Discrete Math Problem Set 6

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March 5th, 2024

All basic arithmetic and algebraic facts about $\mathbb{N}$ *and* $\mathbb{Z}$ *are now yours to use.*

1. (a) Show that $(c \neq 0 \wedge ac \mid bc) \Rightarrow (a \mid b)$ for all $a, b, c \in \mathbb{Z}$.

   Proof. Let $a, b, c \in \mathbb{Z}$. Assume $c \neq 0 \wedge ac \mid bc$. By the definition of *divisibility*, we know $(\exists z \in \mathbb{Z})(ac \cdot z = bc)$.
   Observe, $(acz = bc) \Leftrightarrow (az = b)$ by *multiplicative cancellation*. Thus $(\exists z \in \mathbb{Z})(az = b)$. $(\exists z \in \mathbb{Z})(az = b) \Leftrightarrow a \mid b$. Therefore, $a \mid b$. Because we derived $a \mid b$ from our initial assumption, we know $(c \neq 0 \wedge ac \mid bc) \Rightarrow (a \mid b)$.

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   (b) Show that $(n \mid x \wedge n \mid y) \Rightarrow (n \mid ax + by)$ for all $n, x, y, a, b \in \mathbb{Z}$.

   Proof. Let $n, x, y, a, b \in \mathbb{Z}$. Assume $n \mid x \wedge n \mid y$. By the definition of *divisibility*, we know $(\exists z \in \mathbb{Z})(nz = x)$ and $(\exists h \in \mathbb{Z})(nh = y)$. Thus, we can express x and y in terms of n as: $x = nz$, $y = nh$. Now take the expression $ax + by$.

   $$\begin{aligned} ax + by &= a(nz) + b(nh) && \text{by equivalence} \\ &= n(az + bh) && \text{by basic factoring} \end{aligned}$$

   Since $a, b, z, h \in \mathbb{Z}$, we know that $az + bh \in \mathbb{Z}$. We can then say $m = az + bh$ where $m \in \mathbb{Z}$. So, we can say $ax + by = nm$. Because $m \in \mathbb{Z}$, by the definition of *divisibility*, $n \mid ax + by$. Because we derived this from our initial assumption, we know $(n \mid x \wedge n \mid y) \Rightarrow (n \mid ax + by)$.

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2. For all $z \in \mathbb{Z}$, show that z is even implies z is not odd.

   Proof. Let $z \in \mathbb{Z}$. We want to show (z is even) $\Rightarrow$ (z is not odd). By the definitions of even and odd, we can replace this with wanting to show $(2 \mid z) \Rightarrow (2 \nmid z - 1)$. To do so, assume $2 \mid z$.
   We will now prove that gcd $(z, z - 1) = 1$.
   T.A.C assume gcd $(n, n + 1) > 1$. By the FLA, there must be a prime $p \in \mathbb{N}$ such that $p \mid gcd(n, n + 1)$. So, $p \mid gcd(n, n + 1) \wedge gcd(n, n + 1) \mid n$. This implies

that $p \mid n \wedge p \mid n+1$. We can combine these linearly to say $p \mid n+1-n$ which is equivalent to $p \mid 1$. Therefore $|p| \leq |1| \Rightarrow p \leq 1$. However, $p > 1$ because p is prime. $\lightning$.
So, we know that $gcd(z, z-1) = 1$. Therefore because $2 \mid z$, $2 \nmid z-1$.

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3. (a) For all $n \in \mathbb{N}$, show that n is odd implies $n+1$ is even.
   Proof. Let $n \in \mathbb{N}$. We want to show n is odd $\Rightarrow$ $n+1$ is even. By the definitions of even and odd, we want to show $(2 \nmid n) \Rightarrow (2 \mid n+1)$. To do so, assume $2 \nmid n$. Now we want to show $2 \mid n+1$. As proven in 2, we know gcd $(n, n+1) = 1$. Therefore, since $2 \nmid n$, $2 \mid n+1$.

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   (b) For all $n \in \mathbb{N}$, show that n is even implies $n+1$ is odd.
   Proof. Let $n \in \mathbb{N}$. We want to show n is even $\Rightarrow$ $n+1$ is odd. By the definitions of even and odd, we want to show $2 \mid n \Rightarrow 2 \nmid n+1$. To do so, assume $2 \mid n$. As proven in 2, we know gcd $(n, n+1) = 1$. Since $2 \mid n$, we know that $2 \nmid n+1$.

