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# Novel distributed robust adaptive consensus protocols for linear multi-agent systems with directed graphs and external disturbances

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**ABSTRACT**

This paper addresses the distributed consensus protocol design problem for linear multi-agent systems with directed graphs and external unmatched disturbances. Novel distributed adaptive consensus protocols are proposed to achieve leader–follower consensus for any directed graph containing a directed spanning tree with the leader as the root node and leaderless consensus for strongly connected directed graphs. It is pointed out that the adaptive protocols involve undesirable parameter drift phenomenon when bounded external disturbances exist. By using the $\sigma$ modification technique, distributed robust adaptive consensus protocols are designed to guarantee the ultimate boundedness of both the consensus error and the adaptive coupling weights in the presence of external disturbances. All the adaptive protocols in this paper are fully distributed, relying on only the agent dynamics and the relative states of neighbouring agents.

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## 1. Introduction

In recent years, the consensus problem of multi-agent systems has been an emerging research topic in the field of control, due to its wide applications in many areas such as satellite formation flying, cooperative unmanned systems, and distributed reconfigurable sensor networks (Ren, Beard, & Atkins, 2007). There has been remarkable progress of solving consensus problems for different scenarios; see Hong, Chen, & Bushnell (2008); Guo & Dimarogonas (2013); Li, Fu, Xie, & Zhang (2011); Olfati-Saber & Murray (2004); Ren et al. (2007) and the references therein. For a consensus problem, the critical task is to design distributed consensus protocols based on local information, i.e. local state or output information of each agent and its neighbours.

In this paper, we consider the consensus problem of multi-agent systems with general linear time invariant dynamics. Previous works (Li, Duan, & Chen, 2011; Li, Duan, Chen, & Huang, 2010; Ma & Zhang, 2010; Seo, Shim, & Back, 2009; Tuna, 2009; Zhang, Lewis, & Das, 2011) have presented various static and dynamic consensus protocols, which are proposed in a distributed fashion, using only the local information of each agent and its neighbours. However, those consensus protocols involves some design issues. To be specific, the design of those consensus protocols generally requires the knowledge of some eigenvalue information of the Laplacian matrix associated with the communication graph (the smallest nonzero eigvenvalue of the Laplacian matrix for undirected graphs and the smallest real part of the nonzero eigenvalues of the Laplacian matrix for directed graphs). Note that the nonzero eigenvalue information of the Laplacian matrix is global information in the sense that each agent has to know the entire communication graph to compute it. Therefore, although these consensus protocols are proposed and can be implement in a distributed fashion, they cannot be designed by each agent in a distributed fashion. In other words, those consensus protocols in Li et al. (2010), Li et al. (2011), Tuna (2009), Seo et al. (2009), Zhang et al. (2011), and Ma and Zhang (2010) are not fully distributed.

To remove the limitation of requiring global information of the communication graph, distributed adaptive consensus protocols are reported in Li, Ren, Liu, and Xie (2013), Li, Ren, Liu, and Fu (2013), Su, Chen, Wang, and Lin (2011), and Yu, Ren, Zheng, Chen, and Lv (2013), which, depending on only local information of each agent and its neighbours, are fully distributed. It is worth noting that the adaptive protocols in Li, Ren, Liu, and Xie (2013), Li, Ren, Liu, and Fu (2013), Su et al. (2011), and Yu et al. (2013) are applicable to only undirected communication graphs or leader–follower graphs where the subgraphs among followers are undirected. Due to the asymmetry of the Laplacian matrices, it is much more

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difficult to design distributed adaptive consensus protocols for general directed communication graphs. By including monotonically increasing functions to provide additional design flexibility, a distributed adaptive consensus protocol is derived in Li, Wen, Duan, and Ren (2015) to achieve consensus for general leader–follower directed graphs containing a directed spanning tree. The robustness of the distributed adaptive protocols with respect to uncertainties or external disturbances is an important issue. The adaptive protocol in Li et al. (2015) can only be modified to be applicable to external disturbances satisfying the matching condition (Li & Duan, 2014). To the best of our knowledge, how to design distributed robust adaptive consensus protocols for the case with directed graphs and general disturbances is still open.

In this paper, we extend (Li et al., 2015) to design distributed robust adaptive consensus protocols for linear multi-agent systems with directed communication graphs. Novel distributed adaptive protocols are presented and shown to achieve both leader–follower consensus for directed communication graphs containing a directed spanning tree with leader as the root node and leaderless consensus for strongly connected directed graphs. The novel adaptive protocols are fully distributed, relying on only the agent dynamics and the relative state information of neighbouring agent. In the presence of external disturbances, it is pointed out that the adaptive protocols involve the parameter drift phenomenon (Ioannou & Sun, 1996). Therefore, the adaptive protocols are not robust in the presence of external disturbances. To deal with this instability issue, robust adaptive protocols using the $\sigma$ modification technique are presented, which can guarantee the ultimate boundedness of both the consensus error and the adaptive coupling weights. The existence condition of the proposed adaptive protocols are also discussed. Compared to the previous works of Li et al. (2015) and Li and Duan (2014), the contribution of this paper is at least twofold. First, the adaptive protocols proposed in this paper replace the multiplicative functions in the adaptive protocol in Li et al. (2015) by novel additive functions. In this case, a simple quadratic-like Lyapunov function, rather than the complicated integral-like one in Li et al. (2015), can be used to derive the result. The involved derivations are much simplified. Second, in contrast to the adaptive protocol in Li and Duan (2014) which works only for the case with disturbances satisfying the restrictive matching condition, the robust adaptive consensus protocols given in this paper are applicable to the case of general bounded disturbances. It should be mentioned that the methods used to derive the results in this paper is quite different from those in Li et al. (2015) and Li and Duan (2014).

The rest of this paper is organised as follows. The mathematical preliminaries used in this paper is summarised in Section 2. The distributed adaptive consensus protocol is designed in Section 3 for general directed leader–follower graphs, while for leaderless directed graphs, distributed adaptive consensus protocol is proposed in Section 4. Novel robust adaptive consensus protocols are presented in Section 5 to deal with external disturbances. Simulation results are presented in Section 6. Section 7 concludes this paper.

## 2. Mathematical preliminaries

Let $\mathbf{R}^{n\times m}$ be the set of $n \times m$ real matrices and the superscript $T$ denotes transpose for real matrices. $I_N$ represents the identity matrix of dimension $N$ and $I$ denotes the identity matrix of an appropriate dimension. $\mathbf{1}$ denotes a column vector with all entries equal to 1. diag$(a_1, \dots, a_N)$ represents a diagonal matrix with elements $a_i$, $i = 1, \dots, N$, on its diagonal while $\lambda_{\min}(A)$ denotes the minimal eigenvalue of $A$. The matrix inequality $A > B$ means $A$ and $B$ are symmetric matrices and $A - B$ is positive definite. $A \otimes B$ represents the Kronecker product of matrices $A$ and $B$. A nonsingular $M$-matrix $A = [a_{ij}]$ means that $a_{ij} < 0$, $i \neq j$, and all eigenvalues of $A$ have positive real parts.

A directed graph $\mathcal{G}$ consists a node set $\mathcal{V}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, in which an edge is represented by an ordered pair of distinct nodes. For an edge $(v_i, v_j)$, node $v_i$ is called the parent node, node $v_j$ the child node, and $v_i$ is a neighbour of $v_j$. A path from node $v_{i_1}$ to node $v_{i_l}$ is a sequence of ordered edges of the form $(v_{i_k}, v_{i_{k+1}})$, $k = 1, \dots, l-1$. A directed graph contains a directed spanning tree if there exists a node called the root such that the node has directed paths to all other nodes in the graph. A directed graph is strongly connected if there exists a directed path between every pair of distinct nodes. A directed graph has a directed spanning tree if it is strongly connected, but not vice versa.

