Probability Homework 7

Will Krzastek

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1. (Required) Suppose that a random variable Y has a probability density function given by:

$$f(y) = \begin{cases} ky^3 e^{-y/2} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the value of k that makes $f(y)$ a valid density function.

   Let's look at the pdf of a Gamma distribution:

$$f(y) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   Thus, we need $k = \frac{\lambda^\alpha}{\Gamma(\alpha)}$ for $f(y)$ to be a valid pdf.
   $\alpha - 1 = 3 \Rightarrow \alpha = 4; \lambda = \frac{1}{2}; \Gamma(4) = 3! = 6; k = \frac{1/2^4}{6} = \frac{1}{94}$

   (b) What are the mean and standard deviation of Y?

   The mean of Y is $E[Y] = \frac{\alpha}{\lambda} = \frac{4}{1/2} = 8$.
   The standard deviation of Y is $\sigma(Y) = \frac{\sqrt{\alpha}}{\lambda} = \frac{\sqrt{4}}{\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$

2. (Required) Suppose that a random variable Y has a probability density function given by:

$$f(y) = \begin{cases} 6y(1-y) & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find $F(y)$.

   $F(y) = P(Y \le y) = \int_{-\infty}^{y} f(t)dt$.
   We just care about $0 \le y \le 1$, so our integral is just:

$$F(y) = \int_0^y 6t(1-t)dt = \int_0^y (6t - 6t^2)dt = [3t^2 - 2t^3]|_0^y = 3y^2 - 2y^3$$

So for $0 \le y \le 1$, $F(y) = 3y^2 - 2y^3$, and for $y > 1$, $F(y) = 1$. More formally,

$$F(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

(b) Find $P(0.5 \le Y \le 0.8)$.

We can just use the cdf we created: $P(0.5 \le Y \le 0.8) = F(0.8) - F(0.5)$
Let's calculate both.
$F(0.8) = 3(0.8)^2 - 2(0.8)^3 = 3 \cdot 0.64 - 2 \cdot 0.512 = 0.896$.
$F(0.5) = 3(0.5)^2 - 2(0.5)^3 = 3 \cdot 0.25 - 2 \cdot 0.125 = 0.5$.
So, $P(0.5 \le Y \le 0.8) = 0.896 - 0.5 = 0.396$.

3. (Required) The Weibull cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(x/\alpha)^\beta} & x \ge 0 \end{cases}$$

(a) Find the density function.

$f(x)$ is just $\frac{d}{dx}F(x)$. For $x < 0, F(x) = 0$, so $f(x) = 0$.
For $x \ge 0$,

$$\begin{aligned} f(x) &= \frac{d}{dx}(1 - e^{-(x/\alpha)^\beta}) = \frac{d}{dx}(-e^{-(x/\alpha)^\beta}) \\ &= -e^{-(x/\alpha)^\beta} \cdot \frac{d}{dx}(-(x/\alpha)^\beta) \\ &= e^{-(x/\alpha)^\beta} \cdot \beta\frac{x^{\beta-1}}{\alpha^\beta} \end{aligned}$$

So the pdf is:

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}e^{-(x/\alpha)^\beta} & x \ge 0 \end{cases}$$

(b) Show that if W follows a Weibull distribution, then $X = (W/\alpha)^\beta$ follows an exponential distribution.

Let W follow a Weibull distribution with cdf: $F(w) = 1 - e^{-(w/\alpha)^\beta}$ for $w \ge 0$.
We are looking at the transformation: $X = (W/\alpha)^\beta$ and finding the distribution of

X. Let's look at the cdf of X:

$$\begin{aligned}
F(X) &= P(X \le x) \\
&= P((W/\alpha)^\beta \le x) \\
&= P(W \le \alpha x^{1/\beta}) \\
&= 1 - e^{-(\alpha x^{1/\beta}/\alpha)^\beta} \quad \text{by the Weibull cdf} \\
&= 1 - e^{-x} \quad \text{by simplifying the exponent}
\end{aligned}$$

Since $F(x) = 1 - e^{-x}$, we can conclude that $X = (W/\alpha)^\beta$ follows an exponential distribution with parameter $\lambda = 1$.

4. (Optional) Find the density of cX when X follows a gamma distribution. Show that only λ is affected by such a transformation, which justifies calling λ a rate parameter.

   Let $X \sim Gamma(\alpha, \lambda)$. The pdf is $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$. We're looking at the transformed variable $Y = cX$.
   Observe, $x = \frac{Y}{c}$ and $\frac{dx}{dy} = \frac{1}{c}$.
   We can get the pdf of y by doing: $f_y(y) = f_x(\frac{y}{c})(\frac{dx}{dy}) = f_x(\frac{y}{c}) \cdot \frac{1}{c}$. We can then just substitute $f_x(x)$ with $f_x(y/c)$.

$$\begin{aligned}
f_y(y) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} (\frac{y}{c})^{\alpha-1} e^{-\lambda(y/c)} \frac{1}{c} \\
&= \frac{(\frac{\lambda}{c})^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y/c}
\end{aligned}$$

   This pdf matches the pdf of a gamma distribution with shape parameter α and rate parameter $\frac{\lambda}{c}$, therefore the shape parameter remains unchanged while the rate parameter is scaled by $\frac{1}{c}$ from the transformation $Y = cX$.

5. (Optional) Show that if X has a density function f_x and $Y = aX + b$, then

$$f_y(y) = \frac{1}{|a|} f_x(\frac{y-b}{a})$$

   We can rewrite $Y = aX + b$ as $X = \frac{Y-b}{a}$, so $\frac{dX}{dY} = \frac{1}{a}$ which we can plug into:

$$\begin{aligned}
f_y(y) &= f_x(\frac{y-b}{a}) \cdot \frac{dX}{dY} \\
&= f_x(\frac{y-b}{a}) \cdot |\frac{1}{a}| \\
&= \frac{1}{|a|} f_x(\frac{y-b}{a})
\end{aligned}$$