Discrete Math Problem Set 3

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In addition to the axioms and rules of inference, you may rely on: all proven theorems, *Implication Elimination, Hilbert's First & Second Axioms*.

(10 pts) 1. Prove each of the following statements for any propositions φ, ψ, ξ.

(a) $(\varphi \to \psi, (\psi \to \xi) \vdash (\varphi \to \xi)$

Proof. Observe the following chain of reasoning.
Let φ, ψ, ξ be arbitrary propositions.
Assume $(\varphi \to \psi$, $(\psi \to \xi)$.
Assume φ.

$$\begin{aligned} \varphi, (\varphi \to \psi) &\vdash \psi && \text{by } \textit{Modus Ponens} \\ \psi, \psi \to \xi &\vdash \xi && \text{by } \textit{Modus Ponens} \end{aligned}$$

Therefore, $\varphi \vdash \xi$.

$$(\varphi \vdash \xi) \vdash (\varphi \to \xi) \qquad \text{by } \textit{Deduction Rule}$$

Thus, we conclude that $(\varphi \to \psi, (\psi \to \xi) \vdash (\varphi \to \xi)$.

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(b) $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. Observe the following chain of reasoning.
Let φ, ψ be arbitrary propositions.
Assume φ, ψ.
Towards contradiction, assume $\neg(\varphi \wedge \psi)$.

$$\begin{aligned} \neg(\varphi \wedge \psi) &\equiv \neg\varphi \vee \neg\psi && \text{by } \textit{De Morgan's Rules} \\ &\equiv \neg\neg\varphi \to \neg\psi && \text{by } \textit{conditional disintegration} \\ &\equiv \varphi \to \neg\psi && \text{by } \textit{double negation} \end{aligned}$$

By *modus ponens* we can see that we can derive $\neg\psi$ when we have $\varphi \to \neg\psi$ and φ. This shows that $\neg(\varphi \wedge \psi) \to \neg\psi$ by the *deduction rule*. Since, we assumed ψ already, we can derive $\neg(\varphi \wedge \psi) \to \psi$ by the *deduction rule* again. $\lightning$.
Therefore, we can conclude $\varphi \wedge \psi$ by *reductio ad absurdum*.

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(40 pts) 2. Prove each of the following statements for any propositions φ, ψ, ξ.

(a) $\vdash \varphi \to \varphi$

Bruh. ***Proof.*** Observe the following chain of reasoning.

Let φ be an arbitrary proposition.

We want to show that $\varphi \to \varphi$. Towards that goal, assume φ. Since we have φ, we are able to say that $\varphi \vdash \varphi$ since any proposition proves itself. By the *deduction rule*, we now obtain $\varphi \to \varphi$.

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(b) $\vdash (\neg\varphi \to \varphi) \to \varphi$.

Proof. Observe the following chain of reasoning.

Let φ be an arbitrary proposition.

We want to show that $(\neg\varphi \to \varphi) \to \varphi$. Towards that goal, assume $\neg\varphi \to \varphi$ and separately assume $\neg\varphi$.

Because we know $\neg\varphi, (\neg\varphi \to \varphi)$, we can use *modus ponens* to obtain φ. Because we derived φ from $\neg\varphi \to \varphi$, we can say $(\neg\varphi \to \varphi) \vdash \varphi$. By the *deduction rule*, we then know $(\neg\varphi \to \varphi) \to \varphi$.

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(c) $\vdash \neg\varphi \to (\varphi \to \neg\psi)$

Proof. Observe the following chain of reasoning.

Let φ, ψ be arbitrary propositions. We want to show $\neg\varphi \to (\to \neg\psi)$. Towards that goal, assume $\neg\varphi$.

By *Hilbert's First Axiom*, we know that when we assume φ, we obtain $\vdash \varphi \to (\psi \to \varphi)$. Thus, when we assume $\neg\varphi$ like we just did, we obtain $\neg\varphi \to (\psi \to \neg\varphi)$.

In Problem Set 2, 2A we proved $\neg\varphi \to (\psi \to \neg\varphi) \equiv \neg\varphi \to (\varphi \to \neg\psi)$ via the contrapositive.

Thus, by applying *Hilbert's First Axiom*, we obtain $\neg\varphi \to (\varphi \to \neg\psi)$.

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(d) $\varphi \wedge \psi \vdash \varphi$

Proof. Observe the following chain of reasoning.

Let φ, ψ be arbitrary propositions.

Suppose $\varphi \wedge \psi$.

Towards contradiction, assume $\neg\varphi$.

