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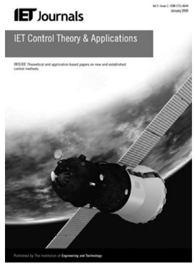

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Time-varying formation control for high-order linear swarm systems with switching interaction topologies

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Abstract: Time-varying formation control problems for high-order linear time-invariant swarm systems with switching interaction topologies are investigated. A general formation control protocol is proposed firstly. Then using a consensus based approach, necessary and sufficient conditions for swarm systems with switching interaction topologies to achieve a given time-varying formation are presented. An explicit expression of the time-varying formation reference function is given. It is revealed that the switching interaction topologies have no effect on the formation reference function and the motion modes of the formation reference can be specified. Furthermore, necessary and sufficient conditions for formation feasibility are presented. An approach to expand the feasible formation set is given and an algorithm to design the protocol for swarm systems with switching interaction topologies to achieve time-varying formations is provided. Finally, numerical simulations are presented to demonstrate theoretical results.

1 Introduction

Recently, formation control of swarm systems has attracted considerable attention because of its broad potential applications in civilian and military areas such as load transportation [1], radiation detection and contour mapping [2], target search and localisation [3], reconnaissance [4], surveillance [5], telecommunication relay [6] and so on. In the past decades, several formation control approaches have been proposed in the robotics community, such as behaviour [7], leader-follower [8] and virtual structure [9] based approaches and so on. However, Beard *et al.* [10] pointed out that behaviour, leader-follower and virtual structure based formation control approaches have their own weaknesses. For example, behaviour based approaches are difficult to be analysed mathematically, and leader-follower approaches lack of robustness because of the existence of the leader, just to name a few.

In recent years, consensus for linear time-invariant (LTI) swarm systems have been studied extensively (see, e.g. [11–18] and references therein). With the development of consensus theory, more and more researchers find that consensus approaches can be used to deal with formation control problems. Using consensus based approaches, Ren [19] discussed formation control problems for second-order swarm systems, and revealed that behaviour, leader-follower and virtual structure based approaches can be treated as special cases of consensus based approaches and the weaknesses of the these approaches can be overcome. In [20], a consensus based formation control strategy was applied to a multi-robot swarm system. Xiao and Wang [21] investigated finite-time formation control problems for first-order swarm systems based on consensus approaches. Sufficient conditions for second-order swarm systems with undirected interaction topologies to achieve formations were presented in [22]. Consensus based formation control problems for second-order swarm systems with time delays were addressed in [23]. Chen *et al.* [24] discussed formation control problems for first-order and second swarm systems with bounded input, disturbance and time delays.

In practical applications, many swarm systems are of high order, so formation control problems for high-order swarm systems make more sense. Based on consensus approaches, Lafferriere *et al.* [25] proposed a necessary and sufficient condition for swarm systems with a special high-order LTI model, which can be regarded as a series of second-order models, to achieve formations. Fax and Murray [26] discussed formation stability problems for general high-order LTI swarm systems. Formation stability problems for general high-order LTI swarm systems with fixed and periodic switching undirected interaction topologies were studied in Porfiri *et al.* [27]. However, both Fax and Murray [26] and Porfiri *et al.* [27] only considered formation stability problems and did not consider the formation feasibility problems. For a swarm system, whether or not a given formation is feasible is a crucial problem. For general high-order LTI swarm systems, Ma and Zhang [28] proposed a necessary and sufficient condition for formation feasibility. However, the formation considered in [28] is time-invariant and the feasible formation set is very limited. Moreover, the interaction topologies in [28] are assumed to be fixed. In practical applications, the interaction topologies may be switching because of the existing of interaction channel failures and creations among agents. When interaction topologies are switching, both the analysis and design for cooperative control of swarm systems become much complicated and challenging than the fixed case [29]. To the best of our knowledge, time-varying formation analysis and feasibility problems for general high-order LTI swarm systems with switching interaction topologies have not been investigated.

In this paper, time-varying formation control problems for general high-order LTI swarm systems with switching interaction topologies are dealt with. Firstly, a consensus based formation protocol is presented. Then the formation problems are transformed into consensus problems. Necessary and sufficient conditions for high-order LTI swarm systems with switching interaction topologies to achieve time-varying formation are proposed. An explicit expression of the time-varying formation reference function is given, and it is shown that switching interaction topologies have no effect on the formation reference function and the motion modes of the formation reference can be assigned. Moreover, necessary and sufficient conditions for time-varying formation feasibility are presented, where the feasible formation set can be expanded. Finally, an algorithm to design the protocol for swarm systems with switching interaction topologies to achieve time-varying formation is proposed.

Compared with the existing works on formation control, the novel features of the current paper are threefold. Firstly, both formation analysis and feasibility problems for general high-order LTI swarm systems with switching interaction topologies are discussed. In [19–27], formation feasibility problems were not considered. The interaction topologies in [28] were fixed. In [19–24], the dynamics of each agent is restricted to be low-order. Secondly, the formation can be time-varying whereas the formations in [25–28] are time-invariant, and a description of the feasible formation set is also shown. Thirdly, an explicit expression of the time-varying formation reference function is given, and it is shown that the motion modes of the formation reference can be specified. Moreover, an algorithm to determine the gain matrices in the protocol is presented. However, protocol design problems were not dealt with in [26, 27].

The rest of this paper is organised as follows. In Section 2, basic concepts and useful results on graph theory are introduced and the problem to be investigated is formulated. In Section 3, necessary and sufficient conditions to achieve time-varying formation are presented. In Section 4, necessary and sufficient conditions for formation feasibility are proposed and an algorithm to design the protocol is given. Numerical simulations are shown in Section 5. Finally, Section 6 concludes the whole work.

Throughout this paper, for simplicity of notation, let 0 denote zero matrices of appropriate size with zero vectors and zero number as special cases, and $\mathbf{1}_N$ be a column vector of size N with 1 as its elements. Let I_N represent an identity matrix with dimension N, and $\otimes$ denote Kronecker product.

2 Preliminaries and problem description

In this section, basic concepts and results on graph theory are introduced and the problem description is presented.

