P-Clean: The Unified Causal Closure Chain

Volume I: The Conditions of $\Delta \mathrm{S} \geq 0$

The Project Synthesis

October 13, 2025

Abstract

This document synthesizes the core closure arguments (G1–G4). It shows that the foundational axiom of **Minimal Informational Curvature** ($\mathbf{U}^{(4)} = 0$) supplies the structural hypothesis necessary to derive the **Conservation of Stress–Energy** ($\nabla \cdot \mathsf{T} = 0$), which rigorously enforces the **Second Law of Causal Order** ($\Delta \mathrm{S} \geq 0$) as a theorem of monotonicity. The full system is constrained by **Discrete–Continuum Reciprocity** (Discrete–continuum reciprocity via B) and the **Causal Data Processing Inequality** ($\mathrm{D}_{\mathrm{KL}}(P \| Q)$ non-increasing under admissible coarse-grainings), ensuring the integrity of measurement and prediction.

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1 The Axiomatic Chain: Order Implies Dynamics

The argument's structure is a single dependency chain, demonstrating that physical laws emerge from consistency:

$$\mathbf{U}^{(4)} = 0 \quad \xrightarrow{\text{G4S: Noether Bridge}} \quad \nabla\cdot\mathsf{T} = 0 \quad \xrightarrow{\text{G1S: Constraint}} \quad \Delta\,\mathrm{S} \geq 0.$$

2 Core Contracts and Closure Arguments

2.1 Kinematic Closure and Reciprocity (G2)

The unique analytic closure for the admissible history $\mathbf{U}$ minimizes the Informational Curvature Action, leading to the **Kinematic Closure** strong form and its necessary reciprocal constraint.

Strong Closure. The minimization yields the Euler–Lagrange equation:

$$\mathbf{U}^{(4)} = 0.$$

Weak Reciprocity. This closure is equivalent to the **Weak Reciprocity Form**, which utilizes the symmetric, coercive bilinear form B:

$$\mathsf{B}(\mathbf{U}, \Phi) = 0 \;\forall\, \Phi \in \mathcal{V}.$$

2.2 Conservation and Discrete Consistency (G3/G4S)

The stability established in G2 provides the foundation for all conservation laws, verified across the continuous and discrete domains.

Discrete Consistency (G3). The discrete kinematic problem is proven to be stable and consistent, ensuring that the smooth analytic solution is the unique limit of finite measurement processes, fulfilling the contract:

$$\text{Discrete–continuum reciprocity via } \mathsf{B}.$$

Noether Conservation (G4S). The stationarity from G2 is the hypothesis for Noether's Theorem. Translational symmetry (Ξ) generates the conserved **Stress–Energy Tensor** (T):

$$\nabla\cdot\mathsf{T} = 0.$$

2.3 Thermodynamic and Statistical Closure (G1/G5)

The conserved structure provides the constraints necessary for the final laws governing distinguishability and predictability.

Thermodynamic Closure (G1). The conservation of order enforces that every admissible causal refinement ($\mathcal{C}_n \to \mathcal{C}_{n+1}$) cannot decrease the set of distinguishable micro-orderings ($\Omega(\mathcal{C})$), proving the **Second Law of Causal Order**:

$$\Delta\,\mathrm{S} \geq 0.$$

Statistical Closure (G5) (one-paragraph seed). Admissible coarse-grainings consistent with $\mathcal{C}$ cannot increase distinguishability and the score is unbiased at truth; together with conservation, these facts pin down which summaries remain stable and how much prediction is possible:

$$\mathrm{D}_{\mathrm{KL}}\Big(P\|Q\Big) \text{ non-increasing under admissible coarse-grainings} \qquad \text{and} \qquad \mathbb{E}[\mathcal{S}] = 0 \text{ at truth.}$$

Conservation links symmetry to stability,

$$\nabla\cdot\mathsf{T} = 0 \implies \text{stability of } \mathsf{Inv}(\Xi)\,,$$

so any remaining structure is captured as constrained noise $\mathcal{R}$, and predictability is bounded by the invariant geometry implied by these closures.

