Discrete Math Problem Set 8

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1. Including the instructor, there are 32 people in our class. Prove that two of these people were born on the same day of the month.

   Proof. Let $P :=$ "The set of 32 people in our class" where $|P| = 32$.
   Let $B := \{1, 2, \dots 31\}$ where B is every possible birthdate. Recall, the largest month in the year has 31 days, so $|B| = 31$.
   Now, we know that $|P| > |B|$.
   Let $f : P \to B$ where $f(P_i) :=$ "the birthdate B_j of P_i ".
   By the *Pigeonhole Principle*, we know that $(\forall f : P \to B)$ f is not injective. So by definition, there exists $P_i, P_j \in P$ such that $P_i \neq P_j \wedge f(P_i) = f(P_j)$.
   Therefore, 2 people in our class were born on the same day of the month.

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2. As of the 28^{th} of March 2024, there are over 8.1 billion people living on Earth. A person's heart will beat no more than 7×10^9 times over their lifespan. Show that there are two currently-living people on Earth whose hearts have beat the same amount of times.

   Proof. Let $E := \{1, 2, \dots 8.1 \times 10^9\}$ where E is the amount of people living on Earth. Observe, $|E| = 8.1 \times 10^9$.
   Let $H := \{1, 2, \dots 7 \times 10^9\}$ where H is the amount of of heartbeats possible in a person's life. Observe, $|H| = 7 \times 10^9$.
   Now, we know that $|E| > |H|$.
   Let $g : E \to H$ where $g(E_i) :=$ "the amount of heartbeats H_j in the life of E_i".
   By the *Pigeonhole Principle*, we know that $(\forall g : E \to H)g$ is not injective. So by definition, there exists $E_i, E_j \in E$ such that $E_i \neq E_j \wedge g(E_i) = g(E_j)$.
   Therefore, there are two people currently living on Earth whose hearts have beat the same amount of times.

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3. Let $n \in \mathbb{N}_+$ and consider $A \subseteq \mathbb{N}$ such that $|A| = n + 1$. Prove that there exist $x, y \in A$ with $x \neq y$ such that $n \mid x - y$.

Proof. Let $A := \{a_1, a_2, \dots a_{n+1}\}$ where $|A| = n+1$ and $A \subseteq \mathbb{N}$.
Let $R := \{0, 1, \dots n-1\}$ where R is the set of possible remainders of any number n divides. Observe, $|R| = n$.
Now, we know that $|A| > |R|$.
Now, let $f : A \to R$ where $f(A_i) :=$ "the remainder R_j of A_i divided by n".
By the *Pigeonhole Principle*, we know that $(\forall f : A \to R) f$ is not injective. So by definition, there exists $A_i, A_j \in A$ such that $A_i \neq A_j \land f(A_i) = f(A_j)$.
Now, let $x := A_i$ and let $y := A_j$. So, we know that $x \neq y$ and the remainder of $n \mid x$ equals the remainder of $n \mid y$. We can then say $x = kn + r$ and $y = mn + r$ where $k, m \in \mathbb{N}$ and $r = r$. By arithmetic, we see that $x - y = (k - m)n$. Since $k, m \in \mathbb{N}$, we know $n \mid x - y$.
Therefore, there exist $x, y \in A$ where $x \neq y \land n \mid x - y$.

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4. Consider $S := \{3, 4, 7, 8, 9, 10, 12, 15, 18, 19, 27, 28\}$ and $X \subseteq S$ with $|X| \geq 9$. Show that there exists *three* distinct elements $x_1, x_2, x_3 \in X$ such that $x_1 + x_2 + x_3 = 40$.
Proof. Consider $S := \{3, 4, 7, 8, 9, 10, 12, 15, 18, 19, 27, 28\}$. Observe, $|S| = 12$.
Let $X \subseteq S$ and $|X| \geq 9$. We want to prove that there must exist $x_1, x_2, x_3 \in X$ such that $x_1 + x_2 + x_3 = 40$ and $x_1 \neq x_2 \neq x_3$.
Now, observe that there are 4 distinct sets of elements of S that sum to 40:

$$\{3, 10, 27\}$$
$$\{4, 8, 28\}$$
$$\{7, 15, 18\}$$
$$\{9, 12, 19\}$$

Now, let P be defined as the set of all 4 of the above sets where:

