1}{2}(x+y), \frac{1}{2i}(x-y)\}$ is L.I.

· Since $\{x,y\}$ is a basis, ∴ dim = 2. ∴ $\{\frac{1}{2}(x+y), \frac{1}{2i}(x-y)\}$ is also a basis ▨

8. $z_1 = a+ib$ & $z_2 = a-ib$ are roots $\Rightarrow y(t) = e^{(a+ib)t}$ or $e^{(a-ib)t}$ are sols. to the homo. linear diff eq.
$= e^{at}(\cos bt + i\sin bt)$ or $e^{at}(\cos bt - i\sin bt)$.

Then by Thm, any linear combination of sols. is still a sol to the homo. linear diff eq.
∴ $e^{at}\cos bt \left(= \frac{1}{2}e^{(a+ib)t} + \frac{1}{2}e^{(a-ib)t}\right)$ & $e^{at}\sin bt \left(= \frac{1}{2i}e^{(a+ib)t} - \frac{1}{2i}e^{(a-ib)t}\right)$ are solutions.
∴ $\{e^{at}\cos bt, e^{at}\sin bt\}$ is a basis for the solution space. by §2-7 #7 ▨

9. If $x \in N(U_i)$ for some i, $U_i(x) = 0$. $\Rightarrow U_1U_2\cdots U_iU_{i+1}((U_{i+2}\cdots U_n)(x))$
$= U_1U_2\cdots U_{i+1}U_i((U_{i+2}\cdots U_n)(x)) = U_1U_2\cdots U_{i+1}U_iU_{i+2}((U_{i+3}\cdots U_n)(x)) = U_1\cdots U_{i+1}U_{i+2}U_i((U_{i+3}\cdots U_n)(x))$
$= \cdots\cdots = U_1\cdots U_{i-1}U_{i+1}U_{i+2}\cdots U_nU_i(x) = U_1\cdots U_{i-1}U_{i+1}U_{i+2}\cdots U_n(0) = 0$. ▨

10. Following the hint. To prove Thm2.33 ($\{e^{c_1t}, \dots, e^{c_nt}\}$ is L.I.), we use the math. induction technique. For $n=1$. $\{e^{c_1t}\}$ is L.I. since $a\cdot e^{c_1t} = 0\ \forall t \Rightarrow a = 0$.

Assume the Thm is correct for $k=n-1$, i.e. $\{e^{c_1t}, \dots, e^{c_{n-1}t}\}$ is L.I.

Claim: The Thm is also correct for $k=n$.

proof. Suppose $b_1e^{c_1t} + \cdots + b_ne^{c_nt} = 0$ (c_i's are all distinct). – (*).

Then $(D - c_nI)(b_1e^{c_1t} + \cdots + b_ne^{c_nt}) = (D-c_nI)(0) = 0$

$\Rightarrow \sum_{i=1}^{n-1}(c_i - c_n)b_ie^{c_it} = 0 \underset{\text{hypothesis}}{\overset{\text{induction}}{\Rightarrow}} \underbrace{(c_i-c_n)}_{\neq 0}b_i = 0\ \forall i = 1,\dots,n-1$.

$\Rightarrow b_i = 0\ \forall i = 1,\dots,n-1$.

Now (*) becomes $b_ne^{c_nt} = 0\ \forall t. \Rightarrow b_n = 0$ ▨

11. (proof of Thm2.34). Following the hint, we first verify that $S = \{e^{c_1t}, te^{c_1t}, \dots, t^{n_1-1}e^{c_1t}, \dots, e^{c_kt}, te^{c_kt}, \dots, t^{n_k-1}e^{c_kt}\}$ lives in the solution space of $p(t) = (t-c_1)^{n_1}\cdots(t-c_k)^{n_k}$:

Given $t^{n_i-l}\cdot e^{c_it}$ for some $i \in \{1,\dots,k\}$ and some $l \in \{1,\dots,n_i-1\}$,

$(D-c_iI)^{n_i}(t^{n_i-l}e^{c_it}) = (D-c_iI)^{n_i-1}((n_i-l)t^{(n_i-l)-1}e^{c_it}) = \cdots = (D-c_iI)^{n_i-(n_i-l)}((n_i-l)!\,t^{(n_i-l)-(n_i-l)}e^{c_it}) = (D-c_iI)^{l}((n_i-l)!\,e^{c_it}) = (D-c_iI)^{l-1}(0) = \cdots = 0$.

∴ S lies in the solution space (續下頁).

Next, verify S is L.I. by math induction: For k=1, Thm 2.34 is just the case for Lemma in p138. Assume Thm 2.34 holds for k-1 distinct c_i's. Then for k distinct c_i's,

Assume $\left(\sum_{i=1}^{k} \sum_{j=0}^{n_i-1} a_{ij} \cdot t^j e^{c_i t} = 0 \quad \forall t, \text{ where } a_{ij} \in \mathbb{C} \right)$ — ① by proof of lemma.

Apply the operator $(D-c_kI)^{n_k} \Rightarrow 0 = (D-c_kI)^{n_k}\left(\sum_{i=1}^{k}\sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t}\right) \overset{\downarrow}{=} \underbrace{(D-c_kI)^{n_k}\left(\sum_{i=1}^{\boxed{k-1}}\sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t}\right)}_{}$ (*)

Now, by induction hypothesis, all the coefficients of the terms $t^j e^{c_i t}$ $\forall \, i=1,\dots,k-1, \; j=0,\dots,n_i-1$ in $(D-c_kI)^{n_k}(\sum\sum a_{ij} t^j e^{c_i t})$ are zero!

直接算太難繁 故用此技巧

Fix i. Observe that the coeffi. of $t^{n_i-1} e^{c_i t}$ is $(c_i-c_k)^{n_k} \cdot a_{i,n_i-1}$. It should be zero.

$\therefore (c_i-c_k)^{n_k} a_{i,n_i-1} = 0 \Rightarrow a_{i,n_i-1} = 0$. Next, reduce (*) to $\underbrace{(D-c_kI)^{n_k}\left(\sum_{i=1}^{k-1}\sum_{j=0}^{n_i-2} a_{ij} t^j e^{c_i t}\right)}$ (**)

since $a_{i,n_i-1} = 0$. Again, obs. the coeffi. of $t^{n_i-2} e^{c_i t}$ in (**) is $(c_i-c_m)^{n_k} a_{i,n_i-2}$.

$\therefore a_{i,n_i-2} = 0$. Repeat the process, we have $a_{i,j} = 0 \;\forall\, i=1,\dots,k-1, \; j=0,\dots,n_i-1$.

Hence, ① becomes $\sum_{j=0}^{n_k-1} a_{k,j} \cdot t^j e^{c_k t} = 0 \Rightarrow a_{k,j} = 0 \;\forall j$ by lemma (p.138). ▨

12. Following the hint, claim: $g(D)(V) \subseteq N(h(D))$.

proof. Given $y \in g(D)(V) = R(g(D_V))$, $\exists\, x \in V$ s.t. $g(D_V)(x) = y$. ($V \subset C^\infty$ is a n-dim sol space).

Then $h(D)(y) = h(D)(g(D_V)(x)) \overset{x \in V}{=} h(D_V)(g(D_V))(x) = p(D_V)(x) \overset{\swarrow}{=} 0$ ▨

Next, "$\dim(N(h(D))) = \dim(R(g(D_V)))$"

proof. Note that $N(g(D_V)) = N(g(D))$ since $N(g(D))$ is a subspace $\subseteq V$.

By lemma 1, $h(D)$ is onto (see exercise 13 of §2-7).

By lemma 2, $\dim(N(p(D))) = \dim(N(g(D))) + \dim(N(h(D)))$.

$\overset{\text{Thm 2.32}}{\Rightarrow} \dim(V) = \dim(N(g(D))) + \deg(h)$.

By dimension thm ($g(D_V): V \to C^\infty$), $\dim(V) = \dim(N(g(D_V))) + \dim(R(g(D_V)))$

Then $\deg(h) = \dim(N(h(D))) = \dim(R(g(D_V)))$ ▨

13. (a). Following the hint; claim: $p(D): C^\infty \to C^\infty$ is onto, $\forall\, p(t) \in P(\mathbb{F})$.

proof. Given $x \in C^\infty$, we want to find y s.t. $P(D)(y) = x$.

Note that by lemma 1, I, D is onto. $\Rightarrow D^n$ is onto $\forall n$.

Then all linear cobinations of D^n and I is also onto.

Hence, P(D) is onto ($p(D) = a_k D^{N_k} + \dots + a_1 D^{N_1} + a_0$). ▨

(b). Given z, a sol. to $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y^{(1)} + a_0 y = x(t)$. for some $x(t) \neq 0$.

Define $p(t) = t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$. V is the sol space to P(D).

§2-7.

If w is another sol i.e. $p(D)(w)=x$. Then $p(D)(w-z)=p(D)(w)-p(D)(z)=x-x=0$
$\Rightarrow w-z=y$ for some $y\in V$ $\therefore$ All the sol(s) s.t. $p(D)(\cdot)=x$. must be of the form $z+y$ for some $y\in V$. $\therefore \{z+y \mid y\in V\}=$ All solutions s.t. $p(D)(\cdot)=x$. ▨

14. Following the hint, Define $p(t)$ to be the auxiliary poly. of order n of the homo. linear diff. eq. Given x a sol., given $t_0\in\mathbb{R}$. For the case $n=1$, $p(D)(x)=0$
$\Rightarrow (D-cI)(x)=0$, $\Rightarrow x'-cx=0$, $\Rightarrow x$ is of the form ae^{ct}. By 題目, $x(t_0)=0=ae^{ct_0}$
$\Rightarrow a=0 \Rightarrow x\equiv 0$ # Suppose the statement in question is true for $n=k-1$.

Now assume $p(t)$ is of degree k. Then $p(D)(x)=0 \underset{\uparrow}{=} ((q(D))(D-cI))(x). = q(D)(x)\cdot(D-cI)(x)$
令 $p(t)=q(t)(t-c)$.
$\underset{\uparrow}{=} z\cdot(D-cI)(x)$.. By 題目, $x(t_0)=x'(t_0)=\cdots=x^{n-1}(t_0)=0$. 則. 因 $q(t)$ is of degree $k-2(=n-1)$.
令 $z=q(D)(x)$
故 $z(t_0)=0$.

另外, $0=p(D)(x)=(q(D)\cdot(D-cI))x=z\cdot(D-cI)x=(z'-cz)(x)$
$\Rightarrow z'-cz=0$ since x is a solution.

Again. $z'-cz=0$ with $z(t_0)=0 \Rightarrow z=ae^{ct}$ s.t. $z(t_0)=0 \Rightarrow z\equiv 0$

Now, $z=q(D)x\equiv 0$ (degree $=k-2(=n-1)$). by induction hypothesis, $x\equiv 0$ ▨

15. (a). • Φ is linear: $\Phi(cx+y)=\begin{pmatrix}(cx+y)(t_0)\\ \vdots \\ (cx+y)^{(n-1)}(t_0)\end{pmatrix}=c\begin{pmatrix}x(t_0)\\ x'(t_0)\\ x^{(n-1)}(t_0)\end{pmatrix}+\begin{pmatrix}y(t_0)\\ y'(t_0)\\ y^{(n-1)}(t_0)\end{pmatrix}$ #

- $N(\Phi)=\{0\}\subseteq V$: If $x\in N(\Phi)$, then $\Phi(x)=0$
  $\Rightarrow$. $x(t_0)=x'(t_0)=\cdots=x^{(n-1)}(t_0)=0$, By exercise 14 (p.143)., $x\equiv 0$ #
- Φ is isomo.: Φ is 1-1 since $N(\Phi)=\{0\}$ (zero function).
  By dimension thm, $\dim(V)(=n)=\text{nullity}(\Phi)^{\to 0}+\text{rank}(\Phi) \Rightarrow$. $\text{rank}(\Phi)=n=\dim(\mathbb{C}^n)$.
  $\Rightarrow \Phi$ is onto. ▨

(b). V is the sol space of an nth-order homo. linear diff eq.
We wanna prove $\exists!\, x\in V$ s.t. $x^{(k)}(t_0)=c_k\ \forall k=0,1,\dots,n-1$, when $\begin{pmatrix}c_0\\ c_1\\ \vdots\\ c_{n-1}\end{pmatrix}\in\mathbb{C}^n$ is given.
But this is true since Φ is an isomorphism. ▨

16. (a). Auxiliary poly $p(t)=t^2+\frac{g}{l}=(t+i\sqrt{\frac{g}{l}})(t-i\sqrt{\frac{g}{l}})$. $\therefore e^{-i\sqrt{\frac{g}{l}}t}$, $e^{+i\sqrt{\frac{g}{l}}t}$ are sols. $\Rightarrow$ (§2-7 #7) $\{\cos(\sqrt{\frac{g}{l}}t), \sin(\sqrt{\frac{g}{l}}t)\}$ is a sol basis ▨

(b). Initial conditions: $\theta(0)=\theta_0>0$; $\theta'(0)=0$. (ICs).
Let $c_1\cos(\sqrt{\frac{g}{l}}t)+c_2\sin(\sqrt{\frac{g}{l}}t)$ be a sol satisfying ICs.
$\Rightarrow$. $c_1=\theta_0$; $c_2=0 \Rightarrow$ the unique sol satisfying ICs is $x(t)=\theta_0\cdot\cos(\sqrt{\frac{g}{l}}t)$ ▨

(C). The period of the unique sol; $x(t) = \theta_0 \cdot \cos(\sqrt{\frac{g}{\ell}}t)$, is $2\pi\sqrt{\frac{\ell}{g}}$.

Since the sol is unique, we have that the period of the pendulum is also the same ▣

17. Solve. $y'' + \frac{k}{m}y = 0$. $p(t) = t^2 + \frac{k}{m}$. $\therefore \{\cos(\sqrt{\frac{k}{m}}t), \sin(\sqrt{\frac{k}{m}}t)\}$ is its sol basis ▣

18. (a). solve $my'' + ry' + ky = 0$. $p(t) = mt^2 + rt + k$ 令 $\alpha = \frac{-r+\sqrt{r^2-4mk}}{2m}$; $\beta = \frac{-r-\sqrt{r^2-4mk}}{2m}$

則 $\alpha . \beta$ are roots of $p(t)$ $\therefore$ The sol is. $y(t) = c_1 \cdot e^{\alpha t} + c_2 e^{\beta t}$ ▣

(b). IC_s: $y(0) = 0$, $y'(0) = v_0$ $\Rightarrow \begin{cases} c_1 + c_2 = 0 \\ \alpha c_1 + \beta c_2 = v_0 \end{cases} \Rightarrow c_1 = \frac{v_0}{\alpha - \beta}$; $c_2 = \frac{v_0}{\beta - \alpha}$

$\Rightarrow y(t) = \frac{mv_0}{\sqrt{r^2-4mk}}(e^{\alpha t} - e^{\beta t})$ ▣

(c). $\lim_{t\to\infty} y(t) = \frac{mv_0}{\sqrt{r^2-4mk}} \lim_{t\to\infty}(e^{\alpha t} - e^{\beta t})$ — (*)

- If $r^2 - 4mk > 0$, then both α, β are negative real numbers. $\therefore (*) \xrightarrow{t\to\infty} 0$ ☆
- If $r^2 - 4mk \le 0$, then $\alpha = \frac{-r}{2m} + i\sqrt{4mk - r^2}$ & $\beta = \bar{\alpha}$,

  $\Rightarrow e^{\alpha t} - e^{\beta t} = e^{\frac{-r}{2m}t} \cdot (2i)\sin(\sqrt{4mk-r^2}\,t) \longrightarrow 0$ as $t \to \infty$ ▣

19. 有時我們總喜欢在複数 field 上看 solutions.(僅管其事實上是 real-valued functions). 是因為，解微分方程時，在複数体上比較容易解(代数基本定理, $p(t)$ 在 $\mathbb{C}$ 上 split). 就算解出来之後發現是 real-valued functions. 把它當作 complex-valued function 亦不影响結果。($\mathcal{F}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C},\mathbb{C})$). ☆

20. (a). (proof of Thm2.27). Let x be a solution to a given homo. linear diff. eq.

We use the math induction technique on the number of derivatives possessed by x.

- For the case $k=1$. $p(t)$ is of degree 1 s.t. $p(D)(x) = 0$, say $p(t) = t - c$.

  $\Rightarrow x' = cx$. This means that $x^{(1)}$ is a linear combination of x. $\Rightarrow x^{(1)}$ must have 1 derivative, $\therefore x^{(2)}$ exists s.t. $x^{(2)} = cx^{(1)}, \dots$ Repeat the process, $x \in C^\infty$ ☆
- Assume Thm2.27 holds for all $n < k$. Assume $p(t)$ is of degree k. $\Rightarrow p(D)(x) = 0$.

  $\Rightarrow x^{(j)}$ exists $\forall j = 0, 1, \cdots, k$. Factor $p(t) = a_k t^k + q(t)$, where $q(t)$ is a poly. of degree less than k. Then $0 = p(D)(x) = a_k D^k(x) + q(D)(x) \Rightarrow x^{(k)} = -\frac{1}{a_k}q(D)(x)$.

  This means that $x^{(k)}$ is a linear combinations of $\{x, x', x^{(2)}, \dots, x^{(k-1)}\}$

  $\therefore x^{(k+1)}$ exists. Repeat the process, $x \in C^\infty$. actually ▣

§2-7.- §3-1. p.53.

(b). Let $c \in \mathbb{C} = a+bi,\ a,b \in \mathbb{R}$. Then $\underset{\sim}{e^{c+d}} = e^{(a+m)+(b+n)i} \doteq e^a e^m(\cos(b+n)+i\sin(b+n))$.
$d \in \mathbb{C} = m+ni,\ m,n \in \mathbb{R}$. $= e^a e^m(\cos b + i\sin b)(\cos n + b\sin n) = e^c e^d$ ☆

- $e^{c+(-c)} = 1 \Rightarrow e^c e^{-c} = 1 \Rightarrow e^{-c} = \frac{1}{e^c}$. ▨

(c).(proof of thm 2.28). Let V be the set of all solutions to the homo. linear diff. eq. with constant coeffi. with auxiliary poly $p(t)$.

- $\forall x$ s.t $p(D)(x)=0$, x is a sol. $\Rightarrow x \in V \Rightarrow N(p(D)) \subset V$.
- If $x \in V$, x is a sol $\Rightarrow p(D)(x)=0 \Rightarrow x \in N(p(D)) \Rightarrow V \subset N(p(D))$ ▨

(d) (proof of thm 2.29).

Given $c \in \mathbb{C}$. Let $c=a+bi,\ a,b \in \mathbb{R}$. Define $f(t) = e^{ct}$.

Then $f'(t) = \frac{d}{dt}e^{ct} = (e^{at+ibt})' = (e^{at}(\cos bt + i\sin bt))'$

$= ae^{at}(\cos bt + i\sin bt) + be^{at}(-\sin bt + i\cos bt)$.

$= e^{at}[a(\cos bt + i\sin bt) + ib(\cos bt + i\sin bt)]$.

$= ce^{at}(\cos bt + i\sin bt) = ce^{at}e^{ibt} = ce^{ct}$ ▨

(e). 略. (Too boring).

(f). Given $x \in \mathcal{F}(\mathbb{R},\mathbb{C})$ 且 $x'=0$. Claim: x = const. function.

proof. Assume that $x = x_1 + ix_2,\ x_1, x_2 \in \mathcal{F}(\mathbb{R},\mathbb{R})$.

$x'=0 \Rightarrow x_1' + ix_2' = 0 \Rightarrow x_1' = x_2' = 0 \Rightarrow x_1(t) = k_1,\ x_2(t) = k_2$ for some $k_1, k_2 \in \mathbb{R} \Rightarrow x(t) = x_1(t) + ix_2(t) = k_1 + ik_2 = c_0$ for some $c_0 \in \mathbb{C}$. ▨

§section 3-1.

1. (a). True, since elementary must be invertible. (b). False. (c). True.

(d). False. 反例: $\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&2\\1&0\end{pmatrix}$, which is not a elementary matrix.

(e). True, by Thm3.2. (f). False, 反例: $\begin{pmatrix}2&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}3&0\\0&2\end{pmatrix}$ ☆

(g). True. (see exercise§3-1#5). (h). False, 反例: $A=\begin{pmatrix}1&0\\1&0\end{pmatrix}$, $B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$.

則 $B=\underbrace{\begin{pmatrix}1&0\\-1&1\end{pmatrix}}_{\text{row operation}}A$, 滿足題意。但 A 不管怎樣操作 column vector operation 都不可能得到 B. ☆

(i). True.

2. Let $E_1 = \begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}$, 即第1欄乘以(-2)加到第2欄。$\Rightarrow B = AE_1$. ☆

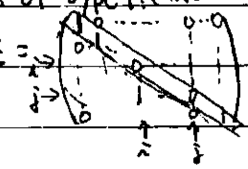
Let $E_2 = \begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$, 即第1列乘以(-1)加到第2列。$\Rightarrow C = E_2A$. ☆

Let $F_1 = \begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}$, $F_2 = \begin{pmatrix}1&0&0\\0&1&-1\\0&0&1\end{pmatrix}$, $F_3 = \begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}$, $F_4 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{2}\end{pmatrix}$, $F_5 = \begin{pmatrix}1&0&-3\\0&1&0\\0&0&1\end{pmatrix}$. ▨

$(F_5F_4F_3F_2F_1C=I)\ C = \begin{pmatrix}1&0&3\\0&-2&-2\\1&-3&1\end{pmatrix} \overset{F_1}{\rightsquigarrow} \begin{pmatrix}1&0&3\\0&-2&-2\\0&-3&-2\end{pmatrix} \overset{F_2}{\rightsquigarrow} \begin{pmatrix}1&0&3\\0&1&0\\0&-3&-2\end{pmatrix} \overset{F_3}{\rightsquigarrow} \begin{pmatrix}1&0&3\\0&1&0\\0&0&-2\end{pmatrix} \overset{F_4}{\rightsquigarrow} \begin{pmatrix}1&0&3\\0&1&0\\0&0&1\end{pmatrix} \overset{F_5}{\rightsquigarrow} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I_3$.

4. Recall that the def of elementary matrix E is to perform either 1 or 2 or 3 type. operation on In.

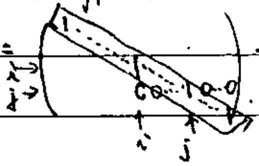
Case 1: If E is of type 1. (interchange ith row & jth row),

then E = 
. clearly, E can also be obtained by interchanging the ith & jth column on In. ※

Case 2 If E is of type 2 (multiplying the ith row by a const c),

then E = 
. Clearly, E can also be obtained by xc on the ith column of In ※

Case 3. If E is of type 3. (adding c×(ith row) to the jth row),

then E = 
. Clearly, E can also be obtained by adding c×(jth column) to the ith column of In ■

5. (⇒) (⇐): Trivially, the elementary matrix of type 1 & 2 are symmetric.

And the transpose of an elementary matrix E, who adds c×(ith row [column]) to jth row [column], is a matrix E^t that adds c×(ith column [row]) to the jth column. ■

6. 已知 B = EA for some E being elementary matrix.

则 $B^t = (EA)^t = (A^t)\cdot E^t$. 故 B^t = 對 A^t 作 column 操作 ※

7. 略 (boring).

8. 已知 Q = EP. for some E being elementary matrix.

By Thm 3.2, E^{-1} exists and is of the same type as E. ∴ $P = E^{-1}Q$ ■

12. 此即高斯消去法...(消成上三角). ■
的变形.

§3-2.

p.55.

1. (a). False. (b). False. 反例: $\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$. (c). True. (d). True.
(e). False, it must preserve rank. (f). True. (g). True. (h) True. (i) True.

3. ($\Leftarrow$). $A=O \Rightarrow \operatorname{rank}(A)=0$ ※

($\Rightarrow$). Suppose A is not a zero matrix $\Rightarrow \exists A_{ij}\neq 0 \Rightarrow$ The ith row is an indep. set
$\Rightarrow$ rank >0. ※ ▥

⟨Another⟩. $\operatorname{rank}(A)=\operatorname{rank}(\overset{:\mathbb{F}^n\to\mathbb{F}^m}{L_A})=0 \Rightarrow L_A(x)=0\ \forall x\in\mathbb{F}^n \Rightarrow L_A$ is a zero function
$\Rightarrow A=$ zero matrix O. ※

6. (a). $[T]_\beta=\begin{pmatrix}-1&2&2\\0&-1&4\\0&0&-1\end{pmatrix}$. $\Rightarrow$ T is invertible. $\left(\begin{array}{ccc|ccc}+1&-2&-2&-1&0&0\\0&+1&-4&0&-1&0\\0&0&+1&0&0&-1\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&-10&-1&-2&0\\0&1&-4&0&-1&0\\0&0&1&0&0&-1\end{array}\right)$

$\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&0&-1&-2&-10\\0&1&0&0&-1&-4\\0&0&1&0&0&-1\end{array}\right)\Rightarrow[T^{-1}]_\beta=\begin{pmatrix}-1&-2&-10\\0&-1&-4\\0&0&-1\end{pmatrix}$. $\therefore T^{-1}(ax^2+bx+c)=(-c-2b-10a)+(-b-4a)x$
$+(-10a)x^2$. ※

(b). $[T]_\beta=\begin{pmatrix}0&1&0\\0&1&2\\0&0&2\end{pmatrix}$. $\Rightarrow$ T is not invertible.

(c). $[T]_\beta=\begin{pmatrix}1&2&1\\-1&1&2\\1&0&1\end{pmatrix}$. $\left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\-1&1&2&0&1&0\\1&0&1&0&0&1\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&3&3&1&1&0\\0&-2&0&-1&0&1\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&3&0&1&1\\0&-2&0&-1&0&1\end{array}\right)$.

$\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&3&0&1&1\\0&0&6&-1&2&3\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&0&-\frac{1}{6}&-\frac{1}{3}&\frac{1}{2}\\0&1&0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&1&\frac{-1}{6}&\frac{1}{3}&\frac{1}{2}\end{array}\right)\Rightarrow[T^{-1}]_\beta=\begin{pmatrix}-\frac{1}{6}&-\frac{1}{3}&\frac{1}{2}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{-1}{6}&\frac{1}{3}&\frac{1}{2}\end{pmatrix}$

$\therefore T^{-1}(a,b,c)=(-\frac{1}{6}a-\frac{1}{3}b+\frac{1}{2}c,\ \frac{1}{2}a-\frac{1}{2}c,\ \frac{-1}{6}a+\frac{1}{3}b+\frac{1}{2}c)$ ※

(d). $[T]_\beta^\gamma=\begin{pmatrix}1&1&1\\1&-1&1\\1&0&0\end{pmatrix}$ $\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&-1&1&0&1&0\\1&0&0&0&0&1\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&+1&-1&0&-1&+1\\0&1&1&1&0&-1\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&-1&0&-1&1\\0&0&2&1&1&-2\end{array}\right)$.

$\rightsquigarrow\left(I_3\left|\begin{array}{ccc}0&0&1\\\frac{1}{2}&-\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&-1\end{array}\right.\right)$ $[T^{-1}]_\gamma^\beta$. $\therefore T^{-1}(ax^2+bx+c)=(a,\ \frac{1}{2}c-\frac{1}{2}b,\ \frac{1}{2}a+\frac{1}{2}b-c)$ ※

(e). $[T]_\beta^\gamma=\begin{pmatrix}1&-1&1\\1&0&0\\1&1&1\end{pmatrix}$. $\left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\1&0&0&0&1&0\\1&1&1&0&0&1\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&0&0&1&0\\0&1&1&0&-1&1\\0&-1&1&+1&-1&0\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|ccc}1&0&0&0&1&0\\0&1&1&0&-1&1\\0&0&2&1&-2&1\end{array}\right)$

$\rightsquigarrow\left(I_3\left|\begin{array}{ccc}0&1&0\\\frac{-1}{2}&0&\frac{1}{2}\\\frac{1}{2}&-1&\frac{1}{2}\end{array}\right.\right)$ $[T^{-1}]_\gamma^\beta$. $\therefore T^{-1}(a,b,c)=(b,\ \frac{-1}{2}a+\frac{1}{2}c,\ \frac{1}{2}a-b+\frac{1}{2}c)$. ※

(f). $[T]_\beta^\gamma=\begin{pmatrix}1&0&0&1\\1&0&0&1\\0&1&1&0\\0&1&1&0\end{pmatrix}$. rank $=2$. $\therefore$ T is not invertible. ▥

8. $c\neq 0$. $R(L_A)=L_A(\mathbb{F}^n)=cL_A(\mathbb{F}^n)=L_{cA}(\mathbb{F}^n)=R(L_{cA})$.
$\therefore \operatorname{rank}(A)=\operatorname{rank}(cA)$ ▥

9. 略 (easy).

10 略.

11. 略 (Tedious) (troublsome).

12 略 (Tedious).

13. trivial.

14. (a). $\forall y \in R(T+U)$., $y = T(x)+U(x)$ for some $x \in V$. $\Rightarrow$. $y \in R(T)+R(U)$. ∎

(b). By part (a), $\mathrm{ran}(T+U) \leq \dim((R(T)+R(U))) \underset{\substack{\uparrow \\ \S 1.6\ \#29\ (a).}}{\leq} \mathrm{rank}(T)+\mathrm{rank}(U) - \dim(R(T)\cap R(U))$.

$\leq \mathrm{rank}(T)+\mathrm{rank}(U)$. ∎

(c). $\mathrm{rank}(A+B) = \mathrm{rank}(L_{A+B}) = \mathrm{rank}(L_A+L_B) \overset{\text{part(b).}}{\leq} \mathrm{rank}(L_A)+\mathrm{rank}(L_B) = \mathrm{rank}(A)+\mathrm{rank}(B)$ ∎

15 略 (trivial)

16 略.(similar to the proof of (a) of Thm3.4)

17. • By Thm3.7.(C) or (d), or by direct row [column] operations, $\mathrm{rank}(BC) \leq 1$ ∎

• Conversely, let A be a 3×3 matrix ~~of~~ with rank 1. $\Rightarrow$. one of the column vectors, say the ith column $v_i = \begin{pmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \end{pmatrix}$ is the maximal linearly indep. set.. Hence, other colums are a const. multiple of v_i, say the coeffi. is b_j for the jth column. Then pick $B = \begin{pmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \end{pmatrix}$. $C = (b_1, b_2, b_3)$. Then $BC = \begin{pmatrix} b_1 A_{i1} & b_2 A_{i1} & b_3 A_{i1} \\ b_1 A_{i2} & b_2 A_{i2} & b_3 A_{i2} \\ b_1 A_{i3}. & b_2 A_{i3} & b_3 A_{i3} \end{pmatrix} = A$ ∎

18. 參#17想法. $\underset{(m\times p)}{(AB)} = \sum_{i=1}^{n} \underset{(m\times1)}{A_{\text{row } i}} \cdot \underset{(1\times p).}{B_{\text{column } i}}$ ∎

19. L_A has rank m. -① ($L_A: \mathbb{F}^n \to \mathbb{F}^m$). $L_B: \mathbb{F}^p \to \mathbb{F}^n$ has rank n. -② (onto).

$\therefore \mathrm{rank}(L_{AB}) = \mathrm{rank}(L_A L_B) = \dim(R(L_A L_B)) = \dim(L_A L_B(\mathbb{F}^p))$.

$\overset{②}{=} \dim(L_A(\mathbb{F}^n)) \overset{①}{=} \dim(\mathbb{F}^m) = m$ ∎

20. (a) solve $Ax=0$, we get the solution space $\{(x_3+3x_5, -2x_3+x_5, x_3, -2x_5, x_5) \mid x_i \in \mathbb{R}\}$

$\therefore \left\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$ is a basis for the sol. space. so we can construct M.

e.g. $M = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$. ∎

(b). $AB = O \Rightarrow Au_j = 0$, u_j denotes the jth column of B.

Since $\mathrm{nullity}(A) = 2$., B has at most 2 linearly indep. column vectors. ∎

21. $L_A: \mathbb{F}^n \to \mathbb{F}^m$ is onto since it's full rank. $\therefore \forall e_i \in \mathbb{F}^m$, $i=1,\dots,m$, $\exists x_i \in \mathbb{F}^n$ s.t. $L_A(x_i) = e_i$. Define B = $n\times m$ matrix with jth column being x_j. Then $AB = I_m$ ∎

22. $B^t \in M_{n\times m}(\mathbb{F})$ having rank m, L_{B^t} is onto $\therefore \forall e_i \in \mathbb{F}^m$, $\exists x_i \in \mathbb{F}^n$ s.t. $L_{B^t}(x_i) = e_i$. Define A^t = $n\times m$ matrix with jth column being x_j.

Then $B^t A^t = I_m \Rightarrow AB = I_m^t = I_m$ ∎

Section §3-3.

1.(a). False. (b). False (c). True, the zero sol. (d). False, eg. $\begin{cases} x_1+x_2=1 \\ 2x_1+2x_2=2 \end{cases}$

(e). False, eg. $\begin{cases} x_1+x_2=1 \\ x_1+x_2=2 \end{cases}$ (f). False, need rank(A) = rank(A|b) i.e. $b\in R(L_A)$ (Thm 3.11)

(g). True, $Ax=0$, A is invertible $\overset{1\text{-}1}{\underset{\text{onto}}{\Rightarrow}}$ $\exists!\, x$ (solution) s.t. $L_A(x)=0$ In fact, $x=0$ ※

(h). False, $Ax=b$ with $b\neq 0$ might not have '$x=0$' as a sol. E.g. p173, Example3 ※

3. (a). $S=\left\{\begin{pmatrix}5\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$. (b). $S=\left\{\begin{pmatrix}(t+2)/3\\(2t+1)/3\\t\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\3\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$

(c). $S=\left\{\begin{pmatrix}3-t\\t\\t\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}=\left\{\begin{pmatrix}3\\0\\0\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$ ※

(d). $\left(\begin{array}{ccc|c}2&1&-1&5\\1&-1&1&1\\1&2&-2&4\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|c}1&-1&1&1\\0&3&-3&3\\0&3&-3&3\end{array}\right)\rightsquigarrow\left(\begin{array}{ccc|c}1&0&0&2\\0&1&-1&1\\0&0&0&0\end{array}\right)$ $\therefore S=\left\{\begin{pmatrix}2\\1+t\\t\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$

$=\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\1\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$.

(e). $S=\left\{\begin{pmatrix}1-2t_2+3t_3-t_4\\t_2\\t_3\\t_4\end{pmatrix} \,\middle|\, t_2,t_3,t_4\in\mathbb{R}\right\}=\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+t_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+t_3\begin{pmatrix}3\\0\\1\\0\end{pmatrix}+t_4\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \,\middle|\, t_2,t_3,t_4\in\mathbb{R}\right\}$ ※

(f). $S=\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ (g). $S=\left\{\begin{pmatrix}-t_2-2t_3\\t_2\\t_3\\1-t_2+t_3\end{pmatrix} \,\middle|\, t_2,t_3\in\mathbb{R}\right\}=\left\{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+t_2\begin{pmatrix}-1\\1\\0\\-1\end{pmatrix}+t_3\begin{pmatrix}-2\\0\\1\\1\end{pmatrix} \,\middle|\, t_2,t_3\in\mathbb{R}\right\}$ ※

4. (a). $A=\begin{pmatrix}1&3\\2&5\end{pmatrix}$. $A^{-1}=\begin{pmatrix}-5&3\\2&-1\end{pmatrix}$ $\therefore$ sol $=A^{-1}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}-11\\5\end{pmatrix}$ ※

(b). $A=\begin{pmatrix}1&2&-1\\1&1&1\\2&-2&1\end{pmatrix}$ $A^{-1}=\frac{1}{9}\begin{pmatrix}3&+1&-4\\0&3&+6\\3&-2&-1\end{pmatrix}^t=\frac{1}{9}\begin{pmatrix}3&0&3\\1&3&-2\\-4&6&-1\end{pmatrix}$. $\therefore$ sol $=A^{-1}\begin{pmatrix}5\\1\\4\end{pmatrix}=\begin{pmatrix}+3\\+0\\-2\end{pmatrix}$ ※

5 略.

6. $[T]_{\gamma\leftarrow\mathbb{R}^3}^{\beta\leftarrow\mathbb{R}^2}=\begin{pmatrix}1&1&0\\2&0&-1\end{pmatrix}_{2\times3}$ Now, solve $\begin{pmatrix}1&1&0\\2&0&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}1\\11\end{pmatrix}$.

$\Rightarrow \begin{cases}a+b=1\\2a-c=11\end{cases}$ $\Rightarrow \begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}t\\1-t\\2t-11\end{pmatrix}=\begin{pmatrix}0\\1\\-11\end{pmatrix}+t\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $t\in\mathbb{R}$ ∎

7. (a). $A=\begin{pmatrix}1&1&-1&2\\1&1&2&0\\2&2&1&2\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&-1&1\\0&1&2&0\\0&2&1&1\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&0&1\\0&0&2&1\\0&1&2&2\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&0&0\\0&0&2&1\\0&1&2&1\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&1&1&0\end{pmatrix}$ has rank 2.

$(A|b)=\begin{pmatrix}1&1&-1&2&2\\1&1&2&0&1\\2&2&1&2&4\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&0&0&0\\0&1&1&+1&+1\\0&2&1&+1&0\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&0&0&0\\0&1&0&1&1\\0&2&0&1&0\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1&0&0&0\\0&0&0&1&1\\0&0&0&1&0\end{pmatrix}$ has rank 3

inconsistent ※

以下略.

8. (a). $[T]_\beta^\gamma=\begin{bmatrix}1&1&0\\0&1&-2\\1&0&2\end{bmatrix}\rightsquigarrow\begin{pmatrix}1&1&0\\0&1&-2\\0&-1&2\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ has rank 2.

Next, $\begin{pmatrix}1&1&0&1\\0&1&-2&3\\1&0&2&-2\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&1&0&1\\0&1&-2&3\\0&-1&2&-3\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&0&0&0\\0&1&-2&3\\0&0&0&0\end{pmatrix}$ has rank 2.

$\therefore$ By Thm 3.11, $v\in R(T)$.

(b). $\begin{pmatrix}1&1&0&2\\0&1&-2&1\\1&0&2&1\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&1&0&2\\0&1&-2&1\\0&-1&2&-1\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&0&0&0\\0&1&-2&1\\0&0&0&0\end{pmatrix}$ has rank 2 $\therefore v\in R(T)$ ∎

§3-3 ~ §3-4.

