Probability Homework 11

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1. (Required) Suppose X and Y are i.i.d. random variables, where $\text{Var}(X) < 1$. Show that

$$P(|X - Y| > 2) \leq \frac{\text{Var}(X)}{2}.$$

Using Chebyshev's inequality, we know:

$$P(|X - Y| > 2) \leq \frac{\text{Var}(X - Y)}{4}.$$

Since X and Y are i.i.d., $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 2\text{Var}(X)$. Thus:

$$P(|X - Y| > 2) \leq \frac{2\text{Var}(X)}{4} = \frac{\text{Var}(X)}{2}.$$

2. (Required) Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Let:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Show that the random variable U_n, defined as:

$$U_n = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}},$$

converges to a standard normal distribution function as $n \to \infty$. (Hint: Consider $W_i = X_i - Y_i$, for $i = 1, 2, \ldots, n$.)

Define $W_i = X_i - Y_i$. Then $\mu_W = \mu_1 - \mu_2$ and $\sigma_W^2 = \sigma_1^2 + \sigma_2^2$. The sample mean of W_i is:

$$\bar{W} = \frac{1}{n}\sum_{i=1}^{n} W_i = \bar{X} - \bar{Y}.$$

The standardized version of $\bar{W}$ is:

$$U_n = \frac{\bar{W} - \mu_W}{\sqrt{\sigma_W^2/n}} = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}.$$

By the Central Limit Theorem, $\bar{W} \sim N(\mu_W, \sigma_W^2/n)$ for large n, so $U_n \sim N(0, 1)$ as $n \to \infty$.

3. (Required) Suppose $\{X_k\}_{k \geq 1}$ are i.i.d. Unif$(0, 1)$ random variables, and for each integer $k \geq 1$, define $Y_n := \min(X_1, \ldots, X_n)$. Show that $Y_n \xrightarrow{P} 0$ as $n \to \infty$.

   The cumulative distribution function of $\min(X_1, \ldots, X_n)$ is given by:

   $$F_{Y_n}(y) = P(\min(X_1, \ldots, X_n) \leq y) = 1 - P(X_1 > y)^n = 1 - (1 - y)^n, \quad 0 \leq y \leq 1.$$

   For $\epsilon > 0$, consider:

   $$P(Y_n > \epsilon) = 1 - F_{Y_n}(\epsilon) = (1 - \epsilon)^n.$$

   As $n \to \infty$, $(1 - \epsilon)^n \to 0$ for any $\epsilon > 0$. Therefore, $Y_n \xrightarrow{P} 0$.

4. (Optional) An experiment is designed to test whether operator A or operator B gets the job of operating a new machine. Each operator is timed on 50 independent trials involving the performance of a certain task using the machine. If the sample means for the 50 trials differ by more than 1 second, the operator with the smaller mean time gets the job. Otherwise, the experiment is considered to end in a tie. If the standard deviations of times for both operators are assumed to be 2 seconds, what is the probability that operator A will get the job even though both operators have equal ability?

   Let $\bar{X}_A$ and $\bar{X}_B$ denote the sample means of operator A and operator B, respectively. Under the null hypothesis (both operators have equal ability), $\bar{X}_A - \bar{X}_B$ follows a normal distribution:

   $$\bar{X}_A - \bar{X}_B \sim N(0, \sigma^2/n + \sigma^2/n) = N(0, 2\sigma^2/n).$$

   Here, $\sigma = 2$ and $n = 50$, so:

   $$\bar{X}_A - \bar{X}_B \sim N(0, 2 \cdot 2^2/50) = N(0, 0.16).$$

   The probability that operator A gets the job is equivalent to $P(\bar{X}_A - \bar{X}_B < -1)$:

   $$P(\bar{X}_A - \bar{X}_B < -1) = P\left(Z < \frac{-1 - 0}{\sqrt{0.16}}\right) = P(Z < -2.5),$$

   where $Z \sim N(0, 1)$. Using standard normal tables:

   $$P(Z < -2.5) \approx 0.0062.$$

   Thus, the probability that operator A gets the job is approximately 0.62%.