Suppose there are $N$ nodes in the directed graph $\mathcal{G}$. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ of $\mathcal{G}$ is defined by $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and 0 otherwise. The Laplacian matrix $L = [l_{ij}] \in \mathbf{R}^{N\times N}$ is defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$.

**Lemma 2.1** (Ren & Beard, 2005)**:** *Zero is an eigenvalue of $L$ with $\mathbf{1}$ as a right eigenvector and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $L$ if and only if $\mathcal{G}$ has a directed spanning tree.*

**Lemma 2.2** (Mei, Ren, & Chen, 2014)**:** *Suppose that $\mathcal{G}$ is strongly connected. Let $r = [r_1, \dots, r_N]$ be the positive left eigenvector of $L$ associated with the zero eigenvalue and*

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*$R = diag(r_1, \ldots, r_N)$. Then, $\hat{L} \triangleq RL + L^T R$ is the symmetric Laplacian matrix associated with an undirected connected graph. Let $\xi$ be any vector with positive entries. Then, $\min_{\xi^T x=0, x\neq 0} \frac{x^T \hat{L} x}{x^T x} > \frac{\lambda_2(\hat{L})}{N}$, where $\lambda_2(\hat{L})$ denotes the smallest nonzero eigenvalue of $\hat{L}$.*

**Lemma 2.3** (Qu, 2009; Zhang, Lewis, & Qu, 2012)**:** *Consider a nonsingular M-matrix L. There exists a diagonal matrix G so that $G \equiv diag(g_1, \ldots, g_N) > 0$ and $GL + L^T G > 0$.*

**Lemma 2.4** (Bernstein, 2009)**:** *If a and b are nonnegative real numbers and p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, equality holds if and only if $a^p = b^q$.*

**Lemma 2.5** (Corless & Leitmann, 1981)**:** *For a system $\dot{x} = f(x, t)$, where $f(\cdot)$ is locally Lipschitz in x and piecewise continuous in t, assume that there exists a continuously differentiable function $V(x, t)$ such that along any trajectory of the system,*

$$\alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|),$$
$$\dot{V}(x,t) \leq -\alpha_3(\|x\|) + \epsilon,$$

*where $\varepsilon > 0$, $\alpha_1$ and $\alpha_2$ are class $\mathcal{K}_\infty$ functions, and $\alpha_3$ is a class $\mathcal{K}$ function. Then, the solution $x(t)$ of $\dot{x} = f(x, t)$ is uniformly ultimately bounded.*

## 3. Leader–follower distributed adaptive consensus protocol design

Consider a group of $N + 1$ identical agents with general linear dynamics, consisting of $N$ followers and a leader. The dynamics of the $i$th agent are described by

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 0, \ldots, N, \tag{1}$$

where $x_i \in \mathbf{R}^n$ is the state, $u_i \in \mathbf{R}^p$ is the control input, $A$ and $B$ are constant matrices with compatible dimensions.

Without loss of generality, let the agent in (1) indexed by 0 be the leader whose control input is assumed to be zero, i.e. $u_0 = 0$, and the other agents be the followers. The communication graph $\mathcal{G}$ among the $N + 1$ agents is assumed to satisfy the following assumption.

**Assumption 3.1:** *The graph $\mathcal{G}$ contains a directed spanning tree with the leader as the root node.*

Under Assumption 3.1, the Laplacian matrix $L$ associated with $\mathcal{G}$ can be partitioned as $L = \begin{bmatrix} 0 & 0_{1\times N} \\ L_2 & L_1 \end{bmatrix}$. In light of Lemma 2.1 and the definition of $M$-matrix, it is easy to verify that $L_1 \in R^{N\times N}$ is a nonsingular $M$-matrix.

The objective of this section is to design distributed consensus protocols such that the $N$ agents in (1) achieve leader–follower consensus in the sense of $\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0$, $\forall\, i = 1, \ldots, N$.

Based on the relative states of neighbouring agents, we propose a distributed adaptive consensus protocol to each follower as

$$u_i = (c_i + \rho_i)K\xi_i,$$
$$\dot{c}_i = \xi_i^T \Gamma \xi_i, \quad i = 1, \ldots, N, \tag{2}$$

where $\xi_i \triangleq \sum_{j=0}^N a_{ij}(x_i - x_j)$, $c_i(t)$ denotes the time-varying coupling weight associated with the $i$th follower with $c_i(0) \geq 0$, $K \in \mathbf{R}^{p\times n}$ and $\Gamma \in \mathbf{R}^{n\times n}$ are the feedback gain matrices, and $\rho_i$ are smooth functions to be determined.

Let $\xi \triangleq [\xi_1^T, \ldots, \xi_N^T]^T$ and $x \triangleq [x_1^T, \ldots, x_N^T]^T$. Then, we get

$$\xi = (L_1 \otimes I_n)(x - \mathbf{1} \otimes x_0). \tag{3}$$

Since the graph $\mathcal{G}$ satisfies Assumption 3.1, it follows from Lemma 2.1 that $L_1$ is a nonsingular $M$-matrix and that the leader–follower consensus problem is solved if and only if $\xi$ asymptotically converges to zero. Hereafter, we refer to $\xi$ as the consensus error. Substituting (2) into (1), we can get that $\xi$ and $c_i$ satisfy the following dynamics:

$$\dot{\xi} = [I_N \otimes A + L_1(C + \rho) \otimes BK]\,\xi,$$
$$\dot{c}_i = \xi_i^T \Gamma \xi_i, \quad i = 1, \ldots, N, \tag{4}$$

where $C \triangleq \mathrm{diag}(c_1, \ldots, c_N)$ and $\rho \triangleq \mathrm{diag}(\rho_1, \ldots, \rho_N)$.

The following result provides a sufficient condition to design the adaptive consensus protocol (2).

**Theorem 3.1:** *Suppose that the communication graph $\mathcal{G}$ satisfies Assumption 3.1. Then, the leader–follower consensus problem of the agents in (1) can be solved under the adaptive protocol (2) with $K = -B^T P^{-1}$, $\Gamma = P^{-1}BB^T P^{-1}$, and $\rho_i = \xi_i^T P^{-1} \xi_i$, where $P > 0$ is a solution to the following linear matrix inequality (LMI):*

$$PA^T + AP - 2BB^T < 0. \tag{5}$$

*Moreover, each coupling weight $c_i$ converges to some finite steady-state value.*

**Proof:** Consider the Lyapunov function candidate:

$$V_1 = \sum_{i=1}^N \frac{1}{2} g_i(2c_i + \rho_i)\rho_i + \frac{1}{2}\sum_{i=1}^N g_i(c_i - \alpha)^2, \tag{6}$$

where $G \triangleq \mathrm{diag}(g_1, \ldots, g_N)$ is a positive definite matrix such that $GL_1 + L_1^T G > 0$, and $\alpha$ is a positive constant to be determined later. Because it follows from Assumption 1 and Lemma 2.1 that $L_1$ is a nonsingular $M$-matrix, we know from Lemma 2.3 that such a positive definite matrix $G$ does exist. Since $c_i(0) > 0$, it follows from $\dot{c}_i \geq 0$ that $c_i(t) > 0$ for any $t > 0$. Then, it is not difficult to see that $V_1$ is positive definite.