9. (⇒). $Ax=b$ has a solution x_0. ⇒ $L_A(x_0)=b$ ⇒ $b\in R(L_A)$ ※

(⇐). $b\in R(L_A)$ ⇒ $\exists\ x\in$ domain s.t. $L_A(x)=b$. ⇒ $Ax=b$ ▨

10. True., proof: $L_A: F^n\to F^m$ has rank $=m$. ∴ L_A is surjective.

∴ $Ax=b$ always has a sol. whatever b is ▨

11. [$Ap=p$.]. ⇒ $[A-I](p)=0$ ⇒ $\frac{1}{16}\begin{pmatrix}-9&8&3\\5&\frac{-40}{8}&5\\4&\frac{16}{3}&-8\end{pmatrix}\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}=0 \sim \left(\begin{array}{ccc|c}-9&8&3&0\\5&-\frac{40}{3}&5&0\\4&\frac{16}{3}&-8&0\end{array}\right)$

$\sim\left(\begin{array}{ccc|c}-9&8&3&0\\15&-40&15&0\\12&16&-24&0\end{array}\right)\sim\left(\begin{array}{ccc|c}-9&8&3&0\\-3&-24&21&0\\3&24&-21&0\end{array}\right)\sim\left(\begin{array}{ccc|c}+9&-8&-3&0\\3&24&-21&0\\0&0&0&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&8&-7&0\\10&0&-10&0\\0&0&0&0\end{array}\right).$

∴ $\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}=\begin{pmatrix}t\\3/4\,t\\t\end{pmatrix}=t\begin{pmatrix}4\\3\\4\end{pmatrix}$, $t\in\mathbb{R}$. ∴ $p_1:p_2:p_3=4:3:4.=\frac{4}{11}:\frac{3}{11}:\frac{4}{11}$ ※▨

12. $A=\begin{pmatrix}0.6&0.3\\0.4&0.7\end{pmatrix}$. solve. $(I-A)x=0$ ⇒ $x=t\begin{pmatrix}3\\4\end{pmatrix}$, $t\in\mathbb{R}$ ▨

13. $(I-A)x=d$. ⇒ $\begin{pmatrix}+\frac{1}{2}&-\frac{1}{5}\\-\frac{1}{3}&+\frac{4}{5}\end{pmatrix}x=\begin{pmatrix}2\\5\end{pmatrix}\sim\left(\begin{array}{cc|c}+5&-2&20\\-5&+12&75\end{array}\right).\sim\left(\begin{array}{cc|c}5&-2&20\\0&10&95\end{array}\right).$

∴ $x=\begin{pmatrix}+7.8\\+9.5\end{pmatrix}$ ▨

14. $d=\begin{pmatrix}90\\20\end{pmatrix}$. (billion) $A=\begin{array}{ccc} & \text{goods.} & \text{ser.}\\ \text{goods} & 0.5 & 0.2\\ \text{ser} & 0.3 & 0.6\end{array}$. Solve $(I-A)x=d$.

⇒ $\left(\begin{array}{cc|c}0.5&-0.2&90\\-0.3&0.4&20\end{array}\right)\sim\left(\begin{array}{cc|c}1&-0.4&180\\0&0.28&74\end{array}\right)$. ⇒ $x=\begin{pmatrix}2000/7\\1850/7\end{pmatrix}$ (billions) ▨

§3-4. section.

1. (a). False, we can't apply elementary "column" operations to $(A|b)$.

(b) True. (C). True, by Thm 3.16. (d). True. (e). Flase, e.g. $(A|b)=\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&0&1\end{array}\right)$ ※

(f). True, $\underset{(m\times n)}{A}x=b$, $(A|b)$ is a reduced echelon form. & consistent.

$r:=$ # of nonzero rows in A.. So we have $\text{rank}(A|b)=\text{rank}(A).=r$.

∴ $\dim(K_H)=\dim(N(L_A))=n-r$ by dimension thm ※

(g). True, By def of reduced echelon form, all nonzero rows in A' are L.I.

By the Corollary to thm 3.4, $\text{rank}(A)=\text{rank}(A')$ ▨

3. (a). (⇒). Note that A' is also reduced echelon form. $\text{rank}(A')\neq\text{rank}(A'|b')$

⇒ $\exists$ nonzero rows in $(A'|b')$ but not in A'. ⇒. This row must has nonzero entry in the last column (b'那行). ※

(⇐). Every nonzero row in A' has its corresponding row in $(A'|b')$ also a nonzero row. And $(A'|b')$ has a row in which the only nonzero lies in the last column. ⇒ it doesn't attribute the rank of A' ∴ $\text{rank}A'\neq\text{rank}(A$

§3-4.

(b). By (a), rank(A')= rank(A'|b') ⇔ (A'|b') contains no row in which the only nonzero entry lies in the last column.

But by Thm 3.11 (p.174) and the fact that rank(A) = rank(A') and rank(A|b) = rank(A'|b'), we complete the proof ▣

5. Use thm 3.16. Let $A=\begin{pmatrix}1&0&u&1&v\\-1&-1&&-2&\\3&1&&0&\end{pmatrix}$.

Then $u=2\begin{pmatrix}1\\-1\\3\end{pmatrix}+(-5)\begin{pmatrix}0\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix}$ & $v=-2\begin{pmatrix}1\\-1\\3\end{pmatrix}-3\begin{pmatrix}0\\-1\\1\end{pmatrix}+6\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}4\\-7\\-9\end{pmatrix}$

∴ $A=\begin{pmatrix}1&0&2&1&4\\-1&-1&3&-2&-7\\3&1&1&0&-9\end{pmatrix}$ ▣

6. Idea: compute a_5, the 5th column vector of A.

$a_6=\begin{pmatrix}3\\-9\\2\\5\end{pmatrix}=5\begin{pmatrix}1\\-2\\-1\\3\end{pmatrix}+2\begin{pmatrix}-1\\1\\2\\-4\end{pmatrix}+(-1)a_5 \Rightarrow a_5=\begin{pmatrix}0\\1\\-3\\2\end{pmatrix}$ &

∴ $a_2=-3\begin{pmatrix}1\\-2\\-1\\3\end{pmatrix}$ and $a_4=4\begin{pmatrix}1\\-2\\-1\\3\end{pmatrix}+3\begin{pmatrix}-1\\1\\2\\-4\end{pmatrix}$ ∴ $A=\begin{pmatrix}1&-3&-1&1&0&3\\-2&6&1&-5&1&-9\\-1&3&2&2&-3&2\\3&-9&4&0&2&5\end{pmatrix}$ ▣

7. $\begin{pmatrix}2&1&-8&1&-3\\-3&4&12&37&-5\\1&-2&-4&-17&8\end{pmatrix}\leadsto\begin{pmatrix}1&\frac{1}{2}&-4&\frac{1}{2}&-\frac{3}{2}\\0&1&0&7&-19/11\\0&0&0&0&1\end{pmatrix}\sim\begin{pmatrix}1&0&-4&-3&0\\0&1&0&7&0\\0&0&0&0&1\end{pmatrix}$ ∴ $\{u_1,u_2,u_5\}$ is a basis ▣

8. 略.

9 略.

10 略. } similar to #12. or 7.

11 略.

12. (a). Trivial.

(b). First find a basis of V. .... $\beta=\left\{\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\1\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\-2\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}-3\\-2\\0\\0\\0\\1\end{pmatrix}\right\}$

Next, do the Gaussian elimination to tranform (S|β) to a reduced echelon form:

$$\begin{pmatrix}0&1&1&-1&1&-3\\-1&0&1&1&-2&-2\\0&1&1&0&0&0\\1&1&0&1&0&0\\1&1&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\leadsto\begin{pmatrix}\boxed{1}&1&0&1&0&0\\0&\boxed{1}&1&0&0&0\\0&0&0&\boxed{1}&-1&0\\0&0&0&0&0&\boxed{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}.$$

S

∴ $\{\underbrace{\begin{pmatrix}0\\-1\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\1\\1\\0\end{pmatrix}}_{S},\begin{pmatrix}-1\\1\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}-3\\-2\\0\\0\\0\\1\end{pmatrix}\}$ is a basis ▣

13 略. - similar to #12.

14. 略. (just check the def. of reduced echelon form).

not so sure. ⑮. [Nontrivial].

No.

Date §4-1.

Secion§4-1.

1. (a). False, 反例: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$, but $\det\left(m\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = m^2(ad-bc)$. ※

(b). True, by Thm 4.1 ▨ (C). False, by Thm 4.2 ※

(d). False, it should be $\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|$. (e). True.

2. (a) 30 (b). −17 (C). −8.

3. (a). $-10+15i$ (b) $36+41i$ (c). −24.

4. [illegible]

5. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{F})$, $\det A = ad-bc$, $\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = bc-ad = -\det(A)$. ▨

6. Let $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \in M_{2\times2}(\mathbb{F})$, $\det A = ab-ab = 0$ ※▨

7. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{F})$, $\det A = ad-bc$. & $\det(A^t) = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad-bc$ ▨

8. Let $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_{2\times2}(\mathbb{F})$., $\det A = ad - 0\cdot b = ad$ = product of the diagonal entries ▨

9. Let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M_{2\times2}(\mathbb{F})$. $\Rightarrow$ $AB = \begin{pmatrix} a_1a_2+c_2b_1 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}$

Then $\det(AB) = (a_1a_2+c_2b_1)(c_1b_2+d_1d_2) - (c_1a_2+d_1c_2)(a_1b_2+b_1d_2)$

$= (\cancel{a_1c_1a_2b_2} + a_1d_1a_2d_2 + b_1c_1b_2c_2 + \cancel{b_1d_1c_2d_2}) - (\cancel{a_1c_1a_2b_2} + b_1c_1a_2d_2 + a_1d_1b_2c_2 + \cancel{b_1d_1c_2d_2})$

$= a_1d_1(a_2d_2-b_2c_2) + b_1c_1(b_2c_2-a_2d_2) = (a_1d_1-b_1c_1)(a_2d_2-b_2c_2)$

$= \det(A)\cdot\det(B)$ ▨

10. (a) Directly check that $cA = AC = \det(A)\cdot I_2$, since we do not know whether A^{-1} exists or not ▨

(b). $\det(AC) = \det(\det(A)\cdot I_2) = \det^2(A)$; but $\det(AC) = \det(A)\cdot\det(C)$ by #9.

$\Rightarrow$. $\det(A) = \det(C)$ ▨

§4-1.-§4-2.

(c). 令 $A^t = B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$.

∴ The classical adjoint of A^t is $\begin{pmatrix} B_{22} & -B_{12} \\ B_{21} & B_{11} \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^t$. ■.

(d). A is invertible ⇒ A^{-1} exists $= \frac{1}{\det(A)} \cdot C = [\det(A)]^{-1} \cdot C$ by Thm 4.2 ■

11. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{F})$, $\det(A) = ad - bc$.

$$\delta(A) = \delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{(i)}{=} a\delta\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} + b\delta\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \overset{(i)}{=} a\left(c\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + d\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) + b\left(c\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

$$\underset{(iii)}{\overset{(ii)}{=}} a(c\cdot 0 + d\cdot 1) + b\left(c\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\cdot 0\right) = ad + bc\cdot\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Claim: $\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$.

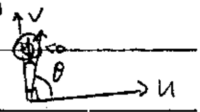
$$\text{pf: } 0 \overset{(ii)}{=} \delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \overset{(i)}{=} \delta\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \overset{(i)}{=} \left(\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) + \left(\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

$$\underset{(iii)}{\overset{(ii)}{=}} (0+1) + \left(\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 0\right) \Rightarrow \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$ ✱ ■

12. Recall that $\{u, v\}$ is called right-handed if u can be rotated in a counterclockwise direction through an angle θ ($0<\theta<\pi$) to coincide with v. (p.202).

(⇒). $O\begin{pmatrix} u \\ v \end{pmatrix} = 1 = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|} \Rightarrow \det\begin{pmatrix} u \\ v \end{pmatrix} > 0$. Let $u = (a, b)$, $v = (c, d)$.

$u' := u\cdot(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})) = (-b, a)$. Then $u'\cdot v = \det\begin{pmatrix} u \\ v \end{pmatrix} > 0$.



That means the angle φ btw u' & v is belonging to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

but $\theta = \frac{\pi}{2} + \phi$. ∴ $\theta \in (0, \pi)$ ⇒ right-handed system ✱

(⇐). Right-handed system ⇒ $\theta \in (0, \pi)$, (θ is the angle btw u & v, oriented counter-clockwise). Define $u' = (-b, a)$, i.e. rotate u by $\frac{\pi}{2}$ (counterclockwise).

Then the angle φ btw u' & v is $\phi = \theta - \frac{\pi}{2}$. Then $u'\cdot v = ad - bc = \det\begin{pmatrix} u \\ v \end{pmatrix}$

and $u'\cdot v = |u'| \times |v| \cdot \cos\phi > 0$ since $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. ■

Section §4-2.

1. (a). False, $\det(cA) = c^n \det(A)$ if $A \in M_{n\times n}(\mathbb{F})$. (b). True, by Thm 4.4

(c). True, by coro. to thm 4.4. (d). True, by Thm 4.5.

(e). False, By Thm 4.3, $\det(B) = k\cdot\det(A)$. (f) False, By Thm 4.6, $\det(B) = \det(A)$ ✱

(g). False, $\det(A)$ might be other values.

(h). True, 依序對 1st 行, 2nd 行 … nth 行 做 cofactor expansion 即可得証 ■

2. $k = 27$. ✱

3. $k = 42$ ✱

No.

Date §4-2 — §4-3.

#4~22 跳過.

23. proof. Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & a_{33} & \vdots \\ O & & \ddots & a_{nn} \end{pmatrix}_{n\times n} \in M_{n\times n}(\mathbb{F})$.

$\det A \overset{\text{1st column of } A}{=} a_{11}\det(\tilde{A}_{11}) + 0\cdot\det(\tilde{A}_{21}) + 0\cdot\det(\tilde{A}_{31}) + \cdots + 0\cdot\det(\tilde{A}_{n1}) = a_{11}\det(\tilde{A}_{11})$.

$\left(B =: \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ & a_{33} & & \vdots \\ O & & \ddots & a_{nn} \end{pmatrix}_{(n-1)\times(n-1)} \right)$.

$\overset{\text{1st column of } B}{=} a_{11}\cdot\left[a_{22}\det(\tilde{B}_{11}) + 0\cdot\det(\tilde{B}_{21}) + \cdots + 0\cdot\det(\tilde{B}_{n-1,1}) \right] = a_{11}\cdot a_{22}\det(\tilde{B}_{11})$.

Repeat the process, we have $\det(A) = \prod^{n} a_{ii}$ ∎

24. Let z be the zero row vector.

Then $\det\begin{pmatrix} \xrightarrow{a_1} \\ \xrightarrow{a_2} \\ \vdots \\ z \\ \vdots \\ \xrightarrow{a_n} \end{pmatrix} = \det\begin{pmatrix} \xrightarrow{a_1} \\ \xrightarrow{a_2} \\ \vdots \\ 0\cdot z \\ \vdots \\ \xrightarrow{a_n} \end{pmatrix} \overset{\text{Thm 4.3}}{=} 0\cdot\det\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ z \\ \vdots \\ a_n \end{pmatrix} = 0$ ∎

25 略.

26. Apply #25, $\det(-A) = (-1)^n\cdot A$, so $\det(-A) \neq \det(A)$ if n is even, or if $\mathbb{F}$ is of characteristic 2. ($\mathbb{Z}_2$ field ~~→~~ so that $-1=1$). ∎

27 略.

28 略.

29. Just check each of the three ~~of~~ types of elementary matrix.

30. We can interchange the i-th row and the $(n+1-i)$-th row, for each $i = 1, 2, \ldots \lfloor \frac{n}{2} \rfloor$

∴ $\det(B) = (-1)^{\lfloor \frac{n}{2} \rfloor}\det(A)$. ∎

Section §4-3.

1. (a). Fake. Type 2 ⇒ $\det(E) = k\det(I_n) = k$ for some $k \in \mathbb{F}$.

(b). True, by thm4.7 (c) False, invertible ⇔ $\det \neq 0$, by coro. to thm4.7.

(d). True. (e). False, $\det(A^t) = \det(A)$, by thm4.8. (f). True.

(g). Fake, $\det(A) \neq 0$ is required. (h). False, M_k 應是換掉 "column k" by b. ※

另外, (h)的 statement 是對的 if the eq. is $xA = b$, where $x = (x_1, x_2, \ldots, x_n)$. ∎

8. Claim: $\det\begin{pmatrix} v_1 & v_2 & \cdots & (x+ky) & \cdots & v_n \end{pmatrix}_{n\times n} = \det\begin{pmatrix} v_1 & \cdots & x & \cdots & v_n \end{pmatrix}_{n\times n} + k\det\begin{pmatrix} v_1 & \cdots & y & \cdots & v_n \end{pmatrix}_{n\times n}$.

proof. Let $A = (v_1\ v_2 \cdots (x+ky) \cdots v_n)_{n\times n}$. Then $\det(A^t) = \det(A)$, so it suffice to check $\det\begin{pmatrix} v_1^t \\ \vdots \\ x^t + ky^t \\ \vdots \\ v_n^t \end{pmatrix} = \det\begin{pmatrix} v_1^t \\ \vdots \\ x^t \\ \vdots \\ v_n^t \end{pmatrix} + k\det\begin{pmatrix} v_1^t \\ \vdots \\ y^t \\ \vdots \\ v_n^t \end{pmatrix}$, but it is exactly Thm 4.3. ∎

9. By coro. to Thm4.7 & by §4.2 #23 (p.222), $\det(A) = \prod_{i=1}^{n} a_{ii}$, if A is an upper triangular matrix, and thus A is invertible ⇔ $\prod a_{ii} \neq 0$ ∎

§4-3. p. 63.

10. M is nilpotent $\Rightarrow M^k = 0$ for some $\overset{(\neq 0)}{k} \in \mathbb{N}$.

Then $0 = \det(M^k) = \det(M \cdot M^{k-1}) = \det(M) \cdot \det(M^{k-1}) = \cdots = \det(M)^k$

$\Rightarrow \det(M) = 0$ ∎

11. M is skew-symmetric $\Rightarrow M^t = -M$, $M \in M_{n\times n}(\mathbb{C})$.

If n is odd, $\det(M) \overset{\text{Thm 4.8}}{=} \det(M^t) = \det(-M) = (-1)^{\overset{\text{odd}}{\textcircled{n}}} \det(M) = -\det(M)$

$\Rightarrow 2\det(M) = 0 \Rightarrow \det(M) = 0$ ∎

if n is even, $\det(M) = (-1)^n \det(M) = \det(M)$, so we can not say anything about M. M may or may not be invertible ∎

12. Q is orthogonal $\Rightarrow QQ^t = I_n \overset{\text{Thm 4.7}}{\Rightarrow} \det(Q)\det(Q^t) = 1$.

$\Rightarrow 1 \overset{\text{Thm 4.8}}{=} \det(Q)^2 \Rightarrow \det(Q) = \pm 1$. ∎

13. (a). For n=1, $\det(\overline{M}) = \overline{M_{11}} = \overline{\det(M_{11})}$ holds.

Suppose the statement is true for all $n < k$, for some $k \overset{\geq 2}{\in} \mathbb{N}$ — (*).

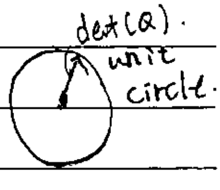
For the case $n = k$, $\det(\overline{M}) = \sum_{j=1}^{n+1} (-1)^{1+j} \det(\widetilde{\overline{M_{1j}}}) \cdot \overline{M_{1j}}$

$\overset{(*)}{=} \sum_{j=1}^{n+1} (-1)^{1+j} \cdot \overline{\det(\widetilde{M_{1j}})} \cdot \overline{M_{1j}} = \overline{\sum_{j=1}^{n+1} (-1)^{1+j} \det(\widetilde{M_{1j}}) \cdot M_{1j}} = \overline{\det(M)}$ ∎

(b). Q is unitary $\Rightarrow Q\overline{Q^t} = I_n \overset{\text{part (a)}}{\Rightarrow} \det(Q)\overline{\det(Q^t)} = 1$.

$\Rightarrow \det(Q)\overline{\det(Q)} = 1$. 令 $\det(Q) = c_1 + ic_2$. Then $(c_1 + ic_2)(c_1 - ic_2) = 1$

$\Rightarrow c_1^2 + c_2^2 = 1 \Rightarrow |\det(Q)| = 1$. ∎



14. $B \in M_{n\times n}(F)$, $B = (u_1, u_2, \ldots, u_n)$.

$\det(B) \neq 0 \Leftrightarrow$ B is invertible by coro. to thm 4.7. $\Leftrightarrow L_B$ is 1-1 & onto

$\Leftrightarrow$ B is full rank (i.e. $\text{rank}(B) = n$). $\Leftrightarrow \beta = \{u_1, \ldots, u_n\}$ is a basis for $R(L_B) = F^n$ ∎

15. A is similar to B $\Leftrightarrow \exists$ invertible matrix Q s.t. $A = Q^{-1}BQ$.

$\Rightarrow \det(A) = \det(Q^{-1}(BQ)) = \det(Q^{-1})\det(BQ) = \det(Q^{-1})\det(B)\det(Q)$

$= \det(B)$ ∎

16. $AB = I_n$ & A, B are square matrix. Claim: A is invertible. (Also, see §2.4 #9, p.107)

proof: Suppose not, A is not full-ranked. $\Rightarrow L_{AB}$ is not onto.

but $L_{AB} = L_I$ is onto $\rightarrow\!\leftarrow$ ∎

Next, $A^{-1}AB = A^{-1}I_n \Rightarrow B = A^{-1}$ ∎

No.

Date §4-3

17. $AB=-BA$ and $A, B \in M_{n\times n}(\mathbb{F})$, where $\mathbb{F}$ is not of characteristic 2.

$\det(AB)=\det(-BA) \Rightarrow \det(A)\det(B) = \det(-B)\det(A) = (-1)^n\det(B)\det(A)$.

$= -\det(B)\det(A)$ since $n=$ odd. $\Rightarrow \det(A)\det(B) = 0$.

$\Rightarrow$. A or B (or both) is not invertible ✱

18. 略.

19. A is a lower triangular matrix.  含对角線的地方.

$\det(A) = \det(A^t) = \prod_{i=1}^{n} A_{ii}$ ∎

20. $M := \left(\begin{array}{c|c} A_{k\times k} & B \\ \hline O & I_{n-k} \end{array}\right)_{n\times n}$ $\det(M) \overset{\text{对第n列展開}}{=} \sum_{j=1}^{n} (-1)^{(n)+j}\cdot M_{n,j}\cdot \det(\tilde{M}_{n,j})$.

$= (-1)^{2n}\times 1 \times \det(\tilde{M}_{n,n}) = \det(\tilde{M}_{n,n})$

Repeat the process, we finally have $\det(M) = \det(A)$ ∎

21. 証法一: If C is not invertible, the set of $\boxed{\text{row}}$ vectors of C is not indep.

$\Rightarrow$. $(O\ C)$ is also not $\text{L.I.}_{(\text{row})}$ $\Rightarrow$ M is not invertible. $\therefore \det(M) = 0 = \det(A)\overset{\overset{0}{\uparrow}}{\det}(C)$ ✱

If C is invertible, observe the identity: $\begin{pmatrix} I & O \\ O & C^{-1} \end{pmatrix}\begin{pmatrix} A & B \\ O & C \end{pmatrix} = \begin{pmatrix} A & B \\ O & I \end{pmatrix}$.

Then $\det\begin{pmatrix} I & O \\ O & C^{-1} \end{pmatrix}\cdot \det M = \det\begin{pmatrix} A & B \\ O & I \end{pmatrix} \underset{(\#20)}{\Rightarrow} \det(C^{-1})\cdot\det(M) = \det(A)$.

$\Rightarrow \det(M) = \det(A)\det(C)$. ∎

証法二: (數歸). For $n=2$, $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. $\det M = ad = \det(A)\det(C)$. ✱

Suppose the statement is true for all $n<k$. $(k\geq 2, k\in\mathbb{N})$.

For the case $n=k$, $\det M = \det\begin{pmatrix} A & B \\ O & C \end{pmatrix}$ (suppose $A\in M_{r\times r}(\mathbb{F})$).

$= \sum_{j=1}^{n} (-1)^{n+j} M_{n,j}\cdot \underbrace{\det(\tilde{M}_{n,j})} \underset{\uparrow}{=} \sum_{j=r+1}^{n} (-1)^{n+j}\cdot C_{n,j}\cdot \underbrace{\det(A)\det(\tilde{C}_{n,j})}$.

↳ induction hypothesis.

$= \det(A)\cdot\det(C)$ ∎

22. (a) $M := [T]_\beta^\gamma = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ 1 & c_2 & c_2^2 & \cdots & c_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}_{(n+1)\times(n+1)}$ (Vandermonde Matrix).

(b). By §2.4 #22, T is an isomorphism. $\Rightarrow [T]_\beta^\gamma = M$ is full rank. $\Rightarrow \det(M) \neq 0$ ∎

✱(c). (Use math induction). For $n=1$. $\det\begin{pmatrix} 1 & c_0 \\ 1 & c_1 \end{pmatrix} = (c_1-c_0) = \prod_{0\leq i<j\leq 1}(c_j-c_i)$ holds.

Assume the equality holds for all $k<n$. For the case $k=n$,

$\det\begin{pmatrix} 1 & c_0 & \cdots & c_0^n \\ 1 & c_1 & \cdots & c_1^n \\ \vdots & \vdots & & \vdots \\ 1 & c_n & \cdots & c_n^n \end{pmatrix} = \det\begin{pmatrix} 1 & c_0 & \cdots & c_0^n \\ 0 & c_1-c_0 & \cdots & c_1^n-c_0^n \\ \vdots & \vdots & & \vdots \\ 0 & c_n-c_0 & & c_n^n-c_0^n \end{pmatrix}_{(n+1)\times(n+1)} = \det\begin{pmatrix} c_1-c_0 & \cdots & c_1^n-c_0^n \\ c_2-c_0 & \cdots & c_2^n-c_0^n \\ \vdots & & \vdots \\ c_n-c_0 & & c_n^n-c_0^n \end{pmatrix}_{n\times n} =$ 續下頁……

§ 4-3.

$$= \det\begin{pmatrix} c_1-c_0 & (c_1-c_0)\cdot(c_1+c_0) & \cdots & (c_1-c_0)\cdot(c_1^{n-1}+c_1^{n-2}c_0+c_1^{n-3}c_0^2+\cdots+c_0^{n-1}) \\ c_2-c_0 & (c_2-c_0)(c_2+c_0) & \cdots & (c_2-c_0)\cdot(c_2^{n-1}+c_2^{n-2}c_0+c_2^{n-3}c_0^2+\cdots+c_0^{n-1}) \\ \vdots & \vdots & & \\ c_n-c_0 & (c_n-c_0)\cdot(c_n+c_0) & \cdots & (c_n-c_0)\cdot(c_n^{n-1}+c_n^{n-2}c_0+c_n^{n-3}c_0^2+\cdots+c_0^{n-1}) \end{pmatrix}_{n\times n}$$

令 $p(x,y,k) = x^{k-1}+x^{k-2}y+\cdots+y^{k-1}$

$$= \prod_{j=1}^{n}(c_j-c_0)\cdot\det\begin{pmatrix} 1 & p(c_1,c_0,2) & \cdots & p(c_1,c_0,n) \\ 1 & p(c_2,c_0,2) & \cdots & p(c_2,c_0,n) \\ \vdots & & & \vdots \\ 1 & p(c_n,c_0,2) & \cdots & p(c_n,c_0,n) \end{pmatrix}_{n\times n}$$

(*).

$$= \prod_{j=1}^{n}(c_j-c_0)\cdot\det\begin{pmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ 1 & c_2 & & c_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & c_n & & c_n^{n-1} \end{pmatrix}_{n\times n}.$$

Thus by induction hypothesis, $= \prod_{j=1}^{n}(c_j-c_0)\cdot\prod_{1\le i<j\le n}(c_j-c_i) = \prod_{0\le i<j\le n}(c_j-c_i)$ $\blacksquare$

Now we claim (*) is correct:

pf. Write $e_i = (c_1^i, c_2^i, \cdots, c_n^i)^t$, for $i=0,1,\cdots,n-1$.

Then $\det(e_0,\ e_1+c_0e_0,\ e_2+c_0e_1+c_0^2e_0,\ \cdots\cdots,\ e_{n-1}+c_0e_{n-2}+\cdots+c_0^{n-1}e_0)$

$= \det(e_0,\ e_1,\ e_2+c_0e_1,\ \cdots,\ e_{n-1}+c_0e_{n-2}+\cdots+c_0^{n-2}e_1)$

$= \det(e_0,\ e_1,\ e_2,\ \cdots,\ e_{n-1}+c_0e_{n-2}+\cdots+c_0^{n-3}e_2)$

$= \cdots\cdots = \det(e_0, e_1, e_2, \cdots\cdots, e_{n-1})$ ※ (claim finished). ※

23.(a). Suppose A' is the largest $k\times k$ submatrix s.t. $\det(A')\neq 0$.

- W.L.O.G., we may assume $A = \left(\begin{array}{c|c} A'_{k\times k} & B \\ \hline C & D_{n-k} \end{array}\right)_{n\times n}$, because we can always interchange any two rows or columns, so we have at most 2k steps to move A' to the most up-left place. (Note that interchanging rows or columns preserves rank). Since $\det(A')\neq 0$, A' is full rank, so $a_1, a_2, \cdots, a_k$ are indep., where a_i is the i-th column of A'. (or A).
- Now, if $\operatorname{rank}(A) > k$, say $= k+l^{\neq 0}$, then $\exists\ \beta = \{a_0, a_1, \cdots, a_k, b_1, \cdots, b_l\}$ s.t. β is L.I., where $a_0, a_1, \cdots, a_k, b_1, \cdots, b_l$ are column vectors of A. Also, there are $k+l$ indep. row vectors. $\therefore \exists$ submatrix $B_{(k+l)\times(k+l)}$ s.t. B has rank $k+l$, $\Rightarrow$ B is invertible. $\Rightarrow$ $\det(B)\neq 0$ ※ (A' should be the largest matrix s.t. $\det(A')\neq 0$).
- $\operatorname{rank}(A) \geq \operatorname{rank}(A') = k$ is clear.

Hence, $\operatorname{rank}(A) = k$ $\blacksquare$

(b). $A_{n\times n}$ has rank k. $\Rightarrow \exists$ k indep. column vectors, $a_1, \cdots, a_k$. Let D be the $n\times k$ matrix, having a_i as its i-th column, $\forall i = 1, \cdots, k$. Then $k \leq \operatorname{rank}(D) \leq \min\{n,k\} = k$.
$\Rightarrow \operatorname{rank}(D) = k$. $\Rightarrow \exists$ k rows being L.I. $\Rightarrow \exists$ submatrix $E_{k\times k}$ of D s.t. $\operatorname{rank}(E) = k$.
$\Rightarrow \det(E)\neq 0$ $\blacksquare$

24. $\det(A+tI) = \det\begin{pmatrix} t&0&0&\cdots&0&a_0\\ -1&t&0&\cdots&0&a_1\\ 0&-1&t&\cdots&0&a_2\\ \vdots&0&-1&\ddots&t&\vdots\\ 0&0&0&\cdots&t&a_{n-2}\\ 0&0&0&\cdots&-1&a_{n-1}+t \end{pmatrix} = \det\begin{pmatrix} t&&&&a_0\\ -1&t&&&\vdots\\ &&\ddots&&\\ &&t&&a_{n-3}\\ &&-1&t&a_{n-2}+(a_{n-1}+t)t\\ &&&-1&0 \end{pmatrix}$

$= \det\begin{pmatrix} t&&&&a_0\\ -1&t&&&\vdots\\ &-1&&&\\ &&\boxed{t}&&a_{n-3}+a_{n-2}t+(a_{n-1}+t)t^2\\ &&\boxed{-1}&t&(0)\\ &&&-1&0 \end{pmatrix} = \cdots = \det\begin{pmatrix} t&\cdots&&&▨\\ -1&t&&&\vdots\\ &-1&\ddots&&0\\ &&-1&t&0\\ &&&-1&0 \end{pmatrix}_{n\times n}$ (, where

共 n-1 項

▨ $= a_0+a_1t+a_2t^2+\cdots+(a_{n-1}+t)t^{n-1}$).

$= (-1)^{n+1}$ ▨ $\cdot \det\begin{pmatrix} -1&t&&\\ 0&-1&t&\\ &&-1&\ddots\\ &&&t\\ 0&&&-1 \end{pmatrix}_{(n-1)\times(n-1)} = ($▨$\cdot(-1)^{n+1})\cdot\underset{\text{upper triangular}}{(-1)^{n-1}} =$ ▨ $= \sum_{i=0}^{n-1} a_it^i + t^n$ ▨

25 (a). $c_{jk} := (-1)^{j+k}\det(\tilde{A}_{jk})$.

$\det(B) \underset{\text{cofactor expansion along the k-th column.}}{=} \sum_{i=0}^{n}(-1)^{i+k}\cdot B_{ik}\cdot\det(\tilde{B}_{ik})$, $(B_{ik}=\delta_{ik})$

$= (-1)^{j+k}\det(\tilde{B}_{jk}) = (-1)^{j+k}\det(\tilde{A}_{jk}) = c_{jk}$ ▨

(b). Following the hint, apply Cramer's rule to $Ax=e_j$.

$\therefore \forall i,\ x_i = \frac{\det(C_i)}{\det(A)}$, where C_i is the matrix obtained from A by replacing column i by e_j.

$\therefore x_i = c_{ji}/\det(A)$ (by part (a)). $\forall i$.

Hence $A\begin{pmatrix} c_{j1}\\ \vdots\\ c_{jn}\end{pmatrix} = \det(A)\cdot e_j$ ▨

(c). $(C_{n\times n})_{ij} = c_{ji},\ \forall i,j$.

By part (b), $A\begin{pmatrix} c_{j1}\\ \vdots\\ c_{jn}\end{pmatrix} = \det(A)\cdot e_j$, but $\begin{pmatrix} c_{j1}\\ \vdots\\ c_{jn}\end{pmatrix}$ is the j-th column of C.

Hence, $AC = \det(A)\cdot I_n$. ▨

(d). $\det(A)\neq 0 \Rightarrow$ A is invertible. $\Rightarrow A^{-1}AC = A^{-1}\det(A)\cdot I_n$

$\Rightarrow C = A^{-1}\cdot I_n\cdot\det(A) = A^{-1}\cdot\det(A). \Rightarrow A^{-1} = C/\det(A)$. ▨

以略 (~~classical adjoint~~ ~~$\begin{pmatrix} \tilde{A}_{11}&\tilde{A}_{12}&\cdots&\tilde{A}_{1n}\\ \vdots&&&\\ \tilde{A}_{n1}&&\cdots&\tilde{A}_{nn}\end{pmatrix}^t$~~), ~~$\tilde{A}_{ij} = (-1)^{i+j}\det(\tilde{A}_{ij})$~~

27. (a). If A is not invertible, $\Rightarrow \det(A)=0$. By #25 part (c), $AC = (\det(A))\cdot I_n = O$

$\Rightarrow$ C is not invertible, o.w., $A = OC^{-1} = O$ and the classical adjoint of $A=O$ is O. ⟶✱ ✱ $\Rightarrow \det(C)=0$ Hence, $0=(\det(A))^{n-1}=\det(C)$.

Next, if A is invertible, $AC = \det(A)\cdot I_n \Rightarrow \det(A)\det(C) = (\det(A))^n$

$\Rightarrow \det(C) = (\det(A))^{n-1}$ ▨

§4-3 – §4-4 .. §5-1.

(b). Let c_{ij} denote the cofactor of position i,j of A.

So the classical adjoint of A would be $\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \ddots & \\ \vdots & & \\ c_{n1} & & c_{nn} \end{pmatrix}^t = \begin{pmatrix} c_{11} & & c_{n1} \\ \vdots & \ddots & \\ c_{1n} & \cdots & c_{nn} \end{pmatrix} := C.$

Then the classical adjoint of A^t would be C^t clearly ▨

(C). For $i>j$, $C_{ij} = c_{ji} = (-1)^{i+j} \det(\tilde{A}_{ji}). = 0$,

Since. $\tilde{A}_{ji}$ is an upper-triangular matrix and will have at least one zero diagonal entry if $i>j$.. (不信的话,畫图就知道了).

$\therefore$ C is an upper triangular matrix., and so is A^{-1} $(= \frac{1}{\det(A)} C)$. ▨

28.(a). Define $v(y(t)) := \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n)}(t) \end{pmatrix}$ and $v_i(y(t)) = v_i(t) = \begin{pmatrix} y_i(t) \\ y_i'(t) \\ \vdots \\ y_i^{(n)}(t) \end{pmatrix}$. Then $\forall x,y \in C^\infty$, and. $c \in F$,

$[T(cx+y)](t) = \det\begin{pmatrix} v(cx+y)(t) & v_1(t) & v_2(t) & \cdots & v_n(t) \end{pmatrix}$

differential operator is linear

$\downarrow$

$= \det\begin{pmatrix} cv(x)(t) + v(y)(t) & v_1(t) & v_2(t) & \cdots & v_n(t) \end{pmatrix}.$

$= c\det(v(x)(t) \ v_1(t) \cdots v_n(t)) + \det(v(y)(t) \ v_1(t) \cdots v_n(t))$ ▨

$\uparrow$

det is linear when the other columns or rows are fixed.

(b). Clearly, by cora to Thm4.4, p.215, $y_i \in N(T)$ $\forall i=1,\dots,n$.

but N(T) is a subspace, $\therefore$ $N(T) \supset \text{span}(\{y_1,\dots,y_n\})$ (by Thm1.5, p30) ▨

Section §4-4.

1. (a). T. (b). T. (c). T. (d). F. (e). F. (f). T. (g). T. (h). F.
(i). T. (j). T. (k). T.

2~4略.

5. the same as #20, p.229.

6. the same as #21, p.229.

Section §5-1.

1. (a). F, (b). T, if v is the eigenvector of the real matrix.. then tv, $\forall t \in \mathbb{R}$. is also an eigenvector. (C). T. example2, p.247. (d) F. (e). F. (f). F, 無任何關聯.
(g). F. eg. $V = P(F)$. $\beta = \{1, x, x^2, \dots\}$ is L.I. Let T be the identiy transformation.
So. $T(x) = x = \lambda \cdot x \Rightarrow \lambda = 1$. is the eigenvalue. (h). T. (i). T. see #12, p.259.
(j). F. (k). F. e.g. $A = \begin{pmatrix} +1 & +1 \\ 0 & 0 \end{pmatrix}$ $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ but $v_1 + v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not an eigenvector. ▨

2.(a). $[T]_\beta = \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$. , β is <u>not</u> a basis consisting of eigenvectors of T.

略.

3.(a). • $p(t) = (1-t)(2-t)-6 = t^2-3t-4 = (t+1)(t-4)$ $\therefore$ $\lambda_1=-1$, $\lambda_2=4$.

•• For λ_1, solve $(A-\lambda_1 I)x=0 \Rightarrow \begin{pmatrix} 1-\lambda_1 & 2 \\ 3 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a+2b=0 \\ 3a+3b=0 \end{cases} \Rightarrow x = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, t \in \mathbb{R}-\{0\}$.

