Types, contexts and terms

Types:

$A, B ::= A \to B \mid A \times B \mid A + B \mid \mathbf{1} \mid \mathbf{0}$

Typing contexts:

$\Gamma ::= \cdot \mid \Gamma, x : A$

Terms

Terms:

$$
\begin{aligned}
e ::= \; & \\
& x \mid \\
& \lambda x.e \mid e_1 \; e_2 \mid \\
& (e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid \\
& \texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid \\
& \texttt{unit} \mid \texttt{exfalso}\ e
\end{aligned}
$$

Declarative typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

Declarative typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \texttt{outl}\ e : A} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \texttt{outr}\ e : B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \texttt{inl}\ e : A + B} \qquad \frac{\Gamma \vdash e : B}{\Gamma \vdash \texttt{inr}\ e : A + B}$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma \vdash f : A \to C \quad \Gamma \vdash g : B \to C}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) : C}$$

$$\frac{}{\Gamma \vdash \texttt{unit} : \mathbf{1}} \qquad \frac{\Gamma \vdash e : \mathbf{0}}{\Gamma \vdash \texttt{exfalso}\ e : A}$$

Metatheory

Extrinsic STLC enjoys strong metatheoretical properties:

- Termination: computation on well-typed terms always terminates.
- Confluence: computing with a well-typed term produces a unique result.
- Type preservation: computing with a well-typed term preserves its type.
- Canonicity: in the empty context, normal forms are inductively generated from term constructors.

However, extrinsic STLC does not enjoy another important property: **uniqueness of typing**. This means that there are terms which can be assigned multiple types.

Terms

Terms:

$e ::=$

$x \mid$

$\lambda x : A.e \mid e_1\ e_2 \mid$

$(e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid$

$\texttt{inl}_A\ e \mid \texttt{inr}_A\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid$

$\texttt{unit} \mid \texttt{exfalso}_A\ e$

Declarative typing – differences

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A.e : A \to B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathtt{inl}_B\ e : A + B} \qquad \frac{\Gamma \vdash e : B}{\Gamma \vdash \mathtt{inr}_A\ e : A + B}$$

$$\frac{\Gamma \vdash e : \mathbf{0}}{\Gamma \vdash \mathtt{exfalso}_A\ e : A}$$

Metatheory

Because of the annotations on lambda, sum constructors and exfalso, intrinsic STLC does enjoy uniqueness of typing. It is easy to prove this by induction: types of most terms are determined by the induction hypothesis, whereas for the aforementioned four we need to supplement the induction hypothesis with the annotation.

Terms

Terms:

$e ::=$

$x \mid (e : A) \mid$

$\lambda x.e \mid e_1\ e_2 \mid$

$(e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid$

$\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid$

$\texttt{unit} \mid \texttt{exfalso}\ e$

Declarative typing – new rules

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash (e : A) : A}\textsc{Annot}$$

Bidirectional typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A}\textsc{Annot}$$

$$\frac{\Gamma \vdash e \Rightarrow B \quad A = B}{\Gamma \vdash e \Leftarrow A}\textsc{Sub}$$

Bidirectional typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x.e \Leftarrow A \to B} \qquad \frac{\Gamma \vdash f \Rightarrow A \to B \quad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash f\ a \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow A \quad \Gamma \vdash b \Leftarrow B}{\Gamma \vdash (a, b) \Leftarrow A \times B} \qquad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \mathtt{outl}\ e \Rightarrow A} \qquad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \mathtt{outr}\ e \Rightarrow B}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash \mathtt{inl}\ e \Leftarrow A + B} \qquad \frac{\Gamma \vdash e \Leftarrow B}{\Gamma \vdash \mathtt{inr}\ e \Leftarrow A + B}$$

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Leftarrow A \to C \quad \Gamma \vdash g \Leftarrow B \to C}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow C}$$

$$\frac{}{\Gamma \vdash \mathtt{unit} \Rightarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0}}{\Gamma \vdash \mathtt{exfalso}\ e \Leftarrow A}$$

Bidirectional typing – additional rules

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Rightarrow A \to C \quad \Gamma \vdash g \Rightarrow B \to C}{\Gamma \vdash \mathtt{case}\ e\ \mathtt{of}\ (f, g) \Rightarrow C}$$

Metatheory

Similarly to extrinsic STLC, typing is not unique in Bidirectional STLC. This is because while we do have annotations in terms, we are not forced to use them. Therefore, we can check terms like $\lambda x.x$ with many types. However, inference is unique.

Hints and terms

Hints:

$H ::= ? \mid H_1 \rightarrow H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$

Terms:

$e ::=$

$\quad x \mid (e : H) \mid$

$\quad \lambda x.e \mid e_1\ e_2 \mid$

$\quad (e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid$

$\quad \texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid$

$\quad \texttt{unit} \mid \texttt{exfalso}\ e$

Typing contexts assign types to variables, but annotations in terms are hints, not necessarily types.