The time derivative of $V_1$ along the trajectory of (4) is given by

$$\begin{aligned}\dot{V}_1 &= \sum_{i=1}^{N}[g_i(c_i+\rho_i)\dot{\rho}_i + g_i\rho_i\dot{c}_i] + \sum_{i=1}^{N} g_i(c_i-\alpha)\dot{c}_i \\ &= \sum_{i=1}^{N} 2g_i(c_i+\rho_i)\xi_i^T P^{-1}\dot{\xi}_i + \sum_{i=1}^{N} g_i(\rho_i+c_i-\alpha)\dot{c}_i.\end{aligned} \tag{7}$$

Note that

$$\sum_{i=1}^{N} g_i(\rho_i+c_i-\alpha)\dot{c}_i = \xi^T[(C+\rho-\alpha I)G\otimes\Gamma]\xi,$$

and

$$\begin{aligned}\sum_{i=1}^{N} 2g_i(c_i+\rho_i)\xi_i^T P^{-1}\dot{\xi}_i &= 2\xi^T[(C+\rho)G\otimes P^{-1}]\dot{\xi} \\ &= \xi^T[(C+\rho)G\otimes(P^{-1}A+A^TP^{-1}) \\ &\quad -(C+\rho)(GL_1+L_1^TG)(C+\rho)\otimes\Gamma]\xi \\ &\leq \xi^T[(C+\rho)G\otimes(P^{-1}A+A^TP^{-1}) \\ &\quad -\lambda_0(C+\rho)^2\otimes\Gamma]\xi,\end{aligned}$$

where $\lambda_0$ denotes the smallest eigenvalue of $GL_1 + L_1^TG$. Thus, we have

$$\begin{aligned}\dot{V}_1 \leq\ & \xi^T[(C+\rho)G\otimes(P^{-1}A+A^TP^{-1}+\Gamma) \\ & -(\lambda_0(C+\rho)^2+\alpha G)\otimes\Gamma]\xi.\end{aligned} \tag{8}$$

By using Lemma 2.4, we can get that

$$\begin{aligned}& -\xi^T[(\lambda_0(C+\rho)^2+\alpha G)\otimes\Gamma]\xi \\ & \leq -2\xi^T[\sqrt{\lambda_0\alpha G}(C+\rho)\otimes\Gamma]\xi.\end{aligned} \tag{9}$$

Substituting (9) into (8) and choosing $\alpha \geq \frac{\max_{i=1,\ldots,N} g_i}{\lambda_0}$ yield

$$\dot{V}_1 \leq \xi^T[(C+\rho)G\otimes(P^{-1}A+A^TP^{-1}-2\Gamma)]\xi. \tag{10}$$

Let $\zeta = (\sqrt{(C+\rho)G}\otimes P^{-1})\xi$. Then, it follows from (10) that

$$\begin{aligned}\dot{V}_1 &\leq \zeta^T(I_N\otimes(AP+PA^T-2BB^T))\zeta \\ &\leq 0,\end{aligned} \tag{11}$$

where the last inequality comes directly from the LMI (5). Therefore, we can get that $V_1(t)$ is bounded and so is each $c_i$. Noting that $\dot{c}_i \geq 0$, we can know that each coupling weight $c_i$ converges to some finite value. Noting that $\dot{V}_1 \equiv 0$ equals to $\zeta \equiv 0$ and thereby $\xi \equiv 0$. By LaSalle's Invariance principle (Krstic, Kanellakopoulos, & Kokotovic, 1995), it follows that the consensus error $\xi$ asymptotically converges to zero. That is, the leader–follower consensus problem is solved. □

**Remark 3.1:** Contrary to the consensus protocols in Li et al. (2010), Li et al. (2011), Seo et al. (2009), and Zhang et al. (2011) which use the nonzero eigenvalues of the Laplacian matrix, the design of the proposed adaptive protocol (2) relies only on the agent dynamics and the relative states of neighbours, which can be conducted by each agent in a fully distributed way. It is known that a necessary and sufficient condition for the existence of the solution $P > 0$ to the LMI (5) is that $(A, B)$ is stabilisable (Li et al., 2010). Therefore, the existence condition of an adaptive protocol (2) satisfying Theorem 3.1 is that $(A, B)$ is stablisable.

**Remark 3.2:** In contrast to the distributed adaptive protocols in Li, Ren, Liu, and Xie (2013), Li, Ren, Liu, and Fu (2013), Su et al. (2011), and Yu et al. (2013) which are applicable to only undirected graphs, the proposed adaptive protocol (2) works for the case with general directed graphs satisfying Assumption 1. It is worth mentioning that similar distributed adaptive protocols were designed in the previous works of Li et al. (2015) and Li and Duan (2014) for directed graphs satisfying Assumption 1. In comparison to the adaptive protocols in Li et al. (2015) and Li and Duan (2014), the novel adaptive protocol (2) has two distinct features. First, different from the adaptive protocol in Li et al. (2015) which uses multiplicative functions to provide additional design flexibility, the adaptive protocol (2) introduces additive functions $\rho_i$ instead. As a consequence, a simple quadratic-like Lyapunov function as in (6), instead of the complicated integral-like Lyapunov function in Li et al. (2015), will be used to show Theorem 3.1. The involved derivations are much simplified. Second, contrary to the adaptive protocol in Li and Duan (2014) which can only deal with external disturbances satisfying the restrictive matching condition, the proposed adaptive protocol (2) can be modified to be applicable to general bounded external disturbances, which will be detailed in the following section.

## 4. Leaderless distributed adaptive consensus protocol design

Theorem 3.1 in the previous section is applicable to any leader–follower directed graph satisfying Assumption 3.1. In this section, we extend to consider the case where there is no leader.

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Consider a group of $N$ identical agents with general linear dynamics described by (1), indexed by $1, \dots, N$. The communication graph $\mathcal{G}$ among the $N$ agents is assumed in this section to satisfy the following assumption.

**Assumption 4.1:** *The communication graph $\mathcal{G}$ is strongly connected.*

The intention of this section is to propose distributed consensus protocols such that the $N$ agents in (1) achieve consensus in the sense of $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, $\forall\, i, j = 1, \dots, N$.

Based on the relative states of neighbouring agents, we propose a distributed adaptive consensus protocol to each agent as

$$\begin{aligned} u_i &= (\tilde{c}_i + \tilde{\rho}_i)K\tilde{\xi}_i, \\ \dot{\tilde{c}}_i &= \tilde{\xi}_i^T\Gamma\tilde{\xi}_i, \quad i = 1, \dots, N, \end{aligned} \tag{12}$$

where $\tilde{\xi}_i \triangleq \sum_{j=1}^N a_{ij}(x_i - x_j)$, $\tilde{c}_i(t)$ denotes the time-varying coupling weight associated with the $i$th agent with $\tilde{c}_i(0) \geq 0$, $K$ and $\Gamma$ are the feedback gain matrices, and $\tilde{\rho}_i$ are smooth functions.

Let $\tilde{\xi} \triangleq [\tilde{\xi}_1^T, \dots, \tilde{\xi}_N^T]^T$ and $x \triangleq [x_1^T, \dots, x_N^T]^T$. Then, we get

$$\tilde{\xi} = (L \otimes I_n)x. \tag{13}$$

Since the graph $\mathcal{G}$ satisfies Assumption 4.1, it follows from Lemma 2.1 that the consensus problem is solved if and only if $\tilde{\xi}$ asymptotically converges to zero. Substituting (12) into (1), we can get that $\tilde{\xi}$ and $\tilde{c}_i$ satisfy the following dynamics:

$$\begin{aligned} \dot{\tilde{\xi}} &= [I_N \otimes A + L(\tilde{C} + \tilde{\rho}) \otimes BK]\tilde{\xi}, \\ \dot{\tilde{c}}_i &= \tilde{\xi}_i^T\Gamma\tilde{\xi}_i, \quad i = 1, \dots, N, \end{aligned} \tag{14}$$

where $\tilde{C} \triangleq \mathrm{diag}(\tilde{c}_1, \dots, \tilde{c}_N)$ and $\tilde{\rho} \triangleq \mathrm{diag}(\tilde{\rho}_1, \dots, \tilde{\rho}_N)$.

