MAT3253 Complex Variables
Lecture Notes

Kenneth Shum

Spring 2025

Contents

References

[Ahlfors] Ahlfors, *Complex Analysis – An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd edition, McGraw-Hill, New York, 1979.

[BakNewman] J. Bak and D. J. Newman, *Complex analysis*, third edition, Springer, 2010.

[BrownChurchill] J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th edition, McGraw-Hill, New York, 2014.

[Gong] S. Gong and Y. Gong, *Concise Complex Analysis*, revised edition, World Scientific2007.

[Miranda] R. Miranda, *Algebraic curves and Riemann surface*, Graduate Studies in Mathematics, vol. 5, American Mathematical Society, 1995.

[Rudin] W. Rudin, *Principles of mathematical analysis*, 3rd edition, McGraw-Hill, 1976.

[Spivak] M. Spivak, *Calculus on manifolds – a modern approach to classical theorems of advanced calculus*, Addison-Wesley, 1965.

[Stein] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, Princeton University Press, New York, 2003.

0.1 Notation

$\mathbb{N}$	Set of integers larger than or equal to 1
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
sup, inf	Supremum and infinum
$\limsup$	Limit superior
$\mathbb{C}$	Set of complex numbers
$\bar{z}$, z^*	Conjugate of complex number z
$\mathrm{Re}(z)$	Real part of z
$\mathrm{Im}(z)$	Imaginary part of z
$\lvert z\rvert$	The modulus/absolute value/magnitude of z
$\arg(z)$	The argument/angle of z
$\mathrm{Arg}(z)$	Principal argument of z
$D(z,r)$	Open disc with radius r centered at z
$(df)_{\mathbf{p}}$	Differential of f at a point $\mathbf{p}$
f', df/dz	Complex derivative of function f
$\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$	Two differential operators
$\int_C f\,dz$	Complex integral of f along a curve C
$\mathrm{Res}(f;z_0)$	Residue of function f at the point z_0
$n(C;z_0)$	Winding number of a closed curve C relative to the point z_0

1 Definitions and arithmetic of complex numbers

Summary:

- Axioms of complex numbers
- Models of complex field

We begin with the rational numbers. The set of rational numbers is denoted by $\mathbb{Q}$. A rational number can be written in the form a/b, where a and b are integer. Formally, we can represent it by a pair (a, b) of integers. Two pairs (a, b) and (c, d) are said to be equal if $ad = bc$. So strictly speaking, a rational number is an equivalence class of pairs of integers.

Addition and multiplication are performed according to the following rules:

$$(a, b) + (c, d) \triangleq (ad + bc, bd)$$
$$(a, b) \cdot (c, d) \triangleq (ac, bd).$$

For example, $1/2$ and $1/3$ are represented by $(1, 2)$ and $(1, 3)$, respectively, and their sum is computed by $(1 \cdot 3 + 2 \cdot 1, 2 \cdot 3) = (5, 6)$. The numbers in the form $(k, 1)$ behave in the same way as the integers. We see that integers are embedded in this number system. This number system has the feature that we can take the reciprocal of any non-zero number. If $a \neq 0 \neq b$, we have $(a, b) \cdot (b, a) = (ab, ab)$, which is equivalent to $(1, 1)$.

The rational number is not closed under taking supremum and infimum. The *supremum* of a set of numbers A is defined as the number s, such that s is larger than any number in A, and if s' is any other upper bound of A, we must have $s \leq s'$. In other words, the supremum of a set A is the least upper bound. Likewikse, the *infimum* of a set A is defined as the greatest lower bound of A. It is well known that the set

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

does not have a smallest upper bound in $\mathbb{Q}$. Indeed, we can find the supremum of this set in the set of real numbers, namely, the irrational number $\sqrt{2}$. The real numbers $\mathbb{R}$ is constructed to guarantee that every subset of $\mathbb{R}$ has supremum and infimum.

1.1 Axioms and models of complex field

The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ have a common algebraic structure.

Definition 1.1. A number system $(F, +, \cdot)$ is called a *field* if it satisfies the following conditions.

1. (closed) $a + b \in F$ for all $a, b \in F$.
2. (associative) $(a + b) + c = a + (b + c)$, for all $a, b, c \in F$.
3. (commutative) $a + b = b + a$, for all $a, b \in F$.
4. (existence of zero) $\exists 0 \in F$ such that $0 + a = a + 0 = a$, for all $a \in F$.
5. (additive inverse) for all $a \in F$, $\exists a' \in F$ such that $a + a' = 0$.
6. (closed) $a \cdot b \in F$ for all $a, b \in F$.
7. (associative) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in F$.
8. (commutative) $a \cdot b = b \cdot a$, for all $a, b \in F$.
9. (existence of one) $\exists 1 \in F$ such that $1 \neq 0$ and $1 \cdot a = a \cdot 1 = a$, for all $a \in F$.
10. (multiplicative inverse) for all $a \in F \setminus \{0\}$, $\exists a'' \in F$ such that $a \cdot a'' = 1$.
11. (distributive) $a \cdot (b + c) = a \cdot b + a \cdot c$, for all $a, b, c \in F$.

A subset K of a field F is called a *subfield* of F if the elements in K satisfy the all axioms of a field, and F is called an *extension* of K.

Example 1.1. The set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$ are fields. Moreover, $\mathbb{Q}$ is a subfield of $\mathbb{R}$.

Definition 1.2. A *complex field* is a field that extends $\mathbb{R}$ and contains a special number i such that $i^2 = -1$, and is generated by $\mathbb{R}$ and i. A complex field is denoted by the symbol $\mathbb{C}$.

Remark. The phrase "generated by $\mathbb{R}$ and i" means that there is no proper subfield that contains all real numbers and the special number i. Consider the rational functions in the form $f(T)/g(T)$, where $f(T)$ and $g(T)$ are polynomial in T with complex coefficients. The set of all complex rational functions is a field, and it contains all real numbers and the number i. However, it is too large to be considered a complex field. If we mix the real numbers and the special number i, we will never see the transcendental variable T.

1.1.1 Complex numbers as points on a plane

We can implement a complex field by regarding a point (x, y), where x and y are real numbers, as a number. The set of all numbers form a plane called the *complex plane* or *Argand diagram*. The point (x, y) can also be regarded as a vector. In this implementation, addition is just defined by vector addition.

Let

$$F_1 \triangleq \{(a, b) :\ a, b \in \mathbb{R}\}. \tag{1.1}$$

Two pairs (a, b) and (c, d) are *equal* if $a = c$ and $b = d$.

The addition and multiplication operators for two points in F_1 are defined by

$$\begin{aligned}(a, b) + (c, d) &\triangleq (a + c,\ b + d),\\ (a, b) \cdot (c, d) &\triangleq (ac - bd,\ ad + bc).\end{aligned}$$

The real numbers are embedded in F_1 by $x \mapsto (x, 0)$. By identifying $x_1 \in \mathbb{R}$ with $(x_1, 0)$ and $x_2 \in \mathbb{R}$ with $(x_2, 0)$, the sum and product of x_1 and x_2 are respectively

$$\begin{aligned}(x_1, 0) + (x_2, 0) &= (x_1 + x_2, 0),\ \text{and}\\ (x_1, 0) \cdot (x_2, 0) &= (x_1 x_2, 0).\end{aligned}$$

Geometrically, it is just saying that the horizontal axis in the complex plane is the real number line. We can show that this number system satisfies all the axioms of field.

The number $(1, 0)$ is the multiplicative identity, as

$$(1, 0)(c, d) = (1 \cdot c - 0 \cdot d, 1 \cdot d + 0 \cdot c) = (c, d).$$

The special number $(0, 1)$ has the property

$$(0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1, 0).$$

This is the number i required in the definition of complex field.

Every number (x, y) in this system can be expressed as a linear combination of $1 = (1, 0)$ and $i = (0, 1)$,

$$(x, y) = x(1, 0) + y(0, 1) = x + iy \tag{1.2}$$

We next show how to divide two complex numbers in principle. Suppose we want to divide the complex number (a, b) by complex number (c, d), where a, b, c and d are real numbers. We want to find a complex number (e, f) such that

$$(e, f) \cdot (c, d) = (a, b).$$

If we multiply the two numbers on the left, and equate the corresponding components on both sides, we obtain a system of linear equations

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

So, the computation of division can be done by solving a system of two linear equations, provided that the determinant of the matrix in the above equation is non zero. As the determinant is equal to $c^2 + d^2$, this can be done when c and d are not both zero.

1.1.2 Complex numbers as square matrices

In the second model, a complex number is represented by a 2×2 matrix.

The set of all complex numbers in the second model is

$$F_2 \triangleq \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}. \tag{1.3}$$

Addition and multiplication are performed using the usual matrix addition and multiplication. The additive and multiplicative identities are the zero matrix and identity matrix, respectively. The arithmetic is exactly the same as in the first model, but now it is expressed in matrix format:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix}$$
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{bmatrix}.$$

The matrices in this model constitute a field.

A real number x is represented by a diagonal matrix.

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}.$$

The subset of diagonal matrices is isomorphic to the real number system

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & 0 \\ 0 & x_1 + x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 & 0 \\ 0 & x_1 x_2 \end{bmatrix}.$$

We let i denote the special matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We can easily check that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A complex number in matrix form can be decomposed as

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.4}$$

1.1.3 Complex numbers as polynomials

A polynomial in $\mathbb{R}$ is in the form

$$a_0 + a_1 T + a_2 T^2 + \cdots + a_d T^d$$

where $a_0, \ldots, a_d$ are real numbers, and d is called the degree of the polynomial. The symbol T is an indeterminate. The set of all polynomials in T with real coefficients is denoted by $\mathbb{R}[T]$.

We impose an equivalence relation on $\mathbb{R}[T]$ by regarding two polynomials $f(T)$ and $g(T)$ to be equivalent if $f(T) - g(T)$ is divisible by $T^2 + 1$. Each equivalence class can be represented by a polynomial in the form $a + bT$, for some choice of real numbers a and b. Formally, the resulting collection of equivalence class is denoted by $\mathbb{R}[T]/(T^2 + 1)$.

We can take $\mathbb{R}[T]/(T^2+1)$ as the carrier set, and perform addition and multiplication using the usual polynomial addition and multiplication, modulo the polynomial T^2+1. The resulting algebraic system is a complex field.

All three representations are isomorphic to each other. The first construction emphasizes that a complex number is a pair of real numbers. The second construction emphasizes that complex multiplication is the same as multiplying matrices in a special form. The one-to-one correspondence is given by

$$(a,b) \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \longleftrightarrow a+bT$$

Definition 1.3. The special number i that satisfies $i^2=-1$ is called the *imaginary unit*. For a complex number $z=a+bi$ in $\mathbb{C}$, define the *real* and *imaginary part* of z as

$$\operatorname{Re}(z) \triangleq a \quad \text{and} \quad \operatorname{Im}(z) \triangleq b.$$

In the three models of complex numbers we described above, the imaginary units are

$$(0,1), \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ and } T$$

respectively.

