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Types, contexts and terms

Types:

$$A, B ::= A \rightarrow B \mid A \times B \mid A + B \mid \mathbf{1} \mid \mathbf{0}$$

Typing contexts:

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

Terms

Terms:

$e ::=$

$x \mid$

$\lambda x.e \mid e_1 \ e_2 \mid$

$(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$

$\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$

$\text{unit} \mid \text{exfalse } e$

Declarative typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

Declarative typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \quad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash f \ a : B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B} \quad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outl } e : A} \quad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \text{outr } e : B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl } e : A + B} \quad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr } e : A + B}$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma \vdash f : A \rightarrow C \quad \Gamma \vdash g : B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) : C}$$

$$\frac{}{\Gamma \vdash \text{unit} : \mathbf{1}} \quad \frac{\Gamma \vdash e : \mathbf{0}}{\Gamma \vdash \text{exfalse } e : A}$$

Metatheory

Extrinsic STLC enjoys strong metatheoretical properties:

- Termination: computation on well-typed terms always terminates.
- Confluence: computing with a well-typed term produces a unique result.
- Type preservation: computing with a well-typed term preserves its type.
- Canonicity: in the empty context, normal forms are inductively generated from term constructors.

However, extrinsic STLC does not enjoy another important property: **uniqueness of typing**. This means that there are terms which can be assigned multiple types.

Terms

Note: red color marks places which differ from Extrinsic STLC.

Terms:

$e ::=$

$x \mid$

$\lambda x : A. e \mid e_1 \ e_2 \mid$

$(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$

$\text{inl}_A e \mid \text{inr}_A e \mid \text{case } e \text{ of } (e_1, e_2) \mid$

$\text{unit} \mid \text{exfalse}_A e$

Declarative typing – differences

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A. e : A \rightarrow B}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{inl}_B e : A + B} \quad \frac{\Gamma \vdash e : B}{\Gamma \vdash \text{inr}_A e : A + B}$$

$$\frac{\Gamma \vdash e : \mathbf{0}}{\Gamma \vdash \text{exfalse}_A e : A}$$

Metatheory

Because of the annotations on lambda, sum constructors and ex falso, intrinsic STLC does enjoy uniqueness of typing. It is easy to prove this by induction: types of most terms are determined by the induction hypothesis, whereas for the aforementioned four we need to supplement the induction hypothesis with the annotation.

Terms

Note: green color marks terms which were not present in Extrinsic STLC.

Terms:

$e ::=$

$x \mid (e : A) \mid$
 $\lambda x. e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{exfalse } e$

Declarative typing – new rules

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash (e : A) : A}^{\text{ANNOT}}$$

Bidirectional typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A} \text{ANNOT}$$

$$\frac{\Gamma \vdash e \Rightarrow B \quad A = B}{\Gamma \vdash e \Leftarrow A} \text{SUB}$$

Bidirectional typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x. e \Leftarrow A \rightarrow B} \quad \frac{\Gamma \vdash f \Rightarrow A \rightarrow B \quad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash f a \Rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow A \quad \Gamma \vdash b \Leftarrow B}{\Gamma \vdash (a, b) \Leftarrow A \times B} \quad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \Rightarrow A} \quad \frac{\Gamma \vdash e \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \Rightarrow B}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash \text{inl } e \Leftarrow A + B} \quad \frac{\Gamma \vdash e \Leftarrow B}{\Gamma \vdash \text{inr } e \Leftarrow A + B}$$

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Leftarrow A \rightarrow C \quad \Gamma \vdash g \Leftarrow B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) \Leftarrow C}$$

$$\frac{}{\Gamma \vdash \text{unit} \Rightarrow 1} \quad \frac{\Gamma \vdash e \Leftarrow 0}{\Gamma \vdash \text{exfalse } e \Leftarrow A}$$

Bidirectional typing – additional rules

$$\frac{\Gamma \vdash e \Rightarrow A + B \quad \Gamma \vdash f \Rightarrow A \rightarrow C \quad \Gamma \vdash g \Rightarrow B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) \Rightarrow C}$$

Metatheory

Similarly to Extrinsic STLC, typing is not unique in Bidirectional STLC. This is because while we do have annotations in terms, we are not forced to use them. Therefore, we can check terms like $\lambda x.x$ with many types. However, inference is unique.

