Positivity Certificates and Polynomial Optimization

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Positive polynomials

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Very hard to answer efficiently: NP-hard.

Sums Of Squares (SOS)

Sufficient condition: f being a sum of squares (SOS), i.e.

$$f = f_1^2 + \cdots + f_m^2, \tag{1}$$

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Example:

$$\begin{aligned} f &= X_1^4 - 2X_1^2X_2X_3 - X_1^2 + X_2^2X_3^2 + 2X_2X_3 + 2 \\ &= 2\left(1 - \tfrac{1}{2}X_1^2 + \tfrac{1}{2}X_2X_3\right)^2 + X_1^2 + \tfrac{1}{2}\left(X_1^2 - X_2X_3\right)^2 \geq 0. \end{aligned}$$

A characterization of SOS-polynomials

Theorem

Let $f \in \mathbb{R}[\underline{X}]_{2d}$ and $z = (\underline{X}^{\alpha})_{\alpha \in \mathbb{N}^n_d}$. f is SOS if and only if there exists a psd matrix $G \in \mathcal{S}_+^{\mathbb{N}^n_d}$ so that $f = z^T G z$.

Proof.

$$f = z^T G z$$

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If $f = \sum_{i=1}^m f_i^2$ then we can choose $B \in \mathbb{R}^{m \times \mathbb{N}^n_d}$ so that $f_i = [Bz]_i$. Then $G = B^T B \succeq 0$ and $f = z^T G z$. $\square$

Where do we look for G?

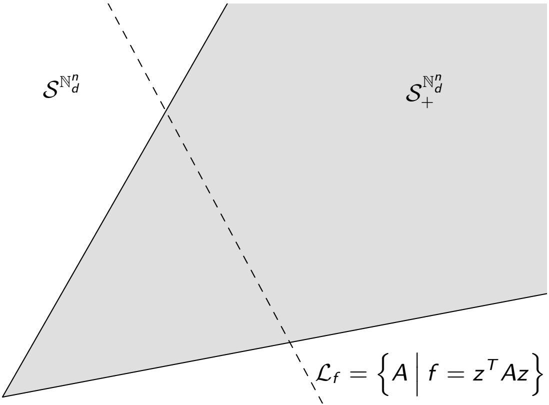

Figure: $\mathcal{L}_f \cap \mathcal{S}_+^{\mathbb{N}^n_d}$ is a conic section known as a *spectrahedron*.

Semi-definite Programming (SDP)

Semi-definite program:

$$\begin{aligned} \text{Minimize}_G \quad & L(G) \\ \text{s.t.} \quad & G \in \mathcal{L} \cap \mathcal{S}^n_+. \end{aligned}$$

$L\colon \mathcal{S}^n \to \mathbb{R}$ is linear and $\mathcal{L} \subseteq \mathcal{S}^n$ is an affine subspace.

Can be solved numericly and efficiently.

Ellipsoid through 4 points

Problem: Find a centered ellipsoid $E \subseteq \mathbb{R}^3$ passing through the points e_1, e_2, e_3, and $v = (3, 4, 5)$.

Formula:

$$u^T G u = 1, \quad G \succ 0.$$

Must find G satisfying:

$$G = G_{xyz} = \begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \succ 0, \qquad v^T G v = 1$$

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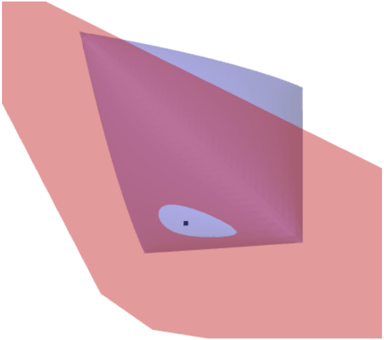

Figure: (x, y, z) such that G_{xyz} yields the desired ellipsoid.

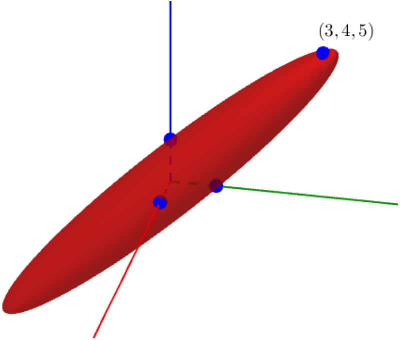

Figure: Associated ellipsoid E through e_1, e_2, e_3, v.

Computer assisted proofs

Theorem (Hadwiger-Finslers inequality)

Let $a, b, c > 0$ be the side lengths of a tiranlge with area K. Then:

$$a^2 + b^2 + c^2 - (a-b)^2 + (a-c)^2 + (b-c)^2 \geq 4\sqrt{3}K.$$

Boils down to show that

$$p = \left(a^2 + b^2 + c^2 - ((a-b)^2 + (a-c)^2 + (b-c)^2)\right)^2 - (4\sqrt{3}K)^2$$

is non-negative for all $a, b, c \in \mathbb{R}$. SOS-decomposition:

$$p = \left(2a^2 + b(a+b-c) + c(a-b+c)\right)^2 + 3\left(b(a-b) + c(a-c)\right)^2.$$

What if no SOS-decomposition exists?