We will use the notation

$$a+bi$$

to represent a complex number. We thus see that the complex numbers form a vector space of dimension 2 over the real numbers. In (1.2) and (1.4), we use1 and i as a basis over $\mathbb{R}$.

With this notation, the multiplication of complex numbers can be written as

$$(a+bi)(c+di) = ac-bd+i(ad+bc). \tag{1.5}$$

We have a coercion of real numbers into the set of complex numbers

$$r \mapsto r+0i.$$

A complex number with zero imaginary part is regarded as a real number. On the other hand, a complex number with zero real part is said to be *purely imaginary*.

2 Geometry of complex numbers

Summary:

- Polar form and conjugate
- DeMoivre formula
- Geometry on the complex plane

2.1 Polar form and complex conjugate

A nonzero complex number, like a vector in $\mathbb{R}^2$, can be written in polar form. Given any nonzero complex number $x + iy$, we can write it in the form

$$x + iy = r(\cos\theta + i\sin\theta)$$

for some nonnegative integer r and angle θ. The number r signifies the length of the vector corresponding to $x + iy$, and θ is the angle from the positive axis to the vector $x + iy$. With the second construction of complex numbers (1.3), we can write the 2×2 matrix corresponding to $a + bi$ as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix that represents the action of rotating counter-clockwise by θ.

Proposition 2.1. *Given $z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$ in polar form, the product of z_1 and z_2 can be computed by*

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Proof. The proof follows from the definition of complex multiplication and basic trigonometric identities,

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\big[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big] \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned}$$

□

The geometric meaning of multiplication by $a + bi = r(\cos\theta + i\sin\theta)$ is thus, (i) first rotate by θ counter-clockwise, then (ii) scale up (or down) by a factor of r.

Definition 2.2. The *argument* of a nonzero complex number z is defined as the angle from the positive real axis to the straight line from 0 to z. We write $\arg(z)$ to denote the argument of z. The argument of $z = 0$ is not defined. The argument of z is also called the *phase* or the *angle*.
The *modulus* of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The modulus of z is also called the *absolute value*, the *magnitude* or the *radius*.

The argument of a complex number is defined up to integral multiple of 2π, because we get the same point on the complex plane if we rotate around the origin by 2π radian. The argument function is thus a *multi-function*. The argument of 0 is undefined.

A nonzero complex number $z = x + iy$ can be written in *polar form*

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

where $r > 0$ is the modulus of z and θ is an argument of z.

Example 2.1. The number -1 in polar form can be written as

$$-1 = 1 \cdot (\cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k)),$$

where k is an integer. The argument of -1 can be any number in the set $\{\pi + 2\pi k : k \in \mathbb{Z}\}$. We may write

$$\arg(-1) = \pi + 2\pi k, \quad \text{for } k \in \mathbb{Z}.$$

To eliminate the ambiguity in the argument function, one may pick a value in a pre-defined range. The common choices of this range is (i) $(-\pi, \pi]$, or (ii) $[0, 2\pi)$. The unique value that falls within the chosen range is called the *principal argument*. We remark that the notion of principal argument is relative to the choice of the range, and hence is a matter of convention, with the purpose of simplifying notation.

Definition 2.3. Suppose we choose the interval $(-\pi, \pi]$ as the range. Given a nonzero complex number z, we define the *principal argument* of z as the unique angle θ_0 (in radian) in $(-\pi, \pi]$ such that

$$\cos(\theta_0) = \frac{\mathrm{Re}(z)}{|z|}$$
$$\sin(\theta_0) = \frac{\mathrm{Im}(z)}{|z|}.$$

In this notes we write $\mathrm{Arg}(z)$ to denote the principal argument, following the use of notation in [BrownChurchill]. The argument function $\arg(z)$ starting with small letter a is a multi-valued function.

Example 2.2. If we take $(\pi, \pi]$ as the range, the principal argument of $-1 - i$ is $-3\pi/4$.

Definition 2.4. Define the *complex conjugate* of a complex number $z = a + bi$ by

$$\bar{z} \triangleq z^* \triangleq a - bi.$$

Geometrically, the complex conjugate of a complex number z is the reflection of z along the real axis. The modulus is the distance between the origin and the point z in the complex plane.

Proposition 2.5. *(i)* $(z^*)^* = z$ *for any* $z \in \mathbb{C}$.
(ii) $z^* = z$ *if and only if* z *is real.*
(iii) $|z|^2 = zz^*$.
(iv) Given any two complex numbers z_1 *and* z_2 *in* $\mathbb{C}$,

$$(z_1 + z_2)^* = z_1^* + z_2^* \quad \textit{and} \quad (z_1 z_2)^* = z_1^* z_2^*.$$

Part (iv) in Prop. 2.5 says that the reflection of the sum (resp. product) of two complex numbers is the same as the sum (resp. product) of the two points obtained by reflection. The proof is simple and is omitted. Using Prop. 2.5, we can show that the modulus is a multiplicative function.

Proposition 2.6. *For any two complex numbers $z_1, z_2 \in \mathbb{C}$,*

$$|z_1 z_2| = |z_1|\,|z_2|.$$

Proof. Use the fact that $|z|^2 = zz^*$ for any $z \in \mathbb{C}$, and complex multiplication is commutative

$$|z_1 z_2|^2 = (z_1 z_2)(z_1 z_2)^* = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2\,|z_2|^2.$$

□

If we substitute z_1 by $a + bi$ and z_2 by $c + di$ in the proposition, we recover a classical identity about sum of squares:

$$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2). \tag{2.1}$$

The relationship between the real part, imaginary part and complex conjugate are

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - z^*}{2i}. \tag{2.2}$$

We can use complex conjugate to calculate division efficiently. Suppose we want to divide $z_1 = a + bi$ by $z_2 = c + di$, where c and d are not zero. We multiply and divide by the conjugate of z_2,

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}. \tag{2.3}$$

We can also do complex division using the 2×2 representation of complex numbers. Division is the same as taking matrix inverse,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2} = \frac{1}{c^2 + d^2} \begin{bmatrix} ac + bd & ad - bc \\ bc - ad & ac + bd \end{bmatrix}.$$

Remark. We note that $c^2 + d^2$ is the determinant of the matrix $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ and is the same as the square of the absolute value of $c + di$. The conjugate operation in matrix representation is just the transpose operation.

2.2 Complex numbers in Python

Complex number is a built-in type in Python. The imaginary unit is represented by the symbol j.

```
# Basic arithemtic of complex numbers
z1 = 2 + 3j
z2 = -1 + 0.2j

print(f"The sum of {z1} and {z2} is {z1+z2}.")
print(f"The difference of {z1} and {z2} is {z1-z2}.")
print(f"The product of {z1} and {z2} is {z1*z2}.")
print(f"The quotient of {z1} and {z2} is {z1/z2}.")
```

The program output is:

```
The sum of (2+3j) and (-1+0.2j) is (1+3.2j).
The difference of (2+3j) and (-1+0.2j) is (3+2.8j).
The product of (2+3j) and (-1+0.2j) is (-2.6-2.6j).
The quotient of (2+3j) and (-1+0.2j) is
   (-1.346153846153846-3.269230769230769j).
```

Commands for the real part, imaginary part and conjugate is part of the data structure.

```
# Real part, imaginary part, and conjugate

z = 3.2-5.3j
print(f"Real Part of {z} = {z.real}")
print(f"Imaginary Part of {z} = {z.imag}")
print(f"Conjugate of {z} = {z.conjugate()}")
```

If we run this program, we get

```
Real Part of (3.2-5.3j) = 3.2
Imaginary Part of (3.2-5.3j) = -5.3
Conjugate of (3.2-5.3j) = (3.2+5.3j)
```

The argument and modulus are called absolute value and phase in Python.

```python
# Absolute value and phase
from cmath import phase
from numpy import abs

z = 12+34j
print (f"z = {z}")
print (f"Argument of z = {phase(z)}")
print (f"Magnitude of z = {abs(z)}")
```

The command phase and abs are imported from library cmath and numpy, respectively. The returned value of phase() is in radian.

```
z = (12+34j)
Argument of z = 1.2315037123408519
Magnitude of z = 36.05551275463989
```

We can check Proposition 2.1 by the Python program below.

```python
from cmath import phase
from numpy import abs, sin, cos

z1 = 12+34j
z2 = 3-6j
r1 = abs(z1)   # absolute value of z1
r2 = abs(z2)   # absolute value of z2
theta1 = phase(z1)   # argument of theta1
theta2 = phase(z2)   # argument of theta2
print (f"z1 = {z1}, z2 = {z2}")
print (f"z1*z2 = {z1*z2}")   # compute the product directly

w = r1*r2*(cos(theta1+theta2) + sin(theta1+theta2)*1j)
print (f"The above product should be equal to {w}")
```

The variables z1 and z2 can be any complex numbers. The result of z1*z2 and the value of w are the same, up to numerical error.

```
z1 = (12+34j), z2 = (3-6j)
z1*z2 = (240+30j)
The above product should be equal to
   (240+30.000000000000007j)
```

2.3 Complex number in LEAN

We can encode the mathematical statements and proofs using LEAN and the Mathlib library. In LEAN, real numbers are constructed by Cauchy sequences. There is no precision issue, and hence, when we say real number x is equal to real number y, we mean exact equality. However, the real number type is not computable in LEAN, which is in contrast to Python and Matlab.

Given that real numbers are available, complex number is represented by a data structure that contains two real numbers: the real part and the imaginary part.

2.4 DeMoivre formula and roots of complex numbers

The DeMoivre formula describes what happens to the argument of a complex number when we raise it to some power.

Theorem 2.7 (DeMoivre formula)**.** *For any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, we have*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{2.4}$$

Proof. The formula is obviously true when $n = 0$ or $n = 1$. We apply mathematical induction to establish (2.4) for all positive integers n and for all real numbers θ. Suppose $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ holds for some positive integer n. Then

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\
&= (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta) \\
&= (\cos n\theta\cos\theta - \sin n\theta\sin\theta) + i(\cos n\theta\sin\theta + \sin n\theta\cos\theta) \\
&= \cos(n+1)\theta + i\sin(n+1)\theta.
\end{aligned}$$

For negative n, we let $m = -n$ and re-use the result we just proved for positive integers.

