Probability Homework 1

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1. (required) Two six sided die are thrown sequentially, and the face values that come up are recorded.

   (a) List the sample space.

   S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

   (b) List the elements that make up the following events: (1) A = the sum of the two values is at least 5, (2) B = the value of the first die is higher than the value of the second, (3) C = the first value is 4.

   A = {(1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}
   B = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}
   C = {(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)}

   (c) List the elements of the following events: (1) $A \cap C$ (2) $B \cup C$ (3) $A \cap (B \cup C)$.

   $A \cap C$ = {(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)}
   $B \cup C$ = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}
   $A \cap (B \cup C)$ = {(3, 2), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}

2. (required) In a game of poker, what is the probability that a five-card hand will contain (a) a straight (defined as five cards in an unbroken numerical sequence, with the exclusion of a straight flush), (b) four of a kind, and (c) a full house (three cards of one value and two cards of another value)?

A deck of cards has 52 cards, and we are choosing 5 of them. So, we can represent the amount of possibilities as:

$$\binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960 \tag{1}$$

(a) Straight (NOT straight flush)

There are 10 possible sequences of straights:
$(A, 2, 3, 4, 5), (2, 3, 4, 5, 6), \ldots (10, J, Q, K, A)$.
There are 4 possible suits we can choose from independently, so there are 4^5 ways to assign 4 suits to 5 cards. Thus, there are $4^5 \times 10 = 1024 \times 10 = 10,240$ ways to make a straight. However, we must account for straight flushes. Since there are 10 possible combinations of straights and 4 suits, we can represent this possibility as $4 \times 10 = 40$. So, there are $10,240 - 40 = 10,200$ straights without straight flushes.
So, P(straight) =

$$\frac{10,200}{2,598,960} = 0.00392465 = 0.3925\% \tag{2}$$

So, the probability of a straight is around 0.3925%.

(b) Four of a kind

There are 13 ranks of cards in the deck. Since we have 5 cards, we are forced to choose 4 cards of the same rank, and then 1 kicker which can be any card. For the first four cards, we have 13 possibilities (the ranks). For the last card, we have the choice of 12 ranks and 4 suits: $12 \times 4 = 48$. Therefore, we can represent the amount of four of a kind hands as $13 \times 48 = 624$.
So, P(four of a kind) =

$$\frac{624}{2,598,960} = 0.0002401 = 0.024\% \tag{3}$$

So, the probability of a four of a kind is around 0.024%.

(c) Full house

For the first 3 cards, we can choose their rank in 13 ways (there are 13 ranks). For these 3 cards, we can choose their suit in one of 4 ways (we are choosing 3 of 4 suits and $\binom{4}{3} = 4$). For the next 2 cards, we can choose their rank in 12 ways (there are 12 ranks we can choose from) and we can choose their suits in $\binom{4}{2} = 6$ ways. We can multiply these all together and get: $13 \times 4 \times 12 \times 6 = 3,744$ possible full house hands.
So, P(full house) =

$$\frac{3,744}{2,598,960} = 0.00144 = 0.144\% \tag{4}$$

So, the probability of a full house is around 0.144%.

3. (required) A fair coin is tossed five times. What is the probability of getting at least 3 consecutive heads?

   A coin toss has 2 outcomes, and we are tossing it 5 times, so there are $2^5 = 32$ possibilities. There are 5 possible outcomes of exactly 3 heads in a row: HHHTT, HHHTH, THHHT, TTHHH, HTHHH.
   There are 2 possible outcomes of 4 heads in a row: HHHHT, THHHH.
   There is 1 outcome of 5 heads in a row: HHHHH.
   Therefore,
   $$\frac{5+2+1}{32} = \frac{8}{32} = 0.25 = 25\% \tag{5}$$
   So, there is a 25% chance there are at least 3 consecutive heads in 5 coin tosses.

4. (choice) A balanced die is tossed six times, and the number on the uppermost face is recorded each time. What is the probability that the numbers recorded are 1, 2, 3, 4, 5, and 6 in any order?

   Since a die has 6 sides and is rolled 6 times, there are 6^6 possibiliites. We are interested in scenarios where the numbers 1, 2, 3, 4, 5, 6 all appear in any order, which is the permutation of 1 through 6, or 6!. Therefore, we can represent this probability as $\frac{6!}{6^6}$.
   $$\frac{6!}{6^6} = \frac{720}{46656} = 0.0154 = 1.54\% \tag{6}$$
   So, there is a 1.54% chance that the numbers 1, 2, 3, 4, 5, 6 appear after 6 rolls.

5. (choice) Six male and six female dancers perform the Virginia reel. This dance requires that they form a line consisting of six male/female pairs. How many such arrangements are there?

   There are 6 males, and they can be arranged in any order, which we represent as 6!.
   The same thing goes for females, as we can arrange 6 females in 6! ways.
   $$6! \times 6! = 720 \times 720 = 518,400 \tag{7}$$
   Therefore, there are 518,400 ways to arrange the six male and six female dancers in male/female pairs.