Observe:

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\begin{aligned}
(\varphi \wedge \psi) \wedge (\neg\varphi) &\qquad \text{by } \textit{conjunction introduction} \\
(\psi \wedge \varphi) \wedge (\neg\varphi) &\qquad \text{by } \textit{commutativity} \\
\psi \wedge (\varphi \wedge \neg\varphi) &\qquad \text{by } \textit{associativity} \\
\psi \wedge \bot &\qquad \text{by } \textit{complement} \\
\bot &\qquad \text{by } \textit{domination}
\end{aligned}
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Therefore $\neg\varphi \vdash \bot$.
By 2A, we know that when we assume $\neg\varphi$, we get $\vdash \neg\varphi \to \neg\varphi$. $\neg\varphi \to \neg\varphi \equiv \top$, so by substitution we can say $\neg\varphi \vdash \top$. $\lightning$.
Because $\neg\varphi \vdash \bot$ and $\neg\varphi \vdash \top$, we can apply *Reductio Ad Absurdum*.
By *Reductio Ad Absurdum*, we get φ.
Therefore, $\varphi \wedge \psi \vdash \varphi$.

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(e) $\vdash \top$

Proof. Observe the following chain of reasoning.
Let φ be an arbitrary proposition.
Observe by 2A, we can write $\vdash \varphi \to \varphi$ when we assume φ.
In Problem Set 2, 3A we proved that $\varphi \to \varphi \equiv \top$, so by substitution we can say $\vdash \top$.

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(30 pts) 3. Prove each of the following statements for any propositions φ, ψ, ξ, χ.

(a) $\varphi \vdash (\varphi \vee \psi)$

Proof. Observe the following chain of reasoning.
Let φ, ψ be arbitary propositions.
Suppose φ.
Let $\gamma := \neg\psi$.
By *Ex Contradictione Quodlibet*, we can say $\neg\varphi \to (\varphi \to \neg\gamma)$. Since $\gamma = \neg\psi$, we can then say $\neg\varphi \to (\varphi \to \psi)$.
$\vdash \neg\varphi \to (\varphi \to \psi) \equiv \varphi \vee (\neg\varphi \vee \psi)$ by *conditional disintegration*. This is then $\equiv \neg\varphi \vee (\varphi \vee \psi)$ by *commutativity and associativity*. We can then conditionally disintegrate again to receive $\varphi \to (\neg\varphi \to \psi)$. We can use *conditional disintegration* again to get $\varphi \to (\varphi \vee \psi)$. By *deduction rule*, we can rewrite this as $\varphi \vdash (\varphi \vee \psi)$. Therefore since we know φ, we obtain $\varphi \vee \psi$. So, $\varphi \vdash (\varphi \vee \psi)$.

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(b) $(\varphi \to \xi), (\psi \to \xi), (\varphi \vee \psi) \vdash \xi$

Proof. Observe the following chain of reasoning.

Let φ, ψ, ξ be arbitrary propositions.

Suppose $(\varphi \to \xi), (\psi \to \xi), (\varphi \vee \psi)$.

Towards a contradiction, now assume $\neg\xi$.

We now have $\neg\xi$ and $\varphi \to \xi$. By *Modus Tollens* we obtain $\neg\varphi$. We also have $\neg\xi$ and $\psi \to \xi$. By *Modus Tollens* again, we obtain $\neg\psi$. By *conjunction introduction*, we can now say $\neg\varphi \wedge \neg\psi$. $\neg\varphi \wedge \neg\psi \equiv \neg(\varphi \vee \psi)$ by *De Morgan's Laws*. Therefore, $\neg\xi \vdash \neg(\varphi \vee \psi)$ since we derived $\neg(\varphi \vee \xi)$ from our assumption of $\neg\xi$. $\neg\xi$ also $\vdash (\varphi \vee \psi)$ because we assume $\varphi \vee \psi$. Since $\neg\xi \vdash \neg(\varphi \vee \psi)$ and $\neg\xi \vdash (\varphi \vee \psi)$, we apply *Reductio Ad Absurdum* and obtain ξ. Therefore, $(\varphi \to \xi), (\psi \to \xi), (\varphi \vee \psi) \vdash \xi$.

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(c) $\varphi, \neg\varphi \vdash \psi$

Proof. Observe the following chain of reasoning.

Let φ, ψ be arbitrary propositions.

Suppose $\varphi, \neg\varphi$.

Since we have $\varphi, \neg\varphi$, by *conjunction introduction* we can say $(\varphi \wedge \neg\varphi)$. We can then say $(\varphi \wedge \neg\varphi) \vee \psi$ by *disjunction introduction*. $(\varphi \wedge \neg\varphi) \vee \psi \equiv \bot \vee \psi$ by *complement*. $\bot \vee \psi \equiv \psi$ by *domination*. Therefore, we obtain ψ from $\varphi, \neg\varphi$. So, $\varphi, \neg\varphi \vdash \psi$.

$\blacksquare$

(d) $(\varphi \vee \psi), \neg\varphi \vdash \psi$

Proof. Observe the following chain of reasoning.

