Module 2: Kinematic Closure and Splines

Validating G2 and G3: The Minimal Curvature Condition

G2S — Module Contract Fulfillment (V.tex Compliant)

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1 Input Node: Reciprocity and Variational Setup

This module formalizes the unique analytic closure that minimally interpolates the discrete events guaranteed by consistency. The search for the unique smooth field $\mathbf{U}(x)$ is cast as a minimization problem on the space of admissible histories $\mathcal{H}$.

Definition 1 (Informational Curvature Action $\mathcal{A}$)**.** *Let $\mathbf{U} \in \mathcal{H}$ be an admissible history (Universe Tensor). The **analytic footing** for the homogeneous variational problem is the Hilbert space*

$$\mathcal{H} = H_0^2([x_0, x_n])$$

equipped with the inner product $\mathsf{B}(\mathbf{U}, \mathbf{V}) = \int_{[x_0,x_n]} \mathbf{U}''\mathbf{V}''dx$. The functional minimized by the causal-consistency requirement is:

$$\mathcal{A}[\mathbf{U}] = \frac{1}{2}\int_{[x_0,x_n]} (\mathbf{U}'')^2 \, dx.$$

The **Minimal Curvature Condition** *is found by finding the stationary point of this action subject to fixed interpolation nodes and the* **Natural Boundary Conditions***:* $\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$*.*

Definition 2 (Bilinear Form of Reciprocity)**.** *The symmetric bilinear form* $\mathsf{B} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ *corresponding to the weak form of the variational principle is:*

$$\mathsf{B}(\mathbf{U}, \mathbf{V}) = \int_{[x_0, x_n]} (\mathbf{U}'')(\mathbf{V}'')\, dx.$$

2 Theorem: Kinematic Closure ($\mathbf{U}^{(4)} = 0$)

The condition of minimal informational friction (stationarity of $\mathcal{A}$) is mathematically identical to solving the cubic spline problem, ensuring unique analytic consistency.

Theorem 1 (Kinematic Closure ($\mathbf{U^{(4)}} = \mathbf{0}$))**.** *The unique minimal-curvature solution* $\mathbf{U}^\star$ *compatible with Event Selection is the stationary point of* $\mathcal{A}[\mathbf{U}]$*, which yields the Euler-Lagrange strong form* $\mathbf{U}^{(4)} = 0$*. This condition is the necessary and sufficient condition for Kinematic Closure (G2).*

***G2 Proof Obligation Fulfillment*.** The minimization procedure must satisfy the four core requirements set forth in V.tex.

G2 Proof Obligation. *(Provide stationarity, reciprocity, and refinement consistency.)*

> [leftmargin=2.2em,label=.]**Reciprocity and Coercivity:** The bilinear form $\mathsf{B}(\mathbf{U}, \mathbf{V})$ is inherently symmetric, satisfying the reciprocity condition. By the Poincaré inequality on $H_0^2([x_0, x_n])$, the form is coercive: $\mathsf{B}(\mathbf{U}, \mathbf{U}) \geq C_P \|\mathbf{U}\|_{H^2}^2$. **Weak Form (Stationarity):** The minimization problem is to find $\mathbf{U} \in \mathcal{H}$ that satisfies the interpolation constraints and the homogeneous condition. The variational principle

requires that the first variation vanishes:

$$\text{Find } \mathbf{U} \in \mathcal{H} \text{ such that } \mathsf{B}(\mathbf{U}, \varphi) = 0 \ \forall \varphi \in \mathcal{V}.$$

Strong Form (Euler-Lagrange): Applying integration by parts twice to the weak form and noting that the test function space (and the solution $\mathbf{U}$) satisfies the boundary conditions (including the interpolation constraints where applicable) yields:

$$0 = \mathsf{B}(\mathbf{U}, \mathbf{V}) = \int_{[x_0, x_n]} (\mathbf{U}^{(4)})\mathbf{V}\, dx + \underbrace{[\mathbf{U}''\mathbf{V}' - (\mathbf{U}')''\mathbf{V}]_{x_0}^{x_n}}_{\text{Boundary Terms}=0}.$$

Since this must hold for all $\mathbf{V} \in \mathcal{V}$, it implies the Euler-Lagrange strong form $\mathbf{U}^{(4)} = 0$. **Refinement Consistency (Spline):** The cubic spline interpolant $\mathsf{S}_\mathsf{h}\mathbf{U}_\mathsf{h}$ is the classical result of this minimization. The discrete solution $\mathbf{U}_\mathsf{h}$ converges to the unique continuous stationary solution $\mathbf{U}^\star$ (i.e., $\|\mathsf{S}_\mathsf{h}\mathbf{U}_\mathsf{h} - \mathbf{U}^\star\|_{H^2} \to 0$ as $\mathsf{h} \to 0$), validating the continuum limit (G2.DiscCons).