•• For λ_2, solve $(A-\lambda_2 I)x=0 \Rightarrow \begin{cases} -3a+2b=0 \\ 3a-2b=0 \end{cases} \Rightarrow x = t\begin{pmatrix} 2 \\ 3 \end{pmatrix}, t \in \mathbb{R}-\{0\}$ is the eigenvector.

••• $\beta = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$

•••• $Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$

略.

5. proof of Thm 5.4.

($\Rightarrow$). Suppose $v \in V$ is an eigenvector of T corresponding to λ.

By def, ① $v \neq 0$. Next, $\overset{\Downarrow}{A}v = \lambda v \Leftrightarrow (A-\lambda I_n)v = 0 \Leftrightarrow v \in N(A-\lambda I_n)$. ※

($\Leftarrow$). $v \neq 0$ & $v \in N(A-\lambda I_n) \Rightarrow (A-\lambda I_n)v=0 \Rightarrow Av=\lambda v$, $v \neq 0$

$\Rightarrow$ v is an eigenvector. ∎

6. $T(v) = \lambda v \Leftrightarrow [T]_\beta [v]_\beta = \lambda [v]_\beta$. because $\underset{x \mapsto [x]_\beta}{\phi_\beta : V \to V}$ is an isomorphism. ∎

(For the detail, $\phi_\beta(T(v)) = [T(v)]_\beta = [\lambda v]_\beta = \lambda [v]_\beta$. , but $[T(v)]_\beta = [T]_\beta [v]_\beta$ by Thm 2.14).

7. (a). let Q be the n×n matrix s.t. $Q = [I_V]_\gamma^\beta$. We know that $[T]_\beta = [I_V]_\gamma^\beta [T]_\gamma [I_V]_\beta^\gamma$
$= Q[T]_\gamma Q^{-1}$.

Then $\det([T]_\beta) = \det(Q[T]_\gamma Q^{-1}) = \det(Q)\cdot\det([T]_\gamma Q^{-1})$

(Thm 4.7) $= \det([T]_\gamma)$ ∎

(b) ($\Rightarrow$). Assume T is invertible. $\overset{\text{Thm 4.7.}}{\Rightarrow}$ $\det(T)\det(T^{-1}) = \det(TT^{-1}) = \det(I_V) = 1$.

(Coro. to Thm 4.7.) $\Rightarrow \det(T) \neq 0$

($\Leftarrow$). Assume $\det(T) \neq 0$. If T is not invertible, T is not of full rank.

$\therefore \det(T) = 0$ ⇸.

(c). $\det(T)\det(T^{-1}) = 1$. $\Rightarrow \det(T^{-1}) = (\det(T))^{-1}$ ∎

(d). By Thm 4.7 ※

(e). $\det(T-\lambda I_V) = \det([T-\lambda I_V]_\beta) = \det([T]_\beta - \lambda[I_V]_\beta) = \det([T]_\beta - \lambda I)$ ∎

8. (a). ($\Rightarrow$). T is invertible $\Rightarrow \det(T) \neq 0$. If $\lambda_0=0$ is an eigenvalue of T, then $T(v) = \lambda_0 v \Rightarrow (T-\underset{0}{\underset{\shortparallel}{\lambda_0}} I_n)\underset{0}{\underset{\nparallel}{v}} = 0$ for some eigenvector $v (\neq 0)$. $\Rightarrow (T-\lambda_0 I_n)$ is not 1-1.

and hence is not full rank. $\Rightarrow \det(T-\lambda_0 I_n) = 0 = \det(T)$. ⇸ ※

($\Leftarrow$). $\lambda_0 = 0$ is not an eigenvalue of T. $\Rightarrow \nexists\ v(\neq 0) \in V$ s.t. $Tv = \lambda_0 v$.
i.e. $Tv = 0$ ∴ T is 1-1 $\Rightarrow$ T is of full rank $\Rightarrow$ $\det(T) \neq 0$.

(b). λ is an eigenvalue of T. $\Leftrightarrow T(v) = \lambda v$ for some $v \neq 0$.
$\Leftrightarrow T^{-1}T(v) = v = T^{-1}(\lambda v) = \lambda T^{-1}(v) \Leftrightarrow \frac{1}{\lambda} v = T^{-1}(v)$. $\Leftrightarrow \lambda^{-1}$ is an eigenvalue of T ▨

(c) 略.

9. Suppose λ is an eigenvalue of M. $\Rightarrow Mx = \lambda x$ for some $x \neq 0 \in \mathbb{F}^n$.
$\Rightarrow (M - \lambda I_n)\underset{\neq 0}{x} = 0 \underset{\text{not 1-1}}{\Rightarrow} \det(M - \lambda I_n) = 0 \Rightarrow \prod_{i=1}^{n}(M_{ii} - \lambda) = 0$.
Hence. λ can be M_{11}, or M_{22} or M_{33}, ..., M_{nn}. ▨

10 (a) just compute directly.
(b). $p(t) = \det(\lambda I_V - tI_n) = \det([\lambda I_V]_\beta - tI_n)$ (here we choose the std. ordered basis β). $= \det((\lambda - t)I_n) = (\lambda - t)^n$.
(c) • The set of eigenvalues = the zeroes of $p(t) = \{t \mid p(t) = 0\} = \lambda$ ※
• $[\lambda I_V]_\beta = \lambda I_n$ is a diagonal matrix. ▨

11. (a). $A = Q^{-1}(\lambda I)Q \Rightarrow A = Q^{-1}\lambda Q = \lambda Q^{-1}Q = \lambda I$ ▨
(b). Let D be a diagonal matrix having only one eigenvalue, say $\bar{\lambda}$.
$\Rightarrow Dv = \bar{\lambda} v$ for some $v \neq 0 \in \mathbb{F}^n$. $\Rightarrow (D - \bar{\lambda} I)v = 0 \Rightarrow \det(D - \bar{\lambda} I) = 0$
$\Rightarrow \prod_{i=1}^{n}(D_{ii} - \bar{\lambda}) = 0 \Rightarrow D_{ii} = \bar{\lambda}\ \forall i = 1, \dots n$. ∴ $D = \lambda I_n$. ▨
(c). $A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $p(t) = \det(A - tI_2) = (t-1)^2$ ∴ $\lambda = 1$ is the only eigenvalue of A.
Observe that $A - \lambda I_2 = A - I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has the nullity $= 1$.
∴ we can only find the L.I. set consisting of eigenvectors of size 1. ∴ By Thm 5.1, the matrix is not diagonalizable. ▨

12. (a). If A and B are similar. ($A, B \in M_{n\times n}(\mathbb{F})$), then $\exists Q \in M_{n\times n}(\mathbb{F})$, invertible, s.t. $A = Q^{-1}BQ$. $\Rightarrow$ $\underbrace{\det(A - tI_n)} = \det(Q^{-1}BQ - tI_n) = \det(Q^{-1}BQ - tQ^{-1}Q)$.
$= \det(Q^{-1}BQ - Q^{-1}tQ) = \det(Q^{-1}(BQ - tQ)) = \det(Q^{-1}(B - tI_n)Q)$
$\overset{\text{Thm 4.7}}{=} \det(Q^{-1})\det((B - tI_n)Q) = \det(Q^{-1})\det(B - tI_n)\det(Q) = \underbrace{\det(B - tI_n)}$. ▨
(b). Suppose β & γ are ordered basis for V. Let $[T]_\beta = A$. $[T]_\gamma = B$.
Let $Q = [I_V]_\gamma^\beta$. Then Q is invertible and $B = Q^{-1}AQ$. By part (a). A & B have the same characteristic poly. ▨

13.(a). $A\phi_\beta(v)=\lambda\phi_\beta(v) \Leftrightarrow [T]_\beta[v]_\beta=\lambda[v]_\beta \Leftrightarrow [T(v)]_\beta=[\lambda v]_\beta$.

Since ϕ_β is an isomorphism, ϕ_β^{-1} exists. $\Rightarrow \phi_\beta^{-1}([T(v)]_\beta)=\phi_\beta^{-1}([\lambda v]_\beta)$

$\Leftrightarrow T(v)=\lambda v$ ▨

(b). $y\in F^n$ is an eigenvector of A corresponding to λ. $\Leftrightarrow Ay=\lambda y. \Leftrightarrow \boxed{L_A(y)}=\lambda y$. $\in F^n$

$\underset{\phi_\beta \text{ isomo.}}{\Leftrightarrow} \phi_\beta^{-1}(L_A(y))=\phi_\beta^{-1}(\lambda y) \Leftrightarrow \phi_\beta^{-1}(\phi_\beta T\phi_\beta^{-1}(y))=\phi_\beta^{-1}(\lambda y)=\lambda\phi_\beta^{-1}(y)$.

$L_A\phi_\beta=\phi_\beta T$

$\Leftrightarrow T(\phi_\beta^{-1}(y))=\lambda\phi_\beta^{-1}(y)$ ▨

14. Recall that $\det(A^t)=\det(A)$ (see Thm 4.8).

Then $\det(A-tI_n)=\det(A^t-tI_n)$. ▨

15 (a) $T(x)=\lambda x$. For any $m\in\mathbb{N}$, $T^m(x)=T^{m-1}T(x)=T^{m-1}(\lambda x)=\lambda T^{m-1}(x)$

$=\cdots=\lambda^m x$. ▨

(b) 略.

16 (a) $A=Q^{-1}BQ \overset{\text{Hint}}{\Rightarrow} tr(A)=tr(Q^{-1}BQ)=tr((BQ)Q^{-1})=tr(B)$ ▨

(b). Define $tr(T)=tr([T]_\beta)$, the det is indep. of the choice of β ▨

17. (a). $T(A)=A^t$. It's hard to compute $\det(T-\lambda I)$, so we use another skill.

Suppose $T(A)=\lambda A$ for some $A\neq 0$. Then $A^t=\lambda A$. $\Rightarrow (A^t)^t=(\lambda A)^t$

$\Rightarrow A=\lambda A^t \Rightarrow A=\lambda A^t=\lambda^2 A. \Rightarrow \lambda=\pm 1$ ▨

(b). If $\lambda=1$. $\Rightarrow T(A)=A^t=1\cdot A$ ∴ the set of eigenvectors w.r.t. $\lambda=1$

is $\{A\in M_{n\times n}(F) \mid A^t=A\}$, i.e. the symm matrices.

· For $\lambda=-1$, $A^t=-1\cdot A$ ∴ the set of eigenvectors w.r.t. $\lambda=-1$

is $\{A\in M_{n\times n}(F) \mid A^t=-A\}$ i.e. the skew-symmetric matrices.

(C). · First, use the std ordered basis α, to diagonalize T.

$[T]_\alpha=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$. Second, find the eigenvectors w.r.t. each eigenvalue of T.

But this work is done by part (b).

Third, let $\beta=\{\underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{\text{basis for symm matrix}},\underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{\text{basis for skew-symm matrix.}}\}$ be our desired basis.

Then $[T]_\beta=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$ ▨

(d). Generally, let β_1 be the basis for symmetric matrix set.

β_2 skew- " " ($\#\beta_2=1+2+\cdots+(n-1)=n(n-1)/2$).

Then the desired basis $\beta=\beta_1\sqcup\beta_2=\{E_{ii}\}_{i=1}^{n}\sqcup\{E_{ij}+E_{ji}\}_{\cancel{i<j}}\sqcup\{E_{ij}-E_{ji}\}_{i>j}$ ▨

§5-1. P.71.

$p_1(t)$ A

18.(a). Let $p_B(t)$ be the characteristic poly. of B. ($p_B(t) = \det(B - tI_n)$).

- B is invertible ⇔ $\det(B) \neq 0$.

$\det(A+cB) = \det(B(B^{-1})(A+cB)) = \det(B\cdot(B^{-1}A + cI_n))$.

$= \det(B)\det(B^{-1}A + cI_n) = \overset{\neq 0}{\det(B)} \cdot \det(B^{-1}A - (-c)I_n)$.

It's the characteristic poly. of $B^{-1}A$, namely $p(-c)$.

We know that $p(t)$ is of the form: (Thm 5.3).

$p(t) = (-1)^n(1\cdot t^n + a_{n-1}t^{n-1} + \cdots + a_0$.

$p(t)$ must split over $\mathbb{C}$. By fundamental thm of algebra,

$\exists$ const. $= a \in \mathbb{C}$ s.t. $p(a) = 0$.

Then let $c = -a$. $\Rightarrow p(-c) = 0 = \det(B^{-1}A + cI_n)$.

$\therefore \det(A+cB) = 0$ ($\Rightarrow A+cB$ is not invertible). ∎

(b). By part (a), B can't be invertible.

Let $A = I_2$ (invertible) Let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Then $A+cB = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ is invertible for all $c \in \mathbb{C}$ since $\det(A+cB) = 1 \neq 0$ ∎

19. Actually, it's the special case of Ex 14, §2.5. Say $B = Q^{-1}AQ$.

Pick $V = F^n$, $T = L_A$, β the std ordered basis. ($\Rightarrow [T]_\beta = [L_A]_\beta = A$).

Define w_j = column j of Q. Then $w_j = \sum Q_{ij} e_i$, where $\beta = \{e_1, \cdots, e_n\}$

$\therefore Q = [I_V]_\gamma^\beta$. Now let $\gamma = \{w_1, \cdots, w_n\}$. Then $[T]_\gamma = [I_V]_\beta^\gamma [T]_\beta [I_V]_\gamma^\beta$

$= Q^{-1}AQ = B$ ∎

20 By def of char poly, $f(t) = \det(A - tI_n)$.

Now, $f(0) = \underline{a_0} = \det(A - 0I_n) = \det(A)$. ∎

21. (a). By def $f(t) = \det(A - tI_n) = \det\begin{pmatrix} A_{11}-t & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-t & & \vdots \\ \vdots & & \ddots & \\ A_{n1} & \cdots & & A_{nn}-t \end{pmatrix}$

cofactor expansion along the first row.

$\overset{\downarrow}{=} (A_{11}-t)\det\begin{pmatrix} A_{22}-t & A_{23} \cdots A_{2n} \\ & \ddots \\ & & A_{nn}-t \end{pmatrix}_{(n-1)\times(n-1)} + g_1(t)$. , $g_1(t)$ has deg $\leq n-2$

again

$\overset{\downarrow}{=} (A_{11}-t)\left((A_{22}-t)\det\begin{pmatrix} A_{33}-t & \\ & \ddots \; A_{nn}-t \end{pmatrix}_{(n-2)\times(n-2)} + h_2(t)\right) + g_1(t)$, $h_2(t)$ has deg $\leq n-4$.

整理

$= (A_{11}-t)(A_{22}-t)\det\begin{pmatrix} A_{33}-t & \\ & \ddots \; A_{nn}-t \end{pmatrix}_{(n-2)\times(n-2)} + g_2(t)$., $\deg(g_2(t)) \leq n-2$.

Repeat the process $\underset{\cdots}{=} \cdots = \prod_{i=1}^{n}(A_{ii}-t) + g(t)$, $\deg(g(t)) \leq n-2$. ∎

(b). By part (a), we know that the coeff of t^{n-1} comes from only the first term, $(A_{11}-t)(A_{22}-t)\cdots(A_{nn}-t)$. $\therefore$ the coeff. of t^{n-1} is $(-1)^{n-1}\cdot\sum_{i=1}^{n} A_{ii} = a_{n-1}$

Then $tr(A) = \sum A_{ii} = (-1)^{n-1} a_{n-1}$ ▨

22. (a). $T(x) = \lambda x. \Rightarrow$

Note that $g(t) = \sum_{i=0}^{N} a_i t^i$ Then $g(T) = \sum_{i=0}^{N} a_i T^i$ and thus, $g(T)(x)$

$= \sum_{i=0}^{N} a_i T^i(x) \underset{\substack{\uparrow \\ \#15(a),\ p.259.}}{=} \sum_{i=0}^{N} a_i \cdot \lambda^i x. = g(\lambda) x.$ ▨

(b). Just change T into A ▨

(c). $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. $g(t) = 2t^2 - t + 1$, $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. $\lambda = 4$.

- $g(A)(x) = \left(2\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^2 - \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + I_2\right)(x) = \left(\begin{pmatrix} 14 & 12 \\ 18 & 20 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}\right)(x)$

  $= \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 58 \\ 87 \end{pmatrix}$. =

- $g(\lambda)(x) = (2\cdot 4^2 - 4 + 1)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 29\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 58 \\ 87 \end{pmatrix}$ ▨

23. By #22(a), $f(T)(x) = f(\lambda)(x)$. But $f(t) = \det([T]_\beta - tI_n)$ and by Thm 5.2, an eigenvalue λ satisfies $\det(T - \lambda I) = 0$ $\therefore f(\lambda) = 0$. $\Rightarrow f(T)(x) = 0x = 0$ $\forall x$ being eigenvector w.r.t. λ. The above argument holds $\forall \lambda$. $-(*)$ $T: V \to V$ is diagonalizable. $\Rightarrow$ By Thm 5.1 $\exists$ ordered basis β consisting of eigenvectors of T, say $\beta = \{v_1, \cdots, v_n\}$. But $f(T)(v_i) = 0$ $\forall i$, by $(*)$. By Thm 2.6, $f(T) = T_0$. ▨

24. (a) By #21(a), p.260, $f(t) = \prod_{i=1}^{n}(A_{ii} - t) + q(t)$, $\deg(q(t)) \le n-2$.

$\therefore$ It's leading coeff. $= (-1)^n$ ▨

(b). $f(t)$ is of degree n, with its coeff $\in \mathbb{F}$.

If $f(t)$ splits over $\mathbb{F}$, it's clear that $f(t)$ has at most n distinct zeroes, i.e. at most n distinct eigenvalues. (Thm 5.2).

If $f(t)$ doesn't split, the number of zeroes of $f(t)$ is still less than n ▨

25 Where's the Corollaries. ?

26. $M_{2\times 2}(\mathbb{Z}_2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}_2\right\}$ $g(t) = \det(A - tI) = (a-t)(d-t) - bc$

$\Rightarrow g(t) = t^2 - (\underset{\substack{0\\1}}{\underbrace{a+d}})t + (\underset{\substack{0\\1}}{\underbrace{ad-bc}}) = t^2 - tr(A)t + \det(A)$. by #21 & #20, p.260.

Ans 4 ▨

Section §5-2.

p.73.

1.(a). F. (b). F. (c). F, zero, $0 \in E_\lambda$ but 0 is not an eigenvector.

(d). T. (e). T, A is diagonalizable and $Av_j = \lambda_j v_j \ \forall j = 1, \dots, n$. Use Thm 5.1 and coro. to Thm 2.23, we have $Q^{-1}AQ = D := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ ※ (note: $\lambda_1, \dots, \lambda_n$ may not have to be distinct.)

(f). F, characteristic poly. splits over F is required ※

(g). T. by Thm 5.1, ∃ basis consisting of eigenvectors of the linear operator T. , and the size of this basis ≥ 1. Since V is a nonzero vector space.

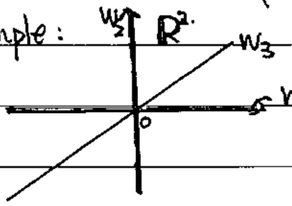
$\Rightarrow \exists$ eigenvalue. ∎

(h). T, by def, $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k \Rightarrow W_j \cap \sum_{i \neq j} W_i = \{0\} \ \forall j = 1, \dots, k$.

$\Rightarrow W_j \cap W_i = \{0\} \ \forall j \neq i$. ∎

*(i). F, $W_j \cap W_i = \{0\} \ \forall i, j, i \neq j \not\Rightarrow W_j \cap \sum_{i \neq j} W_i = \{0\} \ \forall j = 1, \dots, k$.

For example:



Here, $V := \mathbb{R}^2 = W_1 + W_2 + W_3$.

and $W_i \cap W_j = \{0\} \ \forall i \neq j$.

But, $V \neq W_1 \oplus W_2 \oplus W_3$. Since the condition $W_j \cap \sum_{i \neq j} W_i = \{0\}$ fails.

2.(a). $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. $g(t) = (t-1)^2 \Rightarrow \lambda = 1$. $(A - \lambda I) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ has rank 1. ∴ $\dim(E_\lambda) = 1 \neq 2$.

∴ A is not diagonalizable. ※∎

(c). $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ $g(\lambda) = (\lambda-1)(\lambda-2) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2) \Rightarrow \lambda_1 = 5 ; \lambda_2 = -2$.

⇒ A is diagonalizable. (multiplicity = 1 = dim. of the corresponding eigenspaces for λ_i, $i = 1, 2$)

$\therefore (A - \lambda_1 I) = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}$. $E_{\lambda_1} = \text{span}(\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\})$

$\therefore (A - \lambda_2 I) = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$. $E_{\lambda_2} = \text{span}(\{\begin{pmatrix} 4 \\ -3 \end{pmatrix}\})$. 不用算, just by Thm 5.1.

∴ $Q = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}$. and thus, $D = Q^{-1}AQ = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$.

(e). $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ $g(t) = \lambda^2(1-\lambda) + 1 - \lambda = -(\lambda^3 - \lambda^2 + \lambda - 1) = -(\lambda-1)(\lambda^2+1)$

Since g(t) does not split over $\mathbb{R}$, A is not diagonalizable ※

(g). $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$ $g(t) = \det \begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ -1 & -1 & 1-t \end{vmatrix} = (+1)(2-4+t) - (6-2t-2) + (t-1)((t-3)(t-4)-2)$

$\ominus \ \oplus \ \ominus$ $= 3t - 6 + (t-1)(t^2 - 7t + 10) = t^3 - 8t^2 + 20t - 16$

$1 \ \ -8 \ \ 20 \ \ -16 \ |\ 2$

$\quad 2 \ \ -12 \ \ 16$

$\Rightarrow g(t) = (t-2)^2(t-4)$. $\Rightarrow \lambda_1 = 2$, $\lambda_2 = 4$.

$1 \ \ -6 \ \ 8 \ |\ 2$

$\quad 2 \ \ -8$

$1 \ \ -4 \ \ 0$

$\therefore (A - \lambda_1 I) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$ has rank 1 $\Rightarrow \dim(E_{\lambda_1}) = 2$.

∴ A is diagonalizable. $E_{\lambda_1} = \text{span}(\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\})$ ← $a + b + c = 0$

$(A - \lambda_2 I) = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix}$ $\begin{cases} -a + b + c = 0 \\ -a = +c \\ -a = b + 3c \end{cases}$ $\Rightarrow E_{\lambda_2} = \text{span}\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$ $\therefore Q = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & -1 \end{pmatrix}$ ∎

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Date . :

3 (a). Chose the std. ordered basis, β_0, of $V=P_3(\mathbb{R})$. $[T]_{\beta_0}=\begin{pmatrix}0&1&2&0\\0&0&2&6\\0&0&0&3\\0&0&0&0\end{pmatrix}_{4\times4}$.

$g(t)=\det([T]_{\beta_0}-tI_4)=+t^4.\Rightarrow\lambda=0.$ $[T]_{\beta_0}-\lambda I=[T]_{\beta_0}$ has rank 3.

So $\dim(E_\lambda)=4-3=1.\Rightarrow$ T is not diagonalizable.

(C). Chose the std. ordered basis β_0 for $V=\mathbb{R}^3$. $[T]_{\beta_0}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&2\end{pmatrix}_{3\times3}$.

$g(t)=(2-t)\det\begin{pmatrix}-t&1\\-1&-t\end{pmatrix}=(2-t)(t^2+1)$ doesn't split over $\mathbb{R}$. $\Rightarrow$ T is not diagonalizable ∎

(d). $[T]_{\beta_0}=\begin{pmatrix}1&0&0\\1&1&1\\1&1&1\end{pmatrix}_{3\times3}$. $g(t)=(1-t)\cdot\begin{vmatrix}1-t&1\\1&1-t\end{vmatrix}=(1-t)((1-t)^2-1)=(1-t)(t^2-2t)$

$=-(t-1)(t)(t-2)\Rightarrow\lambda_1=0,\ \lambda_2=1,\ \lambda_3=2$, all of which have multiplicity $=1$.

So T is diagonalizable.

$\therefore([T]_{\beta_0}-\lambda_1I_3)=\begin{pmatrix}1&0&0\\1&1&1\\1&1&1\end{pmatrix}$, $E_{\lambda_1}=\text{span}(\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\})$.

$([T]_{\beta_0}-\lambda_2I_3)=\begin{pmatrix}0&0&0\\1&0&1\\1&1&0\end{pmatrix}$, $E_{\lambda_2}=\text{span}(\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\})$.

$([T]_{\beta_0}-\lambda_3I_3)=\begin{pmatrix}-1&0&0\\1&-1&1\\1&1&-1\end{pmatrix}$, $E_{\lambda_3}=\text{span}(\{\begin{pmatrix}0\\1\\1\end{pmatrix}\})$.

$\to$ So choose the basis $\beta=\{x-x^2,\ 1-x-x^2,\ x+x^2\}$. ∎

(e). $[T]_{\beta_0}=\begin{pmatrix}1&i\\i&1\end{pmatrix}_{2\times2}$, where $\beta_0=\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\}$ (over $\mathbb{C}$).

$g(t)=(1-t)^2-i^2=(t-1)^2+1=t^2-2t+2=(t-\alpha)(t-\bar{\alpha})$, where $\alpha=\frac{2\pm\sqrt{4-8}}{2}=1+i$.

$\therefore\lambda_1=\alpha,\ \lambda_2=\bar{\alpha}$. $\therefore$ T is diagonalizable

$\therefore([T]_{\beta_0}-\lambda_1I_2)=\begin{pmatrix}-i&i\\i&-i\end{pmatrix}$. So $E_{\lambda_1}=\text{span}(\{\begin{pmatrix}1\\1\end{pmatrix}\})$.

$([T]_{\beta_0}-\lambda_2I_2)=\begin{pmatrix}i&i\\i&i\end{pmatrix}$ $E_{\lambda_2}=\text{span}(\{\begin{pmatrix}1\\-1\end{pmatrix}\})$.

) choose the desired basis $\beta=\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\}$. ∎

4. "If A has n distinct eigenvalues, A is diagonalizable. ($A\in M_{n\times n}(F)$)"

pf. Let $v_1,\dots,v_n$ be the corresp. eigenvectors of the eigenvalues $\lambda_1,\dots,\lambda_n$.

* For n=1 case, $v_1\neq0$ since v_1 is an eigenvector, and $\beta=\{v_1\}$ is a basis.

By Thm 5.1, A is diagonalizable.

* Assume the statement holds for all $k=1,\dots,n-1$. (induction hypothesis)

* For k=n, Claim: $\beta=\{v_1,\dots,v_n\}$ is L.I.

pf: Assume $\underbrace{\sum_{i=1}^{n}a_iv_i=0}$. Apply $(T-\lambda_nI)$, we have $\sum_{i=1}^{\boxed{n-1}}a_i(T-\lambda_nI)v_i=0$

$\Rightarrow\sum_{i=1}^{n-1}a_i(\lambda_i-\lambda_n)v_i=0$. By the induction hypothesis, $a_i(\lambda_i-\lambda_n)=0$

for all $i=1,\dots,n-1$. But $\lambda_i-\lambda_n\neq0\ \forall i=1,\dots,n-1$, (A has n distinct eigenvalues)

so $a_i=0\ \forall i=1,\dots,n-1\Rightarrow a_nv_n=0\Rightarrow a_n=0$ (v_n is an eigenvector $\Rightarrow v_n\neq0$)

※ (claim finished).

$\therefore\beta$ is L.I., moreover, β is a basis. (by coro. 2 to replacement Thm).

Finally, by Thm 5.1, A is diagonalizable ∎

§5-2. p.75

5. Just the same as the ~~proof~~ in the text.

6. 略.

7. $A=\begin{pmatrix}1&4\\2&3\end{pmatrix}$.

First step, $g(t)=(1-t)(3-t)-8=t^2-4t-5=(t-5)(t+1) \Rightarrow \lambda_1=-1,\ \lambda_2=5$.

$(A-\lambda_1 I_2)=\begin{pmatrix}2&4\\2&4\end{pmatrix}$ $E_{\lambda_1}=\mathrm{span}(\{\begin{pmatrix}2\\-1\end{pmatrix}\})$

$(A-\lambda_2 I_2)=\begin{pmatrix}-4&4\\2&-2\end{pmatrix}$ $E_{\lambda_2}=\mathrm{span}(\{\begin{pmatrix}1\\1\end{pmatrix}\})$ $\Big\} \rightarrow$ choose $Q=\begin{pmatrix}2&1\\-1&1\end{pmatrix}$.

Then we know that $Q^{-1}AQ=D=\begin{pmatrix}\lambda_1&\\&\lambda_2\end{pmatrix}=\begin{pmatrix}-1&0\\0&5\end{pmatrix}$.

Second, $A^n=(QDQ^{-1})^n=\underbrace{(QDQ^{-1})(QDQ^{-1})\cdots(QDQ^{-1})}_{n次}=QD^nQ^{-1}$

$$=\begin{pmatrix}2&1\\-1&1\end{pmatrix}\begin{pmatrix}(-1)^n&0\\0&5^n\end{pmatrix}\begin{pmatrix}1&-1\\1&2\end{pmatrix}\cdot\frac{1}{3}$$

$$=\frac{1}{3}\begin{pmatrix}2&1\\-1&1\end{pmatrix}\begin{pmatrix}(-1)^n&(-1)^{n+1}\\5^n&2\cdot5^n\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2(-1)^n+5^n & 2((-1)^{n+1}+5^n)\\(-1)^{n+1}+5^n & (-1)^n+2\cdot5^n\end{pmatrix}.$$ ▨

8 Easy!

9. (a). characteristic poly. of $T =: g(t) = \det(\underset{\text{upper triangular matrix.}}{\boxed{[T]_\beta}} - tI_n)$.

$\Rightarrow g(t)=\prod_{i=1}^{n}(a_{ii}-t)$. So $g(t)$ splits over $\mathbb{F}$). ▨

(b). 略. (Just replace T by A).

10. Let $\dim(V)=:n$. Then $n=m_1+m_2+\cdots+m_k$.

$g(t)=(\lambda_1-t)^{m_1}(\lambda_2-t)^{m_2}\times\cdots\times(\lambda_k-t)^{m_k}$. by Thm5.2 & the def of multiplicity.

By §5.1 #21, we know that $\mathrm{tr}([T]_\beta)$ = the coefficient of t^{n-1} term $\times(-1)^{n-1}$.

$=\sum^{m_1}\lambda_1+\sum^{m_2}\lambda_2+\cdots+\sum^{m_k}\lambda_k$ ▨

11. (a). By #10, $\mathrm{tr}(A)=\sum_{i=1}^{k}m_i\lambda_i$ ▨ <Another proof: A similar to M, where M is an upper triangular matrix $\Rightarrow \exists$ invertible matrix Q s.t. $A=Q^{-1}MQ$.

Also M has the same eigenvalues as A. (Because the characteristic polys are the same)

$g(t)=\det(M-tI)=\prod_{i=1}^{n}(M_{ii}-t)=\prod_{i=1}^{k}(\lambda_i-t)^{m_i}$ $\Rightarrow$. Each λ_i occurres m_i times in the diag. entries of M.

Hence, $\mathrm{tr}(A)=\mathrm{tr}(Q^{-1}MQ)=\mathrm{tr}(M)=\sum^{k}m_i\lambda_i$ ▨

(b). ∗ $g(t)=\det(A-tI_n)=\det(M-tI_n)$.

$\Rightarrow g(0)=\det(A)=\det(\underset{\text{upper triangular}}{\boxed{M}})=\prod^{n}M_{ii}=\prod_{i=1}^{k}(\lambda_i)^{m_i}$ ▨

12 (a). Given $v\in E_\lambda$, $\Rightarrow Tv=\lambda v$. $\Rightarrow T^{-1}(Tv)=T^{-1}(\lambda v) \Rightarrow v=\lambda T^{-1}(v)$

$\Rightarrow \lambda^{-1}\boxed{v}=T^{-1}(v)$. ∴ v is also an eigenvalue of T^{-1} corresponding to λ^{-1} ∗

It's similar for the converse proof ▨

(b). T is diagonalizable $\Rightarrow$ For each λ_i of T, $\dim(E_{\lambda_i}) = m_i$, where m_i is the multiplicity of λ_i, $i=1,\dots,n$.

Then for each λ_i^{-1} of T^{-1}, by part (a), $\dim(\bar{E}_{\lambda_i^{-1}}) = \dim(\bar{E}_{\lambda_i}) = m_i$; where $\bar{E}_{\lambda_i^{-1}} =: \{v \in V \mid T^{-1}(v) = \lambda_i^{-1} v\}$.

Hence, T^{-1} is diagonalizable by Thm 5.9. $\blacksquare$

13(a). Eg. $A := \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} \Rightarrow g(t) = (2-t)(3-t)$ Take $\lambda_1 = 2$.

$(A - \lambda_1 I) = \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix} \Rightarrow E_{\lambda_1} = \text{span}(\{\begin{pmatrix} 1 \\ -4 \end{pmatrix}\})$.

$(A^t - \lambda_1 I) = \begin{pmatrix} 0 & 4 \\ 0 & 1 \end{pmatrix} \Rightarrow E'_{\lambda_1} = \overset{\#}{\text{span}}(\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\})$. $\blacksquare$

*(b). Let λ be an eigenvalue of $A_{n\times n} \Rightarrow \lambda$ is also an eigenvalue of A^t.

Let E_λ, E'_λ be the eigenspaces of A and A^t, respectively.

$g(t) = \det(A - \lambda I) = \det((A-\lambda I)^t) = \det(A^t - \lambda I)$.

If $\dim(E_\lambda) = k \leq n$, this means $\text{nullity}_{n\times n}(A - \lambda I) = k$.

Since $\text{rank}(A - \lambda I) = \text{rank}((A-\lambda I)^t) = \text{rank}(A^t - \lambda I)$,

we obtain $\text{nullity}(A^t - \lambda I) = k$, i.e. $\dim(E'_\lambda) = k$ $\blacksquare$

14.(a). $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \mathbb{X}' = A\mathbb{X}$.

1° diagonalize A: $g(t) = t^2 - 4 = (t+2)(t-2)$ $\therefore \lambda_1 = -2, \lambda_2 = 2$.

$(A - \lambda_1 I) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$.

$(A - \lambda_2 I) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$ choose $Q = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ s.t. $Q^{-1}AQ = D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

2° $\mathbb{X}' = A\mathbb{X} \Rightarrow \mathbb{X}' = QDQ^{-1}\mathbb{X} \Rightarrow Q^{-1}\mathbb{X}' = (Q^{-1}\mathbb{X})' = D(Q^{-1}\mathbb{X})$

Let $\mathbb{Y} = Q^{-1}\mathbb{X}$. Then $\mathbb{Y}' = D\mathbb{Y}$. $\Rightarrow \begin{cases} y_1'(t) = -2y_1(t) \\ y_2'(t) = +2y_2(t) \end{cases} \Rightarrow \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{2t} \end{pmatrix}$.

So, $\mathbb{X} = Q\mathbb{Y} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix} = \begin{pmatrix} 3c_1 e^{-2t} + c_2 e^{2t} \\ -c_1 e^{-2t} + c_2 e^{2t} \end{pmatrix}$.

Finally $\mathbb{X}(t) = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ $\blacksquare$

(b). $\mathbb{X}' = A\mathbb{X}$, where $A = \begin{pmatrix} 8 & 10 \\ -5 & -7 \end{pmatrix}$.

1° $g(t) = (t-8)(t+7) + 50 = t^2 - t - 6 = (t-3)(t+2)$ $\therefore \lambda_1 = -2, \lambda_2 = 3$.

$(A - \lambda_1 I) = \begin{pmatrix} 10 & 10 \\ -5 & -5 \end{pmatrix}$

$(A - \lambda_2 I) = \begin{pmatrix} 5 & 10 \\ -5 & -10 \end{pmatrix}$ $\therefore$ Choose $Q = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ s.t. $Q^{-1}AQ = D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

2° $\mathbb{Y} =: Q^{-1}\mathbb{X}$. $\Rightarrow \mathbb{Y}' = D\mathbb{Y}$. $\therefore \mathbb{Y}(t) = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{3t} \end{pmatrix}$. Then $\mathbb{X} = Q\mathbb{Y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{3t}$ $\blacksquare$

§5.2. p.77.

(C). $x' = Ax$, where $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. $g(t) = -(t-2)(t-1)^2 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$.

1° $(A - \lambda_1 I) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $E_{\lambda_1} = \text{span}(\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\})$.

$(A - \lambda_2 I) = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $E_{\lambda_2} = \text{span}(\{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\})$. ∴ Choose $Q = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ s.t. $Q^{-1}AQ = D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$.

2° Let $y = Q^{-1}x$. $\Rightarrow y' = Dy \Rightarrow y(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^t \\ c_3 e^{2t} \end{pmatrix} \Rightarrow x(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_3 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} e^{2t}$

$= \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} e^t + c_3 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} e^{2t}$ ∎

15. ($\Rightarrow$). A diagonalizable $\Rightarrow$ $\exists$ invertible matrix Q s.t. $Q^{-1}AQ = D$, Q consists of eigenvecto

Let $y = Q^{-1}x$. Then our problem reduce to: $x' = Ax \Rightarrow x' = QDQ^{-1}x$

$\Rightarrow y' = Dy$. Solve y we get $y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_1 t} \\ \vdots\ e^{\lambda_2 t} \\ c_n e^{\lambda_k t} \end{pmatrix}$, where each λ_i occurres

m_i times. (m_i = multiplicity).

$x(t) = Qy(t) = y_1(t) \cdot v_1 + \cdots + y_n(t) \cdot v_n$, where $Q = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$.

$= c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_k t} v_n$,

∴ $x(t)$ is of the form $x(t) = e^{\lambda_1 t} z_1 + \cdots + e^{\lambda_k t} z_k$, $z_i \in E_{\lambda_i}$ ∎

($\Leftarrow$). Suppose x is of the form: $x(t) = \sum e^{\lambda_i t} z_i$, $z_i \in E_{\lambda_i}$.

Note that for each j, $(e^{\lambda_j t} z_j)' = \lambda_j e^{\lambda_j t} z_j$. and ①

$A(e^{\lambda_j t} z_j) = e^{\lambda_j t} A z_j = e^{\lambda_j t} \lambda_j z_j$. [So each $e^{\lambda_j t} z_j$ is a sol to $x' = Ax$.]