Let φ, ψ be arbitrary propositions.

Suppose $(\varphi \vee \psi), \neg\varphi$.

$\varphi \vee \psi \equiv \neg\varphi \to \psi$ by *conditional disintegration*. Because we know $\neg\varphi$ and $\neg\varphi \to \psi$, we obtain ψ from *Modus Ponens*. So, $(\varphi \vee \psi), \neg\varphi \vdash \psi$.

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(e) $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi) \vdash \xi \vee \chi$

Proof. Observe the following chain of reasoning.

Let φ, ψ, ξ, χ be arbitrary propositions.

Suppose $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi)$.

Towards a contradiction, assume $\neg(\xi \vee \chi)$.

$\neg(\xi \vee \chi) \equiv \neg\xi \wedge \neg\chi$ by *De Morgan's Laws*. Since we have $\neg\xi \wedge \neg\xi$, by *conjunction elimination* we get $\neg\xi$ and separately get $\neg\chi$. Since we now have $\neg\xi$ and $\varphi \to \xi$, by *Modus Tollens* we obtain $\neg\varphi$. Similarly, since we have $\neg\chi$ and $\psi \to \chi$, we obtain $\neg\psi$ by *Modus Tollens*. Since we know $\neg\varphi$ and $\neg\psi$, by *conjunction introduction* we can say $\neg\varphi \wedge \neg\psi$. $\neg\varphi \wedge \neg\psi \equiv \neg(\varphi \vee \psi)$ by *De Morgan's Laws*. So, $\neg(\xi \vee \chi) \vdash \neg(\varphi \vee \psi)$. $\neg(\xi \vee \chi)$ also $\vdash \varphi \vee \psi$ because we assume $\varphi \vee \psi$.

Since $\neg(\xi \vee \chi) \vdash \neg(\varphi \vee \psi)$ and $\neg(\xi \vee \chi) \vdash (\varphi \vee \psi)$, we can apply *Reductio ad Absurdum* and obtain $\xi \vee \chi$. Therefore, $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi) \vdash \xi \vee \chi$.

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(10 pts) 4. Let $\mathscr{L}$ be a binary predicate. Prove the following statement.

$$\vdash \neg\exists x \forall y(\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y))$$

Towards contradiction, assume $\exists x \forall y(\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y))$. This means that there exists an x that for all y $\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y)$. To disprove this statement, we need to find an x where $(\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y)) \equiv \bot$. If the statement is true, then there is one specific value for x that satisfies the rest of the statement, and we will call this value c. Since we asserted $\forall y$, we know that at some point $y = c$. When $x = c$ and $y = c$, we can say $\mathscr{L}(c,c) \leftrightarrow \neg\mathscr{L}(c,c)$. Because a biconditional statement returns $\bot$ when its two inputs have different truth values and $\mathscr{L}(c,c) \not\equiv \neg\mathscr{L}(c,c)$, we obtain $\bot$. By the *truth theorem*, we know that $\exists x \forall y(\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y)) \vdash \top$ because we assume $\exists x \forall y(\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y))$. Therefore by *Reductio ad Absurdum*, we get $\neg\exists x \forall y(\mathscr{L}(x,y) \leftrightarrow \neg\mathscr{L}(y,y))$.

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(10 pts) 5. Consider a universe of discourse consisting of every natural number. Recall that a positive integer is prime when it has exactly two positive divisors: one and itself.
Let $\omega(x)$:= "*x is an odd number*"
Let $\pi(x)$:= "*x is a prime number*"
Further, suppose the following statements only contain propositions.

(a) Prove φ, where φ is the statement $\varphi \vdash \forall x(\omega(x) \to \pi(x))$.
Proof. Let φ := "$\varphi \vdash \forall x(\omega(x) \to \pi(x)))$". Assume φ. Therefore we have $\varphi \vdash \forall x(\omega(x) \to \pi(x)))$ because we have φ. Then by the *deduction rule*, we have $\varphi \to \forall x(\omega(x) \to \pi(x)))$. Since we now have φ and $\varphi \to \forall x(\omega(x) \to \pi(x))$, by *modus ponens* we have $\forall x(\omega(x) \to \pi(x))$. Because we derived $\forall x(\omega(x) \to \pi(x))$ from φ, we obtain $\varphi \vdash \forall x(\omega(x) \to \pi(x))$.

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(b) Prove $\forall x(\omega(x) \to \pi(x))$.
Proof. In $5(a)$, we proved $\varphi \vdash \forall x(\omega(x) \to \pi(x))$. By the *deduction rule*, we then know $\varphi \to \forall x(\omega(x) \to \pi(x))$. In $5(a)$ we also found $\vdash \varphi$. Therefore by *modus ponens*, we obtain $\forall x(\omega(x) \to \pi(x))$.

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