Well-Behaved (Co)algebraic Semantics of Regular Expressions in Dafny

Stefan Zetzsche[1] **Wojciech Różowski**[2]

[1]Amazon Web Services

[2]University College London

March 27, 2025

Introduction

Syntax of regular expressions

The syntax of regular expressions is defined inductively. Namely, it is the least set closed under the following rules:

$$\frac{}{0 \in \mathsf{Exp}} \qquad \frac{}{1 \in \mathsf{Exp}} \qquad \frac{a \in A}{a \in \mathsf{Exp}} \qquad \frac{e \in \mathsf{Exp} \qquad f \in \mathsf{Exp}}{e + f \in \mathsf{Exp}}$$

$$\frac{e \in \mathsf{Exp} \qquad f \in \mathsf{Exp}}{e ; f \in \mathsf{Exp}} \qquad \frac{e \in \mathsf{Exp}}{e^{*} \in \mathsf{Exp}}$$

Regular expressions as an algebra

An algebra for the signature is a set A, equipped with interpretations of each symbol in the signature. Namely, each symbol σ_n of arity n is interpreted as a function $\sigma^A \colon A^n \to A$.

Signature of regular expressions:

$$\Sigma = \{0_0, 1_0, +_2, ;_2, (-)^*_1\} \cup \{a_0 \mid a \in A\}$$

The set Exp is an *initial algebra*, i.e. if you fix a set and a way of interpreting operations of regular expressions, then you can uniquely interpret any regular expression.

An Algebra of Formal Languages

Intrinsically, one can equip $\mathsf{Lang} = \mathsf{P}(A^*)$ with a Σ-algebra structure via:

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{\varepsilon\} \\ L_1 \cdot L_2 &:= \{w_1 \cdot w_2 \mid w_i \in L_i\} \\ L_1 + L_2 &:= L_1 \cup L_2 \\ L^* &:= \bigcup_{n \geq 0} L^n, \quad \text{where} \quad L^{n+1} := L \cdot L^n, \quad L^0 := \{\varepsilon\} \end{aligned}$$

The induced interpretation function $[\![-]\!]$ is a homomorphism, i.e.

$$[\![e + f]\!] = [\![e]\!] \cup [\![f]\!]$$

Coinduction

A slightly less-known dual of induction. For example, infinite streams are the *greatest* set closed under the following rule:

$$\frac{a \in A \qquad \sigma \in \mathsf{Stream}(A)}{a :: \sigma \in \mathsf{Stream}(A)}$$

In other words, an infinite stream is something that we can observe it's first element (head) and the remaining infinite stream (tail).

Formal languages coinductively

Formal languages are usually presented as sets of words, i.e.

$$\mathsf{Lang} = \mathsf{P}(A^*)$$

This set can be characterised extrinsically as the greatest set closed under the following rule:

$$\frac{o \in 2 \qquad t \in A \to \mathsf{Lang}}{\langle o, t \rangle \in \mathsf{Lang}}$$

Formal Languages coinductively

To some, our way of modeling formal languages might seem odd at first sight.

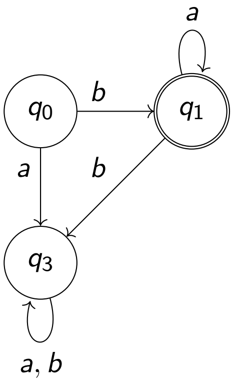
Typically, a formal language is defined intrinsically, as an element of type $\mathcal{P}(A^*)$.

Here, we treat formal languages extrinsically, in terms of their universal property as greatest coalgebraic structure X equipped with functions $E : X \to 2$ and $D : X \to X^A$.

Indeed, for any $U \in \mathcal{P}(A^*)$, we can define $E(U) := [\varepsilon \in U]$ and $D(U)(a) := \{w \mid aw \in U\}$.

