## Types and contexts

Types:

$$A, B ::= A \rightarrow B \mid A \times B \mid A + B \mid \mathbf{1} \mid \mathbf{0}$$

Typing contexts:

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

#### **Terms**

```
Terms: e ::= \\ x \mid \\ \lambda x.e \mid e_1 \mid e_2 \mid \\ (e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid \\ \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid \\ \text{unit } \mid \text{exfalso } e
```

## Declarative typing – basics

$$\frac{(x:A)\in\Gamma}{\Gamma\vdash x:A}\mathrm{Var}$$

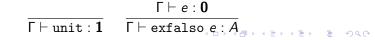
### Declarative typing – type-directed rules

$$\frac{\Gamma, x: A \vdash e: B}{\Gamma \vdash \lambda x. e: A \rightarrow B} \qquad \frac{\Gamma \vdash f: A \rightarrow B \quad \Gamma \vdash a: A}{\Gamma \vdash f \ a: B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outl } e : A} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outr } e : B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl } e : A + B} \qquad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr } e : A + B}$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma \vdash f : A \to C \quad \Gamma \vdash g : B \to C}{\Gamma \vdash \mathsf{case} \ e \ \mathsf{of} \ (f,g) : C}$$



#### **Terms**

```
Terms: e ::= x \mid \\ \lambda x : A.e \mid e_1 \mid e_2 \mid \\ (e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid \\ \text{inl}_A \mid e \mid \text{inr}_A \mid e \mid \text{case } e \mid \text{of } (e_1, e_2) \mid \\ \text{unit} \mid \text{exfalso } e
```

# Declarative typing – basics

$$\frac{(x:A)\in\Gamma}{\Gamma\vdash x:A}\mathrm{Var}$$

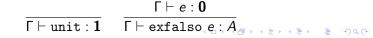
### Declarative typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A.e : A \rightarrow B} \qquad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash f \ a : B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outl } e : A} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outr } e : B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl}_B e : A + B} \qquad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr}_A e : A + B}$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma \vdash f : A \to C \quad \Gamma \vdash g : B \to C}{\Gamma \vdash \text{case } e \text{ of } (f,g) : C}$$



#### **Terms**

```
Terms: e ::= x \mid (e : A) \mid \lambda x.e \mid e_1 e_2 \mid (e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid \text{unit } \mid \text{exfalso } e
```

# Declarative typing - basics

$$\frac{(x:A)\in\Gamma}{\Gamma\vdash x:A}\mathrm{VAR}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash (e : A) : A} Annor$$

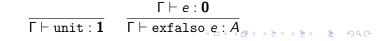
### Declarative typing – type-directed rules

$$\frac{\Gamma, x: A \vdash e: B}{\Gamma \vdash \lambda x. e: A \rightarrow B} \qquad \frac{\Gamma \vdash f: A \rightarrow B \quad \Gamma \vdash a: A}{\Gamma \vdash f \ a: B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outl } e : A} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outr } e : B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl } e : A + B} \qquad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr } e : A + B}$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma \vdash f : A \to C \quad \Gamma \vdash g : B \to C}{\Gamma \vdash \mathsf{case} \ e \ \mathsf{of} \ (f,g) : C}$$



# Bidirectional typing

$$\frac{(x:A) \in \Gamma}{\Gamma \vdash x \Rightarrow A} VAR$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A} A_{\text{NNOT}}$$

$$\frac{\Gamma \vdash e \Rightarrow B \quad A = B}{\Gamma \vdash e \Leftarrow A}$$
SuB

### Bidirectional typing

$$\frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x. e \Leftarrow A \to B} \qquad \frac{\Gamma \vdash f \Rightarrow A \to B \quad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash f \ a \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow A \quad \Gamma \vdash b \Leftarrow B}{\Gamma \vdash (a,b) \Leftarrow A \times B} \qquad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \Rightarrow A} \qquad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \Rightarrow B}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash \text{inl } e \Leftarrow A + B} \qquad \frac{\Gamma \vdash e \Leftarrow B}{\Gamma \vdash \text{inr } e \Leftarrow A + B}$$

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Leftarrow A \rightarrow C \quad \Gamma \vdash g \Leftarrow B \rightarrow C}{\Gamma \vdash \text{case e of } (f,g) \Leftarrow C}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{0}}{\Gamma \vdash \text{unit} \Rightarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0}}{\Gamma \vdash \text{exfalso} \ e \Leftarrow \mathbf{A}}$$

#### Grammar

Hints:

$$H ::= ? \mid H_1 \rightarrow H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$$

Typing contexts assign types to variables, but annotations in terms are hints, not necessarily types.

