Causality
Estimation of Causal Effects

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Identification of Causal Effects

Agenda

- Identification of Causal Effects
  - covariate adjustment
  - Pearl's back door criterion
  - front-door criterion
  - do-calculus
  - do-calculus based algorithms
- Simpson's Paradox

Identification of Causal Effects

- In this lecture we will consider DAGs $G = (\mathbf{V}, \mathbf{E})$ associated with SCMs $M = (S, P)$
- W.l.o.g. we will assume the model is Markovian, i.e. in addition to acyclicity of G, we assume the error terms are jointly independent, e.g.:

$$\begin{tikzcd} U_X \arrow[d, dashed] & U_Y \arrow[d, dashed] \\ X \arrow[r] & Y \end{tikzcd}$$

- Thus, in our DAGs we will not show explicitly the exogene variables, like e.g.

$$\begin{tikzcd} X \arrow[r] & Y \end{tikzcd}$$

- We will assume that some of the variables of $\mathbf{V}$ are unobserved
- The observed variables are denoted as $\mathbf{R} \subseteq \mathbf{V}$
- Recall, that the Identification of Total Causal Effect of variables $\mathbf{X}$ on $\mathbf{Y}$ is to find an expression for

$$P(\mathbf{y} \mid do(\mathbf{x}))$$

  which uses only preintervention (i.e. do-operator free) probabilities involving variables in $\mathbf{R}$, or output that this is not possible

Identification of Causal Effects

- In this lecture we discuss how to solve the identification problem
- Specifically, the popular identifiability by covariate adjustment is cosidered
- For a given DAG $G = (\mathbf{V}, \mathbf{E})$ and observed variables $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \subseteq \mathbf{V}$, $\mathbf{Z}$ is called adjustment set for estimating the causal effect of $\mathbf{X}$ on $\mathbf{Y}$ if for every P consistent with G we have

$$P(\mathbf{y} \mid do(\mathbf{x})) = \begin{cases} \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})\ P(\mathbf{z}) & \text{if } \mathbf{Z} \neq \emptyset \\ P(\mathbf{y} \mid \mathbf{x}) & \text{if } \mathbf{Z} = \emptyset \end{cases}$$

- A crucial problem, here, is which $\mathbf{Z} \subseteq \mathbf{V}$ are *valid* adjustment set?
    - We start with a very simple: adjustment for direct causes
    - Unfortunately, for many instances this approach does not work
    - The famous back-door criterion by Pearl allows to specify more general valid adjustment sets
- Next, we consider estimations of causal effects by using front-door criterion
- Covariate adjustment and front-door criterion are not complete for identification
- The do-calculus proposed by Pearl allows to uncover *all* causal effects that can be identified from a given graph

Identification of Causal Effects

Adjustment for Direct Causes

- In the (Markovian) models we firstly consider the simplest intervention $do(X_i = x_i')$
- Recall, the interventional distribution $P(x_1, \ldots, x_n \mid do(x_i'))$ resulting from any intervention $do(X_i = x_i')$ is defined as a truncated factorization:

$$P(x_1, \ldots, x_n \mid do(x_i')) = \begin{cases} \prod_{j \neq i} P(x_j \mid pa_j) & \text{if } x_i = x_i' \\ 0 & \text{if } x_i \neq x_i' \end{cases}$$

- Multiplying and dividing the equation by $P(x_i' \mid pa_i)$ we get a more expressive relation

$$P(x_1, \ldots, x_n \mid do(x_i')) = \begin{cases} \frac{P(x_1, \ldots, x_n)}{P(x_i' \mid pa_i)} & \text{if } x_i = x_i' \\ 0 & \text{if } x_i \neq x_i' \end{cases}$$

- Next, if we consider the division by $P(x_i' \mid pa_i) = P(x_i', pa_i)/P(pa_i)$ as conditionalization on x_i' and pa_i, we obtain

$$P(x_1, \ldots, x_n \mid do(x_i')) = \begin{cases} P(x_1, \ldots, x_n \mid x_i', pa_i)\ P(pa_i) & \text{if } x_i = x_i' \\ 0 & \text{if } x_i \neq x_i' \end{cases}$$