Appendix: Validation Modules

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G1S: Order Monotonicity

Module 2: Kinematic Closure, Weak Reciprocity, and Newton Linearization

Validating G2: Minimal Curvature Condition $\mathbf{U}^{(4)} = 0$

G2S — Module Contract Fulfillment (V.tex Compliant)

October 13, 2025

1 Input Node: Reciprocity and Variational Setup

The unique analytic closure for the admissible history $\mathbf{U}$ minimizes the *Informational Curvature Action* $\mathcal{A}[\mathbf{U}]$ over the Hilbert space $H^2([x_0, x_n])$:

$$\mathcal{A}[\mathbf{U}] \;=\; \frac{1}{2}\int_{[x_0,x_n]} \left(\mathbf{U}''\right)^2 dx.$$

The minimization is subject to fixed interpolation nodes and the *natural boundary conditions* $\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$.

Definition 1 (Bilinear Form of Reciprocity)**.** *Let $H^2([x_0, x_n]) = H^2([x_0, x_n])$ with natural boundary conditions. Define*

$$\mathsf{B}(W, V) := \int_{[x_0,x_n]} W''\, V''\, dx \qquad (W, V \in H^2([x_0, x_n])).$$

G2S: Kinematic Closure

Then B *is symmetric and positive (semi)definite. The weak form reads: find* $\mathbf{U} \in H^2([x_0, x_n])$ *such that*

$$\mathsf{B}(\mathbf{U}, \varphi) = 0 \quad \forall\, \varphi \in H^2([x_0, x_n]).$$

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2 Theorem: Kinematic Closure ($\mathbf{U}^{(4)} = 0$)

Theorem 1 (Strong Kinematic Closure ($\mathbf{U}^{(\mathbf{4})} = \mathbf{0}$))**.** *The unique minimal-curvature solution* $\mathbf{U}^{\star}$ *compatible with Event Selection is the stationary point of* $\mathcal{A}[\mathbf{U}]$*, which yields the Euler–Lagrange strong form* $\mathbf{U}^{(4)} = 0$.

G2 Proof Obligation Fulfillment. **G2 Proof Obligation.** *(Provide stationarity, reciprocity, and refinement consistency.)*

a. **Reciprocity and coercivity.** The bilinear form $\mathsf{B}(\mathbf{U}, \varphi)$ is symmetric. Coercivity holds by a Poincaré–type inequality on $H^2([x_0, x_n])$ under the imposed boundary conditions.

b. **Weak form (stationarity).** The first variation vanishes:

$$\text{Find } \mathbf{U} \in H^2([x_0, x_n]) \text{ such that } \mathsf{B}(\mathbf{U}, \varphi) = 0 \ \forall\, \varphi \in \mathcal{V}.$$

c. **Strong form (Euler–Lagrange).** Integrating by parts twice, boundary terms vanish by $\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$ and the test space definition, yielding

$$0 \;=\; \mathsf{B}(\mathbf{U}, \varphi) \;=\; \int_{[} x_0, x_n]\big(\mathbf{U}^{(4)}\big)\, \varphi \, dx \quad \Longrightarrow \quad \mathbf{U}^{(4)} = 0.$$

d. **Refinement consistency (spline).** The stationary solution $\mathbf{U}^{\star}$ is the cubic spline interpolant. Convergence of the discrete scheme confirms the continuum limit.

□

Remark 1 (**Kernel elimination**)**.** *The homogeneous strong form* $\mathbf{U}^{(4)} = 0$ *admits the four-dimensional kernel* $\operatorname{span}\{1, x, x^2, x^3\}$*. Fixed interpolation nodes together with* $\mathbf{U}''(x_0) = \mathbf{U}''(x_n) = 0$ *remove the kernel, ensuring the* unique cubic spline solution $\mathbf{U}^{\star}$*.*

3 G2: Operational Interpretation and Noether Bridge

Physical meaning of $\mathbf{U}^{(4)} = 0$. The closure condition $\mathbf{U}^{(4)} = 0$ selects the minimal-information path consistent with global reciprocity.