2.1 Basic concepts and results on graph theory

An undirected graph G consists of a node set $Q = \{q_1, q_2, \ldots, q_N\}$, an edge set $E \subseteq \{(q_i, q_j) : q_i, q_j \in Q, i \neq j\}$, and a symmetric adjacency matrix $W = [w_{ij}] \in \mathbb{R}^{N\times N}$ with non-negative elements w_{ij}. An *edge* of G is denoted by $q_{ij} = (q_i, q_j)$. The adjacency elements associated with the edges of G are positive, i.e. $w_{ji} > 0$ if and only if $q_{ij} \in E$. Moreover, $w_{ii} = 0$ for all $i \in \{1, 2, \ldots, N\}$. The set of *neighbours* of node q_i is denoted by $N_i = \{q_j \in Q : (q_j, q_i) \in E\}$. The *in-degree* of node q_i is defined as $\deg_{in}(q_i) = \sum_{j=1}^{N} w_{ij}$. The *degree matrix* of G is denoted by $D = \operatorname{diag}\{\deg_{in}(q_i), i = 1, 2, \ldots, N\}$. The *Laplacian matrix* of G is defined as $L = D - W$. An undirected graph is said to be *connected* if there is a path from each node to every other nodes. More details on graph theory can be found in [30]. The following lemma is useful in analysing formation problems of swarm systems.

Lemma 1 [30]: Let $L \in \mathbb{R}^{N\times N}$ be the Laplacian matrix of an undirected graph G, then

(i) L has at least one zero eigenvalue, and $\mathbf{1}_N$ is the associated eigenvector; that is, $L\mathbf{1}_N = 0$; and
(ii) If G is connected, then 0 is a simple eigenvalue of L, and all the other $N - 1$ eigenvalues are real and positive.

2.2 Problem description

Consider a swarm system with N agents. Suppose that each agent has the LTI dynamics described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{1}$$

where $i = 1, 2, \ldots, N$, $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^m$ is the control input. The interaction topology of the swarm system can be described by an undirected graph G, and each agent can be treated as a node in G. For $i, j \in \{1, 2, \ldots, N\}$, the interaction channel from agent i to agent j is denoted by the edge q_{ij}, and the corresponding interaction strength is denoted by w_{ji}.

Assumption 1: B is of full column rank.

A time-varying formation is specified by a vector $h(t) = [h_1^{\mathrm{T}}(t), h_2^{\mathrm{T}}(t), \ldots, h_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{nN}$ with $h_i(t)$ $(i = 1, 2, \ldots, N)$ piecewise continuously differentiable.

Definition 1: Swarm system (1) is said to achieve *time-varying formation* $h(t)$ if there exists a vector-valued function $r(t) \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} (x_i(t) - h_i(t) - r(t)) = 0 \ \ (i = 1, 2, \ldots, N)$$

where $r(t)$ is called a *formation reference function*.

Remark 1: The formation definition specified by vectors has been used a lot in previous works on formation control such as [19, 20, 22, 24, 25, 27] and so on. Definition 1 presents a general framework and it can be verified that the definitions in [19, 20, 22, 24, 25, 27] can be treated as special cases of Definition 1.

Definition 2: If there exist control inputs $u_i(t)(i = 1, 2, \ldots, N)$ such that swarm system (1) achieves time-varying formation $h(t)$, then the formation $h(t)$ is feasible for swarm system (1).

Definition 3: Swarm system (1) is said to achieve *consensus* if there exists a vector-valued function $c(t) \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} (x_i(t) - c(t)) = 0 \ \ (i = 1, 2, \ldots, N)$$

where $c(t)$ is called a *consensus function*.

Remark 2: Definitions 1 and 3 imply that consensus problem is just a special case of formation problem in which $h(t) \equiv 0$ and the consensus function is equivalent to the formation reference function. Therefore the results in this paper can be applied to deal with consensus problems for high-order LTI swarm systems with switching interaction topologies.

Consider the following general formation protocol with time-varying interaction topologies

$$u_i(t) = K_1 x_i(t) + K_2(x_i(t) - h_i(t)) + K_3 \sum_{j \in N_i(t)} w_{ij}(t) \times ((x_j(t) - h_j(t)) - (x_i(t) - h_i(t))) + v_i(t) \quad (2)$$

where $i = 1, 2, \ldots, N$, K_1, K_2 and K_3 are constant gain matrices with appropriate dimensions, $N_i(t)$ are the time-varying neighbour sets and $v_i(t) \in \mathbb{R}^m$ denote the external command inputs that depend on $h_i(t)$.

Remark 3: Protocol (2) provides a general framework for consensus based formation protocols. Protocols considered in [19, 20, 22, 25, 27] and so on. can be regarded as special cases of protocol (2). In protocol (2), gain matrices K_1, K_2, K_3 and vectors $v_i(t)$ $(i = 1, 2, \ldots, N)$ have their corresponding roles. K_1 and $v_i(t)$ $(i = 1, 2, \ldots, N)$ are used to expand the set of feasible time-varying formation $h(t)$. K_2 and K_3 can be used to assign the motion modes (see [31]) of the formation reference and ensure that the states of all agents achieve the desired formation, respectively. It should be pointed out that K_1, K_2 and $v_i(t)$ $(i = 1, 2, \ldots, N)$ are not necessary for some time-varying formations.

Consider the case that the interaction topologies are switching. Let the finite set $\bar{S}$ denote all the possible interaction topologies with an index set $\bar{I} \subset \mathbb{N}$, where $\mathbb{N}$ represents the set of natural numbers. Let $\sigma(t) : [0, +\infty) \to \bar{I}$ be a switching signal whose value at time t is the index of the topology at time t, and $G_{\sigma(t)}$ and $L_{\sigma(t)}$ stand for the corresponding interaction topology and Laplacian matrix, respectively.

Assumption 2: The switching time t_i $(i \in \mathbb{N})$ satisfies that $0 < t_1 < \cdots < t_k < \cdots$ and $\inf_k(t_{k+1} - t_k) = T_d > 0$.

Assumption 3: All interaction topologies in $\bar{S}$ are connected.

Definition 4: A time-varying formation $h(t)$ is feasible for swarm system (1) under protocol (2), if there exist K_i $(i = 1, 2, 3)$ and $v_i(t)$ $(i = 1, 2, \ldots, N)$ such that it can be achieved.