Then $x(t)$ is also a sol. to $x' = Ax$, since $x'(t) = (\sum e^{\lambda_i t} z_i)' = \underset{\sim\sim\sim}{\sum \lambda_i e^{\lambda_i t}} z_i$

~~Third, $\beta = \{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_k t}\}$.~~ $\overset{①}{=} \sum A(e^{\lambda_i t} z_i) = A(\sum e^{\lambda_i t} z_i) = Ax$ ∎

~~First, each $e^{\lambda_i t}$ is a solution, ∴ β lies in the sol. space.~~

~~Second, β is linearly indep. (Suppose $\sum a_i \underset{\neq 0}{\boxed{e^{\lambda_i t}}} = 0$. Then $a_i = 0\ \forall i$.)~~

The sol. set to $x' = Ax$ is $x(t) = \underline{c_1} e^{\lambda_1 t} \underline{v_1} + \cdots + \underline{c_n} e^{\lambda_k t} \underline{v_n}$

Thus, the sol. space is an n-dim. real vector space. ∎

16. $C \in M_{m \times n}(\mathbb{R})$ Y $n \times p$ matrix, diff. $C := (v_1\ v_2 \cdots v_n)$, $v_i \in \mathbb{R}^m$, $i = 1, \ldots, n$.

$(CY)' = (u_1\ u_2\ \cdots\ u_p)'_{m \times p}$, $u_j = \sum_{i=1}^{n} Y_{ij} \cdot v_i \in \mathbb{R}^m$, $j = 1, \ldots, p$.

$= (\sum_{i=1}^{n} Y_{i1} v_i, \sum_{i=1}^{n} Y_{i2} v_i, \ldots, \sum_{i=1}^{n} Y_{ip} v_i)'$

$= (\sum_{i=1}^{n} Y'_{i1} v_i, \sum_{i=1}^{n} Y'_{i2} v_i, \ldots, \sum_{i=1}^{n} Y'_{ip} v_i)$.

$= CY'$ ∎

17. (a). T and U are simultaneously diagonalizable $\Rightarrow$ $\exists\ \beta_0$ s.t. $[T]_{\beta_0}$ & $[U]_{\beta_0}$ are diag. matrix. Define $A = [T]_\beta$, $B = [U]_\beta$ for a given $\beta \neq \beta_0$. Define $Q = [I_V]_\beta^{\beta_0}$

Then Q is invertible and $Q^{-1}AQ = [T]_{\beta_0}$, $Q^{-1}BQ = [U]_{\beta_0}$ (Simultaneously diag.) ∎

(b). $\exists Q$, invertible, s.t. $Q^{-1}AQ = D_A$, $Q^{-1}BQ = D_B$, D_A, D_B are diag. matrix

Chose β = the set of column vectors of Q. Let α = s.t.d. ordered basis of $\mathbb{R}^n$.

β is a basis since $\#\beta = n$ and Q is full rank ($\Rightarrow \beta$ L.I.). $\therefore Q = [I_V]_\beta^\alpha$.

Then $[L_A]_\beta = [I_V]_\alpha^\beta \underbrace{[L_A]_\alpha}[I_V]_\beta^\alpha = Q^{-1}\underbrace{A}Q$, is a diag. matrix.

Similarly, $[L_B]_\beta$ is also a diagonal matrix ▨

18(a). $\exists\, \beta$ s.t. $[T]_\beta$, $[U]_\beta$ are digonal matrices.

$[TU]_\beta = \underset{A}{\underset{\parallel}{[T]_\beta}}\underset{B}{\underset{\parallel}{[U]_\beta}} = \sum A_{ii}B_{ii} = \sum B_{ii}A_{ii} = [U]_\beta[T]_\beta = [UT]_\beta$.

And $\phi_\beta : \mathcal{L}(V,V) \to M_{n\times n}(\mathbb{F})$ defined by $T \mapsto [T]_\beta$, is an isomo.

So $TU = UT$ ▨

(b). The same.

19. $m \in \mathbb{N}$. $\exists\, \beta$ basis s.t. $[T]_\beta^\gamma =: A$ is a diagonal matrix.

Then $[T^m]_\beta = [T]_\beta[T]_\beta \cdots [T]_\beta = A^m = D$, where $D_{ii} = (A_{ii})^m$.

So T^m is diagonalizable (using the same β) ▨

20. $\sum^k W_i = V$.

($\Rightarrow$). $V = W_1 \oplus \cdots \oplus W_k \overset{\text{Thm5.10(d)}}{\Longrightarrow} S := \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V,

where γ_i is an ordered basis for W_i. Thus, $\dim(V) = \#S = \sum_{i=1}^k \#\gamma_i = \sum_{i=1}^k \dim(W_i)$ ▨

($\Leftarrow$). $\dim(V) = \sum \dim(W_i)$ and $\sum^k W_i = V$.

$U_1 := \sum_{i=1}^{k-1} W_i \Rightarrow \underbrace{V = U_1 + W_k}$. and $\dim(V) = \sum^k \dim(W_i) = \sum^{k-1} \dim(W_i) + \dim(W_k)$

$\therefore \dim(U_1) = \sum_{k-1} \dim(W_i)$. If $U_1 \cap W_k \neq \{0\}$, $\dim(U_1 + W_k) = \dim(U_1) + \dim(W_k)$

$-\underbrace{\dim(U_1 \cap W_k)}_{>0}$ by §1.6 #29(a), $\Rightarrow \dim(V) < \dim(U_1) + \dim(W_k)$ $\times$.

$\therefore \underbrace{U_1 \cap W_k = \{0\}}$ $\to$ Thus, $V = U_1 \oplus W_k$ by def.

· Again, $U_2 := \sum_{i=1}^{k-2} W_i \Rightarrow U_1 = \sum^{k-1} W_i = U_2 + W_{k-1}$ & $\dim(U_2) = \sum^{k-2} \dim(W_i)$.

By the same argument, $U_1 = U_2 \oplus W_{k-1}$.

· Repeat the process, we finally have $V = U_1 \oplus W_k = (U_2 \oplus W_{k-1}) \oplus W_k$

$= U_2 \oplus W_{k-1} \oplus W_k = (U_3 \oplus W_{k-2}) \oplus W_{k-1} \oplus W_k = \cdots = W_1 \oplus \cdots \oplus W_k$ ▨

21. $\operatorname{span}(\beta) = V$ and $\beta = \bigcup_{i=1}^k \beta_i$, $\beta_i \cap \beta_j = \emptyset$ $\forall i \neq j$.

Let $W_i =: \operatorname{span}(\beta_i)$. $\Rightarrow \dim(W_i) = \dim(V)$ and $\sum W_i = V$

By #20, $V \overset{!}{=} W_1 \oplus \cdots \oplus W_k = \operatorname{span}(\beta_1) \oplus \cdots \oplus \operatorname{span}(\beta_k)$. ▨

§5-2. - §5-3.

22. LHS = $\sum_{i=1}^{k} E_{\lambda_i}$. Let γ_i be the basis for E_{λ_i}, $i=1,\dots,k$.

If $\exists v \neq 0$ s.t $v \in E_{\lambda_1} \cap \sum_{i=2}^{k} E_{\lambda_i}$, then $v \in E_{\lambda_1}$ and $v = c_2v_2 + \cdots + c_kv_2$,

where $v_i \in E_{\lambda_i}$ and c_i's are not all zeroes and v_i's not all zero vectors.

$\Rightarrow T(v) = \underbrace{\lambda_1 v} = T(c_2v_2 + \cdots + c_kv_k) = \underbrace{\sum_{i=2}^{k} c_i\lambda_i v_i}$.

$\Rightarrow \lambda_1(c_2v_2 + \cdots + c_kv_k) = c_2\lambda_2v_2 + \cdots + c_k\lambda_kv_k$.

$\Rightarrow 0 = \underbrace{(\lambda_2-\lambda_1)}_{\neq 0}c_2v_2 + \underbrace{(\lambda_3-\lambda_1)}_{\neq 0}c_3v_3 + \cdots + \underbrace{(\lambda_k-\lambda_1)}_{\neq 0}c_kv_k$ $\nrightarrow$.

$\therefore E_{\lambda_1} \cap \sum_{i=2}^{k} E_{\lambda_i} = \{0\}$. $\Rightarrow$ LHS = $E_{\lambda_1} \oplus \sum_{i=2}^{k} E_{\lambda_i}$

Continue the process. LHS = $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ $\blacksquare$

23. $W_1 = K_1 \oplus \cdots \oplus K_p$; $W_2 = M_1 \oplus \cdots \oplus M_q$; $W_1 \cap W_2 = \{0\}$. — ①

$\Rightarrow W_1 + W_2 \overset{①}{=} W_1 \oplus W_2 = (K_1 \oplus \cdots \oplus K_p) \oplus (M_1 \oplus \cdots \oplus M_q)$

$= K_1 \oplus \cdots \oplus K_p \oplus M_1 \oplus \cdots \oplus M_q$. $\blacksquare$

<Another pf>. WLOG, check. $K_1 \cap (\sum_{i=2}^{p} K_i + W_2) = \{0\}$:

If $\overset{\neq 0}{x} \in K_1 \cap (\sum_{i=2}^{p} K_i + W_2)$, $x = u+v$, $u \in \sum_{i=2}^{p} K_i$, $v \in W_2$.

$\Rightarrow \underbrace{x-u} = \underset{\in W_2}{v} \neq 0$ since $K_1 \cap \sum_{i=2}^{p} K_i = \{0\}$. $\Rightarrow v \neq 0 \in \underset{\overset{\uparrow}{x-u}}{W_1} \cap W_2$. $\nrightarrow$ $\blacksquare$

§ Section 5-3.

1. (a). T. (b) T, by Thm 5.13. (c). F, need $x_i \geq 0$ $\forall i$.

(d). F, sum of each 'column' of transition matrix = 1

(e). T., by coro. to Thm 5.15. (f) T, by Gerschgorin's Thm.

(g). T, by Thm 5.17. (h). F, e.g. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda = -1$. with $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(i). F, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\lim A$ doesn't exist. ($A \to I \to A \to I \dots$).

(j). T, Thm 5.20 $\blacksquare$

2. (a). $A = \begin{pmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{pmatrix}$ $g(t) = (t-0.1)^2 - 0.49 = t^2 - 0.2t - 0.48 = (t-0.8)(t+0.6)$

By Thm 5.13, $\lim_{m\to\infty} A^m$ exists. Choose $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ s.t $Q^{-1}AQ = D = \begin{pmatrix} 0.8 & \\ & -0.6 \end{pmatrix}$

$\lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = Q(\lim D^m)Q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\frac{1}{-2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\blacksquare$

(c). $A = \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$ $g(t) = (t-0.4)(t-0.3) - 0.42 = t^2 - 0.7t - 0.3$.

$\therefore \lambda = \frac{1}{2}(0.7 \pm \sqrt{0.49+1.2}) = 0.35 \pm 0.65 = 1$ or -0.3. $\xRightarrow{\text{Thm 5.13}}$ $\lim_{m\to\infty} A^m$ exists.

Choose $Q = \begin{pmatrix} 0.7 & 1 \\ 0.6 & -1 \end{pmatrix}$ s.t. $Q^{-1}AQ = \underset{=\begin{pmatrix} 1 & \\ & -0.3 \end{pmatrix}}{D}$. Then $\lim A^m = \underbrace{Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}Q^{-1} = \begin{pmatrix} 0.7 & 0 \\ 0.6 & 0 \end{pmatrix} \cdot \frac{1}{-1.3}\begin{pmatrix} -1 & -1 \\ -0.6 & 0.7 \end{pmatrix}$

$= \frac{1}{-1.3}\begin{pmatrix} -0.7 & -0.7 \\ -0.6 & -0.6 \end{pmatrix} = \begin{pmatrix} \frac{7}{13} & \frac{7}{13} \\ \frac{6}{13} & \frac{6}{13} \end{pmatrix}$ $\blacksquare$

(e). $A=\begin{pmatrix}-2&-1\\4&3\end{pmatrix}$ $g(t)=(t+2)(t-3)+4=t^2-t-2=(t-2)(t+1)$.

Since A has $\lambda=-1$, $\lim A$ doesn't exist by Thm 5.13 ▨

*(g). $A=\begin{pmatrix}-1.8&0&-1.4\\-5.6&1&-2.8\\2.8&0&2.4\end{pmatrix}$. $g(t)=(-1)^{2+2}\cdot(1-t)\det\begin{bmatrix}-1.8-t&-1.4\\2.8&+2.4-t\end{bmatrix}=(t-1)\det\begin{bmatrix}t+1.8&1.4\\2.8&+2.4-t\end{bmatrix}$

$=-(t-1)\left((t+1.8)(t-2.4)+2.8\times1.4\right)=-(t-1)(t^2-0.6t-4.32+3.92)$

$=-(t-1)(t^2-0.6t-0.40)=-(t-1)(t-1)(t+0.4)$

$(A-1I)=\begin{pmatrix}-2.8&0&-1.4\\-5.6&0&-2.8\\2.8&0&1.4\end{pmatrix}$. $\therefore E_{\lambda=1}=\text{span}\left(\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right)$. $\therefore \dim(E_{\lambda=1})=2$.

(or $\text{rank}(A-I)=1 \Rightarrow \text{nullity}(A-I)=2=\dim(E_{\lambda=1})$).

$\therefore \lim A^m$ exists by Thm 5.13.

$\lim A^m = Q(\lim_{m\to\infty} D^m)Q^{-1}=\begin{pmatrix}1&0&1\\0&1&2\\-2&0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}-1&0&-1\\-4&1&-2\\+2&0&+1\end{pmatrix}$.

$=\begin{pmatrix}1&0&0\\0&1&0\\-2&0&0\end{pmatrix}\begin{pmatrix}-1&0&-1\\-4&1&-2\\+2&0&+1\end{pmatrix}=\begin{pmatrix}-1&0&-1\\-4&1&-2\\-2&0&-2\end{pmatrix}$. ▨

(i). $g(t)=\det\begin{vmatrix}-\frac{1}{2}-2i-t&4i&\frac{1}{2}+5i\\1+2i&-3i-t&-1-4i\\+1-2i&4i&1+5i-t\end{vmatrix}=\det\begin{vmatrix}+\frac{1}{2}-t&0&-\frac{1}{2}+t\\0&i-t&i-t\\-1-2i&4i&1+5i-t\end{vmatrix}$

$=(\frac{1}{2}-t)\left((i-t)(1+5i-t)+4it+4\right)+(+1+2i)(i-t)(-\frac{1}{2}+t)$

$=(t-\frac{1}{2})\left[(i-t)(\cancel{1}+2i-(\cancel{1}+5i-t))-(4+4it)\right]$

$=(t-\frac{1}{2})\left[(i-t)(t-3i)-(4+4it)\right]$

$=(t-\frac{1}{2})\left[3-t^2+\cancel{4i}t-4-\cancel{4i}t\right]=-(t-\frac{1}{2})(t^2+1)=-(t-\frac{1}{2})(t+i)(t-i)$

$\therefore \lambda=\frac{1}{2}$ or $-i$ or i. $\Rightarrow$ (Thm 5.13)' $\lim A^m$ doesn't exist ▨

3. $\lim_{m\to\infty} A_m=L \Leftrightarrow \lim_{m\to\infty}(A_m)_{ij}=L_{ij}$ $\forall i,j$. $(1\le i\le n, 1\le j\le p)$.

Then $\lim_{m\to\infty}(A_m)^t_{ij}=\lim_{m\to\infty}(A_m)_{ji}=L_{ji}$, $\forall i,j$.

$\therefore \lim_{m\to\infty}(A_m)^t=L^t$ ▨

4. $A\in M_{n\times n}(\mathbb{C})$ is diagonalizable. $L=\lim_{m\to\infty}A^m$ exists.

- Since $L=\lim_{m\to\infty}A^m$ exists & A is diagnalizable, by Thm 5.13, the conditions of Thm 5.13 must hold. Say eigen $=\lambda_1,\dots,\lambda_n$ (may not be distinct values).
- A diagnolizable $\Rightarrow$ $\exists$ invertible matrix Q s.t. $D=Q^{-1}AQ=\begin{pmatrix}\lambda_1&&0\\&\lambda_2&\\0&&\ddots\ \lambda_n\end{pmatrix}$

$\Rightarrow L=\lim_{m\to\infty}A^m=\lim_{m\to\infty}(QDQ^{-1})^m=Q(\lim_{m\to\infty}D)Q^{-1}$.

$\therefore L=QI_nQ^{-1}=I_n$, if all $\lambda_i=1$.

and $\text{rank}(L)<n$ if $\exists\,\lambda_j$ s.t. $\lambda_j<1$. ▨

§5.3. p.81.

5. $A := \frac{1}{\sqrt{2}}P$, $B := \frac{1}{\sqrt{3}}Q$, where $P = \begin{pmatrix}1&0\\1&0\end{pmatrix}$, $Q = \begin{pmatrix}1&1\\0&0\end{pmatrix}$

Note that $\frac{1}{\sqrt{2}}P \cdot \frac{1}{\sqrt{3}}Q = R$, where $R = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\Big\} \to$ 滿足 $\lim_{n\to\infty} M^n = M$.

So, $\lim A^m = O$ & $\lim B^m = 0$,

but $\lim (AB)^m = \lim R^m = R = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ ∎

⑥ Initial vector $= P_0 = \begin{pmatrix}0\\0.3\\0.7\\0\end{pmatrix}$ ← healthy ← ambulatory ← bedridden ← dead.

(Hard)

transition matrix $A = \begin{array}{cccc} \text{healthy} & \text{ambulatory} & \text{bedriden} & \text{dead} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$
$\begin{pmatrix} 1 & 0.6 & 0.1 & 0 \\ 0 & 0.2 & 0.2 & 0 \\ 0 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.2 & 1 \end{pmatrix}_{4\times4}$

- $AP_0 = \begin{pmatrix}0.75\\0.20\\0.41\\0.14\end{pmatrix}$ ※

- $g(t) = \text{Det}(A - t\bar{I}) = \underset{0.96}{(1-t)}\begin{vmatrix} 0.2-t & 0.2 & 0 \\ 0.2 & 0.5-t & 0 \\ 0 & 0.2 & 1-t \end{vmatrix} = (1-t)^2 \begin{vmatrix} 0.2-t & 0.2 \\ 0.2 & 0.5-t \end{vmatrix}$

$= (t-1)^2\left((t^2 - 0.7t + 0.1) - 0.04\right) = (t-1)^2(t^2 - 0.7t + 0.96)$

$t = 0.35 \pm \frac{\sqrt{0.49 - 3.84}}{2}$

$= 0.35 \pm \frac{\sqrt{3.35}}{2} i$, 其 abs. value <1

7. $A = \begin{pmatrix} 1 & \frac13 & 0 & 0 \\ 0 & 0 & \frac13 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1 \end{pmatrix}$. Initial $P_0 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = e_2$.

Then the seq. $e_2, Ae_2, A^2e_2, A^3e_2, \cdots$ would be

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1/3\\0\\2/3\\0\end{pmatrix} \xrightarrow{\times\frac13}, \begin{pmatrix}\overset{+1}{\frac13}\\2/9\\0\\4/9\end{pmatrix} \xrightarrow[\times\frac23,\ 1]{\times\frac13} \begin{pmatrix}11/27\\0\\4/27\\4/9\end{pmatrix} \longrightarrow, \begin{pmatrix}11/27\\4/81\\0\\44/81\end{pmatrix} \xrightarrow[\times\frac23]{\times\frac13}, \cdots$$

$\therefore$ limit of the 1st entry $= \frac13 + \boxed{\frac23}\frac19 + \left(\boxed{\frac23}\frac29\right)\frac19 + \left(\frac23\left(\frac29\right)^2\right)\frac19 + \cdots$

$= \frac13 + \frac23 \cdot \frac19\left(1 + \frac29 + \left(\frac29\right)^2 + \cdots\right) = \frac13 + \frac{2}{27}\cdot\left(\frac{1}{1-\frac29}\right) = \frac13 + \frac{2}{\cancel{27}_3}\cdot\frac{\cancel{9}}{7}$

$= \frac13 + \frac{2}{21} = \frac{9}{21} = \frac37$ ∎

8. No better method to check, so just compute them directly.

9 略

10. (a). 2-stage vector $= \begin{pmatrix}0.225\\0.441\\0.334\end{pmatrix}$.

Next, by Thm 5.20, we could just find the eigenvector corresponding to $\lambda=1$ since the matrix is regular.

$(A-1I_3)=\begin{pmatrix} -0.4 & 0.1 & 0.1 \\ 0.1 & -0.1 & 0.2 \\ 0.3 & 0 & -0.3 \end{pmatrix}$. $E_{\lambda=1}=\left\{\begin{pmatrix}1\\3\\1\end{pmatrix}t \,\middle|\, t\in\mathbb{R}\right\}$.

$\therefore$ fixed vector $v=\begin{pmatrix}1/5\\3/5\\1/5\end{pmatrix}=\begin{pmatrix}0.2\\0.6\\0.2\end{pmatrix}$ ▨

14.* For case $m=1$. $A^m=A^1=A=\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}$.

Suppose the formula holds for all $m<k$.

For the case $m=k$, $A^m=A^k=A^{k-1}A \overset{\curvearrowright}{=} \begin{pmatrix} r_{k-1} & & r_k \\ & r_{k-1} & \\ r_k & & r_{k-1}\end{pmatrix}\begin{pmatrix} 0 & & \frac{1}{2} \\ & 0 & \\ \frac{1}{2} & & 0\end{pmatrix}$.

$=\begin{pmatrix} r_k & & \frac{r_k+r_{k-1}}{2} \\ & r_k & \\ \frac{r_k+r_{k-1}}{2} & & r_k\end{pmatrix}$.

$\frac{r_k+r_{k-1}}{2}=\frac{1}{2}\left(\frac{1}{3}\left(1+\frac{(-1)^k}{2^{k-1}}\right)+\frac{1}{3}\left(1+\frac{(-1)^{k-1}}{2^{k-2}}\right)\right)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(2+\frac{(-1)^{k-1}((-1)+2)}{2^{k-1}}\right)$.

$=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(2+\frac{(-1)^{k-1}}{2^{k-1}}\right)=\frac{1}{3}\left(1+\frac{(-1)^{k+1}}{2^k}\right).=r_{k+1}$ ▨

18. Let $v^{(\neq 0)}$ be the vector with all nonnegative entries. ($v=\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}$).

$W=\text{span}\{v\}$. And $x=\frac{1}{\sum v_i}v$ is a prob. vector in W.

Suppose y is another prob. vector in W, $\Rightarrow$ $y=mv$ for some m=const.

and $\sum mv_i=1$. (prob. vector). $\Rightarrow$ $m\sum v_i=1$. $\Rightarrow$ $m=\frac{1}{\sum v_i}$

So $x=y$ (uniqueness) ▨

16 & 8 (easy).

17. (p.299). Coro1.: proof: Apply Thm5.18 to A^t, we obtain $|\lambda|=\nu(A)=\rho(A^t)$

$\Rightarrow \lambda=\rho(A^t)$. and $\{u\}$ is a basis for $E'_\lambda=\{x \mid A^tx=\lambda x\}$.

- $\lambda=\rho(A^t) \Rightarrow \lambda=\nu(A)$ ☆
- characteristic poly of A & A^t are the same, so $\dim(\overset{\{x|Ax=\lambda x\}}{E_\lambda})=\dim(E'_\lambda)=1$ ▨

Coro2. pf: By Coro3 to Thm5.16, $|\lambda|\le 1$. By 題目, $\lambda\neq 1$.

By Thm5.18, if $|\lambda|=1$ $(=\nu(A))$, $\lambda=1$. Hence, $|\lambda|<1$. #

Again, by Thm5.18, $\{u\}$ is a basis for E_λ, where $|\lambda|=\nu(A)$.

$\Rightarrow \dim(E_{\lambda=1})=1$ ▨

§5.3.

18. By Thm 5.19, $|\lambda| < 1$ or $\lambda = 1$. By Thm 5.14, $\lim A^m$ exists ▨

19. (a). 1° transition matrix: $\left(u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{F}^n\right)$

$$(cM + (1-c)N)^t \cdot u = (cM^t + (1-c)N^t)u = cM^t u + (1-c)N^t u$$

$$= cu + (1-c)u = u \text{ by Coro to Thm 5.15 (p.291).}$$

$\therefore$ $CM + (1-C)N$ is a trasition matrix. by the same coro.

2° regular : Let $k \in \mathbb{N}$ s.t. M^k be a matrix with all positive entries.

Then $(CM + (1-C)N)^k = \underbrace{(CM)^k}_{\text{all entries} > 0 \text{ (positive)}} + \underbrace{(CM)^{k-1}((1-C)N) + \cdots + ((1-C)N)^k}_{\text{all entries} \geq 0 \text{ (nonnegative)}}$.

$\therefore$ $CM + (1-C)N$ must be regular ▨

<1twrd>.
(b). Pick a scalar $d^{\geq 1}$ s.t. $(dM')_{ij} > (M)_{ij}$ $\forall i,j$.

Then choose $c := \frac{1}{d}$ and $N := \frac{1}{1-c}(M' - cM)$.

$\Rightarrow$ N is nonnegative.

Next, $N^t u = N^t \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{1-c}((M')^t u - cM^t u) \overset{\text{題目條件}}{=} \frac{1}{1-c}(u - cu) = u$.

$\Rightarrow$ N is a transition matrix. ▨

(c). ($\Rightarrow$). M is regular. By (b), $M' = CM + (1-C)N$ for some transition matrix N and some const. $C \in (0,1]$. By (a). M' is regular ▨

($\Leftarrow$). Same argument ▨

20. • $e^O = I + O + \frac{O^2}{2!} + \frac{O^3}{3!} + \cdots = I_n$

• $e^I = I + I + \frac{I^2}{2!} + \frac{I^3}{3!} + \cdots = I_n (1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots) = I_n e^1 = e^{e} I_n$ ▨

21. $e^A = e^{PDP^{-1}}$

$$= I + PDP^{-1} + \frac{(PDP^{-1})^2}{2!} + \frac{(PDP^{-1})^3}{3!} + \cdots = \overset{PIP^{-1}}{\boxed{I}} + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots$$

$$= P(I + D + D^2/2! + D^3/3! + \cdots)P^{-1} = Pe^D P^{-1}. \text{ ▨}$$

22. A diagonalizable. $(\in M_{n\times n}(\mathbb{C}))$ $\Rightarrow$ $\exists P$ & P^{-1} s.t $D = P^{-1}AP$ (being diag. matrix)

$e^A = Pe^D P^{-1}$. It suffices to see if e^D exists.

$$e^D = I + D + D^2/2! + \cdots = I_n + \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix} + \cdots$$

So, for each position ii, the value $:= (e^D)_{ii} = 1 + \lambda_i + \frac{\lambda_i^2}{2!} + \frac{\lambda_i^3}{3!} + \cdots$

$= e^{\lambda_i}$. Hence, $e^D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}$, which exists clearly. ▨

No.
Date : § 5.3 - § 5.4.

23. Pick $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Then $e^A = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}$. and $e^B = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$.

- $e^A e^B = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & e-1 \\ 0 & 1 \end{pmatrix}$.
- $e^{A+B} = I + \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 8 & 4 \\ 0 & 0 \end{pmatrix} + \cdots$

$= \begin{pmatrix} e^2 & \frac{e^2-1}{2} \\ 0 & 1 \end{pmatrix}$ ← Note: $0 + 1 + \frac{1}{2!}2 + \frac{4}{3!} + \cdots = \frac{e^2-1}{2}$.

24. • x is a solution s.t. $x' = Ax$, ⇔.

$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$ by Exercise #15, § 5.2.

- Note that $e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots = \overset{PIP^{-1}}{\overset{\shortparallel}{I}} + PDtP^{-1} + \frac{1}{2!}PD^2t^2P^{-1} + \cdots$

$= P(I + Dt + \frac{1}{2!}(Dt)^2 + \cdots)P^{-1} = Pe^{Dt}P^{-1} = P\begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_n t} \end{pmatrix}P^{-1}$,

where $P = (v_1\ v_2\ \cdots\ v_n)$ (consisting of eigenvectors).

- Then $x(t) = Pe^{Dt}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, 令 $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = y \in \mathbb{R}^n$.

$= P\boxed{e^{Dt}}y = P(P^{-1}e^{At}P)y = e^{At}\underbrace{Py}$.

$=: e^{At} \cdot v$, where $v = Py \in \mathbb{R}^n$ $(v = c_1v_1 + \cdots + c_nv_n)$. ∎

Section § 5.4.

1. (a). False, since $T(0) = 0 \Rightarrow \{0\}$ is a T-invariant subspace.

(b). True, by Thm 5.21

*(c) False, let $V = \mathbb{R}$., $T: V \to V$ defined by $T(x) = x$. Let $v = 1$ & $w = 2 \in \mathbb{R}$. Then $v \neq w$, but $W := \text{span}(\{v, T(v), \cdots\}) = \text{span}(\{w, T(w), T^2(w), \cdots\}) =: W'$ #

(d) False, counter eg: Let $T: \overset{\mathbb{R}^2}{V} \to V$ be defined by $T(x,y) = (0,y)$. $v := (1,1)$. Then $W := \text{span}(\{v, T(v), T^2(v), \cdots\}) = \mathbb{R}^2$ (dim = 2) but $W' := \text{span}(\{T(v), T^2(v), \cdots\}) =$ y-axis in $\mathbb{R}^2$ (dim = 1)

(e). True, by Cayley-Hamilton Thm, the characteristic polynomial of T, say $g(t)$, satisfies $g(T) = T_0$ and $\deg(g) = n$. #

(f). True, any characteristic poly. of a n-dim. linear operator is of the form $g(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ #

§5.4. p.85

(g). True, $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. W_i's are T-invariant subspace.

Define $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$, where β_i is the basis of W_i.

By Thm 5.25, β is a basis for V & $[T]_\beta = [T_{W_1}]_{\beta_1} \oplus [T_{W_2}]_{\beta_2} \oplus \cdots \oplus [T_{W_k}]_{\beta_k}$ ★

2. (a). Yes, $W = P_3(\mathbb{R})$ $T(f) = f'$, $f \in V = P_3(\mathbb{R})$.

Then given $f \in W$, $T(f) = f' \in P_2(\mathbb{R}) \subseteq W$. $\therefore T(W) \subseteq W$.

(b). No., $W = P_2(\mathbb{R})$, $T(f) = xf(x)$, $f \in V = P(\mathbb{R})$.

Given $f(x) = ax^2+bx+c$, $a, b, c \in \mathbb{R}$. ; $f \in W$.

Then $T(f) = ax^3+bx^2+cx$. $\notin W$.

(c). Yes., $W = \{(t,t,t) \mid t \in \mathbb{R}\}$ $T(a,b,c) = (a+b+c, a+b+c, a+b+c)$, $V = \mathbb{R}^3$.

Given $v = (t,t,t)$ in W, $t \in \mathbb{R}$. Then $T(v) = (3t, 3t, 3t)$, $\in W$ ★

(d). Yes., $W = \{f \in V \mid f(t) = at+b, a,b \in \mathbb{R}\}$, $T(f(t)) = (\int_0^1 f\,dx) \cdot t$, $V = C[0,1]$.

Given $f \in W$. $(f(t) = at+b)$. $\Rightarrow T(f) = \int_0^1 (ax+b)dx \cdot t$

$= (\frac{1}{2}a+b)t + 0 = ct + 0$ for some c, $\therefore T(f) \in W$.

Hence, $T(W) \subseteq W$.

(e). No, $W = \{A \in V \mid A^t = A\}$ $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A$. , $V = M_{2\times2}(\mathbb{R})$.

Given $A \in W$, then $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$,

$\Rightarrow T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} b & c \\ a & b \end{pmatrix}$.

But $T(A)^t = \begin{pmatrix} b & a \\ c & b \end{pmatrix} \neq T(A)$. ∎

3. $T: V \to V$ (a). • $T(0) = 0 \Rightarrow \{0\}$ is a T-invariant subspace.

• $T(v) \in V$ $\forall v \in V$ since by the def of $T: V \to V$. $\therefore T(V) \subseteq V$ ∎

(b). • If $x \in N(T)$, $T(x) = 0$, But $0 \in N(T)$, so $T(x) \in N(T)$ ★

• Given $x \in R(T)$; $R(T) \subseteq V \Rightarrow x \in V$ but by the def of $R(T)$

$T(\overset{v}{\cancel{x}}) \in R(T)$. ★ (so $T(R(T)) \subseteq R(T)$) ∎

(c). • Given $v \in E_\lambda$. If $v = 0$, $T(v) = T(0) = 0 \in E_\lambda$.

If $v \neq 0$, v is an eigenvector. $\Rightarrow T(v) = \lambda v$, $\in E_\lambda$.

since λv is a linear cobination of v & $v \in E_\lambda$ ∎

4. proof: given $g(t) \in P(\mathbb{R})$, generally $g(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0$.

for some $a_m \neq 0$, $a_i \in \mathbb{R}\ \forall i$. Since W is a T-invariant subspace,

$T^k(W) \subseteq T^{k-1}(W) \subseteq \cdots \subseteq T(W) \subseteq W$. $\therefore g(T)(v) \in W$ for any $v \in W$. ∎

No.
Date : :

5. $\mathcal{F}$:= a collection of T-invariant subspaces of V.

Given a subcollection $S \subseteq \mathcal{F}$, define $W = \bigcap_{A \in S} A$.

We wanna prove that W is a T-invariant subspace.

To see this, for any $w \in W$, $w \in A_i$ $\forall A_i \in S$. Hence, $T(w) \in A_i$ $\forall A_i \in S$, since A_i is a T-invariant subspace. Then $T(w) \in W$ since $T(w) \in A_i$ $\forall A_i \in S$. ∎

6. (a). $z = e_1$, $T(z) = (1,0,1,1)$, $T^2(z) = (1,-1,2,2)$, $T^3(z) = (0,-3,3,3)$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow T^3(z) = -3(T(z) - T^2(z)).$$

The above reduced row echelon form tells us only $\{z, T(z), T^2(z)\}$ forms a L.I. set. ∴ $\beta = \{z, T(z), T^2(z)\}$ is the basis of the T-cyclic subspace generated by z. by Thm 5.22. ☆

(b) $z = x^3$, $T(z) = 6x$, $T^2(z) = 0$.

∴ $\beta = \{z, T(z)\}$. ☆

(c). $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $T(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = z$. ∴ $\beta = \{z\}$ ☆

(d). $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $T(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} z = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. $T^2(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix} = 3\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

∴ $\beta = \{z, T(z)\}$.

7. Let $T: V \to V$ be a linear operator, and $W (\subseteq V)$ is a T-invariant subspace. Restrict T on W, denoted by $T_W : W \to W$ $(T_W = T|_W)$.

Claim: T_W is linear.

pf: Given $x, y \in W$, $c \in F$, $T_W(cx+y) = T(cx+y) \overset{T \text{ linear}}{=} cT(x) + T(y)$

$\underset{\uparrow}{=} cT_W(x) + T_W(y)$ ☆ ∎

since W is T-inarant.

8. v is an eigenvector of T_W w.r.t. λ. $\Rightarrow T_W(v) = \lambda v$.

Then $T(v) \underset{\uparrow}{=} T_W(v) = \lambda v$. ∎

$v \in W \Rightarrow T(v) \in W$ by W being T-invariant.

§5.4.

9. 對應 Exercis 6:

(a). • $T^3(z) = -3(T(z) - T^2(z)) \Rightarrow \overset{a_1}{3}T(z) - \overset{a_2}{3}T^2(z) + T^3(z) = 0$

By Thm 5.22, $g(t) = (-1)^3(t^3 + (-3)t^2 + 3t + 0) = -t^3+3t^2-3t$. ☆

• $\beta = \{z, T(z), T^2(z)\} = \{e_1, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\3\end{pmatrix}\} \Rightarrow [T_W]_\beta = \begin{bmatrix}0&0&0\\1&0&-3\\0&1&3\end{bmatrix}$

$\therefore g(t) = \det(T_W - tI_V) = \det\begin{vmatrix}-t&0&0\\1&-t&-3\\0&1&3-t\end{vmatrix} = (-1)^{1+1}(-t)\det\begin{vmatrix}-t&-3\\1&3-t\end{vmatrix}$

$= (-t)[(-t)(3-t)+3] = (-t)[t(t-3)+3] = -t^3+3t^2-3t$ ☆

(b). • $T^2(z) = 0 = 0z + 0T(z)$ ∴ Thm 5.22 $\Rightarrow g(t) = (-1)^2(t^2+0t+0) = t^2$

• $\beta = \{z, T(z)\} = \{x^3, 6x\} \Rightarrow [T_W]_\beta = \begin{bmatrix}0&0\\1&0\end{bmatrix}$ $\therefore g(t) = \det\begin{vmatrix}-t&0\\1&-t\end{vmatrix} = t^2$ ☆

(c). • $T(z) = z$. Thm 5.22 $\Rightarrow g(t) = (-1)^1(t^1-1) = 1-t$ ☆

• $\beta = \{z\} = \{\begin{pmatrix}0&1\\1&0\end{pmatrix}\} \Rightarrow [T_W]_\beta = [1]_{1\times1}$ $\therefore g(t) = 1-t$ ☆

(d) • $T^2(z) = 3\begin{pmatrix}1&1\\2&2\end{pmatrix} = 3T(z)$ $\therefore -3T(z) + T^2(z) = 0 \Rightarrow g(t) = (-1)^2(t^2-3t+0)$ $= t^2-3t$ ☆

• $\beta = \{z, T(z)\} = \{\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\2&2\end{pmatrix}\} \Rightarrow [T_W]_\beta = \begin{bmatrix}0&0\\1&3\end{bmatrix}$ $\therefore g(t) = (-t)(3-t) = t^2-3t$ ∎

10 (a). $\beta_0 :=$ std ordered basis. $[T]_{\beta_0} = \begin{bmatrix}1&1&0&0\\0&1&-1&0\\1&0&1&0\\1&0&0&1\end{bmatrix}$ $\therefore f(t) = \det(T - tI_V)$.

$= (1-t)\det\begin{vmatrix}1-t&-1&0\\0&1-t&0\\0&0&1-t\end{vmatrix} - 1\cdot\det\begin{vmatrix}0&-1&0\\1&1-t&0\\1&0&1-t\end{vmatrix} = (1-t)^4 - (1-t) = t^4-4t^3+6t^2-3t$
$= (t^3-4t^2+6t-3)t$.

$g(t) = -t^3+3t^2-3t = (-t)(t^2-3t+3)$.

$\Rightarrow f(t)/g(t) = -(t^3-3t^2+6t-3)/t^2-3t+3 = -t+1$ (整除) ☆

(b). $\beta_0 :=$ std ordered basis. $[T]_{\beta_0} = \begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix}$.

$\Rightarrow f(t) = (-t)^4 = t^4$. $f(t)/g(t) = t^4/t^2 = t^2$ ☆

(c). $\beta_0 = \{E^{11}, E^{12}, E^{21}, E^{22}\} \Rightarrow [T]_{\beta_0} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$. $\Rightarrow f(t) = (1-t)(t^2-1)$.
$\begin{pmatrix}1&0\\0&0\end{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} \begin{pmatrix}0&0\\1&0\end{pmatrix} \begin{pmatrix}0&0\\0&1\end{pmatrix}$.

