Probability Homework 3

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1. (Required) A single fair die is tossed once. Let Y be the number facing up. Find the expected value and variance of Y.

   First, we'll find the expected value of Y.

$$E(Y) = \sum_{i=1}^{6} i \cdot P(Y = i)$$
$$E(Y) = \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6)$$
$$E(Y) = \frac{1}{6} \cdot 21$$
$$E(Y) = 3.5$$

   Now, we can find the variance of Y.

$$Var(Y) = E(Y^2) - (E(Y))^2$$
$$E(Y^2) = \sum_{i=1}^{6} i^2 \cdot P(Y = i)$$
$$E(Y^2) = \frac{1}{6} \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{1}{6} \cdot 91 = 15.17$$

   So, $Var(Y) = E(Y^2) - (E(Y))^2 = 15.17 - (3.5)^2 = 2.92$.

2. (Required) A multiple-choice test consists of 20 items, each with four choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices. A passing grade is 18 items or more correct.

   Observe, that each question is a Bernoulli trial (correct or incorrect answer), and the trial occurs 20 times, which will follow a Binomial distribution

   (a) What is the probability that the student passes?

   The student is allowed to remove one incorrect choice, meaning they choose 1 answer out of 3 choices, so the probability of answering correctly is $p = frac13$, and the probability of answering incorrectly is $1 - p = \frac{2}{3}$. Let X be the number of correct answers. We already determined that X follows a Binomial distribution:

$X \sim Binom(20, p = \frac{1}{3})$. It is determined that passing the exam means getting at least 18 questions correct, so we must find $P(X \geq 18)$ which is just $P(X = 18) + P(X = 19) + P(X = 20)$, since $X_{\text{max}} = 20$. To solve this, we'll use the binomial PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.
Observe, $P(X = 18) = \binom{20}{18} (\frac{1}{3})^{18} (\frac{2}{3})^2 = 2.18 \times 10^{-7}$, $P(X = 19) = \binom{20}{19} (\frac{1}{3})^{19} (\frac{2}{3})^1 = 1.15 \times 10^{-8}$, $P(X = 20) = \binom{20}{20} (\frac{1}{3})^{20} = 2.87 \times 10^{-10}$.
So, $P(X \geq 18) = 2.30 \times 10^{-7}$.

(b) Answer the question in part (a) again, assuming that the student can eliminate two of the choices on each question.

Now, we can follow the same process as before, but now probability of success is $p = \frac{1}{2}$ and consequently probability of failure is $1 - p = \frac{1}{2}$.
$P(X \geq 18) = P(X = 18) + P(X = 19) + P(X = 20)$. Again, we'll use binomial PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

$$P(X = 18) = \binom{20}{18} (\frac{1}{2})^{18} (\frac{1}{2})^2 = 1.81 \times 10^{-4}$$
$$P(X = 19) = \binom{20}{19} (\frac{1}{2})^{19} (\frac{1}{2})^1 = 1.91 \times 10^{-5}$$
$$P(X = 20) = \binom{20}{20} (\frac{1}{2})^{20} = 9.54 \times 10^{-7}$$

So, $P(X \geq 18) = 2.01 \times 10^{-4}$.

3. (Required)

(a) Suppose that in a sequence of independent Bernoulli trials, each with probability of success p, the number of failures up to the first success is counted. What is the probability mass function for this random variable?

This is modeled by the Geometric distribution PMF: $P(X = k) = (1-p)^k p$ where k is the number of failures before the first success and p is the probability of success.

(b) Continuing with Part (a), find the probability mass function for the number of failures up to the r-th success.

This is now modeled by the Negative Binomial distribution PMF:
$P(X = k) = \binom{k+r-1}{r-1} (1-p)^k p^r$ where r is the number of successes we are going up to, p is the probability of success, and k is the number of failures before the r-th success. In normal negative binomial pmf, we use $\binom{k-1}{r-1}$, which represents the total trials, but we only care about the number of failures so we use $\binom{k+r-1}{r-1}$ instead.

4. (Choice) In a gambling game a person draws a single card from an ordinary 52-card playing deck. A person is paid \$15 for drawing a jack or a queen and \$5 for drawing a king or an ace. A person who draws any other card pays \$4. If a person plays this game, what is the expected gain?

P(Jack or Queen) $= \frac{8}{52} = \frac{2}{13}$. +\$15 payout.
P(King or Ace) $= \frac{8}{52} = \frac{2}{13}$. +\$5 payout.
P(Other) $= 1 - \frac{2}{13} - \frac{2}{13} = \frac{9}{13}$. -\$4 payout.

$E = (\frac{2}{13} \times 15) + (\frac{2}{13} \times 5) + (\frac{9}{13} \times (-4)) = \frac{30}{13} + \frac{10}{13} - \frac{36}{13} = \frac{4}{13} = 0.3077.$
So, the expected gain is approximately \$0.31 per play.

5. (Choice) An oil prospector will drill a succession of holes in a given area to find a productive well. The probability that he is successful on a given trial is 0.2.

   (a) What is the probability that the third hole drilled is the first to yield a productive well?

   We have a geometric distribution on a random variable X, where the success of each trial is modeled by $p = 0.2$.
   For geometric distributions, we know that $P(X = k) = (1 - p)^{k-1}p$.
   In the case of the third hole, we get $P(X = 3) = (1 - 0.2)^{3-1} \cdot 0.2 = 0.128$.
   So, the probability of the third hole being drilled yielding a productive well is 0.128.

   (b) If the prospector can afford to drill at most ten wells, what is the probability that he will fail to find a productive well?

   The probability of failure is $1 - p = 0.8$. Since all trials are independent, we can just raise the probability of failure to the amount of trials to see the probability of failure in all trials.
   Observe, $(0.8)^{10} = 0.1074$. So, the probability of not finding a productive well in 10 trials is 0.1074.