# Limit-Cycle-Based Decoupled Design of Circle Formation Control with Collision Avoidance for Anonymous Agents in a Plane

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***Abstract*—We study the circle formation problem for a group of anonymous mobile agents in a plane, in which the agents are required to converge onto a circle with a preset target as the center, as well as to maintain the desired relative positions when rotating around the target at the same speed. Each agent is modeled as a kinematic point and can merely perceive the relative positions of the target and its limited neighbors. In order to solve the circle formation problem, a limit-cycle-based decoupled-design approach is delivered. We divide the overall control objective into two subobjectives, where the first is *target circling* that all agents converge onto the circle around the target, and the second is *spacing adjustment* that each agent maintains the desired distance from its neighbors. Then, we propose to use a controller comprised of *converging part* and *layout part* to deal with these two subobjectives, respectively. The former part is based on a limit-cycle oscillator using only the relative position from the target, and the latter is designed by also perceiving the relative position from the agent's neighbors. An important feature of the controller is that it guarantees that no collision between agents ever takes place throughout the system's evolution. Another feature is that some of the parameters in the proposed controller have explicit physical meanings related to the agents' rotating motion around the target, so that they can be set more reasonable and easily in real applications. Numerical simulations are given to show the effectiveness and performance of the proposed circle formation controller.**

***Index Terms*—Circle formation, collision avoidance, distributed control, limit cycle, multi-agent system.**

## I. Introduction

Multirobot systems have captured increasing attention, thanks to their wide applications, such as exploration [2], environmental monitoring [3], pursuit and evasion [4]–[6], and surveillance [7]. In such cooperative tasks, it is of benefit to keep the robots moving in a desired formation with certain geometric shapes in order to successfully complete the tasks [8] and even to improve their performance, such as the quality of the collected data, and the robustness of group motion against random environmental disturbances [9]. One challenge of such a pattern formation problem for multirobot systems arises from the fact that the robots can use only local information to implement their distributed control strategies without centralized coordination.

Forming circle formations is one of the most actively studied topics within the realm of formation control, since on one hand, circle formations are one of the simplest classes of formations with geometric shapes, and on the other hand, they are natural choices of the geometric shapes for a group of robots to exploit an area of interest [10]. In recent years, the circle formation problem has become challenging since the semisynchronous model has been developed and introduced [11]–[13]. To be more specific, the agents are assumed to be oblivious (without memories about past actions and observations), anonymous (not distinguishable from one another), unable to communicate directly, and can only interact through sensing other agents' positions. This model has successfully gained insight into the autonomous agents' key characteristics, which restrict the agents' recognition, sensing, and communication capabilities, and these restrictions make the problem formulation more attractive.

Based on the semisynchronous model, significant efforts have been made on the development of distributed strategies guiding agents to form circle formations. In particular, the focus is how to lead the agents to *distributed evenly on a circle*. In the theoretical computer science community, Défago and Konagaya [12] have proposed algorithms to decompose the circle formation problem into two subproblems and solve them separately. The first is to form a circle in finite time, and the second is to guide the robots to the configuration, where all of them are positioned evenly on the circle. Later, these two subproblems have been solved via a single algorithm proposed by Défago and Souissi [13]. Flocchini *et al.* [14] have investigated under what conditions the circle formation problem is solvable by a group of identical agents without a global coordinate system with the constraint that all the agents can only move on a circle. In the systems and control community, research efforts have also been made on the circle formation problem for multiagent systems, where the dynamics of the agents are modeled as single integrators [15], double integrators [16], and unicycles [17]–[20]. Song *et al.* [15] have developed distributed coverage control laws with input saturation by considering the case where agents' moving speed is upper bounded. Marshall *et al.* [17] have designed a control law under which the multiagent system's equilibrium formations are generalized circular pursuit patterns in the plane. Shi *et al.* [16] have designed control protocols to achieve an equidistant circular formation for a group of agents to enclose a team of targets. The problem of moving-target circular formation for nonholonomic vehicles has been investigated in [18] and [19]. Zheng *et al.* [20] have addressed the problem of

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forming an equidistant circular formation to enclose a target for a group of nonholonomic mobile robots with bearing-only measurements.

However, even distribution may not be the optimal configuration when carrying out some tasks [21], only a few works [22], [23] have considered the problem of forming *arbitrary formations on a circle*. Wang *et al.* [22] have proposed distributed control laws to guide the agents to form the desired circle formation. They also discuss the corresponding sampled-data control laws and finite-time control laws, respectively, to make the control strategies more practical. Wang *et al.* [23] have further considered the scenario when the agents are under locomotion constraints. Note that their works [21]–[23] have restricted all the agents to move in the 1-D space of a circle.

The goal of this paper is to design distributed control laws that can guide a group of anonymous mobile agents to form any given circle formation in a plane. The general control objective of the problem comprises two specific subobjectives. The first is *target circling* that all agents are required to converge onto a circle with a preset target as the center and to rotate around the target at the same speed, and the second is *spacing adjustment* that each agent needs to maintain the desired distance from its neighbors without the requirement that all the desired distances between neighboring agents are equal. We consider a system consisting of multiple mobile agents modeled as kinematic points, all of which move in a plane. ]s are oblivious, anonymous, and unable to communicate directly; they can only sense the relative position information of the target and their neighbors.

To realize the circle formation in a plane, a *limit-cycle-based decoupled-design approach* is delivered in this paper. We propose to use a controller comprised of a *converging part* and a *layout part* to deal with the two subobjectives of target circling and spacing adjustment, respectively. The key idea is to first design the two parts of the controller separately, where the converging part makes each agent converge to a circle around the target with a prescribed radius and rotate counterclockwise on it, while the layout part makes all agents autonomously adjust their positions to maintain the desired distance from their neighbors. Then, with the aid of the properties of limit-cycle oscillators, we are able to combine the two parts into a whole controller for the group of anonymous mobile agents in a plane to solve the circle formation problem asymptotically. We further show that the designed control law guarantees that no collision between agents ever takes place throughout the whole system's evolution, and the parameters in the control law have explicit physical meanings related to the motion characteristics of the agents. The main contribution of the paper is then the decoupled-design methodology based on the properties of limit-cycle oscillators to deal with formation problems.