$$P := \{\{3, 10, 27\}, \{4, 8, 28\}, \{7, 15, 18\}, \{9, 12, 19\}\}. \tag{1}$$

Observe, $|P| = 4$. To show that there always exist x_1, x_2, x_3 where $x_1 + x_2 + x_3 = 40$, we simply need to prove an element of P always exists.
Now, observe $f : X \to P$ such that $f(x_i) := p_j$ where $x_i \in X$ and $p_j \in P$.
By the *Pigeonhole Principle*, we know there exists $p \in P$ such that $|\{x \in X \mid f(x) = p\}| \geq \lfloor \frac{9-1}{4} \rfloor + 1 = \lceil \frac{9}{4} \rceil$.
We know that $\lceil \frac{9}{4} \rceil = 3$ by the definition of the ceiling function.
So, $|p| = 3$. Because $p \in P$, we then know that 3 distinct elements exist such that $x_1 + x_2 + x_3 = 40 \land x_1 \neq x_2 \neq x_3$.

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5. Recall that $\binom{n}{0} = \binom{n}{n} = 1$ for all $n, k \in \mathbb{N}$ when $k \leq n$.

   (a) Show $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ where $k \leq n$.
   Proof. Assume $n, k \in \mathbb{N}$ and $k \leq n$.
   Let $A := \{z \mid z \subseteq n \wedge |z| = k\}$.
   Let $B := \{z \mid z \subseteq n \wedge |z| = n - k\}$.
   To show that $|A| = |B|$, we will prove the existence of a bijection from $A \to B$.
   To do that, we will first show that there exists $f : A \to B$ where f is injective.
   To do that, let $f(a) := n \setminus a$.
   Assume $f(x) = f(y)$. This means that $n \setminus x = n \setminus y$.
   Towards a contradiction, suppose $x \neq y$. This means that there exists an element $b \in x$ where $b \notin y$. Because $x \subseteq n$, $b \in n$ by definition. Because $b \in n$ and $b \notin y$, $b \in n \setminus y$. We know that $n \setminus y = n \setminus x$, so $b \in n \setminus x$. This means that $b \notin x$. However, we assumed $b \in x$↯.
   Therefore, $x = y$, so f is injective.
   Now, we will show that there exists $g : B \to A$ where g is injective.
   To do that, let $g(b) := n \setminus b$.
   Assume $g(x) = g(y)$. This means that $n \setminus x = n \setminus y$.
   Towards a contradiction, suppose $x \neq y$. This means that there exists an element $b \in x$ where $b \notin y$. Because $x \subseteq n$, $b \in n$ by definition. Because $b \in n$ and $b \notin y$, $b \in n \setminus y$. We know that $n \setminus y = n \setminus x$, so $b \in n \setminus x$. This means that $b \notin x$. However, we assumed $b \in x$↯.
   Therefore, $x = y$, so g is injective.
   Thus, there exist $f : A \to B$ and $g : B \to A$ where both are injective.
   Therefore, by the *Cantor-Schroder-Bernstein* Theorem, we know there exists a bijection from $A \to B$.
   Therefore, $|A| = |B|$.
   By definition, we then know $\binom{n}{k} = \binom{n}{n-k}$.

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   (b) Show $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ for all $n, k \in \mathbb{N}$ where $k \leq n$.
   Proof. Assume $n, k \in \mathbb{N}$ and $k \leq n$.
   Let $A := \{z \mid z \subseteq n + 1 \wedge |z| = k + 1\}$.
   Let $B := \{z \mid z \subseteq n \wedge |z| = (k \vee k + 1)\}$.
   To show that $|A| = |B|$, we will show the existence of a bijection from $A \to B$.
   Observe, $f : A \to B$ where $f(a) := a$ if $n \notin a$, and $f(a) := a \setminus n$ if $n \in a$.
   First, we will show that f is injective.
   Assume $f(x) = f(y)$ where $x, y \in A$.

