Probability Homework 4

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1. (Required) Twenty identical looking packets of white power are such that 15 contain cocaine and 5 do not. Four packets were randomly selected, and the contents were tested and found to contain cocaine. Two additional packets were selected from the remainder and sold by undercover police officers to a single buyer. What is the probability that the 6 packets randomly selected are such that the first 4 all contain cocaine and the 2 sold to the buyer do not?

   First, we will observe the probability of the first four packets containing cocaine:
   $\frac{15}{20} \times \frac{14}{19} \times \frac{13}{18} \times \frac{12}{17} = \frac{32760}{116280} = 0.2816.$
   Next, we'll observe the probability that the next two packets do not contain cocaine:
   $\frac{5}{16} \times \frac{4}{15} = \frac{20}{240} = 0.0833.$
   The combined probability is just $0.2816 \times 0.0833 = 0.0235$.

2. (Required) A salesperson has found that the probability of a sale on a single contact is approximately 0.03. If the salesperson contacts 100 prospects, what is the probability of making at least one sale? Please provide an accurate probability and an approximated probability, and comment on whether they are close to each other.

   Observe that $p = 0.03$, so $1 - p = 0.97$. The salesperson contacts $n = 100$ prospects, so observe the exact probability of failing all contacts is:
   $(1-p)^n = (0.97)^{100} = 0.0476.$
   The exact probability of making at least one sale is just $1 - (0.97)^{100} = 0.9524.$

   Now we'll find the approximate probability using a Poisson Approximation since n is large and p is small. Observe,
   $P(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ where $\lambda = np = 100 \times 0.03 = 3$ and $k = 0$.
   So $P(0) = e^{-\lambda} = e^{-3} = 0.0498.$
   So the approximate probability of making at least one sale is $1 - 0.0498 = 0.9502.$

   So, the exact probability $= 0.9524$ and approximate probability $= 0.9502$, which are obviously very close together because the Poisson approximation works very well when n is large and p is small.

3. (Required) Suppose that in a city, the number of suicides can be approximated by a Poisson process with $\mu = 0.33$ per month.

   (a) Find the probability of k suicides in a year for $k = 0, 1, 2, \ldots$ What is the most probable number of suicides?

   We will use a Poisson process where $\lambda = 0.33 \times 12 = 3.96$, as there are 0.33 suicides per month and 12 months in a year.
   $P(k) = \frac{e^{-\lambda}\lambda^k}{k!}$. We can substitute $\lambda = 3.96$:
   $P(k) = \frac{e^{-3.96}3.96^k}{k!}$. We'll use this formula to compute $P(k)$ for $k = 0, 1, 2, \ldots$:

$$
\begin{aligned}
P(0) &= \frac{e^{-3.96}3.96^0}{0!} = 0.019 \\
P(1) &= \frac{e^{-3.96}3.96^1}{1!} = e^{-3.96} \cdot 3.96 = 0.075 \\
P(2) &= \frac{e^{-3.96}3.96^2}{2!} = e^{-3.96} \cdot \frac{3.96^2}{2} = 0.149 \\
P(3) &= \frac{e^{-3.96}3.96^3}{3!} = e^{-3.96} \cdot \frac{3.96^3}{6} = 0.197 \\
P(4) &= \frac{e^{-3.96}3.96^4}{4!} = e^{-3.96} \cdot \frac{3.96^4}{24} = 0.195 \\
P(5) &= \frac{e^{-3.96}3.96^5}{5!} = e^{-3.96} \cdot \frac{3.96^5}{120} = 0.154 \\
P(6) &= \frac{e^{-3.96}3.96^6}{6!} = e^{-3.96} \cdot \frac{3.96^6}{720} = 0.102 \\
P(7) &= \frac{e^{-3.96}3.96^7}{7!} = e^{-3.96} \cdot \frac{3.96^7}{5040} = 0.058
\end{aligned}
$$

   From this, we see that the value of k that maximizes $P(k)$ is $k = 3$, since $P(3) = 0.197$. Also observe that $P(4)$ is only 0.002 less than $P(3)$, so $k = 4$ is nearly just as probable as $k = 3$. However from these calculations, 3 suicides per month is the most probable amount.

   (b) What is the probability of two suicides in one week?

   We'll use the convention there are 4.33 weeks in a month. So, $\lambda = \frac{0.33}{4.33} = 0.0762$.
   We are going to calculate $P(2)$. Observe,
   $P(2) = \frac{e^{-0.0762} \cdot (0.0762)^2}{2!} = \frac{0.9266 \cdot 0.0058}{2} = \frac{0.0054}{2} = 0.0027$.

4. (Optional) Phone calls are received at a certain residence as a Poisson process with parameter $\mu = 2$ per hour.

   (a) If Diane takes a 10-min. shower, what is the probability that the phone rings during that time?

   Observe, 10 minutes $= \frac{1}{6}$ hours. So, $\lambda = 2 \times \frac{1}{6} = \frac{1}{3} = 0.3333$.
   The probability of *not* receiving a phone call is $P(0) = e^{-\lambda} = e^{-0.3333} = 0.7165$.
   So, the probability of receiving a phone call is just $1 - 0.7165 = 0.2835$

(b) How long can her shower be if she wishes the probability of receiving no phone calls to be at most 0.5?

We need to find a λ such that $P(0) = e^{-\lambda} \leq 0.5$. Since she receives 2 phone calls per hours, the Poisson parameter can be represented as $\lambda = 2t$ where t is hours. Thus, $P(0) = e^{-2t}$. Observe,

$$\begin{aligned} e^{-2t} &\leq 0.5 \\ -2t &\leq \ln(0.5) \\ t &\geq \frac{-\ln(0.5)}{2} \\ t &\geq \frac{0.6931}{2} \\ t &\geq 0.3465 \end{aligned}$$

So, $t \geq 0.3465$ hours. $0.3465 \times 60 =$ 20.79 minutes. Thus, her shower can be no longer than 20.79 minutes if she wishes the probability of receiving no phone calls to be at most 0.5.

5. (Optional) Five cards are dealt at random and without replacement from a standard deck of 52 cards. What is the probability that the hand contains all 4 aces if it is known that it contains at least 3 aces?

We will use Bayes Rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ where A is the event that a hand contains 4 aces, and B is the event that a hand contains at least 3 aces.
We know that there are $P(A) = \binom{4}{4}\binom{48}{1} = 48$ hands that contain all 4 aces. We'll now calculate how many hands contain 3 aces.
$P(B) = \binom{4}{3}\binom{48}{2} = 4 \cdot 1128 = 4512$. This is only hands with 3 aces, and we are looking for all cases of at least 3 aces, so we must account for the cases where there are all 4 aces: $4512 + 48 = 4560 = P(B)$. Obviously, if there are 4 aces, we know that there must be at least 3 aces, so $P(B|A) = 1$. Now, we can use Bayes Rule to find $P(A|B)$:
$P(A|B) = \frac{48}{4560} = \frac{1}{95} = 0.01053$.