Hints

Hints:

$$H ::= ? \mid H_1 \rightarrow H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$$

Intuitively, hints are partial types. They are built like types, except that there's one additional way to make hints: $?$, which can be read as “unknown”, “hole” or “whatever”.

We use the letter H for hints. When we use letters like A, B, C which usually stand for types, it means that the hint **is** a type, i.e. it doesn't contain any $?$ s.

Combining hints

$$? \sqcap H = H$$

$$H \sqcap ? = H$$

$$(H_1 \rightarrow H_2) \sqcap (H'_1 \rightarrow H'_2) = (H_1 \sqcap H'_1) \rightarrow (H_2 \sqcap H'_2)$$

$$(H_1 \times H_2) \sqcap (H'_1 \times H'_2) = (H_1 \sqcap H'_1) \times (H_2 \sqcap H'_2)$$

$$(H_1 + H_2) \sqcap (H'_1 + H'_2) = (H_1 \sqcap H'_1) + (H_2 \sqcap H'_2)$$

$$\mathbf{1} \sqcap \mathbf{1} = \mathbf{1}$$

$$\mathbf{0} \sqcap \mathbf{0} = \mathbf{0}$$

Intuitively, \sqcap is a partial binary operation which combines two hints which share the same structure, possibly filling the ?s in the leaves with something more informative coming from the other argument. For hints which have incompatible structure (e.g. function hint and product hint) the result is undefined.

Combining hints – properties

The operation \sqcap :

- Is associative: $(H_1 \sqcap H_2) \sqcap H_3 = H_1 \sqcap (H_2 \sqcap H_3)$
- Is commutative, i.e. $H_1 \sqcap H_2 = H_2 \sqcap H_1$
- Has neutral element: $? \sqcap H = H = H \sqcap ?$
- Is idempotent, i.e. $H \sqcap H = H$.

So, when \sqcap happens to be defined at all, it forms a commutative idempotent monoid. And as we all know, partial monoids are categories. So it seems hints form a thin groupoid... which they obviously do!

Order on hints

The operation \sqcap induces an order on hints in the usual way:

$$H_1 \sqsubseteq H_2 :\equiv (H_1 \sqcap H_2 = H_2).$$

This order can be intuitively interpreted as an information increase: $H_1 \sqsubseteq H_2$ means that hint H_2 is more informative than H_1 , but in a compatible way. In other words, H_1 and H_2 have the same structure, but some ?s from H_1 can be refined to something more informative in H_2 .

Order on hints – explicitly

$$\overline{? \sqsubseteq H}$$

$$\frac{H_1 \sqsubseteq H'_1 \quad H_2 \sqsubseteq H'_2}{H_1 \rightarrow H_2 \sqsubseteq H'_1 \rightarrow H'_2}$$

$$\frac{H_1 \sqsubseteq H'_1 \quad H_2 \sqsubseteq H'_2}{H_1 \times H_2 \sqsubseteq H'_1 \times H'_2}$$

$$\frac{H_1 \sqsubseteq H'_1 \quad H_2 \sqsubseteq H'_2}{H_1 + H_2 \sqsubseteq H'_1 + H'_2}$$

$$\overline{1 \sqsubseteq 1} \quad \overline{0 \sqsubseteq 0}$$

Terms

Note: red color marks differences from Bidirectional STLC.

Terms:

$e ::=$

$x \mid (e : H) \mid$
 $\lambda x. e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{exfalse } e$

Typing contexts assign types to variables, but annotations in terms are hints, not necessarily types.

Notations

$$\lambda x : A. e \equiv (\lambda x. e : A \rightarrow ?)$$
$$\text{inl}_A e \equiv (\text{inl } e : ? + A)$$
$$\text{inr}_A e \equiv (\text{inr } e : A + ?)$$
$$\text{exfalse}_A e \equiv (\text{exfalse } e : A)$$

Hinting – basic rules

$$\frac{(x : A) \in \Gamma \quad H \sqsubseteq A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A} \text{ANNOT}$$