The following result provides a sufficient condition to design the adaptive consensus protocol (12).

**Theorem 4.1:** *Suppose that the communication graph $\mathcal{G}$ satisfies Assumption 4.1. Then, the consensus problem of the agents in (1) can be solved under the adaptive protocol (12) with $K$, $\Gamma$ defined as in Theorem 3.1, and $\tilde{\rho}_i = \tilde{\xi}_i^T P^{-1}\tilde{\xi}_i$, where $P$ is defined as in Theorem 3.1. Moreover, each coupling weight $\tilde{c}_i$ converges to some finite steady-state value.*

**Proof:** Consider the following Lyapunov function candidate

$$V_2 = \frac{1}{2}\sum_{i=1}^N r_i(2\tilde{c}_i + \tilde{\rho}_i)\tilde{\rho}_i + \frac{1}{2}\sum_{i=1}^N r_i(\tilde{c}_i - \beta)^2, \tag{15}$$

where $r = [r_1, \dots, r_N]$ is the left eigenvector of $L$ associated with the zero eigenvalue, $\beta$ is a positive constant to be determined later. Since Assumption 4.1 holds, it follows from Lemma 2.2 that $R \triangleq \mathrm{diag}(r_1, \dots, r_N) > 0$. Since $\tilde{c}_i(0) > 0$ and $\dot{\tilde{c}}_i \geq 0$, it is not difficult to see that $V_2$ is positive definite.

The time derivative of $V_2$ along the trajectory (14) is given by

$$\begin{aligned} \dot{V}_2 &= \sum_{i=1}^N [2r_i(\tilde{c}_i + \tilde{\rho}_i)\tilde{\xi}_i^T P^{-1}\dot{\tilde{\xi}}_i + r_i r\tilde{h}o_i\dot{\tilde{c}}_i] \\ &\quad + \sum_{i=1}^N r_i(\tilde{c}_i - \beta)\dot{\tilde{c}}_i \\ &= \tilde{\xi}^T[(\tilde{C} + \tilde{\rho})R \otimes (P^{-1}A + A^T P^{-1}) \\ &\quad - (\tilde{C} + \tilde{\rho})\hat{L}(\tilde{C} + \tilde{\rho}) \otimes \Gamma]\tilde{\xi} \\ &\quad + \tilde{\xi}^T(\tilde{\rho}R \otimes \Gamma)\tilde{\xi} + \tilde{\xi}^T[(\tilde{C} - \beta I)R \otimes \Gamma]\tilde{\xi}, \end{aligned} \tag{16}$$

where $\hat{L} \triangleq RL + L^T R$.

Let $\bar{\zeta} = ((\tilde{C} + \tilde{\rho}) \otimes I_n)\tilde{\xi}$. Then, we have

$$\begin{aligned} \bar{\zeta}^T((\tilde{C} + \tilde{\rho})^{-1}r \otimes \mathbf{1}) &= \tilde{\xi}^T(r \otimes \mathbf{1}) \\ &= x^T(L^T r \otimes \mathbf{1}) = 0, \end{aligned}$$

where we have used the fact that $r^T L = 0$ to get the last equality. Since every entry of $r$ is positive, it is obvious that every entry of $(\tilde{C} + \tilde{\rho})^{-1}r \otimes \mathbf{1}$ is also positive. In light of Lemma 2.2, we can get that

$$\begin{aligned} \bar{\zeta}^T(\hat{L} \otimes I_n)\bar{\zeta} &> \frac{\lambda_2(\hat{L})}{N}\bar{\zeta}^T\bar{\zeta} \\ &= \frac{\lambda_2(\hat{L})}{N}\tilde{\xi}^T[(\tilde{C} + \tilde{\rho})^2 \otimes I_n]\tilde{\xi}. \end{aligned} \tag{17}$$

By substituting (17) into (16), we can obtain

$$\begin{aligned} \dot{V}_2 \leq \tilde{\xi}^T\Bigg[&(\tilde{C} + \tilde{\rho})R \otimes (P^{-1}A + A^T P^{-1} + \Gamma) \\ &- \left(\frac{\lambda_2(\hat{L})}{N}(\tilde{C} + \tilde{\rho})^2 + \beta R\right) \otimes \Gamma\Bigg]\tilde{\xi}. \end{aligned} \tag{18}$$

Similar to (9), we have

$$\begin{aligned} &-\tilde{\xi}^T\left[\left(\frac{\lambda_2(\hat{L})}{N}(\tilde{C} + \tilde{\rho})^2 + \beta R\right) \otimes \Gamma\right]\tilde{\xi} \\ &\leq -\tilde{\xi}^T\left[\sqrt{\frac{\lambda_2(\hat{L})\beta}{N}}R^{\frac{1}{2}}(\tilde{C} + \tilde{\rho}) \otimes \Gamma\right]\tilde{\xi}. \end{aligned} \tag{19}$$

By choosing $\beta \geq \frac{4N \max_{i=1,\ldots,N} r_i}{\lambda_2(\hat{L})}$, we can get

$$\dot{V}_2 \leq \tilde{\xi}^T[(\tilde{C}+\tilde{\rho})R \otimes (P^{-1}A + A^T P^{-1} - 2\Gamma)]\tilde{\xi}. \quad (20)$$

The rest of the proof is similar to that of Theorem 3.1, which is omitted here for brevity. □

## 5. Distributed robust adaptive consensus protocols

Theorems 3.1 and 4.1 in the previous sections show that the adaptive protocols (2) and (12) are applicable to any directed graph satisfying Assumption 3.1 or Assumption 4.1, respectively, for the case without external disturbances. In many circumstances, the agents might be subject to certain external disturbances, for which case it is necessary and interesting to investigate whether the adaptive protocol (2) or (12) is robust.

The dynamics of the $i$th agent are described by

$$\dot{x}_i = Ax_i + Bu_i + \omega_i, \quad i = 0, \ldots, N, \quad (21)$$

where $\omega_i \in \mathbf{R}^n$ denotes external disturbances associated with the $i$th agent, which satisfies the following assumption.

**Assumption 5.1:** *There exist positive constants $\upsilon_i$ such that $\|\omega_i\| \leq \upsilon_i$, $i = 1, \ldots, N$, and $\|Bu_0 + \omega_0\| \leq \upsilon_0$.*

Note that due to the existence of disturbances $\omega_i$ in (21), the relative states will not converge to zero any more but rather can only be expected to converge into some small neighbourhood of the origin. Since the derivatives of the adaptive gains $c_i$ in (2) are of nonnegative quadratic forms in terms of the relative states, in this case it is easy to see from (2) that $c_i$ will keep growing to infinity, which is called the parameter drift phenomenon in the classic adaptive control literature (Ioannou & Sun, 1996). Therefore, the adaptive protocol (2) is not robust in the presence of external disturbances.

In the following, we aim to make modification on (2) to propose a distributed robust adaptive protocol which can guarantee the ultimate boundedness of the consensus error and adaptive weights for the agents in (21). By utilising the $\sigma$ modification technique (Ioannou & Sun, 1996), we propose a new distributed adaptive consensus protocol as follows:

$$\begin{aligned} u_i &= (d_i + \rho_i)K\xi_i, \\ \dot{d}_i &= -\varphi_i(d_i - 1)^2 + \xi_i^T \Gamma \xi_i, \quad i = 1, \ldots, N, \end{aligned} \quad (22)$$

where $d_i(t)$ denotes the time-varying coupling weight associated with the $i$th follower with $d_i(0) \geq 1$, $\varphi_i$, $i = 1, \ldots, N$, are small positive constants, and the rest of the variables are defined as in (2).

