Probabilistic Guarded Kleene Algebra with Tests

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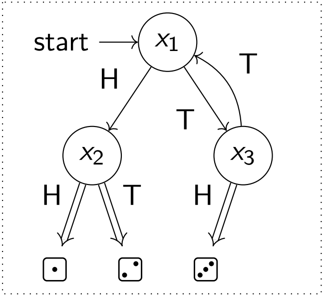
Knuth-Yao algorithm

How to simulate  using   ?

```
while true do
    if flip(0.5) then
        if flip(0.5) then
            return 1 // heads-heads
        else
            return 2 // heads-tails
    else
        if flip(0.5) then
            return 3 // tails-heads
        else
            skip // tails-tails
```

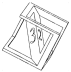


Knuth-Yao algorithm

Correctness?

```
while true do
    if flip(0.5) then
        if flip(0.5) then
            return 1 // heads-heads
        else
            return 2 // heads-tails
    else
        if flip(0.5) then
            return 3 // tails-heads
        else
            skip // tails-tails
```

$\stackrel{?}{\equiv}$



```
if flip(1/3) then
    return 1
else
    if flip(0.5) then
        return 2
    else
        return 3
```

Kleene Algebra with Tests

$$b, c \in \texttt{BExp} ::= \mathbb{0} \mid \mathbb{1} \mid t \in T \mid b \cdot c \mid b + c \mid \bar{b}$$
$$e, f \in \texttt{Exp} ::= b \in \texttt{BExp} \mid p \in \texttt{Act} \mid e + f \mid e \cdot f \mid e^*$$

$$\texttt{e;f} \stackrel{\text{def}}{=} ef$$
$$\texttt{if b then e else f} \stackrel{\text{def}}{=} be + \bar{b}f$$
$$\texttt{while b do e} \stackrel{\text{def}}{=} (be)^* \bar{b}$$

Kleene Algebra with Tests

$\texttt{while } b \texttt{ do } e \equiv \texttt{if } b \texttt{ then } (e;\texttt{while } b \texttt{ do } e)$
$\texttt{if } b \texttt{ then } e \texttt{ else } f \equiv \texttt{if } \bar{b} \texttt{ then } f \texttt{ else } e$

Automata on guarded strings: $Q \to 2^{\texttt{At}} \times Q^{\texttt{At} \times \texttt{Act}}$

Guarded Kleene Algebra with Tests

Replace $(+)$ and $(^*)$ with their Boolean guarded vesions

$$
\begin{array}{rlll}
e, f \in \mathtt{Exp} ::= & & b \in \mathtt{BExp} & \textbf{assert } b \\
& \mid & p \in \mathtt{Act} & \textbf{do } p \\
& \mid & e \cdot f & e\textbf{;}\, f \\
& \mid & {\color{red} e +_b f} & \textbf{if } b \textbf{ then } e \textbf{ else } f \\
& \mid & {\color{red} e^{(b)}} & \textbf{while } b \textbf{ do } e
\end{array}
$$

- Decidable in nearly linear time
- Sound and complete Salomaa-style axiomatisation
- Strictly deterministic automata on guarded strings:
  $Q \times \mathtt{At} \to \{\checkmark, \text{✗}\} + \mathtt{Act} \times Q$

Probabilistic Guarded Kleene Algebra with Tests

$$
\begin{array}{rlll}
e, f \in \texttt{Exp} ::= & & b \in \texttt{BExp} & \textbf{assert } b \\
& \mid & p \in \texttt{Act} & \textbf{do } p \\
& \mid & e \cdot f & e\textbf{;}\, f \\
& \mid & e +_b f & \textbf{if } b \textbf{ then } e \textbf{ else } f \\
& \mid & e^{(b)} & \textbf{while } b \textbf{ do } e \\
& \mid & e \oplus_r f & \textbf{if flip(}r\textbf{) then } e \textbf{ else } f \\
& \mid & e^{[r]} & \textbf{while flip(}r\textbf{) do } e \\
& \mid & v \in V & \textbf{return } v
\end{array}
$$

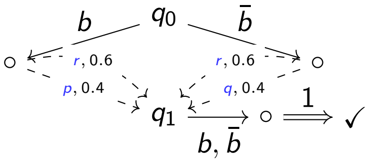
Correctness of Knuth-Yao in ProbGKAT

$$((v_1 \oplus_{\frac{1}{2}} v_2) \oplus_{\frac{1}{2}} (v_3 \oplus_{\frac{1}{2}} \mathbb{1}))^{(\mathbb{1})} \equiv v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3)$$

Operational model

Automata with the transition function of the type

$$Q \times \texttt{At} \to \mathcal{D}_\omega(\{\checkmark, \text{✗}\} + V + \texttt{Act} \times Q)$$

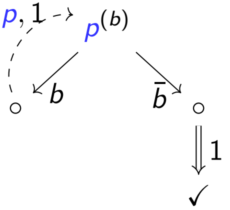


$(p +_b q) \oplus_{0.4} r$

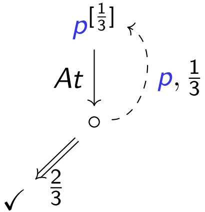
- Notion of equivalence: bisimulation associated with the type functor
- Can be decided in $O(n^2 \log(n))$ using a generic minimization algorithm (Wißmann et al, 2020)