## Combining hints

$$? \sqcap H = H H \sqcap ? = H (H_1 \to H_2) \sqcap (H'_1 \to H'_2) = (H_1 \sqcap H'_1) \to (H_2 \sqcap H'_2) (H_1 \times H_2) \sqcap (H'_1 \times H'_2) = (H_1 \sqcap H'_1) \times (H_2 \sqcap H'_2) (H_1 + H_2) \sqcap (H'_1 + H'_2) = (H_1 \sqcap H'_1) + (H_2 \sqcap H'_2) \mathbf{1} \sqcap \mathbf{1} = \mathbf{1} \mathbf{0} \sqcap \mathbf{0} = \mathbf{0}$$

### Hinting

$$\frac{(x:A) \in \Gamma \quad H \sqcap A = A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} VAR$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e:H_1) \Leftarrow H_2 \Rightarrow A} \text{Annot}$$

$$\frac{H \sqcap \mathbf{1} = \mathbf{1}}{\Gamma \vdash \text{unit} \Leftarrow H \Rightarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \text{exfalso } e \Leftarrow A \Rightarrow A}$$



### Hinting

$$\frac{H \sqcap (? \to ?) = A \to H' \quad \Gamma, x : A \vdash e \Leftarrow H' \Rightarrow B}{\Gamma \vdash \lambda x . e \Leftarrow H \Rightarrow A \to B}$$

$$\frac{\Gamma \vdash f \Leftarrow ? \to H \Rightarrow H' \to B \quad \Gamma \vdash a \Leftarrow H' \Rightarrow A}{\Gamma \vdash f \ a \Leftarrow H \Rightarrow B}$$

$$\frac{H \sqcap (? \times ?) = H_1 \times H_2 \quad \Gamma \vdash a \Leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \Leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \Leftarrow H \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl} \ e \Leftarrow H \Rightarrow A} \quad \frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outr} \ e \Leftarrow H \Rightarrow B}$$

## Hinting

$$\frac{H \sqcap (?+?) = H' + B \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash \text{inl } e \Leftarrow H \Rightarrow A + B}$$

$$\frac{H \sqcap (?+?) = A + H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow B}{\Gamma \vdash \text{inr } e \Leftarrow H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow ? + ? \Rightarrow A + B \quad \Gamma \vdash f \Leftarrow A \rightarrow H \Rightarrow A \rightarrow C \quad \Gamma \vdash g \Leftarrow B \rightarrow H}{\Gamma \vdash \mathsf{case} \ e \ \mathsf{of} \ (f,g) \Leftarrow H \Rightarrow C}$$

$$H \triangle \lambda x.e = H \sqcap (? \rightarrow ?)$$
  
 $H \triangle (e_1, e_2) = H \sqcap (? \times ?)$   
 $H \triangle \text{inl } e = H \sqcap (? + ?)$   
 $H \triangle \text{inr } e = H \sqcap (? + ?)$   
 $H \triangle \text{unit} = H \sqcap \mathbf{1}$ 

$$\frac{(x:A) \in \Gamma \quad H \sqcap A = A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} VAR$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A} \text{Annot}$$

$$\frac{\text{$e$ is a constructor} \quad H \bigtriangleup e = H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash e \Leftarrow H \Rightarrow A}_{\text{HINTFOR}}$$

$$\frac{\Gamma, x : A \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x. e \Leftarrow A \rightarrow H \Rightarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash f \Leftarrow ? \rightarrow H \Rightarrow H' \rightarrow B \quad \Gamma \vdash a \Leftarrow H' \Rightarrow A}{\Gamma \vdash f \Rightarrow a \Leftarrow H \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \Leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \Leftarrow H_1 \times H_2 \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl} e \Leftarrow H \Rightarrow A} \quad \frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl} e \Leftarrow H \Rightarrow B}$$



 $\frac{\Gamma \vdash e \Leftarrow H \Rightarrow A}{\Gamma \vdash \text{inl } e \Leftarrow H + B \Rightarrow A + B}$ 

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \text{inr } e \Leftarrow A + H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow ? + ? \Rightarrow A + B \quad \Gamma \vdash f \Leftarrow A \rightarrow H \Rightarrow A \rightarrow C \quad \Gamma \vdash g \Leftarrow B \rightarrow H}{\Gamma \vdash \mathsf{case} \ e \ \mathsf{of} \ (f,g) \Leftarrow H \Rightarrow C}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \text{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}} \quad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \text{exfalso } e \Leftarrow A \Rightarrow A}$$

