Automatica 73 (2016) 47–55

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Time-varying formation control for general linear multi-agent systems with switching directed topologies☆

Xiwang Dong [a,b], Guoqiang Hu [b]

[a] School of Automation Science and Electrical Engineering, Beihang University, Beijing, 100191, PR China

[b] School of Electrical and Electronic Engineering, Nanyang Technological University, 639798, Singapore

ARTICLE INFO

Article history:
Received 5 April 2015
Received in revised form
5 April 2016
Accepted 5 June 2016

Keywords:
Time-varying formation
Multi-agent system
Switching directed topology
General linear dynamics

ABSTRACT

Time-varying formation analysis and design problems for multi-agent systems with general linear dynamics and switching directed interaction topologies are investigated. Different from the previous results, the formation in this paper can be defined by specified piecewise continuously differentiable vectors and the switching topologies are directed. Firstly, necessary and sufficient conditions for general linear multi-agent systems with switching directed topologies to achieve time-varying formations are proposed, where a description of the feasible time-varying formation set and approaches to expand the feasible formation set are given. Then an explicit expression of the time-varying formation reference function is derived to describe the macroscopic movement of the whole formation. An approach to assign the motion modes of the formation reference is provided. Moreover, an algorithm consisting of four steps to design the formation protocol is presented. In the case where the given time-varying formation belongs to the feasible formation set, it is proven that by designing the formation protocol using the proposed algorithm, time-varying formation can be achieved by multi-agent systems with general linear dynamics and switching directed topologies if the dwell time is larger than a positive threshold. Finally, numerical simulations are presented to demonstrate the effectiveness of the theoretical results.

1. Introduction

Cooperative control of multi-agent systems has received significant attention from both scientific and engineering communities in recent years. This research field includes consensus control (Ren & Beard, 2005; Zhou & Lin, 2014), rendezvous control (Dong & Huang, 2014; Zavlanos, Tanner, Jadbabaie, & Pappas, 2009), containment control (Ji, Ferrari-Trecate, Egerstedt, & Buffa, 2008; Notarstefano, Egerstedt, & Haque, 2011) and formation control (Navaravong, Kan, She, & Dixon, 2012; Wang, Huang, Wen, & Fan, 2014), etc. As one of the most important research topics, formation control of multi-agent systems has broad range of applications in various areas, such as unmanned aerial vehicles (Dong, Yu, Shi, & Zhong, 2015; Karimoddini, Lin, Chen, & Lee, 2013), mobile robots (Yoo & Kim, 2015; Zheng, Lin, Fu, & Sun, 2015), autonomous underwater vehicles (Leonard et al., 2010; Wang, Yan, & Li, 2012). As a matter of fact, formation control problems have been studied a lot in robotics community during the past decades, and three formation control approaches, namely, leader–follower based approach (Das, Fierro, Kumar, & Ostrowski, 2002), behavior based approach (Balch & Arkin, 1998) and virtual structure based approach (Lewis & Tan, 1997), have been proposed.

One of the main challenges in formation control of multi-agent systems lies in the fact that each agent usually cannot rely on centralized coordination and has to use local information to achieve the desired formation (see the latest survey paper (Oh, Park, & Ahn, 2015) for more details). Ren (2007) proposed a consensus based formation control approaches for second-order multi-agent systems and proved that leader–follower, behavior and virtual structure based approaches can be unified in the framework of consensus based approaches. Consensus or graph based formation control problems for first-order and second-order multi-agent systems were studied in Antonelli, Arrichiello, Caccavale, and Marino (2014), Du, Li, and Lin (2013), Guo, Zavlanos, and Dimarogonas (2014), Guzey, Dierks, and Jagannathan (2015), Hawwary (2015), Liu and Jiang (2013), Lin, Wang, Han, and Fu (2015), Mylvaganam and Astolfi (2015), Tian and Wang (2013),

☆ This work was supported by Singapore MOE AcRF Tier 1 Grant RG60/12(2012-T1-001-158). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Dimos V. Dimarogonas under the direction of Editor Christos G. Cassandras.

E-mail addresses: xwdong@buaa.edu.cn (X. Dong), gqhu@ntu.edu.sg (G. Hu).

http://dx.doi.org/10.1016/j.automatica.2016.06.024
0005-1098/

Xiao, Wang, Chen, and Gao (2009) and Wang, Xie, and Cao (2014). In some practical applications, the dynamics of each agent can only be described by high-order model. Results on time-invariant formation control of high-order linear multi-agent systems can be found in Fax and Murray (2004), Lafferriere, Williams, Caughman, and Veerman (2005), Ma and Zhang (2012) and Porfiri, Roberson, and Stilwell (2007). Time-varying formation control problems for high-order linear multi-agent systems with fixed and switching undirected interaction topologies were addressed in Dong, Shi, Lu, and Zhong (2014) and Dong, Xi, Lu, and Zhong (2014), respectively. In practice, undirected interaction topologies mean that the communication among agents is bidirected, which may consume twice the communication and energy resources used in the directed links. Directed interaction among agents is more practical. Therefore, it is meaningful to study time-varying formation control problems for general linear multi-agent systems with switching directed interaction topologies. Moreover, the Laplacian matrix for undirected interaction topology is symmetric while the Laplacian matrix for directed interaction topology does not have the symmetric structure, and the eigenvalues of the Laplacian matrix can be complex, which makes the analysis and design much complicated.

Motivated by the facts and challenges stated above, in this paper, time-varying formation analysis and design problems for multi-agent systems with general linear dynamics and switching directed interaction topologies are investigated. Compared with previous results on formation control, the contributions of the current paper are threefold. Firstly, the formation can be time-varying and each agent has general linear dynamics. In Du et al. (2013), Fax and Murray (2004), Lafferriere et al. (2005), Liu and Jiang (2013) Ma and Zhang (2012), Porfiri et al. (2007), Ren (2007), Tian and Wang (2013) and Xiao et al. (2009), the formation is assumed to be time-invariant. Because the time-varying formation will bring the derivative of the formation information to both the analysis and design, the results for time-invariant formations cannot be directly applied to time-varying formations. In Antonelli et al. (2014), Hawwary (2015) and Wang et al. (2014), the formation can be time-varying, but the dynamics of each agent is first-order. Secondly, the interaction topology can be switching and directed, and each possible topology only needs to have a spanning tree. However, the topologies in Dong et al. (2014) is fixed. Although the topologies in Dong et al. (2014) can be switching, each topology is required to be undirected and connected. The common Lyapunov functional approach used in Dong et al. (2014) cannot be applied to solve the switching directed topology problems in the current paper. Thirdly, a description of the feasible time-varying formation set and an explicit expression of the time-varying formation reference function are derived. It is revealed that switching topologies, dynamics of each agent, initial states of all the agents and the time-varying formation have effects on the macroscopic movement of the whole formation.

2. Preliminaries and problem description

In this section, firstly, basic notations, definitions and useful results on graph theory are introduced. Then the problem description is presented.

2.1. Preliminaries

Some notations used in this paper are given as follows. Let 0 and $\mathbf{1}$ be appropriate zero matrix and column vector of ones with scalar 0 and scalar 1 as special cases. Let I_N represent an identity matrix with dimension N, and $\otimes$ denote the Kronecker product. We use the superscript T to denote the transpose of a matrix.

The interaction topology of the general linear multi-agent system will be described by the directed graph $G = \{Q, E, W\}$, where $Q = \{q_1, q_2, \ldots, q_N\}$ is a node set, $E \subseteq \left\{(q_i, q_j) : q_i, q_j \in Q\right\}$ is an edge set, and $W = [w_{ij}] \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix. An edge in G is denoted by $q_{ij} = (q_i, q_j)$ $(i \neq j)$. The adjacency elements in W satisfy that $w_{ji} > 0$ if and only if $q_{ij} \in E$, and $w_{ij} = 0$ otherwise. Moreover, $w_{ii} = 0$ for all $i \in \{1, 2, \ldots, N\}$. The neighbor set of node q_i is denoted by $N_i = \left\{q_j \in Q : q_{ji} \in E\right\}$. Define the in-degree of node q_i as $\deg_{in}(q_i) = \sum_{j=1}^{N} w_{ij}$. Let $D = \operatorname{diag}\left\{\deg_{in}(q_i), i = 1, 2, \ldots, N\right\}$ be the degree matrix of G. The Laplacian matrix $L \in \mathbb{R}^{N\times N}$ of G is defined as $L = D - W$.

