Hint-based typing for polymorphism

Types and judgements

Types:
$A, B ::= \alpha \mid \forall\alpha.\, A \mid A \to B$

Contexts:
$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha$

Judgements:
$\Gamma \vdash e : A$, $\Gamma \vdash A\ \texttt{type}$

Valid context judgement

$$\frac{}{\cdot\ \mathtt{ctx}}\ \textsc{Ctx-Empty}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad x \notin \Gamma \quad \Gamma \vdash A\ \mathtt{type}}{\Gamma, x : A\ \mathtt{ctx}}\ \textsc{Ctx-Extend}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad \alpha \notin \Gamma}{\Gamma, \alpha\ \mathtt{ctx}}\ \textsc{Ctx-ExtendType}$$

Valid type judgement

$$\frac{\Gamma\ \texttt{ctx} \quad \alpha \in \Gamma}{\Gamma \vdash \alpha\ \texttt{type}}\textsc{TyVar}$$

$$\frac{\Gamma, \alpha \vdash A\ \texttt{type}}{\Gamma \vdash \forall \alpha.\, A\ \texttt{type}}\textsc{All}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash A \to B\ \texttt{type}}\textsc{Fun}$$

Sanity checks

- If $\Gamma \vdash A\ \mathtt{type}$, then $\Gamma\ \mathtt{ctx}$
- If $\Gamma\ \mathtt{ctx}$ and $(x : A) \in \Gamma$, then $\Gamma \vdash A\ \mathtt{type}$
- We will set up the system so that if $\Gamma \vdash e : A$, then $\Gamma \vdash A\ \mathtt{type}$

Terms

Terms:

$e ::=$

$x \mid$

$\lambda x : A.\, e \mid e_1 \;\, e_2 \mid$

$\Lambda\alpha.\, e \mid e \;\texttt{@}A$

Declarative typing

$$\frac{\Gamma\ \mathtt{ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A.\, e : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash \Lambda\alpha.\, e : \forall\alpha.\, A}$$

$$\frac{\Gamma \vdash e : \forall\alpha.\, A \quad \Gamma \vdash B\ \mathtt{type}}{\Gamma \vdash e\ \mathtt{@}B : A[\alpha := B]}$$

Algorithmic typing

$$\frac{\Gamma\ \texttt{ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma, x : A \vdash e \Rightarrow B}{\Gamma \vdash \lambda x : A.\, e \Rightarrow A \to B} \qquad \frac{\Gamma \vdash f \Rightarrow A \to B \quad \Gamma \vdash a \Rightarrow A}{\Gamma \vdash f\ a \Rightarrow B}$$

$$\frac{\Gamma, \alpha \vdash e \Rightarrow A}{\Gamma \vdash \Lambda\alpha.\, e \Rightarrow \forall\alpha.\, A}$$

$$\frac{\Gamma \vdash e \Rightarrow \forall\alpha.\, A \quad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash e\ \texttt{@}B \Rightarrow A[\alpha := B]}$$

Terms

Terms:

$e ::=$

$x \mid$

$\lambda x.\, e \mid \mathrm{app}_A\ e_1\ e_2 \mid$

$e\ @A$

Declarative typing

$$\frac{\Gamma\ \texttt{ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.\, e : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}_A\ f\ a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash e : \forall \alpha.\, A}$$

$$\frac{\Gamma \vdash e : \forall \alpha.\, A \quad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash e\ \texttt{@}B : A[\alpha := B]}$$

Algorithmic typing

$$\frac{\Gamma\ \texttt{ctx} \qquad (x : A) \in \Gamma}{\Gamma \vdash x \Leftarrow A}\ \textsc{Var}$$

$$\frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B} \qquad \frac{\Gamma \vdash f \Leftarrow A \to B \qquad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash \mathsf{app}_A\ f\ a \Leftarrow B}$$

$$\frac{\Gamma, \alpha \vdash e \Leftarrow A}{\Gamma \vdash e \Leftarrow \forall \alpha.\, A}$$

$$\frac{\Gamma \vdash e \Leftarrow \forall \alpha.\, A \qquad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash e\ \texttt{@}B \Leftarrow A[\alpha := B]}$$

Terms

Terms:

$e ::=$

$\quad x \mid$

$\quad \lambda x.\, e \mid e_1 \; e_2 \mid$

Declarative typing

$$\frac{\Gamma\ \mathtt{ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.\, e : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash e : \forall \alpha.\, A}$$

$$\frac{\Gamma \vdash e : \forall \alpha.\, A \quad \Gamma \vdash B\ \mathtt{type}}{\Gamma \vdash e : A[\alpha := B]}$$