Deterministic automata and formal languages

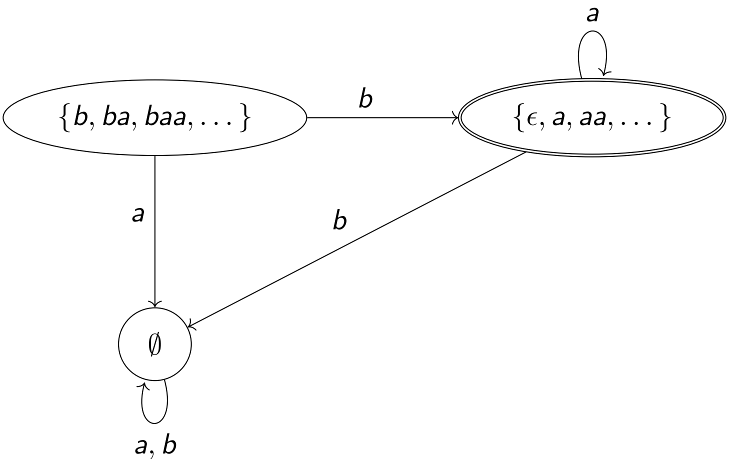
A pair $(Q, \langle o, t\rangle \colon Q \to 2 \times Q^A)$ consisting of *a set of states* Q and *transition function* $\langle o, t\rangle$



Every state $q \in Q$, can be assigned a *language* $L(q) \subseteq A^*$ it accepts. Eg. $L(q_0) = \{b, ba, baa, \dots\}$

Final automaton

There is an automaton whose states are *formal languages*. From any automaton, there is a unique transition-preserving map: the language assigning map.



Operational Semantics of Regular Expressions

We can equip Exp with coalgebraic structure resembling deterministic automata:

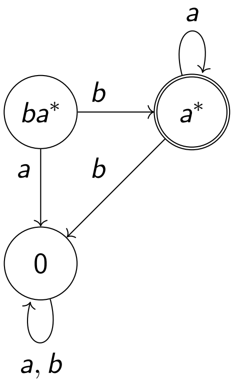
$$\begin{aligned}
E(0) &:= 0 & D_a(0) &:= 0 \\
E(1) &:= 1 & D_a(1) &:= 0 \\
E(b) &:= 0 & D_a(b) &:= [a = b] \\
E(e + f) &:= E(e) + E(f) & D_a(e + f) &:= D_a(e) + D_a(f) \\
E(e \cdot f) &:= E(e) \cdot E(f) & D_a(e \cdot f) &:= D_a(e) \cdot f + E(e) \cdot D_a(f) \\
E(e^*) &:= 1 & D_a(e^*) &:= D_a(e) \cdot e^*.
\end{aligned}$$

Let `operational` be the unique homomorphism

$$(\mathrm{Exp}, E, D_a) \to (\mathrm{Lang}, E, D_a).$$

Brzozowski derivatives

We can give semantics to expressions using automata, by 1) building automaton whose states are regular expressions and 2) getting the corresponding formal language



The set of all formal languages has an automaton (coalgebra) structure. This approach to semantics is: *coinductive*, *operational* and *coalgebraic*.

Well-behaved semantics

- The set Exp has both algebra (initial algebra obtained via induction) and coalgebra structure (Brzozowski automaton)
- The set Lang also has both algebra (operations on languages) and coalgebra structure (final automaton)
- Both approaches induce a mapping from expressions to languages that preserve algebra and coalgebra structures

Semantics of Regular Expressions

$$\begin{tikzcd}
\Sigma(\mathsf{Exp}) \arrow[r, dashed, "\Sigma(\texttt{denotational})"] \arrow[d] & \Sigma(\mathsf{Lang}) \arrow[d] \\
\mathsf{Exp} \arrow[r, dashed, "\texttt{denotational}"] & \mathsf{Lang}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Exp} \arrow[r, dashed, "\texttt{operational}"] \arrow[d] & \mathsf{Lang} \arrow[d] \\
B(\mathsf{Exp}) \arrow[r, dashed, "B(\texttt{operational})"] & B(\mathsf{Lang})
\end{tikzcd}$$

$\mathsf{Exp} ::= 0 \mid 1 \mid a \in A \mid e + f \mid e \cdot f \mid e^*$, $\quad$ $\mathsf{Lang} := \mathcal{P}(A^*)$

Well-Behaved Semantics of Regular Expressions

Some immediate questions:

- Are `denotational` and `operational` well-defined?
- Are `denotational` and `operational` also homomorphisms of the respective other type?
- Are `denotational` and `operational` equal?