Compared to our previous conference paper [1], we mainly have the following three improvements. First, a limit-cycle-based decoupled-design approach is clearly proposed to simultaneously realize both subobjectives, target circling and spacing adjustment, of the circle formation problem. Second, a theoretical proof is given to demonstrate that order preservation guarantees collision avoidance in our problem setting. Third, the explicit physical meanings of the parameters in our proposed control law are analyzed and discussed, so that these parameters can be selected more reasonable and easily according to the request of the robots' motion characteristics when applied to real robot systems in the future.

The rest of this paper is organized as follows. In Section II, we formulate the circle formation problem in a plane. Then, we design a distributed control law in Section III and provide rigorous analysis on its performances in Section IV. Simulation results are given in Section V. Finally, Section VI concludes this paper.

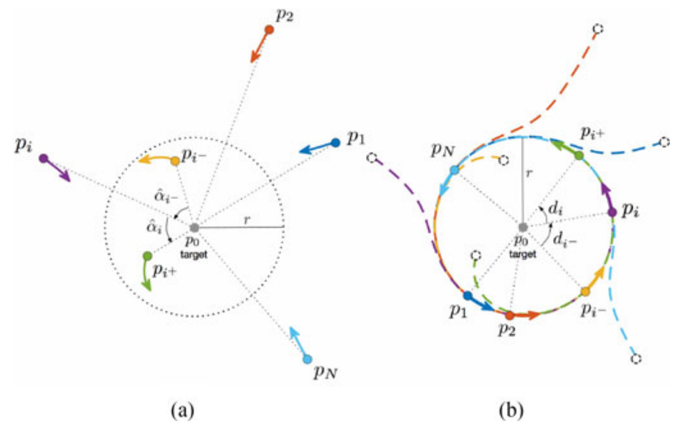


Fig. 1. Circle formation in a plane. (a) Agents are initially located in a plane. (b) Agents form a desired circle formation and rotate counterclockwise around the target.

## II. Problem Formulation

We consider a group of $N$, $N \geq 2$, agents $p_1, \ldots, p_N$ that are initially positioned in a plane that includes a preset target point $p_0$ to be circled around [see Fig. 1(a)]. The $N$ agents are anonymous that they cannot distinguish one from another and can move freely in the plane. The agents' initial positions are NOT required to be distinguished from each other, whereas no agent occupies the same position with the target. For ease of expression, we label the agents based on their initial positions according to the following three rules.

1) The labels are sorted first in ascending order in a counterclockwise manner around the target.
2) For the agents who lie on the same ray extending from the target, their labels are sorted in ascending order by the distance to the target point.
3) For the agents who occupy the same position, their labels are chosen randomly.

Then, we consider the case when the agents' neighbor relationships are described by a ring $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} = \{(1,2), (2,3), \ldots, (N-1, N), (N, 1)\}$. In such a way, each agent only has two neighbors that are immediately in front of or behind itself. We denote the set of agent $i$'s two neighbors by $\mathcal{N}_i = \{i^-, i^+\}$, where

$$i^+ = \begin{cases} i+1 & \text{when } i = 1, 2, \ldots, N-1 \\ 1 & \text{when } i = N \end{cases},$$

and

$$i^- = \begin{cases} N & \text{when } i = 1 \\ i-1 & \text{when } i = 2, 3, \ldots, N. \end{cases} \tag{1}$$

Let $p_i(t) = [x_i(t), y_i(t)]^T \in \mathbb{R}^2$ be the position of agent $i$, $1 \leq i \leq N$, at time $t$, and $p_0 = [x_0, y_0]^T \in \mathbb{R}^2$ is the target's position in the plane. Each agent is described as a kinematic point

$$\dot{p}_i(t) = u_i(t), \qquad i = 1, 2, \ldots, N \tag{2}$$

where $u_i(t) \in \mathbb{R}^2$ is the control input to be designed.

Assume that each agent can only use the relative positions between the target and its two neighbors under the neighbor relationship $\mathbb{G}$. Note that by sensing and using the information about its two immediate neighbors, the agents do NOT need to know the label information. In fact, as to be proved later in Section IV, under our control laws, the agents' spatial ordering will always, except in the initial state, satisfy the first rule we just mentioned above, that is, the $N$ agents are always arranged in such a way that their labels are sorted in ascending order in a counterclockwise manner around the target, so that the neighbor relationship $\mathbb{G}$ is guaranteed to be time invariant. We need to emphasize that the time-invariant neighbor relationship is not an assumption but a nice property of our control law.

In this paper, the *Circle Formation* problem in a plane is formalized as to design local controllers for all agents by only using the relative position between the agent and the target and the relative positions between the agent and its two neighbors such that all the agents asymptotically form a circle with a desired radius to keep the target $p_0$ as its centroid. The circle formation is required to rotate counterclockwise around the target, and to maintain a prescribed distribution pattern *without* the requirement that all the desired distances between neighboring agents are equal [see Fig. 1(b)].