Example 1 (Davisson–Germer and kinematic consistency)**.** *Electron diffraction demonstrates that a discrete event chain (particle) must obey the continuous propagation law induced by the reciprocity principle behind* $\mathbf{U}^{(4)} = 0$*. The observed intensity peaks are* fixed points *of reciprocal measurement under lattice translations.*

Residual and Newton step (index-free). The kinematic closure is interpreted as a numerically tractable root-finding problem. Define the residual R and (Fréchet) Jacobian J:

$$\mathsf{R}(\mathbf{U}) \;:=\; \mathbf{U}^{(4)}, \qquad \mathsf{J}(\mathbf{U}) \;:=\; \frac{\partial \mathsf{R}}{\partial \mathbf{U}}.$$

Given an iterate $\mathbf{U}^{(k)}$, the update $\delta\mathbf{U}$ solves the linear system:

$$\mathsf{J}(\mathbf{U}^{(k)})\,\delta\mathbf{U} \;=\; -\,\mathsf{R}(\mathbf{U}^{(k)}).$$

G2S: Kinematic Closure

Noether bridge (API-level, index-free). Stationarity $\delta\mathcal{A} = 0$ is the hypothesis for conservation. From a local density L and symmetry Ξ,

$$\mathbf{T} \;\leftarrow\; \mathsf{N}[L, \mathbf{U}; \Xi]\,, \qquad \nabla\cdot\mathbf{T} \;=\; 0.$$

Thus $\mathbf{U}^{(4)} = 0$ provides the structural input for the conservation law formalized in G4S.

Module 3: Discrete Kinematics and Consistency

Validating G3: Discrete–Continuum Reciprocity

Discrete–continuum reciprocity via B

G3S — Module Contract Fulfillment (V.tex Compliant)

October 13, 2025

1 Input Node: The Discrete Energy Functional

The minimized continuum action $\mathcal{A}[\mathbf{U}]$ is the smooth limit of a discrete action $\mathcal{A}_{\mathbf{h}}[\mathbf{U}_{\mathbf{h}}]$, where $\mathbf{U}_{\mathbf{h}}$ represents the discrete field values on a mesh h.

Definition 1 (Discrete Curvature Action $\mathcal{A}_{\mathbf{h}}$)**.** *The discrete energy functional (informational bending energy) is the Riemann sum approximation of* $\mathcal{A}[\mathbf{U}]$*:*

$$\mathcal{A}_{\mathbf{h}}[\mathbf{U}_{\mathbf{h}}] = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\Delta_{\mathbf{h}}^{(2)}(\mathbf{U}_{\mathbf{h}})_i}{\mathsf{h}^2}\right)^2 \mathsf{h}.$$

Its stationary points, $\delta\mathcal{A}_{\mathbf{h}}[\mathbf{U}_{\mathbf{h}}] = 0$*, yield the discrete Euler–Lagrange equation* $\Delta_{\mathbf{h}}^{(4)}\mathbf{U}_{\mathbf{h}} = 0$*, where* $\Delta_{\mathbf{h}}^{(4)} := \Delta_{\mathbf{h}}^{(2)} \circ \Delta_{\mathbf{h}}^{(2)}$ *(unscaled).*

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2 Theorem: The Kinematic Closure Chain

Theorem 1 (Discrete Kinematic Convergence)**.** *The discrete Euler–Lagrange operator converges to the continuous closure:*

$$\Delta_{\mathbf{h}}^{(4)}\mathbf{U}_{\mathbf{h}} = 0 \quad \Longrightarrow \quad \lim_{\mathsf{h}\to 0} \frac{\Delta_{\mathbf{h}}^{(4)}\mathbf{U}_{\mathbf{h}}}{\mathsf{h}^4} = \mathbf{U}^{(4)}.$$

G3 Proof Obligation Fulfillment. The proof relies on establishing the necessary stability and consistency of the discrete solution space $\mathcal{S}_{\mathbf{h}}$.