Substituting (22) into (21), it follows that

$$\begin{aligned} \dot{\xi} &= [I_N \otimes A + L_1(D+\rho) \otimes BK]\,\xi + (L_1 \otimes I_n)\omega \\ \dot{d}_i &= -\varphi_i(d_i - 1)^2 + \xi_i^T \Gamma \xi_i, \quad i = 1, \ldots, N, \end{aligned} \quad (23)$$

where $D \triangleq \mathrm{diag}(d_1, \ldots, d_N)$, $\omega \triangleq [\omega_1^T - (Bu_0 + \omega_0)^T, \ldots, \omega_N^T - (Bu_0 + \omega_0)^T]^T$, and the rest of the variables are defined as in (4).

In light of Assumption 5.1, we have

$$\|\omega\| \leq \sqrt{\sum_{i=1}^N (\upsilon_i + \upsilon_0)^2} = \upsilon. \quad (24)$$

Note that $d_i(0) \geq 1$ and $\dot{d}_i \geq 0$ when $d_i = 1$ in (23). Then, it is not difficult to see that $d_i(t) \geq 1$ for any $t > 0$.

The following result provides a sufficient condition to design the robust adaptive consensus protocol (22).

**Theorem 5.1:** *Suppose that Assumptions 3.1 and 5.1 hold. Then, both the consensus error $\xi$ and the coupling weights $d_i$, $i = 1, \ldots, N$, in (23) are uniformly ultimately bounded under the adaptive protocol (22) with $K = -B^T Q^{-1}$, $\Gamma = Q^{-1}BB^T Q^{-1}$, and $\rho_i = \xi_i^T Q^{-1}\xi_i$, where $Q > 0$ is a solution to the LMI:*

$$AQ + QA^T + \varepsilon Q - 2BB^T < 0, \quad (25)$$

*where $\epsilon > 1$. Moreover, the consensus error $\xi$ converges exponentially to the residual set*

$$D_1 = \left\{ \xi : \|\xi\|^2 \leq \frac{2\Pi}{\lambda_{\min}(Q^{-1}) \min\limits_{i=1,\ldots,N} g_i} \right\}, \quad (26)$$

*where $g_i$ is defined as in (6),*

$$\Pi = \sum_{i=1}^N \left[ \frac{2g_i\delta^3}{27\varphi_i^2} + \frac{\delta g_i}{2}(\alpha - 1)^2 \right] + \Pi_1,$$

$$\begin{aligned} \Pi_1 = \sum_{i=1}^N \frac{16}{27} g_i \varphi_i (\alpha - 1)^3 &+ \left\| \left( \sqrt{\varphi^{-1}} G L_1 \otimes \sqrt{Q^{-1}} \right) \upsilon \right\|^2 \\ &+ 2\left\| \left( \sqrt{G} L_1 \otimes \sqrt{Q^{-1}} \right) \upsilon \right\|^2 \\ &+ 2\left\| \left( G^{\frac{1}{4}} L_1 \otimes \sqrt{Q^{-1}} \right) \upsilon \right\|^4, \end{aligned}$$

*and $0 < \delta \leq \epsilon - 1$, with $\alpha$ defined in (10) and $\varphi \triangleq \mathrm{diag}(\varphi_1, \ldots, \varphi_N)$.*

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**Proof:** Consider the Lyapunov function candidate:

$$V_3 = \sum_{i=1}^{N} \frac{1}{2} g_i (2d_i + \rho_i)\rho_i + \frac{1}{2}\sum_{i=1}^{N} g_i (d_i - \alpha)^2, \qquad (27)$$

where $\alpha$ is a positive constant to be determined later, and the rest of the variables are defined as in (6). Since $g_i > 0$, $d_i(t) \geq 1$ for any $t > 0$, and $\rho_i \geq 0$, it can be similarly shown as in the proof of Theorem 3.1 that $V_2$ is positive definite.

By following similar steps in deriving Theorem 3.1, we can obtain the time derivative of $V_3$ along (23) as

$$\begin{aligned}\dot{V}_3 \leq\ & \xi^T[(D+\rho)G \otimes (Q^{-1}A + A^TQ^{-1} - 2\Gamma)]\xi \\ & + 2\xi^T[(D+\rho)GL_1 \otimes Q^{-1}]\omega \\ & - \xi^T[\varphi(D-I)^2 G \otimes Q^{-1}]\xi \\ & - \sum_{i=1}^{N} g_i\varphi_i(d_i-\alpha)(d_i-1)^2,\end{aligned} \qquad (28)$$

where $\alpha$ is chosen to be sufficiently large as in the proof of Theorem 3.1.

By choosing $\alpha > 1$ and using Lemma 2.4, we can get

$$\begin{aligned}& -(d_i-\alpha)(d_i-1)^2 = -(d_i-1)^3 + (\alpha-1)(d_i-1)^2 \\ & = -(d_i-1)^3 + \left[\left(\frac{3}{4}\right)^{\frac{2}{3}}(d_i-1)^2\right]\left[\left(\frac{4}{3}\right)^{\frac{2}{3}}(\alpha-1)\right] \\ & \leq -(d_i-1)^3 + \frac{1}{2}(d_i-1)^3 + \frac{16}{27}(\alpha-1)^3 \\ & = -\frac{1}{2}(d_i-1)^3 + \frac{16}{27}(\alpha-1)^3.\end{aligned} \qquad (29)$$

Note that

$$\begin{aligned}& 2\xi^T[(D+\rho)GL_1 \otimes Q^{-1}]\omega \\ & = 2\xi^T\left[(D-I)\sqrt{\varphi G} \otimes \sqrt{Q^{-1}}\right]\left(\sqrt{\varphi^{-1}G}L_1 \otimes \sqrt{Q^{-1}}\right)\omega \\ & \quad + 2\xi^T\left(\frac{1}{\sqrt{2}}\sqrt{G} \otimes \sqrt{Q^{-1}}\right)\left(\sqrt{2G}L_1 \otimes \sqrt{Q^{-1}}\right)\omega \\ & \quad + 2\xi^T\left(\frac{1}{\sqrt{2}}\sqrt{\rho G} \otimes \sqrt{Q^{-1}}\right)\left(\sqrt{2\rho G}L_1 \otimes \sqrt{Q^{-1}}\right)\omega \\ & \leq \xi^T[(D-I)^2\varphi G \otimes Q^{-1}]\xi + \left\|\left(\sqrt{\varphi^{-1}G}L_1 \otimes \sqrt{Q^{-1}}\right)\omega\right\|^2 \\ & \quad + \frac{1}{2}\xi^T(G \otimes Q^{-1})\xi + 2\left\|\left(\sqrt{G}L_1 \otimes \sqrt{Q^{-1}}\right)\omega\right\|^2 \\ & \quad + \frac{1}{2}\xi^T(\rho G \otimes Q^{-1})\xi + 2\left\|\left(\sqrt{\rho G}L_1 \otimes \sqrt{Q^{-1}}\right)\omega\right\|^2,\end{aligned} \qquad (30)$$

where we have used Lemma 2.4 several times to get the last inequality, and

$$\begin{aligned}& 2\left\|(\sqrt{\rho G}L_1 \otimes \sqrt{Q^{-1}})\omega\right\|^2 \\ & \leq 2\left\|\sqrt{\rho\sqrt{G}} \otimes I_n\right\|^2 \left\|\left(G^{\frac{1}{4}}L_1 \otimes \sqrt{Q^{-1}}\right)\omega\right\|^2 \\ & \leq \frac{1}{2}\left\|\sqrt{\rho\sqrt{G}} \otimes I_n\right\|^4 + 2\left\|\left(G^{\frac{1}{4}}L_1 \otimes \sqrt{Q^{-1}}\right)\omega\right\|^4 \\ & \leq \frac{1}{2}\xi^T(\rho G \otimes Q^{-1})\xi + 2\left\|\left(G^{\frac{1}{4}}L_1 \otimes \sqrt{Q^{-1}}\right)\omega\right\|^4,\end{aligned} \qquad (31)$$

where we have used matrix norm properties to get the first inequality, and Lemma 2.4 to get the second inequality, and to get the last inequality we have used the fact that

$$\begin{aligned}\left\|\sqrt{\rho\sqrt{G}} \otimes I_n\right\|^4 &= \max_{i=1,\ldots,N}\left(\sqrt{\rho_i\sqrt{g_i}}\right)^4 \\ &\leq \sum_{i=1}^{N}\left(\sqrt{\rho_i\sqrt{g_i}}\right)^4 = \xi^T(\rho G \otimes Q^{-1})\xi.\end{aligned}$$