To formulate the problem mathematically, the following variables are introduced. Let

$$\bar{p}_i(t) = p_i(t) - p_0, \qquad i = 1, 2, \ldots, N \tag{3}$$

be the relative position between agent $i$ and the target measured by agent $i$ at time $t$. The relative position between neighboring agents can be derived as

$$\hat{p}_i(t) = p_{i^+}(t) - p_i(t), \qquad i = 1, 2, \ldots, N \tag{4}$$

where $\hat{p}_i$ is the relative position between agents $i$ and $i^+$. It follows that $\hat{p}_{i^-}$ denotes the relative positions between agents $i^-$ and $i$. We further introduce the variables $\hat{\alpha}_i$ as the *angular distance* from agent $i$ to $i^+$, which is formed by counterclockwise rotating the ray extending from the target to agent $i$ until reaching agent $i^+$. It follows that $\hat{\alpha}_{i^-}$ is the angular distance from agent $i^-$ to $i$. Let $d_i$ denote the desired angular spacing from agent $i$ to $i^+$. Then, the desired distribution pattern of the $N$ agents is determined by the vector

$$d = [d_1, d_2, \ldots, d_N]^T. \tag{5}$$

Note that only local information of $d_i$ and $d_{i^-}$ in vector $d$ is available to each agent $i$. We say a prescribed circle formation is *admissible* if $r > 0$, $d_i > 0$ and $\sum_{i=1}^N d_i = 2\pi$, where $r$ is the desired radius of the circle.

We emphasize that, in our problem setting, each agent can only measure the relative position information of the target and its two neighbors. More precisely, agent $i$ can only measure the relative positions $\bar{p}_i$, $\hat{p}_i$ and $\hat{p}_{i^-}$. Furthermore, it is easy to check that agent $i$ can calculate the angular distance $\hat{\alpha}_i$ just using $\bar{p}_i$ and $\hat{p}_i$ by the definition of inner product, since $\hat{\alpha}_i$ happened to be the angle between two vectors $\bar{p}_i$ and $\bar{p}_i + \hat{p}_i$. Similarly, agent $i$ can calculate the angular distance $\hat{\alpha}_{i^-}$ just using $\bar{p}_i$ and $\hat{p}_{i^-}$. Thus, $\hat{\alpha}_i$ and $\hat{\alpha}_{i^-}$ can also be regarded as relative information obtained by agent $i$ (see Fig. 2).

With the above preparation, we are ready to formulate the *Circle Formation Problem in a Plane* of interest. First of all, in order to represent mathematically the $N$ anonymous agents' initial states with their labels, the following two definitions for the agents' spatial ordering are proposed.

*Definition 1 (Counterclockwise order):* The $N$ agents are said to be arranged in a *counterclockwise order* if $\hat{\alpha}_i \in (0, 2\pi)$ for all $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N \hat{\alpha}_i = 2\pi$.

*Definition 2 (Almost counterclockwise order):* The $N$ agents are said to be arranged in an *almost counterclockwise order* if 1) $\hat{\alpha}_i \in [0, 2\pi)$ for all $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N \hat{\alpha}_i = 2\pi$; and 2) $\hat{\alpha}_i = 0$ implies $\|\bar{p}_{i^+}\| \geq \|\bar{p}_i\|$, where $\|\cdot\|$ represents the Euclidean norm.

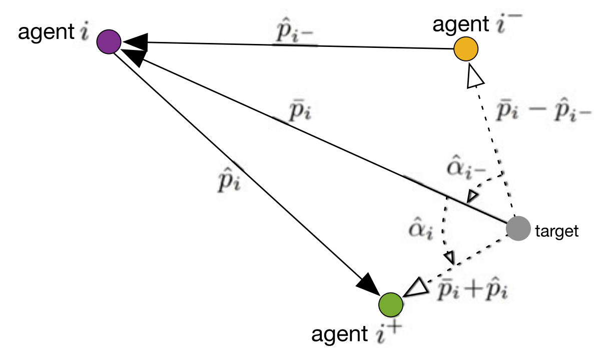


Fig. 2. Locally implementable control.

Then, one can check that adopting the three rules when label these anonymous agents based on their initial positions leads to the fact that the $N$ agents are initially arranged in an almost counterclockwise order.

Now, we are able to present the formal problem formulation.

*Definition 3 (Circle formation problem in a plane):* Given an admissible circle formation characterized by $r$ and $d$ in a plane, design distributed control laws $u_i(t) = u_i(\bar{p}_i, \hat{p}_i, \hat{p}_{i^-}, r, d_i, d_{i^-})$, $i = 1, 2, \ldots, N$, such that under any initial condition when the $N$ agents are arranged in an almost counterclockwise order, the solution to system (2) converges to some equilibrium point $p^*$ satisfying

$$\|\bar{p}_i^*\| = r \qquad i = 1, 2, \ldots, N \qquad \text{(Target circling)} \tag{6}$$

$$\hat{\alpha}^* = d \qquad \text{(Spacing adjustment).} \tag{7}$$

Furthermore, considering the possible applications to multirobot systems, two desired/required properties of the $N$-agent systems are defined as follows.

*Definition 4 (Order preservation):* For the $N$-agent system under consideration, we say the agents' spatial ordering is preserved under control laws $u_i(t)$ if the $N$ agents initially are arranged in an almost counterclockwise order in the plane, the solution to system (2) can ensure the $N$ agents remain in a counterclockwise order for all $t > 0$.

*Definition 5 (Collision avoidance):* For the $N$-agent system under consideration, we say the agents have the property of collision avoidance if the $N$ agents initially are arranged in an almost counterclockwise order in the plane, the solution to system (2) satisfies $\|p_i - p_j\| > 0$ for any pair of $i, j$ $(i \neq j)$ for all $t > 0$.

## III. Limit-Cycle-Based Decoupled Control Design

In this section, using the idea of decoupled design, we propose a control protocol with the aid of the properties of limit-cycle oscillators to solve the circle formation problem in a plane.