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^n &= \frac{1}{(\cos\theta + i\sin\theta)^m} \\
&= \frac{1}{\cos m\theta + i\sin m\theta} \cdot \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta} \\
&= \cos m\theta - i\sin m\theta \\
&= \cos n\theta + i\sin n\theta.
\end{aligned}
$$

This proves the DeMoivre formula for all integers. $\square$

Using the matrix representation of complex numbers, the DeMoivre's formula can be stated as

$$
\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}.
$$

Geometrically speaking, this says that rotating n times by an angle θ is the same as rotating once by angle $n\theta$.

Example 2.3. Compute $(-1 + i\sqrt{3})^8$.

The complex number $-1 + i\sqrt{3}$ in polar form is

$$2(-1/2 + i\sqrt{3}/2) = 2(\cos(2\pi/3) + i\sin(2\pi/3)).$$

Hence, with the use of DeMoivre's formula, we get

$$(-1 + i\sqrt{3})^8 = 2^8(\cos(8 \cdot 2\pi/3) + i\sin(8 \cdot 2\pi/3)) = 256(\cos(4\pi/3) + i\sin(4\pi/3)).$$

In Cartesian form, the answer is $128(-1 - i\sqrt{3})$.

Example 2.4. Compute $(-1 + i)^{20}$.

Express $-1 + i$ in polar form $\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$. By DeMoivre's formula,

$$
\begin{aligned}
(-1 + i)^{20} &= 2^{20/2}(\cos(20 \cdot 3\pi/4) + i\sin(20 \cdot 3\pi/4)) \\
&= 1024(\cos\pi + i\sin\pi) \\
&= -1024.
\end{aligned}
$$

Example 2.5. Express $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

By DeMoivre's formula,

$$
\begin{aligned}
&(\cos(5\theta) + i\sin(5\theta)) = (\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta.
\end{aligned}
$$

Equating the imaginary parts, we obtain

$$
\begin{aligned}
\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\
&= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
&= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.
\end{aligned}
$$

In Python, we can compute the the n-th root of a complex number by the command `z**(1/n)`. Raising to a rational power m/n can be computed by `z**(m/n)`, for integers m and n. Python will return one of the n possible answers. Compare the following calculations in Python.

```
# Computation of roots of complex numbers in Python
z = -1+1j
w1 = (z**3)**(1/4) # raise to the power 3 and then take
   fourth root
w2 = z**(3/4)  # raise to the power 3/4
print(f"w1 = {w1}")
print(f"w2 = {w2}")

print(f"\nThey are both fourth roots of {z*z*z}.")
print(f"w1^4={w1**4}")
print(f"w2^4={w2**4}")
```

The number `w1` is obtained by first raising z to the third power and then taking the fourth root. The number `w2` is computed by directly raising z to the power $3/4$. Mathematically, they are supposed to be the same, but the actually results are not the same. It is because there are four possible answers when we take the fourth root. The fourth power of both of them are equal to $(-1+i)^3 = 2+2i$.

```
w1 = (1.2719211462463909+0.2530008463201178j)
```

```
w2 = (-0.25300084632011777+1.2719211462463909j)

They are both fourth roots of (2+2j).
w1^4=(1.9999999999999991+1.9999999999999991j)
w2^4=(1.9999999999999993+1.9999999999999987j)
```

3 Complex exponential function and Euler's formula

Summary:

- Complex sequence and series
- Complex power series
- Complex exponential function
- Euler's formula

In this lecture we extend the domain of the real exponential function to the complex domain. We will define the complex exponential function by power series, and derive the Euler's formula.

3.1 Complex plane as a metric space

Complex numbers can be interpreted as points on a 2-dimensional plane. The topology of the complex plane is the same as the Euclidean space $\mathbb{R}^2$, because we have the same metric function. As a result, the convergence concepts and limits in the real case and the complex case are basically the same. The main difference is that we now have an extra point at infinity.

The Euclidean distance between two complex numbers z_1 and z_2 can be expressed using the modulus function,

$$d(z_1, z_2) \triangleq |z_1 - z_2|.$$

We verify that $d(z_1, z_2)$ is indeed a metric function. It is clear that $d(z_1, z_2) \geq 0$ with equality if and only if $z_1 = z_2$, and $d(z_1, z_2) = d(z_2, z_1)$. It remains to prove the triangle inequality.

Proposition 3.1 (Triangle inequality)**.**

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

for any two complex numbers z_1 and z_2.

Although the triangle inequality is geometrically obvious, we give an algebraic proof, and show that it is compatible with the arithmetic of complex numbers.

Proof. Take the square of the left-hand side,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1^* + z_2^*) \\ &= |z_1|^2 + 2\,\mathrm{Re}(z_1 z_2^*) + |z_2|^2 \\ &\leq |z_1|^2 + 2|\,\mathrm{Re}(z_1 z_2^*)| + |z_2|^2. \end{aligned} \tag{3.1}$$

Because the real part of any complex number has absolute value less than or equal to the modulus of the complex number, we obtain

$$|\,\mathrm{Re}(z_1 z_2^*)| \leq |z_1 z_2^*| = |z_1|\,|z_2^*| = |z_1|\,|z_2|. \tag{3.2}$$

Substituting (3.2) into (3.1), we get

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.$$

□

The concepts of point-set topology in $\mathbb{R}^2$ can be directly imported to the complex case. For example, open disc is defined as follows.

Definition 3.2. The *open disc* of radius r centered at z_0 is the set

$$D(z_0, r) \triangleq \{z \in \mathbb{C} : |z - z_0| < r\}.$$

A subset A in $\mathbb{C}$ is called an *open set* for every point $z_0 \in A$, we can find a sufficiently small radius r so that $D(z_0, r)$ is a subset of A. A set B is said to be *closed* if the complement of B in $\mathbb{C}$ is open, i.e., given any point w_0 that is not in B, there is radius r such that $D(w_0, r)$ and B are mutually disjoint.

As for the "open discs" centered at the point at infinity, we take the inversion function $w = f(z) = 1/z$. A circle centered at the origin in the w-plane $\{w \in \mathbb{C} : |w| < \epsilon\}$ corresponds to a neighborhood of ∞ in extended complex plane in the z variable. In the Riemann sphere, it is a cap at the top of the sphere. Hence, a small "open disc" centered at the point at infinity has the form

$$\{z \in \mathbb{C} : |z| > R\}$$

for some large radius R.

Notation: A *sequence* of complex numbers $z_1, z_2, z_3, \ldots$ is denoted by $(z_k)_{k=1}^{\infty}$ or simply $\{z_k\}$.

Definition 3.3. Given a complex sequence $(z_n)_{n=1}^{\infty}$, we say that z_n *converges* to a complex number $w \in \mathbb{C}$ if

$$\forall \epsilon > 0 \ \exists N, \ s.t. \ |z_n - w| < \epsilon, \ \forall n \geq N.$$

Geometrically, it is equivalent to saying that for any small ϵ, the points in the complex sequence will eventually stay inside the open disc $D(w, \epsilon)$. A sequence that is *divergent* if it is not converging to any complex number in $\mathbb{C}$.
We write

$$\lim_{n \to \infty} z_n = w$$

if $(z_n)_{n=1}^{\infty}$ converges to the limit w.

Example 3.1. The complex sequence $\{z_n\}$ defined by

$$z_n = \frac{1}{n} + \frac{i}{n^2}$$

converges to 0, because $|z_n| \to 0$ as $n \to \infty$.

Example 3.2. Compute $\lim_{n\to\infty} \frac{n}{n+i}$.

We can first make a guess that the limit should be 1, because when n is large, adding i to n has negligible effect. To make the argument rigorous, we write

$$\frac{n}{n+i} = \frac{n+i-i}{n+i} = 1 - \frac{i}{n+i}.$$

Then

$$\left|\frac{n}{n+i} - 1\right| = \left|\frac{i}{n+i}\right| = \frac{1}{\sqrt{n^2+1}} \to 0, \qquad \text{as } n \to \infty.$$

Therefore $n/(n+i) \to 1$ as $n \to \infty$.

Definition 3.4. A sequence $(z_n)_{n=1}^{\infty}$ is called a *Cauchy sequence* if for all $\epsilon > 0$, there exists an integer N such that

$$|z_m - z_n| < \epsilon \quad \text{whenever } m, n \geq N.$$

We note that the absolute value in the above definition is the absolute value for complex numbers. It is easy to show that a complex sequence is Cauchy if and only if the real and imaginary parts are real Cauchy sequences. As a result, the basic property of Cauchy sequence for real numbers extends to the complex case readily. For example, the followings properties of real sequences are also valid for complex sequences.

Proposition 3.5.
(i) A complex sequence $(z_n)_{n=1}^{\infty}$ converges if and only if both $(\mathrm{Re}(z_n))_{n=1}^{\infty}$ and $(\mathrm{Im}(z_n))_{n=1}^{\infty}$ converge.
(ii) A complex sequence $(z_n)_{n=1}^{\infty}$ converges if and only if $(z_n)_{n=1}^{\infty}$ is Cauchy.

Part (ii) of this proposition says that the set of complex numbers inherits the completeness property of real numbers.

We record some basic properties of complex sequences in the following proposition. The proof is the same as in the real case, and hence is omitted.

Proposition 3.6. *Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be convergent complex sequences, with limits α and β, respectively.*

1. $\lim_{n\to\infty}(az_n + bw_n) = \alpha\alpha + \beta\beta$, *for complex constants a and b.*
2. $\lim_{n\to\infty} z_n w_n = \alpha\beta$.
3. $\lim_{n\to\infty} z_n/w_n = \alpha/\beta$, *if $\beta \neq 0$.*
4. $\lim_{n\to\infty} |z_n| = \left|\alpha\right|$.
5. $\lim_{n\to\infty}(z_n^*) = \alpha^*$.

3.2 Complex series

Definition 3.7. Given a sequence of complex numbers $(z_k)_{k=1}^{\infty}$, we define the infinite series $\sum_{k=1}^{\infty} z_k$ by

$$\sum_{k=1}^{\infty} z_k \triangleq \lim_{n\to\infty} \sum_{k=1}^{n} z_k$$

if the limit exists.

As in calculus, the convergence of infinity series is the same as the convergence of sequence of partial sums.

Example 3.3. (Complex geometric series) Evaluate $\sum_{k=1}^{\infty}(0.5i)^k$.

For any finite n, we have

$$\sum_{k=1}^{n}(0.5i)^k = \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1}.$$

We take limit as $n \to \infty$,

$$\sum_{k=1}^{\infty}(0.5i)^k = \lim_{n\to\infty} \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1} = \frac{-0.5i}{0.5i - 1} = \frac{-1 + 2i}{5}.$$

We record a few elementary properties of complex series in the next theorem.

