Probability Homework 6

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1. (Required) Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = 0.1$.

   This means that the pdf is $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ and the cdf is $F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$.

   (a) Find the probability that the lifetime is less than 10.

   We need to find $P(T < 10)$ which is the cdf at $t = 10$:
   $P(T < 10) = F(10) = 1 - e^{-\lambda \cdot 10} = 1 - e^{-0.1 \cdot 10} = 1 - e^{-1} = 1 - 0.3679 = 0.6321$

   (b) Find the probability that the lifetime is between 5 and 15.

   $P(5 < T < 15) = F(15) - F(5) = (1 - e^{-\lambda \cdot 15}) - (1 - e^{-\lambda \cdot 5}) = (1 - e^{-0.1 \cdot 15}) - (1 - e^{-0.1 \cdot 5}) = (1 - e^{-1.5}) - (1 - e^{-0.5}) = (1 - 0.2231) - (1 - 0.6065) = 0.7769 - 0.3935 = 0.3834$.

   (c) Find t such that the probability that the lifetime is greater than t is 0.01.

   We need to find t such that $P(T > t) = 0.01$. $P(T > t) = 1 - F(t) = e^{-\lambda t} = 0.01$.

$$\begin{aligned}
e^{-\lambda t} &= 0.01 \\
-\lambda t &= \ln(0.01) \\
t &= -\frac{\ln(0.01)}{\lambda} \\
t &= -\frac{\ln(0.01)}{0.1} \\
t &= \frac{4.6052}{0.1} \\
t &= 46.052
\end{aligned}$$

2. (Required) The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 480 with standard deviation 100. The average and standard deviation for ACT mathematics scores are 18 and 6, respectively.

(a) An engineering school sets 550 as the minimum SAT math score for new students. What percentage of students will score below 550 in a typical year?

Observe mean $\mu = 480$ and $\sigma = 100$. We will find the z-score for 550:
$z = \frac{X-\mu}{\sigma} = \frac{550-480}{100} = 0.7$. Then, $P(z < 0.7) = 0.7580$ via the standard distribution table. Thus, 75.80% of students will score below a 550.

(b) What score should the engineering school set as a comparable standard on the ACT math test?

ACT scores are given with a $\mu = 18$ and $\sigma = 6$. We will use the same SAT z-score and convert back to an ACT score:
$X = \mu + z\sigma = 18 + 0.7 \cdot 6 = 18 + 4.2 = 22.2$. Thus, the comparable standard on the ACT math should be a 22.2.

3. (Required) Let T be an exponential random variable with parameter λ. Let X be a discrete random variable defined as $X = k$ if $k \leq T < k+1, k = 0, 1, \ldots$ Find the probability mass function of X.

We know that the pdf of T is as follows: $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.
We are asked to find the pmf of X, which is $P(X = k) = P(k \leq T < k+1) = F(k+1) - F(k)$ where F is the cdf: $F(t) = 1 - e^{-\lambda t}$. Thus,

$$\begin{aligned}
P(X = k) &= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda k}) \\
&= e^{-\lambda k} - e^{-\lambda(k+1)} \\
&= e^{-\lambda k}(1 - e^{-\lambda})
\end{aligned}$$

Therefore, the pmf of X is $P(X = k) = (1 - e^{-\lambda})e^{-\lambda k}$ for $k = 0, 1, 2, \ldots$ which follows a geometric distribution with parameter $p = 1 - e^{-\lambda}$

4. (Optional) The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Find the probability that an earthquake striking this region will

Since it follows an exponential distribution, we know the pdf is given as:
$f(x) = \frac{1}{\beta} e^{-x/\beta}$, for $x \geq 0$ where β is the mean, which is given as $\beta = 2.4$

(a) exceed 3.0 on the Richter scale.

The cdf of an exponential function is given as: $F(x) = 1 - e^{-x/\beta}$.
Observe, $P(X > 3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - (1 - e^{-3/2.4}) = e^{-3/2.4} = 0.2865$

(b) fall between 2.0 and 3.0 on the Richter scale.

We are now asked to find $P(2 \leq X \leq 3)$, which is just $F(3) - F(2)$.
Observe, $F(3) = 1 - e^{-3/2.4} = 1 - 0.2865 = 0.7135$, and $F(2) = 1 - e^{-2/2.4} = 1 - 0.4346 = 0.5654$.
So, $P(2 \leq X \leq 3) = 0.7135 - 0.5654 = 0.1481$.

5. (Optional) If Y has an exponential distribution and $P(Y > 2) = 0.0821$, what is

(a) $E(Y)$?

For an exponential function, we know $P(Y > t) = e^{-t/\beta}$. We are asked to find $E(Y)$, which is just β.
We are given $P(Y > 2) = 0.0821$, and we can use this to find β. Observe,

$$\begin{aligned} e^{-2/\beta} &= 0.0821 \\ -\frac{2}{\beta} &= \ln(0.0821) \\ -\frac{2}{\beta} &= -2.5 \\ \beta &= \frac{2}{2.5} = 0.8 \end{aligned}$$

Therefore, $\beta = E(Y) = 0.8$.

(b) $P(Y \leq 1.7)$?

To find $P(Y \leq 1.7)$ we can use the cdf: $P(Y \leq t) = 1 - e^{-t/\beta}$. We found that $\beta = 0.8$, so we can simply substitute t and β as such:
$P(Y \leq 1.7) = 1 - e^{-1.7/0.8} = 1 - e^{2.125} = 1 - 0.1190 = 0.8810$