The directed interaction topology of the multi-agent system is assumed to be switching and there exists an infinite sequence of uniformly bounded non-overlapping time intervals $[t_k, t_{k+1})$ $(k \in \mathbb{N})$, with $t_1 = 0$, $0 < \tau_0 \leqslant t_{k+1} - t_k \leqslant \tau_1$, and $\mathbb{N}$ being the set of natural numbers. The time sequence t_k $(k \in \mathbb{N})$ is called the switching sequence, at which the interaction topology changes. τ_0 is named as the dwell time, during which the interaction topology keeps fixed. Let $\sigma(t) : [0, +\infty) \to \{1, 2, \ldots, p\}$ be a switching signal whose value at time t is the index of the topology. Define $G_{\sigma(t)}$ and $L_{\sigma(t)}$ as the corresponding interaction topology and Laplacian matrix at $\sigma(t)$. Let $N^i_{\sigma(t)}$ be the neighbor set of the ith agent at $\sigma(t)$. If for a given $\sigma(t) \in \{1, 2, \ldots, p\}$, $G_{\sigma(t)}$ has a spanning tree, then the Laplacian matrix $L_{\sigma(t)}$ has the following property.

Lemma 1 *(Ren & Beard, 2005). If $G_{\sigma(t)}$ contains a spanning tree, then zero is a simple eigenvalue of $L_{\sigma(t)}$ with associated right eigenvector $\mathbf{1}$, and all the other $N-1$ eigenvalues have positive real parts.*

2.2. Problem description

Consider a group of N agents. Suppose that each agent has the general linear dynamics described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \{1, 2, \ldots, N\}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and the control input of the ith agent, respectively.

Assumption 1. The matrix B is of full column rank, i.e., rank$(B) = m$.

Remark 1. As shown in Li, Ren, Liu, and Fu (2013), Yang, Wang, Hung, and Gani (2006) and Zhang, Feng, Qiu, and Shen (2013), Assumption 1 is standard and mild, which means that the columns of B are independent with each other and there exist no redundant control input components.

Assumption 2. Each possible topology $G_{\sigma(t)}$ contains a spanning tree.

The desired time-varying formation is specified by vector $h(t) = [h_1^T(t), h_2^T(t), \ldots, h_N^T(t)]^T \in \mathbb{R}^{nN}$ with $h_i(t)$ $(i = 1, 2, \ldots, N)$ piecewise continuously differentiable. It should be pointed out that $h(t)$ is only used to characterize the desired time-varying formation rather than providing reference trajectory for each agent to follow (see, e.g., Dong et al., 2015 for more details).

Definition 1. Multi-agent system (1) is said to achieve time-varying formation $h(t)$ if for any given bounded initial states, there exists a vector-valued function $r(t) \in \mathbb{R}^n$ such that $\lim_{t\to\infty}(x_i(t) - h_i(t) - r(t)) = 0$ $(i = 1, 2, \ldots, N)$, where $r(t)$ is called the formation reference function.

Consider the following time-varying formation control protocol with switching directed interaction topologies

$$\begin{aligned} u_i(t) &= K_1 x_i(t) + K_2 \left(x_i(t) - h_i(t)\right) \\ &\quad + \alpha K_3 \sum_{j \in N^i_{\sigma(t)}} w_{ij} \left((x_j(t) - h_j(t)) \right. \\ &\quad \left. - (x_i(t) - h_i(t)) \right) + v_i(t), \end{aligned} \tag{2}$$

where $i = 1, 2, \ldots, N$, $K_1, K_2, K_3 \in \mathbb{R}^{m\times n}$ are constant gain matrices, α is the positive coupling strength, and $v_i(t) \in \mathbb{R}^m$ represents the formation compensation signal dependent on $h_i(t)$.

Remark 2. In protocol (2), the gain matrix K_1 and compensation signal $v_i(t)$ $(i = 1, 2, \ldots, N)$ will be used to expand the feasible time-varying formation set. The gain matrix K_2 will be used to specify the motion modes of the time-varying formation reference $r(t)$. The gain matrix K_3 and the constant α can be used to drive the states of multi-agent system (1) to achieve the desired time-varying formation under switching directed topologies. It should be pointed out that K_1, K_2, and $v_i(t)$ $(i = 1, 2, \ldots, N)$ are dispensable for multi-agent system (1) to achieve some time-varying formations.

Let $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$ and $v(t) = [v_1^T(t), v_2^T(t), \ldots, v_N^T(t)]^T$. Under protocol (2) with switching directed topologies, multi-agent system (1) can be written in a compact form as follows

$$\begin{aligned} \dot{x}(t) &= \left(I_N \otimes (A + BK_1 + BK_2) - \alpha L_{\sigma(t)} \otimes BK_3\right) x(t) \\ &\quad + \left(\alpha L_{\sigma(t)} \otimes BK_3 - I_N \otimes BK_2\right) h(t) + (I_N \otimes B)\, v(t). \end{aligned} \tag{3}$$

The current paper mainly focuses on the following two problems for multi-agent system (3) with switching directed interaction topologies: (i) under what conditions the time-varying formation specified by $h(t)$ can be achieved, and (ii) how to design the formation control protocol (2).

3. Time-varying formation analysis

In this section, firstly, necessary and sufficient conditions for multi-agent system (3) with switching directed interaction topologies to achieve time-varying formation specified by $h(t)$ are presented. Then an explicit expression of the formation reference function is given to describe the macroscopic movement of the whole formation.

Let $\phi_i(t) = x_i(t) - h_i(t)$ and $\phi(t) = [\phi_1^T(t), \phi_2^T(t), \ldots, \phi_N^T(t)]^T$. Then multi-agent system (3) with switching directed interaction topologies can be rewritten as

$$\begin{aligned} \dot{\phi}(t) &= \left(I_N \otimes (A + BK_1 + BK_2) - \alpha L_{\sigma(t)} \otimes BK_3\right) \phi(t) \\ &\quad + \left(I_N \otimes (A + BK_1)\right) h(t) \\ &\quad - (I_N \otimes I_n)\, \dot{h}(t) + (I_N \otimes B)\, v(t). \end{aligned} \tag{4}$$

Let $U = [\tilde{u}_1, \tilde{U}] \in \mathbb{R}^{N\times N}$ be a nonsingular matrix with $\tilde{u}_1 = \mathbf{1}_N$ and $\tilde{U} = [\tilde{u}_2, \tilde{u}_3, \ldots, \tilde{u}_N]$. Let $U^{-1} = [\bar{u}_1^T, \bar{U}^T]^T$ with $\bar{U} = [\bar{u}_2^T, \bar{u}_3^T, \ldots, \bar{u}_N^T]^T$ and $\bar{u}_i \in \mathbb{R}^{1\times N}$. Then one has $U^{-1} L_{\sigma(t)} U = \begin{bmatrix} \bar{u}_1 \\ \bar{U} \end{bmatrix} L_{\sigma(t)} \begin{bmatrix} \tilde{u}_1 & \tilde{U} \end{bmatrix} = \begin{bmatrix} 0 & \bar{u}_1 L_{\sigma(t)} \tilde{U} \\ 0 & \bar{U} L_{\sigma(t)} \tilde{U} \end{bmatrix}$. If Assumption 2 is satisfied, it follows from Lemma 1 that all the eigenvalues of $\bar{U} L_{\sigma(t)} \tilde{U}$ have positive real parts, which means that $\bar{U} L_{\sigma(t)} \tilde{U}$ is nonsingular.

Let $\theta(t) = (U^{-1} \otimes I_n)\phi(t) = [\theta_1^T, \theta_2^T, \ldots, \theta_N^T]^T$ and $\vartheta(t) = [\theta_2^T, \theta_3^T, \ldots, \theta_N^T]^T$. Then multi-agent system (4) can be transformed into

$$\begin{aligned} \dot{\theta}_1(t) &= (A + BK_1 + BK_2)\, \theta_1(t) \\ &\quad - \alpha(\bar{u}_1 L_{\sigma(t)} \tilde{U}) \otimes BK_3 \vartheta(t) \\ &\quad + (\bar{u}_1 \otimes (A + BK_1))\, h(t) - (\bar{u}_1 \otimes I_n)\dot{h}(t) \\ &\quad + (\bar{u}_1 \otimes B) v(t), \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{\vartheta}(t) &= \Big(I_{N-1} \otimes (A + BK_1 + BK_2) \\ &\quad - \alpha(\bar{U} L_{\sigma(t)} \tilde{U}) \otimes BK_3\Big)\, \vartheta(t) \\ &\quad + \left(\bar{U} \otimes (A + BK_1)\right) h(t) - (\bar{U} \otimes I_n)\dot{h}(t) \\ &\quad + (\bar{U} \otimes B) v(t). \end{aligned} \tag{6}$$

It follows from Assumption 1 that there exists a nonsingular matrix $T = [\tilde{B}^T, \bar{B}^T]^T$ with $\tilde{B} \in \mathbb{R}^{m\times n}$ and $\bar{B} \in \mathbb{R}^{(n-m)\times n}$ such that $\tilde{B}B = I_m$ and $\bar{B}B = 0$. Let $h_{ij}(t) = h_i(t) - h_j(t)$ and $v_{ij}(t) = v_i(t) - v_j(t)$ $(i, j \in \{1, 2, \ldots, N\})$. The following theorem presents a necessary and sufficient condition for multi-agent system (3) to achieve time-varying formation specified by $h(t)$.