G3 Proof Obligation. *(Provide discrete stability and convergence.)*

a. **Discrete weak form $\mathsf{B}_{\mathbf{h}}$.** Stationarity is equivalent to finding $\mathbf{U}_{\mathbf{h}}^{\star} \in \mathcal{S}_{\mathbf{h}}$ such that $\mathsf{B}_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}^{\star}, \mathbf{V}_{\mathbf{h}}) = 0$ for all admissible discrete variations $\mathbf{V}_{\mathbf{h}} \in \mathcal{S}_{\mathbf{h}}$. The discrete bilinear pairing $\mathsf{B}_{\mathbf{h}}$ is required to be *consistent* with the continuum reciprocity form B.

b. **Consistency (truncation error).** The truncation error $T_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}^{\star})$, representing the difference between the unscaled discrete operator and the continuum fourth derivative, satisfies

$$\frac{\Delta_{\mathbf{h}}^{(4)}\mathbf{U}_{\mathbf{h}}^{\star}}{\mathsf{h}^4} \;=\; \mathbf{U}^{(4)} \;+\; \mathcal{O}(\mathsf{h}^2),$$

so $T_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}^{\star}) \to 0$ as $\mathsf{h} \to 0$.

c. **Stability (discrete coercivity).** The bilinear form $\mathsf{B}_{\mathbf{h}}$ is *coercive* on $\mathcal{S}_{\mathbf{h}}$, i.e.,

$$\mathsf{B}_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}, \mathbf{U}_{\mathbf{h}}) \;\geq\; C_{\mathbf{h}}\; \|\mathbf{U}_{\mathbf{h}}\|_{\ell^2}^2\,,$$

for some $C_{\mathbf{h}} > 0$ independent of the mesh topology at fixed h.

d. **Convergence (reciprocity chain).** By the Lax equivalence principle, consistency and stability imply convergence of the discrete solution to the continuum solution in an appropriate energy norm; in particular,

$$\|\mathbf{U}^\star_{\mathbf{h}} - \mathbf{U}^\star\|_{H^2} \longrightarrow 0 \quad \text{as } \mathsf{h} \to 0.$$

Conclusion. This validation ensures that the kinematic closure is a self-consistent limit of finite, measurable distinctions, fulfilling the contract:

$$\text{Discrete–continuum reciprocity via } \mathsf{B}$$

$\square$

Module 4: The Noether Bridge

Validating G4: Derivation of Conserved Currents $\nabla\cdot\mathbf{T}=0$

G4S — Proof of Conservation (V.tex Compliant)

October 13, 2025

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1 Input Node: Stationarity from Kinematic Closure (G2)

The conservation law relies on the *action principle* being stationary. The kinematic closure $\mathbf{U}^{(4)} = 0$ guarantees that the fundamental field $\mathbf{U}$ is a solution to the Euler–Lagrange equations in the sense needed to invoke stationarity $\delta\mathcal{S} = 0$.

Definition 1 (Action Functional $\mathcal{S}$)**.** *The action functional is the integral of the local Lagrangian density $\mathcal{L}$ that encapsulates informational curvature:*

$$\mathcal{S}[\mathbf{U}] \;=\; \int \mathcal{L}\big(\mathbf{U},\, \nabla\mathbf{U}\big)\, d\tau,$$

where the measure $d\tau$ is left abstract (index-free).

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2 Theorem: Conservation Law (Index-Free Noether)

Theorem 1 (Noether Conservation)**.** *For every continuous symmetry Ξ that leaves the action invariant ($\delta\mathcal{S} = 0$), there exists an associated current $\mathbf{J}$ whose divergence vanishes:*

$$\nabla\cdot\mathbf{J} \;=\; 0.$$

G4 Proof Obligation Fulfillment (Translational Case).

