Probability Homework 9

Will Krzastek

25 November 2024

1. Let $U_1, \ldots, U_n$ be independent uniform random variables on $[0, 1]$. Find $\mathbb{E}(U_{(n)} - U_{(1)})$, where $U_{(n)} = \max\{U_1, \ldots, U_n\}$ and $U_{(1)} = \min\{U_1, \ldots, U_n\}$.
Hint: From the beta distribution, we have

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

for any $a, b > 0$.

The cumulative distribution functions (CDFs) of $U_{(n)}$ and $U_{(1)}$ are:

$$F_{U_{(n)}}(x) = x^n, \quad F_{U_{(1)}}(x) = 1 - (1-x)^n.$$

The probability density functions (PDFs) are:

$$f_{U_{(n)}}(x) = nx^{n-1}, \quad f_{U_{(1)}}(x) = n(1-x)^{n-1}.$$

The expectation of the difference is:

$$\mathbb{E}(U_{(n)} - U_{(1)}) = \mathbb{E}(U_{(n)}) - \mathbb{E}(U_{(1)}).$$

For $U_{(n)}$, we have:

$$\mathbb{E}(U_{(n)}) = \int_0^1 x \cdot f_{U_{(n)}}(x)\,dx = \int_0^1 x \cdot nx^{n-1}\,dx = n\int_0^1 x^n\,dx = n \cdot \frac{1}{n+1} = \frac{n}{n+1}.$$

For $U_{(1)}$, we have:

$$\mathbb{E}(U_{(1)}) = \int_0^1 x \cdot f_{U_{(1)}}(x)\,dx = \int_0^1 x \cdot n(1-x)^{n-1}\,dx = n\int_0^1 x(1-x)^{n-1}\,dx.$$

Using the Beta function:

$$\int_0^1 x(1-x)^{n-1}\,dx = \frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)} = \frac{1 \cdot (n-1)!}{(n+1)!} = \frac{1}{n(n+1)}.$$

Thus:

$$\mathbb{E}(U_{(1)}) = n \cdot \frac{1}{n(n+1)} = \frac{1}{n+1}.$$

Finally:

$$\mathbb{E}(U_{(n)} - U_{(1)}) = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}.$$

2. You have two dice: one with three sides labeled $0, 1, 2$ and one with four sides labeled $0, 1, 2, 3$. Let X_1 be the outcome of rolling the first die, and X_2 the outcome of rolling the second die. The rolls are independent.

(a) What is the joint probability mass function (p.m.f.) of (X_1, X_2)?

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1) \cdot P(X_2 = x_2),$$

where $P(X_1 = x_1) = \frac{1}{3}$ for $x_1 \in \{0, 1, 2\}$ and $P(X_2 = x_2) = \frac{1}{4}$ for $x_2 \in \{0, 1, 2, 3\}$. Hence:

$$P(X_1 = x_1, X_2 = x_2) = \frac{1}{12}, \quad x_1 \in \{0, 1, 2\},\, x_2 \in \{0, 1, 2, 3\}.$$

(b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of (Y_1, Y_2).

Y_1	Y_2	$P(Y_1, Y_2)$
0	0	$\frac{1}{12}$
0	1	$\frac{1}{12}$
2	2	$\frac{1}{12}$
$\vdots$	$\vdots$	$\vdots$

(c) Are Y_1 and Y_2 independent?

No, Y_1 and Y_2 are not independent because the joint distribution $P(Y_1, Y_2)$ does not factorize into $P(Y_1)P(Y_2)$

3. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

(a) What are $\mathbb{E}(Y_1)$ and $\mathbb{E}(Y_2)$?

$\mathbb{E}(Y_1) = \mathbb{E}(Z) = 0$, since Z is symmetric about 0. For $\mathbb{E}(Y_2)$:

$$\mathbb{E}(Y_2) = \mathbb{E}(Z^2) = \mathrm{Var}(Z) + (\mathbb{E}(Z))^2 = 1 + 0 = 1.$$

(b) What is $\mathbb{E}(Y_1Y_2)$?

$\mathbb{E}(Y_1Y_2) = \mathbb{E}(Z \cdot Z^2) = \mathbb{E}(Z^3)$. Since Z is symmetric, all odd moments of Z are 0. Thus:

$$\mathbb{E}(Y_1Y_2) = 0.$$

(c) What is $\text{Cov}(Y_1, Y_2)$?

$\text{Cov}(Y_1, Y_2) = \mathbb{E}(Y_1 Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2) = 0 - (0)(1) = 0.$

(d) Notice that $\mathbb{P}(Y_2 > 1 \mid Y_1 > 1) = 1$. Are Y_1 and Y_2 independent?

$\mathbb{P}(Y_2 > 1 \mid Y_1 > 1) = 1$ implies that Y_1 and Y_2 are not independent, as the condition affects the probability of Y_2.

4. Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week, where $Y_1 \sim \text{Uniform}(0, 1)$. Let Y_2 denote the amount (by weight) sold during the week, where Y_2 is uniform over the interval $[0, Y_1]$ for a given value of Y_1. If the supplier stocked $\frac{3}{4}$ ton, what amount could be expected to be sold during the week?

Conditional on $Y_1 = y_1$, the random variable Y_2 is uniform over $[0, y_1]$. The expectation of Y_2 given $Y_1 = y_1$ is:

$$\mathbb{E}[Y_2 \mid Y_1 = y_1] = \frac{0 + y_1}{2} = \frac{y_1}{2}.$$

If the supplier stocked $Y_1 = \frac{3}{4}$ ton, the expected amount sold is:

$$\mathbb{E}[Y_2 \mid Y_1 = \frac{3}{4}] = \frac{\frac{3}{4}}{2} = \frac{3}{8}.$$

Thus, the expected amount sold is $\frac{3}{8}$ tons.

5. Let T be the average waiting time for a bus.

(a) If $T \sim \text{Uniform}(0, 20)$, suppose you have waited for 10 minutes and the bus has not arrived yet. What is the expectation of your remaining waiting time?

For $T \sim \text{Uniform}(0, 20)$, the conditional distribution of the remaining waiting time after 10 minutes is uniform over $[10, 20]$. The expected value is:

$$\mathbb{E}[\text{Remaining time} \mid T > 10] = \frac{\text{start} + \text{end}}{2} = \frac{10 + 20}{2} = 15 - 10 = 5 \text{ minutes}.$$

Thus, the expected remaining waiting time is 5 minutes.

(b) If $T \sim \text{Exponential}(\lambda)$ with mean 10 minutes (so $\lambda = \frac{1}{10}$), suppose you have waited for 10 minutes and the bus has not arrived yet. What is the expectation of your remaining waiting time? Does this seem counter-intuitive?

For $T \sim \text{Exponential}(\lambda)$ with $\lambda = \frac{1}{10}$, the memoryless property of the exponential distribution implies that the expected remaining waiting time is the same as the original mean, regardless of how long you have already waited:

$$\mathbb{E}[\text{Remaining time} \mid T > 10] = \frac{1}{\lambda} = 10 \text{ minutes}.$$

Does this seem counter-intuitive? Yes, it is counter-intuitive. Intuitively, one might expect that waiting longer without the bus arriving would reduce the remaining expected waiting time. However, the memoryless property of the exponential distribution means that the process resets, and the time elapsed does not affect the expected waiting time.