As described in Definition 3, the problem includes two objectives, target circling and spacing adjustment, which need to be concerned. Thus, we consider a controller consisting two parts

$$u_i(t) = u_i^p(t) f_i(t), \qquad i = 1, 2, \ldots, N \tag{8}$$

where $u_i^p(t) \in \mathbb{R}^2$ is designed to deal with the target circling objective that each agent is required to rotate counterclockwise around the target $p_0$ with a desired radius $r$, and $f_i(t) \in \mathbb{R}^1$ is supposed to mainly focus on the objective of spacing adjustment that the agents need maintain a prescribed distribution pattern $d$.

First, in order to design the part $u_i^p(t)$, we are going to talk a little bit about the limit-cycle oscillators [24]. For an oscillator having a stable limit cycle, it has the property that all trajectories in the vicinity of the limit cycle ultimately tend toward the limit cycle as time goes into infinity. We find, fortunately, that such a property may fit well with the requirement of target circling, if we could construct a second-order nonlinear oscillator having the desired circle, which is with the desired radius and has the target as its centroid, as its limit cycle. Following this idea, we design the first part $u_i^p(t)$ in our controller as a limit-cycle oscillator

$$u_i^p(t) = \lambda \begin{bmatrix} \gamma l_i(t) & -1 \\ 1 & \gamma l_i(t) \end{bmatrix} \bar{p}_i(t), \qquad i = 1, 2, \ldots, N \tag{9}$$

where $\lambda > 0, \gamma > 0$ are constant, and

$$l_i(t) = r^2 - \|\bar{p}_i(t)\|^2 \tag{10}$$

shows the differences between current relative position and the desired one between agent $i$ and the target. Based on such a design of $u_i^p(t)$, we assume the second part $f_i(t)$ to be a coefficient of the limit-cycle controller $u_i^p(t)$ so that $f_i(t) \in \mathbb{R}^1$. To ensure that $u_i^p(t)$ combined with $f_i(t)$ still works for the objective of target circling, a restriction of $f_i(t) > 0$ and $f_i(t)$ is bounded is enforced, which will be discussed in the following section. Moreover, note that the part $u_i^p(t) \in \mathbb{R}^2$ is only using the relative position information between agent $i$ and the given target.

Second, taking into account the objective of spacing adjustment, the $N$ agents have to coordinate with each other using not only the relative position information of the target but also that of their neighbors. Due to the form of (7), which describes the spacing adjustment, we mainly focus on the angular distance $\hat{\alpha}_i$ between pairs of neighbors when designing the part $f_i(t)$ in our controller. Following our previous works [22], [23], we introduce an angular control $u_i^\alpha(t)$ as

$$u_i^\alpha(t) = \frac{d_{i^-}}{d_i + d_{i^-}} \hat{\alpha}_i(t) - \frac{d_i}{d_i + d_{i^-}} \hat{\alpha}_{i^-}(t), \quad i = 1, 2, \ldots, N. \tag{11}$$

Then, we choose the form of $f_i(t)$ as

$$f_i(t) = c_1 + \frac{c_2}{2\pi} u_i^\alpha(t), \qquad i = 1, 2, \ldots, N \tag{12}$$

where $c_1 \geq c_2 > 0$ are constant. We will prove in the following section that the designed $f_i(t)$ in (12) does satisfy the restriction of $f_i(t) > 0$ and $f_i(t)$ is bounded.

Now, we have the complete form of the distributed controller $u_i(t)$ with its two parts $u_i^p(t)$ in (9) and $f_i(t)$ in (12).

It is worth to point out that, in our designing of the controller for each agent $i$, its first part is only using the relative position information between agent $i$ and the target, while the relative position information of its neighbors are only used in the second part of the controller. In other words, the first part of the controller makes each agent *independently* converge to a circle around the target and rotate on it, while the second part is used to *coordinate* the $N$ agents to form and maintain a prescribed distribution pattern when rotating on the circle. Following these two aspects, we provide rigorous analysis for performance of the proposed control law in Section IV.

## IV. Analysis of Convergence and Collision Avoidance

In the previous section, a distributed control protocol (8) with (9) and (12) has been designed. Before analyzing the performance of the proposed control law, we first specify the form of the $N$-agent system under the control law.

Substituting (8) into the dynamic equations of the system (2), we arrive at the resulting closed-loop dynamics of the $N$-agent system

$$\dot{p}_i(t) = \lambda \begin{bmatrix} \gamma l_i(t) & -1 \\ 1 & \gamma l_i(t) \end{bmatrix} \bar{p}_i(t) f_i(t), \quad i = 1, 2, \ldots, N \tag{13}$$

which can be rewritten equivalently using $\bar{p}_i$ as

$$\dot{\bar{p}}_i(t) = \lambda \begin{bmatrix} \gamma l_i(t) & -1 \\ 1 & \gamma l_i(t) \end{bmatrix} \bar{p}_i(t) f_i(t), \quad i = 1, 2, \ldots, N \tag{14}$$

where $f_i(t) \in \mathbb{R}^1$ and $l_i(t) \in \mathbb{R}^1$ are given by (12) and (10), respectively.

Moreover, the second objective of spacing adjustment (7) in the circle formation problem in a plane implies that the angular distance $\hat{\alpha}_i$ between pairs of neighbors becomes the focus of attention. Thus, we treat the variables $\hat{\alpha}_i$ as additional states of the $N$-agent system. It is known from the definition of the angular distance $\hat{\alpha}_i$ that

$$\dot{\hat{\alpha}}_i(t) = \dot{\alpha}_{i^+}(t) - \dot{\alpha}_i(t), \qquad i = 1, 2, \ldots, N \tag{15}$$

where $\alpha_i(t)$ is denoted as the angular of the vector $\bar{p}_i(t)$ for agent $i$.