Combining hints

$$
\begin{gathered}
? \sqcap H = H \\
H \sqcap ? = H \\
(H_1 \to H_2) \sqcap (H_1' \to H_2') = (H_1 \sqcap H_1') \to (H_2 \sqcap H_2') \\
(H_1 \times H_2) \sqcap (H_1' \times H_2') = (H_1 \sqcap H_1') \times (H_2 \sqcap H_2') \\
(H_1 + H_2) \sqcap (H_1' + H_2') = (H_1 \sqcap H_1') + (H_2 \sqcap H_2') \\
\mathbf{1} \sqcap \mathbf{1} = \mathbf{1} \\
\mathbf{0} \sqcap \mathbf{0} = \mathbf{0}
\end{gathered}
$$

Hinting – basic rules

$$\frac{(x : A) \in \Gamma \quad H \sqcap A = A}{\Gamma \vdash x \Leftarrow H \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A}\textsc{Annot}$$

Hinting – type-directed rules 1

$$\frac{H \sqcap (? \to ?) = A \to H' \quad \Gamma, x : A \vdash e \Leftarrow H' \Rightarrow B}{\Gamma \vdash \lambda x.e \Leftarrow H \Rightarrow A \to B}$$

$$\frac{\Gamma \vdash f \Leftarrow ? \to H \Rightarrow H' \to B \quad \Gamma \vdash a \Leftarrow H' \Rightarrow A}{\Gamma \vdash f\ a \Leftarrow H \Rightarrow B}$$

$$\frac{H \sqcap (? \times ?) = H_1 \times H_2 \quad \Gamma \vdash a \Leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \Leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \Leftarrow H \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \mathtt{outl}\ e \Leftarrow H \Rightarrow A} \qquad \frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \mathtt{outr}\ e \Leftarrow H \Rightarrow B}$$

Hinting – type-directed rules 2

$$\frac{H \sqcap (?+?) = H' + B \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash \texttt{inl}\ e \Leftarrow H \Rightarrow A + B}$$

$$\frac{H \sqcap (?+?) = A + H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow B}{\Gamma \vdash \texttt{inr}\ e \Leftarrow H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow ?+? \Rightarrow A + B \quad \begin{array}{c} \Gamma \vdash f \Leftarrow A \to H \Rightarrow A \to C \\ \Gamma \vdash g \Leftarrow B \to H \Rightarrow B \to C \end{array}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow H \Rightarrow C}$$

$$\frac{H \sqcap \mathbf{1} = \mathbf{1}}{\Gamma \vdash \texttt{unit} \Leftarrow H \Rightarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \texttt{exfalso}\ e \Leftarrow A \Rightarrow A}$$

Metatheory

Similarly to extrinsic STLC, STLC with hints v1 does not enjoy uniqueness of typing. This is because we still can have terms like $\lambda x.x$ with hint ?, which can be typed with multiple types. However, if the hint is informative enough, then the type is unique. Moreover, every typable term can be given a hint which makes its type unique.

Hinting v2

$$H \bigtriangleup \lambda x.e = H \sqcap (? \to ?)$$
$$H \bigtriangleup (e_1, e_2) = H \sqcap (? \times ?)$$
$$H \bigtriangleup \texttt{inl } e = H \sqcap (? + ?)$$
$$H \bigtriangleup \texttt{inr } e = H \sqcap (? + ?)$$
$$H \bigtriangleup \texttt{unit} = H \sqcap \mathbf{1}$$

Hinting v2

$$\frac{(x : A) \in \Gamma \quad H \sqcap A = A}{\Gamma \vdash x \Leftarrow H \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A}\textsc{Annot}$$

$$\frac{e \texttt{ is a constructor} \quad H \triangle e = H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash e \Leftarrow H \Rightarrow A}\textsc{HintFor}$$

Hinting v2

$$\frac{\Gamma, x : A \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x.e \Leftarrow A \to H \Rightarrow A \to B}$$

$$\frac{\Gamma \vdash f \Leftarrow ? \to H \Rightarrow H' \to B \quad \Gamma \vdash a \Leftarrow H' \Rightarrow A}{\Gamma \vdash f\ a \Leftarrow H \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \Leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \Leftarrow H_1 \times H_2 \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \texttt{outl}\ e \Leftarrow H \Rightarrow A} \qquad \frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \texttt{outr}\ e \Leftarrow H \Rightarrow B}$$

Hinting v2

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow A}{\Gamma \vdash \texttt{inl}\ e \Leftarrow H + B \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \texttt{inr}\ e \Leftarrow A + H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow ? + ? \Rightarrow A + B \qquad \begin{array}{c}\Gamma \vdash f \Leftarrow A \to H \Rightarrow A \to C \\ \Gamma \vdash g \Leftarrow B \to H \Rightarrow B \to C\end{array}}{\Gamma \vdash \texttt{case}\ e\ \texttt{of}\ (f, g) \Leftarrow H \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \texttt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \texttt{exfalso}\ e \Leftarrow A \Rightarrow A}$$

Metatheory

Metatheoretically, STLC with hints v2 is similar to v1.