Theorem 1. *Multi-agent system (3) with switching directed interaction topologies achieves time-varying formation specified by $h(t)$ if and only if $\forall i \in \{1, 2, \ldots, N\}$, the following formation feasibility condition holds*

$$\lim_{t\to\infty} \left(\bar{B}Ah_{ij}(t) - \bar{B}\dot{h}_{ij}(t)\right) = 0, \quad j \in N^i_{\sigma(t)}; \tag{7}$$

and the switched linear system described by

$$\begin{aligned} \dot{\bar{\vartheta}}(t) &= \Big(I_{N-1} \otimes (A + BK_1 + BK_2) \\ &\quad - \alpha(\bar{U} L_{\sigma(t)} \tilde{U}) \otimes BK_3\Big)\, \bar{\vartheta}(t), \end{aligned} \tag{8}$$

is asymptotically stable.

Proof. Define auxiliary variables $\phi_C(t)$ and $\phi_{\bar{C}}(t)$ as

$$\phi_C(t) = (U \otimes I_n)[\theta_1^T(t), 0]^T, \tag{9}$$

$$\phi_{\bar{C}}(t) = (U \otimes I_n)[0, \vartheta^T(t)]^T. \tag{10}$$

It can be shown that $[\theta_1^T(t), 0]^T = e_1 \otimes \theta_1(t)$, where $e_1 \in \mathbb{R}^N$ has 1 as its first component and 0 elsewhere. Therefore,

$$\phi_C(t) = (U \otimes I_n)\left(e_1 \otimes \theta_1(t)\right) = Ue_1 \otimes \theta_1(t) = \mathbf{1}_N \otimes \theta_1(t). \tag{11}$$

Note that $\theta(t) = [\theta_1^T(t), \vartheta^T(t)]^T$ and $\phi(t) = (U \otimes I_n)\theta(t)$. From (9) and (10), one has

$$\phi(t) = \phi_C(t) + \phi_{\bar{C}}(t). \tag{12}$$

Since $U \otimes I_n$ is nonsingular, it follows from (9) and (10) that $\phi_C(t)$ and $\phi_{\bar{C}}(t)$ are linearly independent. From (11) and (12), one gets

$$\phi_{\bar{C}}(t) = \phi(t) - \mathbf{1}_N \otimes \theta_1(t). \tag{13}$$

From (10), (13) and the fact that $U \otimes I_n$ is nonsingular, one gets that $\lim_{t\to\infty}(\phi(t) - \mathbf{1}_N \otimes \theta_1(t)) = 0$ if and only if $\lim_{t\to\infty} \vartheta(t) = 0$. Note that $\phi(t) - \mathbf{1}_N \otimes \theta_1(t)$ can be rewritten as $x_i(t) - h_i(t) - \theta_1(t)$ $(i = 1, 2, \ldots, N)$. Therefore, multi-agent system (3) achieves time-varying formation if and only if

$$\lim_{t\to\infty} \vartheta(t) = 0, \tag{14}$$

which means that $\vartheta(t)$ describes the time-varying formation error. From (6), one gets that for any given bounded initial states, (14) holds if and only if

$$\begin{aligned} &\lim_{t\to\infty} \left((\bar{U} \otimes B)v(t) + \left(\bar{U} \otimes (A + BK_1)\right) h(t) - (\bar{U} \otimes I_n)\dot{h}(t)\right) \\ &\quad = 0, \end{aligned} \tag{15}$$

and the switched linear system described by (8) is asymptotically stable.

In the following it will be proven that condition (15) is equivalent to condition (7).

Necessity: If condition (7) holds, one has that $\forall i \in \{1, 2, \ldots, N\}$ and $j \in N^i_{\sigma(t)}$

$$\lim_{t\to\infty} \left(\bar{B}(A + BK_1)h_{ij}(t) - \bar{B}\dot{h}_{ij}(t) + \bar{B}Bv_{ij}(t)\right) = 0. \tag{16}$$

For any $i \in \{1, 2, \ldots, N\}$ and $j \in N^i_{\sigma(t)}$, one can find $v_i(t)$ and $v_j(t)$ satisfying

$$\lim_{t\to\infty}\left(\tilde{B}(A+BK_1)h_{ij}(t)-\tilde{B}\dot{h}_{ij}(t)+v_{ij}(t)\right)=0. \tag{17}$$

It follows from (16) and (17) that

$$\lim_{t\to\infty}\left(T(A+BK_1)h_{ij}(t)-T\dot{h}_{ij}(t)+TBv_{ij}(t)\right)=0. \tag{18}$$

Pre-multiplying both sides of (18) by T^{-1}, one gets that $\forall i \in \{1, 2, \ldots, N\}$ and $j \in N^i_{\sigma(t)}$

$$\lim_{t\to\infty}\left((A+BK_1)h_{ij}(t)-\dot{h}_{ij}(t)+Bv_{ij}(t)\right)=0. \tag{19}$$

From (19), one can obtain

$$\lim_{t\to\infty}\left(\left(L_{\sigma(t)}\otimes(A+BK_1)\right)h(t)-\left(L_{\sigma(t)}\otimes I_n\right)\dot{h}(t)\right.$$
$$\left.+\left(L_{\sigma(t)}\otimes B\right)v(t)\right)=0. \tag{20}$$

Substituting $L_{\sigma(t)}=U\begin{bmatrix}0 & \bar{u}_1L_{\sigma(t)}\tilde{U}\\ 0 & \bar{U}L_{\sigma(t)}\tilde{U}\end{bmatrix}U^{-1}$ into (20) and pre-multiplying both sides of (20) by $U^{-1}\otimes I_n$ lead to

$$\lim_{t\to\infty}\left(\left(\bar{U}L_{\sigma(t)}\tilde{U}\bar{U}\otimes(A+BK_1)\right)h(t)\right.$$
$$\left.-\left(\bar{U}L_{\sigma(t)}\tilde{U}\bar{U}\otimes I_n\right)\dot{h}(t)+\left(\bar{U}L_{\sigma(t)}\tilde{U}\bar{U}\otimes B\right)v(t)\right)=0. \tag{21}$$

Since $\bar{U}L_{\sigma(t)}\tilde{U}$ is invertible, pre-multiplying both sides of (21) by $(\bar{U}L_{\sigma(t)}\tilde{U})^{-1}\otimes I_n$ yields

$$\lim_{t\to\infty}\left(\left(\bar{U}\otimes(A+BK_1)\right)h(t)-\left(\bar{U}\otimes I_n\right)\dot{h}(t)+\left(\bar{U}\otimes B\right)v(t)\right)$$
$$=0,$$

that is, condition (15) is required.