Dafny

We use **Dafny**, a verification-aware programming language equipped with a static program verifier to

- Formalise both approaches to semantics
- Show that they are well-behaved

Dafny supports coinduction and codatatypes, allowing us to work with a *final automaton* as a first-class citizen

Dafny: A Verification-Aware Programming Language

Dafny allows a clear distinction between an idealised mathematical specification and an efficient implementation thereof.

```
function Fib(n: nat): nat {
  if n <= 1 then n else Fib(n - 1) + Fib(n - 2)
}

method ComputeFib(n: nat) returns (fib: nat)
{
  var i, currentFib, nextFib := 0, 0, 1;
  while i < n
  {
    i, currentFib, nextFib := i + 1, nextFib, currentFib + nextFib;
  }
  return currentFib;
}
```

Dafny: A Verification-Aware Programming Language

Dafny allows a clear distinction between an idealised mathematical specification and an efficient implementation thereof.

```
function Fib(n: nat): nat {
  if n <= 1 then n else Fib(n - 1) + Fib(n - 2)
}

method ComputeFib(n: nat) returns (fib: nat)
  ensures fib == Fib(n)
{
  var i, currentFib, nextFib := 0, 0, 1;
  while i < n
  {
    i, currentFib, nextFib := i + 1, nextFib, currentFib + nextFib;
  }
  return currentFib;
}
```

Dafny: A Verification-Aware Programming Language

Dafny allows a clear distinction between an idealised mathematical specification and an efficient implementation thereof.

```
function Fib(n: nat): nat {
  if n <= 1 then n else Fib(n - 1) + Fib(n - 2)
}

method ComputeFib(n: nat) returns (fib: nat)
  ensures fib == Fib(n)
{
  var i, currentFib, nextFib := 0, 0, 1;
  while i < n
    invariant i <= n && currentFib == Fib(i) && nextFib == Fib(i + 1)
  {
    i, currentFib, nextFib := i + 1, nextFib, currentFib + nextFib;
  }
  return currentFib;
}
```

Dafny Supports Multiple Paradigms

Polymorphism

Inductive datatypes

- `datatype List<T> = Nil | Cons(T, List)`

Coinductive datatypes

- `codatatype Stream<T> = Cons(T, Stream)`

Lambda expressions

- `var f: int -> int := x => x+1`

Higher-order functions

- `F(f: int -> int, n: int): int { f(n) }`

Classes and traits with mutable states

Some Use Cases of Dafny

Authorization

- AWS Identity and Access Management

Distributed Systems

- IronFleet

Cryptography

- AWS Encryption SDK

Privacy

- DafnyVMC, SampCert

Code Generation

- DafnyBench, DafnyAutopilot, VerMCTS, Laurel, ...

Regular Expressions and Formal Languages

Regular Expressions as Datatype

In Dafny, we define regular expressions as inductive datatype:

```
datatype Exp<A> = | Zero
                  | One
                  | Char(A)
                  | Plus(Exp, Exp)
                  | Comp(Exp, Exp)
                  | Star(Exp)
```

On a high-level, `Exp` is the *smallest* structure closed under the constructors of regular expressions. It is the *initial* Σ-algebra.