$f(t)/g(t) = (1-t)(t^2-1)/(1-t) = t^2-1$ ☆

(d). β_0 同 part (C). $[T]_{\beta_0} = \begin{bmatrix}1&0&1&0\\0&1&0&1\\2&0&2&0\\0&2&0&2\end{bmatrix}$. $f(t) = (1-t)\det\begin{vmatrix}1-t&0&1\\0&2-t&0\\2&0&2-t\end{vmatrix} + \det\begin{vmatrix}0&1-t&1\\2&0&0\\0&2&2-t\end{vmatrix}$

$= (1-t)[(t-2)^2(1-t) - 2(2-t)] + (4 - 2(2-t)(1-t))$.

$= t^4-6t^3+9t^2$ (暴力).

$f(t)/g(t) = t^4-6t^3+9t^2/(t^2-3t) = t^2-3t$ ∎

11 (a). $W = \text{span}(\{v, T(v), T^2(v), \ldots T^k(v)\})$ ← basis. for some $k\in\mathbb{N}$. Given $x \in W$, $x = a_0v + a_1T(v) + \cdots + a_kT^k(v)$

Then $T(x) = a_0T(v) + a_1T^2(v) + \cdots + a_{k-1}T^k(v) + a_kT^{k+1}(v)$, but

$T^{k+1}(v) = \sum_{i=0}^{k} b_iT^i(v)$, so $T(x) \in W$ ∎

(b). If S is a T-invariant subspace of V containing v, $T(v)$, $T^2(v)$, ... are also in S. (T-invariant).
$W=\text{span}(\{v, T(v), \dots, T^k(v)\})$: $\beta_W=\{v, T(v), \dots, T^k(v)\}$ is a basis of W, and is contained in S. Then $\text{span}(\beta_W)\subseteq S$. i.e. $W\subseteq S$. ∎

12. parts of the proof of Thm 5.21:
$\gamma=\{v_1, \dots, v_k\}$ is a basis for W. Extend γ to an ordered basis $\beta=\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V. $A=[T]_\beta$. $B_1=[T_W]_\gamma$.
Since W is a T-invariant subspace, for every $i\in 1, 2, \dots k$, $T(v_i)\in W$.
So $A=\begin{bmatrix}\boxed{\begin{matrix}B_1\\ O\end{matrix}} & \begin{matrix}C\\ D\end{matrix}\end{bmatrix}$ ∎ (很簡略地寫).

13. $w\in W \overset{\swarrow \text{T-cyclic of } v\neq 0}{\Leftrightarrow} \exists$ poly. $g(t)$ s.t. $w=g(T)(v)$.
proof: $(\Rightarrow)$. $w\in W \Rightarrow w=\sum_{i=0}^{k-1} a_iT^i(v)$. ① Define $g(t)=\sum_{i=0}^{k-1} a_it^i$.
We find that ① becomes $w=g(T)(v)$. ∎
$(\Leftarrow)$. $\exists\, g(t)$ (poly.) s.t. $w=g(T)(v)$.
$\Rightarrow w=\sum_{i=0}^n a_iT^i(v)$ for some scalars $a_i\in F$. and $n\in\mathbb{N}$.
$\Rightarrow w\in W=\text{span}(\{v, T(v), \dots\})$. ∎

14. Trivial (Let $\dim(W)=k$., so $W=\text{span}(\underbrace{\{v, T(v), \dots, T^{k-1}(v)\}}_{=\beta,\ \text{basis}})$.
If $g(t)=\sum_{i=0}^{n} a_it^i$ for some $n>k$.,
since $T^m(v)$ can be expressed as a linear combination of β,
$g(t)=\sum_{i=0}^{k-1} b_it^i$ for some scalars $b_i\in F$. ∎).

15. To answer the question in Warning, the def. of $f(A)$ is in page 565, not the $\det(A-AI)$. Also, note that $f(t)=\det(A-tI)$ is a function $f:\mathbb{R}\to\mathbb{R}$, not defined on $M_{n\times n}(\mathbb{R})$.
To prove the matrix version of Caley-H Thm, Let $A\in M_{n\times n}(F)$, $\exists\,\beta$ s.t. $[L_A]_\beta=A$. (In fact, the std. β_0). We know that by Ex 7, p.258, L_A & A have the same char. poly. $f(t)$. By Caley-H Thm, $f(A)=O$. ∎

§5.4. p89.

16.(a). Let $f(t)$ be the char. poly of T, and let $g(t)$ be the char. poly of T_W. By Thm 5.21, $g(t)$ divides $f(t)$. $\therefore f(t) = q(t)g(t)$ for some poly $q(t)$. Since $f(t)$ splits, $q(t)$ and $g(t)$ must split ▨

(b). Let $f(t)$ be the char. poly. of T, $g(t)$ of T_W, where W is a nontrivial T-invariant subspace. So $g(t)$ has degree at least 1.
$(\neq\{0\})$
So there exists at least one eigenvalue λ of T_W, where λ satisfies $g(\lambda)=0$. $\Rightarrow \exists\, v^{\neq 0} \in W$ s.t. $T_W(v) = \lambda v$, But $\underline{T_W(v) = T(v) = \lambda v}$, so W contains an eigenvector v of T. ▨

17. Good!

$= (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$

Let $A \in M_{n\times n}(F)$. define $f(t) = \det(A - tI_n)$ to be its char. poly.

By Caley-Hamilton Thm, $f(A) = O$, $\therefore$

i.e, $(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = O$

$\Rightarrow A^n$ is a linear combination of $S = \{I_n, A, A^2, \cdots, A^{n-1}\}$, and so is A^k for $k \geq n$.

$\Rightarrow \dim(\operatorname{span}\{I_n, A, \cdots\}) \leq \dim(\operatorname{span}\{I_n, A, \cdots, A^{n-1}\}) \leq n$. ▨

18. (a). $a_0 = f(0) = \det(A)$ and A invertible $\Leftrightarrow \det(A) \neq 0$ ▨

(b). $O = f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n$.

$\Rightarrow -a_0 I_n = A((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n)$.

A invertible $\Rightarrow A^{-1}(-a_0 I_n) = -a_0 A^{-1} I_n = -a_0 A^{-1} = (-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n$

$\Rightarrow A^{-1} = \frac{-1}{a_0}((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n)$ ▨

(c). $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$. $f(t) = \det(A - tI_3) = (1-t)(2-t)(-1-t) = (2-t)(t^2-1)$.

$$= -t^3 + \underset{\substack{a_{n-1}\\ \| \\ a_2}}{\boxed{2}}t^2 + \underset{\substack{a_{n-2}\\ \| \\ a_1}}{1}t \underbrace{-2}_{a_0}.$$

Then by part (b), $A^{-1} = \frac{-1}{-2}((-1)^n A^2 + 2A + I_n)$., $n=3$.

$\Rightarrow A^{-1} = \frac{1}{2}\left((-1)\cdot\begin{pmatrix} 1 & 6 & 6 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right)$

$= \frac{1}{2}\left(\begin{pmatrix} -1 & -6 & -6 \\ 0 & -4 & -3 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 6 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right)$

$= \frac{1}{2}\begin{pmatrix} 2 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -1 \end{pmatrix}$. ▨

for $k=1$, $A=(-a_0)$. ∴ $f(t)=(-1)^1\cdot(a_0+t)$ ∴ correct.

19. Following the hint, for $k=2$, $A=\begin{pmatrix}0 & -a_0\\ 1 & -a_1\end{pmatrix}$, $f(t)=(-t)(-a_1-t)+a_0$

$= t^2+a_1t+a_0 = (-1)^2(a_0+a_1t+t^2)$., correct!

Suppose the statement of the question is true until $k=m-1$.

Then for $k=m$, $A=\begin{pmatrix}0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & \cdots & 0 & -a_1\\ 0 & 1 & & \vdots & \vdots\\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & -a_{m-1}\end{pmatrix}_{m\times m}$. $f(t)=\det(A-tI_m)$

cofactor expansion along 1st row

$\overset{\downarrow}{=} (-1)^{1+1}\cdot(-t)\det(\tilde{A}_{11}) + (-1)^{1+m}(-a_0)\det(\tilde{A}_{1m})$.

$= (-t)\det\begin{vmatrix}-t & 0 & \cdots & 0 & -a_1\\ 1 & -t & \cdots & 0 & -a_2\\ \vdots & \vdots & \ddots & & \vdots\\ 0 & 0 & \cdots & 1 & -a_{m-1}\end{vmatrix}_{(m-1)\times(m-1)} + (-1)^m a_0\det\begin{vmatrix}1 & -t & 0 & \cdots & 0\\ 0 & 1 & -t & \cdots & 0\\ \vdots & 0 & 1 & & \vdots\\ \vdots & \vdots & \vdots & & -t\\ 0 & 0 & 0 & & 1\end{vmatrix}_{(m-1)\times(m-1)}$ ↗ $\left(\begin{matrix}\text{上三角}\\ \det = \prod_{i=1}^{m-1} D_{ii}\end{matrix}\right)$.

Induction hypothesis.

$\overset{\downarrow}{=} (-t)\cdot(-1)^{m-1}\cdot(a_1+a_2t+\cdots+a_{m-1}t^{m-2}+t^{m-1}) + (-1)^m a_0$.

$= (-1)^m(a_0+a_1t+a_2t^2+\cdots+a_{m-1}t^{m-1}+t^m)$. ▨

20. V, a T-cyclic subspace of itself.

$UT=TU \Leftrightarrow U=g(T)$ for some $g(t)$ poly.

proof: Following the hint, Suppose V is generated by v. $\Rightarrow V=\text{span}(\overbrace{\{v, T(v), \dots\}}^{\beta})$.

($\Rightarrow$). Let $U(v)=y=\sum_{i=0}^{N} a_iT^i(v)$ for some $a_i\in F$, $i=1\cdots N$, $N \in \mathbb{N}$ since $y\in V$.

Then $U(T(v)) = T(U(v)) = T(y) = \sum_{i=0}^{N} a_iT^{i+1}(v)$.

Define $g(t)=\sum_{i=0}^{N} a_it^i$.

Then $U(v)=g(T)(v)$. ; $U(T(v)) = g(T)(T(v))$. ; ……

So U transforms every vector w_i in β into $g(T)(w_i)$.

$\Rightarrow U=g(T)$. ☆

($\Leftarrow$) $U=g(T)$ for some poly. $g(t)$, say $g(t)=b_0+b_1t+b_2t^2+\cdots+b_nt^n$.

Given $x\in V$, $\exists$ scalars $a_i\in F$ s.t. $x=\sum_{i=0}^{N} a_iT^i(v)$ for some $N\in\mathbb{N}$.

Then $UT(x) = g(T)T(\sum a_iT^i(v)) =$

$= T(g(T)(\sum a_iT^i(v)))$

$= T(U(x)) = TU(x)$. ▨

$g(T)T=Tg(T)$ because $g(T)T(z)=((b_0I_V+b_1T+\cdots+b_kT^k)T)(z)$

$=(b_0T+b_1T^2+\cdots+b_kT^{k+1})(z) \underset{\uparrow}{=} T(b_0I_V+b_1T+\cdots+b_kT^k)(z)$.

for any $z\in V$. T linear.

§5.4.

21. $T: V \to V$, V has $\dim = 2$.

proof: • If $\exists\, x \in V$ s.t. $\{x, T(x)\}$ forms a L.I. set, then V is a T-cyclic subspace of itself generated by x;

• otherwise, given $x \in V$, $T(x) = cx$ for some $c \in F$.
(c is dependent on the choice of x).
x can be arbitrary, so $T = cI$ ■

22. $T: V \to V$, V has $\dim = 2$, $T \neq cI$ $\forall c \in F$.

By §5.4 #21, p.324, V is a T-cyclic subspace of itself, say $V = \text{span}(\{x, T(x)\})$. Given $U: V \to V$, linear, s.t. $UT = TU$.
By §5.4 #20, p.324, $U = g(T)$ for some poly. $g(t)$ ■

23(考古題). ~~For k=1~~, it's trivial. ($v_1 \in W \Rightarrow v_1 \in W$).

For $k=2$, $v_1 + v_2 \in W \overset{T\text{-invariant}}{\Rightarrow} T(v_1+v_2) \in W \Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in W$.

If both v_1 & $v_2 \notin W$, then by the fact that $\begin{cases} v_1 + v_2 \in W \\ \lambda_1 v_1 + \lambda_2 v_2 \in W \end{cases}$

W is a subspace $\Rightarrow \begin{cases} -\lambda_1(v_1+v_2) \in W \\ \lambda_1 v_1 + \lambda_2 v_2 \in W \end{cases} \underset{\text{加法封閉}}{\Rightarrow} (\lambda_2 - \lambda_1) v_2 \in W \underset{\text{乘法封閉}}{\Rightarrow} v_2 \in W$ ⨳
乘法封閉

k=2 的 case 事實上可以不用寫

If either v_1 or $v_2 \notin W$, say $v_2 \notin W$, then we have
$\begin{cases} v_1 \in W \\ v_1 + v_2 \in W \end{cases} \underset{\text{加法}}{\Rightarrow} v_2 \in W$ ⨳. So the statement holds for $k=2$.

Assume the statement holds for all $k = 1, \dots, n-1$.

For the case $k = n$, $v_1 + \cdots + v_n \in W \overset{T\text{-invariant}}{\Rightarrow} T(v_1 + \cdots + v_n) = \sum \lambda_i v_i \in W$.

Also, $v_1 + \cdots + v_n \in W \underset{\text{subspace}}{\Rightarrow} \lambda_n (v_1 + \cdots + v_n) \in W$.
$\underset{\text{subspace}}{\overset{(\text{加法})}{\Rightarrow}} \sum_{i=1}^{n-1} (\lambda_n - \lambda_i) v_i \in W$.

Since $\lambda_1, \dots, \lambda_k$ are distinct, $\lambda_n - \lambda_i \neq 0$ $\forall i = 1, \dots, n-1$.

so $(\lambda_n - \lambda_i) v_i = u_i$ is still an eigenvector to λ_i.

Hence, we may apply the induction hypothesis to $u_1 + \cdots + u_{n-1} \in W$,
$\Rightarrow u_i \in W$ $\forall i = 1, \dots, n-1$. $\Rightarrow (\lambda_n - \lambda_i) v_i \in W \Rightarrow v_i \in W\ \forall i = 1, \dots, n-1$. ①

Now, $v_1 + \cdots + v_n \in W \Rightarrow (v_1 + \cdots + v_n) + (-1)(v_1 + \cdots + v_{n-1}) = v_n \in W$ by ① ■

No.
Date : :

hard.

㉔ T diagonalizable. $W :=$ nontrivial T-invariant subspace.

~~Let $\lambda_1, \dots, \lambda_k$ be the 'distinct' eigenvalues of T, each with multiplicity m_i, $i = 1 \dots k$. Let $\beta = \bigcup_{i=1}^{k} \beta_i$ be the basis consisting of eigenvectors for V, where $\beta_i = \{v_{i1}, v_{i2}, \dots, v_{im_i}\}$ is the basis for E_{λ_i}.~~

Let E_λ be eigenspace of V w.r.t. λ. Let $W_\lambda = E_\lambda \cap W$.

- W_λ is a subspace: (1. If x & $y \in W_\lambda$, $\Rightarrow \begin{cases} x, y \in W \\ x, y \in E_\lambda \end{cases} \Rightarrow \begin{cases} x+y \in W \\ x+y \in E_\lambda \end{cases} \Rightarrow x+y \in W_\lambda$.
  2. If $x \in W_\lambda$, $\Rightarrow \begin{cases} x \in E_\lambda \\ x \in W \end{cases} \Rightarrow \begin{cases} cx \in E_\lambda \\ cx \in W \end{cases} \Rightarrow cx \in W_\lambda$. 3. $0 \in W_\lambda$ clearly).
- W_λ is an eigenspace w.r.t λ: because $W_\lambda \subseteq E_\lambda$.

Assume that β_λ is a basis for W_λ. We wanna prove that $\beta := \bigcup_\lambda \beta_\lambda$ is a basis for W. (Then T_W is diagonalizable on W by def).

- β is L.I. set by Thm 5.8.
- Given $x \in W (\subseteq V)$, $x = \sum_{i=1}^{k} \sum_{j=1}^{m(i)} a_{ij} v_{ij}$, where $\{\overbrace{v_{11}, \dots, v_{1m_1}}^{\in E_{\lambda_1}}, \dots, \overbrace{v_{k1}, \dots, v_{km_k}}^{\in E_{\lambda_k}}\}$ of is a basis consisting of eigenvectors of ~~T for V~~ (since T is diagonalizable)

令 $w_i = \sum_{j=1}^{m_i} a_{ij} v_{ij} \Rightarrow x = \sum_{i=1}^{k} w_i$, & $w_i \in E_{\lambda_i}$ & λ_i's are distinct.

Apply §5.4 #23, p.324, $w_i \in W$ $\forall i = 1 \dots k$. $\Rightarrow w_i \in (E_{\lambda_i} \cap W) = W_{\lambda_i} \Rightarrow w_i \in \operatorname{span}(\beta_\lambda)$ $\forall i$. ▨

hard.

25. (a): Following the hint, given λ eigenvalue of T, for any $x \in E_\lambda = \{x \mid \underset{=0}{(T - \lambda I)(x)}\}$,

$U(x) = \frac{1}{\lambda} U(\lambda x)$ (linear) $= \frac{1}{\lambda} U(T(x)) = \frac{1}{\lambda} T(U(x))$ (commute).

$\Rightarrow T(U(x)) = \lambda \cdot U(x) \Rightarrow U(x) \in E_\lambda$. Hence, E_λ is U-invariant.

- Thus, we may define $U_{E_\lambda} : E_\lambda \to E_\lambda$ and apply §5.4 #24 to get the fact that U_{E_λ} is diagonalizable.
- Hence, $\exists$ basis $\{v_1, \dots, v_n\} \subseteq E_\lambda$ s.t. $U(v_i) = \lambda v_i$.

But by def of E_λ, $T(v_i) = \lambda v_i$.

So T_{E_λ} & U_{E_λ} are simultaneously diagonalizable.

Since λ can be arbitrary, E_λ can be any eigenspace.

$\Rightarrow$ T & U are simultaneously diagonalizable. ▨

(b). Just replace T by L_A & U by L_B ▨

§5.4.

hard:

㉖. Let $v_1, v_2, \dots, v_n$ be the eigenvectors w.r.t. λ_i, where λ_i's are all distinct.
Let $v = \sum_{i=1}^{n} v_i$. Define $W = \text{span}\{v, T(v), T^2(v), \dots\}$, i.e. W is a T-cyclic subspace generated by v. $\Rightarrow$ W is T-invariant.
By §5.4 #23 $v \in W \Rightarrow v_i \in W \ \forall i$.. Note that $\{v_1, \dots, v_n\}$ is L.I. by Thm 5.5. & $\dim(V) = n$. So $\{v_1, \dots, v_n\}$ is a basis for V & $W = V$.
$\Rightarrow$. $\{v, T(v), \dots, T^{n-1}(v)\}$ is a basis for W. ▨

27. (a). If $v + W = v' + W$, then $(v - v') \in W$ and $T(v - v') \in W$ (T-invariant).

$$\bar{T}(v+W) - \bar{T}(v'+W) \overset{\text{def}}{=} (T(v)+W) - (T(v')+W) \underset{\substack{\uparrow \\ \text{by §1.3 \#31. (trivial property).}}}{=} (T(v) - T(v')) + W$$

$= T(v-v') + W = W$. since $T(v-v') \in W$.

$\Rightarrow \bar{T}(v+W) = \bar{T}(v'+W)$. ※ ▨

(b). Given $v_1 + W$ & $v_2 + W \in V/W$. and given $c \in \mathbb{F}$,

$$\bar{T}(c(v_1+W) + (v_2+W)) \underset{\substack{\uparrow \\ §1.3 \#31}}{=} \bar{T}((cv_1+v_2)+W) \overset{\text{def.}}{=} T(cv_1+v_2)+W$$

$$\overset{T \text{ linear}}{=} (cT(v_1) + T(v_2)) + W \overset{§1.3 \#31}{=} cT(v_1) + W + T(v_2) + W \quad ▨$$

(c). $\eta: V \to V/W$, $\eta(v) = v + W$.

Given $x \in V$, $\eta(T(x)) \overset{\text{def}}{=} T(x) + W \overset{\text{def}}{=} \bar{T}(x+W) \overset{\text{def}}{=} \bar{T}(\eta(x))$. ▨

28. Following the hint, $\gamma = \{v_1, \dots, v_k\}$ is a basis for W. Extend γ to a basis $\beta = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V.

- Claim: $\alpha := \{v_{k+1} + W, v_{k+2} + W, \dots, v_n + W\}$ is a basis for V/W.

  pf. (Same as §1.6 #35). Assume $\sum_{i=k+1}^{n} (v_i + W) = 0$ — ①

  ① $\Rightarrow$. $v_{k+1} + W + v_{k+2} + W + \dots + v_n + W = 0 \Rightarrow (v_{k+1} + v_{k+2} + \dots + v_n) + W = 0$

  $\Rightarrow v_{k+1} + \dots + v_n \in W \Rightarrow$. $v_{k+1} + \dots + v_n = \overset{\in W}{x} = \sum_{i=1}^{k} a_i v_i$ for some scalars $a_i \in \mathbb{F}$.

  $\Rightarrow$. $a_i = 0 \ \forall i$ since β is L.I. ※ (claim finished).

- First, for $j = 1, \dots, k$. $T(v_j) \in W$ since W is T-invariant. $\therefore T(v_j) = \sum_{i=1}^{k} a_i v_i$.

  Second, for $j = k+1, \dots, n$, $\bar{T}(v_j + W) \overset{\text{def}}{=} T(v_j) + W = \sum_{\textcircled{i=1}}^{\textcircled{n}} a_i v_i + W$

  $= \sum_{\textcircled{i=k+1}}^{n} a_i v_i + W$ since $\sum_{i=1}^{\textcircled{k}} a_i v_i \in W$. Thus, $[T]_\beta = \begin{bmatrix} B_1 & B_2 \\ O & B_3 \end{bmatrix}$,

  where $B_1 = [T]_\gamma$ & $B_3 = [\bar{T}]_\alpha$. $\Rightarrow$. $f(t) = g(t)h(t)$ ▨

29. T diagonalizable. $\Rightarrow \bar{T}$ diagonalizable. $\underset{\overset{\shortparallel}{\bar{\beta}}}{\{w_1, \dots w_k, w_{k+1}, \dots, w_n\}}$ ⊕

proof: T diagonalizable $\Rightarrow \exists$ ordered basis $\bar{\beta}$ s.t. $[T]_{\bar{\beta}} = D$ diagonal matrix.
Then $\bar{\gamma} := \{w_1, \dots, w_k\}$ is exactly the ordered basis s.t. $[T_W]_{\bar{\gamma}} = D_W$ a diagonal matrix. Now, for each w_j, $j = k+1, \dots, n$, $\bar{T}(w_j + W) = T(w_j) + W \overset{①}{=} D_{jj} w_j + W$
$= D_{jj}(w_j + W)$. $\therefore$ $w_j + W$ is an eigenvector of $\bar{T}: V/W \to V/W$. corresponding to D_{jj} (eigenvalue of T & $\bar{T}$) $\Rightarrow [T]_{\bar{\beta}} = D = \begin{bmatrix} D_W & O \\ O & \bar{D} \end{bmatrix}$, where $\bar{D} = [\bar{T}]_\alpha$,
$\alpha = \{w_{k+1} + W, w_{k+2} + W, \dots, w_n + W\}$. ■

△ 30. (very hard)

31. $T = L_A$. $W = \mathrm{span}(\{e_1, T(e_1), \dots\})$.

(a). $T(e_1) = \begin{pmatrix}1\\2\\1\end{pmatrix}$; $T^2(e_1) = \begin{pmatrix}0\\12\\6\end{pmatrix} = -6e_1 + 6T(e_1)$

$\therefore$ W has a basis $\gamma = \{e_1, T(e_1)\}$. By Thm 5.22, $6e_1 - 6T(e_1) + T^2(e_1) = 0$
$\Rightarrow g(t) = (-1)^2(6 - 6t + t^2)$. ※ ■

(b). $\beta := \{e_3 + W\}$. Assume $ae_3 + be_1 + cT(e_1) = 0$. for some $a, b, c \in \mathbb{R}$.
$\Rightarrow \begin{cases} b + c = 0 \\ a + 2c = 0 \\ c = 0 \end{cases} \Rightarrow a = b = c = 0 \Rightarrow e_3$ is indep. of γ.
$\therefore$ $\alpha := \{e_1, T(e_1), e_3\}$ is a basis for V. Now, given $x \in V$, $x = ae_1 + bT(e_1) + ce_3$ for some $a, b, c \in \mathbb{R}$. Then $V/W = \{x + W \mid x \in V\}$
$= \{ae_1 + bT(e_1) + ce_3 + W \mid a, b, c \in \mathbb{R}\} = \{ce_3 + W \mid c \in \mathbb{R}\}$ (since $ae_1 + bT(e_1) \in W$ for any $a, b \in \mathbb{R}$). $= \{c(e_3 + W) \mid c \in \mathbb{R}\} = \mathrm{span}(\beta)$. Also, β is L.I.
Hence, β is a basis for V/W ※
$\bar{T}(e_3 + W) = T(e_3) + W = \begin{pmatrix}1\\3\end{pmatrix} + W = (-1)e_1 + 2T(e_1) + (-1)e_3 + W = -e_3 + W$.
$\Rightarrow [\bar{T}]_\beta = [-1] \Rightarrow \det[\bar{T} - t\mathbf{I}_{V/W}] (=: h(t)) = -1 - t$. ■

(c) $f(t) := \det[A - tI_n] = g(t)h(t) = (-1)^2(t^2-6t+6)(-1-t) = (-1)(t^3-5t^2+6)$ ∎

32. Following the hint, let $\dim(V) = k \in \mathbb{N}$. $f(t) :=$ char. poly. of T, splits!

- For $k=1$, trivial. ($[T]_\beta$ is an upper triangular matrix for any β).
- For $k=2$, $f(t)$ splits $\overset{\text{Thm 5.2, p.248}}{\Rightarrow}$ $\exists$ eigenvalue λ of T and eigenvector $x \in E_\lambda$ of T

In fact 可以不用寫. Define $W = \mathrm{span}(\{x, T(x), T^2(x), \cdots\}) \Rightarrow W$ is T-invariant & $\dim(W) = 1$

Since $T^i(x) = \lambda^i x \ \forall i = 1, \ldots$. So define $\alpha = \{x\}$ to be the basis for W.

Extend α to $\beta = \{x, y\}$ for some y, s.t. β is a basis for V.

Then $[T]_\beta = \begin{bmatrix} B_1 & B_2 \\ O & B_3 \end{bmatrix}$, where $B_1 = [T_W]$. $\therefore$ $k=2$ holds!

- Suppose the statement holds for all $k<n$.

For $k=n$, $f(t)$ splits $\Rightarrow$ $\exists$ eigenvector w.r.t. eigenvalue λ of T.

$W := \mathrm{span}(\{v\})$. Note that $\bar{T} : V/W \to V/W$ has a char. poly. $g(t)$

s.t. $g(t)$ divides $f(t)$ (by §5.4 #28, p.325) $\Rightarrow g(t)$ splits.

Also, $\dim(V/W) = \dim(V) - \dim(W) = n-1$.

So we can apply the induction hypothesis to $\bar{T}$.

$\Rightarrow$ $\exists\, \beta$ for V/W s.t. $[\bar{T}]_\beta$ is an upper triangular matrix, say

$\beta = \{v_1+W, v_2+W, \cdots, v_{n-1}+W\}$. $\Rightarrow$ If $0 = \sum_{i=1}^{n-1} a_i(v_i+W) = (\sum^{n-1} a_i v_i) + W$,

then $a_i = 0 \ \forall i = 1, \cdots, n-1$. $\Rightarrow$ If $\sum_{i=1}^{n-1} a_i v_i + bv = 0$, then

$0 = (\sum_{i=1}^{n-1} a_i v_i + \underset{\in W}{\boxed{bv}}) + W = \sum_{i=1}^{n-1} a_i v_i + W \Rightarrow a_i = 0 \ \forall i = 1, \ldots, n-1$

代回原式 $\Rightarrow b = 0$ (since $v \neq 0$).

Thus. $\gamma := \{v, v_1, v_2, \cdots, v_{n-1}\}$ is a basis for V.

By §5.4 #28 hint, $[T]_\gamma = \begin{bmatrix} B_1 & B_2 \\ O & B_3 \end{bmatrix}$, where $B_1 = [T_W]_{\alpha=\{v\}}$ (1×1 matrix).

and $B_3 = [\bar{T}]_\beta$, $\Rightarrow$ $[T]_\gamma$ is an upper triangular matrix. ∎

33. $\{W_i\}_{i=1}^k$ are all T-invariant.

Given $x_i \in W_i$, $i = 1, \ldots, k$, define $y = \sum_{i=1}^k x_i \in W_1 + \cdots + W_k$.

$T(y) = T(\sum^k x_i) = \sum^k T(x_i) = \sum^k u_i$, where $u_i := T(x_i) \in W_i$ (T-invariant).

$\Rightarrow T(y) \in W_1 + \cdots + W_k$ So $W_1 + \cdots + W_k$ is also T-invariant ∎

34. For the case $k=2$, $V = W_1 \oplus W_2$, W_i is T-invariant, $i=1,2$.

Let $\beta = \beta_1 \cup \beta_2$, β_i is a basis for W_i $\{A := [T]_\beta,\ B_i := [T_{W_i}]_{\beta_i},\ i=1,2.\} \leftarrow ①$

Then $A = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix}$. by ①, $\Rightarrow$ $A = B_1 \oplus B_2$. ∎

35. Suppose the statement holds for all $n=1,\cdots,k-1$, where $V=W_1\oplus\cdots\oplus W_n$.

For $n=k$, let $W=: W_1\oplus\cdots\oplus W_{k-1} \Rightarrow V=W\oplus W_k$.

$\beta := \bigcup_{i=1}^{k}\beta_i$ & $\gamma := \bigcup_{i=1}^{k-1}\beta_i\,(=\beta-\beta_k)$ Apply the induction hypothesis,

$[T_W]_\gamma = B_1\oplus\cdots\oplus B_{k-1}$, where $B_i=[T_{W_i}]_{\beta_i}$

By §5.4 #34, $[T]_\beta = \begin{bmatrix} [T_W]_\gamma & O \\ O & [T_{W_k}]_{\beta_k}\end{bmatrix}$

$\Rightarrow [T]_\beta = (B_1\oplus\cdots\oplus B_{k-1})\oplus B_k = B_1\oplus B_2\oplus\cdots\oplus B_k$

36. T is diagonalizable $\Leftrightarrow$ $V=W_1\oplus\cdots\oplus W_k$ for some $k\in\mathbb{N}$, where W_i is the 1-dimensional T-invariant subspace of V $\forall i$.

proof ($\Rightarrow$). Let $\lambda_1,\dots,\lambda_k$ be the 'distinct' eigenvalues of T. Let m_i be the multiplicity of λ_i. T diagonalizable $\Rightarrow \exists\beta$ consisting of eigenvectors s.t. $[T]_\beta=D$, a diagonal matrix.

Chose $W_i=E_{\lambda_i}$, $i=1,\dots,k$. Note that W_i is T-invariant (easy).

and $\left(\bigcup_{\substack{i=1\\ i\neq j}}^{k}E_{\lambda_i}\right)\cap E_{\lambda_j}=\{0\}$ $\forall j$ and $W_1+\cdots+W_k=V$ (easy).

Thus, by def (p.275), $V=W_1\oplus\cdots\oplus W_k$, $\dim(W_i)=m_i$, $i=1,\cdots,k$.

- Now, given any i, $T_{E_{\lambda_i}}$ is also diagonalizable. (a $m_i\times m_i$ submatrix of D). $\Rightarrow \exists\beta_i$, say $\beta_i=\{v_{i1},\dots,v_{im_i}\}$ s.t. $[T_{E_{\lambda_i}}]$ is a diagonal matrix and $\beta=\bigcup_{i=1}^{k}\beta_i$, $[T]_\beta=D$. Let $\gamma_{ij}=\{v_{ij}\}$ & $L_{ij}=:\mathrm{span}(\gamma_{ij})$.

Then each L_{ij}, $j=1,\dots,m_i$ is a subspace of $W_i\,(=E_{\lambda_i})$ with $\dim(L_{ij})=1$.

Claim: $\left(\bigcup_{\substack{j=1\\ j\neq p}}^{m_i}L_{ij}\right)\cap L_{ip}=\{0\}$ $\forall p=1,\dots,m_i$.

pf. Assume $x\in\bigcup_{\substack{j=1\\ j\neq p}}^{m_i}L_{ij}$ & $x\in L_{ij}$. $\Rightarrow x=\sum_{\substack{j=1\\ j\neq p}}^{m_i}a_j v_{ij}=b\,v_{ip}$.

But β_i is L.I. $\Rightarrow a_j=b=0$ $\forall j=1,\dots,m_i$, $j\neq p$. ※ (claim finished).

$E_{\lambda_i}=L_{i1}+L_{i2}+\cdots+L_{im_i}$ since β_i is a basis.

$\therefore E_{\lambda_i}=L_{i1}\oplus L_{i2}\oplus\cdots\oplus L_{im_i}$ by claim ※

($\Leftarrow$). $V=W_1\oplus\cdots\oplus W_k$, W_i is T-invariant & $\dim(W_i)=1$.

Let $\{v_i\}=:\beta_i$ to be the basis for W_i. So $\beta:=\bigcup_{i=1}^{k}\beta_i$ is a basis for V (Thm 5.10). Given v_i, $T(v_i)=a_iv_i$ for some $a_i\in F$. since W_i is T-invariant. $\Rightarrow [T]_\beta=\begin{bmatrix}a_1 & & \\ & a_2 & \\ & & \ddots a_k\end{bmatrix}$ $\Rightarrow$ T is diagonalizable by def. ■

37. $V = W_1 \oplus \cdots \oplus W_k$.

By Thm 5.25, p.321, $[T]_\beta = B_1 \oplus \cdots \oplus B_k$, where $B_i = [T_{W_i}]_{\beta_i}$.

For $k=2$, $[T]_\beta = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix}$. $\Rightarrow \det([T]_\beta) = \det(B_1)\det(B_2)$. by §4.3 #21, p.229.
$\Rightarrow \det(T) = \det(T_{W_1})\det(T_{W_2})$ since $\det(T)$ is indep. of β.

Suppose the statement hold for $k = 1, \ldots, n-1$. ~~$[T_{W_1 \oplus \cdots \oplus W_{n-1}}]$~~

For $k=n$, $[T]_\beta = B_1 \oplus \cdots \oplus B_{n-1} \oplus B_{n} = M \oplus B_n$, where $\overset{n}{M} := B_1 \oplus \cdots \oplus B_{n-1}$.

By induction hypothesis, $\det(M) = \det(B_1)\det(B_2)\cdots\det(B_{n-1})$.

Again, this is the $k=2$ case for $[T]_\beta = M \oplus B_n$. So $\det([T]_\beta) = \det(M)\det(B_n)$

$\Rightarrow \det([T]_\beta) = \det(B_1)\det(B_2)\cdots\det(B_n)$.

$\Rightarrow \det(T) = \det(T_{W_1}) \cdots \det(T_{W_n})$ since $\det(T)$ is indep. of the choice of β. (see. §5.1 #7, p.258). ∎

38. T is diagonalizable. $\Leftrightarrow$ T_{W_i} is diagonalizable for all i.

pf ($\Rightarrow$). T diagonalizable. $\Rightarrow \exists \beta$ s.t. $[T]_\beta = D$, diagonal matrix.

$\beta_i =: \beta \cap W_i$. Then $[T_{W_i}]_{\beta_i}$ is also a diagonal matrix since $[T_{W_i}]_{\beta_i}$ is a submatrix of D, containing some diagonal entries of D. ※

($\Leftarrow$). Let β_i be an ordered basis for W_i s.t. $[T_{W_i}]_{\beta_i} \overset{B_i}{=}$ is a diagonal matrix. By Thm 5.10, $\beta := \bigcup_{i=1}^{k} \beta_i$ is a basis for $V = W_1 \oplus \cdots \oplus W_k$.

Then $[T]_\beta = \begin{bmatrix} B_1 & & O \\ & B_2 & \\ O & & \ddots \; B_k \end{bmatrix}$ is a diagonal matrix since each B_i is a diagonal matrix ∎

39. $C :=$ collection of diagonalizable linear operators on a finite dim. V.

$\exists \beta$ s.t. $[T]_\beta$ is a diagonal matrix. $\forall T \in C$ $\Leftrightarrow$ the operators of C commute under composition.

pf. ($\Rightarrow$). $\forall T \in C$, $\exists \beta$ s.t. $[T]_\beta = D_T$ is a diagonal matrix. say $\beta = \{v_1, \ldots, v_n\}$.

Given $U, T \in C$, $\forall x \in V$, $UT(x) = UT(\sum a_i v_i) = U(\sum a_i D_{ii} v_i) = \sum a_i D_{T,ii} D_{U,ii} v_i$

$TU(x) = TU(\sum a_i v_i) = T(\sum a_i D_{U,ii} v_i) = \sum a_i \cdot D_{U,ii} \cdot D_{T,ii} \cdot v_i$ $\therefore UT(x) = TU(x)$ $\forall x \in V$. $\Rightarrow$ commute ※

($\Leftarrow$). Let $n = \dim(V)$. For $n=1$, given $T \in C$, T is diagonalizable & $\dim(V) = 1$

(hard). $\Rightarrow$. T has exactly 1 eigenvalue λ and an eigenvector v. Let $\beta = \{v\}$.

Given $U \in C$, $UT = TU \Rightarrow T(U(v)) = U(T(v)) = \lambda U(v)$. $\Rightarrow U(v) \in E_\lambda$.

$\Rightarrow U(v) = av$ for some $a \in F$. $\Rightarrow [U]_\beta = a$ is a 1×1 diagonal matrix.

(Continued).

- Assume the statement holds for all $n < k$ $\overset{\in \mathbb{N}}{}$.
- For $n=k$, given $T \in C$, T diagonalizable $\overset{\text{Thm5.11}}{\Leftrightarrow}$ $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$. $(E_{\lambda_i} = \{x \mid (T-\lambda_i I)(x)=0\})$

  Let $\beta_i := \{v_{i1}, v_{i2}, \cdots v_{im_i}\}$ be a basis of E_{λ_i} & $\beta := \bigcup^{k} \beta_i$ a basis for V.