Theorem 3.8. *Let $\sum_{k=1}^{\infty} z_k$ denote a convergent complex series.*

1. $\sum_{k=1}^{\infty}(z_k^*) = \left(\sum_{k=1}^{\infty} z_k\right)^*$.
2. $\mathrm{Re}\left(\sum_{k=1}^{\infty} z_k\right) = \sum_{k=1}^{\infty} \mathrm{Re}(z_k)$.
3. $\mathrm{Im}\left(\sum_{k=1}^{\infty} z_k\right) = \sum_{k=1}^{\infty} \mathrm{Im}(z_k)$.

We give a proof of the first part of this theorem. The proof of the other parts are exercise.

Proof. Let L be the limit of $\sum_{k=1}^{\infty} z_k$. By definition, for any positive real number ϵ, there exists a positive integer N such that

$$\Big|\sum_{k=1}^{n} z_k - L\Big| \leq \epsilon$$

for all $n \geq N$. Because a complex number and its conjugate have the same absolute value, we can write

$$\Big|\sum_{k=1}^{n} z_k^* - L^*\Big| \leq \epsilon$$

for any $n \geq N$. Hence by the definition of limit,

$$\sum_{k=1}^{\infty} z_k^* = L^* = \big(\sum_{k=1}^{\infty} z_k\big)^*.$$

□

As in the real case, we have the linearity property for infinite series.

Theorem 3.9. *If $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} w_k$ are convergent complex series, and α and β are complex constants, then*

$$\sum_{k=1}^{\infty} (\alpha z_k + \beta w_k) = \alpha \sum_{k=1}^{\infty} z_k + \beta \sum_{k=1}^{\infty} w_k.$$

The next two proposition provides two tests for convergence. The first one is a test for divergence, and the second one is a test for convergence.

Proposition 3.10. *(n-th term test) If $\sum_{k=1}^{\infty} z_k$ converges, then $|z_n| \to 0$ as $n \to \infty$. In other words, if $(|z_n|)_{n=1}^{\infty}$ does not converge to 0, then $\sum_{k=1}^{\infty} z_k$ does not converge.*

Proof. Suppose $\sum_{k=1}^{\infty} z_k$ converges to w. The sequence of partial sums $(\sum_{k=1}^{n} z_k)_{n=1}^{\infty}$ converges, and is thus a Cauchy sequence (Prop. 3.5 part (ii)). By the definition of Cauchy sequence, given any $\epsilon > 0$, there exists an integer N such that

$$|\sum_{k=1}^{m} z_k - \sum_{k=1}^{n} z_k| < \epsilon$$

for any $n, m \geq N$. By picking $m = n+1$, we obtain $|z_{n+1}| < \epsilon$ for all $n \geq N$. □

Proposition 3.11. *(Absolute convergence test) Given a sequence of complex numbers $(z_k)_{k=1}^{\infty}$, if the real infinite series $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ also converges.*

Proof. Suppose $\sum_{k=1}^{\infty} |z_k|$ converges. Since $|\operatorname{Re}(z_k)| \leq |z_k|$ for all $k \geq 1$, the real series $\sum_{k=1}^{\infty} |\operatorname{Re}(z_k)|$ converges. By the absolute convergence property for real series, $\sum_{k=1}^{\infty} \operatorname{Re}(z_k)$ converges. Similarly, $\sum_{k=1}^{\infty} \operatorname{Im}(z_k)$ is convergent. The convergence of $\sum_{k=1}^{\infty} z_k$ then follows by applying part (i) of Prop. 3.5. □

Example 3.4. The series

$$\sum_{n=1}^{\infty} \frac{2 + i/n}{n^2 + 2ni} \tag{3.3}$$

is convergent because we can upper bound the magnitude of each term

$$\left| \frac{2 + i/n}{n^2 + 2ni} \right| \leq c/n^2$$

for some real constant c. Because the series $\sum_{n=1}^{\infty} n^{-2}$ is convergent, by Prop. 3.11, the series in (3.3) converges.

In view of Prop. 3.11, we make the following definition.

Definition 3.12. We say that a series $\sum_{k=1}^{\infty} z_k$ is *absolutely convergent* if $\sum_{k=1}^{\infty} |z_k|$ is convergent. If $\sum_{k=1}^{\infty} z_k$ converges but $\sum_{k=1}^{\infty} |z_k|$ does not converge, then we say that it *converges conditionally*.

Proposition 3.13. *(limit ratio test) Suppose a_k, for $k = 1, 2, 3, \ldots$ are complex numbers and $\lim_{k\to\infty} |a_{k+1}/a_k|$ exists and is equal to L.*

1. *If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges;*

2. *If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.*

Proof. First consider the case $L > 1$. By the meaning of converging to L, which is larger than 1 by assumption, there exists an integer N such that $|a_{k+1}/a_k| > 1$ for all $k \geq N$. Therefore

$$|a_N| < |a_{N+1}| < |a_{N+2}| < \cdots$$

By applying the n-th term test, we see that $\sum a_k$ is divergent.

Next consider the case $L < 1$. Let ϵ be a small positive real number such that $L < 1 - \epsilon < 1$. By the definition of convergence, there exists an integer N such that

$$\left|\frac{a_{k+1}}{a_k}\right| < 1 - \epsilon < 1$$

for all $k \geq N$. Then, we have

$$\begin{aligned} |a_{N+1}| &< |a_N|(1-\epsilon) \\ |a_{N+2}| &< |a_{N+1}|(1-\epsilon) < |a_N|(1-\epsilon)^2, \end{aligned}$$

and in general

$$|a_{N+k}| < |a_N|(1-\epsilon)^k$$

for $k \geq 1$. Therefore, for each positive number m, we have

$$\sum_{j=0}^{m} |a_{N+j}| \leq |a_N| \sum_{j=0}^{m} (1-\epsilon)^j.$$

Since $\sum_{j=0}^{\infty}(1-\epsilon)^j$ is finite, the sequence

$$\Big(\sum_{j=0}^{m} |a_{N+j}|\Big)_{m=0}^{\infty}$$

is a bounded and monotonically increasing sequence. Hence it must be convergent. By Prop. 3.11, $\sum_{j=0}^{\infty} a_{N+j}$ is also convergent. This proves that $\sum_{k=1}^{\infty} a_k$ is convergent. □

We quote two properties of absolutely convergent series.

Proposition 3.14. *If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent series, then*

$$\Big(\sum_{k=0}^{\infty} a_k\Big)\Big(\sum_{k=0}^{\infty} b_k\Big) = \sum_{k=0}^{\infty} c_k,$$

where

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0.$$

Proposition 3.15. *If a series converges absolutely, then a series obtained by rearranging the terms converges and converges to the same limit.*

The proofs can be found in [Rudin, Theorem 3.50] and [Rudin, Theorem 3.55]. Although the corresponding theorems in [Rudin] are stated for real series, but the proofs works for the complex case in a straightforward manner.

3.3 Complex exponential function

Definition 3.16. A *complex power series* centered at the origin is a series in the form $\sum_{k=0}^{\infty} a_k z^k$, where $a_k \in \mathbb{C}$.

The power series in the next theorem is the most important power series.

Theorem 3.17. *For any complex number $z \in \mathbb{C}$, the power series $\sum_{k=0}^{\infty} z^k/k!$ converges. Moreover, it converges absolutely.*

Proof. It follows from the limit ratio test. The ratio of two consecutive terms has absolute value

$$\frac{\left|\frac{z^{k+1}}{(k+1)!}\right|}{\left|\frac{z^k}{k!}\right|} = \frac{|z|}{k+1}.$$

For any fixed complex number z, the absolute value $|z|$ is fixed, and hence the above ratio has limit zero. By part 2 of Theorem 3.13, the power series converges absolutely. □

In view of the previous result, we can define

Definition 3.18. Define the *complex exponential function* e^z by the complex power series

$$e^z \triangleq \exp(z) \triangleq \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

for $z \in \mathbb{C}$.

When z is a real number, the above definition is the same as the power series of the exponential function with real domain. For example, we have $\exp(0) = 1$.

We establish below a fundamental property of the function $\exp(z)$.

Theorem 3.19. *For any complex numbers z_1 and z_2,*

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2).$$

Proof. We know that both series $\sum_{k=0}^{\infty} z_1^k/k!$ and $\sum_{k=0}^{\infty} z_2^k/k!$ converge absolutely. We can apply Prop. 3.14 to $a_k = z_1^k/k!$ and $b_k = z_2^k/k!$. This gives

$$\exp(z_1)\exp(z_2) = \sum_{k=0}^{\infty} c_k$$

where c_k is defined by

$$c_k \triangleq \sum_{\ell=0}^{k} \frac{z_1^\ell}{\ell!}\frac{z_2^{k-\ell}}{(k-\ell)!}$$

for $k \geq 0$. By binomial theorem, we can simplify it to

$$c_k = \sum_{\ell=0}^{k} \frac{z_1^\ell}{\ell!}\frac{z_2^{k-\ell}}{(k-\ell)!} = \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell} z_1^\ell z_2^{k-\ell} = \frac{1}{k!}(z_1+z_2)^k.$$

Hence,

$$\exp(z_1)\exp(z_2) = \sum_{k=0}^{\infty} \frac{(z_1+z_2)^k}{k!} \triangleq \exp(z_1+z_2).$$

□

Using this theorem we can derive some immediate properties

Theorem 3.20.

- *For any $z \in \mathbb{C}$, $e^{-z} = (e^z)^{-1}$.*
- *For any nonzero $z \in \mathbb{C}$, $\exp(z) \neq 0$.*
- *if $z = x + iy$, then $\exp(z) = e^x e^{iy}$.*
- *For real number y, we have $|e^{iy}| = 1$.*

Proof. To see that $e^z \neq 0$ for all $z \in \mathbb{C}$, we take $z_1 = z$ and $z_2 = -z$ in Theorem 3.19,

$$e^z e^{-z} = e^{z-z} = e^0 = 1.$$

Since 0 times any complex number w is equal to 0, we cannot have $e^z = 0$, otherwise it would contradict the above equality, which says that there exists a complex number, namely e^{-z}, such that e^z time e^{-z} is equal to a nonzero value. We can also deduce from this equality that the multiplicative inverse of e^z is equal to e^{-z}.

For any real numbers x and y, we let $z_1 = x$ and $z_2 = iy$ and apply Theorem 3.19 to get

$$e^{x+iy} = e^x e^{iy}.$$

The factor e^x is a real number because x is real. On the other hand, e^{iy} lies on the unit circle, because

$$|e^{iy}|^2 = e^{iy}(e^{iy})^* = e^{iy}e^{-iy} = e^{iy-iy} = e^0 = 1.$$

We have used part 1 of Theorem 3.8 to replace $(e^{iy})^*$ by e^{-iy}. This proves that e^{iy} has modulus 1. $\square$

Although the complex exponential function is available from the numpy library, we can implement it from scratch using power series. The function my_exp below add the first 50 terms in the power series that defines the complex exponential function, starting from the higher degree to the lower degree.

```
# Computing complex exponential function by power series
def my_exp(z):
  answer=1
  for k in range(50,0,-1):
    answer = answer*z/k + 1
  return answer
```

3.4 Euler's formula

From Theorem 3.20, we know that e^{iy} lies on the unit circle for any real number y. The precise location is given by Euler's formula.