Identification of Causal Effects

Adjustment for Direct Causes

- Using the formula

$$P(x_1, \ldots, x_n \mid do(x_i')) = \begin{cases} P(x_1, \ldots, x_n \mid x_i', pa_i)\ P(pa_i) & \text{if } x_i = x_i' \\ 0 & \text{if } x_i \neq x_i' \end{cases}$$

  we are able to compute the effect of an intervention $do(X_i = x_i')$ on a set of variables $\mathbf{Y}$ disjoint from $\{X_i\} \cup Pa_i$ by summing:

$$\begin{aligned} P(\mathbf{y} \mid do(x_i')) &= \textstyle\sum_{X_j \notin \mathbf{Y} \cup \{X_i\}} P(x_1, \ldots, x_n \mid do(x_i')) \\ &= \textstyle\sum_{X_j \notin \mathbf{Y} \cup \{X_i\}} P(x_1, \ldots, x_n \mid x_i', pa_i)\ P(pa_i) \\ &= \textstyle\sum_{pa_i} P(\mathbf{y} \mid x_i', pa_i)\ P(pa_i) \end{aligned}$$

- This yields the following Theorem

Identification of Causal Effects

Adjustment for Direct Causes

Theorem (Adjustment for Direct Causes)

Let Pa_i denote the set of direct causes of variable X_i and let $\mathbf{Y}$ be any set of variables disjoint of $\{X_i\} \cup Pa_i$. Assuming the variables Pa_i are observed, the effect of the intervention $do(X_i = x_i')$ on $\mathbf{Y}$ is given by

$$P(\mathbf{y} \mid do(x_i')) \;=\; \sum_{pa_i} P(\mathbf{y} \mid x_i', pa_i)\; P(pa_i)$$

where $P(\mathbf{y} \mid x_i', pa_i)$ and $P(pa_i)$ represent preintervention probabilities.

- **Remark:** If we replace the single variable X_i by a set of variables $\mathbf{X}$ in the theorem above, then the statement is not true, in general
- **Example:** Consider the following model, involving the observed variables:
  S – size of stone (small, large), T – treatment (A or B), R – recovery (0,1)

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ T \arrow[rr] & & R \end{tikzcd}$$

The causal effect of treatment (T) on recovery (R) is identifiable and is given by the formula $P(R = 1 \mid do(t)) \;=\; \sum_s P(R = 1 \mid t, s)\; P(s)$

Identification of Causal Effects

The Back-Door Criterion

- Consider a graphical model representing the relationship between
  - a new drug (D),
  - recovery (R),
  - weight (W), and
  - an unmeasured variable (socioeconomic status (S))

$$\begin{tikzcd}
\text{unobserved } S \arrow[r] \arrow[d] & W \arrow[d] \\
D \arrow[r] & R
\end{tikzcd}$$

- Our task is to decide if the causal effect of the new drug D on recovery R is identifiable and if yeas to give a formula
- Notice that we can not use the adjustment for direct causes since the parent S of D is unobserved

Identification of Causal Effects

The Back-Door Criterion

Definition (Back-Door)

A set of variables $\mathbf{Z}$ satisfies the back-door criterion relative to an ordered pair of variables (X_i, X_j) in a DAG G if:

(i) no node in $\mathbf{Z}$ is a descendant of X_i and

(ii) $\mathbf{Z}$ blocks every path between X_i and X_j that contains an arrow into X_i.

Similarly, if $\mathbf{X}$ and $\mathbf{Y}$ are two disjoint subsets of nodes in G, then $\mathbf{Z}$ is said to satisfy the back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ if it satisfies the criterion relative to any pair (X_i, X_j) such that $X_i \in \mathbf{X}$ and $X_j \in \mathbf{Y}$.

