STLC with Subtyping

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Types, contexts and judgements

Types:
$A, B ::= A \to B \mid A \times B \mid A + B \mid \top \mid \bot$

Typing contexts:
$\Gamma ::= \cdot \mid \Gamma, x : A$

Judgements:
$\Gamma \vdash e : A$

Subtyping relation

$$\frac{}{\bot <: A} \qquad \frac{}{A <: \top}$$

$$\frac{A' <: A \quad B <: B'}{A \to B <: A' \to B'}$$

$$\frac{A <: A' \quad B <: B'}{A \times B <: A' \times B'}$$

$$\frac{A <: A' \quad B <: B'}{A + B <: A' + B'}$$

The subtyping relation is the reflexive-transitive closure of the above rules.

Subtyping – properties

Subtyping is a partial order with top and bottom, congruent with type constructors.

- Reflexivity: $A <: A$
- Transitivity: $A <: B \implies B <: C \implies A <: C$
- Antisymmetry: $A <: B \implies B <: A \implies A = B$

Terms

Terms:

$$
\begin{aligned}
e ::= \; & \\
& x \mid \\
& \lambda x.e \mid e_1 \; e_2 \mid \\
& (e_1, e_2) \mid \texttt{outl}\; e \mid \texttt{outr}\; e \mid \\
& \texttt{inl}\; e \mid \texttt{inr}\; e \mid \texttt{case}\; e \; \texttt{of}\; (e_1, e_2) \mid \\
& \texttt{unit} \mid \texttt{exfalso}\; e
\end{aligned}
$$

Note: `unit` is the term of the $\top$ type, whereas `exfalso` is the eliminator of the $\perp$ type.

Declarative typing – differences

$$\frac{\Gamma \vdash e : A \quad A <: B}{\Gamma \vdash e : B}\textsc{Sub}$$

Comments

The only difference between Extrinsic STLC with Subtyping and the original Extrinsic STLC is the addition of the subsumtpion rule. We invoke this rule every time we need subtyping to happen. However, this rule is stationary, and thus non-algorithmic, so we need to seek a better system.

Bidirectional typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A}\textsc{Annot}$$

$$\frac{\Gamma \vdash e \Rightarrow B \quad A <: B}{\Gamma \vdash e \Leftarrow A}\textsc{Sub}$$

Metatheory

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Hints

We no longer need holes – because of $\top$ and $\bot$, types suffice. We also don't need a separate order on hints, because subtyping suffices.

Least upper bound of two types

$$
\begin{aligned}
\bot \sqcup H &= H \\
H \sqcup \bot &= H \\
\top \sqcup H &= \top \\
H \sqcup \top &= \top \\
(H_1 \to H_2) \sqcup (H_1' \to H_2') &= (H_1 \sqcap H_1') \to (H_2 \sqcup H_2') \\
(H_1 \times H_2) \sqcup (H_1' \times H_2') &= (H_1 \sqcup H_1') \times (H_2 \sqcup H_2') \\
(H_1 + H_2) \sqcup (H_1' + H_2') &= (H_1 \sqcup H_1') + (H_2 \sqcup H_2')
\end{aligned}
$$

Greatest lower bound of two types

$$
\begin{aligned}
\bot \sqcap H &= \bot \\
H \sqcap \bot &= \bot \\
\top \sqcap H &= H \\
H \sqcap \top &= H \\
(H_1 \to H_2) \sqcap (H_1' \to H_2') &= (H_1 \sqcup H_1') \to (H_2 \sqcap H_2') \\
(H_1 \times H_2) \sqcap (H_1' \times H_2') &= (H_1 \sqcap H_1') \times (H_2 \sqcap H_2') \\
(H_1 + H_2) \sqcap (H_1' + H_2') &= (H_1 \sqcap H_1') + (H_2 \sqcap H_2')
\end{aligned}
$$

Terms and judgements

Terms:

$e ::=$

$x \mid (e : A) \mid$

$\lambda x.e \mid e_1\ e_2 \mid$

$(e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid$

$\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid$

$\texttt{unit} \mid \texttt{exfalso}\ e$

Note: terms are the same as in Bidirectional STLC, i.e. annotations are types, not hints.

Judgements:

$\Gamma \vdash e \Leftarrow A \Rightarrow B$ – in context Γ, term e checks with type A as hint and infers type B

Declarative typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash (e : A) : A}\textsc{Annot}$$

$$\frac{\Gamma \vdash e : A \quad A <: B}{\Gamma \vdash e : B}\textsc{Sub}$$

Algorithmic typing – basic rules

$$\frac{(x : B) \in \Gamma \quad B <: A}{\Gamma \vdash x \Leftarrow A \Rightarrow B}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow C \quad B <: A}{\Gamma \vdash (e : B) \Leftarrow A \Rightarrow C}\textsc{Annot}$$

$$\frac{\Gamma \vdash e \Leftarrow \texttt{hint}(e) \Rightarrow A \quad e\ \texttt{constructor}}{\Gamma \vdash e \Leftarrow \top \Rightarrow A}\textsc{Hole}$$

Note that the rule Hole can only be applied once, because $\texttt{hint}(e)$ can never be **?**. After applying Hole, the only applicable rules are the type-directed ones.

Algorithmic typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x.e \Leftarrow A \to H \Rightarrow A \to B}$$

$$\frac{\Gamma \vdash f \Leftarrow \mathbf{?} \to H \Rightarrow A \to B \qquad \Gamma \vdash a \Leftarrow A \Rightarrow A}{\Gamma \vdash f \ a \Leftarrow H \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow H_1 \Rightarrow A \qquad \Gamma \vdash b \Leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \Leftarrow H_1 \times H_2 \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times \mathbf{?} \Rightarrow A \times B}{\Gamma \vdash \texttt{outl} \ e \Leftarrow H \Rightarrow A} \qquad \frac{\Gamma \vdash e \Leftarrow H \times \mathbf{?} \Rightarrow A \times B}{\Gamma \vdash \texttt{outr} \ e \Leftarrow H \Rightarrow B}$$

Algorithmic typing – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow A}{\Gamma \vdash \mathtt{inl}\ e \Leftarrow H + B \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \mathtt{inr}\ e \Leftarrow A + H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \qquad \begin{array}{c}\Gamma \vdash f \Leftarrow A \to H \Rightarrow A \to C \\ \Gamma \vdash g \Leftarrow B \to C \Rightarrow B \to C\end{array}}{\Gamma \vdash \mathtt{case}\ e\ \mathtt{of}\ (f, g) \Leftarrow H \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \mathtt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \mathtt{exfalso}\ e \Leftarrow A \Rightarrow A}$$