

Types, contexts and terms

Types:

$$A, B ::= A \rightarrow B \mid A \times B \mid A + B \mid \mathbf{1} \mid \mathbf{0}$$

Typing contexts:

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

Terms

Terms:

$e ::=$

$x \mid$
 $\lambda x.e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{exfalse } e$

Declarative typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

Declarative typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \quad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash f a : B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B} \quad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outl } e : A} \quad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outr } e : B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl } e : A + B} \quad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr } e : A + B}$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma \vdash f : A \rightarrow C \quad \Gamma \vdash g : B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) : C}$$

$$\frac{}{\Gamma \vdash \text{unit} : \mathbf{1}} \quad \frac{\Gamma \vdash e : \mathbf{0}}{\Gamma \vdash \text{exfalse } e : A}$$

Terms

Terms:

$e ::=$

$x \mid$

$\lambda x : A. e \mid e_1 \ e_2 \mid$

$(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$

$\text{inl}_A e \mid \text{inr}_A e \mid \text{case } e \text{ of } (e_1, e_2) \mid$

$\text{unit} \mid \text{exfalse}_A e$

Declarative typing – differences

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A. e : A \rightarrow B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl}_B e : A + B} \quad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr}_A e : A + B}$$

$$\frac{\Gamma \vdash e : \mathbf{0}}{\Gamma \vdash \text{exfalse}_A e : A}$$

Terms

Terms:

$e ::=$

$x \mid (e : A) \mid$
 $\lambda x. e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{exfalse } e$

Declarative typing – new rules

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash (e : A) : A}^{\text{ANNOT}}$$

Bidirectional typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A} \text{ANNOT}$$

$$\frac{\Gamma \vdash e \Rightarrow B \quad A = B}{\Gamma \vdash e \Leftarrow A} \text{SUB}$$

Bidirectional typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x. e \Leftarrow A \rightarrow B} \quad \frac{\Gamma \vdash f \Rightarrow A \rightarrow B \quad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash f a \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow A \quad \Gamma \vdash b \Leftarrow B}{\Gamma \vdash (a, b) \Leftarrow A \times B} \quad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \Rightarrow A} \quad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \Rightarrow B}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash \text{inl } e \Leftarrow A + B} \quad \frac{\Gamma \vdash e \Leftarrow B}{\Gamma \vdash \text{inr } e \Leftarrow A + B}$$

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Leftarrow A \rightarrow C \quad \Gamma \vdash g \Leftarrow B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) \Leftarrow C}$$

$$\frac{}{\Gamma \vdash \text{unit} \Rightarrow 1} \quad \frac{\Gamma \vdash e \Leftarrow 0}{\Gamma \vdash \text{exfalse } e \Leftarrow A}$$

Bidirectional typing – additional rules

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Rightarrow A \rightarrow C \quad \Gamma \vdash g \Rightarrow B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) \Rightarrow C}$$

Hints and terms

Hints:

$H ::= ? \mid H_1 \rightarrow H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$

Terms:

$e ::=$

$x \mid (e : H) \mid$
 $\lambda x. e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{exfalse } e$

Typing contexts assign types to variables, but annotations in terms are hints, not necessarily types.

Combining hints

$$? \sqcap H = H$$

$$H \sqcap ? = H$$

$$(H_1 \rightarrow H_2) \sqcap (H'_1 \rightarrow H'_2) = (H_1 \sqcap H'_1) \rightarrow (H_2 \sqcap H'_2)$$

$$(H_1 \times H_2) \sqcap (H'_1 \times H'_2) = (H_1 \sqcap H'_1) \times (H_2 \sqcap H'_2)$$

$$(H_1 + H_2) \sqcap (H'_1 + H'_2) = (H_1 \sqcap H'_1) + (H_2 \sqcap H'_2)$$

$$\mathbf{1} \sqcap \mathbf{1} = \mathbf{1}$$

$$\mathbf{0} \sqcap \mathbf{0} = \mathbf{0}$$

Hinting – basic rules

$$\frac{(x : A) \in \Gamma \quad H \sqcap A = A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A} \text{ANNOT}$$