Let $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $v(t) = [v_1^{\mathrm{T}}(t), v_2^{\mathrm{T}}(t), \ldots, v_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Under protocol (2), swarm system (1) can be written in a compact form as follows

$$\dot{x}(t) = \left(I_N \otimes (A + BK_1 + BK_2) - L_{\sigma(t)} \otimes BK_3\right) x(t) + \left(L_{\sigma(t)} \otimes BK_3 - I_N \otimes BK_2\right) h(t) + (I_N \otimes B)\, v(t) \quad (3)$$

The current paper mainly focuses on the following three problems for swarm system (3) with switching interaction topologies: (i) under what conditions the time-varying formation $h(t)$ can be achieved; (ii) under what conditions a time-varying formation $h(t)$ is feasible; and (iii) how to design protocol (2) to achieve time-varying formation $h(t)$.

3 Time-varying formation analysis

In this section, firstly, necessary and sufficient conditions for swarm system (3) with switching interaction topologies to achieve time-varying formation $h(t)$ are presented. Then an explicit expression of the formation reference function is given.

Let $z_i(t) = x_i(t) - h_i(t)$ and $z(t) = [z_1^{\mathrm{T}}(t), z_2^{\mathrm{T}}(t), \ldots, z_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then swarm system (3) with switching interaction topologies can be rewritten as

$$\dot{z}(t) = (I_N \otimes (A + BK_1 + BK_2) - L_{\sigma(t)} \otimes BK_3)\, z(t) + (I_N \otimes B)\, v(t) + (I_N \otimes (A + BK_1))\, h(t) - (I_N \otimes I_n)\, \dot{h}(t) \quad (4)$$

The following lemma holds directly.

Lemma 2: Swarm system (3) with switching interaction topologies achieves time-varying formation $h(t)$ if and only if swarm system (4) achieves consensus.

Let $U = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N]$ be an orthogonal constant matrix with $\bar{u}_1 = \mathbf{1}_N/\sqrt{N}$, then one obtains $U^{\mathrm{T}} L_{\sigma(t)} U = \mathrm{diag}\{0, \tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}\}$, where $\tilde{U} = [\bar{u}_2, \bar{u}_3, \ldots, \bar{u}_N]$. Let $\theta(t) = (\bar{u}_1^{\mathrm{T}} \otimes I_n) z(t)$ and $\varsigma(t) = (\tilde{U}^{\mathrm{T}} \otimes I_n) z(t)$, then swarm system (4) can be transformed into

$$\dot{\theta}(t) = (A + BK_1 + BK_2)\, \theta(t) + \frac{1}{\sqrt{N}} (\mathbf{1}_N^{\mathrm{T}} \otimes B) v(t) + \frac{1}{\sqrt{N}} \left(\mathbf{1}_N^{\mathrm{T}} \otimes (A + BK_1)\right) h(t) - \frac{1}{\sqrt{N}} (\mathbf{1}_N^{\mathrm{T}} \otimes I_n) \dot{h}(t) \quad (5)$$

$$\dot{\varsigma}(t) = \left(I_{N-1} \otimes (A + BK_1 + BK_2) - (\tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}) \otimes BK_3\right) \times \varsigma(t) + (\tilde{U}^{\mathrm{T}} \otimes B) v(t) + \left(\tilde{U}^{\mathrm{T}} \otimes (A + BK_1)\right) h(t) - (\tilde{U}^{\mathrm{T}} \otimes I_n) \dot{h}(t) \quad (6)$$

The following theorem presents a necessary and sufficient condition for swarm system (3) to achieve formation $h(t)$.

Theorem 1: Swarm system (3) with switching interaction topologies achieves time-varying formation $h(t)$ if and only if

$$\lim_{t \to \infty} \varsigma(t) = 0$$

Proof: Let

$$z_C(t) = \frac{1}{\sqrt{N}} \mathbf{1}_N \otimes \theta(t) \quad (7)$$

$$z_{\bar{C}}(t) = z(t) - z_C(t) \quad (8)$$

Let $e_1 \in \mathbb{R}^N$ be a vector with 1 as its first component and 0 elsewhere. Note that $[\theta^{\mathrm{T}}(t), 0]^{\mathrm{T}} = e_1 \otimes \theta(t)$, one has

$$z_C(t) = (U \otimes I_n)[\theta^{\mathrm{T}}(t), 0]^{\mathrm{T}} \quad (9)$$

Since $[\theta^{\mathrm{T}}(t), \varsigma^{\mathrm{T}}(t)]^{\mathrm{T}} = (U^{\mathrm{T}} \otimes I_n) z(t)$, by (7)–(9), it can be obtained that

$$z_{\bar{C}}(t) = (U \otimes I_n)[0, \varsigma^{\mathrm{T}}(t)]^{\mathrm{T}} \quad (10)$$

Owing to the fact that $U^{\mathrm{T}} \otimes I_n$ is non-singular, by (9) and (10), one knows that $z_C(t)$ and $z_{\bar{C}}(t)$ are linearly independent. Therefore from (7) and (8), one sees that the

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subsystems with states $z_C(t)$ and $z_{\bar{C}}(t)$ describe the consensus dynamics and disagreement dynamics of swarm system (4), respectively. From Lemma 2, it follows that swarm system (3) achieves time-varying formation $h(t)$ if and only if $\lim_{t\to\infty} z_{\bar{C}}(t) = 0$; that is, $\lim_{t\to\infty} \varsigma(t) = 0$. This completes the proof. □

Remark 4: Xie and Wang [22] studied formation problems for second-order swarm systems with fixed undirected interaction topologies and proposed a sufficient condition for swarm systems to achieve constant formation. It can be verified by Theorem 1 that the problems in [22] is a special case of the current paper, and the condition in Theorem 3.1 of [22] is not only sufficient but also necessary.

In the sequel, based on the above analysis, an explicit expression of the formation reference function is given.