Sufficiency: Recall that $\operatorname{rank}(\bar{U})=N-1$. Without loss of generality, let $\bar{U}=[\hat{u},\hat{U}]$, where $\hat{u}\in\mathbb{R}^{(N-1)\times 1}$ and $\hat{U}\in\mathbb{R}^{(N-1)\times(N-1)}$ is of full rank. If condition (15) holds, one has

$$\lim_{t\to\infty}([\hat{u},\hat{U}]\otimes B)v(t)+\left([\hat{u},\hat{U}]\otimes(A+BK_1)\right)h(t)$$
$$-([\hat{u},\hat{U}]\otimes I_n)\dot{h}(t)=0. \tag{22}$$

Due to $\bar{U}\mathbf{1}=0$, one gets $\hat{u}=-\hat{U}\mathbf{1}$. Let $\hat{h}(t)=[h_2^T(t),h_3^T(t),\ldots,h_N^T(t)]^T$ and $\hat{v}(t)=[v_2^T(t),v_3^T(t),\ldots,v_N^T(t)]^T$. Then it follows from (22) that

$$\lim_{t\to\infty}\left(\hat{U}\otimes I_n\right)\left(\bar{\Upsilon}-\tilde{\Upsilon}\right)=0, \tag{23}$$

where $\bar{\Upsilon}=(I_{N-1}\otimes(A+BK_1))\hat{h}(t)-(I_{N-1}\otimes I_n)\dot{\hat{h}}(t)+(I_{N-1}\otimes B)\hat{v}(t)$ and $\tilde{\Upsilon}=(\mathbf{1}\otimes(A+BK_1))h_1(t)-(\mathbf{1}\otimes I_n)\dot{h}_1(t)+(\mathbf{1}\otimes B)v_1(t)$. Note that $\hat{U}$ is invertible. Pre-multiplying both sides of (23) by $\hat{U}^{-1}\otimes I_n$ yields

$$\lim_{t\to\infty}\left((A+BK_1)h_{i1}(t)-\dot{h}_{i1}(t)+Bv_{i1}(t)\right)=0$$
$$(i=2,3,\ldots,N). \tag{24}$$

From (24), it can be obtained that for any $i\in\{1,2,\ldots,N\}$ and $j\in N^i_{\sigma(t)}$

$$\lim_{t\to\infty}\left((A+BK_1)h_{ij}(t)-\dot{h}_{ij}(t)+Bv_{ij}(t)\right)=0. \tag{25}$$

Pre-multiplying the both sides of (25) by T gives $\lim_{t\to\infty}\left(\bar{B}Ah_{ij}(t)-\bar{B}\dot{h}_{ij}(t)\right)=0$ $(i=1,2,\ldots,N;\ j\in N^i_{\sigma(t)})$. Therefore, condition (15) is equivalent to condition (7). Based on the above analysis, the conclusion of Theorem 1 can be obtained.

Remark 3. Condition (7) reveals that for any $i\in\{1,2,\ldots,N\}$ and $j\in N^i_{\sigma(t)}$, $Ah_{ij}(t)-\dot{h}_{ij}(t)$ must belong to the right null space or the kernel of $\bar{B}$, which means that not all the time-varying formation can be achieved by general high-order multi-agent systems with switching directed topologies. In other words, constraint (7) describes the feasible time-varying formation set which is determined by the dynamics of each agent and the switching directed topologies, and only the ones belonging to the feasible formation set can be achieved. From (7) and (19), one sees that the application of $v(t)$ expands the feasible formation set. Condition (8) is an asymptotic stability constraint for a switched linear system. For the switched linear system like (8), no testable necessary and sufficient criteria for the asymptotic stability of the system have been obtained in the literature, and the best stability result attained so far is that if for any $\sigma(t)$, $I_{N-1}\otimes(A+BK_1+BK_2)-\alpha(\bar{U}L_{\sigma(t)}\tilde{U})\otimes BK_3$ is Hurwitz and the dwell time is large enough, then system (8) is asymptotically stable (Allerhand & Shaked, 2011; Xiang, 2015). Based on the results of Theorem 1, testable sufficient conditions for multi-agent system (1) under protocol (2) to achieve time-varying formation will be further presented in Section 4.

Remark 4. Time-varying formation control problems for general linear multi-agent systems with fixed topologies and switching undirected topologies were investigated in Dong et al. (2014) and Dong et al. (2014), respectively. Different from Dong et al. (2014), the Laplacian matrix in this paper is not symmetric and it is impossible to find an orthogonal matrix U used in Dong et al. (2014). It can be verified that by setting the topologies in this paper to be fixed or switching undirected, the formation feasibility conditions in Dong et al. (2014) or Dong et al. (2014) are just special cases of the ones in Theorem 1 of this paper.

In the case where $v(t)\equiv 0$, the following corollary can be obtained directly from Theorem 1.

Corollary 1. *In the case where $v(t)\equiv 0$, multi-agent system (3) with switching directed interaction topologies achieves time-varying formation specified by $h(t)$ if and only if $\forall i\in\{1,2,\ldots,N\}$*

$$\lim_{t\to\infty}\left((A+BK_1)h_{ij}(t)-\dot{h}_{ij}(t)\right)=0,\quad j\in N^i_{\sigma(t)}, \tag{26}$$

and the switched linear system

$$\dot{\bar{\vartheta}}(t)=\left(I_{N-1}\otimes(A+BK_1+BK_2)-\alpha(\bar{U}L_{\sigma(t)}\tilde{U})\otimes BK_3\right)\bar{\vartheta}(t),$$

is asymptotically stable.

Remark 5. From constraint (26) in Corollary 1, one sees that in the case where $v(t)\equiv 0$, K_1 can be applied to expand the feasible time-varying formation set. Formation feasibility problems for general linear multi-agent systems to achieve time-invariant formations with fixed topologies were discussed in Ma and Zhang (2012). By choosing $v(t)\equiv 0$, $\dot{h}(t)\equiv 0$, $K_1=0$, $K_2=0$, $\alpha=1$, fixed topologies and appropriate U, Theorem 1 in Ma and Zhang (2012) can be treated as a special case of Corollary 1.

The formation reference represents the macroscopic movement of the whole formation. The following theorem reveals the effects of switching directed interaction topologies, dynamics of each agent, initial states of all the agents and time-varying formation on the evolution of the formation reference.

Theorem 2. *If multi-agent system (3) with switching directed interaction topologies achieves time-varying formation specified by $h(t)$, then the formation reference function $r(t)$ satisfies $\lim_{t\to\infty}(r(t)-(r_0(t)+r_\vartheta(t)+r_v(t)+r_h(t)))=0$, where $r_0(t)=e^{(A+BK_1+BK_2)t}(\bar{u}_1\otimes I_n)x(0)$, $r_\vartheta(t)=-\int_0^t e^{(A+BK_1+BK_2)(t-\tau)}\alpha$*

$\left(\bar{u}_1 L_{\sigma(t)}\tilde{U}\right) \otimes (BK_3)\,\vartheta(\tau)d\tau,\ r_v(t) = \int_0^t \left(e^{(A+BK_1+BK_2)(t-\tau)}(\bar{u}_1 \otimes B)\, v(\tau)\right)d\tau,\ r_h(t) = -(\bar{u}_1 \otimes I_n)h(t) - \int_0^t e^{(A+BK_1+BK_2)(t-\tau)}(\bar{u}_1 \otimes BK_2)\, h(\tau)d\tau$.

Proof. If multi-agent system (3) achieves time-varying formation specified by $h(t)$, it follows from Theorem 1 that the formation error converges to zero at $t \to \infty$; that is, $\lim_{t\to\infty} \vartheta(t) = 0$. From (10) and (13), one gets

$$\lim_{t\to\infty} (\phi_i(t) - \theta_1(t)) = 0 \quad (i = 1, 2, \ldots, N). \tag{27}$$

It holds that

$$\theta_1(0) = (\bar{u}_1 \otimes I_n)\,(x(0) - h(0))\,. \tag{28}$$

It can be obtained that

$$\begin{aligned}
&\int_0^t e^{(A+BK_1+BK_2)(t-\tau)}(\bar{u}_1 \otimes I_n)\dot{h}(\tau)d\tau \\
&= e^{(A+BK_1+BK_2)(t-\tau)}(\bar{u}_1 \otimes I_n)h(\tau)\Big|_{\tau=0}^{\tau=t} \\
&\quad - \int_0^t \frac{d}{d\tau}\left(e^{(A+BK_1+BK_2)(t-\tau)}\right)(\bar{u}_1 \otimes I_n)h(\tau)d\tau \\
&= (\bar{u}_1 \otimes I_n)h(t) - e^{(A+BK_1+BK_2)t}(\bar{u}_1 \otimes I_n)h(0) \\
&\quad - \int_0^t e^{(A+BK_1+BK_2)(t-\tau)}\left(-(A + BK_1 + BK_2)\right) \\
&\quad \times (\bar{u}_1 \otimes I_n)h(\tau)d\tau,
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
&(A + BK_1 + BK_2)(\bar{u}_1 \otimes I_n)h(\tau) \\
&= (\bar{u}_1 \otimes (A + BK_1 + BK_2))\,h(\tau).
\end{aligned} \tag{30}$$

In virtue of (5) and (27)–(30), one can derive the conclusions of Theorem 2.

