Discrete Math Problem Set 4

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(20 pts) 1. (a) Show $\forall x(\varnothing \subseteq x)$.

Proof. Let x be an arbitrary set. Towards a contradiction, suppose $\varnothing \nsubseteq x$. Then there exists some z such that $z \subseteq \varnothing \wedge z \nsubseteq x$ by definition. This claims that $z \in \varnothing$. However, we know $\forall y(y \notin \varnothing)$. $\lightning$. Therefore, $\varnothing \subseteq x$.

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(b) Show $\forall x(x \subseteq x)$.

Proof. Let x and z be arbitrary sets. We know that $z \in x \Rightarrow z \in x$. Therefore by the definition of a subset, $x \subseteq x$.

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(c) Show $\forall x(\varnothing \in \mathbb{P}(x))$.

Proof. Let x be an arbitrary set. By Axiom 4, $\mathbb{P}(x) := \{z \mid z \subseteq x\}$. In $1a$, we showed that $\forall x(\varnothing \subseteq x)$. So, $\varnothing \in z$. Therefore, $\varnothing \in \mathbb{P}(x)$.

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(d) Show $\forall x(x \in \mathbb{P}(x))$.

Proof. Let x be an arbitrary set. By Axiom 4, $\mathbb{P}(x) := \{z \mid z \subseteq x\}$. In *1(b)*, we showed that $\forall x(x \subseteq x)$. So, $x \in z$. Therefore, $x \in \mathbb{P}(x)$.

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(e) Show $\forall x \forall y \forall z(((x \subseteq y) \wedge (y \subseteq z)) \Rightarrow x \subseteq z)$.

Proof. Let x, y, z be arbitrary sets. To show $((x \subseteq y) \wedge (y \subseteq z)) \Rightarrow x \subseteq z$, assume $(x \subseteq y) \wedge (y \subseteq z)$. Now suppose $a \in x$. Because $x \subseteq y$, we know $a \in y$. Because $y \subseteq z$, we know $a \in z$. Because we chose an arbitrary element of x and proved it must be an element of z, we can say $\forall s(s \in x \Rightarrow s \in z)$. By the definition of a subset, we can then say $x \subseteq z$. Because we derived $x \subseteq z$, from our assumption of $(x \subseteq y) \wedge (y \subseteq z)$, we can say $(x \subseteq y) \wedge (y \subseteq z) \Rightarrow x \subseteq z$.

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(10 pts) 2. We define the *intersection* and *difference* of any two sets x and y below.

$$x \cap y := \{z \mid z \in x \wedge z \in y\}$$

$$x \backslash y := \{z \mid z \in x \wedge z \notin y\}$$

(a) Show $\forall x \forall y \exists z (z = x \cap y)$.

Proof. Let x, y, z be arbitrary sets. By *Schema of Separation*, we know that $\forall x \exists z (z = \{t \mid t \in x \wedge \varphi(t)\})$. Now, let the predicate $\varphi(t) := t \in y$. We can then equivalently say $\forall x \forall y \exists z (z = \{t \mid t \in x \wedge t \in y\})$. By the definition of *intersection*, we can then say $\forall x \forall y \exists z (z = x \cap y)$.

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(b) Show $\forall x \forall y \exists z (z = x \backslash y)$.

Proof. Let x, y, z be arbitrary sets. By *Schema of Separation*, we know that $\forall x \exists z (z = \{t \mid t \in x \wedge \varphi(t)\})$. Now let the predicate $\varphi(t) := t \notin y$. We can then equivalently say $\forall x \forall y \exists z (z = \{t \mid t \in x \wedge t \notin y\})$. By the definition of *difference*, we can then say $\forall x \forall y \exists z (z = x \backslash y)$.

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(20 pts) 3. We define the *union* of two sets x and y below as:

$$x \cup y := \{z \mid z \in x \vee z \in y\}.$$

(a) Show $\forall x \forall y (x \cap y \subseteq x)$.

Proof. Let x, y be arbitrary sets.

Suppose $a \in x \cap y$. We now know that $a = \varnothing \vee a \neq \varnothing$. In the case $a = \varnothing$, we know $\varnothing \subseteq x$ because we proved it in *1a*. In the case $a \neq \varnothing$, we know $a \in x \wedge a \in y$ by the definition of *intersection*. By *conjunciton elimination*, we know $a \in x$. Because we showed this for an arbitrary element of $x \cap y$, we can say $\forall b (b \in x \cap y \Rightarrow b \in x)$. By the definition of a subset, we now obtain $x \cap y \subseteq x$.

