Probability Homework 8

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1. (Required) The joint probability mass (frequency) function of two discrete random variables, X and Y, is given in the following table: Find the marginal probability mass

	X			
Y	1	2	3	4
1	0.10	0.05	0.02	0.02
2	0.05	0.20	0.05	0.02
3	0.02	0.05	0.20	0.04
4	0.02	0.02	0.04	0.10

Table 1: Joint Probability Mass Function of X and Y

functions of X and Y.

The marginal probability mass function of X is calculated by summing each row's values:

$$P(X = 1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$$

$$P(X = 2) = 0.05 + 0.20 + 0.05 + 0.02 = 0.32$$

$$P(X = 3) = 0.02 + 0.05 + 0.20 + 0.04 = 0.31$$

$$P(X = 4) = 0.02 + 0.02 + 0.04 + 0.10 = 0.18$$

The marginal probability mass function of Y is calculated by summing each column's values:

$$P(Y = 1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$$

$$P(Y = 2) = 0.05 + 0.20 + 0.05 + 0.02 = 0.32$$

$$P(Y = 3) = 0.02 + 0.05 + 0.20 + 0.04 = 0.31$$

$$P(Y = 4) = 0.02 + 0.02 + 0.04 + 0.10 = 0.18$$

2. (Required) Find the joint and marginal densities corresponding to the cdf

$$F(x,y) = (1 - e^{-\alpha x})(1 - e^{-\beta y}), \quad x \geq 0, y \geq 0, \alpha > 0, \beta > 0$$

The joint probability density function (PDF) $f(x,y)$ is obtained by taking the mixed partial derivative of $F(x,y)$ with respect to x and y:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \, \partial y}.$$

Start by differentiating $F(x,y)$ with respect to x:

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x} \left(1 - e^{-\alpha x}\right)\left(1 - e^{-\beta y}\right) = \alpha e^{-\alpha x}(1 - e^{-\beta y}).$$

Now differentiate this result with respect to y:

$$f(x,y) = \frac{\partial}{\partial y} \left(\alpha e^{-\alpha x}(1 - e^{-\beta y})\right) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} = \alpha\beta e^{-\alpha x} e^{-\beta y}.$$

Thus, the joint density is

$$f(x,y) = \alpha\beta e^{-\alpha x} e^{-\beta y}, \quad x \geq 0, y \geq 0.$$

The marginal density $f_X(x)$ is obtained by integrating the joint density over y:

$$f_X(x) = \int_0^\infty f(x,y) \, dy = \int_0^\infty \alpha\beta e^{-\alpha x} e^{-\beta y} \, dy.$$

Since $e^{-\alpha x}$ does not depend on y, we can factor it out:

$$f_X(x) = \alpha e^{-\alpha x} \int_0^\infty \beta e^{-\beta y} \, dy.$$

Now, integrate with respect to y:

$$\int_0^\infty \beta e^{-\beta y} \, dy = \beta \int_0^\infty e^{-\beta y} \, dy = \beta \cdot \frac{1}{\beta} = 1.$$

Thus,

$$f_X(x) = \alpha e^{-\alpha x}, \quad x \geq 0.$$

Similarly, the marginal density $f_Y(y)$ is obtained by integrating the joint density over x:

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty \alpha\beta e^{-\alpha x} e^{-\beta y}\, dx.$$

Since $e^{-\beta y}$ does not depend on x, we can factor it out:

$$f_Y(y) = \beta e^{-\beta y} \int_0^\infty \alpha e^{-\alpha x}\, dx.$$

Now, integrate with respect to x:

$$\int_0^\infty \alpha e^{-\alpha x}\, dx = \alpha \int_0^\infty e^{-\alpha x}\, dx = \alpha \cdot \frac{1}{\alpha} = 1.$$

Thus,

$$f_Y(y) = \beta e^{-\beta y}, \quad y \geq 0.$$

3. (Required) The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

with time measured in minutes. Find

   (a) $P(Y_1 < 2, Y_2 > 1)$,

   To calculate $P(Y_1 < 2, Y_2 > 1)$, we set up the double integral over the region $0 \leq y_2 \leq y_1 < \infty$, with the conditions $Y_1 < 2$ and $Y_2 > 1$:

$$P(Y_1 < 2, Y_2 > 1) = \int_1^2 \int_{y_2}^2 e^{-y_1}\, dy_1\, dy_2.$$

   1. Integrate with respect to y_1:

$$\int_{y_2}^2 e^{-y_1}\, dy_1 = \left[-e^{-y_1}\right]_{y_1=y_2}^{y_1=2} = e^{-y_2} - e^{-2}.$$

   2. Integrate with respect to y_2:

$$\int_1^2 \left(e^{-y_2} - e^{-2}\right) dy_2 = \int_1^2 e^{-y_2}\, dy_2 - \int_1^2 e^{-2}\, dy_2.$$