The Motzkin polynomial

$$M = X^4Y^2 + X^2Y^4 - 3X^2Y^2 + 1.$$

is globally non-negative but it is *not* SOS.

- How do we certify non-negativity when no SOS-decomposition exists?
  $\rightsquigarrow$ Hilbert's 17th problem.
- How do we certify non-negativity on a *subset* $S \subseteq \mathbb{R}^n$? $\rightsquigarrow$ Positivstellensatz.

Semialgebraic sets and pre-orderings

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$$\mathcal{W}(B) = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_t(x) \geq 0\} \subseteq \mathbb{R}^n$$

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We associate to $\mathcal{W}(B)$ the *pre-ordering generated by* B:

$$T = T[B] = \left\{ \sum_{\nu \in \{0,1\}^t} \sigma_\nu g_1^{\nu_1} \ldots g_t^{\nu_t} \;\middle|\; \sigma_\nu \in \sum \mathbb{R}[\underline{X}]^{(2)} \right\}$$

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T consists of polynomials which are “obviously” non-negative on $\mathcal{W}(B)$.

The Semialgebraic Nullstellensatz

Theorem (Semialgebraic Nullstellensatz)

Let $I \subseteq \mathbb{R}[\underline{X}]$ be an ideal and $T = T[B]$. Then:

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Here

$$\sqrt[T]{I} = \{f \in \mathbb{R}[\underline{X}] \mid f^{2m} + a \in I \text{ for et } m > 0 \text{ og } a \in T\}$$

is the *T-radical of I*.

The easy direction $\mathbb{I}(\mathbb{V}_{\mathbb{R}}(I) \cap \mathcal{W}(B)) \supseteq \sqrt[T]{I}$

Suppose that $f \in \sqrt[T]{I}$ and choose $a \in T$ and $m > 0$ so that $f^{2m} + a \in I$.

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I.e. $f(x) = 0$.
Hence $f \in \mathbb{I}(\mathbb{V}_{\mathbb{R}}(I) \cap \mathcal{W}(B))$.

The difficult direction: $\mathbb{I}(\mathbb{V}_{\mathbb{R}}(I) \cap \mathcal{W}(B)) \subseteq \sqrt[T]{I}$

First ingredient:

Lemma

$\sqrt[T]{I}$ is the intersection of all T-radical prime ideals containing I.

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Second ingredient:

Lemma

There exists an ordering $\leq_Q$ on Q which extends the ordering on $\mathbb{R}$ and and satisfies $\bar{g}_i = g_i + I \geq_Q 0$ for $i = 1, \ldots, t$.

Finishing the proof

Proof that $\mathbb{I}(\mathbb{V}_{\mathbb{R}}(I) \cap \mathcal{W}(B)) \subseteq \sqrt[\mathbb{R}]{I} = I$.

Suppose $h \notin I = \langle f_1, \ldots, f_s \rangle$. We must show that $h \notin \mathbb{I}(\mathbb{V}_{\mathbb{R}}(I) \cap \mathcal{W}(B))$. Amounts to find a solution $x \in \mathbb{R}^n$ to the system:

$$\begin{aligned} f_i &= 0, \quad i = 1, \ldots, s, \\ g_j &\geq 0, \quad j = 1, \ldots, t, \\ h &\neq 0. \end{aligned} \tag{2}$$

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The Transfer-principle: It is sufficient to find a solution in Q^n!

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I.e. $(\overline{X_1}, \ldots, \overline{X_n}) \in Q^n$ is a solution to (2). $\square$

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I.e. if and only if there exists $a_1, a_2 \in T$ and $m > 0$, so that

$$f^{2m} + a_1 - fa_2 = 0.$$

$\square$

Hilbert's 17th problem

Corollary (Artin's Theorem)

If $f \geq 0$ on $\mathbb{R}^n$, then f can be written on the form

$$f = \frac{f^{2m} + \sigma_1}{\sigma_2}, \qquad \sigma_1, \sigma_2 \in \sum \mathbb{R}[\underline{X}]^{(2)}.$$

In particular f is a sum of squares of rational functions.

Improvements when S is compact.