Let $\rho_i(t) \triangleq \|\bar{p}_i\|$ be the radius of the vector $\bar{p}_i(t)$ for agent $i$. Then, the system (14) can be represented in the polar coordinates

$$\bar{p}_i(t) = \rho_i(t) \begin{bmatrix} \cos\alpha_i(t) \\ \sin\alpha_i(t) \end{bmatrix}$$

by

$$\dot{\rho}_i(t) = \gamma\lambda\rho_i(r^2 - \rho_i^2) f_i(t) \tag{16}$$

$$\dot{\alpha}_i(t) = \lambda f_i(t). \tag{17}$$

Consequently, we arrive at the dynamical equation of the additional states

$$\dot{\hat{\alpha}}_i(t) = \lambda[f_{i^+}(t) - f_i(t)], \qquad i = 1, 2, \ldots, N \tag{18}$$

where $f_i(t)$ is given by (12). It is worth to emphasize that $\alpha_i(t)$ here is only used for analysis purposes and is not known to the agents.

Now, we have the overall closed-loop system in the polar coordinates with states $\rho_i(t)$ and $\hat{\alpha}_i(t)$ described by (16) and (18). We emphasize that the overall system is a time-invariant autonomous system.

In the following two lemmas, we first prove that order preservation implies collision avoidance in our problem setting, and then prove the first objective of target circling in the circle formation problem in a plane can be achieved for the closed-loop system.

*Lemma 1:* For the $N$-agent system under consideration, it has the property of collision avoidance if the property of order preservation is achieved.

*Proof*: We prove by contradiction. Suppose that there exists a $t_c > 0$ such that at least two agents $i, j$ collide with each other, namely $\|p_i - p_j\| = 0$ at $t_c > 0$. Then, it must be true that $\mod(\theta_{ij}(t_c), 2\pi) = 0$, where $\theta_{ij}$ denotes the angular distance from agent $i$ to $j$ (i.e., $\theta_{ij}$ is formed by counterclockwise rotating the ray extending from the target to agent $i$ until reaching agent $j$), and $\mod(\cdot)$ is the modular operation. Consider separately the following two cases.

*Case A:* $j \in \mathcal{N}_i$. From Definitions 1 and 4, we know that $\hat{\alpha}_i(t_c), \hat{\alpha}_{i-}(t_c) \in (0, 2\pi)$. One can easily get $\theta_{ij}(t_c) \in (0, 2\pi)$ due to the definition of $\theta_{ij}$ and $j \in \mathcal{N}_i$. So we have arrive at a contradiction.

*Case B:* $j \notin \mathcal{N}_i$. Without loss of generality, we assume the labels of agents to be $1, \ldots, i, k_1, k_2, \ldots, j-1, j, \ldots$.

From Definitions 1 and 4 again, we have $\sum_{i=1}^{N} \hat{\alpha}_i(t_c) = 2\pi$ and $\theta_{ik_1}(t_c), \theta_{k_1 k_2}(t_c), \ldots, \theta_{j-1,j}(t_c) \in (0, 2\pi)$. Thus, one can have $0 < \theta_{ik_1}(t_c) + \theta_{k_1 k_2}(t_c) + \ldots + \theta_{j-1,j}(t_c) < 2\pi$, or equivalent $\theta_{ij}(t_c) \in (0, 2\pi)$. However, this contradicts the assumption that $\bmod\ (\theta_{ij}(t_c), 2\pi) = 0$.

Hence, in both of the two cases, we have encountered contradiction and this completes the proof. ■

*Lemma 2:* For each agent $i$, under the control law (8), the solution to system (16) asymptotically converges to an equilibrium point $\rho_i^*$ satisfying $\|\rho_i{}^*\| = r$ if $f_i(t) > 0$ and $f_i(t)$ is bounded for all $t \geq 0$.

*Proof*: From the equation $0 = \gamma\lambda\rho_i(r^2 - \rho_i^2) f_i(t)$, we can determine that the system has two equilibrium points, $\rho_i = 0$ and $\rho_i = r$.

We first check the stability of the equilibrium point $\rho_i = 0$. A Lyapunov function candidate may be taken as

$$V_1(\rho_i) = \rho_i^2.$$

One can check that $V_1(\rho_i)$ is continuously differentiable and positive definite. The derivative of $V_1(\rho_i)$ along the trajectories of the system is given by

$$\dot{V}_1(\rho_i) = 2\rho_i \dot{\rho}_i = 2\gamma\lambda(r^2 - \rho_i^2)\rho_i^2 f_i(t).$$

In a certain neighborhood $\Omega$ of the origin ($\rho_i = 0$), both $V_1$ and $\dot{V}_1$ are positive definite and bounded, since $\gamma > 0, \lambda > 0, f_i(t) > 0$ and $f_i(t)$ is bounded for all $t \geq 0$. It follows the fact that the equilibrium point $\rho_i = 0$ is unstable.

In order to show the stability of the equilibrium point $\rho_i = r$, consider the Lyapunov function candidate

$$V_2(\rho_i) = (r^2 - \rho_i^2)^2.$$

One can check that $V_2(\rho_i)$ is continuously differentiable, positive definite, and radially unbounded. The derivative of $V_2(\rho_i)$ along the trajectories of the system is given by

$$\dot{V}_2(\rho_i) = -4(r^2 - \rho_i^2)\rho_i \dot{\rho}_i = -4\gamma\lambda(r^2 - \rho_i^2)^2 \rho_i^2 f_i(t) \leq 0.$$

Since $f_i(t) > 0$ and $f_i(t)$ is bounded for all $t \geq 0$, one can further check that $\{\rho_i = 0\} \bigcup \{\rho_i = r\}$ is the largest invariant set in $\{\rho_i(t) \in \mathbb{R}^1 | \dot{V}_2 = 0\}$. From the LaSalle's theorem (see [24, Th. 4.4]), we know that every solution starting in $\{\rho_i(t) \in \mathbb{R}^1\}$ approaches $\{\rho_i = 0\} \bigcup \{\rho_i = r\}$ as $t \to \infty$.