Theorem 2: If swarm system (3) achieves time-varying formation $h(t)$, then

$$\lim_{t\to\infty} (r(t) - r_0(t) - r_v(t) - r_h(t)) = 0$$

where

$$r_0(t) = e^{(A+BK_1+BK_2)t}\left(\frac{1}{N}\sum_{i=1}^{N} x_i(0)\right)$$

$$r_v(t) = \int_0^t \left(e^{(A+BK_1+BK_2)(t-\tau)} B\left(\frac{1}{N}\sum_{i=1}^{N} v_i(\tau)\right)\right) \mathrm{d}\tau$$

$$r_h(t) = -\frac{1}{N}\sum_{i=1}^{N} h_i(t) - \int_0^t e^{(A+BK_1+BK_2)(t-\tau)} BK_2 \times \left(\frac{1}{N}\sum_{i=1}^{N} h_i(\tau)\right) \mathrm{d}\tau$$

Proof: If swarm system (3) achieves formation $h(t)$, then $\lim_{t\to\infty} \varsigma(t) = 0$. From (7)–(10), one has

$$\lim_{t\to\infty}\left(z_i(t) - \frac{1}{\sqrt{N}}\theta(t)\right) = 0 \tag{11}$$

It can be shown that

$$\theta(0) = \frac{1}{\sqrt{N}}\left(\mathbf{1}_N^{\mathrm{T}} \otimes I_n\right) z(0) = \frac{1}{\sqrt{N}}\left(\mathbf{1}_N^{\mathrm{T}} \otimes I_n\right)(x(0) - h(0)) \tag{12}$$

and

$$\begin{aligned}\int_0^t & e^{(A+BK_1+BK_2)(t-\tau)}(\bar{u}_1^{\mathrm{T}} \otimes I_n)\dot{h}(\tau)d\tau \\ &= (\bar{u}_1^{\mathrm{T}} \otimes I_n)h(t) - e^{(A+BK_1+BK_2)t}(\bar{u}_1^{\mathrm{T}} \otimes I_n)h(0) \\ &\quad + \int_0^t e^{(A+BK_1+BK_2)(t-\tau)}\left(\bar{u}_1^{\mathrm{T}} \otimes (A + BK_1 + BK_2)\right) h(\tau)\,\mathrm{d}\tau\end{aligned} \tag{13}$$

From (5) and (11)–(13), the conclusion of Theorem 2 can be obtained. □

Remark 5: In Theorem 2, $r_0(t)$ is said to be the consensus function which describes the formation reference of the swarm system without $v(t)$ and $h(t)$. $r_v(t)$ and $r_h(t)$ describe the impacts of $v(t)$ and $h(t)$, respectively. If $h(t) \equiv 0$, $r(t)$ becomes the explicit expression of the consensus function. Moreover, from Theorem 2, one sees that the switching interaction topologies have no effect on $r(t)$ and K_2 can be used to design the motion modes of the formation reference.

4 Time-varying formation feasibility and protocol design

In this section, necessary and sufficient conditions for time-varying formation feasibility are presented and an algorithm to design the protocol for swarm systems with switching interaction topologies to achieve time-varying formation is given.

By Assumption 1, there exists a non-singular matrix $\hat{B} = [\bar{B}^{\mathrm{T}}, \tilde{B}^{\mathrm{T}}]^{\mathrm{T}}$ with $\bar{B} \in \mathbb{R}^{m\times n}$ and $\tilde{B} \in \mathbb{R}^{(n-m)\times n}$ such that $\bar{B}B = I_m$ and $\tilde{B}B = 0$.

Theorem 3: A time-varying formation $h(t)$ is feasible for swarm system (3) with any bounded initial states if and only if the following conditions hold simultaneously

(i) For $\forall i \in \{1, 2, \ldots, N\}$

$$\lim_{t\to\infty}\left(\tilde{B}A\left(h_i(t) - h_j(t)\right) - \tilde{B}\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\right) = 0, \quad j \in N_i(t) \tag{14}$$

(ii) The following system is asymptotically stable

$$\begin{aligned}\dot{\varphi}(t) = \Big(&I_{N-1} \otimes (A + BK_1 + BK_2) \\ &- (\tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}) \otimes BK_3\Big)\varphi(t)\end{aligned} \tag{15}$$

Proof: See the appendix. □

Remark 6: Theorem 3 indicates that the feasibility of the time-varying formation $h(t)$ depends on the dynamics of each agent, switching interaction topologies and external command input $v(t)$. From (23) and (24), one knows that the application of $v(t)$ can expand the set of feasible time-varying formation $h(t)$, and K_1 has no directly effect on the feasible set of $h(t)$ when $v(t)$ is applied.

From Theorem 3, the following corollaries can be obtained directly.

Corollary 1: If $v(t) \equiv 0$, a time-varying formation $h(t)$ is feasible for swarm system (3) with any bounded initial states if and only if condition (ii) in Theorem 3 holds, and for $\forall i \in \{1, 2, \ldots, N\}$

$$\lim_{t\to\infty}\left((A + BK_1)\left(h_i(t) - h_j(t)\right) - \left(\dot{h}_i(t) - \dot{h}_j(t)\right)\right) = 0, \quad j \in N_i(t) \tag{16}$$

Remark 7: From (16), it can be found that that K_1 can be used to expand the set of feasible time-varying formation $h(t)$ in the case that $v(t) \equiv 0$. Moreover, if $h(t) \equiv 0$, Corollary 1 presents necessary and sufficient conditions for swarm system (3) with switching interaction topologies to achieve consensus.

Corollary 2: If $v(t) \equiv 0$ and the formation is given by a constant vector h, then the formation is feasible for swarm system (3) with any bounded initial states if and only if condition (ii) in Theorem 3 holds, and for $\forall i \in \{1, 2, \ldots, N\}$

$$(A + BK_1)(h_i - h_j) = 0, \quad j \in N_i(t) \tag{17}$$

Remark 8: In Corollary 2, if formation h is feasible for swarm system (3) with switching interaction topologies, then for $\forall i \in \{1, 2, \ldots, N\}$ and $j \in N_i(t)$, $h_i - h_j$ must belong to the right null space of $A + BK_1$. Moreover, K_1 can be used to expand the set of feasible formation h.

Let $\lambda^i_{\sigma(t)}$ $(i = 1, 2, \ldots, N)$ be the eigenvalues of the Laplacian matrix $L_{\sigma(t)}$, where $\lambda^1_{\sigma(t)} = 0$ with the associated eigenvector $\bar{u}_1 = \mathbf{1}/\sqrt{N}$ and $0 < \lambda^2_{\sigma(t)} \leq \cdots \leq \lambda^N_{\sigma(t)}$. Let $\lambda_{\min} = \min\{\lambda^i_k(\forall k \in \bar{I}; i = 2, 3, \ldots, N)\}$. The following theorem presents an approach to determine K_3.