40. For $k=2$, $A = B_1 \oplus B_2$. Then $f_A(t) := \det(A - tI) = \det\begin{pmatrix} B_1 - tI_\ell & O \\ O & B_2 - tI_{n-\ell} \end{pmatrix}$

$\overset{\S 4.3\#21}{=} \det(B_1 - tI) \cdot \det(B_2 - tI) = f_{B_1}(t) \cdot f_{B_2}(t)$ ☒

Suppose the statement holds for $k = n-1$.

For $k=n$, $A = B_1 \oplus \cdots \oplus B_n$. Let $B := B_1 \oplus \cdots \oplus B_{n-1}$. $(\Rightarrow A = B \oplus B_n)$.

Then by induction hypothesis, $\det(B - tI) =: f_B(t) = f_{B_1}(t) \cdot f_{B_2}(t) \cdot \cdots \cdot f_{B_{n-1}}(t)$.

Since $A = B \oplus B_n$, it's the $k=2$ case, so $f_A(t) = f_B(t) \cdot f_{B_n}(t)$

$= (f_{B_1}(t) \cdot \cdots \cdot f_{B_{n-1}}(t)) \cdot f_{B_n}(t)$ ▨

41. Following the hint, • rank(A) = ? : $\begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & & & \\ n^2-n+1 & \cdots & & n^2 \end{pmatrix}_{n\times n} \to \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 2n & 2n & \cdots & 2n \\ n^2-n & n^2-n & \cdots & n^2-n \end{pmatrix}$ $\times -2$ $\times(-n+1)$

$$\to \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & -n & -2n & \cdots & n-n^2 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -n & -2n & \cdots & n-n^2 \\ 0 & 0 & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \end{pmatrix} \Rightarrow \text{rank}(A) = 2.$$

- $\beta := \{(1,1,\cdots,1), (1,2,\cdots,n)\}$ Clearly, β is L.I.. $W := \operatorname{span}(\beta)$.

(*)

$$\therefore A \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} (n+1)n/2 \\ n^2 + (n+1)n/2 \\ 2n^2 + (n+1)n/2 \\ \vdots \\ n^3 - n^2 + (n+1)n/2 \end{pmatrix} = \frac{n(n+1)}{2} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + n^2 \left(\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right). \Rightarrow A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in W.$$

$$\therefore A \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} 1+4+\cdots+n^2 \\ \frac{(n+n^2)n}{2} + 1+4+\cdots+n^2 \\ 2 \cdot \frac{(n+n^2)n}{2} + 1+4+\cdots+n^2 \\ \vdots \\ (n-1)\frac{(n+n^2)n}{2} + 1+4+\cdots+n^2 \end{pmatrix} = \underset{\downarrow\ \frac{n(n+1)(2n+1)}{6}}{\left(\sum_{k=1}^{n} k^2\right)} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{(n+n^2)n}{2} \left(\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) \Rightarrow A \cdot \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} \in W$$

Thus, W is L_A-invariant (Continued).

§5-4 - §6.1

③ $\#(\gamma-\beta)=n-2$.

② $\text{rank}(A)=2$. ($\Rightarrow$ $\text{nullity}(A)=n-2$.)

By (*), ① $L_A\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\neq 0$ & $L_A\begin{pmatrix}1\\ 2\\ \vdots\\ n\end{pmatrix}\neq 0$, $\Rightarrow [A]_{\gamma\supseteq\beta} = \begin{bmatrix}[L_A|_W]_\beta & O\\ O & O\end{bmatrix}$, where γ is a basis for F^n extended from β.

Than $f_A(t) := \det(A-tI_n) = \det(L_A|_W - tI_2)\cdot(-t)^{n-2}$

By (*), $[L_A|_W]_\beta = \begin{pmatrix}\frac{n-n^2}{2} & n^2\\ \frac{-n^3+n}{6} & \frac{n^2+n^3}{2}\end{pmatrix}_{2\times 2}$.

$\Rightarrow \det(L_A|_W - tI_2) = (t-\frac{n-n^2}{2})(t-\frac{n^2+n^3}{2}) + n^2\cdot\frac{n^3+n}{6}$

$= t^2 - \frac{n+n^3}{2}t + \frac{n^3(1-n)(1+n)}{4} + \frac{n^3(1+n^2)}{6}$

$= t^2 - \frac{n+n^3}{2}t + \frac{n^3(5-n^2)}{12}$

Thus, $f_A(t) = (-1)^n\left(t^n - \frac{n(1+n^2)}{2}t^{n-1} + \frac{n^3(5-n^2)}{12}\right)$. ▨

42. $A = \begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & \ddots & & \\ \vdots & & \ddots & \\ 1 & & & 1\end{pmatrix}$. $\text{rank}(A)=1$.

Choose $\beta = \left\{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\right\} \in F^n$ and define $W = \text{span}(\beta)$. Extend β to γ, a basis for F^n.

$A\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}n\\ \vdots\\ n\end{pmatrix} = n\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ $\therefore$ W is L_A-invariant. & $[L_A|_W]_\beta = [n]_{1\times 1}$.

$\text{nullity}(A) = n-1$ and $L_A\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\neq 0$ and $\#(\gamma-\beta) = n-1$ $\Rightarrow$ $\forall v\in\gamma-\beta$, $v\in N(L_A)$.

Thus, $f_A(t) = \det(A-tI_n) = \det\begin{bmatrix}[L_A|_W]_\beta - tI_1 & O\\ O & O-tI_{n-1}\end{bmatrix}$

$= (-t)^{n-1}(n-t) = (-1)^n(t^n - nt^{n-1})$. ▨

Ch6. §6.1

Section §6.1

1 (a). T. (b) ~~F~~ T.

(c). F, because $\langle x, cy\rangle \neq c\langle x,y\rangle$ if $c\in\mathbb{C}$. (d) F., see p.330 E.g 1 & E.g.2.

(e). F. (f). F. (g) F, 要 $\forall x$, $\langle x,y\rangle = \langle x,z\rangle$ 才可以. (h) T.

2. $\langle x,y\rangle = 2\overline{(2-i)} + (1+i)\overline{2} + i\overline{(1+2i)} = 2(2+i) + (1+i)2 + i(1-2i)$

$= 4+2i+2+2i+i+2 = 8+5i$

$\|x\| = \langle x,x\rangle^{\frac{1}{2}} = (4+1^2+1^2+1)^{\frac{1}{2}} = \sqrt{7}$

$\|y\| = \langle y,y\rangle^{\frac{1}{2}} = ((4+1)+4+1^2+4)^{\frac{1}{2}} = \sqrt{14}$.

$\|x+y\| = \|(4-i, 3+i, 1+3i)\| = (1+16+10+10)^{\frac{1}{2}} = \sqrt{37}$

$\sqrt{64} \leq \sqrt{98}$.

$16 \overset{\wedge}{\leq} 2\sqrt{98}$

Cauchy-Schwarz ineq.: $|\langle x,y\rangle| = \sqrt{89} \leq \sqrt{7}\times\sqrt{14} = \|x\|\cdot\|y\|$.

Triangle-inequality: $\|x+y\| = \sqrt{37} \leq \|x\|+\|y\| = \sqrt{7}+\sqrt{14}$ since $37 \leq 7+14+2\sqrt{98}$ ▨

No.

Date : §6.1

3. $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 te^t\,dt = \int_0^1 t\,de^t = te^t\big|_0^1 - \int_0^1 e^t\,dt = e-(e-1)=1$ #

$\|f\| = (\int_0^1 t^2\,dt)^{\frac12} = \sqrt{\frac13}$; $\|g\| = (\int_0^1 e^{2t}\,dt)^{\frac12} = (\frac12(e^2-1))^{\frac12}$;

$\|f+g\| = (\int_0^1 t^2+2te^t+e^{2t}\,dt)^{\frac12} = (\frac13+2+\frac12e^2-\frac12)^{\frac12} = (\frac12e^2+\frac{11}{6})^{\frac12}$

Cauchy-Schwarz ineq.: $|\langle f,g\rangle| = 1 \le (\frac13)^{\frac12}\cdot(\frac12e^2-\frac12)^{\frac12} = \sqrt{\frac{e^2-1}{6}} = \|f\|\cdot\|g\|$.

Since $1\le e^2-1$, where $e^2 \approx (2.718)^2 \approx 7.3875$

Triangle ineq.: $\|f+g\| = (\frac{3e^2+11}{6})^{\frac12} \le \|f\|+\|g\| = (\frac13)^{\frac12} + (\frac12(e^2-1))^{\frac12}$

Since $\frac{\cancel{3e^2}+11}{6} \le (\frac13)+(\underbrace{\frac12\cancel{e^2}}-\frac12)+2\sqrt{\frac13\cdot\frac{e^2-1}{2}} \Leftrightarrow 2\le 2\sqrt{\frac{e^2-1}{6}} \Leftrightarrow 1\le\sqrt{\frac{e^2-1}{6}}$. ■

4. (a) proof of Eg 5.

Check (b) of the defn.: $\langle cA,B\rangle = tr(B^*cA) = c\,tr(B^*A) = c\langle A,B\rangle$ #

Check (c) of the defn.: $\overline{\langle A,B\rangle} = \overline{tr(B^*A)} = \overline{\sum_{i=1}^n (B^*A)_{ii}} = \overline{\sum_i\sum_k (B^*)_{ik}(A)_{ki}}$

$\overset{\text{def of }B^*}{=} \sum_i\sum_k \overline{\overline{(B)_{ki}}\cdot(A)_{ki}} = \sum_i\sum_k \overline{\overline{(B)_{ki}}}\,\overline{(A)_{ki}} = \sum_i\sum_k (B)_{ki}\overline{(A)_{ki}} \overset{\text{def of }A^*}{=}$

$= \sum_i\sum_k B_{ki}(A^*)_{ik} = \sum_i\sum_k (A^*)_{ik}B_{ki} = tr(A^*B) = \langle B,A\rangle$ ■

(b) $\|A\| = (tr(A^*A))^{\frac12} = \left(tr\begin{bmatrix}10 & 2+4i\\ 2-4i & 6\end{bmatrix}\right)^{\frac12} = \sqrt{16} = 4$.

$(A^* = \begin{bmatrix}1&3\\2-i&-i\end{bmatrix})$

$\|B\| = (tr(B^*B))^{\frac12} = \left(tr\begin{bmatrix}3&-1\\-1&1\end{bmatrix}\right)^{\frac12} = \sqrt4 = 2$.

$(B^*=\begin{bmatrix}1-i&-i\\0&+i\end{bmatrix})$

$\langle A,B\rangle = tr(B^*A) = tr\begin{bmatrix}1-4i&4-i\\3i&-1\end{bmatrix} = -4i$.

5. • $\langle x,y\rangle := x\begin{pmatrix}1&i\\-i&2\end{pmatrix}^* y$ is an inner product:

∵ Given $x,y,z\in\mathbb{C}^2$, $\langle x+z,y\rangle = (x+z)(Ay^*) = x(Ay^*)+z(Ay^*) = \langle x,y\rangle+\langle z,y\rangle$.

∵ Given $x,y\in\mathbb{C}^2$ & $c\in\mathbb{C}$, $\langle cx,y\rangle = cx(Ay^*) = c(xAy^*) = c\langle x,y\rangle$.

∵ Given $x,y\in\mathbb{C}^2$, $\overline{\langle x,y\rangle} = \overline{xAy^*} = \overline{x\cdot\begin{pmatrix}\sum(A)_{1j}(y^*)_j\\ \sum(A)_{2j}(y^*)_j\end{pmatrix}} = \overline{\sum_{i=1}^2\sum_{j=1}^2 (x)_i(A)_{ij}(y^*)_j}$

$= \sum_i\sum_j \overline{x_i}\,\overline{A_{ij}}\,y_j = \sum_j\sum_i y_j\overline{A_{ij}}\,\overline{x_i} = \sum_j\sum_i y_j(A^*)_{ji}(x^*)_i \overset{*(A^*=A)}{=} \sum_j\sum_i y_jA_{ji}(x^*)_i$

$= yAx^*$

∵ Given $x\in\mathbb{C}^2$, $\langle x,x\rangle = xAx^* = (x_1\ x_2)\begin{pmatrix}1&i\\-i&2\end{pmatrix}\begin{pmatrix}\overline{x_1}\\ \overline{x_2}\end{pmatrix} = (x_1\ x_2)\begin{pmatrix}\overline{x_1}+i\overline{x_2}\\ 2\overline{x_2}-i\overline{x_1}\end{pmatrix}$

$= \|x_1\|^2 + ix_1\overline{x_2} + 2\|x_2\|^2 - i\overline{x_1}x_2 = \underbrace{\|x_1\|^2+2\|x_2\|^2}_{\text{Real part}} + \underbrace{i(x_1\overline{x_2}-\overline{x_1}x_2)}_{\text{Imaginary part}} > 0 \Leftrightarrow x_1 \,\&\, x_2 \neq 0$. #

• $\langle(1-i,2+3i),(2+i,3-2i)\rangle = (1-i,2+3i)\begin{pmatrix}1&i\\-i&2\end{pmatrix}\begin{pmatrix}2-i\\3+2i\end{pmatrix} = (1-i\ \ 2+3i)\begin{pmatrix}2i\\5+2i\end{pmatrix}$

$= (2i+2)+(10-6+19i) = 6+21i$ # ■

§6.1

6. Complete the proof of Thm6.1:

(b): $\langle x, cy\rangle = \overline{\langle cy, x\rangle} = \overline{c\langle y, x\rangle} = \bar{c}\,\overline{\langle y, x\rangle} = \bar{c}\langle x, y\rangle$. ※

(c): $\langle x, 2\cdot\vec{0}\rangle = \bar{2}\langle x, \vec{0}\rangle = 2\langle x, \vec{0}\rangle \Rightarrow \langle x, \vec{0}\rangle = 0$ ※

(d): If $\langle x, x\rangle = 0 \Rightarrow x = 0$ by defn. (d) in p.330.

(e): $\langle x, y\rangle = \langle x, z\rangle\ \forall x \in V \Rightarrow \langle x, y-z\rangle = 0\ \forall x \in V$.

Then choose $x = y - z \Rightarrow \langle y-z, y-z\rangle = 0 \overset{\text{Defn.}}{\Rightarrow} y - z = 0 \Rightarrow y = z$ ※▨

7. Complete the proof of Thm6.2.

(a): $\|cx\|^2 = \langle cx, cx\rangle = c\langle x, cx\rangle = c\bar{c}\langle x, x\rangle = |c|^2\|x\|^2$.

$\Rightarrow \|cx\| = |c|\cdot\|x\| \geq 0$ ▨

(b): By defn. $\|x\| = 0 \Leftrightarrow x = 0$; and by defn. $\langle x, x\rangle > 0$ if $x \neq 0$. ▨

8. (a): $\langle (a,b), (c,d)\rangle := ac - bd$ on $\mathbb{R}^2$:

Then choose $x = (1, 2)$, $y = (2, 1) \Rightarrow \langle (1,2), (2,1)\rangle = 2 - 2 = 0$ ($\times$ to (d) of defn).

(b): $\langle A, B\rangle := tr(A+B)$ on $M_{2\times 2}(\mathbb{R})$.

Then choose $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $R = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \langle A, B\rangle = 0$, $\times$ to (d) of defn.

(C): $\langle f(x), g(x)\rangle := \int_0^1 f(t)g(t)\,dt$ on $P(\mathbb{R})$

Then choose $f(x) = 3$ & $g(x) = x^2 \Rightarrow \langle f, g\rangle = \int_0^1 0\cdot x^2\,dx = 0$, $\times$ to (d) of defn. ▨

9. (a): $\langle x, z\rangle = 0\ \forall z \in \beta \Rightarrow \langle x, \sum a_i z_i\rangle = \sum \bar{a_i}\langle x, z_i\rangle = 0$, $z_i \in \beta$.

$\Rightarrow \langle x, y\rangle = 0\ \forall y \in V \overset{\text{Thm6.1(e)}}{\Rightarrow} x = 0$ ▨

(b): $\langle x, z\rangle = \langle y, z\rangle\ \forall z \in \beta \Rightarrow \langle x-y, z\rangle = 0\ \forall z \in \beta \Rightarrow \langle x-y, \sum a_i z_i\rangle$

$= \sum \bar{a_i}\langle x-y, z_i\rangle = 0$, where $z_i \in \beta\ \forall i$. $\Rightarrow \langle x-y, z\rangle = 0\ \forall z \in V$.

$\overset{\text{Thm6.1(e)}}{\Rightarrow} x - y = 0 \Rightarrow x = y$. ▨

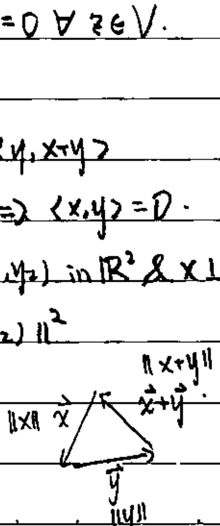
10. • $\|x+y\|^2 = \|x\|^2 + \|y\|^2$: $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2$ since $x \perp y \Rightarrow \langle x, y\rangle = 0$.

• 畢氏定理 ⟨Pythagorean Thm.⟩ in $\mathbb{R}^2$: Given $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$ & $x \perp y$,

$\|x+y\|^2 = \|(x_1+y_1, x_2+y_2)\|^2 = \|x\|^2 + \|y\|^2 = \|(x_1, x_2)\|^2 + \|(y_1, y_2)\|^2$

$\Rightarrow \|(x_1+y_1, x_2+y_2)\|^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2$ ▨



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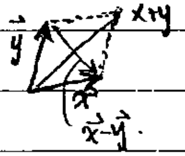
Date : §6.1

11. $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \left(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\right) + \left(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle\right) = 2\|x\|^2 + 2\|y\|^2$ ※

- 說明: ⇒ 平行四邊形的對角線平方和 = 四邊平方和. ▨



12. $\|\sum^{k} a_i v_i\|^2 = \langle \sum a_i v_i, \sum a_i v_i\rangle = \sum a_i \langle v_i, \sum a_i v_i\rangle = \sum_i a_i\left(\sum_j \bar{a_j}\langle v_i, v_j\rangle\right)$
$\overset{\text{orthogonal}}{=} \sum a_i\left(\bar{a_i}\langle v_i, v_i\rangle\right) = \sum a_i\bar{a_i}\|v_i\|^2 = \sum |a_i|^2\|v_i\|^2$ ▨

13. $\langle \cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$. We check the defn. of inner product.

(a). $\langle x+z, y\rangle = \langle x+z, y\rangle_1 + \langle x+z, y\rangle_2 = \langle x,y\rangle_1 + \langle z,y\rangle_1 + \langle x,y\rangle_2 + \langle z,y\rangle_2$
$= \langle x,y\rangle_1 + \langle x,y\rangle_2 + \langle z,y\rangle_1 + \langle z,y\rangle_2 = \langle x,y\rangle + \langle z,y\rangle$ ※

(b). $\langle x, cy\rangle = \langle x, cy\rangle_1 + \langle x, cy\rangle_2 = \bar{c}\langle x,y\rangle_1 + \bar{c}\langle x,y\rangle_2 = \bar{c}\left(\langle x,y\rangle_1 + \langle x,y\rangle_2\right) = \bar{c}\langle x,y\rangle$ ※

(c). $\overline{\langle x,y\rangle} - \overline{\langle x,y\rangle_1 + \langle x,y\rangle_2} = \overline{\langle x,y\rangle_1} + \overline{\langle x,y\rangle_2} = \langle y,x\rangle_1 + \langle y,x\rangle_2 = \langle y,x\rangle$ ※

(d). $\langle x,x\rangle = \langle x,x\rangle_1 + \langle x,x\rangle_2$. So if $x=0$, $\langle x,x\rangle = \langle x,x\rangle_1 + \langle x,x\rangle_2 = 0+0=0$.
If $x\neq 0$, $\langle x,x\rangle = \underset{\vee 0}{\langle x,x\rangle_1} + \underset{\vee 0}{\langle x,x\rangle_2} > 0$ ▨

14. $A, B \in M_{n\times n}(\mathbb{F})$, $c\in\mathbb{F}$.
$\forall i,j$, $(A+cB)^*_{ij} = \overline{(A+cB)_{ji}} = \overline{A_{ji} + cB_{ji}} = \overline{A_{ji}} + \bar{c}\overline{B_{ji}}$
$= (A)^*_{ij} + \bar{c}(B)^*_{ij}$ ▨

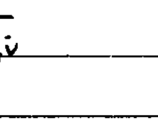
15.(a) (⇒) Following the hint, $(y\neq 0)$ let $a = \frac{\langle x,y\rangle}{\|y\|^2}$, $z = x-ay$ (case for $y=0$ is trivial).

- Claim: y & z are orthogonal.

pf: $\langle y,z\rangle = \langle y, x-ay\rangle = \langle y,x\rangle - \bar{a}\langle y,y\rangle = \langle y,x\rangle - \frac{\langle y,x\rangle}{\cancel{\|y\|^2}}\cancel{\langle y,y\rangle} = 0$
& $y\neq 0$, $z\neq 0$., ⇒ $y\perp z$ ※

- $|a| = \frac{|\langle x,y\rangle|}{\|y\|^2} \overset{\text{condition}}{=} \frac{\|x\|\cdot\|y\|}{\|y\|^2} = \frac{\|x\|}{\|y\|}$.

By §6.1#10
畢氏 $(z\perp ay)$.



- $z=x-ay \Rightarrow x = z+ay \Rightarrow \|x\|^2 = \|z+ay\|^2 = \|z\|^2 + \|ay\|^2$,

but $\|x\| = |a|\|y\|$, ⇒ $\|z\|^2 = 0$ ⇒ $z=0$ by defn of inner product.
Thus, $x = ay$. ※

(⇐). Say $x=ay$, ⇒ $|\langle x,y\rangle| = |\langle ay, y\rangle| = |a|\,\|y\|^2 = |a|\|y\|\cdot\|y\| = \|x\|\cdot\|y\|$ ▨

(b). $\|x+y\| = \|x\|+\|y\| \Leftrightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \Leftrightarrow |\langle x,y\rangle + \langle y,x\rangle| = 2\|x\|\|y\|$
$\Leftrightarrow$ $\mathrm{Re}(\langle x,y\rangle) = \|x\|\|y\|$. ① Note that by Cauchy-Schwarz inequality,

§6.1

$Re(\langle x,y\rangle) \leq |\langle x,y\rangle| \leq \|x\|\cdot\|y\|$. Thus, $Re(\langle x,y\rangle) = |\langle x,y\rangle| = \|x\|\cdot\|y\|$ by ①, $\Leftrightarrow$ one of the vectors x or y is a multiple of the other. ☆

Next, $\|x_1+x_2+\cdots+x_n\| \underset{\text{2-case}}{=} \|x_1\| + \|x_2+\cdots+x_n\| \underset{\text{Repeat}}{=} \cdots\cdots = \sum_{i=1}^{n}\|x_i\|$ ▨

16.(a). H with $\langle\cdot,\cdot\rangle : H \to \mathbb{C}$ defined by $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$.

之后记得 (b).

Check the defn of inner product.

(a): $\langle f+h, g\rangle = \frac{1}{2\pi}\int_0^{2\pi}(f+h)\bar{g}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\,dt + \frac{1}{2\pi}\int_0^{\pi} h\bar{g}\,dt = \langle f,g\rangle + \langle h,g\rangle$ ☆

(b): $\langle f, cg\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\,\bar{c}\bar{g}\,dt = \bar{c}\langle f,g\rangle$ ☆

(c): $\overline{\langle f,g\rangle} = \overline{\frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\,dt} = \frac{1}{2\pi}\int_0^{2\pi}\bar{f} g\,dt = \langle g,f\rangle$ ☆

(d): $\langle f,f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\cdot\bar{f}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\|f\|^2\,dt \begin{cases} >0 & \text{o.w.} \\ =0 & \text{if } f=0 \end{cases}$ ▨

17. $\|T(x)\| = \|x\|\ \forall x$. Prove that T is 1-1.

pf. If $T(x)=0$, then $\|T(x)\| = \|0\| = \|x\| \Leftrightarrow x=0 \Leftrightarrow N(T)=\{0\}$ ▨

18 ($\Rightarrow$) $\langle x,y\rangle' =: \langle T(x),T(y)\rangle$ is an inner product on V.

If $T(x)=0, \Rightarrow \langle x,x\rangle' = \langle T(x),T(x)\rangle = 0 \overset{\text{defn.}}{\Leftrightarrow} x=0$ ▨

($\Leftarrow$). T is 1-1. & Define $\langle x,y\rangle' = \langle T(x),T(y)\rangle$. We check the defn. of inner product:

(a). $\langle x+z,y\rangle' = \langle T(x+z),T(y)\rangle = \langle T(x)+T(z),T(y)\rangle = \langle T(x),T(y)\rangle + \langle T(z),T(y)\rangle = \langle x,y\rangle' + \langle z,y\rangle'$ ☆

(b). $\langle x,cy\rangle' = \langle T(x),T(cy)\rangle = \langle T(x),cT(y)\rangle = \bar{c}\langle T(x),T(y)\rangle = \bar{c}\langle x,y\rangle'$ ☆

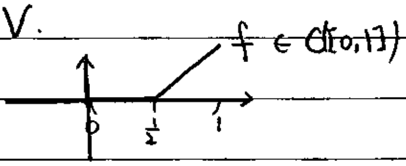
(c). $\overline{\langle x,y\rangle'} = \overline{\langle T(x),T(y)\rangle} = \langle T(y),T(x)\rangle = \langle y,x\rangle'$ ☆

(d). $\langle x,x\rangle' = \langle T(x),T(x)\rangle = \|T(x)\|^2 \begin{cases} >0 & \text{o.w.} \\ =0 & \text{if } T(x)=0 \overset{T \text{ is 1-1}}{\Leftrightarrow} x=0 \end{cases}$ ▨

16 (b) $V = C([0,1])$ $\langle f,g\rangle := \int_0^{\frac{1}{2}} f(x)g\,dt$.

It's not an inner product on V.

Eg. Let $f(x) = \begin{cases} 0, & \text{if } x < \frac{1}{2} \\ x-\frac{1}{2}, & \text{if } x \geq \frac{1}{2} \end{cases}$



Then $\langle f,f\rangle = 0$ but $f \neq 0$. ▨

19 (a) $\|x \pm y\|^2 = \langle x\pm y, x\pm y\rangle = \langle x,x\rangle \pm \langle x,y\rangle \pm \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 \pm 2Re(\langle x,y\rangle)$. ▨

(b). ~~$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle + \langle y,y\rangle - (\langle x,y\rangle + \langle y,x\rangle)$.~~ ~~2Re⟨x,y⟩).~~ $\|x-y\| \geq |\|x\|-\|y\||$ ▨

By Triangle ineq., $\begin{cases} \|x-y\| + \|y\| \geq \|x-y+y\| = \|x\|. \Rightarrow \|x-y\| \geq \|x\|-\|y\|. \\ \|x-y\| + \|x\| = \|y-x\| + \|x\| \geq \|y-x+x\| = \|y\| \Rightarrow \|x-y\| \geq \|y\|-\|x\| \end{cases}$

No.

Date §6.1 :

20.(a). $\mathbb{F}=\mathbb{R}$. $\frac{1}{4}(\|x+y\|^2-\|x-y\|^2) = \frac{1}{4}\left(\begin{pmatrix}\cancel{\langle x,x\rangle}+\langle x,y\rangle \\ +\langle y,x\rangle+\cancel{\langle y,y\rangle}\end{pmatrix} - \begin{pmatrix}\cancel{\langle x,x\rangle}-\langle x,y\rangle \\ -\langle y,x\rangle+\cancel{\langle y,y\rangle}\end{pmatrix}\right)$.

$= \frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle) = \frac{1}{2}(\langle x,y\rangle+\langle y,x\rangle) \underset{\mathbb{F}=\mathbb{R}}{=} \langle x,y\rangle$ ■

(b). $\mathbb{F}=\mathbb{C}$. $\frac{1}{4}\sum_{k=1}^{4} i^k \|x+i^k y\|^2 = \frac{1}{4}(i\|x+iy\|^2-\|x-y\|^2-i\|x-iy\|^2+\|x+y\|^2)$

$= \frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle+i(\langle x,iy\rangle+\langle iy,x\rangle-(\langle x,-iy\rangle+\langle -iy,x\rangle)))$

$= \frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle+(\langle x,y\rangle-\langle y,x\rangle+\langle x,y\rangle-\langle y,x\rangle))$

$= \frac{1}{4}(4\langle x,y\rangle) = \langle x,y\rangle$ ■

21.(a). $A_1 := \frac{1}{2}(A+A^*)$. $A_2 := \frac{1}{2i}(A-A^*)$, $A\in M_{n\times n}(\mathbb{F})$.

- $A_1^* = \left(\frac{1}{2}(A+A^*)\right)^* = \frac{1}{2}(A^*+A^{**}) = \frac{1}{2}(A^*+A)$. ☆
- $A_2^* = \left(\frac{1}{2i}(A-A^*)\right)^* = \frac{1}{-2i}(A^*-A^{**}) = \frac{1}{2i}(A-A^*) = A_2$. ☆
- $A_1+iA_2 = \frac{1}{2}(A+A^*) + \cancel{i}\left(\frac{1}{2\cancel{i}}(A-A^*)\right) = A$ ☆

?• Usually, for each i,j, $A_{ij} = b_{ij}+ic_{ij}$. Define $(B)_{ij}=b_{ij}$ & $(C)_{ij}=c_{ij}$ $\forall i,j$. Then we define $Re(A)=B$, $Img(A)=C$, where B & $C \in M_{n\times n}(\mathbb{R})$. But here, $A=A_1+iA_2$, where A_1 & A_2 may still contain some complex entries. It could be another defn. of $Re(A)$ & $Img(A)$, but I don't think it's reasonable. ■

(b). [Uniqueness property]. Suppose $A = A_1+iA_2 = B_1+iB_2$, $A_i^*=A_i$ & $B_i^*=B_i$.

$\Rightarrow$ $A_1-B_1+i(A_2-B_2)=0$ —① $\Rightarrow$ $A_1^*-B_1^*-i(A_2^*-B_2^*)=0$

$\Rightarrow$ $A_1-B_1-i(A_2-B_2)=0$ —②

①+② $\Rightarrow$ $2(A_1-B_1)=0 \Rightarrow A_1=B_1$; ①−② $\Rightarrow 2i(A_2-B_2)=0 \Rightarrow A_2=B_2$ ■

22.(a). • proof for ~~defn. of~~ inner product.

- (a). $\langle x,y\rangle+\langle z,y\rangle = \sum a_i\bar{b}_i + \sum c_i\bar{b}_i = \sum(a_i+c_i)\bar{b}_i = \langle x+z,y\rangle$.
- (b). $\langle x,cy\rangle = \sum a_i\bar{c}\bar{b}_i = \bar{c}\sum a_i\bar{b}_i = \bar{c}\langle x,y\rangle$
- (c). $\overline{\langle x,y\rangle} = \overline{\sum a_i\bar{b}_i} = \sum \bar{a}_i b_i = \langle y,x\rangle$.
- (d). $\langle x,x\rangle = \sum a_i\bar{a}_i = \sum|a_i|^2 \geq 0$ & "=" holds $\overset{(\Leftrightarrow)}{\text{iff.}}$ $a_i=0\ \forall i$, iff $x=0$. ☆

• β is an orthonormal basis for V.

- Given $v_i\in\beta$, $\langle v_i,v_i\rangle = 1\cdot\bar{1} = 1$. $\Rightarrow$ unit vector.
- Given $v_i, v_j\in\beta$, $\langle v_i,v_j\rangle = \sum_{k=1}^{n} a_k\bar{b}_k$, where $a_k=\delta_{ki}$, $b_k=\delta_{kj}$.

$\Rightarrow \langle v_i,v_j\rangle = 0$ if $i\neq j$. ■

§6.1

:(b). Trivial.

23 (a). Note that $\langle x, y\rangle = y^* x$. Thus, $\langle x, Ay\rangle = \underset{1\times n\ \ n\times 1}{(Ay)^* x} = \underset{~~1\times n\ \ n\times n\ \ n\times 1~~}{y^* A^* x} = y^*(A^* x)$.

$= \langle A^* x, y\rangle$ ※ ▨

(b). $\langle x, Ay\rangle = \langle Bx, y\rangle\ \forall x,y \overset{(a)}{\Leftrightarrow} \langle A^* x, y\rangle = \langle Bx, y\rangle\ \forall x,y \Leftrightarrow \langle (A^*-B)x, y\rangle = 0\ \forall x,y$.

$\Leftrightarrow \langle \underbrace{(A^*-B)e_i}_{\text{column } i \text{ of } A^*-B,\text{ say } v_i}, y\rangle = 0\ \forall e_i \in \mathbb{F}^n, 1\le i\le n. \Leftrightarrow \langle v_i, y\rangle = 0\ \forall i = 1,\dots,n. \overset{\text{Thm 6.1}}{\Leftrightarrow} v_i = 0\ \forall i$ ▨

~~(Another proof)~~ Alternative: $\langle A^* x, y\rangle = \langle Bx, y\rangle \underset{\forall x,y}{\overset{\text{Thm 6.1}}{\Rightarrow}} A^* x = Bx\ \forall x. \Leftrightarrow A^* = B.$ ※

(C) By the statement, $Q = [I_V]_\beta^\alpha = [v_1\ v_2 \cdots v_n]$, where $v_i \in \beta\ \forall i$, β: orthonormal.

$\forall i,j\ (Q^*Q)_{ij} = v_i^* v_j = \langle v_j, v_i\rangle = \delta_{ij}. \Rightarrow Q^*Q = I_n. \Leftrightarrow Q^* = Q^{-1}$ ▨

(d). $T(x) = Ax$; $U(x) = A^*(x)$. Let β be an orthonormal basis for $V = \mathbb{F}^n$, say $\beta = \{v_1, \dots, v_n\}$. Note that $([U]_\beta)_{ij} = \langle A^* v_j, v_i\rangle$ &

$([T]_\beta)_{ji} = \langle Av_i, v_j\rangle \Rightarrow ([T]_\beta^*)_{ij} = \overline{([T]_\beta)_{ji}} = \overline{\langle Av_i, v_j\rangle} = \langle v_j, Av_i\rangle$

$\overset{(a)}{=} \langle A^* v_j, v_i\rangle = ([U]_\beta)_{ij}$ ▨

24.(a). $V = M_{m\times n}(\mathbb{F})$ $\|A\| := \max_{i,j} |A_{ij}|\ \forall A \in V$.

(1) $\|A\| = \max_{i,j} |A_{ij}| \ge 0$. & "=" holds $\Leftrightarrow \max |A_{ij}| = 0 \Leftrightarrow A_{ij} = 0\ \forall i,j$.

$\Leftrightarrow A = O$ ※

(2) $\|aA\| = \max_{i,j} |aA_{ij}| = |a| \max_{i,j} |A_{ij}| = |a|\cdot\|A\|$.

(3) $\|A+B\| = \max_{i,j} |A_{ij}+B_{ij}| \le \max_{i,j}(|A_{ij}|+|B_{ij}|) \le \max_{i,j}|A_{ij}| + \max_{i,j}|B_{ij}| = \|A\|+\|B\|$ ▨

(b). $V = C([0,1])$ $\|f\| := \max_{t\in[0,1]} |f(t)|\ \forall f \in V$.

(1). $\|f\| = \max_{t\in[0,1]} |f(t)| \ge 0$ & "=" holds $\Leftrightarrow \max_t |f(t)| = 0 \Leftrightarrow f(t) = 0\ \forall t \in [0,1]$

$\Leftrightarrow f \equiv 0$ ※

(2). $\|af\| = \max_t |af(t)| = |a| \max_t |f| = |a|\,\|f\|$ ※

(3). $\|f+g\| = \max_t |f+g| \le \underset{[illegible]}{\max_t(|f|+|g|)} \le \max_t |f| \underset{[illegible]}{+} \max_t |g| = \|f\| + \|g\|$. ▨

(C). $V = C([0,1])$ $\|f\| = \int_0^1 |f|\,dt\ \forall f \in V$.

(1). $\|f\| = \int_0^1 \underset{\ge 0}{\boxed{|f|}}\,dt \ge 0$ & "=" holds $\Leftrightarrow |f| = 0\ \forall t \Leftrightarrow f \equiv 0$.

(2). $\|af\| = \int_0^1 |af|\,dt = |a| \int_0^1 |f|\,dt = |a|\cdot\|f\|$.

(3). $\|f+g\| = \int_0^1 |f+g|\,dt \le \int_0^1 |f|+|g|\,dt = \|f\| + \|g\|$. ▨

(d). $V=\mathbb{R}^2$ $\|(a,b)\| = \max\{|a|,|b|\}$ $\forall (a,b)\in V$

(1). $\|(a,b)\| = \max\{\overset{\geq 0}{|a|},\overset{\geq 0}{|b|}\} \geq 0$ & "=" holds $\Leftrightarrow$ $|a|=|b|=0$ $\Leftrightarrow$ $(a,b)=(0,0)$ ✱

(2). $\|m(a,b)\| = \|(ma,mb)\| = \max\{|ma|,|mb|\} = \max\{|m|\cdot|a|,|m|\cdot|b|\}$
$= |m|\cdot\max\{|a|,|b|\} = |m|\cdot\|(a,b)\|$ ✱

(3) $\|(a,b)+(c,d)\| = \|(a+c,b+d)\| = \max\{|a+c|,|b+d|\} \leq \max\{|a|+|c|,|b|+|d|\}$
$\leq \max\{|a|,|b|\} + \max\{|c|,|d|\}$. ▨

25. $V:=\mathbb{R}^2$ $\|(a,b)\| := \max\{|a|,|b|\}$.

If $\langle\cdot,\cdot\rangle$ is an inner product on V s.t. $\|x\|^2 = \langle x,x\rangle$, then $\|(x_1,x_2)\|^2$
$= (\max\{|x_1|,|x_2|\})^2$ WLOG, say $x_1 \geq x_2$, so $\|x\|^2 = |x_1|^2 = x_1^2 = \langle x,x\rangle$.

Then by §6.1#20. we know that $\langle x,y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$

Choose $x=(2,0)$, $y=(1,3)$

We have $\langle x,y\rangle = \frac{1}{4}\|(3,3)\|^2 - \frac{1}{4}\|(1,-3)\|^2 = 0$

and $\langle 2x,y\rangle = \frac{1}{4}\|(5,3)\|^2 - \frac{1}{4}\|(3,-3)\|^2 = \frac{1}{4}(25-9) = 4$.

$\Rightarrow \langle 2x,y\rangle \neq \langle x,y\rangle + \langle x,y\rangle$, violating the defn (a)(b) of inner product ▨

26. (a). $d(x,y) = \|x-y\| \geq 0$ trivially. & "=" holds $\Leftrightarrow$ $x=y$. by defn of norm.

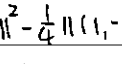
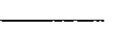
(b). $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$.

