Discrete Math Problem Set 5

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(15 pts) 1. Find and explain the flaw(s) in the argument:

We prove every nonempty set of people all have the same age.

Proof. We denote the age of a person p by $\alpha(p)$.

Basis step:

Suppose $P = \{p\}$ is a set with one person in it. Clearly, all the people in P have the same age as each other.

Inductive step:

Let $k \in \mathbb{N}_+$ and suppose any set of k-many people all have the same age. Let $P = \{p_1, p_2, \ldots p_k, p_{k+1}\}$ be a set with $k+1$ people in it. Consider $L := \{p_1, \ldots p_k\}$ and $R := \{p_2, \ldots p_{k+1}\}$. Since L and R both have k people, we know everyone in these sets has the same age by the *inductive hypothesis.*

Let $\ell, r \in P$. If $\ell \in L \wedge r \in L$, then $\alpha(\ell) = \alpha(r)$. Similarly, if $\ell \in R \wedge r \in R$, then $\alpha(\ell) = \alpha(r)$. Now, suppose $\ell \in L \wedge r \in R$.

$$\alpha(\ell) = \alpha(p_1) = \alpha(p_2) = \alpha(p_{k+1}) = \alpha(r)$$

So, all people in P have the same age.

Therefore, everyone on Earth has the same age.

Flaws in the argument:

This argument does not hold because $\alpha(p_1)$ does not have to $= \alpha p_2$.

Consider the case where $k = 1$ and P therefore has two elements. A problem arises in the step: "Since L and R both have k people, we know everyone in these sets has the same age by the *inductive hypothesis*". We define L and R as $L := \{p_1\}$ and $R := \{p_2\}$. By the inductive hypothesis, everyone in L and R has the same age since they are sets with k people in them. However, there is no relationship between $\alpha(p_1)$ and $\alpha(p_2)$. So, p_1 and p_2 do not have to have the same age. Therefore, when $k = 1$, everyone in L and R has the same age, but there can be people with different ages in P. Therefore, this argument is incorrect because it does not hold when $k = 1$.

(20 pts) 2. Show that $\forall x(x \neq x \cup \{x\})$.

Proof. Let x be an arbitrary set. $x \cup \{x\} := \{z \mid z \in x \wedge z \in \{x\}\}$. Let $z = x$.

Towards a contradiction, assume $x = x \cup \{x\}$. By definition of union, we can then say $x = x \in x \wedge x \in \{x\}$. By *conjunction elimination*, we get $x = x \in x$. However by *Russell's Paradox*, we know $x \notin x$ $\lightning$. Therefore, $x \neq x \cup \{x\}$.

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(15 pts) 3. We will work up to a proof of the commutativity of addition on $\mathbb{N}$.

(a) Show $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.
To show $x + 0 = 0 + x$ we will use *induction.*
Basis step:
By definition of *addition*, we know $0 + 0 = 0$. By definition of *addition* again, we know $0 + 0 = 0 + 0$. Therefore, we know $x + 0 = 0 + x$ when $x = 0$.
Inductive step:
Let $k \in \mathbb{N}$. Assume $k + 0 = 0 + k$ because this is our inductive hypothesis. Now, we need to show $S(k) + 0 = 0 + S(k)$.
Observe $0 + S(k) = S(0 + k)$ by the definition of *addition.* By the inductive hypothesis, we know $0 + k = k + 0$. By the definition of *addition*, we know $k + 0 = k$. Therefore, we obtain $0 + S(k) = S(k)$. By the definition of *addition*, we obtain $S(k) + 0$. Therefore, $S(k) + 0 = 0 + S(k)$.
By *mathematical induction*, $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.