Theorem 3.21 (Euler's formula)**.** *For any real number θ, we have*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{3.4}$$

Proof. The proof follows from re-arranging the terms in the power series that defines $e^{i\theta}$. This is permissible because

$$e^{i\theta} = 1 + \frac{i\theta}{1} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots$$

is an absolutely convergent power series. By separating the real and imaginary parts by terms re-arrangement, we obtain

$$e^{i\theta} = \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\Big) + i\Big(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\Big)$$

the first term is precisely the power series expansion of $\cos(\theta)$, and the second term is i times the power series expansion of $\sin(\theta)$. $\square$

Combining Theorems 3.20 and 3.21, we can evaluate the complex exponential function in terms of the real exponential function and the real sine and cosine functions,

$$e^{x+iy} = e^x(\cos(y) + i\sin(y)). \tag{3.5}$$

Remark. Some books, such as [BrownChurchill] defines the complex exponential function by (3.5), as an example of holomorphic function, which requires the definition of complex differentiable function. If we define complex exponential function by (3.5), then the Euler's formula is true by definition. In this notes, we took the path of defining the complex exponential function by power series. With this approach, Euler's formula is magic.

Remark. The Euler's formula is a generalization of DeMoivre formula. We can recover DeMoivre formula from Euler's formula by considering $e^{in\theta}$, which can be written as

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

By Prop. 3.19, we can also write it as

$$e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n.$$

Example 3.5. We can compute $e^{\sqrt{2}+i}$ by

$$e^{\sqrt{2}+i} = e^{\sqrt{2}}(\cos(1) + i\sin(1)).$$

Example 3.6. When $z = \pi i$, we obtain

$$e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1.$$

From Euler's formula, we see that the complex exponential function is a periodic function with complex period $2\pi i$.

Theorem 3.22. *For any $z \in \mathbb{C}$, we have $\exp(z + 2\pi ik) = \exp(z)$ for all integers k.*

Example 3.7. Consider the infinite series

$$\sum_{k=0}^{\infty} \frac{\cos k\theta}{2^k}$$

as a function of real number θ. This series is convergent by applying the comparison test and comparing with $\sum_{k=0}^{\infty} 1/2^k$. We can evaluate the exact value by defining $z \triangleq \cos\theta + i\sin\theta$ and considering the series as the real part of the complex series

$$\sum_{k=0}^{\infty} \frac{z^k}{2^k}.$$

This is a complex geometric series, and the limit is

$$\frac{1}{1 - \frac{z}{2}}.$$

The remaining task is to obtain the real part of $(1-z/2)^{-1}$. To this end, we write $(1-z/2)^{-1}$ as

$$\begin{aligned}\frac{2}{2 - \cos\theta - i\sin\theta} &= \frac{2}{2 - \cos\theta - i\sin\theta} \cdot \frac{2 - \cos\theta + i\sin\theta}{2 - \cos\theta + i\sin\theta} \\ &= \frac{4 - 2\cos\theta + 2\sin\theta}{5 - 4\cos\theta}.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{\cos k\theta}{2^k} = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}.$$

This is a periodic function of θ.

Application to engineering and physics. We can use the complex exponential function to represent a sinusoidal function. Given amplitude A, frequency ω and phase θ, we can write

$$A\cos(\omega t+\theta)=\mathrm{Re}(Ae^{i(\omega t+\theta)})=\mathrm{Re}(e^{i\omega t}z_0),$$

where z_0 is a complex number $z_0 = Ae^{i\theta}$.

Example 3.8. Consider n sinusoidal functions with the same frequency ω,

$$A_k\cos(\omega t+\theta_k),$$

where amplitudes A_k and phase θ_k, for $k = 1, 2, \dots, n$, are real numbers. The sum of them is another sinusoidal function with the same frequency ω. To find the amplitude and phase of the sum, we can simply add the n complex numbers $A_k e^{i\theta_k}$, for $k = 1, 2, \dots, n$,

$$\sum_{k=1}^{n} A_k e^{i\theta_k} \triangleq z_0.$$

The modulus and argument of z_0 is then the amplitude and the phase of the resulting sinusoidal function

$$\sum_{k=1}^{n} A_k\cos(\omega t+\theta_k)=\mathrm{Re}\left(e^{i\omega t}\sum_{k=1}^{n} A_k e^{i\theta_k}\right)=\mathrm{Re}(e^{i\omega t}z_0)=|z_0|\cos(\omega t+\arg(z_0)).$$

4 Complex trigonometric and log function

Summary

- Complex trigonometric functions
- Complex log function
- Complex power function

4.1 Complex trigonometric functions

We extend the definition of trigonometric function and hyperbolic trigonometric functions to the complex plane.

Definition 4.1. For $z \in \mathbb{C}$, define the *complex sine complex cosine*, *complex hyperbolic sine*, and *complex hyperbolic cosine* functions by

$$\sin(z) \triangleq \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},$$
$$\cos(z) \triangleq \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!},$$
$$\sinh(z) \triangleq \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$
$$\cosh(z) \triangleq \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

By the limit ratio test (Prop. 3.13), they all converge absolutely on the whole complex plane, and reduce to the real trigonometric and real hyperbolic trigonometric function when z is a real number. In terms of the complex sine and complex cosine function, we can express the Euler's formula as

Theorem 4.2. *For $z \in \mathbb{C}$,*

$$e^{iz} = \cos(z) + i\sin(z). \tag{4.1}$$

Proof. The proof is basically the same as the proof of Theorem 3.21. We just need to apply the definition of $\cos(z)$ and $\sin(z)$ for complex number z.

Starting from the power series expansion of e^{iz}

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} + \cdots$$

because of absolute convergence, we can re-arrange the terms and separate the terms without i and the terms with i,

$$\begin{aligned} e^{iz} &\triangleq \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= \cos(z) + i\sin(z). \end{aligned}$$

The last equality is true by the definition of complex sine and complex cosine functions. □

Using Theorem 4.2, we can express sine function and cosine function in terms of the complex exponential function.

Theorem 4.3. *For any $z \in \mathbb{C}$,*

$$\begin{aligned} \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i}. \end{aligned}$$

Proof. The complex cosine function is an even function, i.e., $\cos(-z) = \cos(z)$, because the power series that define $\cos(z)$ consists of even powers. Similarly, because all terms in the power series that define $\sin(z)$ have odd powers, the complex sine function is an odd function, i.e., $\sin(-z) = -\sin(z)$. This gives

$$e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z) \tag{4.2}$$

By adding (4.2) to (4.1), we get

$$e^{iz} + e^{-iz} = 2\cos(z).$$

By subtracting (4.2) from (4.1), we get

$$e^{iz} - e^{-iz} = 2i \sin(z).$$

□

The result in Theorem 4.3 holds for any complex numbers. In particular, when we restrict z to a real number θ, this yields

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Remark. Some people take the two identities in Theorem 4.3 as the definition of complex sine and complex cosine, and then deduce Definition 4.1 as a corollary. Which definitions we adopt is a matter of taste. The resulting theory is the same.

Example 4.1. Evaluate $\sin(i)$. By Theorem 4.3,

$$\sin(i) = \frac{e^{i\cdot i} - e^{-i\cdot i}}{2i} = \frac{e^{-1} - e^{1}}{2i} = i\frac{e^{1} - e^{-1}}{2} = i \sinh(1).$$

The trigonometric functions in the `numpy` library can compute complex number. We can verify the calculations in this example as follows.

```
In [0]: from numpy import sin, sinh
In [1]: sin(1j)
Out[1]: 1.1752011936438014j

In [2]: sinh(1)
Out[2]: 1.1752011936438014
```

Example 4.2. For any real number a, we can evaluate $\cos(\pi + ai)$ by using the formula in Theorem 4.3,

$$\cos(\pi + ai) = \frac{e^{i(\pi+ai)} + e^{-i(\pi+ai)}}{2} = \frac{e^{-a+i\pi} + e^{a-i\pi}}{2} = -\frac{e^{-a} + e^{a}}{2} = -\cosh(a).$$

The next theorem states the relationship between trigonometric functions and hyperbolic functions.

Theorem 4.4. *For all complex number $z \in \mathbb{C}$,*

$$\sin(iz) = i\sinh(z),$$
$$\cos(iz) = \cosh(z).$$

Proof. The first identity can be proved by considering the power series expansion of $\sin(iz)$,

$$\sin(iz) \triangleq \sum_{k=0}^{\infty}(-1)^k \frac{(iz)^{2k+1}}{(2k+1)!} = i\sum_{k=0}^{\infty}(-1)^k \frac{i^{2k}z^{2k+1}}{(2k+1)!} = i\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = i\sinh(z).$$

The proof of the second identity is similar. □

In particular, for any real number x, we have

$$\sin(ix) = i\sinh(x), \quad \text{and } \cos(ix) = \cosh(x).$$

Some trigonometric identities in the real case continue to hold when the variables are complex numbers.

Theorem 4.5. *For any complex numbers z and w, the following hold:*

- $\cos^2 z + \sin^2 z = 1.$
- $\sin(z+w) = \sin z \cos w + \sin w \cos z.$
- $\cos(z+w) = \cos z \cos w - \sin z \sin w.$

Proof. We prove the first one. The proofs of the rest are similar.

$$\begin{aligned}
L.H.S. &= \cos^2 z + \sin^2 z \\
&= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\
&= \frac{1}{4}[(e^{iz}+e^{-iz})^2 - (e^{iz}-e^{-iz})^2] \\
&= \frac{1}{4}[e^{2iz}+2+e^{-2iz}-e^{2iz}+2-e^{-2iz}] \\
&= 1 = R.H.S.
\end{aligned}$$

□

4.2 Complex log function

The complex exponential function has a complex period $2\pi i$; that is, for any $z \in \mathbb{C}$,

$$e^{z+2\pi k i} = e^z, \quad \text{for } k \in \mathbb{Z}.$$

As a result, the inverse function of e^z is multi-valued.

Notation: We will use the notation $\ln x$ for the natural log function with positive real number as input.

Example 4.3. Find $z \in \mathbb{C}$ such that $e^z = 1 + i$.