- Notice: $\mathbf{Z}$ can be empty
- **Example:** in the DAG below, $\mathbf{Z} = \{W\}$ satisfies the back-door criterion relative to (D, R)

$$\begin{tikzcd}
\text{unobserved } S \arrow[r] \arrow[d] & W \arrow[d] \\
D \arrow[r] & R
\end{tikzcd}$$

Identification of Causal Effects

The Back-Door Criterion

Theorem (Back-Door Adjustment)

If a set of variables $\mathbf{Z}$ satisfies the back-door criterion relative $(\mathbf{X},\mathbf{Y})$ in a DAG G and variables $\mathbf{Z}$ are observed then the causal effect of $\mathbf{X}$ on $\mathbf{Y}$ is identifiable and is given by the formula

$$P(\mathbf{y} \mid do(\mathbf{x})) = \begin{cases} \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x},\mathbf{z})\ P(\mathbf{z}) & \text{if } \mathbf{Z} \neq \emptyset \\ P(\mathbf{y} \mid \mathbf{x}) & \text{if } \mathbf{Z} = \emptyset. \end{cases}$$

Example: In the model involving the observed variables: a new drug (D), recovery $R)$, weight (W), and an unmeasured variable (socioeconomic status (S))

$$\begin{tikzcd} \text{unobserved } S \arrow[r] \arrow[d] & W \arrow[d] \\ D \arrow[r] & R \end{tikzcd}$$

the causal effect of a new drug (D) on recovery (R) is identifiable and is given by the formula

$$P(R=1 \mid do(d)) \ = \ \sum_{w} P(R=1 \mid d,w)\ P(w)$$

Identification of Causal Effects

The Back-Door Criterion

Proof (of the Back-Door-Adjustment Theorem)

- Consider a Markovian Model G
- We show only the case of the singleton variable X_i and $\mathbf{Z} \neq \emptyset$, $Pa_i \neq \emptyset$
- From the Adjustment-for-Direct-Causes Theorem we have (the formula is true even if Pa_i are unobserved)

$$P(\mathbf{y} \mid do(x_i')) \;=\; \sum_{pa_i} P(\mathbf{y} \mid x_i', pa_i)\; P(pa_i) \;=\; \sum_{pa_i} P(pa_i) \sum_{\mathbf{z}} P(\mathbf{y} \mid x_i', pa_i, \mathbf{z})\; P(\mathbf{z} \mid x_i', pa_i)$$

- From our assumptions it follows, that $\mathbf{Z}$ saisfies:

$$(\mathbf{Y} \perp\!\!\!\perp Pa_i \mid X_i, \mathbf{Z}) \quad \text{and} \quad (X_i \perp\!\!\!\perp \mathbf{Z} \mid Pa_i)$$

- Thus, due to the first CI, we can rewrite the equation as

$$P(\mathbf{y} \mid do(x_i')) \;=\; \sum_{pa_i} P(pa_i) \sum_{\mathbf{z}} P(\mathbf{y} \mid x_i', \mathbf{z})\; P(\mathbf{z} \mid x_i', pa_i)$$

- By the second CI, we have $P(\mathbf{z} \mid x_i', pa_i) = P(\mathbf{z} \mid pa_i)$; Thus, we can conclude:

$$P(\mathbf{y} \mid do(x_i')) = \sum_{\mathbf{z}} P(\mathbf{y} \mid x_i', \mathbf{z})\; P(\mathbf{z})$$

- The cases $\mathbf{Z} = \emptyset$ or $Pa_i = \emptyset$ can be analysed in the similar way
- The proof for $\mathbf{X}$, with $|\mathbf{X}| \geq 2$, is much more involved $\square$

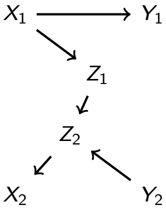
Identification of Causal Effects

Covariate Adjustment: a Generalized Criterion

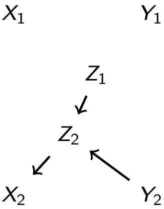
- In summary: $\mathbf{Z}$ satisfies Pearl's *back-door* criterion if
  - (i) no element in $\mathbf{Z}$ is a descendant of $\mathbf{X}$ and
  - (ii) $\mathbf{Z}$ d-separates $\mathbf{X}$ and $\mathbf{Y}$ in $G_{\underline{\mathbf{X}}}$