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(b) Show $(\forall x, y \in \mathbb{N})(x + s(y) = s(y) + x)$.
Let $y \in \mathbb{N}$. To show $x + S(y) = S(y) + x$ we will use *induction.*
Basis step:
As we proved in *3(b)*, $0 + S(y) = S(y) + 0$, so $x + S(y) = S(y) + x$.
Inductive step:
Let $k \in \mathbb{N}$. Assume $k + S(y) = S(y) + k$ because this is our inductive hypothesis. Now, we need to show $S(k) + S(y) = S(y) + S(k)$.
Observe, $S(y) + S(k) = S(S(y) + k)$. By our inductive hypothesis, we know $S(y) + k = k + S(y)$. Therefore, $S(k + S(y))$. By the definition of *addition*, we get $k + S(S(y))$. By definition, we know $S(S(y)) = 1 + S(y)$. So we can say $k + 1 + S(y)$. By definition again, we know $k + 1 = S(k)$. So, we obtain $S(k) + S(y)$. Therefore, we obtain $S(k) + S(y) = S(y) + S(k)$.
By *mathematical induction*, $(\forall x, y \in \mathbb{N})(x + S(y) = S(y) + x)$.

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(c) Show $(\forall x, y \in \mathbb{N})(x + y = y + x)$.
Let $y \in \mathbb{N}$. To show $x + y = y + x$ we will use *induction.*
Basis step:

As we proved in *3(a)*, $0 + y = y + 0$. So, $x + y = y + x$.
Inductive step:
Let $k \in \mathbb{N}$. Assume $k + y = y + k$ because this is our inductive hypothesis.
Now we need to show $S(k) + y = y + S(k)$.
As we proved in *3(b)*, $S(k) + y = y + S(k)$.
By *mathematical induction*, $(\forall x, y \in \mathbb{N})(x + y = y + x)$.

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(15 pts) 4. Show $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.
Let $x, y \in \mathbb{N}$. To show $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, we will use *induction.*
Basis step:

$$\begin{aligned} x \cdot (y + 0) &= x \cdot y && \text{by } \textit{definition of addition} \\ &= (x \cdot y) + 0 && \text{by } \textit{definition of addition} \\ &= (x \cdot y) + (x \cdot 0) && \text{by } \textit{definition of multiplication} \end{aligned}$$

Thus, $(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.
Inductive step:
Let $k \in \mathbb{N}$. Assume $(x \cdot (y + k) = (x \cdot y) + (x \cdot k))$ because this is our inductive hypothesis.
Now we need to show $(x \cdot (y + S(k)) = (x \cdot y) + (x \cdot S(k)))$.

$$\begin{aligned} x \cdot (y + S(k)) &= x \cdot (S(y + k)) && \text{by } \textit{definition of addition} \\ &= x \cdot (y + k) + x && \text{by } \textit{definition of multiplication} \\ &= ((x \cdot y) + (x \cdot k)) + x && \text{by } \textit{inductive hypothesis} \\ &= (x \cdot y) + ((x \cdot k) + x) && \text{by } \textit{associativity of addition} \\ &= (x \cdot y) + (x \cdot S(k)) && \text{by } \textit{definition of multiplication} \end{aligned}$$

Thus, $(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.
By *mathematical induction*, $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.

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(15 pts) 5. For this problem, you may assume the commutativity and associativity of addition and multiplication over $\mathbb{N}$. You may also assume multiplication distributes over addition on $\mathbb{N}$. Prove the following statement for all $n \in \mathbb{N}$.

$$1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$$

To show this, we will use *induction.*
Basis step:

Observe,

$$\begin{aligned}
1 + \sum_{i=0}^{0} 2^i &= 1 + 2^0 && \text{by } \textit{definition of summation} \\
&= 1 + 1 && \text{by } \textit{definition of exponenentiation} \\
&= 2 && \text{as proven in class} \\
&= 2 + 0 && \text{by } \textit{definition of addition} \\
&= 0 + 2 && \text{by } \textit{commutativity of addition} \\
&= (2 \cdot 0) + 2 && \text{by } \textit{definition of multiplication} \\
&= 2 \cdot S(0) && \text{by } \textit{definition of multiplication} \\
&= 2 \cdot 1 && \text{by } \textit{definition of successor} \\
&= 2 \cdot 2^0 && \text{by } \textit{definition of exponentiation} \\
&= 2^{S(0)} && \text{by } \textit{definition of exponentiation} \\
&= 2^1 && \text{by } \textit{definition of successor} \\
&= 2^{1+0} && \text{by } \textit{definition of addition} \\
&= 2^{0+1} && \text{by } \textit{commutativity of addition}
\end{aligned}$$