The complex number $1 + i$ in polar coordinates is

$$\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4)).$$

We want to find all complex numbers $x + iy$ such that

$$e^{x+iy} = e^x(\cos y + i \sin y) = \sqrt{2}(\cos(\pi/4) + i \sin(\pi/4)).$$

By comparing the moduli and arguments, we have

$$\begin{aligned} x &= \frac{1}{2}\ln(2) \\ y &= \pi/4 + 2\pi k, \quad \text{for } k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hence

$$\log(1+i) = \frac{1}{2}\ln(2) + i(\pi/4 + 2\pi k)$$

for $k \in \mathbb{Z}$.

In general, given a complex number $r(\cos\theta + i\sin\theta)$ in polar form, the complex log of $r(\cos\theta + i\sin\theta)$ can take values

$$\ln r + i(\theta + 2\pi k)$$

for any integer k. In general, we have the following

Definition 4.6. For nonzero $w \in \mathbb{C}$, the *complex log function* is defined as

$$\log(w) \triangleq \ln|w| + i(\arg(w) + 2\pi k)$$

where $k = 0, \pm 1, \pm 2, \ldots$

The function $\arg(w)$ in Definition 4.6 is the multi-valued argument function. If we specify a range for the angle by take the principal argument (Definition 2.3), we have a uniquely defined function.

Definition 4.7. The function

$$\mathrm{Log}(w) \triangleq \ln|w| + i\,\mathrm{Arg}(w),$$

is called the *principal complex log function*. We will denote the principal log function by $\mathrm{Log}(z)$.

The complex log function extends the domain of the real log function to the whole complex plane except the origin. We note that it is not possible to define the $\log(0)$ even if we extend the number system to complex numbers, because the complex exponential function is never equal to zero. One way to see it is by the following general fact

$$\forall z \in \mathbb{C}, \quad e^z e^{-z} = e^{z-z} = e^0 = 1.$$

This will yield a contradiction if $e^z = 0$.

Using the complex log function we can compute the log of a negative number.

Example 4.4. Compute the principal log of -10, using a principal argument function with range $[0, 2\pi)$.

We first write -10 as

$$-10 = 10(\cos\pi + i\sin\pi)$$

If $e^{x+iy} = 10(\cos\pi + i\sin\pi)$, we must have $10 = e^x$, giving $x = \ln 10$. The imaginary part y equals π, which is the unique argument in the range $[0, 2\pi)$. Therefore, the principal log of -10 is

$$\mathrm{Log}(-10) = \ln(10) + i\pi.$$

The log function from the numpy library in Python is able to compute the complex log function. However, we need to activate the mode of complex computation by providing a complex number as the input. If we directly ask Python to compute $\log(-10)$, we will get a warning, saying that there is a type mismatch.

```
In [0]: from numpy import log
In [1]: log(-10)
```

```
RuntimeWarning: invalid value encountered in log
Out[1]: nan
```

One way to make it works is to manually convert the input to a complex number. Then we can compare with $\log(10) + i\pi$, and verify that the answer is correct.

```
In [2]: from math import log, pi
In [3]: log( complex(-10) )
Out[3]: (2.302585092994046+3.141592653589793j)

In [4]: log(10)+1j*pi
Out[5]: (2.302585092994046+3.141592653589793j)
```

An alternate way is to use the complex log function in the cmath library. It automatically converts the input value to the complex number data type.

```
In [2]: from cmath import log
In [3]: log(-10)
Out[3]: (2.302585092994046+3.141592653589793j)
```

4.3 Complex power function

We can define the complex power function via the complex log function.

Definition 4.8. Given a nonzero complex number z and a complex number w, we define the *complex power* z^w by

$$z^w \triangleq e^{w \log(z)}$$

where $\log(z)$ is the complex log function.

Example 4.5. Compute 2^i.

$$\begin{aligned} 2^i \triangleq \exp(i \log 2) &= \exp(i(\ln 2 + i2\pi k)) \\ &= \exp(-2\pi k + i \log 2) \\ &= e^{-2\pi k}(\cos(\ln 2) + i \sin(\ln 2)), \end{aligned}$$

where k can take any integer as its value. We can consider the value when $k = 0$ as the principal value. If we want to write down just one answer, we can write

$$2^i = \cos(\ln 2) + i \sin(\ln 2).$$

The following Python commands compute the principal value of 2^i.

```
In [0]: from numpy import cos, sin, log
In [1]: 2**1j
Out[1]: (0.7692389013639721+0.6389612763136348j)

In [2]: cos(log(2))+1j*sin(log(2))
Out[2]: (0.7692389013639721+0.6389612763136348j)
```

We can use the complex power function to compute the n-th roots of a complex numbers. We can compare this method with the DeMoivre formula in (2.4).

Example 4.6. Calculate the cube roots of $2i$ by $(2i)^{1/3} \triangleq \exp(\log(2i)/3)$.

We compute

$$\begin{aligned} \log(2i) &= \ln 2 + i\big(\pi/2 + 2\pi k\big), \qquad \text{for } k \in \mathbb{Z} \\ \frac{\log(2i)}{3} &= \frac{\ln 2}{3} + i\big(\pi/6 + 2\pi k/3\big). \end{aligned}$$

Exponentiating both sides, we get

$$\exp\big(\frac{\log(2i)}{3}\big) = \sqrt[3]{2}\Big(\cos\big(\frac{\pi}{6} + \frac{2\pi k}{3}\big) + i \sin\big(\frac{\pi}{6} + \frac{2\pi k}{3}\big)\Big)$$

for $k = 0, 1, 2$. There are three answers, all with magnitude $\sqrt[3]{2}$.

The next example has complex base and complex exponents at the same time.

Example 4.7. Compute i^i by calculating $e^{i \log i}$.

$$\begin{aligned} \exp(i \log i) &= \exp\Big(i(\log 1 + i(\frac{\pi}{2} + 2\pi k))\Big) \\ &= \exp\big(-(\frac{\pi}{2} + 2\pi k)\big), \end{aligned}$$

where $k \in \mathbb{Z}$. All possible values are real numbers. By taking $k = 0$, we can say that the principal value of i^i is $e^{-\pi/2}$.

We double check the answer by Python.

```
In [0]: from numpy import exp, pi
In [1]: 1j**1j
Out[1]: (0.20787957635076193+0j)

In [2]: exp(-pi/2)
Out[2]: 0.20787957635076193
```

4.4 Inverse of complex trigonometric functions

Inverse trigonometry function can be computed by solving quadratic equation.

Example 4.8. Find all complex numbers z such that $\cos(z) = 2$.

Recall from Theorem 4.3 that we can express $\cos(z)$ as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

Let $w = e^{iz}$. This amounts to solve

$$\frac{w + w^{-1}}{2} = 2.$$

This is a quadratic equation $w^2 - 4w + 1 = 0$. The roots are $2 \pm \sqrt{3}$. Hence

$$e^{iz} = 2 \pm \sqrt{3}.$$

By taking the complex log function, we obtain

$$z = -i\big[\ln(2 \pm \sqrt{3}) + 2\pi ik\big] = -i\ln(2 \pm \sqrt{3}) + 2\pi k,$$

where $k \in \mathbb{Z}$.

The arccos function in Python is able to compute the complex arc cosine function. We can check that it returns $-i\log(2 + \sqrt{3})$ as the answer.

```
In [0]: from numpy import arccos, log
In [1]: arccos(2+0j)
Out[1]: -1.3169578969248166j

In [2]: -log(2+sqrt(3))*1j
Out[2]: (-0-1.3169578969248166j)
```

However, as we have shown in the example, the other solution is $-i\log(2-\sqrt{3})$, which differs from the first one by a minus sign. We can verify this by the following Python script.

```
In [3]: -log(2-sqrt(3))*1j
Out[3]: 1.3169578969248164j

In [4]: cos(-log(2-sqrt(3))*1j)
Out[4]: (1.9999999999999996-0j)
In [5]: cos(-log(2+sqrt(3))*1j)
Out[5]: (1.9999999999999998-0j)
```

Example 4.9. Solve $\cot(z) = 2i$ for $z \in \mathbb{C}$.

This is equivalent to solving

$$\frac{e^{iz}+e^{-iz}}{2}\frac{2i}{e^{iz}-e^{-iz}} = 2i.$$

By letting $w = e^{iz}$, we can simplify it to

$$\frac{w+w^{-1}}{w-w^{-1}} = 2$$
$$\frac{w^2+1}{w^2-1} = 2.$$

The values for w is $\pm\sqrt{3}$. Then, we can obtain z by taking complex log

$$z = -i\log(\pm\sqrt{3}) = -i(\ln\sqrt{3} + i\pi k) = \pi k - i\ln\sqrt{3},$$

where k is any integer.

We double check the answer by Python. Because the arc cotangent function is not available in Python, we import the arc tangent function instead.

```
In [0]: from numpy import log, arctan

In [1]: -1j*log(3**(1/2))
Out[1]: -0.5493061443340548j

In [2]: arctan(1/2j)
Out[2]: -0.5493061443340549j
```

5 Riemann sphere and the point at infinity

Summary:

- Complex division and circle inversion
- Riemann sphere
- Stereographic projection

5.1 Equations of circle and straight line

We can specify a circle in the complex plane using the modulus. Given a point z_0 in the complex plane and a positive real number, the circle with radius r and center z_0 has equation

$$|z - z_0| = r.$$

Using the fact that $|z|^2 = zz^*$, we can write the equation in an alternate form:

$$(z - z_0)(z - z_0)^* = r^2.$$

If we write $z = x + iy$ and $z_0 = x_0 + iy_0$, the above equation becomes the usual equation of circle:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

A straight line in the x-y plane can be represented by equation

$$ax + by = c, \tag{5.1}$$

for some real numbers a, b and c. Using the relation $x = \mathrm{Re}(z) = (z + z^*)/2$ and $y = \mathrm{Im}(z) = (z - z^*)/(2i)$, we can write the equation as

$$a\frac{z + z^*}{2} + b\frac{z - z^*}{2i} = c.$$

We can simplify this equation to

$$\begin{aligned} ai(z + z^*) + b(z - z^*) &= 2ic \\ (b + ai)z - (b - ai)z^* &= 2ic \end{aligned} \tag{5.2}$$

If we define $\alpha = b + ai$, then we can further simplify it to

$$\operatorname{Im}(\alpha z) = c. \tag{5.3}$$

Alternately, we can write the equation of a straight line using Re instead of Im. We let $\beta = a - ib$ in (5.2), we obtain

$$\operatorname{Re}(\beta z) = c. \tag{5.4}$$

A straight line with equation (5.1) can be represented by (5.3) or (5.4).

Circles and straight lines belong to the same family of geometric shapes described by equation

$$A(x^2 + y^2) + Bx + Cy + D = 0, \tag{5.5}$$

where A, B, C and D are real numbers such that not all of them are zero. If the coefficient A is nonzero, it is the equation of a circle. If A is zero, it is an equation of a straight line.