Hinting – type-directed rules 1

$$\frac{H \sqcap (? \rightarrow ?) = A \rightarrow H' \quad \Gamma, x : A \vdash e \Leftarrow H' \Rightarrow B}{\Gamma \vdash \lambda x. e \Leftarrow H \Rightarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash f \Leftarrow ? \rightarrow H \Rightarrow H' \rightarrow B \quad \Gamma \vdash a \Leftarrow H' \Rightarrow A}{\Gamma \vdash f \ a \Leftarrow H \Rightarrow B}$$

$$\frac{H \sqcap (? \times ?) = H_1 \times H_2 \quad \Gamma \vdash a \Leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \Leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \Leftarrow H \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \Leftarrow H \Rightarrow A}$$

$$\frac{\Gamma \vdash e \Leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \Leftarrow H \Rightarrow B}$$

Hinting – type-directed rules 2

$$\frac{H \sqcap (? + ?) = H' + B \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash \text{inl } e \Leftarrow H \Rightarrow A + B}$$

$$\frac{H \sqcap (? + ?) = A + H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow B}{\Gamma \vdash \text{inr } e \Leftarrow H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow ? + ? \Rightarrow A + B \quad \begin{array}{l} \Gamma \vdash f \Leftarrow A \rightarrow H \Rightarrow A \rightarrow C \\ \Gamma \vdash g \Leftarrow B \rightarrow H \Rightarrow B \rightarrow C \end{array}}{\Gamma \vdash \text{case } e \text{ of } (f, g) \Leftarrow H \Rightarrow C}$$

$$\frac{H \sqsubseteq \mathbf{1}}{\Gamma \vdash \text{unit} \Leftarrow H \Rightarrow \mathbf{1}}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \text{exfalse } e \Leftarrow A \Rightarrow A}$$

Metatheory

Similarly to extrinsic STLC, STLC with hints v1 does not enjoy uniqueness of typing. This is because we still can have terms like $\lambda x.x$ with hint $?$, which can be typed with multiple types. However, if the hint is informative enough, then the type is unique. Moreover, every typable term can be given a hint which makes its type unique.

Hinting v2 – hints for term constructors

$$\begin{aligned}H \triangle \lambda x.e &= H \sqcap (? \rightarrow ?) \\H \triangle (e_1, e_2) &= H \sqcap (? \times ?) \\H \triangle \text{inl } e &= H \sqcap (? + ?) \\H \triangle \text{inr } e &= H \sqcap (? + ?) \\H \triangle \text{unit} &= H \sqcap \mathbf{1}\end{aligned}$$

Hinting v2 – basic rules

$$\frac{(x : A) \in \Gamma \quad H \sqsubseteq A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcap H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A} \text{ANNOT}$$

$$\frac{e \text{ is a constructor} \quad H \triangle e = H' \quad \Gamma \vdash e \Leftarrow H' \Rightarrow A}{\Gamma \vdash e \Leftarrow H \Rightarrow A} \text{HINTFOR}$$

Hinting v2 – basic rules

$$\frac{\Gamma, x : A \vdash e \leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x. e \leftarrow A \rightarrow H \Rightarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash f \leftarrow ? \rightarrow H \Rightarrow H' \rightarrow B \quad \Gamma \vdash a \leftarrow H' \Rightarrow A}{\Gamma \vdash f \ a \leftarrow H \Rightarrow B}$$

$$\frac{\Gamma \vdash a \leftarrow H_1 \Rightarrow A \quad \Gamma \vdash b \leftarrow H_2 \Rightarrow B}{\Gamma \vdash (a, b) \leftarrow H_1 \times H_2 \Rightarrow A \times B}$$

$$\frac{\Gamma \vdash e \leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outl } e \leftarrow H \Rightarrow A}$$

$$\frac{\Gamma \vdash e \leftarrow H \times ? \Rightarrow A \times B}{\Gamma \vdash \text{outr } e \leftarrow H \Rightarrow B}$$

Hinting v2 – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow A}{\Gamma \vdash \text{inl } e \Leftarrow H + B \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \text{inr } e \Leftarrow A + H \Rightarrow A + B}$$

$$\frac{\Gamma \vdash e \Leftarrow ? + ? \Rightarrow A + B \quad \begin{array}{l} \Gamma \vdash f \Leftarrow A \rightarrow H \Rightarrow A \rightarrow C \\ \Gamma \vdash g \Leftarrow B \rightarrow H \Rightarrow B \rightarrow C \end{array}}{\Gamma \vdash \text{case } e \text{ of } (f, g) \Leftarrow H \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \text{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}} \quad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma \vdash \text{exfalse } e \Leftarrow A \Rightarrow A}$$

Metatheory

Metatheoretically, STLC with hints v2 is similar to v1.