Hints

Hints:

$$H ::= \mathbf{?} \mid H_1 \to H_2 \mid \forall \alpha.\, H \mid \alpha$$

Valid hint judgement

$$\frac{\Gamma\ \texttt{ctx}}{\Gamma \vdash \mathbf{?}\ \texttt{hint}}\textsc{Hint-Hole}$$

$$\frac{\Gamma\ \texttt{ctx} \quad \alpha \in \Gamma}{\Gamma \vdash \alpha\ \texttt{hint}}\textsc{Hint-TyVar}$$

$$\frac{\Gamma, \alpha \vdash H\ \texttt{hint}}{\Gamma \vdash \forall \alpha.\, H\ \texttt{hint}}\textsc{Hint-All}$$

$$\frac{\Gamma \vdash H_1\ \texttt{hint} \quad \Gamma \vdash H_2\ \texttt{hint}}{\Gamma \vdash H_1 \to H_2\ \texttt{hint}}\textsc{Hint-Fun}$$

Order on hints

$$\frac{\Gamma \vdash H\ \mathtt{hint}}{\Gamma \vdash \mathbf{?} \sqsubseteq H}$$

$$\frac{\Gamma\ \mathtt{ctx}}{\Gamma \vdash \alpha \sqsubseteq \alpha}$$

$$\frac{\Gamma, \alpha \vdash H_1 \sqsubseteq H_2}{\Gamma \vdash \forall \alpha.\, H_1 \sqsubseteq \forall \alpha.\, H_2}$$

$$\frac{\Gamma \vdash H_1 \sqsubseteq H_1' \quad \Gamma \vdash H_2 \sqsubseteq H_2'}{\Gamma \vdash H_1 \to H_2 \sqsubseteq H_1' \to H_2'}$$

Combining hints

$$
\begin{gathered}
\mathbf{?} \sqcup H = H \\
H \sqcup \mathbf{?} = H \\
(H_1 \to H_2) \sqcup (H_1' \to H_2') = (H_1 \sqcup H_1') \to (H_2 \sqcup H_2') \\
(\forall \alpha.\, H_1) \sqcup (\forall \alpha.\, H_2) = \forall \alpha.\, H_1 \sqcup H_2 \\
\alpha \sqcup \alpha = \alpha
\end{gathered}
$$

Terms

Terms:

$$
\begin{aligned}
e ::= &\\
&x \mid (e : H) \mid \\
&\lambda x.\, e \mid e_1 \; e_2 \mid
\end{aligned}
$$

Declarative typing

$$\frac{\Gamma\ \mathtt{ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash H \sqsubseteq A}{\Gamma \vdash (e : H) : A}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.\, e : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash e : \forall \alpha.\, A}$$

$$\frac{\Gamma \vdash e : \forall \alpha.\, A \quad \Gamma \vdash B\ \mathtt{type}}{\Gamma \vdash e : A[\alpha := B]}$$

Algorithmic typing – basic rules

$$\frac{(x : A) \in \Gamma \quad \Gamma \vdash H \sqsubseteq A}{\Gamma \vdash x \Leftarrow H \Rightarrow A}\,\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcup H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A}\,\textsc{Annot}$$

Algorithmic typing – other rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to H \Rightarrow A \to B}$$

$$\frac{\Gamma \vdash a \Leftarrow \mathbf{?} \Rightarrow A \quad \Gamma \vdash f \Leftarrow A \to H \Rightarrow A \to B}{\Gamma \vdash f\ a \Leftarrow H \Rightarrow B}$$

$$\frac{\Gamma, \alpha \vdash e \Leftarrow H \Rightarrow A}{\Gamma \vdash e \Leftarrow \forall \alpha.\, H \Rightarrow \forall \alpha.\, A}$$

Embedding

- $\lambda x : A.\, e :\equiv (\lambda x.\, e : A \to \mathbf{?})$
- $\mathrm{app}_A\ f\ \ a :\equiv (f\ :\ A \to \mathbf{?})\ a$
- $\Lambda\alpha.\, e :\equiv (e : \forall\alpha.\, \mathbf{?})$
- $e\ \texttt{@}A :\equiv\ ???$

As we can see, we can think of type abstraction as just another kind of annotation, but type application doesn't admit such thinking.