Remark 6. Theorem 2 shows an explicit expression of the formation reference function $r(t)$ which describes the macroscopic movement of the whole time-varying formation. From Theorem 2, one sees that $r(t)$ is jointly determined by $r_0(t)$, $r_\vartheta(t)$, $r_v(t)$ and $r_h(t)$, where $r_0(t)$ is the nominal component determined by the dynamics of each agent and initial states, $r_\vartheta(t)$ describes the effect of switching directed topologies and the time-varying formation error, $r_v(t)$ and $r_h(t)$ represent the contributions of $v(t)$ and $h(t)$ to $r(t)$, respectively. It should be pointed out that although we can obtain the explicit expression of the formation reference, the trajectory of the formation reference cannot be specified arbitrarily in advance. However, from Theorem 2, K_2 can be used to specify the motion modes of the formation reference by assigning the eigenvalues of $A + BK_1 + BK_2$ at the desired places in the complex plane. Moreover, if $h(t) \equiv 0$, $r(t)$ becomes the explicit expression of the consensus function for general linear multi-agent systems with switching directed interaction topologies, which has not been obtained before.

4. Time-varying formation protocol design

In this section, firstly an algorithm to design the time-varying formation protocol (2) is proposed. Then it is proven that using the algorithm, time-varying formation can be achieved by multi-agent system (3) with switching directed topologies if the formation feasibility condition is satisfied and the dwell time is larger than a positive threshold.

Since the interaction topology $G_{\sigma(t)}$ has a spanning tree, from Lemma 1 and the structure of U, one knows that the real parts of all the eigenvalues of $\bar{U}L_{\sigma(t)}\tilde{U}$ are positive. Let $\hat{\mu}_{\sigma(t)} = \min\{\mathrm{Re}(\lambda_i(\bar{U}L_{\sigma(t)}\tilde{U})), i = 1, 2, \ldots, N-1\}$, where $\lambda_i(\bar{U}L_{\sigma(t)}\tilde{U})$ represents the ith eigenvalue of $\bar{U}L_{\sigma(t)}\tilde{U}$. Then from Lemma 3 in Saboori and Khorasani (2014), it can be obtained that for any $0 < \mu_{\sigma(t)} < \hat{\mu}_{\sigma(t)}$, there exists a symmetric positive definite matrix $\Xi_{\sigma(t)} \in \mathbb{R}^{(N-1)\times(N-1)}$ such that

$$\left(\bar{U}L_{\sigma(t)}\tilde{U}\right)^T \Xi_{\sigma(t)} + \Xi_{\sigma(t)}\left(\bar{U}L_{\sigma(t)}\tilde{U}\right) > 2\mu_{\sigma(t)}\Xi_{\sigma(t)}. \tag{31}$$

Lemma 2 (*Huang (1984)*). *For any positive definite matrix $M_1 \in \mathbb{R}^{n\times n}$ and symmetric matrix $M_2 \in \mathbb{R}^{n\times n}$, it holds that $x^T(t)M_2x(t) \leqslant \lambda_{\max}(M_1^{-1}M_2)x^T(t)M_1x(t)$.*

In the following, a design procedure with four steps is presented to determine the control parameters in time-varying formation control protocol (2).

Algorithm 1. The time-varying formation control protocol (2) with switching directed topologies can be designed in the following procedure:

Step 1: Check the time-varying formation feasibility condition (7). If it is satisfied, $v_i(t)$ $(i = 1, 2, \ldots, N)$ can be determined by solving Eq. (17) and K_1 can be any constant matrix with appropriate dimension (e.g., $K_1 = 0$). From (17), $v_i(t)$ $(i = 1, 2, \ldots, N)$ are not unique. One can firstly specify a $v_k(t)$ $(k \in \{1, 2, \ldots, N\})$, and then determine the other $v_j(t)$ $(j \in \{1, 2, \ldots, N\}, j \neq k)$ by Eq. (17). If the feasibility condition (7) is not satisfied, then the time-varying formation specified by $h(t)$ is not feasible and the algorithm stops.

If it is required that $v(t) \equiv 0$, solve the time-varying formation feasibility condition (26) for K_1. If there exists a K_1 satisfying (26), then continue, otherwise the time-varying formation specified by $h(t)$ is not feasible and the algorithm stops.

Step 2: Choose K_2 to specify the motion modes of the formation reference $r(t)$ by placing the eigenvalues of $A + BK_1 + BK_2$ at the desired places in the complex plane. If (A, B) is controllable, the existence of K_2 can be guaranteed.

Step 3: For a given $\beta > 0$, solve the following linear matrix inequality for a symmetric positive definite matrix P

$$\begin{aligned}
&(A + BK_1 + BK_2)P + P(A + BK_1 + BK_2)^T \\
&- BB^T + \beta P < 0.
\end{aligned} \tag{32}$$

Then K_3 can be given by $K_3 = B^TP^{-1}$. It can be verified that if (A, B) is controllable, then inequality (32) is feasible for any given $\beta > 0$.

Step 4: Choose a coupling strength α satisfying that $\alpha > 1/(2\bar{\mu})$ where $\bar{\mu} = \min\{\mu_{\sigma(t)}, \sigma(t) \in \{1, 2, \ldots, p\}\}$.

Based on Algorithm 1, the following theorem can be obtained.

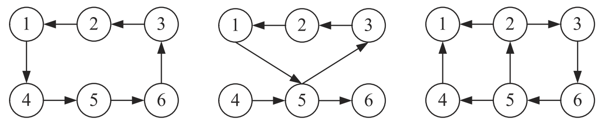
Theorem 3. *In the case where the time-varying formation feasibility condition (7) in Theorem 1 is satisfied, multi-agent system (3) with switching directed interaction topologies achieves time-varying formation specified by $h(t)$ if the formation control protocol (2) is designed by Algorithm 1 and the dwell time of the switching directed topologies satisfies*

$$\tau_0 > \frac{\ln\gamma}{\beta}, \tag{33}$$

where $\gamma = \max\{\lambda_{\max}(\Xi_i^{-1}\Xi_j), i, j \in \{1, 2, \ldots, p\}, i \neq j\}$ with $\lambda_{\max}(\Xi_i^{-1}\Xi_j)$ being the largest eigenvalue of $\Xi_i^{-1}\Xi_j$.

Proof. Consider the stability of the switched linear system (8). Choose the following piecewise Lyapunov functional candidate

$$\begin{aligned}
&V(t) = \bar{\vartheta}^T(t)\left(\Xi_{\sigma(t)} \otimes P^{-1}\right)\bar{\vartheta}(t), \\
&\Xi_{\sigma(t)} \in \{\Xi_1, \Xi_2, \ldots, \Xi_p\},
\end{aligned} \tag{34}$$



(a) G_1. (b) G_2. (c) G_3.

Fig. 1. Switching directed interaction topologies.

where $\Xi_{\sigma(t)}$ and P are defined in (31) and (32). Note that the interaction topology $G_{\sigma(t)}$ is fixed for $t \in [t_1, t_2)$. Taking the time derivative of $V(t)$ along the trajectories of switched linear system (8), one has that $\forall t \in [t_1, t_2)$,

$$\dot{V}(t) = \bar{\vartheta}^T(t)\left(\Xi_{\sigma(t)} \otimes \Psi - \alpha\Phi_{\sigma(t)}\right)\bar{\vartheta}(t), \tag{35}$$

where $\Psi = (A + BK_1 + BK_2)^T P^{-1} + P^{-1}(A + BK_1 + BK_2)$ and $\Phi_{\sigma(t)} = (\bar{U}L_{\sigma(t)}\tilde{U})^T \Xi_{\sigma(t)} \otimes (BK_3)^T P^{-1} + \Xi_{\sigma(t)}(\bar{U}L_{\sigma(t)}\tilde{U}) \otimes P^{-1}BK_3$. Substituting $K_3 = B^T P^{-1}$ into (35) gives

$$\dot{V}(t) = \bar{\vartheta}^T(t)\left(\Xi_{\sigma(t)} \otimes \Psi - \alpha\bar{\Phi}_{\sigma(t)}\right)\bar{\vartheta}(t), \tag{36}$$

where $\bar{\Phi}_{\sigma(t)} = \left((\bar{U}L_{\sigma(t)}\tilde{U})^T \Xi_{\sigma(t)} + \Xi_{\sigma(t)}(\bar{U}L_{\sigma(t)}\tilde{U})\right) \otimes P^{-1}BB^T P^{-1}$. Let $\bar{\vartheta}(t) = (I_{N-1} \otimes P)\tilde{\vartheta}(t)$. It follows from (36) that

$$\dot{V}(t) = \tilde{\vartheta}^T(t)\left(\Xi_{\sigma(t)} \otimes \bar{\Psi} - \alpha\tilde{\Phi}_{\sigma(t)}\right)\tilde{\vartheta}(t), \tag{37}$$

where $\bar{\Psi} = P(A + BK_1 + BK_2)^T + (A + BK_1 + BK_2)P$ and $\tilde{\Phi}_{\sigma(t)} = \left((\bar{U}L_{\sigma(t)}\tilde{U})^T \Xi_{\sigma(t)} + \Xi_{\sigma(t)}(\bar{U}L_{\sigma(t)}\tilde{U})\right) \otimes BB^T$. From (31), (32) and (37), one gets

$$\begin{aligned}\dot{V}(t) \leqslant{}& \tilde{\vartheta}^T(t)\left(\Xi_{\sigma(t)} \otimes \left(BB^T - \beta P\right)\right.\\ &\left. - \alpha\left(2\mu_{\sigma(t)}\Xi_{\sigma(t)}\right) \otimes BB^T\right)\tilde{\vartheta}(t).\end{aligned} \tag{38}$$

