A Proof of the Front-Door Adjustment Formula

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Abstract

We provide a proof of the the Front-Door adjustment formula using the do-calculus.

Keywords: causal Bayesian networks; semi-Markovian models; identifiability; latent variables; causal effect; causal inference.

1. Introduction

In (Pearl, 2009), a formula for computing the causal effect of X on Y in the causal model of figure 1 is derived and used to motivate the definition of front-door criterion. Pearl then states, without proof, the Front-Door Adjustment Theorem (Pearl, 2009, Theorem 3.3.4). In section 3.4.3, he provides a symbolic derivation of the front door adjustment formula for the same example from the do-calculus. In this short technical report, we provide a proof of Theorem 3.3.4 using the do-calculus. The next section consists of the proof of the front-door adjustment formula; the theorem

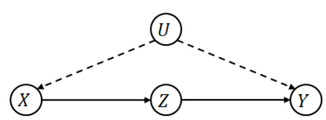


Figure 1: A causal Bayesian network with a latent variable U.

is restated for the reader's convenience. The do-calculus rules, the back-door criterion, the back-door adjustment formula, and the front-door criterion are in the slide set provided as an ancillary document.

2. Front-Door Adjustment Theorem

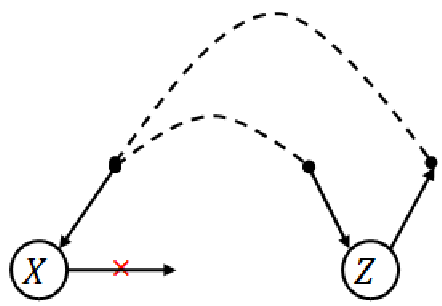
Theorem 1 (Front-Door Adjustment) *If a set of variables Z satisfies the front-door criterion relative to (X, Y) and if $P(x, z) > 0$, then the causal effect of X on Y is identifiable and is given by the formula*

$$P(y|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x') \tag{1}$$

.

Proof By well known probability identities (for example, the Fundamental Rule and the Theorem of Total Probability), $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$. In Step 1, below, we show how to compute $P(z|\hat{x})$ using only observed quantities. In Steps 2 and 3, we show how to compute $P(y|z, \hat{x})$ using only observed quantities; this part of the proof is by far the hardest.

- Step 1: Compute $P(z|\hat{x})$
  - $X \perp\!\!\!\perp Z$ in $G_{\underline{X}}$ because there is no outgoing edge from X in $G_{\underline{X}}$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from X to Z are blocked.



  - G satisfies the applicability condition for Rule 2:

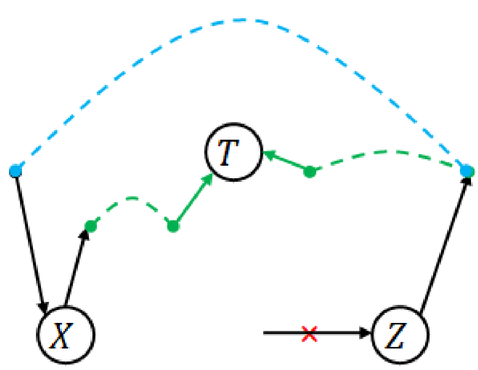
$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}\underline{Z}}}.$$

  - In Rule 2, set $y = z, x = \emptyset, z = x, w = \emptyset$:

$$P(z|\hat{x}) = P(z|x) \tag{2}$$

    because $\quad (Z \perp\!\!\!\perp X)_{G_{\underline{X}}}$

- Step 2: $P(y|\hat{z})$
  - $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
  - $X \perp\!\!\!\perp Z$ in $G_{\overline{Z}}$ because there is no incoming edge to Z in $G_{\overline{Z}}$, and also all paths from X to Z either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.