Theorem 4: If condition (i) in Theorem 3 holds and (A, B) is stabilisable, then swarm system (1) achieves time-varying formation $h(t)$ by protocol (2) with $K_3 = \lambda^{-1}_{\min} R_o^{-1} B^{\mathrm{T}} P_o / 2$ where P_o is the positive definite solution to the algebraic Riccati equation

$$\begin{aligned} &P_o(A + BK_1 + BK_2) + (A + BK_1 + BK_2)^{\mathrm{T}} P_o \\ &\quad - P_o B R_o^{-1} B^{\mathrm{T}} P_o + Q_o = 0 \end{aligned} \tag{18}$$

for $R_o = R_o^{\mathrm{T}} > 0$ and $Q_o = D_o^{\mathrm{T}} D_o \geq 0$ with $(A + BK_1 + BK_2, D_o)$ detectable.

Proof: If (A, B) is stabilisable, so is $(A + BK_1 + BK_2, B)$. Thus, for any given $R_o^{\mathrm{T}} = R_o > 0$ and $Q_o = D_o^{\mathrm{T}} D_o \geq 0$ with $(A + BK_1 + BK_2, D_o)$ detectable, algebraic Riccati equation (18) has a unique solution $P_o^{\mathrm{T}} = P_o > 0$.

Consider the following Lyapunov function candidate

$$\bar{V}(t) = \varphi^{\mathrm{T}}(t)\left(I_{N-1} \otimes P_o\right)\varphi(t) \tag{19}$$

Taking the derivative of $\bar{V}(t)$ with respect to t along the solution to system (15), one has

$$\begin{aligned} \dot{\bar{V}}(t) = \varphi^{\mathrm{T}}(t)\Big(&I_{N-1} \otimes \Xi - (\tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}) \\ &\otimes \left(K_3^{\mathrm{T}} B^{\mathrm{T}} P_o + P_o B K_3\right)\Big)\varphi(t) \end{aligned} \tag{20}$$

where $\Xi = (A + BK_1 + BK_2)^{\mathrm{T}} P_o + P_o(A + BK_1 + BK_2)$.

Since $U^{\mathrm{T}} L_{\sigma(t)} U = \mathrm{diag}\{0, \tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}\}$ and U is orthogonal, one knows that the eigenvalues of $\tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}$ are $\lambda^2_{\sigma(t)}, \lambda^3_{\sigma(t)}, \ldots, \lambda^N_{\sigma(t)}$. Note that $\tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U}$ is symmetric, then there exists an orthogonal matrix $\bar{U}_{\sigma(t)}$ such that

$$\bar{U}^{\mathrm{T}}_{\sigma(t)} \tilde{U}^{\mathrm{T}} L_{\sigma(t)} \tilde{U} \bar{U}_{\sigma(t)} = \mathrm{diag}\left\{\lambda^2_{\sigma(t)}, \lambda^3_{\sigma(t)}, \ldots, \lambda^N_{\sigma(t)}\right\} \tag{21}$$

Let

$$\begin{aligned} \xi_{\sigma(t)}(t) &= \left(\bar{U}^{\mathrm{T}}_{\sigma(t)} \otimes I_n\right)\varphi(t) \\ &= \left[\left(\xi^2_{\sigma(t)}(t)\right)^{\mathrm{T}}, \left(\xi^3_{\sigma(t)}(t)\right)^{\mathrm{T}}, \ldots, \left(\xi^N_{\sigma(t)}(t)\right)^{\mathrm{T}}\right]^{\mathrm{T}} \end{aligned}$$

Then by (20) and (21), it can be obtained that

$$\dot{\bar{V}}(t) = \sum_{i=2}^{N} \left(\xi^i_{\sigma(t)}(t)\right)^{\mathrm{T}} \left(\Xi - \lambda^i_{\sigma(t)}\left(K_3^{\mathrm{T}} B^{\mathrm{T}} P_o + P_o B K_3\right)\right) \xi^i_{\sigma(t)}(t) \tag{22}$$

Substituting $K_3 = \lambda^{-1}_{\min} R_o^{-1} B^{\mathrm{T}} P_o / 2$ into (22), from (18), one has

$$\begin{aligned} \dot{\bar{V}}(t) = \sum_{i=2}^{N} \left(\xi^i_{\sigma(t)}(t)\right)^{\mathrm{T}} &\left(-Q_o + \left(1 - \lambda^i_{\sigma(t)} \lambda^{-1}_{\min}\right)\right. \\ &\times \left. P_o B R_o^{-1} B^{\mathrm{T}} P_o\right) \xi^i_{\sigma(t)}(t) \end{aligned}$$

Because $Q_o^{\mathrm{T}} = Q_o > 0$, $R_o^{\mathrm{T}} = R_o > 0$ and $1 - \lambda^i_{\sigma(t)} \lambda^{-1}_{\min} < 0$, by Assumption 2, one obtains that $\dot{\bar{V}}(t) \equiv 0$ if and only if $\xi^i_{\sigma(t)}(t) \equiv 0$ $(i = 2, 3, \ldots, N)$, which means that $\xi_{\sigma(t)}(t) \equiv 0$. Note that $\xi_{\sigma(t)}(t) = \left(\bar{U}^{\mathrm{T}}_{\sigma(t)} \otimes I_n\right)\varphi(t)$, $\dot{\bar{V}}(t) \equiv 0$ if and only if $\varphi(t) \equiv 0$. Therefore system (15) is asymptotically stable. From Theorem 3, one knows that swarm system (1) under protocol (2) achieves time-varying formation $h(t)$. The proof for Theorem 4 is completed. □

Based on the above results, an algorithm to design protocol (2) for swarm system (1) to achieve time-varying formation $h(t)$ can be summarised as follows.

Algorithm: For swarm system (3) to achieve time-varying formation $h(t)$, K_i $(i = 1, 2, 3)$ and $v_i(t)$ $(i = 1, 2, \ldots, N)$ can be designed in the following procedure:

Step 1: Check the feasible condition (14). If it is satisfied, then $v_i(t)$ $(i = 1, 2, \ldots, N)$ can be determined by equation (30), and K_1 can be any constant matrix with appropriate dimension, for example, $K_1 = 0$; else $h(t)$ is not feasible and stop.

If it is required that $v(t) \equiv 0$, solve the feasible condition (16) for K_1. If there exists constant gain matrix K_1 to satisfy condition (16), then continue; else $h(t)$ is not feasible and stop.

Step 2: Choose K_2 to specify the motion modes of the formation reference by assigning the eigenvalues of $A + BK_1 + BK_2$ at the desired locations in the complex plane. Since (A, B) is controllable, the existence of K_2 can be guaranteed.