Substituting $\alpha > 1/(2\bar{\mu})$ into (38) yields $\dot{V}(t) \leqslant -\beta\tilde{\vartheta}^T(t)\left(\Xi_{\sigma(t)} \otimes P\right)\tilde{\vartheta}(t)$. Note that $\tilde{\vartheta}(t) = (I_{N-1} \otimes P^{-1})\bar{\vartheta}(t)$. One has that $\forall t \in [t_1, t_2)$

$$\dot{V}(t) \leq -\beta\bar{\vartheta}^H(t)\left(\Xi_{\sigma(t)} \otimes P^{-1}\right)\bar{\vartheta}(t) = -\beta V(t). \tag{39}$$

Since multi-agent system (3) switches at $t = t_2$, it follows from (39) that

$$V(t_2^-) < e^{-\beta(t_2 - t_1)}V(t_1) < e^{-\beta\tau_0}V(t_1). \tag{40}$$

Because $\bar{\vartheta}(t)$ is continuous, from (34) and Lemma 2, it can be obtained that

$$V(t_2) \leqslant \gamma V(t_2^-). \tag{41}$$

From (40) and (41), one gets $V(t_2) < \gamma e^{-\beta\tau_0}V(t_1) = e^{(\ln\gamma - \beta\tau_0)} V(0)$. Let $\nu = \beta - (\ln\gamma)/\tau_0$. If inequality (33) holds, then $\nu > 0$ and $V(t_2) < e^{-\nu\tau_0}V(0)$. For an arbitrarily given $t > t_2$, there exists a positive integer b satisfying $b \geqslant 2$. When $t \in (t_b, t_{b+1})$, using recursion approach, one has

$$V(t) < e^{-(\beta(t - t_b) + (b-1)\nu\tau_0)}V(0) < e^{-(b-1)\nu\tau_0}V(0). \tag{42}$$

Note that $t \leqslant b\tau_1$ and $b \geqslant 2$. It follows from (42) that $\forall t \in (t_b, t_{b+1})$

$$V(t) < e^{-\frac{(b-1)\tau_0\nu}{b\tau_1}t}V(0) < e^{-\frac{\tau_0\nu}{2\tau_1}t}V(0). \tag{43}$$

If $t = t_{b+1}$, it can be obtained that

$$V(t) < e^{-\frac{\tau_0\nu}{\tau_1}t}V(0). \tag{44}$$

From (43) and (44), one gets $\lim_{t\to\infty}\bar{\vartheta}(t) = 0$. Since the formation feasibility condition (7) is satisfied and the switched linear system (8) is asymptotically stable, it follows from Theorem 1 that multi-agent system (3) with switching directed interaction topologies achieves time-varying formation specified by $h(t)$. This completes the proof of Theorem 3.

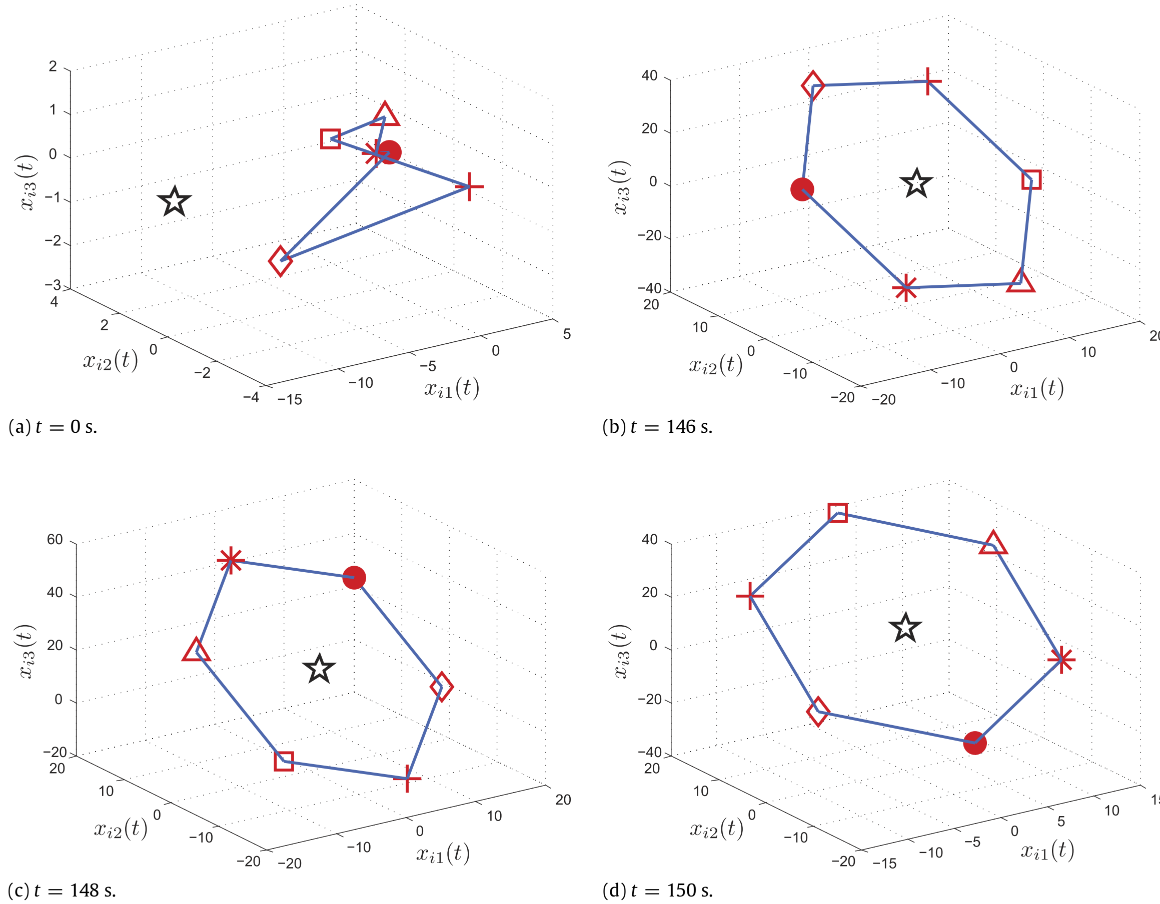
Remark 7. In the case where $h(t) \equiv 0$, the problems discussed in this paper become consensus problems. Necessary and sufficient conditions for general linear multi-agent systems with switching directed topologies to achieve consensus and the consensus protocol design procedure can be obtained from Theorem 1 and Algorithm 1 directly. An explicit expression of the consensus function and a positive threshold for the dwell time can be derived from Theorems 2 and 3 respectively. Furthermore, if $h(t) \equiv 0$ and all the possible topologies have the same root, then all the results in this paper can be applied to deal with the consensus tracking problems for general linear multi-agent systems with switching directed topologies. Consensus and consensus tracking problems for general linear multi-agent systems with switching topologies were studied in Feng, Hu, and Wen (2016), Su and Huang (2012) and Wen, Hu, Yu, and Chen (2014). However, in Su and Huang (2012) the system matrix A is required to be marginally stable and the topology is undirected. The topology in Wen et al. (2014) is assumed to be strongly connected, and explicit expressions of the consensus function are not given in Feng et al. (2016) and Wen et al. (2014).

Remark 8. From Theorems 1–3, one sees that the nonsingular transformation matrix U is a useful tool to deal with the time-varying formation control problems in this paper. The construction of U utilizes the common property for all the possible switching topologies $G_{\sigma(t)}$ ($\sigma(t) = 1, 2, \ldots, p$) containing a spanning tree that 0 is the simple eigenvalue of the corresponding Laplacian matrices $L_{\sigma(t)}$ with the associated right eigenvector $\mathbf{1}_N$. For theoretical analysis, U can be any nonsingular matrix with $\mathbf{1}_N$ as one of its columns. Without loss of generality, we choose the first column $\tilde{u}_1 = \mathbf{1}_N$ in the current paper. Note that the choice of U is not unique. For simplicity, one can choose $U = \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix}$ as used in the numerical simulation of Section 5.

