

Module 3: Discrete Kinematics and Consistency

Validating G3: Discrete–Continuum Reciprocity

Discrete–continuum reciprocity via B

G3S — Module Contract Fulfillment (V.tex Compliant)

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1 Input Node: The Discrete Energy Functional

The minimized continuum action $\mathcal{A}[\mathbf{U}]$ is the smooth limit of a discrete action $\mathcal{A}_{\mathbf{h}}[\mathbf{U}_{\mathbf{h}}]$, where $\mathbf{U}_{\mathbf{h}}$ represents the discrete field values on a mesh \mathbf{h} .

Definition 1 (Discrete Curvature Action $\mathcal{A}_{\mathbf{h}}$). *The discrete energy functional (informational bending energy) is the Riemann sum approximation of $\mathcal{A}[\mathbf{U}]$:*

$$\mathcal{A}_{\mathbf{h}}[\mathbf{U}_{\mathbf{h}}] = \frac{1}{2} \sum_{i=1}^N \left(\frac{\Delta_{\mathbf{h}}^{(2)}(\mathbf{U}_{\mathbf{h}})_i}{h^2} \right)^2 h.$$

Its stationary points, $\delta\mathcal{A}_{\mathbf{h}}[\mathbf{U}_{\mathbf{h}}] = 0$, yield the discrete Euler–Lagrange equation $\Delta_{\mathbf{h}}^{(4)}\mathbf{U}_{\mathbf{h}} = 0$, where $\Delta_{\mathbf{h}}^{(4)} := \Delta_{\mathbf{h}}^{(2)} \circ \Delta_{\mathbf{h}}^{(2)}$ (unscaled).

2 Theorem: The Kinematic Closure Chain

Theorem 1 (Discrete Kinematic Convergence). *The discrete Euler–Lagrange operator converges to the continuous closure:*

$$\Delta_{\mathbf{h}}^{(4)} \mathbf{U}_{\mathbf{h}} = 0 \quad \implies \quad \lim_{\mathbf{h} \rightarrow 0} \frac{\Delta_{\mathbf{h}}^{(4)} \mathbf{U}_{\mathbf{h}}}{\mathbf{h}^4} = \mathbf{U}^{(4)}.$$

G3 Proof Obligation Fulfillment. The proof relies on establishing the necessary stability and consistency of the discrete solution space $\mathcal{S}_{\mathbf{h}}$.

G3 Proof Obligation. *(Provide discrete stability and convergence.)*

- a. **Discrete weak form $\mathbf{B}_{\mathbf{h}}$.** Stationarity is equivalent to finding $\mathbf{U}_{\mathbf{h}}^* \in \mathcal{S}_{\mathbf{h}}$ such that $\mathbf{B}_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}^*, \mathbf{V}_{\mathbf{h}}) = 0$ for all admissible discrete variations $\mathbf{V}_{\mathbf{h}} \in \mathcal{S}_{\mathbf{h}}$. The discrete bilinear pairing $\mathbf{B}_{\mathbf{h}}$ is required to be *consistent* with the continuum reciprocity form \mathbf{B} .
- b. **Consistency (truncation error).** The truncation error $T_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}^*)$, representing the difference between the unscaled discrete operator and the continuum fourth derivative, satisfies

$$\frac{\Delta_{\mathbf{h}}^{(4)} \mathbf{U}_{\mathbf{h}}^*}{\mathbf{h}^4} = \mathbf{U}^{(4)} + \mathcal{O}(\mathbf{h}^2),$$

so $T_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}^*) \rightarrow 0$ as $\mathbf{h} \rightarrow 0$.

- c. **Stability (discrete coercivity).** The bilinear form $\mathbf{B}_{\mathbf{h}}$ is *coercive* on $\mathcal{S}_{\mathbf{h}}$, i.e.,

$$\mathbf{B}_{\mathbf{h}}(\mathbf{U}_{\mathbf{h}}, \mathbf{U}_{\mathbf{h}}) \geq C_{\mathbf{h}} \|\mathbf{U}_{\mathbf{h}}\|_{\ell^2}^2,$$

for some $C_{\mathbf{h}} > 0$ independent of the mesh topology at fixed \mathbf{h} .

- d. **Convergence (reciprocity chain).** By the Lax equivalence principle, consistency and stability imply convergence of the discrete solution to

the continuum solution in an appropriate energy norm; in particular,

$$\|\mathbf{U}_{\mathbf{h}}^* - \mathbf{U}^*\|_{H^2} \longrightarrow 0 \quad \text{as } h \rightarrow 0.$$

Conclusion. This validation ensures that the kinematic closure is a self-consistent limit of finite, measurable distinctions, fulfilling the contract:

Discrete–continuum reciprocity via \mathcal{B}

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