Discrete Math Problem Set 10

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1. Construct the explicit functions requested below.

   (a) A bijection from $\{x \in \mathbb{R} \mid -1 < x < 1\}$ to $\{x \in \mathbb{R} \mid -\pi < x < \pi\}$.

   Proof. Let $A := \{x \in \mathbb{R} \mid -1 < x < 1\}$ and let $B := \{x \in \mathbb{R} \mid -\pi < x < \pi\}$. Let $f : A \to B$. We will define f as $f(a) = a \cdot \pi$ where $a \in A$. Now, we will show that f is a bijection.
   Injective:
   Let $a, b \in A$. Assume $f(a) = f(b)$. Thus by the definition of f, $a\pi = b\pi$. So, $a = b$. Thus, f is injective.
   Surjective:
   Let $c \in B$. Let $d := \frac{c}{\pi}$. By the definition of B, we know that $-\pi < c < \pi$, thus $-1 < d < 1$. So, $d \in A$. Thus, we obtain $f(d) = d\pi = \frac{c}{\pi}\pi = c$. Because we chose an arbitrary element of B, we know that every element of B has an element of the domain which maps to it.
   Thus, f is surjective.
   Thus, f is a bijection.

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   (b) A surjection from $\mathbb{N}$ to $\{p \mid p \text{ is prime}\}$.

   Proof. Let $P := \{p \mid p \text{ is prime}\}$.
   Let $f : \mathbb{N} \to P$ be defined as $f(n) := p$ such that $|\{q \in P \mid q \leq p\}| = n$. We will show that f is a surjection.
   Let $k \in P$ where $k \in \mathbb{N}$. This means that k is prime by definition of P. Observe, k is finite. This lets us know that there are finitely many prime numbers less than k. As such, $|\{q \in P \mid q \leq k\}|$ is finite, which allows us to say that $|\{q \in P \mid q \leq k\}| = m$ for some $m \in \mathbb{N}$. Then by the definition of f, we know that $f(m) = k$. Because we showed this for an arbitrary element of P, we know that every element of P has an element of the domain which maps to it.
   Therefore, f is surjective.

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(c) An injection from X to $\mathbb{P}(X)$ for every set X.

Proof. Let X be a set.
Let $f : X \to \mathbb{P}(X)$ be defined as $f(x) := \{x\}$. We will show that f is injective.
Let $a, b \in X$. Assume $f(a) = f(b)$.
This means that $\{a\} = \{b\}$. By definition, we know $\forall c(c \in \{a\} \iff c \in \{y\})$.
Therefore, all elements of a and b are the same. So, $a = b$.
Therefore, f is injective.

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(d) A surjection from $\mathbb{P}(X)$ to X for every set $X \neq \varnothing$.

Proof. Let X be a set where $X \neq \varnothing$. Let $x \in X$.
Let $f : \mathbb{P}(X) \to X$ be defined

$$f(a) = \begin{cases} \cup a & |a| = 1 \\ x & |a| \neq 1 \end{cases}$$

We will show that f is surjective.
Let $y \in X$. By the definition of a subset, we know that $\{y\} \subseteq X$, which means that $\{y\} \in \mathbb{P}(X)$ by the definition of a power set. We also know that $|\{y\}| = 1$ because there is only one element. Therefore by the definition of f, we know that $f(\{y\}) = \cup\{y\} = y$.
Because we showed this for an arbitrary element of X, we know that f is surjective for every set X where $X \neq \varnothing$.

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2. Let A be an arbitrary finite set of cardinality $|A| = n$, where $n \in \mathbb{N}$. How many finite strings over A are there?