Look at the case where $|f(x)| = k + 1$. This also means that $f(y) = k + 1$. Suppose towards a contradiction that $n \in x$. This means that $f(x) = x \setminus n$, so $|f(x)| = k$. However, we know $|f(x)| = k + 1 \lightning$. Thus, $n \notin x$. So, $f(x) = x$ and $f(y) = y$. We know $f(x) = f(y)$, so $x = y$.
Now, look at the case where $|f(x)| = k$. Assume towards a contradiction that $n \notin x$. This means $|f(x)| = k + 1$ as we showed in the first case. However, we assumed $|f(x)| = k \lightning$. Therefore, we know $n \in x, y$. So, $f(x) = x \setminus n$ and $f(y) = y \setminus n$. Since we know $x \setminus n = y \setminus n$ and $n \in x, y$, we then know $x = y$.
Therefore, f is injective.
Now, we will show that f is surjective.
Let $b \in B$. $|b| = k + 1 \vee k$ by the definition of B.
First, look at the case $|b| = k + 1$.
We know that $b \subseteq A$ and $|b| = k + 1$. Therefore, we also know $b \in A$. Because $|b| = k + 1$, we know $f(b) = b$ as we showed in the injective proof. Therefore, there exists an input in A for every $b \in B$ where $|b| = k + 1$.
Now look at the case $|b| = k$.
We know that $b \subseteq A$ and $|b| = k$. Because $|b| = k$, we know $n \in b$. So, $b \cup n \in A$ and $f(b) = b \setminus n$. Therefore, there exists an input in A for every $b \in B$ where $|b| = k$.
Therefore, f is surjective.
Because we proved f is injective and surjective, we know f is bijective. So, $|A| = |B|$.
We know that $|A| = \binom{n+1}{k+1}$. We can define $|B|$ as $|C \cup D|$ where $C := \{z \mid z \subseteq n \wedge |z| = k\}$ and $D := \{z \mid z \subseteq n \wedge |z| = k + 1\}$. Observe, $|C \cap D| = 0$ because C and D do not contain any of the same elements. Thus, $|C \cup D| = |C| + |D|$. We can define $|C|$ as $\binom{n}{k+1}$ and $|D|$ as $\binom{n}{k}$. Therefore, $|B| = \binom{n}{k+1} + \binom{n}{k}$.
We know $|A| = |B|$, so $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.

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6. Prove that $|\mathbb{P}(X)| = 2^{|X|}$.

   Proof. Let X be a set. We will prove $|\mathbb{P}(X)| = 2^{|X|}$ this by induction on $|X| \in \mathbb{N}$.

   Basis step:

   Observe, $|X| = 0$, so $X := \varnothing$. Therefore, $\mathbb{P}(X) = \{\varnothing\}$, so $|\mathbb{P}(X)| = 1$. We also know $2^{|\varnothing|} = 2^0 = 1$.

   Therefore, $|\mathbb{P}(\varnothing)| = 2^{|\varnothing|}$.

   Inductive step:

   Let $k \in \mathbb{N}$. Let $k := |X|$. Assume $|\mathbb{P}(X)| = 2^{|X|}$ as our inductive hypothesis.

Now, let m be a set such that $m \notin X$. Let Y also be a set. Let $Y := X \cup m$. Now, observe that $|Y| = k + 1$ because we just added one element to X.
Now, we want to show that $|\mathbb{P}(Y)| = 2^{|Y|}$.
Now, let's define a set S that is every subset of Y that contains m.
$S := \{s \mid (\forall x \in \mathbb{P}(X))(s = x \cup m)\}$. Now, we see that $\mathbb{P}(Y) = \mathbb{P}(X) \cup S$, because S is just every subset of Y that contains m, and $\mathbb{P}(X)$ is all the subsets of Y that do not. Because they do not share any elements, we can say $|\mathbb{P}(Y)| = |\mathbb{P}(X)| + |S|$. We want to show that $|\mathbb{P}(X)| = |S|$. To do so, let's define a function $f : \mathbb{P}(X) \to S$ where $f(x) = x \cup m$. We will show that f is bijective. First, let's show f is injective. Let $x_1, x_2 \in \mathbb{P}(X)$. Assume $f(x_1) = f(x_2)$. This means that $x_1 \cup m = x_2 \cup m$. Since $m \notin \mathbb{P}(X)$, we know $a \notin x_1, x_2$. So, we know $x_1 = x_2$.
Now, we will show f is surjective. Let $s \in S$. By the definition of S, $\exists x \in \mathbb{P}(X)$ where $s = x \cup m$. So, every element of S has an input value. Therefore f is surjective.
So, f is bijective, and $|\mathbb{P}(X)| = |S|$.
So, we can represent $|\mathbb{P}(X)| + |S|$ as $2 \cdot |\mathbb{P}(X)|$. This means that $|\mathbb{P}(Y)| = 2 \cdot \mathbb{P}(X)$.
By the *Inductive hypothesis*, we know $2 \cdot |\mathbb{P}(X)| = 2 \cdot 2^{|X|}$.
Observe, $2^{|Y|} = 2^{|X|+1} = 2 \cdot 2^{|X|}$.
Therefore, $|\mathbb{P}(Y)| = 2^{|Y|}$.
Therefore, by *mathematical induction*, we know $|\mathbb{P}(X)| = 2^{|X|}$.

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