5. Numerical simulations

In this section, a numerical example is given to illustrate the effectiveness of theoretical results obtained in the previous sections.

Consider a third-order multi-agent system with six agents, where the dynamics of each agent is described by (1) with $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$ ($i = 1, 2, \ldots, 6$) and $A = \begin{bmatrix} 0 & -4 & 1 \\ 2 & 2 & -1 \\ 3 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Suppose that there are three different 0–1 weighted directed topologies, namely, G_1, G_2 and G_3 as shown in Fig. 1. These six agents are required to keep a periodic time-varying parallel hexagon formation and at the same time keep rotation around the time-varying formation reference $r(t) = [r_1(t), r_2(t), r_3(t)]^T$. The time-varying formation is specified by $h_i(t) = \begin{bmatrix} 15\cos\left(2t + \frac{(i-1)\pi}{3}\right) \\ 15\sin\left(2t + \frac{(i-1)\pi}{3}\right) \\ 30\cos\left(2t + \frac{(i-1)\pi}{3}\right) \end{bmatrix}$ ($i = 1, 2, \ldots, 6$).



(a) $t = 0$ s. (b) $t = 146$ s. (c) $t = 148$ s. (d) $t = 150$ s.

Fig. 2. State snapshots of the six agents and the formation reference.

As $B \in \mathbb{R}^{3\times 1}$ and $\mathrm{rank}(B) = 1$, one gets that B is of full column rank. Choose $\tilde{B} = [0, 0, 1]$, $\bar{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix}$. It can be verified that the formation feasibility constraint (7) in Theorem 1 is satisfied. According to Algorithm 1, gain matrix K_1 can be chosen as $K_1 = [0, 0, 0]$ and $v_i(t)$ can be solved from (17) as $v_i(t) = -285 \sin\left(2t + \frac{\pi}{3}(i-1)\right) + 15 \cos\left(2t + \frac{\pi}{3}(i-1)\right)$, where $i = 1, 2, \ldots, 6$. In the case where $K_2 = 0$, one can obtain the eigenvalues of $A + BK_1 + BK_2$ are 7.0439, $0.9780 + 3.0438j$ and $0.9780 - 3.0438j$ with $j^2 = -1$, which means that the motion modes of the formation reference are unstable and the whole formation will diverge exponentially. To keep the whole time-varying formation moving in a visual range, one can assign the motion modes of the formation reference to be oscillated using the approach in Step 2 of Algorithm 1. To this end, choose $K_2 = [3.8125, 0.0625, -10]$ to assign the eigenvalues of $A + BK_1 + BK_2$ at -1, $0.5j$ and $-0.5j$. Choose $\beta = 0.2$. Solving the inequality (32), one gets $K_3 = [-1.0241, -9.974, 2.9112]$. It can be obtained that $\alpha > 2.7651$ and $\tau_0 > 12.6762$ s. Therefore, choose $\alpha = 3$ and the dwell time to be 15 s.

Let the initial states of the six agents generated by $x_{ij}(0) = i(\Theta - 0.5)$ $(i = 1, 2, \ldots, 6; j = 1, 2, 3)$ with Θ being a random value between 0 and 1. Fig. 2 displays the snapshots of the six agents and the formation reference at $t = 0$ s, $t = 146$ s, $t = 148$ s and $t = 150$ s, where the states of the six agents and the formation reference are denoted by the triangle, asterisk, dot, plus, square, diamond and pentagram respectively. Fig. 3 shows the trajectory of the formation reference, where the initial state is denoted by the circle. Fig. 4 depicts the curve of the formation error $\vartheta(t)$. From Figs. 2–4, the following phenomena can be observed: (i) the states of the six agents keep a parallel hexagon formation, (ii) the edge of the parallel hexagon is time-varying, (iii) the formation reference moves along a circle and lies in the center of the time-varying formation, and (iv) the parallel hexagon is keeping rotation around the formation reference. Therefore, the desired time-varying formation is achieved by multi-agent system (3) under switching directed interaction topologies.

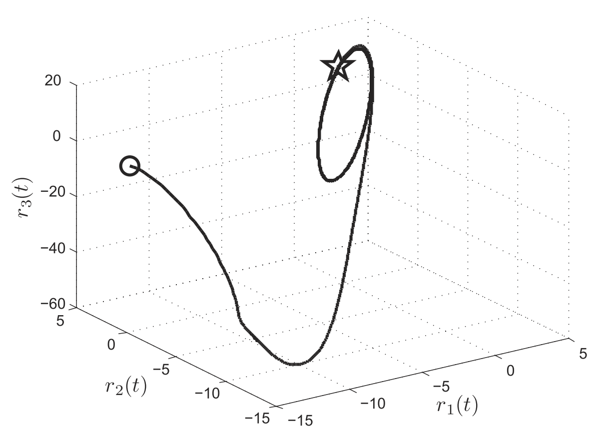


Fig. 3. Trajectory of $r(t)$.

6. Conclusions

Time-varying formation control problems for general linear multi-agent systems with switching directed interaction topologies were studied. Necessary and sufficient conditions for general linear multi-agent systems with switching directed topologies to achieve time-varying formations were presented. A description of the feasible time-varying formation set and an explicit expression

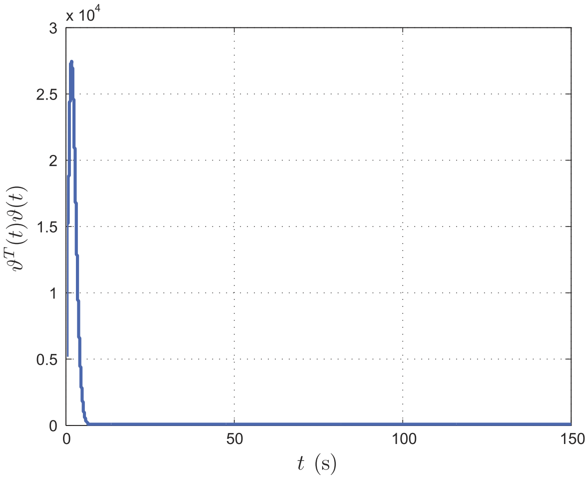


Fig. 4. Curve of the formation error.

of the time-varying formation reference function were proposed. Approaches to expand the feasible formation set and assign the motion modes of the formation reference were given. An algorithm to design the formation protocol was presented. By designing the formation protocol using the proposed algorithm, time-varying formation can be achieved by multi-agent systems with general linear dynamics and switching directed topologies under the condition that the time-varying formation belongs to the feasible formation set and the dwell time is larger than a positive threshold. Based on this result, it is of interest to further study time-varying formation control problems for general linear multi-agent systems with switching directed topologies, measurement noises and model uncertainties. Another meaningful future research topic is time-varying formation tracking control in which the states of all agents form predefined time-varying formation while the whole formation (formation reference) tracks the specified reference trajectory.

References

Allerhand, L., & Shaked, U. (2011). Robust stability and stabilization of linear switched systems with dwell time. *IEEE Transactions on Automatic Control*, *56*(2), 381–386.

Antonelli, G., Arrichiello, F., Caccavale, F., & Marino, A. (2014). Decentralized time-varying formation control for multi-robot systems. *The International Journal of Robotics Research*, http://dx.doi.org/10.1177/0278364913519149, Published online before print.

Balch, T., & Arkin, R. (1998). Behavior-based formation control for multi robot teams. *IEEE Transactions on Robotics and Automation*, *14*(6), 926–939.

Das, A. K., Fierro, R., Kumar, V., & Ostrowski, J. P. (2002). A vision-based formation control framework. *IEEE Transactions on Robotics and Automation*, *18*(5), 813–825.

Dong, Y., & Huang, J. (2014). Leader-following connectivity preservation rendezvous of multiple double integrator systems based on position measurement only. *IEEE Transactions on Automatic Control*, *59*(9), 2598–2603.

Dong, X. W., Shi, Z. Y., Lu, G., & Zhong, Y. S. (2014). Time-varying formation control for high-order linear swarm systems with switching interaction topologies. *IET Control Theory & Applications*, *8*(18), 2162–2170.

Dong, X. W., Xi, J. X., Lu, G., & Zhong, Y. S. (2014). Formation control for high-order linear time-invariant multi-agent systems with time delays. *IEEE Transactions on Control of Network Systems*, *1*(3), 232–240.