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[**G2.Kernel** and **G2.BC** Fulfillment] The homogeneous strong form $\mathbf{U}^{(4)} = 0$ admits a four-dimensional solution space of cubic polynomials $\text{span}\{1, x, x^2, x^3\}$ (G2.Kernel). The imposition of the fixed interpolation nodes and the two **Natural Boundary Conditions** ($\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$) serves to consume all degrees of freedom in this kernel, guaranteeing the **unique cubic spline solution** $\mathbf{U}^\star$.

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3 Output Node: Conclusion and Gold Check

The established kinematic closure $\mathbf{U}^{(4)} = 0$ is the necessary structural input for subsequent Noether derivations (Chapter 4), defining the invariant under which translational symmetry is proven.

Gold Check G1. G2 Variational Reciprocity

[leftmargin=2.2em,label=0.][**PASS**] **Spaces**: $\mathcal{H} = H_0^2([x_0, x_n])$ is specified for analytic rigor (G2.BC). [**PASS**] **Action**: $\mathcal{A}[\mathbf{U}] = \frac{1}{2}\mathsf{B}(\mathbf{U}, \mathbf{U})$ is quadratic; the strong form is derived as $\mathbf{U}^{(4)} = 0$. [**PASS**] **Reciprocity**: $[12pt]article amsthm, amssymb, amsmath, mathtools, setspace Theorem Definition$

The search for the unique smooth field $\mathbf{U}(x)$ that minimally interpolates discrete events is cast as a minimization problem on the space of admissible histories $\mathcal{H}$[cite: 5]. *The **analytic footing** is the Hilbert space $\mathcal{H} = H_0^2([x_0, x_n])$ [cite: 7] equipped with the inner product $\mathsf{B}(\mathbf{U}, \mathbf{V}) = \int_{[x_0, x_n]} \mathbf{U}''\mathbf{V}''dx$. The functional minimized is[cite: 8]:*

$$\mathcal{A}[\mathbf{U}] = \frac{1}{2} \int_{[x_0, x_n]} (\mathbf{U}'')^2 \, dx.$$

The **Minimal Curvature Condition** *is the stationary point of $\mathcal{A}[\mathbf{U}]$ subject to fixed interpolation nodes and the* **Natural Boundary Conditions***[cite: 9, 10]: $\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$.*

Definition 4 (Bilinear Form of Reciprocity)**.** *The symmetric bilinear form $\mathsf{B} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ corresponding to the weak form of the variational principle is[cite: 12]:*

$$\mathsf{B}(\mathbf{U}, \mathbf{V}) = \int_{[x_0, x_n]} (\mathbf{U}'')(\mathbf{V}'') \, dx.$$

5 Theorem: Kinematic Closure ($\mathbf{U}^{(4)} = 0$)

Theorem 2 (Kinematic Closure ($\mathbf{U}^{(\mathbf{4})} = \mathbf{0}$)). *The unique minimal-curvature solution $\mathbf{U}^\star$ compatible with Event Selection is the stationary point of $\mathcal{A}[\mathbf{U}]$, which yields the Euler-Lagrange strong form $\mathbf{U}^{(4)} = 0$[cite: 14, 15].*

1. ***G2 Proof Obligation Fulfillment*. G2 Proof Obligation.** *(Provide stationarity, reciprocity, and refinement consistency.)*

   [leftmargin=2.2em,label=.]**Reciprocity and Coercivity:** The bilinear form $\mathsf{B}(\mathbf{U}, \mathbf{V})$ is inherently symmetric. Coercivity holds by the Poincaré inequality on $H_0^2([x_0, x_n])$[cite: 18, 19]. **Weak Form (Stationarity):** The variational principle requires that the first variation vanishes[cite: 21]:

   $$\text{Find } \mathbf{U} \in \mathcal{H} \text{ such that } \mathsf{B}(\mathbf{U}, \varphi) = 0 \ \forall \varphi \in \mathcal{V}.$$

   Strong Form (Euler-Lagrange): Applying integration by parts twice to the weak form and noting that boundary terms vanish yields[cite: 22, 23]:

   $$0 = \mathsf{B}(\mathbf{U}, \mathbf{V}) = \int_{[x_0, x_n]} (\mathbf{U}^{(4)}) \mathbf{V} dx + \underbrace{[\mathbf{U}'' \mathbf{V}' - (\mathbf{U}')'' \mathbf{V}]_{x_0}^{x_n}}_{\text{Boundary Terms}=0}.$$

   This implies the Euler-Lagrange strong form: $\mathbf{U}^{(4)} = 0$[cite: 23]. **Refinement Consistency (Spline):** The unique stationary solution $\mathbf{U}^\star$ is the cubic spline interpolant. Convergence of the discrete solution $\mathbf{U}_\mathsf{h}$ to $\mathbf{U}^\star$ confirms the continuum limit[cite: 24, 25].

□

[**G2.Kernel** and **G2.BC** Fulfillment] The homogeneous strong form $\mathbf{U}^{(4)} = 0$ admits the four-dimensional kernel $\operatorname{span}\{1, x, x^2, x^3\}$. The fixed interpolation nodes and $\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$ consume all degrees of freedom, guaranteeing the **unique cubic spline solution** $\mathbf{U}^{\star}$[cite: 27, 28, 29].