$\Sigma X = 1 + 1 + A + X^2 + X^2 + X$

Formal Languages as Codatatype

In Dafny, we define formal languages extrinsically, as coinductive datatype:

```
codatatype Lang<!A> = Alpha(eps: bool, delta: A -> Lang<A>)
```

On a high-level, `Lang` is the *greatest* structure closed under the destructors of deterministic automata. It is the *final* B-coalgebra.

$BX = 2 \times X^A$

An Algebra of Formal Languages

An Algebra of Formal Languages, Extrinsically

Extrinsically, one can equip Lang with a Σ-algebra structure via:

```
function Zero<A>(): Lang {
  Alpha(false, (a: A) => Zero())
}

function One<A>(): Lang {
  Alpha(true, (a: A) => Zero())
}

function Singleton<A(==)>(a: A): Lang {
  Alpha(false, (b: A) => if a == b then One() else Zero())
}

function {:abstemious} Plus<A>(L1: Lang, L2: Lang): Lang {
  Alpha(L1.eps || L2.eps, (a: A) => Plus(L1.delta(a), L2.delta(a)))
}

function {:abstemious} Comp<A>(L1: Lang, L2: Lang): Lang {
  Alpha(L1.eps && L2.eps,
        (a: A) => Plus(Comp(L1.delta(a), L2),
                       Comp(if L1.eps then One() else Zero(), L2.delta(a))))
}

function Star<A>(L: Lang): Lang {
  Alpha(true, (a: A) => Comp(L.delta(a), Star(L)))
}
```

Denotational Semantics

Denotational Semantics as Induced Morphism

```
function Denotational<A(==)>(e: Exp): Lang {
  match e
  case Zero => Languages.Zero()
  case One => Languages.One()
  case Char(a) => Languages.Singleton(a)
  case Plus(e1, e2) => Languages.Plus(Denotational(e1), Denotational(e2))
  case Comp(e1, e2) => Languages.Comp(Denotational(e1), Denotational(e2))
  case Star(e1) => Languages.Star(Denotational(e1))
}
```

The Σ-algebra structures of `Exp` and `Lang` allow us to define `Denotational`.

Bisimilarity and Coinduction

```
greatest predicate Bisimilar<A(!new)>[nat](L1: Lang, L2: Lang) {
  && (L1.eps == L2.eps)
  && (forall a :: Bisimilar(L1.delta(a), L2.delta(a)))
}
```

Two languages that are equal are also bisimilar, but the reverse is not necessarily true, since there is no extensional equality for functions in Dafny.

Dafny automatically proves that e.g. bisimilarity is reflexive.

Bisimilarity and Coinduction 2

```
/* Pseudo code for illustration purposes */

predicate Bisimilar#<A(!new)>[k: nat](L1: Lang, L2: Lang)
  decreases k
{
  if k == 0 then
    true
  else
    && (L1.eps == L2.eps)
    && (forall a :: Bisimilar#[k-1](L1.delta(a), L2.delta(a)))
}

predicate Bisimilar<A(!new)>(L1: Lang, L2: Lang) {
  forall k: nat :: Bisimilar#[k](L1, L2)
}
```

Under the hood, Dafny transforms the *greatest* predicate `Bisimilar` into the above data.

Bisimilarity and Coinduction 2

```
greatest lemma BisimilarityIsReflexive<A(!new)>[nat](L: Lang)
  ensures Bisimilar(L, L)
{}
```

Dafny automatically proves that bisimilarity is reflexive.

Bisimilarity and Coinduction 4

```
/* Pseudo code for illustration purposes */

lemma BisimilarityIsReflexive#<A(!new)>[k: nat](L: Lang)
  ensures Bisimilar#[k](L, L)
  decreases k
{
  if k == 0 {
  } else {
    forall a ensures Bisimilar#[k-1](L.delta(a), L.delta(a)) {
      BisimilarityIsReflexive#[k-1](L.delta(a));
    }
  }
}

lemma BisimilarityIsReflexive<A(!new)>(L: Lang)
  ensures Bisimilar(L, L)
{
  forall k: nat ensures Bisimilar#[k](L, L) {
    BisimilarityIsReflexive#[k](L);
  }
}
```

Under the hood, Dafny transforms the *greatest* lemma `BisimilarityIsReflexive` into the above data.