Therefore:

$$1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$$

Inductive step:

Let $k \in \mathbb{N}$. By the inductive hypothesis, assume:

$$1 + \sum_{i=0}^{k} 2^i = 2^{k+1}$$

Now, we want to show:

$$1 + \sum_{i=0}^{S(k)} 2^i = 2^{S(k)+1}$$

Observe,

$$
\begin{aligned}
1 + \sum_{i=0}^{S(k)} 2^i &= 1 + (\sum_{i=0}^{k} 2^i + 2^{S(k)}) && \text{by } \textit{definition of summation} \\
&= (1 + \sum_{i=0}^{k} 2^i) + 2^{S(k)} && \text{by } \textit{associavity of addition} \\
&= 2^{k+1} + 2^{S(k)} && \text{by } \textit{inductive hypothesis} \\
&= 2^{S(k)} + 2^{S(k)} && \text{by } \textit{definition of successor} \\
&= (2^{S(k)} \cdot 1) + 2^{S(k)} && \text{by } \textit{mult. identity (proved in basis step)} \\
&= 2^{S(k)} \cdot S(1) && \text{by } \textit{definition of multiplication} \\
&= 2^{S(k)} \cdot 2 && \text{by } \textit{definition of successor} \\
&= 2 \cdot 2^{S(k)} && \text{by } \textit{commutativity of multiplication} \\
&= 2^{S(S(k))} && \text{by } \textit{definition of exponentiation} \\
&= 2^{S(k)+1} && \text{by } \textit{definition of successor}
\end{aligned}
$$

Therefore:

$$1 + \sum_{i=0}^{S(k)} 2^i = 2^{S(k)+1}$$

By *mathematical induction*, $\forall n \in \mathbb{N}$:

$$1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$$

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(20 pts) 6. We say x is $\in$-*transitive* by definition when $(\forall y \in x)(\forall z \in y)(z \in x)$. Show that every natural number is $\in$-*transitive*.

Let $y, z \in \mathbb{N}$. To show every natural number is $\in$-transitive, we will use *induction*.

Basis step:

Observe, $(\forall y \in 0)(\forall z \in y)(z \in 0)$.

Because $0 := \varnothing$, we know $y \in \varnothing$. However, since the empty set is empty, $y \notin \varnothing$ $\lightning$. Therefore by the *explosion theorem*, we conclude that $z \in \varnothing$.

Inductive step:

Let $k \in \mathbb{N}$. Assume $(\forall y \in k)(\forall z \in y)(z \in k)$ because this is our inductive hypothesis.

We want to show that $(\forall y \in S(k))(\forall z \in y)(z \in S(k))$.

Observe:

$S(k) := k \cup \{k\}$.

Let $a \in S(k)$. By definition, $a \in k \vee a \in \{k\}$.

In the case $a \in k$, by the *Inductive Hypothesis* we know $(\forall z \in a)(z \in k)$.

In the case $a \in \{k\}$, by *extensionality* we know $a = k$. By *extensionality* again, since we know $(\forall z \in a)$, we then know $(z \in k)$.
In either case, $(\forall z \in a)(z \in k)$. Since we chose an arbitrary element a of $S(k)$ and proved $(\forall z \in a)(z \in k)$, we can say $(\forall b \in S(k)(\forall z \in b)(z \in S(k)))$.
Therefore, by *mathematical induction*, we know $(\forall y \in x)(\forall z \in y)(z \in x)$ and consequently know that every natural number is $\in$-transitive.

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