Dong, X. W., Yu, B. C., Shi, Z. Y., & Zhong, Y. S. (2015). Time-varying formation control for unmanned aerial vehicles: Theories and applications. *IEEE Transactions on Control Systems Technology*, *23*(1), 340–348.

Du, H., Li, S. H., & Lin, X. (2013). Finite-time formation control of multiagent systems via dynamic output feedback. *International Journal of Robust and Nonlinear Control*, *23*(14), 1609–1628.

Fax, J., & Murray, R. (2004). Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, *49*(9), 1465–1476.

Feng, Z., Hu, G. Q., & Wen, G. H. (2016). Distributed consensus tracking for multi-agent systems under two types of attacks. *International Journal of Robust and Nonlinear Control*, *26*(5), 896–918.

Guo, M., Zavlanos, M. M., & Dimarogonas, D. V. (2014). Controlling the relative agent motion in multi-agent formation stabilization. *IEEE Transactions on Automatic Control*, *59*(3), 820–826.

Guzey, H.M., Dierks, T., & Jagannathan, S. (2015). Hybrid consensus-based formation control of agents with second order dynamics. In *Proceedings of the 2015 American control conference* (pp. 4386–4391).

Hawwary, M. E. (2015). Three-dimensional circular formations via set stabilization. *Automatica*, *54*, 374–381.

Huang, L. (1984). *Linear algebra in system and control theory*. Beijing, China: Science Press.

Ji, M., Ferrari-Trecate, G., Egerstedt, M., & Buffa, A. (2008). Containment control in mobile networks. *IEEE Transactions on Automatic Control*, *53*(8), 1972–1975.

Karimoddini, A., Lin, H., Chen, B. M., & Lee, T. H. (2013). Hybrid three-dimensional formation control for unmanned helicopters. *Automatica*, *49*(2), 424–433.

Lafferriere, G., Williams, A., Caughman, J., & Veerman, J. (2005). Decentralized control of vehicle formations. *Systems and Control Letters*, *54*(9), 899–910.

Leonard, N. E., Paley, D. A., Davis, R. E., Fratantoni, D. M., Lekien, F., & Zhang, F. M. (2010). Coordinated control of an underwater glider fleet in an adaptive ocean sampling field experiment in Monterey Bay. *Journal of Field Robotics*, *27*(6), 718–740.

Lewis, M., & Tan, K. (1997). High precision formation control of mobile robots using virtual structures. *Autonomous Robots*, *4*(4), 387–403.

Li, Z. K., Ren, W., Liu, X. D., & Fu, M. Y. (2013). Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders. *International Journal of Robust and Nonlinear Control*, *23*(5), 534–547.

Lin, Z. Y., Wang, L. L., Han, Z. M., & Fu, M. Y. (2015). A graph Laplacian approach to coordinate-free formation stabilization for directed networks. *IEEE Transactions on Automatic Control*, http://dx.doi.org/10.1109/TAC.2015.2454711, Published online before print.

Liu, T. F., & Jiang, Z. P. (2013). Distributed formation control of nonholonomic mobile robots without global position measurements. *Automatica*, *49*(2), 592–600.

Ma, C. Q., & Zhang, J. F. (2012). On formability of linear continuous-time multi-agent systems. *Journal of Systems Science and Complexity*, *25*(1), 13–29.

Mylvaganam, T., & Astolfi, A. (2015). A differential game approach to formation control for a team of agents with one leader. In *Proceedings of the 2015 American control conference* (pp. 1469–1474).

Navaravong, L., Kan, Z., She, J. M., & Dixon, W. E. (2012). Formation reconfiguration for mobile robots with network connectivity constraints. *IEEE Network*, *26*(4), 18–24.

Notarstefano, G., Egerstedt, M., & Haque, M. (2011). Containment in leader-follower networks with switching communication topologies. *Automatica*, *47*(5), 1035–1040.

Oh, K. K., Park, M. C., & Ahn, H. S. (2015). A survey of multi-agent formation control. *Automatica*, *53*, 424–440.

Porfiri, M., Roberson, D., & Stilwell, D. (2007). Tracking and formation control of multiple autonomous agents: A two-level consensus approach. *Automatica*, *43*(8), 1318–1328.

Ren, W. (2007). Consensus strategies for cooperative control of vehicle formations. *IET Control Theory & Applications*, *1*(2), 505–512.

Ren, W., & Beard, R. (2005). Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661.

Saboori, I., & Khorasani, K. (2014). H_∞ consensus achievement of multi-agent systems with directed and switching topology networks. *IEEE Transactions on Automatic Control*, *59*(11), 3104–3109.

Su, Y. F., & Huang, J. (2012). Stability of a class of linear switching systems with applications to two consensus problems. *IEEE Transactions on Automatic Control*, *57*(6), 1420–1430.

Tian, Y. P., & Wang, Q. (2013). Global stabilization of rigid formations in the plane. *Automatica*, *49*(5), 1436–1441.

Wang, W., Huang, J. S., Wen, C. Y., & Fan, H. J. (2014). Distributed adaptive control for consensus tracking with application to formation control of nonholonomic mobile robots. *Automatica*, *50*(4), 1254–1263.

Wang, C., Xie, G. M., & Cao, M. (2014). Controlling anonymous mobile agents with unidirectional locomotion to form formations on a circle. *Automatica*, *50*(4), 1100–1108.

Wang, Y., Yan, W., & Li, J. (2012). Passivity-based formation control of autonomous underwater vehicles. *IET Control Theory & Applications*, *6*(4), 518–525.

Wen, G. H., Hu, G. Q., Yu, W. W., & Chen, G. R. (2014). Distributed H_∞ consensus of higher order multiagent systems with switching topologies. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *61*(5), 359–363.

Xiang, W. M. (2015). On equivalence of two stability criteria for continuous-time switched systems with dwell time constraint. *Automatica*, *54*, 36–40.

Xiao, F., Wang, L., Chen, J., & Gao, Y. (2009). Finite-time formation control for multi-agent systems. *Automatica*, *45*(11), 2605–2611.

Yang, F. W., Wang, Z. D., Hung, Y. S., & Gani, M. (2006). H_∞ control for networked systems with random communication delays. *IEEE Transactions on Automatic Control*, *51*(3), 511–518.

Yoo, S. J., & Kim, T. H. (2015). Distributed formation tracking of networked mobile robots under unknown slippage effects. *Automatica*, *54*, 100–106.

Zavlanos, M. M., Tanner, H. G., Jadbabaie, A., & Pappas, G. J. (2009). Hybrid control for connectivity preserving flocking. *IEEE Transactions on Automatic Control*, *54*(12), 2869–2875.

Zhang, C. Z., Feng, G., Qiu, J. B., & Shen, Y. Y. (2013). Control synthesis for a class of linear network-based systems with communication constraints. *IEEE Transactions on Industrial Electronics*, *60*(8), 3339–3348.

Zheng, R. H., Lin, Z. Y., Fu, M. Y., & Sun, D. (2015). Distributed control for uniform circumnavigation of ring-coupled unicycles. *Automatica*, *53*, 23–29.

Zhou, B., & Lin, Z. L. (2014). Consensus of high-order multi-agent systems with large input and communication delays. *Automatica*, *50*(2), 452–464.

Xiwang Dong received his B.E. degree in Automation from Chongqing University, Chongqing, China, in 2009, and Ph.D. degree in Control Science and Engineering from Tsinghua University, Beijing, China, in 2014. From December 2014 to December 2015, he was a Research Fellow in the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. He is now a Lecturer in the School of Automation Science and Electrical Engineering, Beihang University, Beijing, China. His research interests include consensus control, formation control and containment control of multi-agent systems. Dr. Dong is the recipient of the Academic Rookie Award of the Department of Automation, Tsinghua University in 2014, Outstanding Doctoral Dissertation Award of the Tsinghua University in 2014 and Springer Theses Award in 2015.

Guoqiang Hu received the B.Eng. degree in Automation from the University of Science and Technology of China, Hefei, China, in 2002, the M.Phil. degree in Automation and Computer-Aided Engineering from the Chinese University of Hong Kong, Hong Kong, in 2004, and the Ph.D. degree in Mechanical Engineering from the University of Florida, Gainesville, FL, USA, in 2007. He is currently with the School of Electrical and Electronic Engineering at Nanyang Technological University, Singapore. Prior to his current position, he was a Post-doctoral Research Associate at University of Florida, Gainesville, FL, USA, in 2008, and an Assistant Professor at Kansas State University, Manhattan KS, USA, from 2008 to 2011. His current research interests include analysis, control and design of distributed intelligent systems.