Theorem (Schmüdgen)

Suppose that $S = \mathcal{W}(B)$ is compact. Then

$$f > 0 \text{ på } S \implies f \in T[B]$$

The *quadratic module generated by* B:

$$M[B] = \left\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_t g_t \;\middle|\; \sigma_i \in \sum \mathbb{R}[\underline{X}]^{(2)} \right\}.$$

Theorem (Putinar)

If $N - \sum_{i=1}^{n} X_i^2 \in M[B]$ for some $N \in \mathbb{N}$, then

$$f > 0 \text{ på } S \implies f \in M[B].$$

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Decide on a maximal degree k for the certificates in M:

$$M_k = \left\{ \sum_{i=0}^{t} \sigma_i g_i \,\middle|\, \sigma_i \in \sum \mathbb{R}[\underline{X}]^{(2)},\ \deg(\sigma_i g_i) \le k \right\} \subseteq M \cap \mathbb{R}[\underline{X}]_k,$$

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$$M_k = \left\{ \sum_{i=0}^{t} \sigma_i g_i \,\middle|\, \sigma_i \in \sum \mathbb{R}[\underline{X}]^{(2)},\ \deg(\sigma_i g_i) \le k \right\} \subseteq M \cap \mathbb{R}[\underline{X}]_k,$$

This yields an SDP:

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Constrained minimization

Minimize f on S:

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By increasing k we can approximate f^*:

$$\bar{f}_k^* \leq \bar{f}_{k+1}^*,\ \forall k, \qquad \lim_{k\to\infty} \bar{f}_k^* = f^*.$$

Example

Find the smallest circle enclosing the budderfly curve:

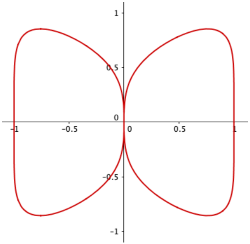

Figure: $\mathbb{V}_{\mathbb{R}}(f)$, $f = X^6 + Y^6 - X^2$.

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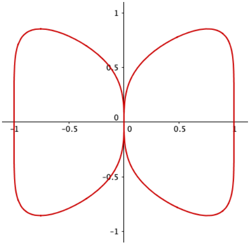

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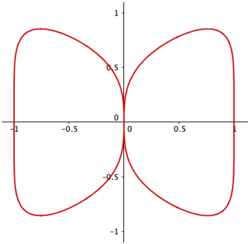

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\end{aligned}
$$

Example continued ...

$$\bar{f}_2^* \approx 1.4679$$

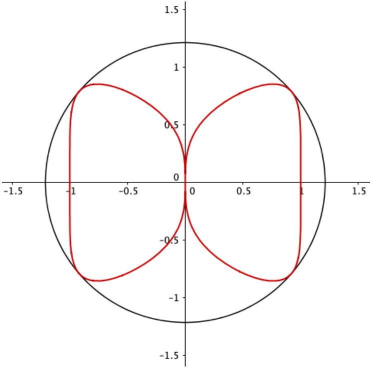

Figure: $1.4679 = X^2 + Y^2$.

Lasserre: dual perspective

$$f^* = \inf\{f(x) \mid x \in S\}$$

,

Lasserre: dual perspective

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\begin{aligned}
f^* &= \inf\{f(x) \mid x \in S\} \\
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Decide on maximal degree k:

$$\chi_k = \{L \colon \mathbb{R}[\underline{X}]_k \to \mathbb{R} \mid L(M_k) \subseteq [0,\infty), L(1) = 1\}.$$

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Yields another SDP:

$$\bar{f}_k^* \leq f_k^* := \inf\{L(f) \mid L \in \chi_k\} \leq f^*.$$

The MaxCut-problem

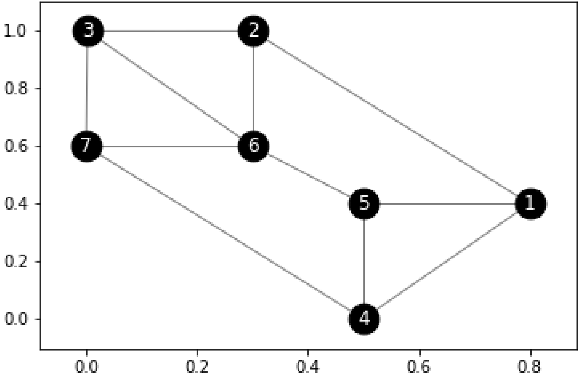

Figure: Find a maximal cut.

MaxCut-problemet

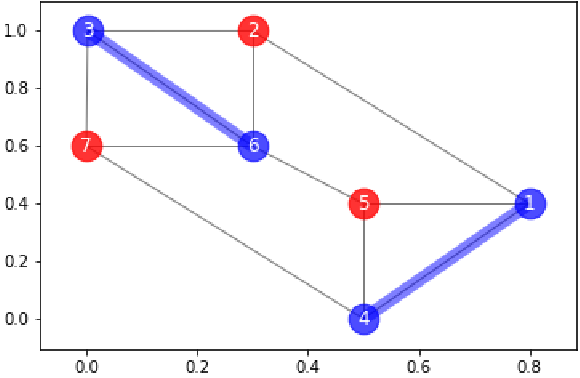

Figure: A maximal cut of size 9.

Approksimation

Quadratic integer problem:
Maximise the polynomial

$$f = \sum_{(i,j)\in E} \frac{1}{2}(1 - X_i X_j) = \frac{1}{2}(11 - X_1X_2 - X_1X_4 - \cdots - X_6X_7)$$

over the set

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Lasserre approximation with $k = 2$:

$$f_2^* \approx 9.3231.$$