Probability Homework 5

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1. (Required) Let $F(x) = 1 - \exp(-\alpha x^\beta)$ for $x \geq 0, \alpha > 0, \beta > 0$, and $F(x) = 0$ for $x < 0$. Show that F is a cdf, and find the corresponding density.

   Property 1: Non-decreasing. The exp function is always positive and decreases as the argument inside increases, so $\exp(-\alpha x^\beta)$ is decreasing as x is increasing. Therefore, $F(x)$ is non-decreasing for $x \geq 0$. For $x < 0$, $F(x) = 0$, so it is constant and non-decreasing.
   Property 2: $\lim_{x\to-\infty} F(x) = 0$. For $x < 0$, $F(x) = 0$, so obviously $\lim_{x\to-\infty} F(x) = 0$.
   Property 3: $\lim_{x\to+\infty} F(x) = 1$. As x approaches $+\infty$, we have: $\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty}(1 - \exp(-\alpha x^\beta)) = 1 - 0 = 1$.
   Property 4: Right continous. Since $F(x) = 0$ for $x < 0$ and $F(x) = 1 - \exp(-\alpha x^\beta)$ is a continous function for $x \geq 0$, so $F(x)$ is right continous.
   Therefore $F(x)$ satisfies all the properties of a cdf.
   Now we will show the corresponding density.
   We find the pdf by taking $\frac{d}{dx}F(x)$ when $F(x)$ is a cdf. For $x \geq 0$:

   $$\begin{aligned} f(x) &= \frac{d}{dx}(1 - \exp(-\alpha x^\beta)) \\ &= \exp(-\alpha x^\beta) \cdot \frac{d}{dx}(-\alpha x^\beta) \\ &= \alpha\beta x^{\beta-1}\exp(-\alpha x^\beta) \end{aligned}$$

   This is the pdf for $x \geq 0$. When $x < 0$, the pdf is just 0.

2. (Required) Suppose that X has the density function $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise.

   (a) Find c.

   Given $f(x) = cx^2$ for $0 \leq x \leq 1$. Observe,

   $$\begin{aligned} &\int_0^1 f(x)dx = 1 \\ &c\int_0^1 x^2 dx = 1 \\ &c[\frac{x^3}{3}]\bigg|_0^1 = c(\frac{1^3}{3} - \frac{0^3}{3}) = \frac{c}{3} \end{aligned}$$

Setting $\frac{c}{3} = 1$ yields $c = 3$.

(b) Find the cdf.

$$F(x) = \int_0^x f(t)dt = \int_0^x 3t^2 dt$$
$$= 3[\frac{t^3}{3}]\Big|_0^x = x^3$$

Thus, the cdf is:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(c) What is $P(0.1 \leq X < 0.5)$?

To find $P(0.1 \leq X < 0.5)$, we will calculate the difference in cdf values between 0.5 and 0.1: $F(0.5) - F(0.1)$.
Using our cdf: $F(x) = x^3$, we compute:
$F(0.5) = (0.5)^3 = 0.125$
$F(0.1) = (0.1)^3 = 0.001$
Thus, $P(0.1 \leq X < 0.5) = 0.125 - 0.001 = 0.124$

(d) Find $E(X)$.

$$E(X) = \int_0^1 xf(x)dx$$
$$= \int_0^1 x(3x^2)dx$$
$$= 3\int_0^1 x^3 dx$$
$$= 3[\frac{x^4}{4}]\Big|_0^1$$
$$= 3 \cdot \frac{1^4}{4}$$
$$E(X) = \frac{3}{4}$$

(e) Find $\text{Var}(X)$.

$\text{Var}(X) = E(X^2) - (E(X))^2$. First, we'll find $E(X^2)$:

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 f(x)dx \\
&= \int_0^1 x^2(3x^2)dx \\
&= 3\int_0^1 x^4 dx \\
&= 3[\frac{x^5}{5}]\Big|_0^1 \\
&= 3 \cdot \frac{1^5}{5} \\
&= \frac{3}{5}
\end{aligned}$$

Now, $\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{5} - \frac{9}{16} = \frac{48}{80} - \frac{45}{80} = \frac{3}{80}$.

3. (Required) Suppose that Y has the density function:

$$f(y) = \begin{cases} ky(1-y) & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of k that makes $f(y)$ a probability density function

To make $f(y)$ a valid probability density, the total probability must be one:

$$\begin{aligned}
&\int_0^1 ky(1-y)dy = 1 \\
&k\int_0^1 y(1-y)dy = k\int_0^1 (y-y^2)dy \\
&= k(\int_0^1 ydy - \int_0^1 y^2 dy) \\
&= k([\frac{y^2}{2} - \frac{y^3}{3}]\Big|_0^1) \\
&= k(\frac{1}{2} - \frac{1}{3}) = k \cdot \frac{1}{6} = 1
\end{aligned}$$

$k \cdot \frac{1}{6} = 1 \implies k = 6$. Therefore, $k = 6$.