Substituting (29), (30), and (31) into (28) yields

$$\begin{aligned}\dot{V}_3 \leq\ & \xi^T[(D+\rho)G \otimes (Q^{-1}A + A^TQ^{-1} - 2\Gamma)]\xi \\ & + \xi^T\left[\left(\rho + \frac{1}{2}\right)G \otimes Q^{-1}\right]\xi - \sum_{i=1}^{N}\frac{1}{2}g_i\varphi_i(d_i-1)^3 + \Pi_1 \\ \leq\ & W(\xi) - \frac{1}{2}\xi^T(G \otimes Q^{-1})\xi - \sum_{i=1}^{N}\frac{1}{2}g_i\varphi_i(d_i-1)^3 + \Pi_1,\end{aligned} \qquad (32)$$

where we have used the fact that $D \geq I$, and

$$\begin{aligned}W(\xi) \triangleq\ & \xi^T[(D+\rho)G \otimes (Q^{-1}A + A^TQ^{-1} + Q^{-1} - 2\Gamma)]\xi \\ \leq\ & 0.\end{aligned}$$

Note that for any positive $\delta$, we have the following assertion:

$$\begin{aligned}\frac{\delta}{2}(d_i-\alpha)^2 &\leq \frac{\delta}{2}(d_i-1)^2 + \frac{\delta}{2}(\alpha-1)^2 \\ &= \left[\left(\frac{3\varphi_i}{4}\right)^{\frac{2}{3}}(d_i-1)^2\right]\left[\left(\frac{2}{9\varphi_i^2}\right)^{\frac{1}{3}}\delta\right] \\ &\quad + \frac{\delta}{2}(\alpha-1)^2 \\ &\leq \frac{1}{2}\varphi_i(d_i-1)^3 + \frac{2\delta^3}{27\varphi_i^2} + \frac{\delta}{2}(\alpha-1)^2,\end{aligned} \qquad (33)$$

where we have used the fact that $d_i > 1$ and $\alpha > 1$ to get the first inequality, and Lemma 2.4 to get the last inequality.

From (32) and (33), we can obtain that

$$\begin{aligned}\dot{V}_3 &\le -\delta V_3 + \delta V_3 + W(\xi) - \frac{1}{2}\xi^T(G\otimes Q^{-1})\xi \\ &\quad - \sum_{i=1}^{N}\frac{1}{2}g_i\varphi_i(d_i-1)^3 + \Pi_1 \\ &\le -\delta V_3 + \bar{W}(\xi) - \frac{1}{2}\xi^T(G\otimes Q^{-1})\xi + \Pi,\end{aligned} \tag{34}$$

where

$$\begin{aligned}\bar{W} &= \xi^T[(D+\rho)G\otimes(Q^{-1}A + A^TQ^{-1} \\ &\quad + (1+\delta)Q^{-1} - 2Q^{-1}BB^TQ^{-1})]\xi,\end{aligned}$$

By choosing $\delta$ such that $\epsilon \ge 1+\delta$, we can obtain that $\bar{W}(\xi) \le 0$. Then, it follows from (34) that

$$\dot{V}_3 \le -\delta V_3 - \frac{1}{2}\xi^T(G\otimes Q^{-1})\xi + \Pi. \tag{35}$$

In light of Lemma 2.5, we can derive from (35) that both the consensus error $\xi$ and the adaptive gains $d_i$ are uniformly ultimately bounded. Further, from (35), we can get that $\dot{V}_3 \le -\delta V_3$ if $\|\xi\|^2 \ge \frac{2\Pi}{\lambda_{\min}(Q^{-1})\min_{i=1,\dots,N} g_i}$. Therefore, $\xi$ converges to the set (26) with a convergence rate faster than $e^{-\delta t}$. □

For the case, the communication graph is leaderless, the adaptive protocol (12) can be modified to be

$$\begin{aligned}u_i &= (\tilde{d}_i + \tilde{\rho}_i)K\tilde{\xi}_i, \\ \dot{\tilde{d}}_i &= -\tilde{\varphi}_i(\tilde{d}_i - 1)^2 + \tilde{\xi}_i^T\Gamma\tilde{\xi}_i, \qquad i = 1,\dots,N,\end{aligned} \tag{36}$$

where $\tilde{\varphi}_i$ are small positive constants.

Substituting (36) into (21), we have that $\tilde{\xi}$ and $\tilde{d}_i$ satisfy the following dynamics:

$$\begin{aligned}\dot{\tilde{\xi}} &= [I_N\otimes A + L(\tilde{D}+\tilde{\rho})\otimes BK]\xi + (L\otimes I_n)\tilde{\omega} \\ \dot{\tilde{d}}_i &= -\tilde{\varphi}_i(\tilde{d}_i-1)^2 + \tilde{\xi}_i^T\Gamma\tilde{\xi}_i, \quad i=1,\dots,N,\end{aligned} \tag{37}$$

where $\tilde{\omega} \triangleq [\omega_1^T,\dots,\omega_N^T]^T$.

In light of Assumption 5.1, we have

$$\|\tilde{\omega}\| \le \sqrt{\sum_{i=1}^{N}\upsilon_i^2} = \tilde{\upsilon}. \tag{38}$$

**Theorem 5.2:** *Suppose that Assumptions 4.1 and 5.1 hold. Then, both the consensus error $\tilde{\xi}$ and the coupling weights $\tilde{d}_i$, $i = 1,\dots,N$, are uniformly ultimately bounded under the adaptive protocol (36) with $K$, $\Gamma$, designed as in Theorem 5.1, $\tilde{\rho}_i = \tilde{\xi}_i^TQ^{-1}\tilde{\xi}_i$. Moreover, $\tilde{\xi}$ converges exponentially to the residual set*

$$D_2 = \left\{\tilde{\xi} : \|\tilde{\xi}\|^2 \le \frac{2\tilde{\Pi}}{\lambda_{\min}(Q^{-1})\min_{i=1,\dots,N} r_i}\right\}, \tag{39}$$

*where $r_i$ is defined as in (15),*

$$\begin{aligned}\tilde{\Pi} &= \sum_{i=1}^{N}\left[\frac{2r_i\delta^3}{27\tilde{\varphi}_i^2} + \frac{\delta r_i}{2}(\beta-1)^2\right] + \tilde{\Pi}_1, \\ \tilde{\Pi}_1 &= \sum_{i=1}^{N}\frac{16}{27}r_i\tilde{\varphi}_i(\beta-1)^3 + \left\|\left(\sqrt{\tilde{\varphi}^{-1}}RL\otimes\sqrt{Q^{-1}}\right)\tilde{\upsilon}\right\|^2 \\ &\quad + 2\|(\sqrt{R}L\otimes\sqrt{Q^{-1}})\tilde{\upsilon}\|^2 + 2\left\|\left(R^{\frac{1}{4}}L\otimes\sqrt{Q^{-1}}\right)\tilde{\upsilon}\right\|^4,\end{aligned}$$

*and $0 < \delta \le \epsilon - 1$, with $\beta$ defined in (20) and $\tilde{\varphi} \triangleq \mathrm{diag}(\tilde{\varphi}_1,\dots,\tilde{\varphi}_N)$.*

