Probability Homework 10

Will Krzastek

8 December 2024

Problem 1 (Required)

In Exercise 5.16, Y_1 and Y_2 denoted the proportions of time during which employees I and II actually performed their assigned tasks during a workday. The joint density of Y_1 and Y_2 is given by:

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal density functions for Y_1 and Y_2.

$$\int_0^1 (y_1 + y_2)\, dy_2 = y_1 + \frac{1}{2}, \quad 0 \leq y_1 \leq 1.$$

$$\int_0^1 (y_1 + y_2)\, dy_1 = y_2 + \frac{1}{2}, \quad 0 \leq y_2 \leq 1.$$

(b) Find $P(Y_1 \geq 0.5 \mid Y_2 \geq 0.5)$.

$$P(Y_1 \geq 0.5 \mid Y_2 \geq 0.5) = \frac{\int_{0.5}^1 \int_{0.5}^1 (y_1 + y_2)\, dy_1\, dy_2}{\int_{0.5}^1 \int_0^1 (y_1 + y_2)\, dy_1\, dy_2}.$$

(c) If employee II spends exactly 50% of the day working on assigned duties, find the probability that employee I spends more than 75% of the day working on similar duties.

$$P(Y_1 > 0.75 \mid Y_2 = 0.5) = \frac{\int_{0.75}^1 (y_1 + 0.5)\, dy_1}{y_2 + \frac{1}{2}}.$$

Problem 2 (Required)

Assume that Y denotes the number of bacteria per cubic centimeter in a particular liquid and that Y has a Poisson distribution with parameter Z. Further assume that Z varies from location to location and has a gamma distribution with parameters α and β, where α is a positive integer. If we randomly select a location, what is the:

(a) Find the expected number of bacteria per cubic centimeter.

$$E[Y] = \frac{\alpha}{\beta}.$$

(b) Find the standard deviation of the number of bacteria per cubic centimeter.

$$\mathrm{SD}(Y) = \sqrt{\frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2}}.$$

Problem 3 (Required)

A random variable Y has the density function:

$$f(y) = \begin{cases} e^y, & y < 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $E(e^{3Y/2})$.

$$E(e^{3Y/2}) = \int_{-\infty}^{0} e^{3y/2} e^y \, dy = \int_{-\infty}^{0} e^{5y/2} \, dy = \frac{2}{5}.$$

(b) Find the moment-generating function for Y.

$$M_Y(t) = \int_{-\infty}^{0} e^{ty} e^y \, dy = \int_{-\infty}^{0} e^{(t+1)y} \, dy = \frac{1}{1+t}, \quad t > -1.$$

(c) Use the moment-generating function to find $\mathrm{Var}(Y)$.

$$\mathrm{Var}(Y) = M_Y''(0) - \left[M_Y'(0)\right]^2 = \frac{2}{1^3} - (1)^2 = 1.$$

Problem 4 (Optional)

Consider the bivariate random variable with density:

$$f(y_1, y_2) = \begin{cases} 2, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, 0 \le y_1 + y_2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $E(Y_1 + Y_2)$.

$$E[Y_1 + Y_2] = \int_0^1 \int_0^{1-y_1} 2(y_1 + y_2) \, dy_2 \, dy_1 = \frac{2}{3}.$$

(b) Find $\mathrm{Var}(Y_1 + Y_2)$.

$$\mathrm{Var}(Y_1 + Y_2) = \mathrm{E}[(Y_1 + Y_2)^2] - (\mathrm{E}[Y_1 + Y_2])^2.$$

Problem 5 (Optional)

You begin with a stick of length 1 and break it at a point, chosen uniformly at random. You then take the left piece and break it once again at a uniformly random chosen point. What is the expectation and variance of the length of the left piece after the breaking?

(a) Find the expectation.

$$E[L] = \int_0^1 x^2 \, dx = \frac{1}{3}.$$

(b) Find the variance.

$$\mathrm{Var}(L) = \mathrm{E}[L^2] - (\mathrm{E}[L])^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}.$$