Denotational Semantics as Algebra Homomorphism

```
ghost predicate IsAlgebraHomomorphism<A(!new)>(f: Exp -> Lang) {
  forall e :: IsAlgebraHomomorphismPointwise(f, e)
}

ghost predicate IsAlgebraHomomorphismPointwise<A(!new)>(f: Exp -> Lang, e: Exp) {
  Bisimilar<A>(
    f(e),
    match e
    case Zero => Languages.Zero()
    case One => Languages.One()
    case Char(a) => Languages.Singleton(a)
    case Plus(e1, e2) => Languages.Plus(f(e1), f(e2))
    case Comp(e1, e2) => Languages.Comp(f(e1), f(e2))
    case Star(e1) => Languages.Star(f(e1))
  )
}

lemma DenotationalIsAlgebraHomomorphism<A(!new)>()
  ensures IsAlgebraHomomorphism<A>(Denotational)
{
  forall e ensures IsAlgebraHomomorphismPointwise<A>(Denotational, e) {
    BisimilarityIsReflexive<A>(Denotational(e));
  }
}
```

`Denotational` is a Σ-algebra homomorphism (up to pointwise bisimilarity).

Operational Semantics

A Coalgebra of Regular Expressions

```
function Eps<A>(e: Exp): bool {
  match e
  case Zero => false
  case One => true
  case Char(a) => false
  case Plus(e1, e2) => Eps(e1) || Eps(e2)
  case Comp(e1, e2) => Eps(e1) && Eps(e2)
  case Star(e1) => true
}

function Delta<A(==)>(e: Exp): A -> Exp {
  (a: A) =>
    match e
    case Zero => Zero
    case One => Zero
    case Char(b) => if a == b then One else Zero
    case Plus(e1, e2) => Plus(Delta(e1)(a), Delta(e2)(a))
    case Comp(e1, e2) => Plus(Comp(Delta(e1)(a), e2),
                              Comp(if Eps(e1) then One else Zero, Delta(e2)(a)))
    case Star(e1) => Comp(Delta(e1)(a), Star(e1))
}
```

Using Brzozowski's derivatives, we equip `Exp` with a B-coalgebra structure.

Operational Semantics as Induced Morphism

```
function Operational<A(==)>(e: Exp): Lang {
  Alpha(Eps(e), (a: A) => Operational(Delta(e)(a)))
}
```

The B-coalgebra structures of `Exp` and `Lang` allow us to define `Operational`.

Operational Semantics as Coalgebra Homomorphism

```
ghost predicate IsCoalgebraHomomorphism<A(!new)>(f: Exp -> Lang) {
  && (forall e :: f(e).eps == Eps(e))
  && (forall e, a :: Bisimilar(f(e).delta(a), f(Delta(e)(a))))
}

lemma OperationalIsCoalgebraHomomorphism<A(!new)>()
  ensures IsCoalgebraHomomorphism<A>(Operational)
{
  forall e, a ensures Bisimilar<A>(Operational(e).delta(a), Operational(Delta(e)(a))) {
    BisimilarityIsReflexive(Operational(e).delta(a));
  }
}
```

`Operational` is a B-coalgebra homomorphism (up to pointwise bisimilarity).