3. Evaluate each integral: For $\int_1^2 e^{-y_2}\,dy_2$:

$$\int_1^2 e^{-y_2}\,dy_2 = \left[-e^{-y_2}\right]_1^2 = e^{-1} - e^{-2}.$$

For $\int_1^2 e^{-2}\,dy_2$:

$$\int_1^2 e^{-2}\,dy_2 = e^{-2} \cdot (2-1) = e^{-2}.$$

So,

$$P(Y_1 < 2, Y_2 > 1) = (e^{-1} - e^{-2}) - e^{-2} = e^{-1} - 2e^{-2}.$$

(b) $P(Y_1 \geq 2Y_2)$,

We want $P(Y_1 \geq 2Y_2)$, which is the probability over the region $0 \leq y_2 \leq y_1/2$:

$$P(Y_1 \geq 2Y_2) = \int_0^\infty \int_0^{y_1/2} e^{-y_1}\,dy_2\,dy_1.$$

1. Integrate with respect to y_2:

$$\int_0^{y_1/2} e^{-y_1}\,dy_2 = e^{-y_1} \cdot \frac{y_1}{2}.$$

2. Integrate with respect to y_1:

$$P(Y_1 \geq 2Y_2) = \int_0^\infty \frac{y_1}{2} e^{-y_1}\,dy_1 = \frac{1}{2}\int_0^\infty y_1 e^{-y_1}\,dy_1.$$

3. Evaluate the integral using integration by parts:

$$\int_0^\infty y_1 e^{-y_1}\,dy_1 = 1.$$

So,

$$P(Y_1 \geq 2Y_2) = \frac{1}{2}.$$

(c) $P(Y_1 - Y_2 > 1)$. (Notice that $Y_1 - Y_2$ denotes the time spent at the service window.)

To calculate $P(Y_1 - Y_2 > 1)$, rewrite it as $P(Y_1 > Y_2 + 1)$:

$$P(Y_1 - Y_2 > 1) = \int_0^\infty \int_{y_2+1}^\infty e^{-y_1}\,dy_1\,dy_2.$$

1. Integrate with respect to y_1:

$$\int_{y_2+1}^{\infty} e^{-y_1}\, dy_1 = \left[-e^{-y_1}\right]_{y_1=y_2+1}^{\infty} = e^{-(y_2+1)} = e^{-y_2}e^{-1}.$$

2. Integrate with respect to y_2:

$$P(Y_1 - Y_2 > 1) = e^{-1}\int_0^{\infty} e^{-y_2}\, dy_2 = e^{-1} \cdot \left[-e^{-y_2}\right]_0^{\infty} = e^{-1} \cdot 1 = e^{-1}.$$

4. (Optional) Three players play 10 independent rounds of a game, and each player has probability 1/3 of winning each round. Find the joint distribution of the numbers of games won by each of the three players.

   Let X_1, X_2, and X_3 represent the numbers of games won by Players 1, 2, and 3, respectively. Since each round has a single winner and each player has an equal probability of winning, the total number of games won across all players must add up to 10. Thus, we have:

$$X_1 + X_2 + X_3 = 10.$$

   Each game can result in a win for Player 1, Player 2, or Player 3, with probability $\frac{1}{3}$ for each player. Since the rounds are independent, the number of wins for each player follows a multinomial distribution.

   The joint distribution of X_1, X_2, and X_3 is given by the multinomial distribution:

$$P(X_1 = k_1, X_2 = k_2, X_3 = k_3) = \frac{10!}{k_1!\, k_2!\, k_3!}\left(\frac{1}{3}\right)^{k_1}\left(\frac{1}{3}\right)^{k_2}\left(\frac{1}{3}\right)^{k_3},$$

   where $k_1 + k_2 + k_3 = 10$ and $k_1, k_2, k_3 \geq 0$.

   Multinomial Coefficient: The term $\frac{10!}{k_1!\, k_2!\, k_3!}$ represents the number of distinct ways to assign k_1 wins to Player 1, k_2 wins to Player 2, and k_3 wins to Player 3 out of 10 rounds.

   Probability of Wins: Since each player has a probability of $\frac{1}{3}$ of winning a single round, the probability of Player 1 winning exactly k_1 games, Player 2 winning exactly k_2 games, and Player 3 winning exactly k_3 games is $\left(\frac{1}{3}\right)^{k_1}\left(\frac{1}{3}\right)^{k_2}\left(\frac{1}{3}\right)^{k_3} = \left(\frac{1}{3}\right)^{10}$.

   Thus, the joint distribution of the numbers of games won by each player is:

$$P(X_1 = k_1, X_2 = k_2, X_3 = k_3) = \frac{10!}{k_1!\, k_2!\, k_3!}\left(\frac{1}{3}\right)^{10},$$

   subject to $k_1 + k_2 + k_3 = 10$ and $k_1, k_2, k_3 \geq 0$.

So, the joint distribution of the numbers of games won by each of the three players is a multinomial distribution with parameters $n = 10$ and probabilities $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{3}$, and $p_3 = \frac{1}{3}$.

5. (Optional) Let $F(x, y)$ be the cdf for a bivariate random variable (X, Y). For any $a_1 < a_2$ and $b_1 < b_2$, we want to prove that

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)$$

By the definition of the CDF $F(x, y)$, we can rewrite each of these probabilities as:

$$\begin{aligned}
P(X \leq a_2, Y \leq b_2) &= F(a_2, b_2), \\
P(X \leq a_1, Y \leq b_2) &= F(a_1, b_2), \\
P(X \leq a_2, Y \leq b_1) &= F(a_2, b_1), \\
P(X \leq a_1, Y \leq b_1) &= F(a_1, b_1).
\end{aligned}$$

Substituting these expressions into the equation, we obtain

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1).$$