Step 3: Design K_3 to make system (15) asymptotically stable using the approach in Theorem 4.

Remark 9: It should be emphasised that by equation (30), $v_i(t)$ $(i = 1, 2, \ldots, N)$ cannot be uniquely determined. One can first specify a $v_i(t)$ $(i \in \{1, 2, \ldots, N\})$, then determine the other $v_j(t)$ $(j \in \{1, 2, \ldots, N\}, j \neq i)$ by equation (30).

5 Numerical simulations

In this section, a numerical example is given to illustrate the effectiveness of theoretical results obtained in the previous sections. For simplicity of description, assume that in the example the interaction topologies of the swarm system are randomly chosen from $\bar{S}$ with interval T_d, where $\bar{S}$ consists of four undirected interaction topologies as shown in Fig. 1, and the interaction topologies have 0–1 weights.

Consider a third-order swarm system with eight agents, where the dynamics of each agent is described by (1) with $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^{\mathrm{T}}$ $(i = 1, 2, \ldots, 8)$, and

$$A = \begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 5 \\ 2 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

These eight agents are required to preserve a periodic time-varying parallel octagon formation and keep rotation

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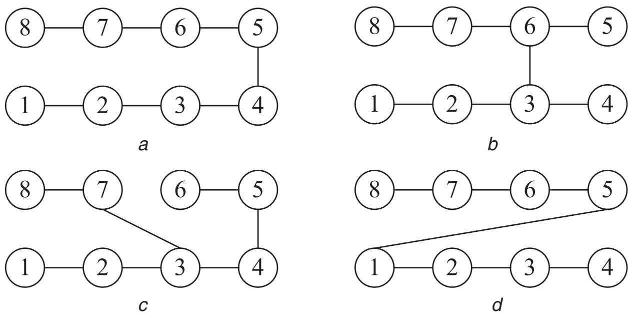

Fig. 1 *Interaction topologies*
a G_1
b G_2
c G_3
d G_4

around the predefined time-varying formation reference. The formation is defined as follows

$$h_i(t) = \begin{bmatrix} r\sin\left(\omega t + \dfrac{(i-1)\pi}{3}\right) \\ 2r\sin\left(\omega t + \dfrac{(i-1)\pi}{3}\right) \\ r\cos\left(\omega t + \dfrac{(i-1)\pi}{3}\right) \end{bmatrix} \quad (i = 1, 2, \ldots, 8)$$

If the formation specified by the above $h_i(t)$ $(i = 1, 2, \ldots, 8)$ is achieved, the eight agents will locate on the eight vertices of a parallel octagon respectively and keep rotation with an angular velocity of ω. Moreover, the edge length of the desired parallel octagon is periodic time-varying.

Choose $r = 6$ and $\omega = 2$. According to the Algorithm, K_1 can be any constant matrix with appropriate dimension,

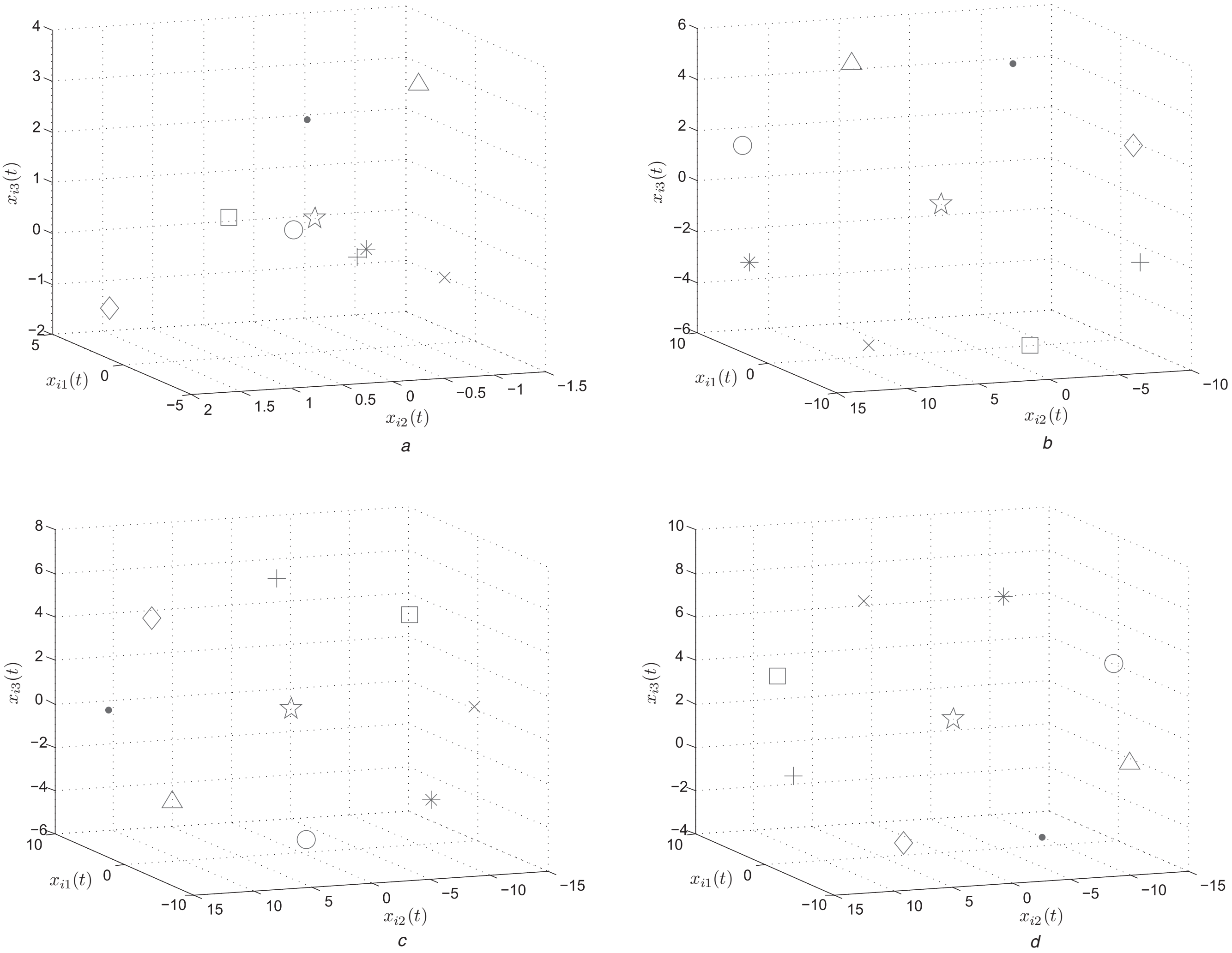

Fig. 2 *State snapshots of eight agents and* $r(t)$
a $t = 0$ *s*
b $t = 98$ *s*
c $t = 99$ *s*
d $t = 100$ *s*