Well-Behaved Semantics

The Status

$$\begin{tikzcd}
\Sigma(\mathtt{Exp}) \arrow[r, dashed, "\Sigma(\mathtt{Denotational})"] \arrow[d] & \Sigma(\mathtt{Lang}) \arrow[d, "{[\mathtt{Zero},\mathtt{One},\mathtt{Singleton},\mathtt{Plus},\mathtt{Comp},\mathtt{Star}]}"'] \\
\mathtt{Exp} \arrow[r, dashed, "\mathtt{Denotational}"] & \mathtt{Lang}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathtt{Exp} \arrow[r, dashed, "\mathtt{Operational}"] \arrow[d, "\langle \mathtt{Eps},\mathtt{Delta} \rangle"] & \mathtt{Lang} \arrow[d] \\
B(\mathtt{Exp}) \arrow[r, dashed, "B(\mathtt{Operational})"] & B(\mathtt{Lang})
\end{tikzcd}$$

The Goal

$$\begin{tikzcd}
\Sigma(\texttt{Exp}) \arrow[r, dashed, "\Sigma(\texttt{denotational}{\cong}\texttt{operational})"] \arrow[d] & \Sigma(\texttt{Lang}) \arrow[d] \\
\texttt{Exp} \arrow[r, dashed, "\texttt{denotational}{\cong}\texttt{operational}"] \arrow[d] & \texttt{Lang} \arrow[d] \\
B(\texttt{Exp}) \arrow[r, dashed, "B(\texttt{denotational}{\cong}\texttt{operational})"] & B(\texttt{Lang})
\end{tikzcd}$$

The Steps We Take

1. `denotational` is also a B-coalgebra homomorphism.
2. B-coalgebra homomorphisms are unique up to pointwise bisimilarity.
3. `denotational` and `operational` are equal up to pointwise bisimilarity.
4. `operational` is also a Σ-algebra homomorphism.

The Steps We Take: Highlights

1. `denotational` is also a B-coalgebra homomorphism.
   - Uses that bisimilarity is reflexive and congruence w.r.t. `Plus` and `Comp`.
2. B-coalgebra homomorphisms are unique up to pointwise bisimilarity.
   - Uses that bisimilarity is transitive.
3. `denotational` and `operational` are equal up to pointwise bisimilarity.
4. `operational` is also a Σ-algebra homomorphism.
   - Uses that bisimilarity is symmetric and a congruence w.r.t. `Star`.

(1a/4) Denotational Semantics as Coalgebra Homomorphism

```
lemma DenotationalIsCoalgebraHomomorphism<A(!new)>()
  ensures IsCoalgebraHomomorphism<A>(Denotational)
```

The proof is a bit more elaborate than the previous ones. We again use that bisimilarity is reflexive, but also that it is a congruence relation with respect to `Plus` and `Comp`:

```
greatest lemma PlusCongruence<A(!new)>[nat]
  (L1a: Lang, L1b: Lang, L2a: Lang, L2b: Lang)
  requires Bisimilar(L1a, L1b)
  requires Bisimilar(L2a, L2b)
  ensures Bisimilar(Plus(L1a, L2a), Plus(L1b, L2b))
{}
```

(1b/4) Denotational Semantics as Coalgebra Homomorphism

```
lemma CompCongruence<A(!new)>(L1a: Lang, L1b: Lang, L2a: Lang, L2b: Lang)
  requires Bisimilar(L1a, L1b)
  requires Bisimilar(L2a, L2b)
  ensures Bisimilar(Comp(L1a, L2a), Comp(L1b, L2b))
{
  ...
}
```

(1b/4) Denotational Semantics as Coalgebra Homomorphism

```
lemma CompCongruence<A(!new)>(L1a: Lang, L1b: Lang, L2a: Lang, L2b: Lang)
  requires Bisimilar(L1a, L1b)
  requires Bisimilar(L2a, L2b)
  ensures Bisimilar(Comp(L1a, L2a), Comp(L1b, L2b))
{
  forall k ensures Bisimilar#[k](Comp(L1a, L2a), Comp(L1b, L2b)) {
    ...
  }
}
```

(1b/4) Denotational Semantics as Coalgebra Homomorphism

```
lemma CompCongruence<A(!new)>(L1a: Lang, L1b: Lang, L2a: Lang, L2b: Lang)
  requires Bisimilar(L1a, L1b)
  requires Bisimilar(L2a, L2b)
  ensures Bisimilar(Comp(L1a, L2a), Comp(L1b, L2b))
{
  forall k ensures Bisimilar#[k](Comp(L1a, L2a), Comp(L1b, L2b)) {
    if k != 0 {
      ...
    }
  }
}
```

