

# Module 1/2 Bridge: Variational Closure

Validating G4: Boundary Terms and Sign Convention

Independent Logical Section: Validating G4

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## 1 Input Node: The Reciprocity Functional

The system seeks minimal informational curvature, defined by the functional  $\mathcal{R}[\mathbf{U}]$ . The core principle of Reciprocity (Module 1, G5) requires that every variation  $\delta\mathcal{R}$  must correspond to a measurable distinction  $\Phi$ . The canonical closure is achieved when the first variation vanishes,  $\delta\mathcal{R} = 0$ .

The functional is defined as the integral of the squared informational curvature  $\mathcal{R}[\mathbf{U}] = \int \mathcal{L}(\mathbf{U}'') dx$ , where  $\mathcal{L}(\mathbf{U}'') = \frac{1}{2}(\mathbf{U}'')^2$ .

## 2 Formal Variation and Integration by Parts

To obtain the Euler-Lagrange condition (the input for G2/G3), we compute the first variation  $\delta\mathcal{R}$  over an interval  $[x_1, x_2]$  and apply integration by parts. This step formally validates the necessary boundary terms and signs for **G4**.

**Proposition 1** (Closure of the Reciprocity Functional (**G4**)). *The first vari-*

ation of the functional  $\mathcal{R}[\mathbf{U}]$  is:

$$\delta\mathcal{R} = \int_{x_1}^{x_2} \left( \frac{\partial\mathcal{L}}{\partial\mathbf{U}} - \frac{d}{dx} \left( \frac{\partial\mathcal{L}}{\partial\mathbf{U}'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial\mathcal{L}}{\partial\mathbf{U}''} \right) \right) \delta\mathbf{U} dx + \text{Boundary Terms.}$$

For  $\mathcal{L} = \frac{1}{2}(\mathbf{U}'')^2$ , the Euler-Lagrange equation (the integrand) reduces to  $\mathbf{U}^{(4)} = 0$  (the input to G2).

### 3 The Boundary Terms and Physical Constraint

The \*\*Boundary Terms\*\* resulting from the integration by parts must vanish for  $\delta\mathcal{R} = 0$  to hold for arbitrary internal variations ( $\delta\mathbf{U}$ ):

$$\text{Boundary Terms} = \left[ \frac{\partial\mathcal{L}}{\partial\mathbf{U}'} - \frac{d}{dx} \left( \frac{\partial\mathcal{L}}{\partial\mathbf{U}''} \right) \right]_{x_1}^{x_2} \delta\mathbf{U} + \left[ \frac{\partial\mathcal{L}}{\partial\mathbf{U}''} \right]_{x_1}^{x_2} \delta\mathbf{U}'$$

Substituting  $\mathcal{L} = \frac{1}{2}(\mathbf{U}'')^2$ :

$$\text{Boundary Terms} = [-\mathbf{U}''']_{x_1}^{x_2} \delta\mathbf{U} + [\mathbf{U}''']_{x_1}^{x_2} \delta\mathbf{U}' = 0$$

**Interpretation (Validating G4):** The vanishing of these terms enforces the dual system's closure at the measurement anchors ( $x_1, x_2$ ). They confirm that the variation ( $\delta\mathbf{U}$ ) and its slope ( $\delta\mathbf{U}'$ ) at the boundaries must be either \*\*fixed by measurement\*\* (Dirichlet/Neumann conditions) or that the \*\*natural conditions\*\* ( $\mathbf{U}'' = 0$  and  $\mathbf{U}''' = 0$ ) hold. This ensures the correct sign convention and closure needed to enforce  $\mathbf{U}^{(4)} = 0$  (G2) consistently across finite intervals.

## 4 Output Node: Input to Kinematic Closure

The successful closure of  $\delta\mathcal{R} = 0$  via the necessary boundary terms (G4) confirms the validity of the \*\*Euler-Lagrange Equation\*\*  $\mathbf{U}^{(4)} = \mathbf{0}$ , which is the foundational starting point for Module 2 (Chapter 3, G2/G3).

*Conclusion for G4: The Reciprocity Law closes exactly at the boundaries of observation.*