$$v_i(t) = \begin{bmatrix} -42\sin\left(2t + \dfrac{\pi}{4}(i-1)\right) - 6\cos\left(2t + \dfrac{\pi}{4}(i-1)\right) \\ -108\sin\left(2t + \dfrac{\pi}{4}(i-1)\right) - 24\cos\left(2t + \dfrac{\pi}{4}(i-1)\right) \end{bmatrix} \quad (i = 1, 2, \ldots, 8)$$

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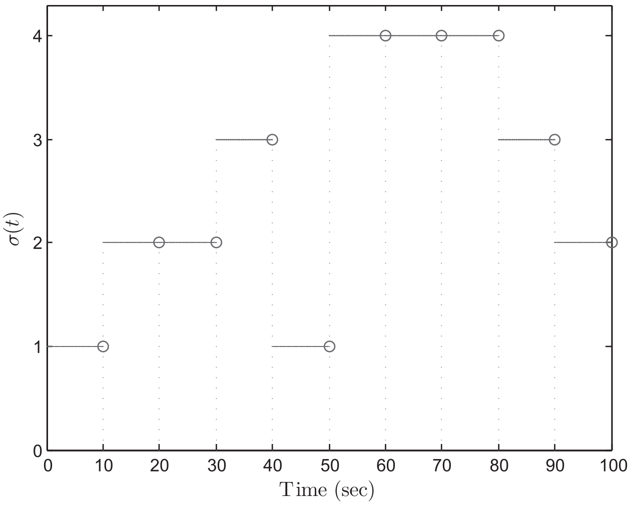

Fig. 3 *Switching signal*

for example, $K_1 = 0$ and (see equation at the bottom of the page)

The motion modes of the formation reference are placed at $-2j$, $2j$ and 0.01 with $j^2 = -1$ by

$$K_2 = \begin{bmatrix} 3.9988 & -4.9905 & -3.0122 \\ -7.0005 & -4.9783 & -5.9995 \end{bmatrix}$$

Using the approach in Theorem 4, K_3 can be obtained to make system (15) asymptotically stable as

$$K_3 = \begin{bmatrix} -13.9520 & 8.5232 & -2.4500 \\ 7.0585 & -2.4500 & 5.0634 \end{bmatrix}$$

For simplicity of description, let the initial states of each agent be $x_{ij}(0) = i\left(\bar{\Theta} - 0.5\right)$ $(i = 1, 2, \ldots, 8;\ j = 1, 2, 3)$, where $\bar{\Theta}$ is a pseudo-random value with a uniform distribution on the interval $(0, 1)$. Choose $T_d = 10$ s. Fig. 2 shows the snapshots of the agents and the predefined formation reference at different time, where the states of eight agents are denoted by the point, triangle, circle, asterisk, x-mark, square, plus and diamond respectively and the state of the predefined formation reference are marked by the pentagram. Fig. 3 displays the switching signal. From Figs. 2*a* and *b*, one sees that the swarm system achieves a parallel octagon formation and the point corresponding to the predefined formation reference lies in the centre of the formation. From Figs. 2(b), (c) and (d), it can be seen that the achieved formation keeps rotating around the predefined formation reference, and both the edge length of parallel octagon formation and the formation reference are time-varying. Therefore the time-varying formation is achieved under the switching interaction topologies.

6 Conclusions

Time-varying formation analysis and feasibility problems for general high-order LTI swarm system with switching interaction topologies were studied. Necessary and sufficient conditions for swarm systems with switching interaction topologies to achieve a given time-varying formation were presented. An explicit expression of the time-varying formation reference function was derived. Necessary and sufficient conditions for formation feasibility were proposed and approaches to expand the feasible formation set were presented. An algorithm to design the protocol for swarm systems with switching interaction topologies to achieve a given time-varying formation was proposed. This paper focuses on switching undirected interaction topologies. As interesting and important research topics, one can aim at analysing the time-varying formation problems for high-order LTI swarm systems with switching directed interaction topologies.

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9 Appendix

Proof: (Theorem 3) Necessity: If a time-varying formation $h(t)$ is feasible for swarm system (3), then there exist K_i $(i = 1, 2, 3)$ and $v(t)$ such that the formation $h(t)$ is achieved; that is

$$\lim_{t\to\infty}(z_i(t) - c(t)) = 0 \ (i = 1, 2, \ldots, N) \tag{23}$$

From Theorem 1 and (6), one knows the facts that

$$\begin{aligned}\lim_{t\to\infty}\Big(\Big(\tilde{U}^{\mathrm{T}} \otimes (A + BK_1)\Big) h(t) \\ - (\tilde{U}^{\mathrm{T}} \otimes I_n)\dot{h}(t) + (\tilde{U}^{\mathrm{T}} \otimes B)v(t)\Big) = 0\end{aligned} \tag{24}$$

and that the system described by (15) is asymptotically stable are necessary conditions for (23). Hence, condition (ii) is required.