Operational semantics

$f \stackrel{\text{def}}{=} p^{(b)}$



$g \stackrel{\text{def}}{=} p^{[\frac{1}{3}]}$



Axiomatisation of bisimulation equivalence

Guarded Choice Axioms

G1. $e +_b e \equiv e$

G2. $e +_{\mathbb{1}} f \equiv e$

G3. $e +_b f \equiv f +_{\bar{b}} e$

G4. $(e +_b f) +_c g \equiv e +_{bc} (f +_c g)$

G5. $(e +_b f) \equiv (be +_b f)$

Probabilistic Choice Axioms

P1. $e \oplus_r e \equiv e$

P2. $e \oplus_1 f \equiv e$

P3. $e \oplus_r f \equiv f \oplus_{(1-r)} e$

P4. $(e \oplus_r f) \oplus_s g \equiv e \oplus_{rs} (f \oplus_{\frac{(1-r)s}{1-rs}} g)$

Sequencing axioms

AS. $(ef)g \equiv e(fg)$

AL. $\mathbb{0}e \equiv \mathbb{0}$

VS. $ve \equiv v$

NL. $\mathbb{1}e \equiv e$

NR. $e\mathbb{1} \equiv e$

GDR. $(e +_b f)g \equiv eg +_b fg$

PDR. $(e \oplus_r f)g \equiv eg \oplus_r fg$

Distributivity axiom

D. $(e \oplus_r f) +_b (e \oplus_r g) \equiv e \oplus_r (f +_b g)$

Loop axioms

GU. $e^{(b)} \equiv ee^{(b)} +_b \mathbb{1}$

PU. $e^{[r]} \equiv ee^{[r]} \oplus_r \mathbb{1}$

GT. $(e +_c \mathbb{1})^{(b)} \equiv (ce)^{(b)}$

PT. $(e \oplus_s \mathbb{1})^{[r]} \equiv e^{[\frac{rs}{1-r(1-s)}]}$

PB. $e^{[1]} \equiv e^{(\mathbb{1})}$

PGT. $(e \oplus_r \mathbb{1})^{(b)} \equiv e^{(b)} \quad (r \neq 0)$

GF. $\dfrac{E(e) \equiv \mathbb{0} \quad g \equiv eg +_b f}{g \equiv e^{(b)} f}$

PF. $\dfrac{E(e) \equiv \mathbb{0} \quad g \equiv eg \oplus_r f}{g \equiv e^{[r]} f}$

Laws involving division apply when the denominator is not zero.

- Two Salomaa-like inference rules for introducing the loops
- Sound: $e \equiv f \implies e \underline{\leftrightarrow} f$
- Complete if we use the generalised version of GF and UA allowing to obtain unique solutions to arbitrary guarded systems

Knuth-Yao example revisited: axiomatic reasoning

$$d = v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3) \text{ and } g = (v_1 \oplus_{\frac{1}{2}} v_2) \oplus_{\frac{1}{2}} (v_3 \oplus_{\frac{1}{2}} \mathbb{1})$$

$g^{(\mathbb{1})} \equiv \left((v_1 \oplus_{\frac{1}{2}} v_2) \oplus_{\frac{1}{2}} (v_3 \oplus_{\frac{1}{2}} \mathbb{1})\right)^{(\mathbb{1})}$	Definition of g
$\equiv (((v_1 \oplus_{\frac{1}{2}} v_2) \oplus_{\frac{2}{3}} v_3) \oplus_{\frac{3}{4}} \mathbb{1})^{(\mathbb{1})}$	Probabilistic skew associativity
$\equiv ((v_1 \oplus_{\frac{1}{2}} v_2) \oplus_{\frac{2}{3}} v_3)^{(\mathbb{1})}$	Loop tightening: $(e \oplus_r \mathbb{1})^{(b)} \equiv e^{(b)}$
$\equiv (v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3))^{(\mathbb{1})}$	Probabilistic skew associativity
$\equiv (v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3))(v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3))^{(\mathbb{1})} +_{\mathbb{1}} \mathbb{1}$	Loop unrolling: $e^{(b)} = ee^{(b)} +_b \mathbb{1}$
$\equiv (v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3))d^{(\mathbb{1})}$	Definition of d and $e +_{\mathbb{1}} f \equiv e$
$\equiv (v_1 d^{(\mathbb{1})} \oplus_{\frac{1}{3}} (v_2 d^{(\mathbb{1})} \oplus_{\frac{1}{2}} v_3 d^{(\mathbb{1})}))$	Right distributivity of ; over $\oplus$
$\equiv (v_1 \oplus_{\frac{1}{3}} (v_2 \oplus_{\frac{1}{2}} v_3))$	Sequencing after **return**: $ve \equiv v$
$= d$	Definition of d

Summary

- GKAT + probabilistic choice and loops + return variables
- Operational semantics: $\texttt{Exp} \times \texttt{At} \to \mathcal{D}_\omega(\{\checkmark, \text{✗}\} + V + \texttt{Act} \times \texttt{Exp})$
- Decidable in $O(n^2 \log(n))$ time
- A sound axiomatisation of bisimulation equivalence

Things I haven't talked about

- Coalgebraic semantics
- Model checking ProbGKAT automata with Prism (Kwiatkowska et al, 2011) or STORM (Hensel et al, 2022)
- Fundamental Theorem
- Verifying discrete simulation protocols

Some references

(Knuth & Yao, 1976) "The complexity of nonuniform random number generation"

(Kozen, 1997) "Kleene Algebra with Tests"

(Smolka, Foster, Hsu, Kappé, Kozen & Silva, 2019) "Guarded Kleene Algebra with Tests: Verification of Uninterpreted Programs in Nearly Linear Time"

(Schmid, Kappé, Kozen & Silva., 2021) "Guarded Kleene Algebra with Tets: Coequations, Coinduction and Completeness"

(Wißmann, Dorsch, Milius & Schröder, 2020) "Efficient and Modular Coalgebraic Partition Refinement"