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(b) Show $\forall x \forall y (x \subseteq x \cup y)$.

Proof. Let x, y be arbitrary sets.

Suppose $a \in x$.

$x \cup y := \{b \mid b \in x \vee b \in y\}$. Since we know $a \in x$, by *disjunction introduction*, we can say $a \in x \vee a \in y$. By the definition of *union* we then know $a \in x \cup y$. Because we derived $a \in x \cup y$ from our assumption $a \in x$. We can say $a \in x \Rightarrow a \in x \cup y$. Since we chose an arbitrary element of x, we can say $\forall c (c \in x \Rightarrow c \in x \cup y)$. This is the definition of a subset, so we obtain $x \subseteq x \cup y$.

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(c) Show $\forall x \forall y(\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y))$.

Proof. Let x, y be arbitrary sets.

Suppose $a \subseteq x$ and separately suppose $b \subseteq y$.

Because $\mathbb{P}(x)$ is defined as a set of all the subsets of x, we know that $a \in \mathbb{P}(x)$. Similarly, $\mathbb{P}(y)$ is defined as a set of all the subsets of y, so we know that $b \in \mathbb{P}(y)$. By *2b* we know that $x \subseteq x \cup y$. Because $a \subseteq x$, we can say $a \subseteq x \cup y$. Because $b \subseteq y$, we can say $b \subseteq x \cup y$. Since a and b are both subsets of $x \cup y$, we can say $a \cup b \subseteq x \cup y$. Because $\mathbb{P}(x \cup y)$ is the set of all the subsets of $x \cup y$ and $a \cup b$ is a subset of $x \cup y$, we can say $a \cup b \in \mathbb{P}(x \cup y)$. Because we derived $a \cup b \in \mathbb{P}(x \cup y)$ from $a \in \mathbb{P}(x) \cup b \in \mathbb{P}(y)$, we can say $a \in \mathbb{P}(x) \cup b \in \mathbb{P}(y) \Rightarrow a \cup b \in \mathbb{P}(x \cup y)$. Because we chose arbitrary elements of x and y, we can say $\forall c \forall d(c \in \mathbb{P}(x) \cup d \in \mathbb{P}(y) \Rightarrow c \cup d \in \mathbb{P}(x \cup y))$. This is the definition of a subset, so we obtain $\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y)$.

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(d) Show $\forall x \forall y(x \cap y = x \Leftrightarrow x \in \mathbb{P}(y))$.

Proof. Let x, y be arbitrary sets.

To show $x \cap y = x \Leftrightarrow x \in \mathbb{P}(y)$, we need to show: (1) $x \cap y = x \Rightarrow x \in \mathbb{P}(y)$ and (2) $x \in \mathbb{P}(y) \Rightarrow x \cap y = x$

i. We want to show that $x \cap y = x \Rightarrow x \in \mathbb{P}(y)$.
   Assume $x \cap y = x$.
   $x \cap y := \{z \mid z \in x \wedge z \in y\}$. By *extensionality*, we then know $\forall z(z \in x \Leftrightarrow z \in x \cap y)$. By the definition of *intersection*, we can equivalently say $\forall z(z \in x \Leftrightarrow z \in x \wedge z \in y)$. Now assume $z \in x$. By *modus ponens* we get $z \in x \wedge z \in y$. By *conjunction elimination*, we get $z \in y$. Because $z \Leftrightarrow z \in x \wedge z \in y$, we know $z \in x \Rightarrow z \in x \wedge z \in y$. Because we chose an arbitrary z, and derived $z \in y$ from $z \in x$, so we can say $\forall h(h \in x \Rightarrow h \in y)$. This is the definition of a subset, so we can say $x \subseteq y$. By the definition of a power set, we know $x \in \mathbb{P}(y)$. So, $x \cap y = x \Rightarrow x \in \mathbb{P}(y)$.

ii. We now want to show that $x \in \mathbb{P}(y) \Rightarrow x \cap y = x$.
   Assume $x \in \mathbb{P}(y)$.
   Because x is in the power set of y, we know that $x \subseteq y$. $x \subseteq y := \{z \mid z \in x \Rightarrow z \in y\}$. So, we know that $z \in x \Rightarrow z \in y$. To show $x \cap y = x$, we want to show $z \in x \Leftrightarrow z \in x \wedge z \in y$. To do so, we will show (1) $z \in x \Rightarrow z \in x \wedge z \in y$ and (2) $z \in x \wedge z \in y \Rightarrow z \in x$.