  ($G_{\underline{\mathbf{X}}}$ denotes the graph obtained by deleting from G all arrows emerging from nodes in $\mathbf{X}$)
- Its a simple and easily implementable rule; But it is not complete
- Consider $\mathbf{X} = \{X_1, X_2\}$, $\mathbf{Y} = \{Y_1, Y_2\}$, and the DAG G

G:



$G_{\underline{\mathbf{X}}}$:



- In G there exists no $\mathbf{Z}$ which satisfies back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$
- On the other hand, one can prove that for $\mathbf{Z} = \{Z_1, Z_2\}$ it is true:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})\ P(\mathbf{z})$$

Sound and complete criterion to characterize adjustments in DAGs

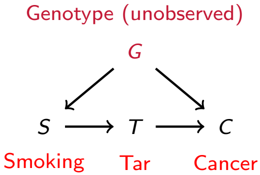
- Let $G_{\overline{\mathbf{X}}}$ denote the graph obtained by deleting from G all arrows pointing to nodes in $\mathbf{X}$
- Sound and Complete Adjustment Criterion (Shpitser, VanderWeele, Robins):
  $\mathbf{Z}$ satisfies the criterion relative to $(\mathbf{X}, \mathbf{Y})$ in G if
  - (a) no element in $\mathbf{Z}$ is a descendant in $G_{\overline{\mathbf{X}}}$ of any $W \in \mathbf{V} \setminus \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$ and
  - (b) all proper non-causal paths in G from $\mathbf{X}$ to $\mathbf{Y}$ are blocked by $\mathbf{Z}$

  (a path from from $\mathbf{X}$ to $\mathbf{Y}$ is called proper if it does not intersect $\mathbf{X}$ except at the start point)
- The drawback: it does not yield a practical algorithm for adjustment set construction
- The sound and complete *constructive* criterion (van der Zander, Liśkiewicz, Textor) allows identification by adjustment in linear time (`www.dagitty.net`)

Identification of Causal Effects

The Front-Door Criterion

- Consider or model for the "Smoking and the Genotype" problem:



- Our task is to decide if the causal effect of S on C is identifiable and if yeas to give a formula
- Notice that the we can not use the (back-door) adjustment for $\mathbf{Z} = \{G\}$, since G is unobserved

Definition (Front-Door)

A set of variables $\mathbf{Z}$ satisfies the front-door criterion relative to an ordered pair of variables $(\mathbf{X}, \mathbf{Y})$ in a DAG G if:

(i) $\mathbf{Z}$ intercepts all directed paths from $\mathbf{X}$ to $\mathbf{Y}$

(ii) there is no unblocked back-door path from $\mathbf{X}$ to $\mathbf{Z}$ and

(iii) all back-door paths from $\mathbf{Z}$ to $\mathbf{Y}$ are blocked by $\mathbf{X}$

Theorem (Front-Door Adjustment)

If a set of variables $\mathbf{Z}$ satisfies the front-door criterion relative $(\mathbf{X}, \mathbf{Y})$ in a DAG G, the variables $\mathbf{Z}$ are observed, and $P(\mathbf{x}, \mathbf{z}) > 0$ then the causal effect of $\mathbf{X}$ on $\mathbf{Y}$ is identifiable and is given by the formula

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{z} \mid \mathbf{x}) \sum_{\mathbf{x}'} P(\mathbf{y} \mid \mathbf{x}', \mathbf{z}) \; P(\mathbf{x}')$$

Example

- $\mathbf{Z} = \{T\}$ satisfies the front-door criterion relative to (S, C) and T is observed

$$\begin{tikzcd}
& G \;\; \text{unobserved} \arrow[dl] \arrow[dr] & \\
S \arrow[r] & T \arrow[r] & C
\end{tikzcd}$$

- From the theorem we get the formula

$$P(C = 1 \mid do(s)) = \sum_{t} P(t \mid s) \sum_{s'} P(C = 1 \mid s', t) \; P(s')$$

  used in our previous lecture to estimate the total effect of smoking (S) on lung cancer (C)

A Calculus of Intervention

- Covariate adjustment and front-door criterion are sound but not complete methods for identification of causal effects
- The do-calculus, we will discuss now, is a powerful *symbolic* machinery that allows to uncover *all* causal effects that can be identified from a given graph
- The do-calculus is the basis for sound and complete methods for identification
- Particularly it can permit identification even if covariate adjustment or the use of front-door criterion is impossible