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4. Show that $3 \mid n^3 - n$ for all $n \in \mathbb{N}$.
   Proof. We will use a proof by *induction.*
   Basis step:
   We know that every number divides 0 by *Lemma 5.1.* Therefore, $3 \mid 0$. By basic arithmetic we know $3 \mid 0 = 3 \mid (1-1)$. We know $1^3 - 1 = 1 - 1$ by more arithmetic. So, we know $3 \mid 0 \Leftrightarrow 3 \mid 1^3 - 1$.
   Inductive step:
   Let $k \in \mathbb{N}$. Assume $3 \mid k^3 - k$. Now, we need to show $3 \mid (k+1)^3 - (k+1)$. We can say $(k+1)^3 - (k+1) \Leftrightarrow (k^3 - k) + 3k^2 + 3k$ by basic factoring. In our *inductive hypothesis*, we assumed $3 \mid k^3 - k$. Thus, $3p = k^3 - k$ where $p \in \mathbb{Z}$. So, we can substitute our expression and say $(k^3 - k) + 3k^2 + 3k \Leftrightarrow 3p + 3k^2 + 3k$. We can factor 3 out here and see $3p + 3k^2 + 3k \Leftrightarrow 3(p + k^2 + k)$. Since $p, k \in \mathbb{Z}$, $(p + k^2 + k) \in \mathbb{Z}$. Now let $m = (p + k^2 + k)$ where $m \in \mathbb{Z}$. We then obtain $3(p + k^2 + k) \Leftrightarrow 3m$. We know that $3 \mid 3m$ because $3 \cdot m = 3m$. Thus, $3 \mid (k+1)^3 - (k+1)$.
   Therefore by *mathematical induction*, we know $(\forall n \in \mathbb{N})(3 \mid n^3 - n)$.

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5. The *Fibonacci Sequence* is the recursive function $\mathcal{F} : \mathbb{N} \to \mathbb{N}$.

$$\begin{aligned}\mathcal{F}(0) &:= 0\\ \mathcal{F}(1) &:= 1\\ \mathcal{F}(n+2) &:= \mathcal{F}(n+1) + \mathcal{F}(n)\end{aligned}$$

Show that $1 + \sum_{i=0}^{n} \mathcal{F}(i) = \mathcal{F}(n+2)$ for all $n \in \mathbb{N}$.

Proof. We will use a proof by *induction.*

Basis step:

$1 + \sum_{i=0}^{0} \mathcal{F}(i) = 1 + 0 = 1$ by the definitions of *summation and addition.*

$\mathcal{F}(0+2) := \mathcal{F}(0+1) + \mathcal{F}(0)$. By the definition of the *Fibonacci Sequence*, we know $\mathcal{F}(1) = 1$ and $\mathcal{F}(0) = 0$. Therefore, $\mathcal{F}(0+2) = 1 + 0 = 1$ by simple addition.

Therefore, $1 + \sum_{i=0}^{0} \mathcal{F}(i) = \mathcal{F}(0+2)$.

Inductive step:

Let $k \in \mathbb{N}$. Assume $1 + \sum_{i=0}^{k} \mathcal{F}(i) = \mathcal{F}(k+2)$ because this is our inductive hypothesis. Now, we need to show $1 + \sum_{i=0}^{k+1} \mathcal{F}(i) = \mathcal{F}(k+1+2)$.

By the definition of *summation*, $1 + \sum_{i=0}^{k+1} \mathcal{F}(i) = 1 + \sum_{i=0}^{k} \mathcal{F}(i) + \mathcal{F}(k+1)$. By our *inductive hypothesis*, we know $1 + \sum_{i=0}^{k} \mathcal{F}(i) = \mathcal{F}(k+2)$. So, $1 + \sum_{i=0}^{k+1} \mathcal{F}(i) = \mathcal{F}(k+2) + \mathcal{F}(k+1)$. Now, by the definition of the *Fibonacci Sequence*, we know $\mathcal{F}(k+3) := \mathcal{F}(k+2) + \mathcal{F}(k+1)$. Because $k+3 = k+1+2$ by addition, we know $1 + \sum_{i=0}^{k+1} \mathcal{F}(i) = \mathcal{F}(k+1+2)$.

Therefore, by *mathematical induction*, we know $1 + \sum_{i=0}^{n} \mathcal{F}(i) = \mathcal{F}(n+2)$.

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