(b) Find $P(0.4 \leq Y \leq 1)$

$P(0.4 \leq Y \leq 1) = F(1) - F(0.4) = (3(1)^2 - 2(1)^3) - (3(0.4)^2 - 2(0.4)^3) = 0.648$

(c) Find $P(0.4 \leq Y < 1)$

Since $P(0.4 \leq Y \leq 1) = 0.648$ and $f(y)$ is continous, the probability at $Y = 1$ is 0. Therefore, $P(0.4 \leq Y < 1) = P(0.4 \leq Y \leq 1) = 0.648$

(d) Find $P(Y \leq 0.4|Y \leq 0.8)$

We will use the law of conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
$P(Y \leq 0.4 \cap Y \leq 0.8) = P(Y \leq 0.4)$
$P(Y \leq 0.4) = F(0.4) = 0.352$
$P(Y \leq 0.8) = F(0.8) = 0.896$.
Therefore, $P(Y \leq 0.4|Y \leq 0.8) = \frac{P(Y \leq 0.4)}{P(Y \leq 0.8)} = \frac{0.352}{0.896} = 0.393$.

(e) Find $P(Y \leq 0.4|Y < 0.8)$

Since $P(Y \leq 0.8) = P(Y < 0.8)$ for a continous random variable, we know $P(Y \leq 0.4|Y < 0.8) = 0.393$

4. (Optional) Find $f(x) = (1+\alpha x)/2$ for $-1 \leq 1$ and $f(x) = 0$ otherwise, where $-1 \leq \alpha \leq 1$:

(a) Show that f is a density.

Property 1: $f(x) \geq 0$ for all x in the domain. This is satisfied because the range of $f(x)$ from $-1 < x < 1$ is from $(1-\alpha)/2$ to $(1+\alpha)/2$, and since $-1 \leq \alpha \leq 1$, we get that $(1-\alpha) \geq 0$ for all x, so $f(x) \geq 0$ for all x in the domain.

Property 2: The total probability = 1: $\int_{-\infty}^{+\infty} f(x)dx = 1$.

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)dx &= \int_{-1}^{1} \frac{1+\alpha x}{2} dx \\
&= \left(\frac{x}{2} + \frac{\alpha x^2}{4}\right)\Big|_{-1}^{1} \\
&= \frac{1}{2} - \frac{-1}{2} = 1
\end{aligned}$$

Therefore the total probability is 1.
Therefore, f satisfies both properties and is a density function

(b) Find the corresponding cdf.

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{-1}^{x} \frac{1+\alpha t}{2} dt \\
&= \left(\frac{t}{2} + \frac{\alpha t^2}{4}\right)\Big|_{-1}^{1} = \frac{\alpha x^2}{4} + \frac{x}{2} + \frac{1-\alpha}{4}
\end{aligned}$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{\alpha x^2}{4} + \frac{x}{2} + \frac{1-\alpha}{4} & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

5. (Optional) A circle of area $A = \pi r^2$. If a random circle has a radius that is uniformly distributed on the interval (0, 1), what are the mean and variance of the area of the circle?

We will first compute the expected value of the area: $E(A) = E(\pi r^2) = \pi E(r^2)$

$$E(r^2) = \int_0^1 r^2 f(r)dr = \int_0^1 r^2 dr = [\frac{r^3}{3}]\Big|_0^1 = \frac{1}{3}$$

Thus, $E(A) = \pi E(r^2) = \pi(\frac{1}{3}) = \frac{\pi}{3}$.

Now, we will compute the variance of the area: $\text{Var}(A) = E(A^2) - (E(A))^2$.
$E(A^2) = E(\pi^2 r^4) = \pi^2 E(r^4)$.

$$E(r^4) = \int_0^1 r^4 f(r)dr = \int_0^1 r^4 dr = [\frac{r^5}{5}]\Big|_0^1 = \frac{1}{5}$$

$E(A^2) = \pi^2 E(r^4) = \pi^2(\frac{1}{5}) = \frac{\pi^2}{5}$.
$\text{Var}(A) = \frac{\pi^2}{5} - (\frac{\pi}{3})^2 = \frac{\pi^2}{5} - \frac{\pi^2}{9} = \frac{4\pi^2}{45}$.