A Calculus of Intervention

Notation

- Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be arbitrary disjoint sets of nodes in a causal DAG G
- We denote by $G_{\overline{\mathbf{X}}}$ the graph obtained by deleting from G all arrows pointing to nodes in $\mathbf{X}$
- $G_{\underline{\mathbf{X}}}$ denotes the graph obtained by deleting from G all arrows emerging from nodes in $\mathbf{X}$
- To represent the deletion of both incoming and outgoing arrows, we use the notation $G_{\overline{\mathbf{X}},\, \underline{\mathbf{Z}}}$

$$G \quad \begin{tikzcd} & A \arrow[rr] \arrow[dl] \arrow[dr] & & Z \arrow[dr] & \\ D \arrow[rr] & & X \arrow[rr] \arrow[ur] & & Y \end{tikzcd} \qquad\qquad G_{\overline{\mathbf{X}},\, \underline{\mathbf{Z}}} \quad \begin{tikzcd} & A \arrow[rr] \arrow[dl] & & Z & \\ D & & X \arrow[rr] \arrow[ur] & & Y \end{tikzcd}$$

- The expression
$$P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) = \frac{P(\mathbf{y}, \mathbf{z} \mid do(\mathbf{x}))}{P(\mathbf{z} \mid do(\mathbf{x}))}$$
stands for the probability of $\mathbf{Y} = \mathbf{y}$ given that $\mathbf{X}$ is held constant at $\mathbf{x}$ and that (under this condition) $\mathbf{Z} = \mathbf{z}$ is observed

A Calculus of Intervention

Theorem (Rules of do Calculus)

Let G be the DAG associated with an SCM $M = (S, P)$. For any disjoint subsets of variables $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, and $\mathbf{W}$, we have the following rules

Rule 1 (Insertion/deletion of observations):

$$P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w}) \quad \text{if } (\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}}}}$$

Rule 2 (Action/observation exchange):

$$P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w}) \quad \text{if } (\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\ \underline{\mathbf{Z}}}}$$

Rule 3 (Insertion/deletion of actions):

$$P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w}) \quad \text{if } (\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\ \overline{\mathbf{Z}(\mathbf{W})}}}$$

where $\mathbf{Z}(\mathbf{W})$ is the set of $\mathbf{Z}$-nodes that are not ancestors of any $\mathbf{W}$-node in $G_{\overline{\mathbf{X}}}$

A Calculus of Intervention

Theorem (Soundness and Completeness of do-Calculus) [Shpitser/Pearl and Huang/Valtorta)]

A causal effect

$$q = P(y_1, \ldots, y_k \mid do(x_1), \ldots, do(x_m))$$

is identifiable in a model characterized by a graph G *if and only if* there exists a finite sequence of transformations, each conforming to one of the inference rules in Rules-of-do-Calculus Theorem that reduces q into a standard (i.e., do-free) probability expression involving observed quantities.

A Calculus of Intervention

Rule 1 $P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}}}}$

Rule 2 $P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\, \underline{\mathbf{Z}}}}$

Rule 3 $P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\, \overline{\mathbf{Z(W)}}}}$

Examples: Consider the following DAG:

$$\begin{tikzcd} & U \text{ unobserved} \arrow[dl] \arrow[dr] & \\ X \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

- Task 1: Compute $P(z \mid do(x))$

$$P(z \mid do(x)) \quad = \quad P(z \mid x) \qquad \text{Rule 2}$$

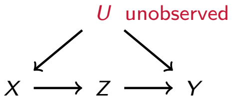
A Calculus of Intervention

Rule 1 $P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}}}}$

Rule 2 $P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\, \underline{\mathbf{Z}}}}$

Rule 3 $P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\, \overline{\mathbf{Z}(\mathbf{W})}}}$

Examples: Consider the following DAG:



- Task 2: Compute $P(y \mid do(z))$

$$
\begin{aligned}
P(y \mid do(z)) &= \textstyle\sum_x P(y \mid x, do(z))\; P(x \mid do(z))) && \text{probabilistic calculus}\\
&= \textstyle\sum_x P(y \mid x, do(z))\; P(x) && \text{Rule 3}\\
&= \textstyle\sum_x P(y \mid x, z)\; P(x) && \text{Rule 2}
\end{aligned}
$$

A Calculus of Intervention

Rule 1 $P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}}}}$

Rule 2 $P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\ \underline{\mathbf{Z}}}}$

Rule 3 $P(\mathbf{y} \mid do(\mathbf{x}), do(\mathbf{z}), \mathbf{w}) \;=\; P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{X}},\ \overline{\mathbf{Z(W)}}}}$

Examples: Consider the following DAG:

$$\begin{tikzcd} & U \ \text{unobserved} \arrow[dl] \arrow[dr] & \\ X \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

- Task 3: Compute $P(y \mid do(x))$

$$\begin{aligned}
P(y \mid do(x)) &= \textstyle\sum_z P(y \mid z, do(x))\ P(z \mid do(x))) && \text{probabilistic calculus} \\
&= \textstyle\sum_z P(y \mid do(z), do(x))\ P(z \mid x)) && \text{Rule 2, Task 1} \\
&= \textstyle\sum_z P(y \mid do(z))\ P(z \mid x)) && \text{Rule 3} \\
&= \textstyle\sum_z \left(\sum_{x'} P(y \mid x', z)\ P(x')\right)\ P(z \mid x)) && \text{Task 2} \\
&= \textstyle\sum_z P(z \mid x) \sum_{x'} P(y \mid x', z)\ P(x') && \text{rearrangement of sum terms}
\end{aligned}$$

A Calculus of Intervention

Do-Calculus Based Identification Algorithm

- The IDC algorithm (Shpitser and Pearl): based on do-calculus solves the identification problem in polynomial time.
- More precisely, for a given G and $\mathbf{X}, \mathbf{Y}$ it computes a do-free formula for

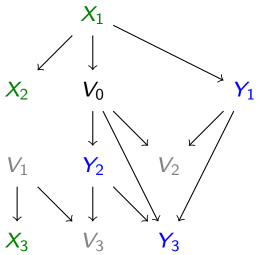
$$P(\mathbf{y} \mid do(\mathbf{x}))$$

  if and only if the effect is identifiable; otherwise it outputs that identification is impossible.

A Calculus of Intervention

Do-Calculus Based Identification Algorithm

- Two major drawbacks of the IDC algorithm:
  - polynomial time of high degree
  - computes complex expressions, even if simple ones exists, e.g.



The Instance is identified by using the empty set as an AS and by the formula

$$\sum_{v_0}[P(y_1|x_1)P(v_0|x_1)P(y_3|x_1,y_1,v_0,y_2)P(y_2|x_1,v_0)]$$

found by the IDC algorithm

A Calculus of Intervention

Do-Calculus Based Identification Algorithm

In practice the vast majority of causal effect estimations are computed either by

- covariate adjustment or
- instrumental variable (discussed later on in this course)

method

Simpson's Paradox

A classic example of Simpson's paradox

- Consider the famous dataset, comparing the success rates of two treatments (a and b) for kidney stones

	Treatment a	Treatment b
Small Stones $\left(\frac{357}{700} = 0.51\right)$	$\frac{81}{87} = 0.93$	$\frac{234}{270} = 0.87$
Large Stones $\left(\frac{343}{700} = 0.49\right)$	$\frac{192}{263} = 0.73$	$\frac{55}{80} = 0.69$
	$\frac{273}{350} = 0.78$	$\frac{289}{350} = 0.83$
	$\frac{562}{700} = 0.80$	

Charig et al.: *Comparison of treatment of renal calculi by open surgery (...)*, British Medical Journal, 1986

- Let S denote the size of stone (small, large), T – treatment (a or b), R – recovery (0,1)
- If we know only overall recovery rates (and nothing else) we would prefer treatment b, if we had a choice:

