Module 1: Order Monotonicity and The Second Law

Validating G1: Thermodynamic Closure $\Delta \mathrm{S} \geq 0$

G1S — Module Contract Fulfillment (V.tex Compliant)

October 13, 2025

1 Axiomatic Foundation: Causal Order and Event Selection

The proof begins by grounding the concepts of entropy and distinction within ZFC set theory, as outlined in `P.tex`[cite: 202, 211].

Definition 1 (Causal Set and Admissible Micro-Orderings)**.** *Let be the causal set of distinguishable events accessible to an observer[cite: 881, 1004]. Let $\Omega()$ be the set of all **admissible micro-orderings** of those events consistent with the Axiom of Event Selection (Martin-like consistency)[cite: 81, 881, 1328].*

Definition 2 (Causal Entropy S)**.** *The* **Entropy** *associated with a causal set is the logarithm of this count, where k_{B} is Boltzmann's constant:*

$$\mathrm{S} = k_{\mathrm{B}} \ln |\Omega()|.$$

Operationally, S *measures the number of distinct internal configurations that yield the same observable causal invariants[cite: 883].*

2 Theorem: The Second Law of Causal Order ($\Delta\,\mathrm{S} \geq 0$)

The principle that information, and thus entropy, can never decrease is proved as a theorem of causal consistency rather than assumed as a postulate[cite: 78, 79].

Theorem 1 (Monotonicity of Causal Entropy ($\boldsymbol{\Delta}\,\mathrm{S} \geq \mathbf{0}$))**.** *In any extension of a finite causal order that remains globally consistent (via Event Selection/Martin-like consistency), the count of distinguishable states cannot decrease.*

$$\Delta\,\mathrm{S} \geq 0.$$

G1 Proof Obligation Fulfillment.

1. **Causal Refinement:** Let $_n$ be a causal set, and $_{n+1}$ be an admissible extension (refinement) that introduces new distinctions while remaining consistent with the Axiom of Event Selection[cite: 1306]. By definition, $_n \subseteq_{n+1}$.

2. **Monotonicity of Ω:** Since the refinement preserves the partial order and is globally consistent, every micro-ordering that was admissible in $_n$ must remain admissible in $_{n+1}$[cite: 92]. Therefore, the set of admissible orderings cannot shrink:

$$\Omega(_n) \subseteq \Omega(_{n+1}).$$

3. **Logarithmic Closure:** Taking the natural logarithm and multiplying by k_B:

$$k_\mathrm{B} \ln |\Omega(_n)| \leq k_\mathrm{B} \ln |\Omega(_{n+1})|.$$

4. **Conclusion (Entropy Invariance):** This directly implies that the

entropy is non-decreasing[cite: 1307, 1308]:

$$\mathrm{S}_{n+1} - \mathrm{S}_{n} = \Delta \mathrm{S} \geq 0.$$

The equality holds only if the refinement $_{n+1}$ does not expose any previously indistinguishable configurations. Thus, the Arrow of Time is a theorem of order monotonicity[cite: 93]. $\square$