   1. Assume $z \in x$. Because $x \subseteq y$, we know $z \in x \Rightarrow z \in y$. By *conjunction introduction*, we then get $z \in x \Rightarrow z \in x \wedge z \in y$.

2. Assume $z \in x \land z \in y$. By *conjunction elimination*, we get $z \in x$. Since we derived $z \in x$ from our assumption, we can say $z \in x \land z \in y \Rightarrow z \in x$.

So, $z \in x \Leftrightarrow z \in x \land z \in y$. Because this was an arbitrary z, we can say $\forall a(a \in x \Leftrightarrow a \in x \land a \in y)$. By the definition of *intersection*, we can say $\forall a(a \in x \Leftrightarrow a \in x \cap y)$. By *existentiality*, we can reduce this to $x = x \cap y$. So, $x \in \mathbb{P}(y) \Rightarrow x = x \cap y$.

Because we proved $x \cap y = x \Rightarrow x \in \mathbb{P}(y)$ and $x \in \mathbb{P}(y) \Rightarrow x \cap y = x$, we know $x \cap y = x \Leftrightarrow x \in \mathbb{P}(y)$.

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(50 pts) 4. We define the *union over* x and *intersection over* x for any set x below.

$$\cup x := \{z \mid \exists y(y \in x \land z \in y)\}$$

$$\cap x := \{z \mid \forall y(y \in x \Rightarrow z \in y)\}$$

(a) Show that $\forall x(\cup\mathbb{P}(x) = x)$.

Proof. Let x be an arbitrary set.

By *1(d)*, we know $x \in \mathbb{P}(x)$. We also know $\cup\mathbb{P}(x) := \{z \mid \exists y(y \in \mathbb{P}(x) \land z \in y)\}$. By *existential elimination*, we can then say $\{z \mid x \in \mathbb{P}(x) \land z \in x\}$. Since we know $x \in \mathbb{P}(x)$, by *conjunction elimination* we can say $\{z \mid z \in x\}$. Since all elements of $\{z \mid z \in x\}$ are in x, and all elements of x are in $\{z \mid z \in x\}$, by *existentiality* we can conclude $\{z \mid z \in x\} = x$. Since we determined $\{z \mid z \in x\} = \cup\mathbb{P}(x)$, we conclude $\cup\mathbb{P}(x) = x$.

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(b) ***Proof.*** What is $\cup\varnothing$? Justify your answer with a proof.

We know that $\cup\varnothing := \{z \mid \exists y(y \in \varnothing \land z \in y)\}$. However, by the definition of the empty set, we know there is no y such that $y \in \varnothing$. As such, $\cup\varnothing$ is an empty set because the predicate is false for every possible element of $\cup\varnothing$. Therefore, $\cup\varnothing = \varnothing$.

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(c) ***Proof.*** What is $\cap\varnothing$? Justify your answer with a proof.

We know that $\cap\varnothing := \{z \mid \forall y(y \in \varnothing \Rightarrow z \in y)\}$. By the definition of the empty set, we know that $\neg\exists y(y \in \varnothing)$. So, the consequent in the conditional statement is always false. Because the consequent of the conditional statement is false for all y, the conditional statement is then true for all y. So, we can conclude that every element exists in $\cap\varnothing$. Therefore, $\cap\varnothing \in \cap\varnothing$. However by *Russell's Paradox*, we know this to be false. Therefore, $\cap\varnothing$ does not exist.

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(d) Is $\varnothing = \{z \mid z \in \varnothing\}$? Justify your anwer with a proof.

Proof. Let z be a set. By the definition of the empty set, we know $\neg\exists z(z \in \varnothing)$. So, there are no elements of z that satisfy $z \in \varnothing$. So, the set of all elements that satisfy $\{z \mid z \in \varnothing\}$ is empty because the empty set contains no elements. Thus, $\varnothing = \{z \mid z \in \varnothing\}$.

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(e) Is $\varnothing = \{z \mid z \notin \varnothing\}$? Justify your answer with a proof.

Proof. Let z be a set. By the definition of the empty set, we know $\neg\exists z(z \in \varnothing)$. By *existential elimination*, we then know $\forall z(z \notin \varnothing)$. Because the empty set cannot contain any elements, the set of all elements that satisfy $\{z \mid z \notin \varnothing\}$ is every element. Thus, $\varnothing \neq \{z \mid z \notin \varnothing\}$ because the empty set is not equal to a set with infinite elements.

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