Furthermore, since the equilibrium point $\rho_i = 0$ is unstable, $\rho_i = r$ is asymptotically stable. So under the control law (8), for each agent $i$, the every solution starting in $\{\mathbb{R}^1 - 0\}$ approaches the equilibrium point $\{\rho_i = r\}$ as $t \to \infty$. ■

In view of Lemma 2, we have proven the first objective of target circling in the problem that each agent independently converges to a circle around the target with the desired radius $r$ when $f_i(t) > 0$ and $f_i(t)$ is bounded for all $i = 1, 2, \ldots, N$ and all $t \geq 0$.

We go ahead analyzing the complete circle formation problem in a plane with both the first objective of target circling and the second objective of forming and maintaining a desired distribution pattern for the $N$-agent system. For this purpose, we have to take into account the $N$ agents' neighbor relationships and mainly focus on the introduced additional states $\hat{\alpha}_i(t)$.

The dynamical equation of $\hat{\alpha}_i(t)$ (18) can be further written into

$$\begin{aligned} \dot{\hat{\alpha}}_i(t) &= \lambda[f_{i^+}(t) - f_i(t)] = \frac{\lambda c_2}{2\pi}[u_{i^+}^{\alpha}(t) - u_i^{\alpha}(t)], \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{20}$$

We summarize the system dynamics into a compact form

$$\dot{\hat{\alpha}}(t) = -\frac{\lambda c_2}{2\pi} L(d)\hat{\alpha}(t) \tag{21}$$

where $L(d)$ is given by (19) shown at the bottom of the this page.

In order to analyze the closed-loop system (21), we first list some useful matrix analysis results. For a positive integer $n$, we use $\mathcal{M}_n$ to denote the set of all $n$-by-$n$ real matrices. We say a matrix $A$ is *nonnegative* (respectively, *positive*), denoted by $A \geq 0$ (respectively, $A > 0$), if all its entries are nonnegative (respectively, positive). The directed graph of a matrix $A \in \mathcal{M}_n$, denoted by $\mathbb{G}(A)$, is the directed graph with the vertex set $\{v_i\}$, $i \in \{1, 2, \ldots, n\}$, such that there is a directed edge in $\mathbb{G}(A)$ from $v_j$ to $v_i$ if and only if $a_{ij} \neq 0$. A directed graph is said to be *strongly connected* if there is a directed path between any pair of distinct vertices [25], [26].

*Lemma 3 (see[22, Lemma 5]):* It holds that:

1) $L(d)$ is diagonalizable and $\lambda_i \in [0, 2]$, $i = 1, 2, \ldots, N$.
2) 0 is a single eigenvalue.
3) When $N$ is even, 2 is an eigenvalue, while when $N$ is odd, 2 is not.

Now, we prove the main result in this paper.

*Theorem 1:* Given an admissible circle formation characterized by $r$ and $d$ in a plane, the circle formation problem in a plane is solved with collision avoidance under the proposed control law (8) with (9) and (12).

*Proof*: We first prove the second objective of spacing adjustment in the problem using the similar idea as in [22, Th. 1]. Let $D = \text{diag}\{d_1, d_2, \ldots, d_N\}$, and then, $D^{-1}L(d)D = L^T(d)$. Let $\delta = D^{-1}\hat{\alpha}$, and then, we have

$$\dot{\delta}(t) = -\frac{\lambda c_2}{2\pi} L^T(d)\delta(t).$$

Since $L^T(d)$ is the Laplacian matrix of $\mathbb{G}(L^T(d))$, which is strongly connected, we have $\lim_{t\to\infty} \delta(t) = c\mathbf{1}_N$, where $c$ is a constant. It follows then from the definition of $\delta$ that $\lim_{t\to\infty} \hat{\alpha}(t) = cd$.

Furthermore, the solution to system (21) is

$$\hat{\alpha}(t) = e^{-\frac{\lambda c_2}{2\pi} L(d)t}\hat{\alpha}(0), \qquad t \geq 0.$$

Since $\lambda, c_2 > 0$ and from [22, Th. 1] and Lemma 3, one can know that $e^{-\frac{\lambda c_2}{2\pi} L(d)t}$ is positive for all $t \geq 0$. The initial condition when the $N$ agents arranged in an almost counterclockwise order (see Definition 2) ensures that $\hat{\alpha}(0) \geq 0, \hat{\alpha}(0) \neq 0$ and $\sum_{i=1}^{N} \hat{\alpha}_i(0) = 2\pi$. So under the initial condition, any solution to system (21) satisfies $\hat{\alpha}(t) > 0$ and $\sum_{i=1}^{N} \hat{\alpha}_i(t) = 2\pi$ for all $t \geq 0$. Thus, the $N$ agents are arranged in

$$L(d) = \begin{bmatrix} \frac{d_2}{d_2+d_1} + \frac{d_N}{d_1+d_N} & -\frac{d_1}{d_2+d_1} & 0 & \cdots & 0 & -\frac{d_1}{d_1+d_N} \\ -\frac{d_2}{d_2+d_1} & \frac{d_3}{d_3+d_2} + \frac{d_1}{d_2+d_1} & -\frac{d_2}{d_3+d_2} & \cdots & 0 & 0 \\ 0 & -\frac{d_3}{d_3+d_2} & \frac{d_4}{d_4+d_3} + \frac{d_2}{d_3+d_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{d_N}{d_N+d_{N-1}} + \frac{d_{N-2}}{d_{N-1}+d_{N-2}} & -\frac{d_{N-1}}{d_N+d_{N-1}} \\ -\frac{d_N}{d_1+d_N} & 0 & 0 & \cdots & -\frac{d_N}{d_N+d_{N-1}} & \frac{d_1}{d_1+d_N} + \frac{d_{N-1}}{d_N+d_{N-1}} \end{bmatrix}. \tag{19}$$