**Proof:** Consider the Lyapunov function candidate

$$V_4 = \frac{1}{2}\sum_{i=1}^{N} r_i(2\tilde{d}_i + \tilde{\rho}_i)\tilde{\rho}_i + \frac{1}{2}\sum_{i=1}^{N} r_i(\tilde{d}_i - \beta)^2. \tag{40}$$

By following the similar steps in the proof of Theorem 5.1, we can obtain that

$$\begin{aligned}\dot{V}_4 &\le \tilde{\xi}^T[(\tilde{D}+\tilde{\rho})R\otimes(Q^{-1}A + A^TQ^{-1} + Q^{-1} \\ &\quad - 2Q^{-1}BB^TQ^{-1})]\tilde{\xi} \\ &\quad - \frac{1}{2}\tilde{\xi}^T(R\otimes Q^{-1})\tilde{\xi} - \sum_{i=1}^{N}\frac{1}{2}r_i\tilde{\varphi}_i(\tilde{d}_i-1)^3 + \tilde{\Pi}_1.\end{aligned} \tag{41}$$

The upper bound of the consensus error $\tilde{\xi}$ can be obtained by following the last part of the proof of Theorem 5.1. The details are omitted here for conciseness. □

**Remark 5.1:** As shown in Proposition 1 in Li et al. (2011), there exists a $Q > 0$ satisfying (25) if and only if $(A, B)$ is controllable. Thus, a sufficient condition for the existence of (22) satisfying Theorem 5.1 or (36) satisfying Theorem 5.2 is that $(A, B)$ is controllable, which, compared to the existence condition of (2) satisfying Theorem 3.1 or (22) satisfying Theorem 4.1, is more stringent. Theorems 5.1 and 5.2 show the modified adaptive protocols (22) and (36) do ensure the ultimate boundedness of both the consensus error and the adaptive gains. That is, the adaptive protocols (22) and (36) are robust in the presence of external disturbances. The upper bound of the consensus error $\xi$ or $\tilde{\xi}$ as given in (26) or (39) depends on the dynamics of each agent, the communication graph, the upper bounds of the disturbances, and the parameters $\varphi_i$ or $\tilde{\varphi}_i$.

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We should choose appropriately small $\varphi_i$ or $\tilde{\varphi}_i$ to get an acceptable upper bound of $\xi$ or $\tilde{\xi}$.

**Remark 5.2:** Compared to the robust adaptive protocol in Li and Duan (2014) which are only applicable to the case with matching disturbances, the adaptive protcol (22) works for general external disturbances. This is a favourable consequence of introducing novel additive functions $\rho_i$, rather than multiplicative ones as in Ioannou and Sun (1996), into (22). It is worth noting that the steps in proving Theorem 2 is quite different from those in Li and Duan (2014).

## 6. Simulation

Consider a network of second-order integrators, described by (1), with

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The communication graph is given as in Figure 1, which clearly satisfies Assumption 3.1.

Solving the LMI (5) by using the LMI toolbox of Matlab gives a feasible solution matrix $P = \begin{bmatrix} 1.7559 & -0.5853 \\ -0.5853 & 0.5853 \end{bmatrix}$. Then, the feedback gain matrices of (2) are given by

$$K = \begin{bmatrix} -0.8543 & -2.5628 \end{bmatrix},$$
$$\Gamma = \begin{bmatrix} 0.7298 & 2.1893 & 2.1893 & 6.5678 \end{bmatrix}.$$

Let $c_i(0) = 1, i = 1, \dots, 6$. The consensus errors $x_i - x_0, i = 1, \dots, 6$, of the second-order integrators under the adaptive protocol (2) are depicted in Figure 2 and the adaptive coupling weights $c_i$ are shown in Figure 3.

Further, consider the case where the second-order integrators are perturbed by external disturbances. For illustration, the disturbances associated with the agents are assumed to be $\omega_0 = \begin{bmatrix} 0.1\sin(2t) \\ 0.1\sin(4t) \end{bmatrix}$, $\omega_1 = \begin{bmatrix} 0.2\sin(3.5t) \\ 0.2\cos(2.5t) \end{bmatrix}$, $\omega_2 = \begin{bmatrix} 0.15\cos(4t) \\ 0.15\sin(5t) \end{bmatrix}$, $\omega_3 = \begin{bmatrix} 0.3\sin(2t) \\ 0.3\sin(3t) \end{bmatrix}$, $\omega_4 = \begin{bmatrix} 0.15\cos(3t) \\ 0.15\cos(3t) \end{bmatrix}$, $\omega_5 = \begin{bmatrix} 0.25\sin(4t) \\ 0.25\cos(3t) \end{bmatrix}$, $\omega_6 = \begin{bmatrix} 0.3\sin(5t) \\ 0.3\cos(4t) \end{bmatrix}$, and the control input of the leader is assumed to be $u_0 = e^{-0.1t}$. Solving the LMI (25) with $\epsilon = 2$ gives $Q = \begin{bmatrix} 0.2622 & -0.3517 \\ -0.3517 & 0.7395 \end{bmatrix}$ and then $K = [\,-5.0141 \; -3.7372\,]$, $\Gamma = \begin{bmatrix} 25.1412 & 18.7386 \\ 18.7386 & 13.9665 \end{bmatrix}$. In (22), let $\varphi_i = 0.1$ and $d_i(0) = 1.5$, $i = 1, \dots, 6$. The consensus errors $x_i - x_0$, $i = 1, \dots, 6$, under the robust adaptive protocol (22) are depicted in Figure 4 and the coupling weights $d_i$ are shown in Figure 5, both of which are obviously bounded.

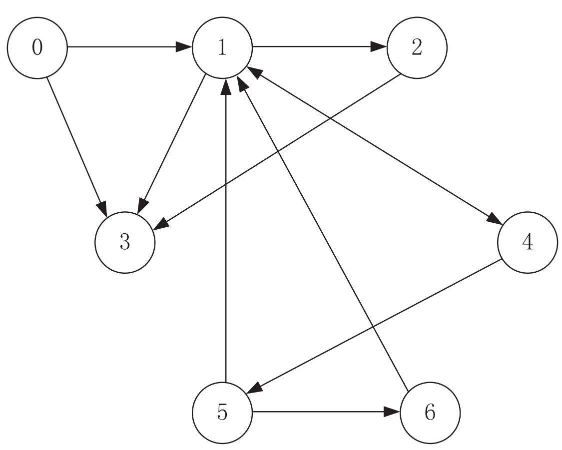

Figure 1. The leader–follower directed communication graph.

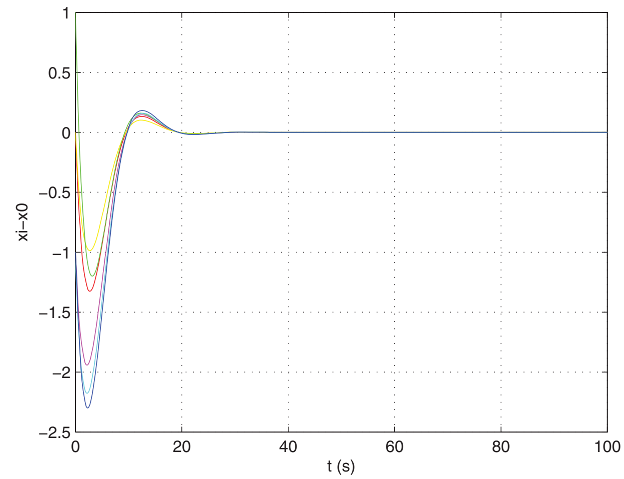

Figure 2. The consensus errors $x_i - x_0, i = 1, \dots, 6$ under the adaptive protocol (2).