5.2 Complex division

In Euclidean geometry, *circle inversion* (a.k.a. geometric inversion or circular inversion) is an operation

$$(x, y) \mapsto (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}).$$

The origin is excluded from the domain of this mapping because $x^2 + y^2$ has no inverse when $x = y = 0$. This mapping can be performed using straight line and compass. In this section we will see that a circle/straight line is transformed to a circle/straight, by making a link to the complex inversion mapping.

The circle inversion is closely related to complex division. Geometrically, the complex inversion function $f(z) = 1/z$ can be interpreted as a circle inversion function followed by a reflection. In terms of polar coordinates, the point $z = r(\cos\theta + i\sin\theta)$ is mapped to the point $(1/r)(\cos\theta + i\sin\theta)$. We can then take the complex conjugate of $(1/r)(\cos\theta + i\sin\theta)$ to obtain

$$(1/r)(\cos\theta - i\sin\theta) = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{z}.$$

Hence, the function $z \mapsto 1/z$ and the circle inversion function differ only by a complex conjugate.

We note that the major difference between the two maps is: the complex inversion map preserves orientation, but the circle inversion reverses orientation.

Theorem 5.1. *The inversion function $f(z) = 1/z$ transforms circles and straight lines to circles and straight lines.*

Before giving a formal proof, we illustrate this feature of the inversion map by examples.

Example 5.1. Find the image of the circle $|z-2i| = 1$ under the transformation $f(z) = 1/z$.

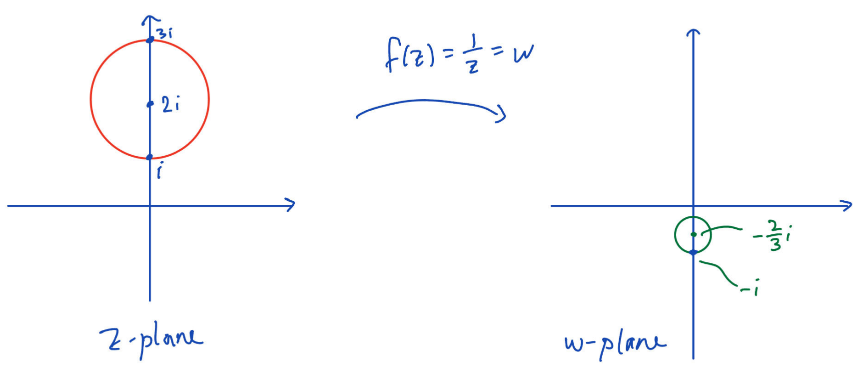
Let $w = 1/z$.

$$\begin{aligned} \left|\frac{1}{w} - 2i\right| &= 1 \\ \Leftrightarrow \quad |1 - 2wi|^2 &= |w|^2 \\ \Leftrightarrow \quad 3|w|^2 - 2wi + 2w^*i &= -1. \end{aligned}$$

By completing square, the last equation is equivalent to

$$|w + \frac{2i}{3}|^2 = \frac{1}{9},$$

which is the equation of a circle with radius $1/3$ and center $-2i/3$.



Example 5.2. Equation $|z-z_0| = |z_0|$ describes a circle that passes through the origin. Find the image of this circle under the transformation $f(z) = 1/z$.

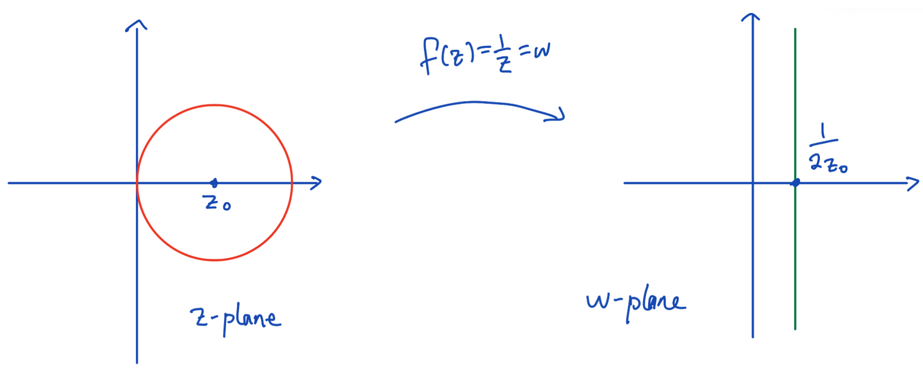
Define a variable $w = 1/z$.

$$\begin{aligned} \left|\frac{1}{w} - z_0\right| &= |z_0| \\ \Leftrightarrow \quad |1 - wz_0|^2 &= |wz_0|^2 \\ \Leftrightarrow \quad 1 - wz_0 - w^* {z_0}^* &= 0 \\ \Leftrightarrow \mathrm{Re}(wz_0) &= 1/2. \end{aligned}$$

This equation defines a straight line in the w-plane. Indeed, if we write $w = u + iv$ and $z_0 = x_0 + iy_0$, then

$$\mathrm{Re}(wz_0) = 1/2 \quad \Longleftrightarrow \quad ux_0 - vy_0 = 0.5.$$

Hence, the image of the circle $|z - z_0| = |z_0|$ is a straight line with equation $ux_0 - vy_0 = 0.5$.



Proof of Theorem 5.1. A circle or straight line in $\mathbb{R}^2$ can be described by equation

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

where A, B, C and D are real numbers, not all equal zero. (This equation also describe a single point or empty set.) Represent x and y by complex variable $z = x + iy$, and re-write the above equation as

$$azz^* + bz + b^*z^* + c = 0 \tag{5.6}$$

where $a = A$, $b = (B - iC)/2$, and $c = D$. The coefficients a and c are real numbers, and the coefficient b is a complex number. If we map z to $w = 1/z$, the equation becomes

$$a\frac{1}{ww^*} + b\frac{1}{w} + b^*\frac{1}{w^*} + c = 0,$$

which is the same as

$$a + bw^* + b^*w + cww^* = 0.$$

This is in the same form as in (5.6), and hence it represents a circle or a straight line. $\square$

5.3 Riemann sphere and spherical representation of complex numbers

We extend the domain of the complex inverse function by adding an extra point.

Definition 5.2. We identify a complex number $x + iy$ with a point (x, y) in $\mathbb{R}^2$. Consider the sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - r)^2 = r^2\},$$

which has radius r and touches x-y plane at one point. The *stereographic projection* is a function that maps a complex number $x + iy$ to the point $P(x, y)$ on the sphere such that (x, y), $P(x, y)$ and the north pole $(0, 0, 2r)$ lie on a straight line. The north pole is called the *point at infinity*, and is denoted by the symbol ∞. The sphere is referred to as the *Riemann sphere*.
The *extended complex number system*, as a set, is defined as $\mathbb{C} \cup \{\infty\}$. It is also called the *extended complex plane*. We will use the notation $\bar{\mathbb{C}}$. Other common notation for the extended complex plane includes $\hat{\mathbb{C}}$ and $\mathbb{C}_\infty$.

There is no easy way to perform addition and multiplication on the Riemann sphere. The main purpose of introducing the Riemann sphere is to bring in the point of infinity, which can be treated as an ordinary point like the other complex numbers, and making the resulting Riemann sphere a compact set. Hence, the Riemann sphere is often called the *one-point compactification* of the complex plane.

Remark. In contrast to the extended complex number system, the *extended real number system* is obtained by adding two special points to the real number system, i.e., $\mathbb{R} \cup \{\infty, -\infty\}$. In the real case, we have both $+\infty$ and $-\infty$ due to the total ordering of the real number line. However, since there is no ordering in the complex case, we only have one point at infinity.

We identify a complex number $x + iy$ on the horizontal plane in a 3D space by $x + iy \mapsto (x, y, 0)$. Write a general point in the 3D space as (α, β, γ) (See Fig. 1). Place a sphere S

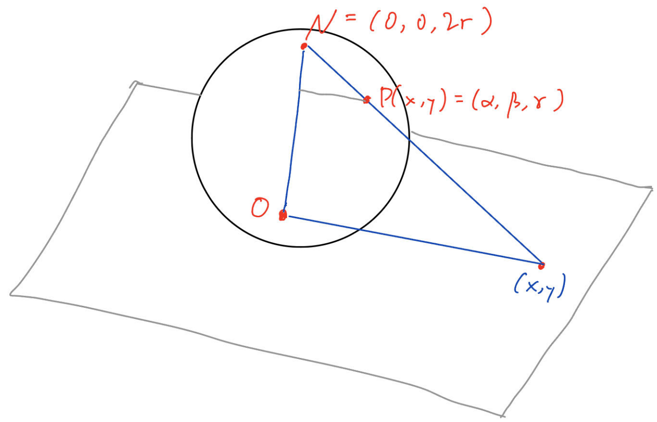


Figure 1: Riemann sphere and stereographic projection

of radius r centered at $(0,0,r)$ in the 3D space. The equation of S is

$$S: \quad \alpha^2+\beta^2+(\gamma-r)^2=r^2. \tag{5.7}$$

The sphere S is called the Riemann sphere. We call $N=(0,0,2r)$ the north pole of S. Given any point $(x,y,0)$, let $P(x,y)$ be the point of S such that $(0,0,2r)$, $(x,y,0)$ and $P(x,y)$ are collinear.

An open set on the Riemann sphere containing the north pole is called a *neighborhood* of the point infinity. Given any such neighborhood $\mathcal{N}$ of ∞, we can find a sufficiently small radius r such that the cap with radius r on the Riemann sphere with the north pole as the center is contained inside $\mathcal{N}$. The image of this cap under the stereographic projection is a complement of a large disc centered at the origin in the complex plane. Hence, on the complex plane, the collection of sets in the form

$$\{z\in\mathbb{C}: \, |z|>R\}$$

form a basis of the neigborhoods of the point at infinity.

If we consider the complex sequence as points on the Riemann sphere, then it may converge to the point at infinity, even though it does not converge to any point in the complex plane.

Definition 5.3. A sequence of complex numbers $\{z_k\}_{k=1}^{\infty}$ is said to *converge to the point at infinity* if for any positive number r, there exists a sufficiently large integer N such that $|z_k| > r$ for all $k \geq N$.

When the sequence $\{z_k\}_{k=1}^{\infty}$ converges to the point at infinity, we write $z_n \to \infty$ as $n \to \infty$, or

$$\lim_{n\to\infty} z_n = \infty.$$

Example 5.3. The sequence $(n(1+i))_{n=1}^{\infty}$ is divergent, but is converging to the point at infinity on the Riemann sphere.