(c). $d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \underbrace{\|x-z\| + \|z-y\|}_{d(x,z)+d(z,y)}$ by defn (3) of norm ✱

(d). $d(x,x) = \|x-x\| = \|0\| = 0$ ✱

hard & boring (e). $d(x,y) = \|x-y\| \neq 0$ if $x\neq y$. by defn (1). of norm. ▨

17. (a). "$\langle x,2y\rangle = 2\langle x,y\rangle$": Note that $\langle x,y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2] = \langle y,x\rangle$ $\forall x,y\in V$, & $\overset{\|x\|^2}{\langle x,x\rangle} > 0$ if $x\neq 0$.

Now, by the parallelogram law,  $2\|x+y\|^2 + 2\|y\|^2 = \|x+2y\|^2 + \|x\|^2$ —① & 

$2\|x-y\|^2 + 2\|y\|^2 = \|x\|^2 + \|x-2y\|^2$ —② Then ①−② $\Rightarrow$ $\|x+2y\|^2 - \|x-2y\|^2 = 2\|x+y\|^2 - 2\|x-y\|^2$

$\Rightarrow \langle x,2y\rangle = \frac{1}{4}(\|x+2y\|^2 + \|x-2y\|^2) = \frac{1}{4}(2\|x+y\|^2 - 2\|x-y\|^2) = 2\langle x,y\rangle$. ▨

(b) "$\langle x+u,y\rangle = \langle x,y\rangle + \langle u,y\rangle$": $\langle x,y\rangle + \langle u,y\rangle = ?$
$= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + \|u+y\|^2 - \|u-y\|^2) = \frac{1}{4}(\frac{1}{2}\|x+u+2y\|^2 + \frac{1}{2}\|x-u\|^2 - \frac{1}{2}\|x+u-2y\|^2 - \frac{1}{2}\|x-u\|^2)$
$= \frac{1}{2}\langle x+u,2y\rangle \underset{(a)}{=} \langle x+u,y\rangle$ ▨.

(C). "$\langle nx,y\rangle = n\langle x,y\rangle$ $\forall n\in\mathbb{N}$.": $\langle nx,y\rangle = \langle x+\overset{u}{\boxed{(n-1)x}},y\rangle \overset{(b)}{=} \langle x,y\rangle + \langle (n-1)x,y\rangle$
$= \langle x,y\rangle + \langle x+(n-2)x,y\rangle \overset{(b)}{=} 2\langle x,y\rangle + \langle (n-2)x,y\rangle = \cdots\cdots = n\langle x,y\rangle$ ▨

§6.1

p107.

(d). "$m\langle \overset{z}{\boxed{\tfrac{1}{m}x}}, y\rangle = \langle x,y\rangle \ \forall m\in\mathbb{N}$" : $(\Leftrightarrow \langle \frac{1}{m}x, y\rangle = \frac{1}{m}\langle x,y\rangle)$.

By (c), $m\langle z,y\rangle = \langle mz, y\rangle = \langle m(\frac{1}{m}x), y\rangle = \langle x,y\rangle$ ▨

(e). "$r\langle x,y\rangle = \langle rx,y\rangle \ \forall r\in\mathbb{Q}$": Let $r = \frac{q}{p}\in\mathbb{N}$, $p,q\in\mathbb{Z}$, & $p\neq 0$.

- By (c),(d), $\langle rx,y\rangle = \langle \frac{q}{p}x, y\rangle = \langle q(\frac{1}{p}x), y\rangle \overset{(c)}{=} q\langle \frac{1}{p}x, y\rangle \overset{(d)}{=} \frac{q}{p}\langle x,y\rangle$ ☆ $\Big\}$ (*)
- For the case $r=0$, $\langle rx,y\rangle = \langle 0,y\rangle = \frac{1}{4}(\|y\|^2 - \|-y\|^2) = 0 = 0\cdot\langle x,y\rangle$ ☆
- For r being negative rational number,

$$\langle rx,y\rangle = \langle \underset{\vee 0}{(-r)}(-x), y\rangle \overset{(*)}{=} (-r)\langle -x,y\rangle = (-r)\frac{1}{4}(\|-x+y\|^2 - \|-x-y\|^2)$$

$$= (-r)\frac{1}{4}(\|x-y\|^2 - \|x+y\|^2) = \frac{r}{4}(\|x+y\|^2 - \|x-y\|^2) = r\langle x,y\rangle \text{ ☆ ▨}$$

(f) • $\|x\|^2 + 2\langle x,y\rangle + \|y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle x,y\rangle + \langle y,y\rangle \overset{(b)}{=} \langle x,x\rangle + \langle x,y\rangle + \langle x+y, y\rangle$

$= \langle x,x\rangle + \langle y,x\rangle + \langle x+y,y\rangle$ since $\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2) = \langle y,x\rangle$

$\overset{(b)}{=} \langle x+y, x\rangle + \langle x+y, y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle \overset{(b)}{=} \langle x+y, x+y\rangle = \|x+y\|^2.$

$\leq (\|x\| + \|y\|)^2$ (by defn (3) of norm). $= \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$

$\Rightarrow \langle x,y\rangle \leq \|x\|\cdot\|y\|.$

- $\|x\|^2 - 2\langle x,y\rangle + \|y\|^2 = \langle x,x\rangle - \langle x,y\rangle - \langle x,y\rangle + \langle y,y\rangle \overset{\text{similarly,}}{=} \langle x, x-y\rangle - \langle x-y, y\rangle = \langle x-y, x\rangle$

$-\langle x-y, y\rangle = \langle x-y, x-y\rangle = \|x-y\|^2 \underset{\text{defn of norm.}}{\leq} (\|x\| + \|-y\|)^2 = \|x\| + 2\|x\|\|y\| + \|y\|^2$

$\Rightarrow -\langle x,y\rangle \leq \|x\|\cdot\|y\|$

Combine these 2 results together, we conclude that $|\langle x,y\rangle| \leq \|x\|\cdot\|y\|$. ▨

(g). • $(c-r)\langle x,y\rangle - \langle (c-r)x, y\rangle = (c\langle x,y\rangle - r\langle x,y\rangle) - \langle cx - rx, y\rangle$

$\overset{(b)}{=} (c\langle x,y\rangle - r\langle x,y\rangle) - (\langle cx,y\rangle - \langle rx,y\rangle) \overset{r\in\mathbb{Q}\,(e)}{=} c\langle x,y\rangle - \langle cx,y\rangle$

Thus, the first equality holds.

- Note that $-\|x\|\|y\| \leq \langle x,y\rangle \leq \|x\|\|y\|$ by (f).

So, $(c-r)\langle x,y\rangle \leq |c-r|\|x\|\|y\|$ Let $z = (c-r)x$.

$\langle z,y\rangle \geq -\|z\|\|y\| = -\|(c-r)x\|\|y\| = -|c-r|\|x\|\|y\|.$

Then, $|(c-r)\langle x,y\rangle - \langle (c-r)x, y\rangle| \leq 2|c-r|\|x\|\|y\|$. ▨

(h). Given $c\in\mathbb{R}$, $\exists r\in\mathbb{Q}$ s.t. $|c-r|<\varepsilon$. for a sufficiently small, fixed ε.

Then by (g), $|c\langle x,y\rangle - \langle cx,y\rangle| \leq 2|c-r|\|x\|\|y\| < 2\varepsilon\|x\|\|y\|.$

Take $\varepsilon' = 2\varepsilon\|x\|\|y\|$. $\Rightarrow |c\langle x,y\rangle - \langle cx,y\rangle| < \varepsilon'$ can be arbitrary small

$\Rightarrow c\langle x,y\rangle = \langle cx,y\rangle$. ▨

No.

Date §6.1 :

28. • Check the defn. of inner product:

(a). $[x+z,y] = Re(\langle x+z,y\rangle) = Re(\langle x,y\rangle + \langle z,y\rangle) = Re(\langle x,y\rangle) + Re(\langle z,y\rangle) = [x,y] + [z,y]$ ※

(b). $\forall c \in \mathbb{R}$, $[cx,y] = Re(\langle cx,y\rangle) = c\,Re(\langle x,y\rangle) = c[x,y]$ ※

(c). $\overline{[x,y]} = \overline{Re(\langle x,y\rangle)} = \overset{=[x,y].}{\widetilde{Re(\overline{\langle x,y\rangle})}} = Re(\overline{\langle y,x\rangle}) = Re(\langle y,x\rangle) .= [y,x]$.

(d). $[x,x] = Re(\langle x,x\rangle) = Re(\|x\|^2) = \|x\|^2 > 0$ iff $x \neq 0$.

• "$[x,ix] = 0\ \forall x \in V$": $[x,ix] = Re(\langle x,ix\rangle) = Re(-i\langle x,x\rangle) = Re(\underbrace{-\|x\|^2}_{\in\mathbb{R}.}\textcircled{i}) = 0$ ▨

29. (与§6.1 #28相關)

•• $\langle x+z,y\rangle = [x+z,y] + i[x+z,iy] = ([x,y]+[z,y]) + i([x,iy]+[z,iy]) = \langle x,y\rangle + \langle z,y\rangle$ ※

•• $\langle cx,y\rangle = [cx,y] + i[cx,iy] = c[x,y] + ci[x,iy] = c\langle x,y\rangle$ ※

•• $\overline{\langle x,y\rangle} = \overline{[x,y]} + \overline{i[x,iy]} = [x,y] - i[x,iy] \underset{\uparrow}{=} [y,x] - i[iy,x]$. —①

[,] is a real inner product.

Claim: $[x,y] = [ix,iy]$.

pf: $0 = [x+iy; i(x+iy)] = [x+iy, ix-y] = \cancel{[x,ix]}^{\to 0} + [iy,ix] + [x,-y] + \cancel{[iy,-y]}^{\to 0}$

$\Rightarrow [ix,iy] = -[x,-y] = +[x,y]$ ※

Then ① $= [y,x] - i[i(iy), i(x)] = [y,x] + i[y,ix] = \langle y,x\rangle$ ※

•• $\langle x,x\rangle = [x,x] + i\cancel{[x,ix]}^{\to 0} = [x,x] > 0$ iff $x \neq 0$. by defn of [·,·]. ※ ▨

30. • $\|\cdot\| : V \text{ over } \mathbb{C} \to \mathbb{R}$.

Then $\|\cdot\|\big|_{\overset{\equiv V_R.}{V \text{ over } \mathbb{R}}}$ would also satisfy the parallelogram law..

By §6.1 #27, $[x,y] =: \frac{1}{4}[(\|x+y\||_{V_R})^2 - (\|x-y\||_{V_R})^2]$ is a real inner product. (i.e. $[,] : V \text{ over } \mathbb{R} \to \mathbb{R}$).

.. $\forall x \in V$, $[x,ix] = \frac{1}{4}[\|x+ix\|^2 - \|x-ix\|^2] = \frac{1}{4}[\|x+ix\|^2 - \|(-i)(x+ix)\|^2]$

$= \frac{1}{4}[\|x+ix\|^2 - \underbrace{|-i|^2}_{1}\|x+ix\|^2] = 0$.

Then apply §6.1 #29, $\langle x,y\rangle =: [x,y] + i[x,iy]$ is a complex inner product on V over $\mathbb{C}$. ($\langle , \rangle : V \text{ over } \mathbb{C} \to \mathbb{C}$). ▨

§6.2 p.109.

1.(a). Fi, from a 'linearly independent' set of vectors. ※

(b). T, just use Gramm-Schmidt process to any basis for V

(c). T, proof: given a set S, if ① $x, y \in S^{\perp}$, then $\langle x+y, v\rangle = \langle x,v\rangle + \langle y,v\rangle = 0\ \forall v \in S$
$\Rightarrow x+y \in S^{\perp}$; ② $cy \in S^{\perp}$ trivially.; ③ $0 \in S^{\perp}$ ※

(d) Fi, need "orthonormal".

(e). T.

(f). Fi. an orthogonal set might contain a zero vector.

(g). T.

2. (a). $V = \mathbb{R}^3$. $S = \{(1,0,1), (0,1,1), (1,3,3)\}$; $x = (1,1,2)$.

- $v_1 = (1,0,1)$. $v_2 = (0,1,1) - \frac{\langle (0,1,1), v_1\rangle}{(\sqrt{2})^2} \times v_1 = (0,1,1) - \frac{1}{2} v_1 = (\frac{-1}{2}, 1, \frac{1}{2})$

  $v_3 = (1,3,3) - (\frac{4}{2} v_1 + \frac{4}{3/2} v_2) = (1,3,3) - (2,0,2) - (\frac{-4}{3}, \frac{8}{3}, \frac{4}{3}) = (\frac{1}{3}, \frac{1}{3}, \frac{-1}{3})$

  ∴ orthogonal basis $= \{(1,0,1), (\frac{-1}{2}, 1, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}, \frac{-1}{3})\}$.
- orthonormal basis $= \{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{6}}(-1,2,1), \frac{1}{\sqrt{3}}(1,1,-1)\} =: \beta$.
- Fourier coefficients of x w.r.t. β: $\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{6}}, 0$.
- Check Thm 6.5: $\frac{3}{\sqrt{2}}(\frac{1}{\sqrt{2}}(1,0,1)) + \frac{1}{\sqrt{6}}(\frac{3}{\sqrt{6}}(-1,2,1)) + 0 \cdot v_3 = \frac{3}{2}(1,0,1) + \frac{1}{2}(-1,2,1) = (1,1,2) = x$ ※

(b). $V = \mathbb{R}^3$, $S = \{(1,1,1), (0,1,1), (0,0,1)\}$; $x = (1,0,1)$

- $v_1 = (1,1,1)$, $v_2 = (0,1,1) - \frac{2}{3} v_1 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, $v_3 = (0,0,1) - \frac{1}{3} v_1 - \frac{\frac{1}{3}}{\frac{6}{9}} v_2 = (0,0,1) - (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
  $- \frac{1}{2}(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}) = (0, \frac{-1}{2}, \frac{1}{2})$. ∴ orthogonal basis $= \{(1,1,1), (\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}), (0, \frac{-1}{2}, \frac{1}{2})\}$.
- orthonormal basis $= \{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(-2,1,1), \frac{1}{\sqrt{2}}(0,-1,1)\} =: \beta$.
- Fourier coefficients: $\frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{2}}$
- Check Thm 6.5: $\frac{2}{3}(1,1,1) + \frac{-1}{6}(-2,1,1) + \frac{1}{2}(0,-1,1) = (1,0,1) = x$ ※

(c). $V = P_2(\mathbb{R})$. $\langle f, g\rangle := \int_0^1 fg\,dt$. $S = \{1, x, x^2\}$, $h(x) = 1 + x$.

- $v_1 = 1$, $v_2 = x - \frac{1}{2} \times v_1 = x - \frac{1}{2}$, $v_3 = x^2 - \frac{1}{3} v_1 - \frac{\frac{1}{12}}{\frac{1}{12}} v_2 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$

  ∴ orthogonal basis $= \{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$.
- Orthonormal basis $=: \beta = \{1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})\}$
- Fourier coeffi.: ① $\int_0^1 1 + x\,dx = \frac{3}{2}$, ② $\sqrt{12}\int_0^1 (1+x)(x - \frac{1}{2})\,dx = \frac{\sqrt{12}}{12}$; ③ $\sqrt{180}\int_0^1 (1+x)(x^2 - x + \frac{1}{6})\,dx = 0$
- Check Thm 6.5: $\frac{3}{2} \times 1 + \frac{\sqrt{12}}{12} \times (\sqrt{12}(x - \frac{1}{2})) + 0 = \frac{3}{2} + x - \frac{1}{2} = x + 1$ ※

No.

Date : §6:2.

(d). $V=\text{Span}(S)$, where $S=\{(1,i,0),(1-i,2,4i)\}$; $x=(3+i,4i,-4)$.

inner product $\langle x,y\rangle=\sum_{i=1}^{3}x_i\overline{y_i}$.
$=y^*x$.

- $v_1=(1,i,0)$. $v_2=(1-i,2,4i)-\frac{(1-3i)}{2}(1,i,0)=(\frac{1+i}{2},\frac{1-i}{2},4i)$.

  orthogonal basis $=\{(1,i,0),(\frac{1+i}{2},\frac{1-i}{2},4i)\}$.
- Orthonormal basis $:=\beta=\{\frac{1}{\sqrt2}(1,i,0),\frac{1}{\sqrt{68}}(1+i,1-i,8i)\}$.
- Fourier coeffi.: $\frac{1}{\sqrt2}(7+i)$, $\frac{1}{\sqrt{68}}(34i)$.
- Check Thm6.5: $\frac{1}{\sqrt2}(7+i)\times\frac{1}{\sqrt2}(1,i,0)+\frac{1}{\sqrt{68}}(34i)\times\frac{1}{\sqrt{68}}(1+i,1-i,8i)$

  $=\frac12(7+i,-1+7i,0)+\frac12(-1+i,1+i,-8)=(3+i,4i,-4)$. #

(e). $V=\mathbb{R}^4$, $S=\{(2,-1,-2,4),(-2,1,-5,5),(-1,3,7,11)\}$; $x=(-11,8,-4,18)$.

- $v_1=\begin{pmatrix}2\\-1\\-2\\4\end{pmatrix}^t$, $v_2=\begin{pmatrix}-2\\1\\-5\\5\end{pmatrix}^t-\frac{-25}{25}\begin{pmatrix}2\\-1\\-2\\4\end{pmatrix}^t=(-4,2,-3,1)$, $v_3=\begin{pmatrix}-1\\3\\7\\11\end{pmatrix}^t-\frac{25}{25}\begin{pmatrix}2\\-1\\-2\\4\end{pmatrix}^t-0v_2=\begin{pmatrix}-3\\4\\9\\7\end{pmatrix}^t$

  orthogonal basis $=\{(2,-1,-2,4),(-4,2,-3,1),(-3,4,9,7)\}$.
- Orthonormal basis $=\{\frac15(2,-1,-2,4),\frac{1}{\sqrt{30}}(-4,2,-3,1),\frac{1}{\sqrt{155}}(-3,4,9,7)\}$.
- Fourier coeffi.: $\frac{50}{5}=10$, $3\sqrt{30}$, $\sqrt{155}$
- Check Thm6.5: $10\times(\frac15(2,-1,-2,4))+\frac{3\sqrt{30}}{\sqrt{30}}(-4,2,-3,1)+\frac{\sqrt{155}}{\sqrt{155}}(-3,4,9,7)$

  $=(4,-2,-4,8)+(-12,6,-9,3)+(-3,4,9,7)=(-11,8,-4,18)$ #

(f) $V=\mathbb{R}^4$, $S=\{(1,-2,-1,3),(3,6,3,-1),(1,4,2,8)\}$; $x=(-1,2,1,1)$

- $v_1=(1,-2,-1,3)$, $v_2=(3,6,3,-1)-\frac{(-15)}{15}(1,-2,-1,3)=(4,4,2,2)$

  $v_3=(1,4,2,8)-\frac{15}{15}(1,-2,-1,3)-\frac{40}{40}(4,4,2,2)=(-4,2,1,3)$

  Orthogonal basis $=\{(1,-2,-1,3),(4,4,2,2),(-4,2,1,3)\}$.
- Orthonormal basis $=\{\frac{1}{\sqrt{15}}(1,-2,-1,3),\frac{1}{\sqrt{10}}(2,2,1,1),\frac{1}{\sqrt{30}}(-4,2,1,3)\}$
- Fourier coeffi.: $\frac{-3}{\sqrt{15}}$, $\frac{4}{\sqrt{10}}$, $\frac{12}{\sqrt{30}}$
- Check Thm6.5: $\frac{-3}{\sqrt{15}}(\frac{1}{\sqrt{15}}(1,-2,-1,3))+\frac{4}{\sqrt{10}}(\frac{1}{\sqrt{10}}(2,2,1,1))+\frac{12}{\sqrt{30}}(\frac{1}{\sqrt{30}}(-4,2,1,3))$

  $=(\frac{-1}{5},\frac{+2}{5},\frac{+1}{5},\frac{-3}{5})+(\frac45,\frac45,\frac25,\frac25)+(\frac{-8}{5},\frac45,\frac25,\frac65)=(-1,2,1,1)$. #

(g) $V=M_{2\times2}(\mathbb{R})$, $S=\{\begin{pmatrix}3&5\\-1&1\end{pmatrix},\begin{pmatrix}-1&9\\5&-1\end{pmatrix},\begin{pmatrix}7&-17\\2&-6\end{pmatrix}\}$; $A=\begin{pmatrix}-1&27\\-4&8\end{pmatrix}$.

- $A_1=\begin{pmatrix}3&5\\-1&1\end{pmatrix}$, $A_2=\begin{pmatrix}-1&9\\5&-1\end{pmatrix}-tr(\begin{pmatrix}-1&5\\9&-1\end{pmatrix}\begin{pmatrix}3&5\\-1&1\end{pmatrix})\div\frac{1}{36}A=\begin{pmatrix}-1&9\\5&-1\end{pmatrix}-A_1=\begin{pmatrix}-4&4\\6&-2\end{pmatrix}$

  $A_3=\begin{pmatrix}7&-17\\2&-6\end{pmatrix}-\frac{\boxed{tr(\begin{pmatrix}7&2\\-17&-6\end{pmatrix}A_1)}^{=-72}}{36}A_1-\frac{\boxed{tr(\begin{pmatrix}7&2\\-17&-6\end{pmatrix}A_2)}^{=-72}}{72}A_2=\begin{pmatrix}7&-17\\2&-6\end{pmatrix}+2A_1+A_2=\begin{pmatrix}9&-3\\6&-6\end{pmatrix}$

  orthogonal basis $=\{\begin{pmatrix}3&5\\-1&1\end{pmatrix},\begin{pmatrix}-4&4\\6&-2\end{pmatrix},\begin{pmatrix}9&-3\\6&-6\end{pmatrix}\}$
- Orthonormal basis $=\{\frac16\begin{pmatrix}3&5\\-1&1\end{pmatrix},\frac{1}{\sqrt{18}}\begin{pmatrix}-2&2\\3&-1\end{pmatrix},\frac{1}{\sqrt{18}}\begin{pmatrix}3&-1\\2&-2\end{pmatrix}\}$
- Fourier coeffi.: 24, $6\sqrt2$, $-9\sqrt2$.
- Check Thm6.5.: $24\times\frac16\begin{pmatrix}3&5\\-1&1\end{pmatrix}+\frac{6\sqrt2}{\sqrt{18}}\begin{pmatrix}-2&2\\3&-1\end{pmatrix}+\frac{-9\sqrt2}{\sqrt{18}}\begin{pmatrix}3&-1\\2&-2\end{pmatrix}=\begin{pmatrix}12&20\\-4&4\end{pmatrix}+\begin{pmatrix}-4&4\\6&-2\end{pmatrix}+\begin{pmatrix}-9&+3\\-6&+6\end{pmatrix}=\begin{pmatrix}-1&27\\-4&8\end{pmatrix}$ #

§6.2

(h). $V = M_{2\times2}(\mathbb{R})$, $S = \left\{ \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} \right\}$; $A = \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$

- $A_1 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix} - \frac{39}{13}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} - \frac{(-26)}{13}A_1 - \frac{0}{49}A_2 = \begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix}$

Orthogonal basis : $\left\{ \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix} \right\}$

- Orthonormal basis : $\left\{ \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}, \frac{1}{\sqrt{373}}\begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix} \right\}$ $\begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$
- Fourier coeffi. : $5\sqrt{13}$, -14, $\sqrt{373}$.   ※ $\parallel$ -
- Check Thm 6.5 : $(5\sqrt{13})\frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} + \frac{(-14)}{7}\begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} + \frac{(\sqrt{373})}{\sqrt{373}}\begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 10 & 5 \end{pmatrix} + \begin{pmatrix} -10 & 4 \\ 8 & -4 \end{pmatrix} + \begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix}$

(i). $V = \text{span}(S)$ with $\langle f, g \rangle := \int_0^\pi f(t)g(t)\,dt$. $S = \{\sin t, \cos t, 1, t\}$; $h(t) = 2t+1$.

- $v_1 = \sin t$, $v_2 = \cos t - \int_0^\pi \sin t \underbrace{\cos t\, dt}_{d\sin t} = \cos t - 0 = \cos t$.

$v_3 = 1 - \frac{\int_0^\pi 1\cdot\sin t\,dt}{\int_0^\pi \sin^2 t\,dt} \times \sin t - \frac{\int_0^\pi 1\cdot\cos t\,dt}{\int_0^\pi \cos^2 t\,dt}\cos t = 1 - \frac{2}{\frac{\pi}{2}}\sin t - \frac{0}{\frac{\pi}{2}}\cos t = 1 - \frac{4}{\pi}\sin t$.

$v_4 = t - \frac{\int_0^\pi t\sin t\,dt}{\pi/2}\sin t - \frac{\int_0^\pi t\cos t\,dt}{\pi/2}\cos t - \frac{\int_0^\pi t(1-\frac{4}{\pi}\sin t)\,dt}{\int_0^\pi (1-\frac{4}{\pi}\sin t)^2\,dt}(1 - \frac{4}{\pi}\sin t)$.

$= t - \frac{\pi}{\pi/2}\sin t + \frac{2}{\pi/2}\cos t - \frac{\frac{\pi^2}{2} - 4}{\pi - \frac{8}{\pi}}(1 - \frac{4}{\pi}\sin t)$.

$= t - 2\sin t + \frac{4}{\pi}\cos t - \frac{\pi\cancel{(\pi^2-8)}}{2\cancel{(\pi^2-8)}}(1 - \frac{4}{\pi}\sin t) = t + \frac{4}{\pi}\cos t - \frac{\pi}{2}$

Orthogonal basis = $\{\sin t, \cos t, 1 - \frac{4}{\pi}\sin t, t + \frac{4}{\pi}\cos t - \frac{\pi}{2}\}$

- Orthonormal basis = $\left\{\sqrt{\frac{2}{\pi}}\sin t, \sqrt{\frac{2}{\pi}}\cos t, \sqrt{\frac{\pi}{\pi^2-8}}(1 - \frac{4}{\pi}\sin t), \sqrt{\frac{12\pi}{\pi^4-96}}(t + \frac{4}{\pi}\cos t - \frac{\pi}{2})\right\}$
- Fourier coeffi. : $\sqrt{\frac{2}{\pi}}(2\pi+2)$, $-4\sqrt{\frac{2}{\pi}}$, $\sqrt{\frac{\pi^2-8}{\pi}}(1+\pi)$, $\sqrt{\frac{\pi^4-96}{3\pi}}$

∴ Neglect. ※

(j) ~ (m) 略.

3. $\beta = \{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\}$ $x := (3,4)$

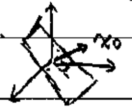
Fourier coefficient : $\frac{7}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$. ※

4. $S = \{(1,0,i), (1,2,1)\} \subset \mathbb{C}^3$. $S^\perp = \{x \in \mathbb{C}^3 \mid \langle x, y \rangle = 0\ \forall y \in S\}$ 令 $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{C}^3$.

Then $\begin{cases} \sum x_i \overline{y_{1i}} = 0 \\ \sum x_i \overline{y_{2i}} = 0 \end{cases} \Rightarrow \begin{cases} x_1 - i x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = i x_3 \\ x_2 = -(\frac{1+i}{2})x_3 \end{cases} \Rightarrow x = \begin{pmatrix} -b+ai \\ -(\frac{1+i}{2})(a+bi) \\ a+bi \end{pmatrix}$, $a, b \in \mathbb{R}$.

$\Rightarrow S^\perp = \left\{ \begin{pmatrix} i \\ -(\frac{1+i}{2}) \\ 1 \end{pmatrix} a + \begin{pmatrix} -1 \\ \frac{i-1}{2} \\ i \end{pmatrix} b \;\middle|\; a, b \in \mathbb{R} \right\}$ $\left(= \text{span}(\{(i, -\frac{(1+i)}{2}, 1)\})\right)$ ※

5. $S_0 = \{x_0\} (\neq 0) \in \mathbb{R}^3$.



- Intepretation : $S_0^\perp$ is a plane whose normal vector is $\pm x_0$.

$S =: \{x_1, x_2\} \subseteq \mathbb{R}^3$ is a L.I. set.

- Intepretation : $S^\perp = \{x \in \mathbb{R}^3 \mid \langle x, x_i \rangle = 0\ \forall i\}$ = a line L, which is perpendicular to the plane span(S).

No.

Date : §6.2

6. $x \notin W$. (Thm 6.6). $\exists! u \in W$ & $\exists! y \in W^\perp$ s.t. $x = u + y$. $x \notin W \Rightarrow \underbrace{y = x - \overset{\in W}{u} \neq 0}$ — ①

Also, $\langle x, \overset{\in W^\perp}{\textcircled{y}} \rangle = \langle y, y \rangle = \|y\|^2 > 0$ by ① ■

7. "$z \in W^\perp \Leftrightarrow \langle z, v \rangle = 0$ for every $v \in \beta$," (β is a basis for W).

($\Rightarrow$). $z \in W^\perp \overset{\text{defn.}}{\Rightarrow} \langle z, y \rangle = 0 \ \forall y \in W \Rightarrow \langle z, v \rangle = 0 \ \forall v \in \beta \subset W$. *

($\Leftarrow$). Let $\beta = \{v_1, \ldots, v_n\}$ since $\dim(W) \leq \dim(V) < \infty$. Given $y \in W$, $\exists$ scalars a_i, $1 \leq i \leq n$ s.t. $y = \sum a_i v_i$. Then $\langle z, y \rangle = \langle z, \sum a_i v_i \rangle = \sum \bar{a_i} \langle z, \underset{\in \beta}{\textcircled{v_i}} \rangle = \sum \bar{a_i} \cdot 0 = 0 \overset{\text{defn.}}{\Rightarrow} z \in W^\perp$ ■

8. Following the hint. For the case $n=2$, $\{w_1, w_2\}$ is an orthogonal set of nonzero vectors. Apply the Gram-Schmidt process, $v_1 \triangleq w_1$, $v_2 \triangleq w_2 - \frac{\overset{=0}{\langle w_2, v_1 \rangle}}{\|v_1\|^2} = w_2$. So the statement holds for $n=2$. Assume that the statement holds $\forall n \leq k-1$, $n \in \mathbb{N}$. Now, for $n=k$, $\{w_1, \ldots, w_k\}$ is an orthogonal set of nonzero vectors. Let $S' = \{w_1, \ldots, w_{k-1}\}$. Then by induction hypothesis, $v_i = w_i \ \forall i = 1, \ldots, k-1$, where v_i is derived from Gram-Schmidt process. Then $v_k := w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$

$\Rightarrow v_k \doteq w_k$ since $v_j = w_j \ \forall j = 1, \ldots, k-1$ & $\langle w_k, w_j \rangle = 0 \ \forall j = 1, \ldots, k-1$ ■

9 $W = \text{span}(\{(i, 0, 1)\})$ in $\mathbb{C}^3$. $\Rightarrow$ W has $\dim = 1$.

- Orthonormal basis for W is $\frac{1}{\sqrt{2}}(i, 0, 1)$.
- $W^\perp = \{x \mid \langle x, \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \rangle = 0\}$ Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{C}^3$. $\Rightarrow -ix_1 + 0x_2 + x_3 = 0 \Rightarrow x_3 = ix_1$

$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ ix_1 \end{pmatrix}$, $x_1, x_2 \in \mathbb{C}$. $\Rightarrow$ The basis for $W^\perp$ is $\beta' = \{(1, 0, i), (0, 1, 0)\}$

Apply Gram-Schmidt process. $v_1 = (1, 0, i)$, $v_2 := (0, 1, 0) - \frac{\langle (0,1,0), (1,0,i) \rangle}{\|(1,0,i)\|^2}(1, 0, i)$.

$\Rightarrow v_2 = (0, 1, 0) - 0 = (0, 1, 0)$. $\therefore$ β' is the desired orthonormal basis ■

10. • Thm 6.7 $\Rightarrow$ $V = W \oplus W^\perp$. Let $\beta_1 = \{v_1, \ldots, v_k\}$ be an ordered basis for W, and $\beta_2 = \{v_{k+1}, \ldots, v_n\}$ for $W^\perp$. & $\beta = \beta_1 \cup \beta_2$ for V.

By Thm 2.6 (Ch2) (p.72), $\exists!$ linear operator $T: V \to V$ s.t. $\begin{cases} w_i \mapsto w_i, & \text{if } i = 1 \ldots k. \\ w_j \mapsto 0, & \text{if } j = k+1 \ldots n \end{cases}$

Now, $\forall x \in V$, $\exists! x_1 \in W$ & $x_2 \in W^\perp$ s.t. $x = x_1 + x_2$. Then $T(x) = \underbrace{T(x_1)}_{x_1} + \overset{0}{\cancel{T(x_2)}} = x_1$

$\Rightarrow$ $N(T) = W^\perp$. *

- $\overset{\in V}{x} = \overset{\in W}{x_1} + \overset{\in W^\perp}{x_2}$. $\Rightarrow \|T(x)\|^2 = \|T(x_1) + \overset{0}{\cancel{T(x_2)}}\|^2 = \|T(x_1)\|^2 = \|x_1\|^2 \leq \|x_1\|^2 + \|x_2\|^2$

$\Rightarrow$ $\|T(x)\| \leq \|x\| \ \forall x \in V$ ■

11. $A \in M_{n \times n}(\mathbb{C})$. "$AA^* = I \Leftrightarrow$ the rows of A form an orthonormal basis for $\mathbb{C}^n$".

($\Rightarrow$). Let $a_j =:$ row j of A. $AA^* = I \Rightarrow \sum_{k=1}^{n} A_{jk}(A^*)_{ki} = (I)_{ji} = \delta_{ji}$ 續下頁

$j \rightarrow (\circ\circ\circ\circ\circ)\ (\ \vdots\ )A^*$

§6.2

$(x)_k$ denotes the kth component of x.

$\Rightarrow \sum_{k}^{n} A_{jk}\overline{(A)_{ik}} = \delta_{ji} \Rightarrow \sum_{k}^{n} (a_j)_k(\overline{a_i})_k = \delta_{ji} \Rightarrow \langle a_j, a_i \rangle = \delta_{ji}$.

$\Rightarrow \beta := \{a_1, \dots, a_n\}$ forms an orthogonal set $\subset \mathbb{C}^n$.

Next, assume $\sum_{i=1}^{n} p_i a_i = 0$ for some scalars $p_i \in \mathbb{C}$, $i=1, \dots, n$.

Then $P_j = \langle \sum_{i=1}^{n} P_i a_i, a_j \rangle = \langle 0, a_j \rangle = 0$. (see Coro 2, p.342). ✲

(⇐). Let a_j be the jth row of A. So $\{a_1, \dots, a_n\}$ is a basis for $\mathbb{C}^n$.

Then given $i \in \mathbb{N}$ $(1 \le i \le n)$, $\langle a_i, a_j \rangle = \delta_{ij} \Rightarrow AA^* = I$ ▨

12. $A \in M_{m \times n}(\mathbb{F})$. $(R(L_{A^*}))^{\perp} = (\underset{\text{Thm}}{A^*}(\mathbb{F}^m))^{\perp} = \{x \in \mathbb{F}^n \mid \langle x, y \rangle = 0\ \forall y \in R(L_{A^*})\}$ — ①

(Note that if $y \in R(L_{A^*})$, $\exists\ w \in \mathbb{F}^m$ s.t. $y = A^* w$.)

So, ① $= \{x \in \mathbb{F}^n \mid \underset{\langle Ax, w \rangle}{\underset{\parallel}{\langle x, A^* w \rangle}} = 0\ \forall w \in \mathbb{F}^m\}$

By Thm 6.1(e), $\langle Ax, w \rangle = 0\ \forall w \in \mathbb{F}^m \Rightarrow Ax = 0. \Rightarrow x \in N(L_A)$.

$\Rightarrow (R(L_{A^*}))^{\perp} = N(L_A)$ ▨

13. (a). $S_0 \subseteq S$. If $x \in S^{\perp}$, $\langle x, y \rangle = 0\ \forall y \in S \Rightarrow \langle x, y \rangle = 0\ \forall y \in S_0 \Rightarrow x \in S_0^{\perp}$ ✲

(b). $S \subseteq (S^{\perp})^{\perp}$. If $x \in \text{span}(S)$, $x = \sum_{i=1}^{n} a_i v_i$ for some $a_i, i=1,\dots,n \in \mathbb{F}$; $\{v_i\}_{i=1}^{n} \in S$, $n \in \mathbb{N}$.

It suffices to show that $\langle x, z \rangle = 0\ \forall z \in S^{\perp}$

pf. $\langle x, z \rangle = \langle \sum a_i v_i, z \rangle$, $v_i \in S\ \forall i = 1, \dots, n. = \sum a_i \langle v_i, z \rangle = 0$ ▨

(c). Prove that $W = (W^{\perp})^{\perp}$.

pf. "⊃". If $x \in (W^{\perp})^{\perp}$, $\langle x, z \rangle = 0\ \forall z \in W^{\perp}$. By §6.2 #6, p.354, $x \in W$ ✲

"⊂". Let $x \in W$. It suffices to show that $\langle x, z \rangle = 0\ \forall z \in W^{\perp}$.

But by def. of $W^{\perp}$, $\langle z, y \rangle = 0\ \forall y \in W$ if $z \in W^{\perp}$.

$\therefore \langle x, z \rangle = 0\ \forall z \in W^{\perp}$ since $x \in W$. ▨

(d). By Thm 6.7 ▨

<Another/Alternative proof>: Thm 6.6 $\Rightarrow \forall x \in V$, $\exists!\ u^{\in W}$ & $y^{\in W^{\perp}}$ s.t. $x = u + y$.

$\Rightarrow V \subseteq W + W^{\perp}$. Next, given $u \in W$ & $v \in W^{\perp} \Rightarrow u, v \in V \Rightarrow u + v \in V \Rightarrow W + W^{\perp} \subseteq V$.

So $V = W + W^{\perp}$. Third, if $x \in W \cap W^{\perp}$, $\langle x^{\in W}, x^{\in W^{\perp}} \rangle = 0 \Rightarrow x = 0$. So $V = W \oplus W^{\perp}$ ▨

14. • "$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$": If $x \in (W_1 + W_2)^{\perp}$, $\langle x, w_1 + w_2 \rangle = 0\ \forall w_i \in W_i$, $i = 1, 2$. Then

$\langle x, w_1 + 0 \rangle = 0\ \forall w_1$ & $\langle x, w_2 \rangle = 0\ \forall w_2 \in W_2$. $\Rightarrow x \in W_1^{\perp} \cap W_2^{\perp}$ ✲ For the converse, if

$x \in W_1^{\perp} \cap W_2^{\perp}$, then $\langle x, w_1 \rangle = 0\ \forall w_1 \in W_1$, & $\langle x, w_2 \rangle = 0\ \forall w_2 \in W_2$ $\Rightarrow \forall w_1 \in W_1, w_2 \in W_2$, $\langle x, w_1 + w_2 \rangle = \langle x, w_1 \rangle + \langle x, w_2 \rangle = 0$

$\Rightarrow x \in (W_1 + W_2)^{\perp}$ ✲ (續下頁。).