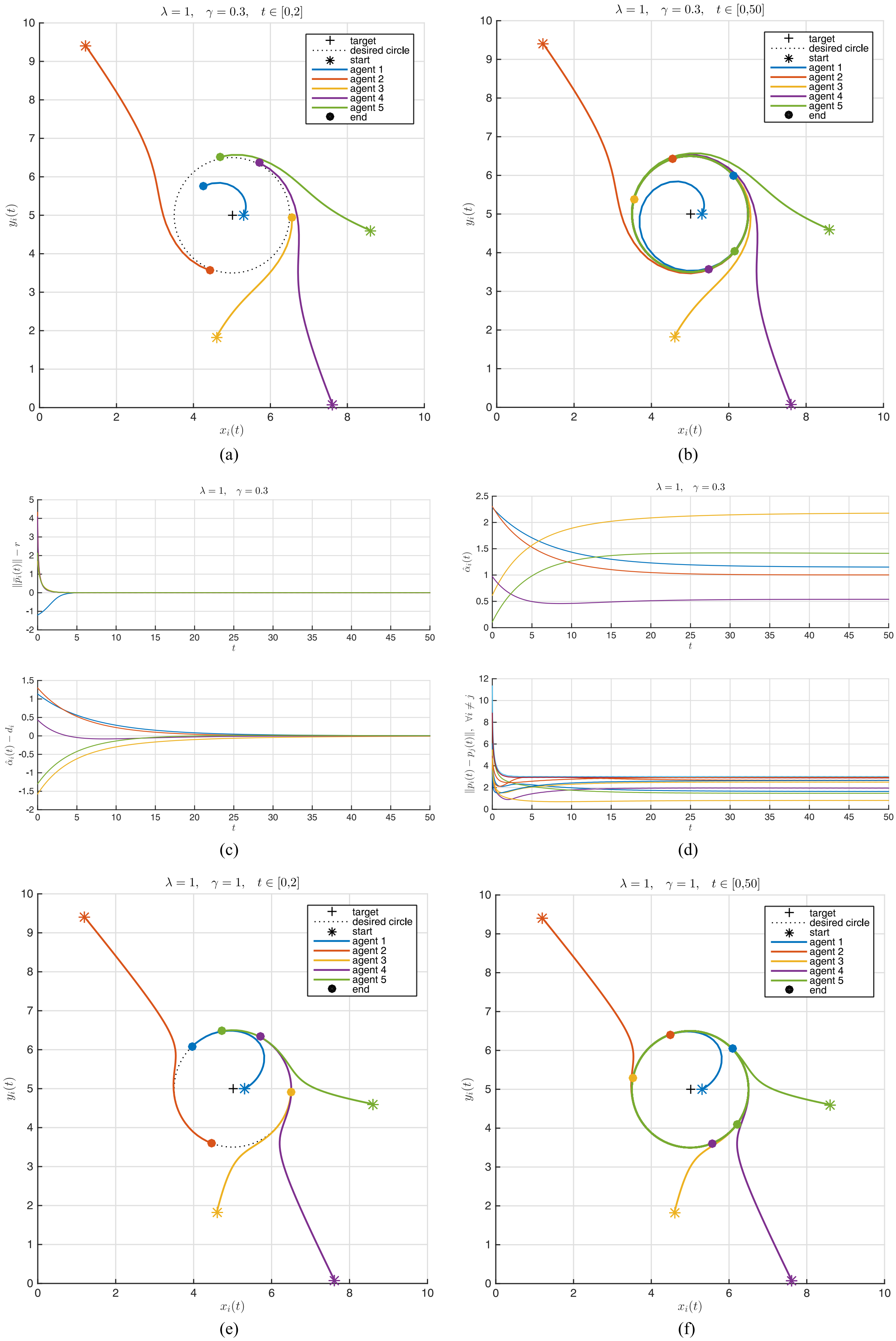


Fig. 3. Simulation results of the proposed control laws solving the circle formation problem in a plane when $N = 5$, $c_1 = c_2 = 1$, $\lambda = 1$. (a)–(d) $\gamma = 0.3$. (e) and (f) $\gamma = 1$. (a) and (e) Trajectories of five agents in the plane at $t \in [0, 2]$. (b) and (f) Trajectories of five agents in the plane at $t \in [0, 50]$. (c) Evolution of $\|\bar{p}_i(t)\| - r$, $\hat{\alpha}_i(t) - d_i$ for $i = 1, \ldots, 5$. (d) Evolution of $\hat{\alpha}_i(t)$, $\|p_i(t) - p_j(t)\|, j \in \mathcal{N}_i$ for $i = 1, \ldots, 5$.

a counterclockwise order (see Definition 1) for all $t > 0$. Now, order preservation has been proven to be achieved using the same techniques as in [22, Th. 1]. Together with Lemma 1, one can have that collision avoidance is achieved.

Then, we recall the fact we have got that $\lim_{t\to\infty} \hat{\alpha}(t) = cd$, where $c$ is a constant. Noticing $\sum_{i=1}^{N} \hat{\alpha}_i(t) = 2\pi$ for all $t$ and $\sum_{i=1}^{N} d_i = 2\pi$, it must be true that $c = 1$ and thus

$$\lim_{t\to\infty} \hat{\alpha}(t) = d.$$

So we have proven the second objective of the problem that the requirement of spacing adjustment is achieved with order preservation and collision avoidance.

Now, we prove that the designed $f_i(t)$ satisfies $f_i(t) > 0$ and $f_i(t)$ is bounded for all $i = 1, 2, \ldots, N$ and all $t \geq 0$ to ensure that the result of Lemma 2 still works here. Since the $N$ agents have the property of order preservation, based on Definitions 1, 2, and 4, one can check that $u_i^{\alpha}(t) \in (-2\pi, 2\pi)$ for all $i = 1, 2, \ldots, N$ and all $t \geq 0$. It follows that $0 \leq c_1 - c_2 < f_i(t) < c_1 + c_2$ for all $t \geq 0$. Further from Lemma 2, we know that the first objective of target circling is achieved.