(1b/4) Denotational Semantics as Coalgebra Homomorphism

```
lemma CompCongruence<A(!new)>(L1a: Lang, L1b: Lang, L2a: Lang, L2b: Lang)
  requires Bisimilar(L1a, L1b)
  requires Bisimilar(L2a, L2b)
  ensures Bisimilar(Comp(L1a, L2a), Comp(L1b, L2b))
{
  forall k ensures Bisimilar#[k](Comp(L1a, L2a), Comp(L1b, L2b)) {
    if k != 0 {
      var k' :| k' + 1 == k;
      CompCongruenceHelper(k', L1a, L1b, L2a, L2b);
    }
  }
}

lemma CompCongruenceHelper<A(!new)>(k: nat, L1a: Lang, L1b: Lang, L2a: Lang, L2b: Lang)
  requires forall n : nat :: n <= k + 1 ==> Bisimilar#[n](L1a, L1b)
  requires forall n : nat :: n <= k + 1 ==> Bisimilar#[n](L2a, L2b)
  ensures Bisimilar#[k+1](Comp(L1a, L2a), Comp(L1b, L2b))
{
  ...
}
```

(2/4) Coalgebra Homomorphisms Are Unique

```
lemma UniqueCoalgebraHomomorphism<A(!new)>
(f: Exp -> Lang, g: Exp -> Lang, e: Exp)
  requires IsCoalgebraHomomorphism(f)
  requires IsCoalgebraHomomorphism(g)
  ensures Bisimilar(f(e), g(e))
```

At the heart of the proof lies the observation that bisimilarity is transitive:

```
greatest lemma BisimilarityIsTransitive<A(!new)>[nat]
(L1: Lang, L2: Lang, L3: Lang)
  requires Bisimilar(L1, L2) && Bisimilar(L2, L3)
  ensures Bisimilar(L1, L3)
{}
```

(3/4) Denotational and Operational Semantics Are Bisimilar

```
lemma OperationalAndDenotationalAreBisimilar<A(!new)>(e: Exp)
  ensures Bisimilar<A>(Operational(e), Denotational(e))
{
  OperationalIsCoalgebraHomomorphism<A>();
  DenotationalIsCoalgebraHomomorphism<A>();
  UniqueCoalgebraHomomorphism<A>(Operational, Denotational, e);
}
```

From the previous results, we can immediately deduce our main claim.

(4/4) Operational Semantics as Algebra Homomorphism

```
lemma OperationalIsAlgebraHomomorphism<A(!new)>()
  ensures IsAlgebraHomomorphism<A>(Operational)
```

The main idea of the proof is to take advantage of that `denotational` is an algebra homomorphism and pointwise bisimilar to `operational`.

We also use that bisimilarity is symmetric and a congruence with respect to `Star`.

Related and Future Work

Some Related Work

Derivatives of Regular Expressions (Brzozowski, 1964)

- Introduces the B-coalgebra structure on `Lang`.

Towards a Mathematical Operational Semantics (Turi; Plotkin, 1997)

- Introduces the concept of well-behaved semantics in the context of bialgebras.

A Bialgebraic Review of Deterministic Automata, Regular Expressions and Languages (Jacobs, 2006)

- Discusses a bialgebraic perspective on regular expressions.

Formal Languages, Formally and Coinductively (Traytel, 2017)

- Represents languages coinductively in Isabelle.

Some Future Work

Kleene Algebra With Tests

- Extends Kleene Algebra with primitives in a Boolean algebra.

Guarded Kleene Algebra with Tests

- Axiomatises an efficiently decidable fragment of Kleene Algebra with Tests.

NetKAT

- Extends Kleene Algebra with Tests with primitives for network verification.