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Mamy $X \sim N(\mu_1, \sigma_1^2)$ i $Y \sim N(\mu_2, \sigma_2^2)$ niezależne.

Gęstość (X,Y):

$$f(x,y) \overset{(*)}{=} f(x)\cdot f(y) = \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_1\sigma_2}\cdot \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)\exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right)$$

$(*)$ z niezależności X, Y.

Niech $Z = X+Y$, $T = Y$ $\iff$ $X = Z - Y$, $Y = T$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$\cdot$	z	$-t$	$-\mu_1$
z	z^2	$-zt$	$-z\mu_1$
$-t$	$-zt$	t^2	$+t\mu_1$
$-\mu_1$	$-z\mu_1$	$+t\mu_1$	μ_1^2

Gęstość zmiennej (Z,T):

$$g(z,t) = f(x(z,t), y(z,t))\cdot|J| =$$

$$\frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_1\sigma_2}\cdot\exp\left(-\frac{(z-t-\mu_1)^2}{2\sigma_1^2}\right)\exp\left(-\frac{(t-\mu_2)^2}{2\sigma_2^2}\right) =$$

$$\frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_1\sigma_2}\cdot\exp\left(-\frac{\sigma_2^2(z^2-2tz-2z\mu_1+t^2+2t\mu_1+\mu_1^2)+\sigma_1^2(t^2-2t\mu_2+\mu_2^2)}{2\sigma_1^2\sigma_2^2}\right) =$$

$$\frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_1\sigma_2}\cdot\exp\left(-\frac{\sigma_2^2z^2-\sigma_2^2 2tz-\sigma_2^2 2z\mu_1+\sigma_2^2t^2+\sigma_2^2 2t\mu_1+\sigma_2^2\mu_1^2+\sigma_1^2t^2-2\sigma_1^2t\mu_2+\sigma_1^2\mu_2^2}{2\sigma_1^2\sigma_2^2}\right) =$$

$$\frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_1\sigma_2}\cdot\exp\left(-\frac{t^2(\sigma_2^2+\sigma_1^2)-2t(\sigma_2^2z+\sigma_1^2\mu_2-\sigma_2^2\mu_1)+\sigma_2^2(z^2+\mu_1^2-2z\mu_1)+\sigma_1^2\mu_2^2}{2\sigma_1^2\sigma_2^2}\right)$$

Wiemy, że $x,y\in\mathbb{R}$. Skoro $z=x+y$, $t=y$ to również $z,t\in\mathbb{R}$.
Aby znaleźć gęstość zmiennej Z musimy policzyć gęstość brzegową zmiennej (Z,T), czyli:

$$g_Z(z) = \int_{\mathbb{R}} g(z,t)\,dt \qquad z\in\mathbb{R}$$

Ponadto, oznaczmy $\sigma_3 = \sqrt{\sigma_1^2+\sigma_2^2}$, mamy wtedy:

$$g_Z(z) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\,\sigma_3} \cdot \frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}} \cdot \exp\left(-\frac{\frac{t^2(\sigma_2^2+\sigma_1^2)}{\sigma_3^2} - \frac{2t(\sigma_2^2 z + \sigma_1^2\mu_2 - \sigma_2^2\mu_1)}{\sigma_3^2} + \frac{\sigma_2^2(z^2+\mu_1^2-2z\mu_1)+\sigma_1^2\mu_2^2}{\sigma_3^2}}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right) dt =$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\,\sigma_3}\,\frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}}\, \exp\left(-\frac{\left(t - \frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2 - \left(\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2 + \frac{\sigma_2^2(z-\mu_1)^2+\sigma_1^2\mu_2^2}{\sigma_3^2}}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)dt =$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\,\sigma_3}\cdot\exp\left(-\frac{-\left(\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2+\frac{\sigma_2^2(z-\mu_1)^2+\sigma_1^2\mu_2^2}{\sigma_3^2}}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)\cdot\frac{1}{\sqrt{2\pi}\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)}\exp\left(-\frac{\left(t-\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)dt =$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\,\sigma_3}\exp\left(-\frac{-(\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2)^2+\sigma_3^2(\sigma_2^2(z-\mu_1)^2+\sigma_1^2\mu_2^2)}{2\sigma_3^2(\sigma_1\sigma_2)^2}\right)\cdot\frac{1}{\sqrt{2\pi}\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)}\exp\left(-\frac{\left(t-\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)dt =$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_3}\cdot\exp\left(-\frac{-\sigma_2^4 z^2+2\sigma_2^4 z\mu_1-2\sigma_1^2\sigma_3^2 z\mu_2-\sigma_2^4\mu_1^2+2\sigma_1^2\sigma_2^2\mu_1\mu_2-\sigma_1^4\mu_2^2+\sigma_3^2\sigma_2^2 z^2-\sigma_3^2\sigma_2^2\, 2z\mu_1+\sigma_3^2\sigma_2^2\mu_1^2+\sigma_3^2\sigma_1^2\mu_2^2}{2\sigma_3^2(\sigma_1\sigma_2)^2}\right)\cdot$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}}\exp\left(-\frac{\left(t-\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)dt =$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_3}\exp\left(-\frac{z^2\left(-\frac{\sigma_2^2}{\sigma_1^2}+\frac{\sigma_3^2}{\sigma_1^2}\right)-2z\mu_1\left(-\frac{\sigma_2^2}{\sigma_1^2}+\frac{\sigma_3^2}{\sigma_1^2}\right)^{-2z\mu_2}+\mu_1^2\left(-\frac{\sigma_2^2}{\sigma_1^2}+\frac{\sigma_3^2}{\sigma_1^2}\right)+\mu_2^2\left(-\frac{\sigma_1^2}{\sigma_2^2}+\frac{\sigma_3^2}{\sigma_2^2}\right)-2\mu_1\mu_2}{2\sigma_3^2}\right)\cdot\underbrace{\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}}\exp\left(-\frac{\left(t-\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)dt}_{(**)} =$$

Zauważmy, że (**) jest całką z gęstości zmiennej, której rozkład wynosi $N\left(\frac{\sigma_2^2(z-\mu_1)+\sigma_1^2\mu_2}{\sigma_3^2}, \left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2\right)$, zatem jej wartość wyniesie 1.

Mamy więc:

$$g_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_3}\exp\left(-\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_3^2}\right)$$

Widzimy, że $Z \sim N(\mu_1+\mu_2, \sigma_3^2)$, a skoro $\sigma_3 = \sqrt{\sigma_1^2+\sigma_2^2}$ to $Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ ∎.