Module 1/2 Bridge: Variational Closure

Validating G4/G5: Existence and Boundary Conditions

Independent Logical Section: Validating G4

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1 Input Node: The Reciprocity Functional

The system seeks minimal informational curvature, defined by the functional $\mathcal{R}[\mathbf{U}]$. The core principle of Reciprocity (Module 1) requires that every variation $\delta\mathcal{R}$ must correspond to a measurable distinction Φ. The canonical closure is achieved when the first variation vanishes, $\delta\mathcal{R} = 0$.

The functional is defined as the integral of the squared informational curvature $\mathcal{R}[\mathbf{U}] = \int \mathcal{L}(\mathbf{U}'')\, dx$, where $\mathcal{L}(\mathbf{U}'') = \frac{1}{2}(\mathbf{U}'')^2$.

Proposition 1 (License for Variational Principle (Validating **G5**))**.** *The use of the continuous integral $\mathcal{R}[\mathbf{U}]$ is licensed by the **Axiom of Event Selection** (Martin-like consistency,* **G5***), which guarantees the non-constructive existence of a globally consistent extension. This existence allows the discrete reciprocal measure to be approximated by a continuous, minimal-curvature functional $\mathcal{R}[\mathbf{U}]$. This axiom is the required **Continuity Boundary**.*

2 Formal Variation and Integration by Parts

To obtain the Euler-Lagrange condition (the input for G2/G3), we compute the first variation $\delta\mathcal{R}$ over an interval $[x_1, x_2]$ and apply integration by parts. This step formally validates the necessary boundary terms and signs for **G4**.

Proposition 2 (Closure of the Reciprocity Functional (**G4**)). *The first variation of the functional $\mathcal{R}[\mathbf{U}]$ is:*

$$\delta\mathcal{R} = \int_{x_1}^{x_2} \left(\frac{\partial\mathcal{L}}{\partial\mathbf{U}} - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{U}'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{U}''}\right)\right) \delta\mathbf{U}\, dx + \textit{Boundary Terms.}$$

For $\mathcal{L} = \frac{1}{2}(\mathbf{U}'')^2$, the Euler-Lagrange equation (the integrand) reduces to $\mathbf{U}^{(4)} = 0$ (the input to G2).

3 The Boundary Terms and Physical Constraint

The **Boundary Terms** resulting from the integration by parts must vanish for $\delta\mathcal{R} = 0$ to hold for arbitrary internal variations ($\delta\mathbf{U}$):

$$\text{Boundary Terms} = \left[\frac{\partial\mathcal{L}}{\partial\mathbf{U}'} - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{U}''}\right)\right]_{x_1}^{x_2} \delta\mathbf{U} + \left[\frac{\partial\mathcal{L}}{\partial\mathbf{U}''}\right]_{x_1}^{x_2} \delta\mathbf{U}'$$

Substituting $\mathcal{L} = \frac{1}{2}(\mathbf{U}'')^2$:

$$\text{Boundary Terms} = [-\mathbf{U}''']_{x_1}^{x_2}\, \delta\mathbf{U} + [\mathbf{U}'']_{x_1}^{x_2}\, \delta\mathbf{U}' = 0$$

Interpretation (Validating G4): The vanishing of these terms enforces the dual system's closure at the measurement anchors (x_1, x_2). They confirm that the variation ($\delta\mathbf{U}$) and its slope ($\delta\mathbf{U}'$) at the boundaries must be either **fixed by measurement** (Dirichlet/Neumann conditions) or that the **natural conditions** ($\mathbf{U}'' = 0$ and $\mathbf{U}''' = 0$) hold. This ensures the correct sign convention and closure needed to enforce $\mathbf{U}^{(4)} = 0$ (G2) consistently across finite intervals.

4 Output Node: Input to Kinematic Closure

The successful closure of $\delta\mathcal{R} = 0$ via the necessary boundary terms (G4) confirms the validity of the **Euler-Lagrange Equation** $\mathbf{U}^{(\mathbf{4})} = \mathbf{0}$, which is the foundational starting point for Module 2 (Chapter 3, G2/G3).

Conclusion for G4/G5: The continuity of the system is logically guaranteed and mechanically closed exactly at the boundaries of observation.