Probability Homework 2

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1. (required) A fair coin is tossed three times.

   (a) What is the probability of two or more heads given that there was at least one head?

   Here, we can use Bayes Rule: $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$. In this case, $P(B) = 7/8$, as there are 7 cases where a coin can be flipped 8 times and has at least one head. $P(A) = 4/8 = 1/2$ as there are 4 out of 8 cases with 2 or more heads: {(HHH), (HHT), (HTH), (THH)}. We know that $P(B|A) = 1$ because if there is at least two heads there has to be at least one head. So by Bayes Rule, $P(A|B) = \frac{4/8}{7/8} = \frac{4}{7} =$ **0.57143**.

   (b) What is the probability of two or more heads given that there was at least one tail?

   Here, we will use the conditional probability formula. $P(A|B) = \frac{P(A \cap B)}{P(B)}$. We know that $P(B) = 7/8$ as there are 7 cases with at least one tail (everything except HHH). Now, $P(A) = 4/8$, so $P(A \cap B) = 3/8$, as these are the cases: {(HHT), (HTH), (THH)}. So by the rule of conditional probability: $P(A|B) = \frac{3/8}{7/8} = \frac{3}{7} =$ **0.4286**.

2. (required) If a parent has genotype Aa, he transmits either A or a to an offspring (each with a $1/2$ chance). The gene he transmits to one offspring is independent of the one he transmits to another. Consider a parent with three children and the following events: A={children 1 and 2 have the same gene}, B={children 1 and 3 have the same gene}, C={children 2 and 3 have the same gene}. Show that these events are pairwise independent but not mutually independent.

   Observe the definition of pairwise independence: $(P(A \cap B) = P(A) \cdot P(B)) \wedge (P(B \cap C) = P(B) \cdot P(C)) \wedge (P(A \cap C) = P(A) \cdot P(C))$, and the definition of mutual indepence: $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$. So, to prove that these events are pairwise independent and not mutually independent, we need to show that any two events are independent, but all three are not independent with each other.

   Observe that for 2 children to have the same genes, they can either both have A or both have a. There are $2^2 = 4$ total combinations, so $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$. Thus, $P(A) \cdot P(B) = \frac{1}{4}$. Observe that for $P(A \cap B)$, all 3 children must have the same gene, which happens when all 3 have A or all 3 have a. There are $2^3 = 8$ total combinations, so $P(A \cap B) = \frac{2}{8} = \frac{1}{4}$. Thus, $P(A \cap B) = P(A) \cdot P(B)$. This same logic can be applied to A and C, and B and C because $P(A) = P(B) = P(C)$. So, these events are pairwise independent.

Now, observe that $P(A \cap B \cap C) = \frac{1}{4}$ because it again implies that all 3 children have the same gene, which we defined as $\frac{1}{4}$ above. However, $P(A) \cdot P(B) \cdot P(C) = \frac{1}{8} \neq \frac{1}{4}$, so $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$, so these events are pairwise independent but not mutually independent.

3. (required) An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

   (a) What is the probability that the second ball drawn is white?

   We will use the law of total probability here: $P(W_2) = P(W_2|R_1) \cdot P(R_1) + P(W_2|W_1) \cdot P(W_1)$. $P(R_1)$ and $P(W_1)$ are very simple to calculate, as there are 5 balls initially with 3 being red and 2 being white, so $P(R_1) = \frac{3}{5}$ and $P(W_1) = \frac{2}{5}$.

   Now, observe that $P(W_2|R_1)$ is the probability of drawing a white ball after a red ball, which means that there are now 6 balls with 4 being red (so 2 are white), so $P(W_2|R_1) = \frac{2}{6} = \frac{1}{3}$. Also observe that $P(W_2|W_1)$ is the probability of drawing a white ball after a white ball, which means there are 6 balls with 3 being white and 3 being red, so $P(W_2|W_1) = \frac{3}{6} = \frac{1}{2}$.

   Now we can combine all this and see that $P(W_2) = \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = \mathbf{0.40}$.

   (b) If the second ball drawn is white, what is the probability that the first ball drawn was red?

   To calculate $P(R_1|W_2)$ we will use Bayes Rule: $P(R_1|W_2) = \frac{P(W_2|R_1)P(R_1)}{P(W_2)}$. We calculated a lot of these above: $P(W_2|R_1) = \frac{1}{3}$, $P(R_1) = \frac{3}{5}$, and $P(W_2) = \frac{2}{5}$. Combining all that, we see that $P(R_1|W_2) = \frac{1/3 \cdot 3/5}{2/5} = \frac{1/5}{2/5} = \frac{1}{2} = \mathbf{0.50}$.

4. (optional) Suppose that A and B are mutually disjoint events, with P(A) $>$ 0 and P(B) $<$ 1. Are A and B independent? Prove your answer.

   A and B being mutually disjoint events means that $P(A \cap B) = 0$. To be independent, $P(A \cap B) = P(A) \cdot P(B)$, so in this case $P(A) \cdot P(B) = 0$. We know that then $P(A) = 0 \vee P(B) = 0$. In the question, we learn that $P(A) > 0$, so for these two events to be independent, we need $P(B) = 0$. **Therefore, these events are independent if and only if P(B) $=$ 0**.

5. (optional) A communications network has a built-in safeguard system against failures. In this system if line I fails, it is bypassed and line II is used. If line II also fails, it is

bypassed and line III is used. The probability of failure of any one of these three lines is .01, and the failures of these lines are independent events. What is the probability that this system of three lines does not completely fail?

Observe that the chances of each line failing are $P(L_1) = P(L_2) = P(L_3) = 0.01$. These events are independent, so $P(L_1 \cap L_2 \cap L_3) = P(L_1) \cdot P(L_2) \cdot P(L_3) = 0.01 \cdot 0.01 \cdot 0.01 = 0.000001$. This is then the chance of the system failing. So the chancof the system not failing is $1 - P(L_1 \cap L_2 \cap L_3) =$**0.999999**.