Proof. Let A be a set, where $|A| = n$ for some $n \in \mathbb{N}$. Let $s := \{z \mid (\exists c \in \mathbb{N})(z = \{s \mid s : c \to A\})\}$. Observe, $\cup s = \bigcup\limits_{i \in \mathbb{N}}^{\infty} \{s \mid s : i \to A\}$. Let's define $B \in s$ for some $m \in \mathbb{N}$. So, $B = \{s \mid s : m \to A\}$. Thus, $|B| = |A|^m = n^m$ by *Theorem 6.10*. So, $|B|$ is finite thus B is countable. This means that $\cup s$ is a countable collection of countable sets. Therefore by *Theorem 8.8*, we know that $|\cup s|$ is countable, meaning $|\cup s| \leq |\mathbb{N}|$.
Now, we would like to prove that $\cup s$ is an infinite set. Towards a contradiction, assume $\cup s$ is finite. This means that for some $l \in \mathbb{N}$, $|\cup s| = l$. So, there exists some surjection $f : l \to \cup s$.
Let $a \in A$. Let $h := f(0) +\!\!+ f(1) +\!\!+ \ldots f(l-1) +\!\!+ a$. So, $|h| = \sum\limits_{i=0}^{l-1} |f(i)| + 1$. So by definition, $(\forall k \in l)(|f(k)|) \leq \sum\limits_{i=0}^{l-1} |f(i)| < \sum\limits_{i=0}^{l-1} |f(i)| + 1 = |h|$. This tells us that $(\forall k \in l)(|f(k)| \neq |h|)$.
Therefore, $(\forall k \in l)(f(k) \neq h)$. So, h is a finite string over A where $h \in \cup s$, but f does

not map to h. Therefore f is not a surjection. However, we assumed f was a surjection $\lightning$.
Therefore, $(\forall l \in \mathbb{N})(\cup s \neq l)$, so $\cup s$ is infinite.
Therefore, we know that $\cup s$ is both countable and infinite.
Therefore by definition, we know $|\cup s| = |\mathbb{N}| = \aleph_0$. So, there are $\aleph_0$ finite strings over A.

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3. Imagine that, one day, you encounter a library. At the entrance of this library is an enormous tome $\mathscr{B}$ listing *all* of the *possible* sentences in the English language, indexed by natural numbers.

   Walking past the entrance, you see that the library has rows of bookshelves numbered $0, 1, 2, \ldots$, so that there is exactly one row of books for each $n \in \mathbb{N}$. A great owl—perched on the pedestal that supports $\mathscr{B}$—informs you that, for each $n \in \mathbb{N}$, the n^{th} row of bookshelves contains *all* of the books that could possibly ever be that begin with the n^{th} sentence in $\mathscr{B}$. With the sound of granite scraping against marble, the doors to the library close behind you. The owl makes you the following proposal: you will be free to go *if and only if* you can read every book in the library *in a countable amount of time.*

   Will you be set free? The owl demands a proof to justify your answer.

   Proof. Observe a few things. There are countibly many sentences in $\mathscr{B}$ because they are numbered. $\mathscr{B}$ is a set of cardinality $\mathbb{N}$ where $n \in \mathbb{N}$ is a sentence at the n^{th} position of $\mathscr{B}$. We know that every book contains an infinitely countable amount of sentences, because they are indexed. Thus, we will represent the amount of sentences in each book with $|\mathbb{N}|$. Let S be $S := \{s \mid s : \mathbb{N} \to \mathbb{N}\}$ where S is the set of all books of infinite length in the library. Thus, $S \subseteq$ "all of the books in the library".
   Towards a contradiction, assume $|\mathbb{N}| \geq |S|$. This means there exists some $f : \mathbb{N} \to S$ such that f is surjective. Let $g \in S$. Consider, $g : \mathbb{N} \to \mathbb{N}$ where:

$$g(i) := \begin{cases} 1 & f(i)(i) \neq 1 \\ 2 & f(i)(i) = 1 \end{cases}$$

   Because f is surjective, $(\exists t \in \mathbb{N})(f(t) = g)$. So, $(\forall i \in \mathbb{N})(f(t)(i) = g(i))$. However, observe $f(t)(t) = 1 \Leftrightarrow g(t) = 2 \neq 1$, and $f(t)(t) \neq 1 \Leftrightarrow g(t) = 1$. This means that $f(t)(t) \neq g(t)$, so $(\exists i \in \mathbb{N}) f(t)(i) \neq g(i) \lightning$.
   Therefore, $|\mathbb{N}| < |S|$. Because $|S| > |\mathbb{N}|$, we know $|S| > \aleph_0$, which means by definition that S is uncountibly infinite. So I will sadly *not* be set free and the owl will own me forever :/

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