Summarizing the above results, we conclude that the circle formation problem in a plane is solved with order preservation and collision avoidance. ■

We want to emphasize that the parameters $\lambda$, $c_1$, and $\gamma$ in our proposed control law (8) have explicit physical meanings, which play an important role in the motion characteristics of each agent $i$ when solving the circle formation problem in a plane. Toward this end, we first calculate agent $i$'s linear velocity as $v_i(t) = \|\dot{p}_i(t)\| = \lambda f_i(t)\|\bar{p}_i(t)\|\sqrt{\gamma^2 l_i^2 + 1}$ and its angular velocity relative to the target as $\omega_i(t) = \dot{\alpha}_i(t) = \lambda f_i(t)$. Moreover, the curvature of agent $i$'s trajectory at time $t$ can be calculated as

$$\kappa_i(t) = \frac{1}{(\gamma^2 l_i^2 + 1)^{\frac{1}{2}} (r^2 - l_i)^{\frac{1}{2}}} + \frac{2\gamma^2 l_i (r^2 - l_i)^{\frac{1}{2}}}{(\gamma^2 l_i^2 + 1)^{\frac{3}{2}}}.$$

Then, we can also have their corresponding stable states when the $N$ agents solve the circle formation problem as

$$\lim_{t\to\infty} v_i(t) = \lambda r c_1, \ \lim_{t\to\infty} \omega_i(t) = \lambda c_1, \ \lim_{t\to\infty} \kappa_i(t) = \frac{1}{r}.$$

In view of above equations, we can conclude that the product of the parameters $\lambda c_1$ affects agent $i$'s linear velocity and its angular velocity relative to the target and further determines the final stable states of the velocity, while the parameter $\gamma$ affects the curvature of agent $i$'s trajectory that the larger $\gamma$ implies smaller curvature. In other words, the parameter $\gamma$ determines the trajectory of each agent $i$, while the product of the parameters $\lambda c_1$ determines how fast the agent moves on the trajectory. Benefit from these features, the parameters $\lambda$, $c_1$, and $\gamma$ in our proposed control law can be selected more reasonable and easily according to the request of the agents' motion characteristics when applies the control law to real robot systems in the future.

In the next section, we present simulation results that validate our theoretical analysis in this section.

## V. Simulations

In the simulations, we consider a system consisting of five agents. The target is set at the point of $(5, 5)$ in the plane without loss of generality. The radius of the desired enclosing circle is $r = 1.5$. Both the initial positions of the five agents and the desired distribution pattern $d = [1.1469,\ 1.0034,\ 2.1867,\ 0.5395,\ 1.4068]^T$ are generated randomly.

We first choose the parameters of the control laws (8) as $c_1 = c_2 = 1$, $\lambda = 1$, and $\gamma = 0.3$. We run the simulations and show the results in Fig. 3(a)–(d). Fig. 3(a) and (b) shows the trajectories of five agents in the plane at $t \in [0, 2]$ and $t \in [0, 50]$, respectively. Fig. 3(c) shows the distances between the agents to the desired circle for all the five agents, and the differences between current angular distance and the prescribed distances between all pairs of neighboring agents. In Fig. 3(a)–(c), the simulation results clearly indicate that the group of agents generates asymptotically the prescribed circle formation characterized by $d$ and $r$ around the given target under the control law (8). In addition, we show the angular distances between all pairs of neighboring agents and the distances between any two agents of the group in Fig. 3(d). One can see that the multiagent system under the control law (8) has the properties of order preservation and collision avoidance.

Moreover, in order to show the physical meanings of the parameter $\gamma$ in our proposed control, we reset the parameter as $\gamma = 1$. For ease of comparison, we use the same initial positions of the five agents and the desired distribution pattern $d$ and $r$ as in the first case. We run the simulation and show the trajectories of five agents in the plane at $t \in [0, 2]$ and $t \in [0, 50]$ in Fig. 3(e) and (f), respectively. Compared with the trajectories of the first case shown in Fig. 3(a) and (b), it is indicated that the parameter $\gamma$ determines the trajectory of each agent. Due to a space limit, we omit the simulation results of different $\lambda$ and $c_1$ here.

## VI. Conclusion

In this paper, we have studied the circle formation problem for a group of anonymous mobile agents in a plane. The problem includes two subobjectives of target circling, for which all agents are required to form a circle surrounding a preset target and to rotate around the target, and spacing adjustment, for which each agent has to maintain the desired distance from its neighbors. A limit-cycle-based decoupled-design approach has been delivered to solve the circle formation problem. We have proposed a distributed controller comprised of the converging part and the layout part to deal with the two subobjectives of target circling and spacing adjustment, respectively. The former part is based on a limit cycle using only the relative position from the target, and the latter has been designed by also using the relative position from the agent's neighbors. With the aid of the properties of limit-cycle oscillators, we have separately designed the two parts and finally obtained a whole controller for the groups of anonymous agents. It has been proven that our proposed control law can solve the circle formation problem in a plane asymptotically with the additional guarantee that no collision between agents ever takes place. It is worth to point out that our controller works and guarantees collision avoidance among agents even if some or all agents occupy the same position at the beginning.

Moreover, the parameters in the control law have explicit physical meanings that they determine the trajectory of each agent and how fast the agent moves on the trajectory. Thus, they can be selected more reasonable and easily according to the request of the robots' motion characteristics when applied to real robot systems. We are conducting experiments using mobile robots to implement the designed control strategies. We are also interested in extending the work of forming arbitrary formations on a circle in two aspects of allowing for a moving target and modeling the agents as unicycles.

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