Let $\tilde{U}^{\mathrm{T}} = [\hat{U}, \hat{u}]$ with $\hat{U} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\hat{u} \in \mathbb{R}^{(N-1)\times 1}$. Since $\operatorname{rank}(\tilde{U}^{\mathrm{T}}) = N - 1$, without loss of generality, it is assumed that $\operatorname{rank}(\hat{U}) = N - 1$. From (24), one obtains

$$\begin{aligned}\lim_{t\to\infty}\Big(\Big([\hat{U}, \hat{u}] \otimes (A + BK_1)\Big) h(t) \\ - ([\hat{U}, \hat{u}] \otimes I_n)\dot{h}(t) + ([\hat{U}, \hat{u}] \otimes B)v(t)\Big) = 0\end{aligned} \tag{25}$$

Note that $\tilde{U}^{\mathrm{T}}\mathbf{1}_N = 0$, it follows that

$$\hat{u} = -\hat{U}\mathbf{1}_{N-1} \tag{26}$$

Let $\bar{h}(t) = [h_1^{\mathrm{T}}(t), h_2^{\mathrm{T}}(t), \ldots, h_{N-1}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\bar{v}(t) = [v_1^{\mathrm{T}}(t), v_2^{\mathrm{T}}(t), \ldots, v_{N-1}^{\mathrm{T}}(t)]^{\mathrm{T}}$. From (25) and (26), one has

$$\lim_{t\to\infty}\left(\hat{U} \otimes I_n\right)\left(\Xi_{\bar{h}} - \Xi_{h_N}\right) = 0 \tag{27}$$

where

$$\begin{aligned}\Xi_{\bar{h}} &= (I_{N-1} \otimes (A + BK_1))\,\bar{h}(t) \\ &\quad - (I_{N-1} \otimes I_n)\,\dot{\bar{h}}(t) + (I_{N-1} \otimes B)\,\bar{v}(t) \\ \Xi_{h_N} &= (\mathbf{1}_{N-1} \otimes (A + BK_1))\,h_N(t) \\ &\quad - (\mathbf{1}_{N-1} \otimes I_n)\,\dot{h}_N(t) + (\mathbf{1}_{N-1} \otimes B)\,v_N(t)\end{aligned}$$

Since $\hat{U}$ is non-singular, from (27), it follows that for $\forall i \in \{1, 2, \ldots, N-1\}$

$$\begin{aligned}\lim_{t\to\infty}\Big((A + BK_1)\,(h_i(t) - h_N(t)) - \left(\dot{h}_i(t) - \dot{h}_N(t)\right) \\ + B\,(v_i(t) - v_N(t))\Big) = 0\end{aligned} \tag{28}$$

Equation (28) implies that for $\forall i \in \{1, 2, \ldots, N\}$ and $j \in N_i(t)$

$$\begin{aligned}\lim_{t\to\infty}\Big((A + BK_1)\left(h_i(t) - h_j(t)\right) - \left(\dot{h}_i(t) - \dot{h}_j(t)\right) \\ + B\left(v_i(t) - v_j(t)\right)\Big) = 0\end{aligned} \tag{29}$$

Pre-multiplying the both sides of (29) by $\hat{B}$ results in

$$\begin{aligned}\lim_{t\to\infty}\Big(\bar{B}(A + BK_1)\left(h_i(t) - h_j(t)\right) \\ - \bar{B}\left(\dot{h}_i(t) - \dot{h}_j(t)\right) + v_i(t) - v_j(t)\Big) = 0\end{aligned} \tag{30}$$

and

$$\lim_{t\to\infty}\left(\tilde{B}A\left(h_i(t) - h_j(t)\right) - \tilde{B}\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\right) = 0 \tag{31}$$

Choosing appropriate $v(t)$ can guarantee (30) holds for all $i, j \in \{1, 2, \ldots, N\}$. From (31), one knows that condition (i) is necessary.

Sufficiency: If condition (i) holds for time-varying formation $h(t)$, one has that for $\forall i \in \{1, 2, \ldots, N\}$ and $j \in N_i(t)$

$$\lim_{t\to\infty}\left(\tilde{B}(A + BK_1)\left(h_i(t) - h_j(t)\right) - \tilde{B}\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\right) = 0 \tag{32}$$

For $\forall i, j \in \{1, 2, \ldots, N\}$, one can find $v_i(t) - v_j(t)$ satisfying (30). From (30) and (32), it can be shown that

$$\begin{aligned}\lim_{t\to\infty}\Big(\hat{B}(A + BK_1)\left(h_i(t) - h_j(t)\right) - \hat{B}\left(\dot{h}_i(t) - \dot{h}_j(t)\right) \\ + \hat{B}B\left(v_i(t) - v_j(t)\right)\Big) = 0\end{aligned} \tag{33}$$

Pre-multiplying the both sides of (33) by $\hat{B}^{-1}$, one has

$$\begin{aligned}\lim_{t\to\infty}\Big((A + BK_1)\left(h_i(t) - h_j(t)\right) - \left(\dot{h}_i(t) - \dot{h}_j(t)\right) \\ + B\left(v_i(t) - v_j(t)\right)\Big) = 0\end{aligned} \tag{34}$$

From (34), one knows

$$\lim_{t\to\infty}\left(\left(L_{\sigma(t)}\otimes(A+BK_1)\right)h(t)-\left(L_{\sigma(t)}\otimes I_n\right)\dot{h}(t)\right.$$
$$\left.+\left(L_{\sigma(t)}\otimes B\right)v(t)\right)=0 \tag{35}$$

Substituting $L_{\sigma(t)}=U\mathrm{diag}\{0,\tilde{U}^{\mathrm{T}}L_{\sigma(t)}\tilde{U}\}U^{\mathrm{T}}$ into (35) and pre-multiplying the both sides of (35) by $U^{\mathrm{T}}\otimes I_n$ lead to

$$\lim_{t\to\infty}\left((\tilde{U}^{\mathrm{T}}L_{\sigma(t)}\tilde{U})\otimes I_n\right)\left(\left(\tilde{U}^{\mathrm{T}}\otimes(A+BK_1)\right)h(t)\right.$$
$$\left.-\left(\tilde{U}^{\mathrm{T}}\otimes I_n\right)\dot{h}(t)+\left(\tilde{U}^{\mathrm{T}}\otimes B\right)v(t)\right)=0 \tag{36}$$

Since the interaction topology is connected, from Lemma 1 and the structure of U, one has that $\tilde{U}^{\mathrm{T}}L_{\sigma(t)}\tilde{U}$ is non-singular and

$$\lim_{t\to\infty}\left(\left(\tilde{U}^{\mathrm{T}}\otimes(A+BK_1)\right)h(t)-\left(\tilde{U}^{\mathrm{T}}\otimes I_n\right)\dot{h}(t)\right.$$
$$\left.+\left(\tilde{U}^{\mathrm{T}}\otimes B\right)v(t)\right)=0 \tag{37}$$

Equations (6), (37) and condition (ii) ensure that $\lim_{t\to\infty}\varsigma(t)=0$. Then from Theorem 1, one can conclude that the time-varying formation $h(t)$ is feasible for swarm system (3) with any bounded initial states. This completes the proof. □

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