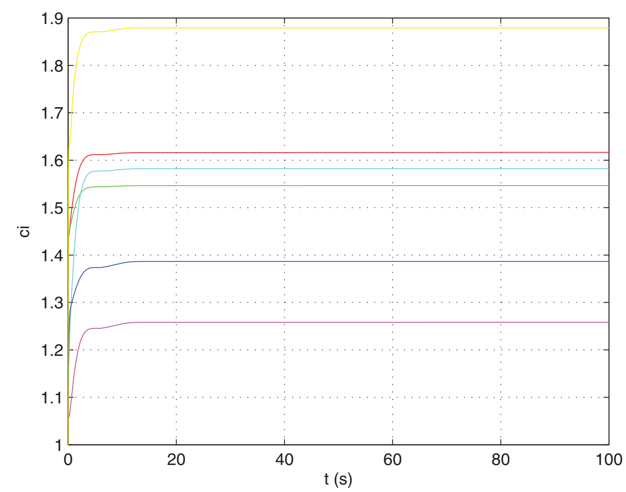

Figure 3. The adaptive coupling weights $c_i$ under the adaptive protocol (2).

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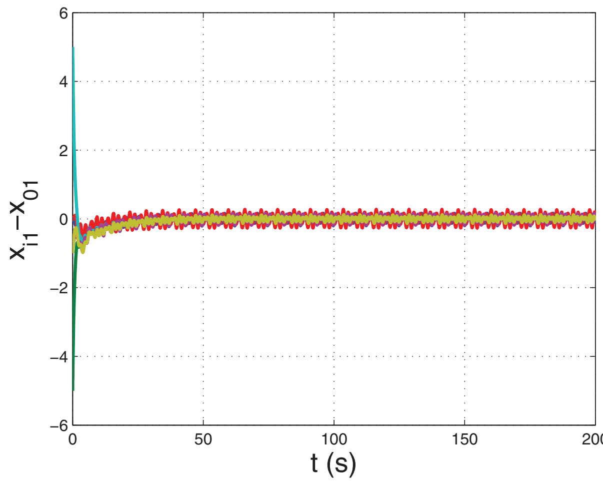


**Figure 4.** The consensus errors under the robust adaptive protocol (22).

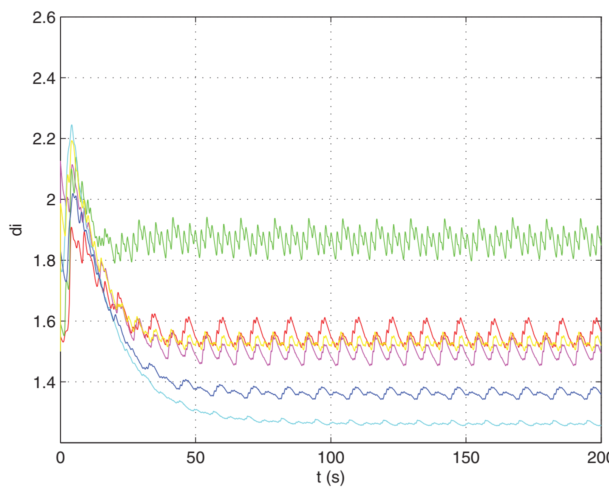


**Figure 5.** The coupling weights $d_i$ in the presence of disturbances.

## 7. Conclusion

In this paper, we have presented novel distributed adaptive consensus protocols for linear multi-agent systems with external disturbances and leader–follower directed graphs containing a directed spanning tree with the leader as the root or leaderless directed graphs which are strongly connected. The adaptive consensus protocols, depending on only the agent dynamics an the relative state information of neighbouring agents, can be designed and implemented in a fully distributed way. One contribution of this paper is that the a new distributed adaptive protocol has been derived, which is robust in the presence of general bounded external disturbances. An interesting topic for future investigation is to design fully distributed adaptive protocols for nonlinear multi-agent systems or the case with local output information of each agent and its neighbours.


## Disclosure statement

No potential conflict of interest was reported by the authors.

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## References


Bernstein, D.S. (2009). *Matrix mathematics: Theory, facts, and formulas*. Princeton, NJ: Princeton University Press.

Corless, M., & Leitmann, G. (1981). Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems. *IEEE Transactions on Automatic Control*, *26*(5), 1139–1144.

Guo, M., & Dimarogonas, D.V. (2013). Consensus with quantized relative state measurements. *Automatica, 49*(8), 2531–2537.

Hong, Y., Chen, G., & Bushnell, L. (2008). Distributed observers design for leader-following control of multi-agent networks. *Automatica, 44*(3), 846–850.

Ioannou, P.A., & Sun, J. (1996). *Robust adaptive control*. New York, NY: Prentice-Hall.

Krstic, M., Kanellakopoulos, I., & Kokotovic, P. (1995). *Nonlinear and adaptive control design*. New York, NY: John Wiley Sons.

Li, T., Fu, M., Xie, L., & Zhang, J. (2011). Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control, 56*(2), 279–292.

Li, Z., & Duan, Z. (2014, July). Distributed adaptive consensus protocols for linear multi-agent systems with directed graphs in the presence of external disturbances. In *Proceedings of the 33rd Chinese control conference* (pp. 1632–1637). Nanjing, China.

Li, Z., Duan, Z., & Chen, G. (2011). Dynamic consensus of linear multi-agent systems. *IET Control Theory and Applications, 5*(1), 19–28.

Li, Z., Duan, Z., Chen, G., & Huang, L. (2010). Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems, 57*(1), 213–224.

Li, Z., Ren, W., Liu, X., & Fu, M. (2013). Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols. *IEEE Transactions on Automatic Control, 58*(7), 1786–1791.

Li, Z., Ren, W., Liu, X., & Xie, L. (2013, July). Distributed consensus of linear multi-agent systems with adaptive dynamic protocols. *Automatica, 49*(7), 1986–1995.

Li, Z., Wen, G., Duan, Z., & Ren, W. (2015). Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs. *IEEE Transactions on Automatic Control, 60*(4), 1152–1157.

Ma, C., & Zhang, J. (2010). Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control, 55*(5), 1263–1268.


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Mei, J., Ren, W., & Chen, J. (2014, June). Consensus of second-order heterogeneous multi-agent systems under a directed graph. In *American control conference* (pp. 802–807).

Olfati-Saber, R., & Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control, 49*(9), 1520–1533.

Qu, Z. (2009). *Cooperative control of dynamical systems: Applications to autonomous vehicles*. London: Springer-Verlag.

Ren, W., Beard, R.W., & Atkins, E.M. (2007). Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine, 27*(2), 71–82.

Ren, W., & Beard, R. (2005, May). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control, 50*(5), 655–661.

Seo, J.H., Shim, H., & Back, J. (2009). Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach. *Automatica, 45*(11), 2659–2664.

Su, H., Chen, G., Wang, X., & Lin, Z. (2011). Adaptive second-order consensus of networked mobile agents with nonlinear dynamics. *Automatica, 47*(2), 368–375.

Tuna, S.E. (2009). Conditions for synchronizability in arrays of coupled linear systems. *IEEE Transactions on Automatic Control, 54*(10), 2416–2420.

Yu, W., Ren, W., Zheng, W., Chen, G., & Lv, J. (2013). Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics. *Automatica, 49*(7), 2107–2115.

Zhang, H., Lewis, F.L., & Das, A. (2011). Optimal Design for synchronization of cooperative systems: State feedback, observer and output feedback. *IEEE Transactions on Automatic Control, 56*(8), 1948–1952.

Zhang, H., Lewis, F.L., & Qu, Z. (2012, July). Lyapunov, adaptive, and optimal design techniques for cooperative systems on directed communication graphs. *IEEE Transactions on Industrial Electronics, 59*(7), 3026–3041.

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