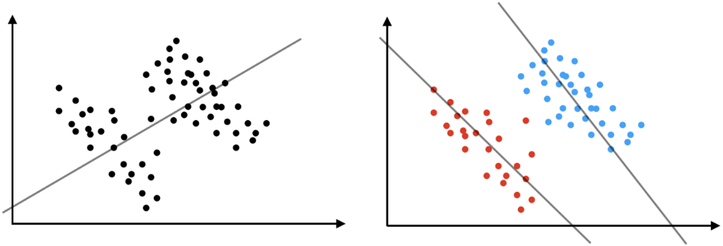
$$P(R = 1 \mid T = a) \;=\; 0.78 \quad < \quad 0.83 \;=\; P(R = 1 \mid T = b)$$

- But observing the data in both categories, we realize that a performs better in each of them

$$P(R = 1 \mid T = a, S = \text{small}) \;=\; 0.93 \quad > \quad 0.87 \;=\; P(R = 1 \mid T = b, S = \text{small}))$$
$$P(R = 1 \mid T = a, S = \text{large}) \;=\; 0.73 \quad > \quad 0.69 \;=\; P(R = 1 \mid T = b, S = \text{laerge}))$$

- How to explain this inversion of conclusion?

Simpson's Paradox



Simpson's Paradox

Simpson's Paradox

In general, Simpson's paradox refers to the phenomenon whereby

- an event C increases the probability of event E in a given population p:

$$P(E \mid C) \quad > \quad P(E \mid \neg C)$$

- and, at the same time, decreases the probability of E in every subpopulation of p described by two complementary properties F and $\neg F$

$$\begin{aligned} P(E \mid C, F) \quad &< \quad P(E \mid \neg C, F) \\ P(E \mid C, \neg F) \quad &< \quad P(E \mid \neg C, \neg F) \end{aligned}$$

Remark Although such inversion of conclusion might not surprise in probability studies, it is paradoxical when considering causal (C) / effect (E) interpretation

Simpson's Paradox

- Let's go back to our example data

	Treatment a	Treatment b
Small Stones $\left(\frac{357}{700} = 0.51\right)$	$\frac{81}{87} = 0.93$	$\frac{234}{270} = 0.87$
Large Stones $\left(\frac{343}{700} = 0.49\right)$	$\frac{192}{263} = 0.73$	$\frac{55}{80} = 0.69$
	$\frac{273}{350} = 0.78$	$\frac{289}{350} = 0.83$
	$\frac{562}{700} = 0.80$	

- We analyze the "paradox", using the language of interventions that particularly allows distinguishing between *seeing* from *doing*
- Let, as before, S – size of stone (small, large), T – treatment (a or b), R – recovery (0,1) and assume the following causal model

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ T \arrow[rr] & & R \end{tikzcd}$$

- We compare the treatments when we force *all* patients to take treatment a or treatment b, respectively

$$\begin{tikzcd} & S \arrow[d] \\ T = a \arrow[r] & R \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[d] \\ T = b \arrow[r] & R \end{tikzcd}$$

Simpson's Paradox

- For the data

	Treatment a	Treatment b
Small Stones $\left(\frac{357}{700} = 0.51\right)$	$\frac{81}{87} = 0.93$	$\frac{234}{270} = 0.87$
Large Stones $\left(\frac{343}{700} = 0.49\right)$	$\frac{192}{263} = 0.73$	$\frac{55}{80} = 0.69$
	$\frac{273}{350} = 0.78$	$\frac{289}{350} = 0.83$
	$\frac{562}{700} = 0.80$	

  the situations

$$\begin{tikzcd} & S \arrow[d] \\ T = a \arrow[r] & R \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[d] \\ T = b \arrow[r] & R \end{tikzcd}$$

  concern intervention distribution

- Then we estimate the causal effects of T on R via adjustment for direct causes:

$$P(R = 1 \mid do(T = a)) \; = \; \sum_s P(R = 1 \mid s, T = a) \; P(s) \; = \; 0.832$$

  and

$$P(R = 1 \mid do(T = b)) \; = \; \sum_s P(R = 1 \mid s, T = b) \; P(s) \; = \; 0.782 \; < 0.832$$

- This explains why we should prefer treatment a over treatment b

Literature

- J. Pearl (2009), Ch. 3
- J. Pearl, M. Glymour, and N.P. Jewell (2016), Ch. 3