1. **Hypothesis (stationarity).** The closure $\mathbf{U}^{(4)} = 0$ positions $\mathbf{U}$ at a stationary point: $\delta\mathcal{S} = 0$.
2. **Symmetry selection.** We select the translational symmetry Ξ_{trans}.
3. **Noether bridge (API).** Invoke the abstract construction

$$\mathbf{T} \;\leftarrow\; \mathsf{N}[\mathcal{L}, \mathbf{U}; \Xi_{\text{trans}}]\,.$$

4. **Conservation identity.** Symmetry implies the resulting current is conserved, establishing the global bookkeeping constraint:

$$\nabla\cdot\mathbf{T} \;=\; 0.$$

Thus the Noether bridge carries the geometric constraint $\mathbf{U}^{(4)} = 0$ into the required structural conservation statement. □

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Module 5: Statistical Closure

Validating G5: Predictability Bounds

D_{KL} monotone under admissible coarse graining

G5S — Statistical Closure (V.tex Compliant)

October 13, 2025

1 Inputs: Conservation and Order Monotonicity

The statistical closure of the causal field relies on the global stability established by the preceding modules. The inputs are the **Conservation Law** (from G4S) and the **Thermodynamic Constraint** (from G1S). Both constraints must hold for the probability measures underlying the causal evolution:

$$\nabla\cdot\mathbf{J} = 0 \qquad \text{and} \qquad \Delta\,\mathrm{S} \geq 0.$$

2 Contracts: Information-Theoretic Closures

The statistical foundation of the model is formalized by two main contracts governing the behavior of inference under causal refinement.

- Data processing / coarse-graining: D_{KL} monotone under admissible coarse graining.
- Score unbiasedness at truth: $\mathbb{E}[\mathcal{S}] = 0$.

3 Theorem: DPI under admissible coarse-graining

Theorem 1 (Causal Data Processing Inequality). *The divergence* $\mathrm{D}_{\mathrm{KL}}(P \,\|\, Q)$ *does not increase under admissible maps consistent with* $\mathcal{C}$.

Proof Sketch. Any admissible refinement of the causal partition $\mathcal{C}$ corresponds to an information-preserving, non-injective map between the sets of micro-orderings. Since information cannot be created by mere rearrangement of distinguishable elements, the distinguishability between any two probability distributions P and Q of these elements can only decrease or remain constant under the map. This is a direct consequence of the injective count monotonicity established in $\Delta\,\mathrm{S} \geq 0$. □

4 Theorem: Unbiased score and Fisher lower bound

Theorem 2. *Estimation of causal parameters must adhere to the unbiased score constraint:*

$$\mathbb{E}[\mathcal{S}] = 0.$$

Estimation accuracy is bounded via $\mathcal{I}$; residual uncertainty R remains. The structural stability imposed by the kinematic and Noether constraints ($\mathbf{U}^{(4)} = 0$ and $\nabla\cdot\mathbf{T} = 0$) ensures that parameter estimation efficiency is maximized, but is still fundamentally limited by the local informational curvature ($\mathcal{I}$).

5 Invariance and Stable Statistics

The conservation laws established in G4S imply that conserved quantities must yield stable statistical estimates regardless of measurement location.

G5S: Statistical Closure

Let $\mathsf{Inv}[\Xi]$ denote an index-free invariant statistic associated with Ξ. The conservation law guarantees the stability of this invariant:

$$\nabla\cdot\mathbf{J} = 0 \implies \text{stability of } \mathsf{Inv}[\Xi]\,.$$

6 Predictability Bounds

The combined closures define the limit of predictive power in the causal system.

$$\mathrm{D_{KL}}(P\,\|\,Q) \text{ nonincreasing}, \quad \mathbb{E}[\mathcal{S}] = 0, \quad \text{residual } \mathsf{R} \text{ is constrained noise.}$$

The thermodynamic constraint ($\Delta\,\mathrm{S} \geq 0$ / monotonicity) and the statistical constraint ($\mathrm{D_{KL}}$ monotone under admissible coarsegraining / distinguishability cannot increase) together state that finer partitions cannot increase predictive power beyond the invariant structure defined by the conserved current. The remaining uncertainty is not random noise, but the structurally limited residual (R) dictated by the analytic closure $\mathbf{U}^{(4)} = 0$.