Hinting – type-directed rules 1

$$\frac{H \sqcap (? \rightarrow ?) = A \rightarrow H' \quad \Gamma, x : A \vdash e \leftarrow H' \Rightarrow B}{\Gamma \vdash \lambda x. e \leftarrow H \Rightarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash f \leftarrow ? \rightarrow H \Rightarrow H' \rightarrow B \quad \Gamma \vdash a \leftarrow H' \Rightarrow A}{\Gamma \vdash f \ a \leftarrow H \Rightarrow B}$$

$$\frac{H \sqcap (? \times ?) = H_1 \times H_2 \quad \Gamma \vdash a \leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \leftarrow H \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \leftarrow H \Rightarrow A}$$

$$\frac{\Gamma \vdash e \leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \leftarrow H \Rightarrow B}$$

Hinting – type-directed rules 2

$$\frac{H \sqcap (? + ?) = H' + B \quad \Gamma \vdash e \leftarrow H' \Rightarrow A}{\Gamma \vdash \text{inl } e \leftarrow H \Rightarrow A + B}$$

$$\frac{H \sqcap (? + ?) = A + H' \quad \Gamma \vdash e \leftarrow H' \Rightarrow B}{\Gamma \vdash \text{inr } e \leftarrow H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \leftarrow ? + ? \Rightarrow A + B \quad \begin{array}{l} \Gamma \vdash f \leftarrow A \rightarrow H \Rightarrow A \rightarrow C \\ \Gamma \vdash g \leftarrow B \rightarrow H \Rightarrow B \rightarrow C \end{array}}{\Gamma \vdash \text{case } e \text{ of } (f, g) \leftarrow H \Rightarrow C}$$

$$\frac{H \sqcap \mathbf{1} = \mathbf{1}}{\Gamma \vdash \text{unit} \leftarrow H \Rightarrow \mathbf{1}}$$

$$\frac{\Gamma \vdash e \leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \text{exfalse } e \leftarrow A \Rightarrow A}$$

Hinting v2

$$\begin{aligned}H \triangle \lambda x.e &= H \sqcap (? \rightarrow ?) \\H \triangle (e_1, e_2) &= H \sqcap (? \times ?) \\H \triangle \text{inl } e &= H \sqcap (? + ?) \\H \triangle \text{inr } e &= H \sqcap (? + ?) \\H \triangle \text{unit} &= H \sqcap \mathbf{1}\end{aligned}$$

Hinting v2

$$\frac{(x : A) \in \Gamma \quad H \sqcap A = A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A} \text{ANNOT}$$

$$\frac{e \text{ is a constructor} \quad H \triangle e = H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash e \Leftarrow H \Rightarrow A} \text{HINTFOR}$$

Hinting v2

$$\frac{\Gamma, x : A \vdash e \leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x. e \leftarrow A \rightarrow H \Rightarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash f \leftarrow ? \rightarrow H \Rightarrow H' \rightarrow B \quad \Gamma \vdash a \leftarrow H' \Rightarrow A}{\Gamma \vdash f \ a \leftarrow H \Rightarrow B}$$

$$\frac{\Gamma \vdash a \leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \leftarrow H_1 \times H_2 \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \leftarrow H \Rightarrow A}$$

$$\frac{\Gamma \vdash e \leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \leftarrow H \Rightarrow B}$$

Hinting v2

$$\frac{\Gamma \vdash e \leftarrow H \Rightarrow A}{\Gamma \vdash \text{inl } e \leftarrow H + B \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \leftarrow H \Rightarrow B}{\Gamma \vdash \text{inr } e \leftarrow A + H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \leftarrow ? + ? \Rightarrow A + B \quad \begin{array}{l} \Gamma \vdash f \leftarrow A \rightarrow H \Rightarrow A \rightarrow C \\ \Gamma \vdash g \leftarrow B \rightarrow H \Rightarrow B \rightarrow C \end{array}}{\Gamma \vdash \text{case } e \text{ of } (f, g) \leftarrow H \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \text{unit} \leftarrow 1 \Rightarrow 1} \quad \frac{\Gamma \vdash e \leftarrow 0 \Rightarrow 0}{\Gamma \vdash \text{exfalse } e \leftarrow A \Rightarrow A}$$