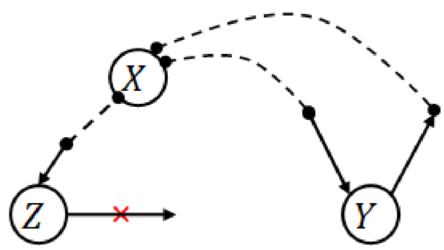
  - G satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X},\overline{Z(W)}}}.$$

- In Rule 3, set $y = x, x = \emptyset, z = z, w = \emptyset$:

$$P(x|\hat{z}) = P(x) \quad \text{because} \quad (Z \perp\!\!\!\perp X)_{G_{\overline{Z}}}.$$

- $(Z \perp\!\!\!\perp Y|X)_{G_{\underline{Z}}}$ because there is no outgoing edge from Z in $G_{\underline{Z}}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from Z to Y are blocked by X.



- G satisfies the applicability condition for Rule 2: $P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}\underline{Z}}}$.
- In Rule 2, set $y = y, x = \emptyset, z = z, w = x$:

$$P(y|x, \hat{z}) = P(y|x, z) \quad \text{because} \quad (Z \perp\!\!\!\perp Y|X)_{G_{\underline{Z}}}.$$

- $$P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z}) = \sum_x P(y|x, z)P(x) \tag{3}$$

  This formula is a special case of the back-door formula.

- Step 3: Compute $P(y|\hat{x})$

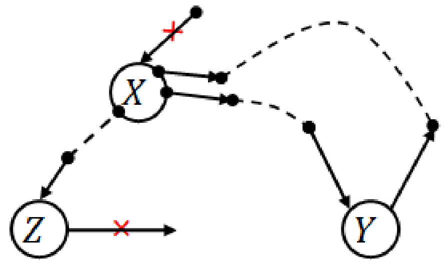
  As already noted at the beginning of the proof, $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.

  - $P(z|\hat{x}) = P(z|x)$, as shown in Step 1 (see equation (2))

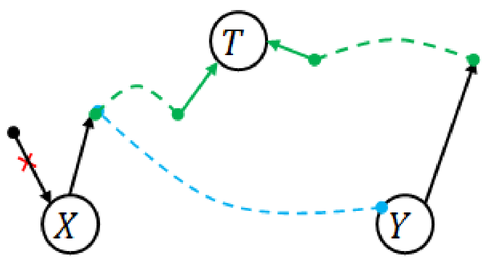
  There is no rule of the *do*-calculus that allows the elimination of the hat from $P(y|z, \hat{x})$, so we take a circuitous route: we first replace an observation (z) with an intervention ($\hat{z}$) using Rule 2, and then remove an intervention variable ($\hat{z}$) using Rule 3.

  - $(Y \perp\!\!\!\perp Z|X)_{G_{\overline{X}\underline{Z}}}$ because there is no outgoing edge from Z in $G_{\overline{X}\underline{Z}}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from Z to Y are blocked by X.
  - G satisfies the applicability condition for Rule 2: $P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}\underline{Z}}}$.
  - In Rule 2, set $y = y, x = x, z = z, w = \emptyset$:

$$P(y|z, \hat{x}) = P(y|\hat{z}, \hat{x}) \quad \text{because} \quad (Y \perp\!\!\!\perp Z|X)_{G_{\overline{X}\underline{Z}}}.$$



- $(Y \perp\!\!\!\perp X|Z)_{G_{\overline{XZ}}}$ because there is no incoming edge to X in $G_{\overline{XZ}}$, and also all paths from X to Y are blocked either because of condition (i) of the definition of the front-door criterion (blue-type paths)[directed paths from X to Y], or because of the existence of a collider on the path (green-type paths) (note that the case $T \in Z$ cannot happen because there is no incoming edge to Z in $G_{\overline{XZ}}$).



- G satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}, \overline{Z(W)}}}.$$

- In Rule 3, set $y = y, x = z, z = x, w = \emptyset$:

$$P(y|\hat{z}, \hat{x}) = P(y|\hat{z}) \quad \text{because} \quad (Y \perp\!\!\!\perp Z|X)_{G_{\overline{XZ}}}.$$

Now, by equations (2) and (3),

$$P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x').$$

■

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