• "$(W_1\cap W_2)^\perp = W_1^\perp + W_2^\perp$": Following to hint, by §6.2 #13(C), we have:

$$(W_1\cap W_2)^\perp = ((W_1^\perp)^\perp \cap (W_2^\perp)^\perp)^\perp \underset{\substack{\uparrow\\ \text{first equation}}}{=} ((W_1^\perp + W_2^\perp)^\perp)^\perp = W_1^\perp + W_2^\perp$$ ▨

15.(a). By Thm6.3, $x=\sum_i \frac{\langle x,v_i\rangle}{\boxed{\|v_i\|^2}_{=1}} v_i = \sum_{i=1}^n \langle x,v_i\rangle v_i$; $y=\sum_j \langle y,v_j\rangle v_j$. —(*)

$\langle x,y\rangle = \langle \sum_i \langle x,v_i\rangle v_i, \sum_j \langle y,v_j\rangle v_j\rangle = \sum_{i,j}(\langle x,v_i\rangle \times \overline{\langle y,v_j\rangle})\overset{\delta_{ij}}{\boxed{\langle v_i,v_j\rangle}}$

$= \sum_{i=1}^n \langle x,v_i\rangle \overline{\langle y,v_i\rangle}$ ※▨

(b). By (a), $\langle x,y\rangle = \sum_{i=1}^n \langle x,v_i\rangle \overline{\langle y,v_i\rangle} \overset{\text{by}(*)}{=} \sum_{i=1}^n ([x]_\beta)_{\boxed{i}} \cdot \overline{([y]_\beta)_i}$

$\overset{\text{defn.}}{=} \langle [x]_\beta, [y]_\beta\rangle \overset{\text{defn}}{=} \langle \phi_\beta(x), \phi_\beta(y)\rangle$ ▨ i-th component.

16 (a). (Bassel's inequality). $W:=\text{span}(S)\subset V$. $x\in V \overset{\text{Thm}6.6}{\Longrightarrow} \exists!\ u\in W\ \&\ z\in W^\perp$ s.t. $x=u+z$, and $u=\sum_{i=1}^n \langle x,v_i\rangle v_i$

$\|x\|^2=\langle x,x\rangle = \langle u+z,u+z\rangle = \|u\|^2+\|z\|^2+\langle \underset{W}{u},\underset{W^\perp}{z}\rangle^{\nearrow 0}+\langle z,u\rangle^{\nearrow 0} = \|z\|^2+\langle \sum \langle x,v_i\rangle v_i, \sum\langle x,v_j\rangle v_j\rangle$

$= \|z\|^2 + \sum_{i,j}(\langle x,v_i\rangle \overline{\langle x,v_j\rangle})\overset{\delta_{ij}}{\boxed{\langle v_i,v_j\rangle}} = \|z\|^2 + \sum_{i=1}^n |\langle x,v_i\rangle|^2 \geq \sum_{i=1}^n |\langle x,v_i\rangle|^2$ ▨

(b). 由(a), "=" holds $\iff \|z\|^2=0 \iff z=0 \iff x=u\in\text{span}(S)$ ▨

17. $\langle T(x),y\rangle = 0\ \forall x,\boxed{y}$ $\overset{\text{Thm}6.1}{\Longrightarrow} T(x)=0\ \forall x \Rightarrow T=T_0$ ※

• Say $\beta=\{v_1,\dots,v_n\}$ is a basis for V.

$\langle T(v_i),v_j\rangle = 0\ \forall i,j \Rightarrow \langle T(v_i),\sum a_iv_i\rangle = 0\ \forall i$ & scalars $a_1,\dots,a_n$.

$\Rightarrow \langle T(v_i),z\rangle = 0\ \underset{\forall i}{\forall z\in V} \overset{\text{Thm}6.1}{\Rightarrow} T(v_i)=0\ \forall i \overset{\text{Thm}2.6}{\Rightarrow} T=T_0$. ▨

18. $V:=C([-1,1])$ with $\langle f,g\rangle := \int_{-1}^1 f(t)g(t)\,dt$.

※ If g is even, f is odd, then $f(t)g(t)$ is odd. ∵ $\int_{-1}^1$ odd function $dt=0$.

$\Rightarrow W_e^\perp \supseteq W_o$.

• $\forall f\in V$, f can be written as $f=g+h$, where $\left\{\begin{array}{l} g(x)=:\frac{1}{2}(f(x)+f(-x)) \in W_e \\ h(x)=:\frac{1}{2}(f(x)-f(-x)) \in W_o\end{array}\right.$

Suppose $f\in W_e^\perp \Rightarrow \langle f,l\rangle = 0\ \forall l\in W_e \Rightarrow \langle g,l\rangle + \langle \underset{\in W_o}{h},l\rangle = 0\ \forall l\in W_e$.

$\Rightarrow \langle g,l\rangle = 0\ \forall l\in W_e$, especially, $\langle g,g\rangle=0$. Thus, $g=0$ & $f\in W_o$. ▨

19(a) $V=\mathbb{R}^2$. $u=(2,6)$. $W=\{(x,y)\mid y=4x\}$. $\Rightarrow W=\text{span}(\{(1,4)\})$

Thm6.6 $\Rightarrow \exists!$ vectors $w\in W$ & $z\in W^\perp$ s.t. $u=w+z$, and $w=\sum_{i=1}^n \frac{\langle u,v_i\rangle}{\|v_i\|^2}v_i$, where

$\text{span}(\{v_1,\dots,v_n\})=W$. So here we have $w=\frac{\langle (2,6),(1,4)\rangle}{\|(1,4)\|^2}(1,4) = \frac{26}{17}(1,4)$ ※

(b). $V=\mathbb{R}^3$. $u=(2,1,3)$. $W=\{(x,y,z)\mid x+3y-2z=0$. $\Rightarrow W=\text{span}(\{\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\3\end{pmatrix}\})$

An orthogonal basis for W is required if we wish to use Thm6.6..

(續下頁).

§6.2. p.115.

So apply the Gramn-Schmidt process. $v_1 = \begin{pmatrix}2\\0\\1\end{pmatrix}$. $v_2 = \begin{pmatrix}0\\2\\3\end{pmatrix} - \frac{3}{5}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-6/5\\2\\12/5\end{pmatrix}$.

So $\left\{\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}-3\\5\\6\end{pmatrix}\right\}$ is an orthogonal basis for W.

Then by Thm6.6, the orthogonal projection of u on W is $\omega = \sum_i \frac{\langle \overset{\begin{pmatrix}2\\1\\3\end{pmatrix}}{u}, v_i\rangle}{\|v_i\|^2} v_i$.

$\Rightarrow \omega = \frac{7}{5}\begin{pmatrix}2\\0\\1\end{pmatrix} + \frac{17}{70}\begin{pmatrix}-3\\5\\6\end{pmatrix} = \frac{1}{70}\begin{pmatrix}145\\17\times 5\\200\end{pmatrix} = \frac{1}{14}\begin{pmatrix}29\\17\\40\end{pmatrix}$ ※ ▨

(c). $V = P(\mathbb{R})$ with $\langle f,g\rangle := \int_0^1 fg\,dt$; $h(x) = 4+3x-2x^2$. $W = P_1(\mathbb{R})$.

Following the same concept as in (b), let $\{1, x\}$ be a basis for W.

$v_1 := 1$. $v_2 := x - \frac{\langle 1,x\rangle}{\underbrace{\|1\|^2}_{=1}}\cdot 1 = x - \frac{1}{2}$. $\Rightarrow \{1, x-\frac{1}{2}\}$ is an orthogonal basis.

The orthog. projection of h(x) on W is $\omega(x) = \sum_i \frac{\langle h(x), v_i\rangle}{\|v_i\|^2} v_i$

$\Rightarrow \omega(x) = \frac{(4+\frac{3}{2}-\frac{2}{3})}{1}\times 1 + \frac{(2+1-\frac{1}{2})-\frac{1}{2}(4+\frac{3}{2}-\frac{2}{3})}{(\frac{1}{3}-\frac{1}{2}+\frac{1}{4})}(x-\frac{1}{2}) = \frac{29}{6} + \frac{\frac{5}{2}-\frac{29}{12}}{\frac{1}{12}}(x-\frac{1}{2})$

$= \frac{29}{6} + (30-29)(x-\frac{1}{2}) = \frac{29}{6} + x - \frac{1}{2} = x - \frac{26}{6} = x - \frac{13}{3}$ ※ ▨

20. 承(19).

(a). distance $= \|(2,6) - \frac{26}{17}(1,4)\| = \|(\frac{8}{17}, \frac{-2}{17})\| = \frac{\sqrt{68}}{17} = \frac{2}{\sqrt{17}}$ ※

(b). distance $= \|(2,1,3) - \frac{1}{14}(29,17,40)\| = \frac{1}{14}\|(-1,-3,2)\| = \frac{1}{\sqrt{14}}$ ※

(c). distance $= \|(4+3x-2x^2) - (x-\frac{13}{3})\| = \|\frac{25}{3} + 2x - 2x^2\|$

$= \left(\int_0^1 \frac{625}{9} + 4x^2 + 4x^4 + \frac{100}{3}x - 8x^3 - \frac{100}{3}x^2\, dx\right)^{\frac{1}{2}}$

$= \left(\frac{625}{9} + \frac{4}{3} + \frac{4}{5} + \frac{50}{3} - 2 - \frac{100}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{3381}{45}} = \sqrt{\frac{1127}{15}}$ ※ ▨.

21. $V = C([-1,1])$ with $\langle f,g\rangle := \int_{-1}^1 f(t)g(t)\,dt$. $W := P_2(\mathbb{R})$, viewed as a subspace of V.

$h(t) := e^t$ on $[-1,1]$. Orthonormal basis for W is $\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2-1)\}$.

Thm6.6 $\Rightarrow \exists!$ vectors $w \in W$ & $l \in W^\perp$ s.t. $h = w + l$, and

$w = \sum_{i=1}^{n} \frac{\langle h, u_i\rangle u_i}{\underbrace{\|u_i\|^2}_{=1}}$ is the best approximation of h by Coro to Thm6.6.

$w = \sum_i \langle h, u_i\rangle u_i = \frac{1}{\sqrt{2}}(e-e^{-1})\times\frac{1}{\sqrt{2}} + (\sqrt{6}e^{-1})(\sqrt{\frac{3}{2}}x) + (\frac{\sqrt{5}}{\sqrt{2}}(e-7e^{-1}))(\sqrt{\frac{5}{8}}(3x^2-1))$

$= \frac{1}{2}e - \frac{1}{2}e^{-1} + 3e^{-1}x - \frac{5}{4}(e-7e^{-1})(3x^2-1) = (\frac{7}{4}e - \frac{37}{4}e^{-1}) + 3e^{-1}x - \frac{15}{4}(e-7e^{-1})x^2$. ▨

22. (a). Gramn-Schmidt process: $v_1 := t$, $v_2 := \sqrt{t} - \frac{\langle\sqrt{t}, t\rangle}{\langle t,t\rangle}t = \sqrt{t} - \frac{\frac{2}{5}}{\frac{1}{3}}t = \sqrt{t} - \frac{6}{5}t$

∴ Orthonormal basis $= \{\sqrt{3}t, \sqrt{50}(\sqrt{t} - \frac{6}{5}t)\}$

(b). $h(t) = t^2$. best approximation of h(t) in W is $\omega(t) = \frac{\langle h, v_1\rangle}{\|v_1\|^2}v_1 + \frac{\langle h, v_2\rangle}{\|v_2\|^2}v_2$

$= \frac{\sqrt{3}}{4}(\sqrt{3}t) + (\sqrt{50})(\frac{2}{7} - \frac{6}{20})(\sqrt{50}(\sqrt{t} - \frac{6}{5}t)) = \frac{3}{4}t - \frac{5}{7}(\sqrt{t} - \frac{6}{5}t) = \frac{45}{28}t - \frac{5}{7}\sqrt{t}$ ▨

No.

Date $\S 6.2$

23. $V := \{\{a_n\} \in \mathbb{F} \mid \{a_n\}_{n=1,2,\cdots}$ has only finite number of nonzero terms $a_n\}$.

Define $\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}$ if $\sigma, \mu \in V$. The series converges.

(a). • Given $\sigma, \mu, \tau \in V$. & $c \in \mathbb{F}$,

$\cdot\cdot\ \langle \sigma+\tau, \mu \rangle = \sum_{n=1}^{\infty} (\sigma+\tau)(n)\overline{\mu(n)} = \sum (\sigma(n)+\tau(n))\overline{\mu(n)} = \sum \sigma(n)\overline{\mu(n)} + \sum \tau(n)\overline{\mu(n)} \overset{\langle \sigma,\mu\rangle + \langle \tau,\mu\rangle}{=}$ ※

$\cdot\cdot\ \langle c\sigma, \mu \rangle = \sum_{n=1}^{\infty} (c\sigma)(n)\overline{\mu(n)} = \sum c\sigma(n)\overline{\mu(n)} = c\sum \sigma(n)\overline{\mu(n)} = c\langle \sigma, \mu \rangle$. ※

$\cdot\cdot\ \overline{\langle \sigma, \mu \rangle} = \overline{\sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}} = \sum \overline{\sigma(n)}\mu(n) = \langle \mu, \sigma \rangle$ ※

$\cdot\cdot\ \langle \sigma, \sigma \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\sigma(n)} = \sum_{n=1}^{\infty} |\sigma(n)|^2, \begin{cases} = 0 & \text{if } \sigma(n) \equiv 0\ \forall n. \\ > 0 & \text{o.w.} \end{cases}$ ※ ▨

(b). Suppose to the contrary. $\Rightarrow \exists\ i \neq j$ s.t. $\langle e_i, e_j \rangle \neq 0$.

$\langle e_i, e_j \rangle \overset{\text{defn.}}{=} \sum_{n=1}^{\infty} e_i(n)\overline{e_j(n)} = \sum_{n=1}^{\infty} \delta_{in}\overline{\delta_{jn}} = \sum_{n=1}^{\infty} \delta_{in}\delta_{jn} = 0$ if $i \neq j$. ※

Next, $\|e_i\|^2 = \langle e_i, e_i \rangle = \sum_{n=1}^{\infty} \delta_{in}\overline{\delta_{in}} \overset{(n=i)}{=} 1$. ▨

(c). $\sigma_n := e_1 + e_n$. $W := \operatorname{span}(\{\sigma_n \mid n \geq 2\})$

pf.(i). Suppose $\exists\, k \in \mathbb{N}$ s.t. $e_1 = \sum_{i=2}^{k} c_i\sigma_i$ for some $\{c_i\}_{i=2}^{k} \in \mathbb{F}$.

~~For~~ $2 \leq j \leq k, j \in \mathbb{N}$., $0 \overset{(b)}{=} \langle e_1, e_j \rangle = \langle \sum_{2}^{k} c_i\sigma_i, e_j \rangle = \sum_{i=2}^{k} c_i \langle e_1 + e_i, e_j \rangle$

$= \sum_{i=2}^{k} c_i(\cancelto{0}{\langle e_1, e_j \rangle} + \langle e_i, e_j \rangle) = \sum_{i=2}^{k} c_i \langle e_i, e_j \rangle = c_j$

$\therefore c_2 = c_3 = \cdots = c_k = 0$. ※$(e_1 \neq 0) \Rightarrow e_1 \notin W. \Rightarrow V \neq W$.

(ii). If $x \in W^{\perp}$, $\langle x, \sum_{i=2}^{n} c_i\sigma_i \rangle = 0\ \forall n \in \mathbb{N}, \forall c_i \in \mathbb{F}, 2 \leq i \leq n$.

$\Rightarrow \langle x, \sigma_i \rangle = 0\ \forall i = 2, 3, \cdots$.

$\Rightarrow \langle x, e_1 + e_i \rangle = 0 = \langle x, e_1 \rangle + \langle x, e_i \rangle\ \ \forall i = 2, \cdots$.

$\therefore -\langle x, e_1 \rangle = \langle x, e_i \rangle\ \forall i = 2, \cdots \Rightarrow -x(1) = x(i)\ \forall i = 2, \cdots$

if $x(1) \neq 0$, then $x(k) \neq 0\ \forall k = 1, 2, \cdots$, contradicting the defn. of the vector space V.

Thus $x(1) = 0 = -x(i)\ \forall i \geq 2 \in \mathbb{N}$. $\Rightarrow x = 0$.

$\therefore W^{\perp} \subseteq \{0\}$. $\{0\} \subseteq W^{\perp}$ is trivial $\Rightarrow W^{\perp} = \{0\}$ ※

$(W^{\perp})^{\perp} = (\{0\})^{\perp} = V$, by (i) $\Rightarrow V \neq W$. $\Rightarrow (W^{\perp})^{\perp} \neq W$ ▨

(Compare this result with $\S 6.2$ #13(c).).

§7-1. Linear Algebra

Chapter 7.

Section §7-1.

1. (a). T. (b) F, if x is an generalized eigenvector of T, ∃ smallest $p \in \mathbb{N}$ s.t. $(T-\lambda I)^p(x)=0$ for some $\lambda \in \mathbb{F}$. Then λ is an eigenvalue of T since $T(y)=\lambda y$, where $y=(T-\lambda I)^{p-1}(x)$

(c) F, split of the char. poly. of T is needed. (d) T, proof: Assume $\gamma=\{(T-\lambda I)^{p-1}(x), (T-\lambda I)^{p-2}(x), \dots, x\}$ is the cycle of generalized eigenvectors. and assume $0=\sum_{i=0}^{p-1} a_i (T-\lambda I)^i(x)$, then $0=(T-\lambda I)^k(0)=(T-\lambda I)^k(\sum a_i (T-\lambda I)^i(x))$.

$=\sum_{i=0}^{p-1} a_i (T-\lambda I)^{i+k}(x)$. Choose $k=p-1 \Rightarrow 0=a_0\cdot(T-\lambda I)^{p-1}(x) \Rightarrow a_0=0$.

Repeat the process, we have $a_0=a_1=\cdots=a_{p-1}=0$. ※

(e) F, e.g. $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. $\lambda=1$ is an eigenvalue of A. & $m(\lambda)=2$.

$E_\lambda=\mathbb{R}^2 \Rightarrow \beta=\{\underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\1\end{pmatrix}}\}$ for E_λ. So in this case, there are 2 cycles $\{v_1\}$ & $\{v_2\}$ corresponding to $\lambda=1$. ※

(f). F, it requires additional condition, the basis β_i for K_{λ_i} should consist of cycles of generalized eigenvectors of T.

- Counterexample: $A=\begin{pmatrix}1&1\\0&1\end{pmatrix} \Rightarrow \lambda=1, m(\lambda)=2$. $K_\lambda=\{x \mid \begin{pmatrix}0&1\\0&0\end{pmatrix}^p x=0, p\in\mathbb{N}\}$ $=\mathbb{R}^2$ Let $\beta=\{\underset{\begin{pmatrix}1\\0\end{pmatrix}}{(A-\lambda I)\begin{pmatrix}0\\1\end{pmatrix}}, \begin{pmatrix}0\\1\end{pmatrix}\}$ & $\gamma=\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\}$ be bases of K_λ.

  Then $[A]_\beta=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ is a Jordan canonical form,

  but $[A]_\gamma=\begin{bmatrix}\frac{3}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}$ is not!

(g). T, choose std ordered basis β_0, then $[L_J]_{\beta_0}=J$ ※

(h). T.

2. (a). $A=\begin{pmatrix}1&1\\-1&3\end{pmatrix}$. $\Rightarrow f(t)=(t-1)(t-3)+1=t^2-4t+4=(t-2)^2 \Rightarrow \lambda=2$.

- $[\dim(K_\lambda)=m(\lambda)=2]$ 1° $(A-\lambda I_2)(x)=0 \Rightarrow \begin{pmatrix}-1&1\\-1&1\end{pmatrix}x=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x=\begin{pmatrix}a\\a\end{pmatrix}$, choose $x=\begin{pmatrix}1\\1\end{pmatrix}$.

  2° $(A-\lambda I_2)(y)=x \Rightarrow \begin{pmatrix}-1&1\\-1&1\end{pmatrix}y=\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow y=\begin{pmatrix}a\\1+a\end{pmatrix}$, choose $y=\begin{pmatrix}1\\2\end{pmatrix}$.

  So $\beta=\{\underbrace{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}}\}$ forms a Jordan canonical basis of A, and

  $[A]_\beta=J=\begin{bmatrix}2&1\\0&2\end{bmatrix}$. ※

(b). $A=\begin{pmatrix}1&2\\3&2\end{pmatrix} \Rightarrow f(t)=(t-1)(t-2)-6=t^2-3t-4=(t-4)(t+1) \Rightarrow \lambda_1=-1, \lambda_2=4$.

∵ $\dim(K_{\lambda_1})=m(\lambda_1)=1$. $(A-\lambda_1 I_2)(x)=0 \Rightarrow \begin{pmatrix}2&2\\3&3\end{pmatrix}x=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x=\begin{pmatrix}a\\-a\end{pmatrix}$, Choose $x=\begin{pmatrix}1\\-1\end{pmatrix}$.

∵ $\dim(K_{\lambda_2})=m(\lambda_2)=1$. $(A-\lambda_2 I_2)(x)=0 \Rightarrow \begin{pmatrix}-3&2\\3&-2\end{pmatrix}x=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x=\begin{pmatrix}2a\\3a\end{pmatrix}$, Choose $x=\begin{pmatrix}2\\3\end{pmatrix}$.

$\beta:=\{\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}, \underbrace{\begin{pmatrix}2\\3\end{pmatrix}}\}$ is a Jordan basis. & $J=[A]_\beta=\begin{bmatrix}-1&0\\0&4\end{bmatrix}$. ※

(c). $A=\begin{pmatrix}11&-4&-5\\21&-8&-11\\3&-1&0\end{pmatrix} \Rightarrow f(t)=\det\begin{vmatrix}11-t&-4&-5\\21&-8-t&-11\\3&-1&-t\end{vmatrix} = 3(44-(40+5t))+1\cdot(11(t-11)+105)$

$-t((t-11)(t+8)+84) = 12-15t+11t-16-t^3+3t^2+4t = -t^3+3t^2-4$

$=(-1)^3(t^3-3t^2+4) = (-1)^3(t+1)(t^2-4t+4) = (-1)^3(t+1)(t-2)^2 \Rightarrow \lambda_1=-1 ; \lambda_2=2.$

- $\dim(K_{\lambda_1}) = m(\lambda_1) = 1.$ $(A-\lambda_1 I_3)x=0 \Rightarrow \begin{pmatrix}12&-4&-5\\21&-7&-11\\3&-1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$

$\left[\begin{array}{ccc|c}12&-4&-5&0\\21&-7&-11&0\\3&-1&1&0\end{array}\right] \to \left[\begin{array}{ccc|c}0&0&-9&0\\0&0&-18&0\\3&-1&1&0\end{array}\right] \Rightarrow x=\begin{pmatrix}a\\3a\\0\end{pmatrix}.$ Choose $x=\begin{pmatrix}1\\3\\0\end{pmatrix}.$

- $\dim(K_{\lambda_2}) = m(\lambda_2) = 2.$ $(A-\lambda_2 I_3)x=0 \Rightarrow \begin{pmatrix}9&-4&-5\\21&-10&-11\\3&-1&-2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$

$\left[\begin{array}{ccc|c}9&-4&-5&0\\21&-10&-11&0\\3&-1&-2&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}0&1&-1&0\\0&3&-3&0\\3&-1&-2&0\end{array}\right] \Rightarrow x=\begin{pmatrix}c\\c\\c\end{pmatrix}.$ Choose $x=\begin{pmatrix}1\\1\\1\end{pmatrix}.$

•• $(A-\lambda_2 I_3)y = x \Rightarrow \left[\begin{array}{ccc|c}9&-4&-5&1\\21&-10&-11&1\\3&-1&-2&1\end{array}\right] \to \left[\begin{array}{ccc|c}0&1&-1&-2\\0&3&-3&6\\3&-1&-2&1\end{array}\right] \Rightarrow y=\begin{pmatrix}1+c\\2+c\\c\end{pmatrix}.$ Choose $y=\begin{pmatrix}1\\2\\0\end{pmatrix}.$

Then $\beta=\{\underbrace{\begin{pmatrix}1\\3\\0\end{pmatrix}}_{\beta_1}, \underbrace{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}}_{\beta_2}\}$ forms a Jordan canonical basis & $[A]_\beta = J = \begin{bmatrix}-1&0&0\\0&2&1\\0&0&2\end{bmatrix}$ ※

(d). 略!

3. (a). $T(f(x)) = 2f-f'$, $\forall f\in P_2(\mathbb{R})$. $f(t)$ = char. poly = $\det(T-tI) = \det([T]_{\beta_0}-tI)$

$= \det\left(\begin{bmatrix}2&-1&0\\0&2&-2\\0&0&2\end{bmatrix}-tI_3\right) = (-1)^3(t-2)^3 \Rightarrow \lambda=2.$

- $\dim(K_\lambda) = m(\lambda) = 3.$

•• $([T]_{\beta_0}-\lambda I_3)x=0 \Rightarrow \left[\begin{array}{ccc|c}0&-1&0&0\\0&0&-2&0\\0&0&0&0\end{array}\right] \Rightarrow x=\begin{pmatrix}a\\0\\0\end{pmatrix}.$ Choose $x=\begin{pmatrix}1\\0\\0\end{pmatrix}.$

•• $([T]_{\beta_0}-\lambda I_3)y = x \Rightarrow \left[\begin{array}{ccc|c}0&-1&0&1\\0&0&-2&0\\0&0&0&0\end{array}\right] \Rightarrow y=\begin{pmatrix}a\\-1\\0\end{pmatrix}.$ Choose $y=\begin{pmatrix}1\\-1\\0\end{pmatrix}.$

•• $([T]_{\beta_0}-\lambda I_3)z = y \Rightarrow \left[\begin{array}{ccc|c}0&-1&0&1\\0&0&-2&-1\\0&0&0&0\end{array}\right] \Rightarrow z=\begin{pmatrix}a\\-1\\\frac{1}{2}\end{pmatrix}.$ Choose $z=\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}.$

$\beta := \{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}\}$ forms a Jordan basis. & $J=[T]_\beta = \begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$ ※

(b) $V=\operatorname{span}\{1,t,t^2,e^t,te^t\}$. $T: V\to V$ $f\mapsto f'$ $\beta_0 := \{1,t,t^2,e^t,te^t\}.$

$f(t) = \det(T-tI) = \det\left(\begin{bmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}_{\leftarrow\beta_0} - tI_5\right) = \det\begin{bmatrix}-t&1&&&\\&-t&2&&\\&&-t&&\\&\text{O}&&1-t&1\\&&&&1-t\end{bmatrix}.$

$\begin{matrix}-1&1&&&\\&-1&2&&\\&&&&\\&&&0&1\\&&&&0\end{matrix}\left|\begin{matrix}0\\0\\0\\a\\0\end{matrix}\right.$

上三角公式

$= (-t)^3(1-t)^2 = (-1)^3(t)^3(t-1)^2 \Rightarrow \lambda_1=0, \lambda_2=1.$

- $\dim(K_{\lambda_1})=3.$ •• $([T]_{\beta_0}-\lambda_1 I_5)x=0 \Rightarrow x=\begin{pmatrix}a\\0\\0\\0\\0\end{pmatrix}, a\in\mathbb{R}.$

•• $([T]_{\beta_0}-\lambda_1 I_5)y=x \Rightarrow y=\begin{pmatrix}b\\a\\0\\0\\0\end{pmatrix}, b\in\mathbb{R}.$ •• $([T]_{\beta_0}-\lambda_1 I_5)z=y \Rightarrow z=\begin{pmatrix}c\\b\\a/2\\0\\0\end{pmatrix}, c\in\mathbb{R}.$

Choose $a=2, b=1, c=1 \Rightarrow \beta_1=\{\begin{pmatrix}2\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}\}$ s.t. $[T|_{K_{\lambda_1}}]_{\beta_1} = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

- $\dim(K_{\lambda_2}) = 2.$ •• $([T]_{\beta_0}-\lambda_2 I_5)x=0 \Rightarrow x=\begin{pmatrix}0\\0\\0\\a\\0\end{pmatrix}, a\in\mathbb{R}.$

•• $([T]_{\beta_0}-\lambda_2 I_5)y=x \Rightarrow y=\begin{pmatrix}0\\0\\0\\b\\a\end{pmatrix}.$ Choose $a=1, b=1 \Rightarrow \beta_2=\{\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}\}$ 續下頁.

§7.1

Define $\beta := \beta_1 \cup \beta_2$. By Thm 7.4(b), β is a basis for V & By Thm 7.5, β is a Jordan canonical basis. $J = [T]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ※

(c) 略.

(d) 略.

4. Let $\gamma := \{(T-\lambda I)^{p-1}(x), \cdots, (T-\lambda I)(x), x\}$ be a cycle of generalized eigenvectors.

解答解法更佳 • γ is L.I. : Assume $0 = \sum_{k=0}^{p-1} ((T-\lambda I)^k(x)) a_k \Rightarrow 0 = (T-\lambda I)^{p-1}(\sum_0^{p-1} a_k (T-\lambda I)^k(x)) = \sum_{k=0}^{p-1} a_k (T-\lambda I)^{k+p-1}(x)$

$= a_0$ since $(T-\lambda I)^{p+m}(x) = 0 \ \forall\, m \geq 0 \in \mathbb{N}$.

Repeat the process, we get $a_0 = a_1 = \cdots = a_{p-1} = 0$.

Next:

or (Thm 7.5(a) proof) • span(γ) is T-invariant : by def (p.488) $(T-\lambda I)(\overbrace{(T-\lambda I)^{p-1}(x)}^{=: y}) = 0$

$\Rightarrow \quad Ty = \lambda y \in \text{span}(\gamma)$

Next, for any $k = 0, 1, 2, \ldots, p-2$,

$(T-\lambda I)(\underbrace{(T-\lambda I)^k(x)}_{=: y_k}) = \underbrace{(T-\lambda I)^{k+1}(x)}_{=: y_{k+1}} \in \gamma$.

$\Rightarrow T(y_k) = y_{k+1} + \lambda y_k \in \text{span}(\gamma)$. ■

5. Let v_{ij} be the end vector of γ_i, $i = 1, 2, \ldots, p$.

解答解法更佳 Let $(T-\lambda I)^{m_i}(v_{ij})$ be the initial vector of γ_i, $i \in \{1, \ldots, p\}$ and they are distinct. Then given $x := (T-\lambda I)^{k_i}(v_{ij})$ and $y = (T-\lambda I)^{l_{\tilde{i}}}(v_{\tilde{i}\tilde{j}})$,

Claim: x & y are distinct.

pf. Suppose $x = y$, then $(T-\lambda I)^{m_i - k_i}(x) = (T-\lambda I)^{m_i - k_i}(y)$,

$\Rightarrow (T-\lambda I)^{m_i}(v_{ij}) = (T-\lambda I)^{m_i - k_i + l_{\tilde{i}}}(v_{\tilde{i}\tilde{j}})$ — ①

If $m_i - k_i + l_{\tilde{i}} > m_{\tilde{i}}$, then RHS of ① is zero ※

If $m_i - k_i + l_{\tilde{i}} = m_{\tilde{i}}$, then two initial vectors of γ_i & $\gamma_{\tilde{i}}$ are the same, ※

If $m_i - k_i + l_{\tilde{i}} < m_{\tilde{i}}$, take $(T-\lambda I)^{m_{\tilde{i}} - (m_i - k_i + l_{\tilde{i}})}$ on both sides of ①, we find that LHS = 0 & RHS = initial vector of $\gamma_{\tilde{i}}$, ※. ■

6. (a) $N(-T) = \{x \mid -T(x) = 0\} = \{x \mid T(x) = 0\} (= N(T))$ ※

(b) $N((-T)^k) = \{x \mid (-T)^k(x) = 0\} = \{x \mid -T(-T(\cdots(-T(x)))) = 0\} = \{x \mid (-1)^k T^k(x) = 0\}$

$= \{x \mid T^k(x) = 0\}$ ※

(c). $T: V \to V$ and λ is an eigenvalue.

$N((T-\lambda I_V)^k) = \{x \mid (T-\lambda I_V)^k(x)=0\} = \{x \mid (-1)^k (T-\lambda I_V)^k(x)=0\}$

$= \{x \mid (\lambda I_V - T)^k(x)=0\} = N((\lambda I_V - T)^k)$. , for all $k \in \mathbb{N}$ ∎

7. $U: V \to V$ is linear.

(a). Given $k \in \mathbb{N}$, if $x \in N(U^k)$, $U^k(x)=0 \Rightarrow U(U^k(x)) = U^{k+1}(x) = 0$.

$\Rightarrow x \in N(U^{k+1})$. Since k is arbitrary, $N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$ ∎

(b) Given $k \geq m$, $k \in \mathbb{N}$.

$\text{rank}(U^m) = \text{rank}(U^{m+1}) \Rightarrow \dim(R(U^m)) = \dim(R(U^{m+1}))$.. Since both $R(U^m)$ and $R(U^{m+1})$ are subspaces of V and they have the same dimension, $R(U^m) = R(U^{m+1})$, i.e. $U^m(V) = U^{m+1}(V) = U(U^m(V))$. Let $W = U^m(V)$. Then $\underbrace{U(W)=W}_{①}$. Thus, $U^k(V) = U^{k-m}(U^m(V)) = U^{k-m}(W) = \underbrace{U\cdots U}_{k-m\text{次}}(W)$
$= W$ by ① ∎

(c) By part (b), and by Dimension Thm; $\text{nullity}(U^m) = \text{nullity}(U^k)$ $\forall k \geq m$. Since $N(U^m)$ & $N(U^k)$ are subspaces of V, and have the same dim,
$N(U^m) = N(U^k)$. ∎

(d) By def, $K_\lambda = \{x \mid \exists p \in \mathbb{N} \text{ s.t. } (T-\lambda I_V)^p(x)=0\}$.

Let $U = T - \lambda I : V \to V$. By (C), $N(U^m) = N(U^k)$ $\forall k \geq m$.

$\Rightarrow N((T-\lambda I)^m) = N((T-\lambda I)^k)$. $\forall k \geq m$. Thus, if $\exists\, x$ s.t. $(T-\lambda I)^k(x)=0$, for some $k > m$, then $(T-\lambda I)^m(x)=0$. $\Rightarrow K_\lambda = N((T-\lambda I)^m)$. ∎

(e). Second Test for diagonalizability.

$\text{rank}(T-\lambda_i I) = \text{rank}((T-\lambda_i I)^2) \overset{(b)}{\Leftrightarrow} \text{rank}(T-\lambda_i I) = \text{rank}((T-\lambda_i I)^k)$ $\forall k \geq 1$

$\overset{\text{Dimension Thm.}}{\Leftrightarrow} \underset{\overset{\|}{\dim(E_{\lambda_i})}}{\text{nullity}(T-\lambda_i I)} = \text{nullity}((T-\lambda_i I)^k)$ $\forall k \geq 1$.

$\Leftrightarrow$. $\dim(E_{\lambda_i}) = \dim(K_{\lambda_i})$; $\underset{E_{\lambda_i} \subseteq K_{\lambda_i}}{\Leftrightarrow}$. $E_{\lambda_i} = K_{\lambda_i}$ ∎

(f). $T: V \to V$ is diagonalizable.

If λ is an eigenvalue of T_W, so is T by Thm 5.21.

T diagonalizable $\overset{(e)}{\Rightarrow} \text{rank}(T-\lambda I) = \text{rank}((T-\lambda I)^2) \overset{(c)}{\Rightarrow} N(T-\lambda I) = N((T-\lambda I)^2)$ (—①).

$\Rightarrow N(T_W - \lambda I) = N((T_W - \lambda I)^2)$. since every $x \in W \subseteq V$ satisfying ① would naturally satisfy $(T_W - \lambda I)^2(x) = (T_W - \lambda I)(x) = 0$.. $\overset{\text{Dimension Thm.}}{\Longrightarrow}$ $\text{rank}(T_W - \lambda I) = \text{rank}((T_W - \lambda I)^2)$. ∎

§7.1.

8. By Thm 7.3, given $x \in V$, $\exists\ v_i \in K_{\lambda_i}$, $i = 1, \cdots, k$, s.t. $x = v_1 + v_2 + \cdots + v_k$.

By Thm 5.10, x is uniquely expressed as $v_1 + v_2 + \cdots + v_k$, $v_i \in K_{\lambda_i}$. ■

9. (a). 略.

(b). $\beta' = \beta \cap K_\lambda$. Claim: β' is a basis for K_λ, where β = Jordan basis for T.

proof. • β' is L.I. since $\beta' \subseteq \beta$.

• $[T]_\beta = J$, upper triangular matrix. $\Big) \Rightarrow$ multiplicity of λ

$\therefore m := \#$ of λ appearing in J. $=: m(\lambda)$ is m.

$\therefore \beta'$ is exactly those v_i's such that $(J)_{ii} = \lambda$. $\Rightarrow \#(\beta') = m$.

Now, Thm 7.4(c) $\Rightarrow$ $\dim(K_\lambda) = m\ (= \#(\beta'))$

$\star$ L.I. + same dim. $\Rightarrow$ β' is a basis for K_λ. ■

10. (a). γ has q disjoint cycles. and thus have q distinct initial vectors.

We know that by def, all initial vectors are in E_λ.

Also, γ is a basis $\Rightarrow$ initial vectors form a L.I. set.

So $\dim(E_\lambda) \geq q$. ■

(b) J has q Jordan blocks with λ. $\Rightarrow$ There are q initial vectors corresponding to λ and form a L.I. set since these q cycles form a basis. Thus, $q \leq \dim(E_\lambda)$.

11. By Coro. 1 to Thm 7.7, L_A has a Jordan canonical form, say $[L_A]_\gamma = J$ for some γ, a basis for F^n. Note that $[L_A]_{\beta_0} = A$, where β_0 is the s.t.d. ordered basis. Then $[L_A]_\gamma = [I]_{\beta_0}^{\gamma} [L_A]_{\beta_0} [I]_{\gamma}^{\beta_0}$

Let $Q = [I]_\gamma^{\beta_0}$ $\Rightarrow$ $J = Q^{-1} A Q$. $\Rightarrow$ $A \sim J$. ■

12 Thm 7.3 $\Rightarrow$ $V = \sum_\lambda K_\lambda$.

Thm 7.4 $\overset{\text{Thm 5.10}}{\Longrightarrow}$ $V = \bigoplus_\lambda K_\lambda$ ■

13. By §7.1 #12, $V = \bigoplus_\lambda K_\lambda$ and $\{K_\lambda\}_\lambda$ are T-invariant.

Then by Thm 5.25, $J = \bigoplus_i J_i$. ■

No.
Date §7.2.

Section §7.2.