Example 5.4. Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that

$$\lim_{n\to\infty} z_n = \infty.$$

Consider another sequence of complex numbers defined by $w_k = -z_k$. The sequence $\{w_k\}_{k=1}^{\infty}$ converges to the point at infinity as well,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} (-z_n) = \infty.$$

5.4 A unified approach to limit and convergence using filter

If we include the point at infinity, we need to consider four types of limits in complex variables:

$$\lim_{z\to z_0} f(z) = w_0, \qquad \lim_{z\to z_0} f(z) = \infty, \qquad \lim_{z\to\infty} f(z) = w_0, \qquad \lim_{z\to\infty} f(z) = \infty, \tag{5.8}$$

where z_0 and w_0 are finite complex numbers. The four definitions share a lot of similarity, but are not exactly the same. One way to understand these four notions of limit is to use the inversion function, and reduce the last three to the first one.

The sentence "the limit of a complex function $f(z)$ at z_0 is equal to w_0" is defined formally as

$$\forall\epsilon > 0, \exists r > 0 \ s.t. \ |f(z) - w_0| \leq \epsilon \text{ for all } |z - z_0| < r.$$

- $\lim_{z\to z_0} f(z) = \infty$ means $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.
- $\lim_{z\to\infty} f(z) = w_0$ means $\lim_{z\to 0} f(1/z) = w_0$.

- $\lim_{z\to\infty} f(z) = \infty$ means $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$.

While this approach is logically valid, it is not convenient when we come to proof. For instance, suppose $f(z)$ and $g(z)$ are functions whose domains and ranges are the extended complex system. If $g(z)$ is continuous at z_0, converging to the value w_0, and $f(z)$ is continuous at w_0, converging to the value u_0, then we expect that the composite function $f(g(z))$ is continuous at z_0, converting to the value u_0,

$$\lim_{z\to z_0} f(g(z)) = u_0. \tag{5.9}$$

If we use the above definitions of limits, a proof of this fact requires the consideration of eight cases, depending on whether z_0, w_0 and u_0 are the point at infinity or not. This is certainly not the best way to get things done.

Remark. The same problem is even more severe in calculus. In addition to the usual meaning that a function is continuous a point, we also have continuity from the left, continuity from the right, convergence to infinity, and convergence to negative infinity. Basically the idea is the same, but an elementary book on calculus would list all combinations one by one, give one proof in the simplest case, and then say that the other cases can be proved similarly.

A better way is to have a single definition of limit that includes all limits in (5.8). Then we just need one high-level proof to cover all cases. We introduce the notion of "filter" from topology. This is relatively more abstract, but it is the precise level of abstraction that is required to simplify things. We recall that a topological space is a pair $(X, O(X))$, consisting of a ground set X and the collection $O(X)$ of all open subsets in X. In our application, we are primarily interested in the complex plane $\mathbb{C}$ as the ground set. The collection of all open sets in $\mathbb{C}$ is denoted by $O(\mathbb{C})$. It is known that the intersection of finitely many open sets is an open set, and an arbitrary union of open sets is also open.

Definition 5.4. Given a collection of open sets $O(\mathbb{X})$ in a topological space X, a *filter* $\mathcal{F}$ is a nonempty collection of subsets of X that is closed under taking finite intersection and going upward. This is, $\mathcal{F}$ is a nonempty subset of the power set $\mathcal{P}(X)$ of X, satisfying the following requirements:

1. If A and B are in $\mathcal{F}$, then $A \cap B$ is in $\mathcal{F}$, and
2. If A is in $\mathcal{F}$ and B is in $O(\mathbb{C})$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Remark. We note that conditions 1 and 2 in the above definition involve set operation $\cap$ and $\subseteq$. In general, the notion of filter is defined with $O(\mathbb{C})$ replaced by a lattice $(L, \wedge, \vee)$. However, in this note, we only discuss filter in the context of complex plane.

Example 5.5. On the real number line $\mathbb{R}$, all open sets that contained a given point x_0 form a filter.

Example 5.6. Given a set B in X, the collection of sets

$$\{S \subseteq X : S \supseteq B\}$$

is called the *principal filter* of B.

Example 5.7. On the complex plane $\mathbb{C}$, all open sets that contained a given complex number z_0 form a filter. We will call this filter $\mathcal{F}_{z_0}$. We can use this filter to study the behavior of a function near z_0.

Example 5.8. On the complex plane $\mathbb{C}$, the collection of sets

$$\{A \subseteq \mathbb{C} : \exists R > 0 \text{ s.t. } (|z| > R \Rightarrow z \in A)\}$$

is a filter. We remark that the radius R may depend on the set A. A set A is in this collection of sets if the complement A^c is bounded. The sets in this filter are the neighborhoods of the point at infinity. We call this filter $\mathcal{F}_\infty$.

From the above examples, we can regard a filter as a "generalized element" in X, or a "generalized set" in X. Given a principal filter $\mathcal{F}$, the intersection of all subsets in $\mathcal{F}$ is the corresponding point in X.

Definition 5.5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function in complex variable, $\mathcal{F}$ and $\mathcal{G}$ are filters in $\mathbb{C}$. We say that f *converges to filter $\mathcal{G}$ along filter $\mathcal{F}$* if for all $V \in \mathcal{G}$, the pre-image $f^{-1}(V)$ of V under the function f is a member of $\mathcal{F}$.

We note that the special symbol "∞" is not in the above definition. Using this terminology, we have a unified method to define various modes of convergence:

1. $\lim_{z\to z_0} f(z) = w_0$ is the same as "$f(z)$ converges to filter $\mathcal{F}_{w_0}$ along filter $\mathcal{F}_{z_0}$".
2. $\lim_{z\to z_0} f(z) = \infty$ is the same as "$f(z)$ converges to filter $\mathcal{F}_\infty$ along filter $\mathcal{F}_{z_0}$".
3. $\lim_{z\to \infty} f(z) = w_0$ is the same as "$f(z)$ converges to filter $\mathcal{F}_{w_0}$ along filter $\mathcal{F}_\infty$".
4. $\lim_{z\to \infty} f(z) = \infty$ is the same as "$f(z)$ converges to filter $\mathcal{F}_\infty$ along filter $\mathcal{F}_\infty$".

We can now give a simple proof of (5.9).

Proposition 5.6. *Let f and g denote two complex functions from $\mathbb{C}$ to $\mathbb{C}$. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be three filters on $\mathbb{C}$. If g converges to $\mathcal{G}$ along $\mathcal{F}$, and f converges to $\mathcal{H}$ along $\mathcal{G}$, then the composite function $f(g(z))$ converges to $\mathcal{H}$ along $\mathcal{F}$.*

Proof. Let V be a set in $\mathcal{H}$. Since f converges to $\mathcal{H}$ along $\mathcal{G}$, we have $f^{-1}(V) \in \mathcal{G}$. Now, by the assumption that g converges to $\mathcal{G}$ along $\mathcal{F}$, we have $g^{-1}(f^{-1}(V)) \in \mathcal{F}$. Finally, we note that the inverse of the composition $f \circ g$ is $g^{-1} \circ f^{-1}$ and hence $g^{-1}(f^{-1}(V)) = (f \circ g)^{-1}(V)$. This proves that $(f \circ g)^{-1}(V)$ is a set in $\mathcal{F}$. $\square$

5.5 Appendix: Details of stereographic projection

We give the formula for stereographic projection in the next proposition.

Proposition 5.7. *The functions $P : \mathbb{R}^2 \to S \setminus \{(0, 0, 2r)\}$ described in the previous paragraph is given by*

$$P(x, y) = \Big(\frac{4r^2x}{4r^2 + x^2 + y^2}, \frac{4r^2y}{4r^2 + x^2 + y^2}, \frac{2r(x^2 + y^2)}{4r^2 + x^2 + y^2}\Big).$$

This gives a bijection between the complex plane $\mathbb{R}^2$ and the punctured sphere $S \setminus \{(0, 0, 2r)\}$.

We may alternately write the projection function P with $z = x + iy$ as input,

$$P(x + iy) = \Big(\frac{4r^2 \operatorname{Re} z}{4r^2 + |z|^2}, \frac{4r^2 \operatorname{Im} z}{4r^2 + |z|^2}, \frac{2r|z|^2}{4r^2 + |z|^2}\Big).$$

When the radius of the sphere is equal to $r = 1$, we can further simplify it to

$$P(x + iy) = \Big(\frac{4 \operatorname{Re} z}{4 + |z|^2}, \frac{4 \operatorname{Im} z}{4 + |z|^2}, \frac{2|z|^2}{4 + |z|^2}\Big).$$

Proof. Let N denote the north pole $(0, 0, 2r)$ of S. The triangle with vertices N, $(0, 0, 0)$ and $(x, y, 0)$ and the triangle with vertices N, $(0, 0, \gamma)$, $P(x, y)$ are similar. This gives

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{2r}{2r - \gamma}. \tag{5.10}$$

Substitute $\alpha = \frac{2r-\gamma}{2r}x$ and $\beta = \frac{2r-\gamma}{2r}y$ into the equation of sphere S (Equation 5.7), we get

$$\begin{aligned} \big(\frac{2r - \gamma}{2r}\big)^2(x^2 + y^2) + (\gamma - r)^2 &= r^2 \\ (2r - \gamma)(x^2 + y^2) &= 4r^2\gamma. \end{aligned}$$

In the last step, we cancel the factor $2r-\gamma$ from both sides. This is possible because $\gamma < 2r$. We can express γ in terms of x and y

$$\gamma = \frac{2r(x^2 + y^2)}{4r^2 + x^2 + y^2}.$$

Hence,

$$2r - \gamma = \frac{8r^3}{4r^2 + x^2 + y^2}.$$

Substituting the expression of $2r - \gamma$ back to (5.10) gives

$$\alpha = \frac{4r^2 x}{4r^2 + x^2 + y^2} \text{ and } \beta = \frac{4r^2 y}{4r^2 + x^2 + y^2}.$$

This proves that $P(x, y)$ can be written as in the proposition.

We now verify that P is a bijection by checking that P has an inverse function

$$Q(\alpha, \beta, \gamma) \triangleq \big(\frac{2r\alpha}{2r - \gamma}, \frac{2r\beta}{2r - \gamma}\big). \tag{5.11}$$

For any point (α, β, γ) on $S \setminus \{N\}$, we can compute

$$\begin{aligned} P(Q(\alpha, \beta, \gamma)) &= P\Big(\frac{2r\alpha}{2r-\gamma}, \frac{2r\beta}{2r-\gamma}\Big) \\ &= \Big(\alpha\frac{2r(2r-\gamma)}{(2r-\gamma)^2+\alpha^2+\beta^2}, \beta\frac{2r(2r-\gamma)}{(2r-\gamma)^2+\alpha^2+\beta^2}, \frac{2r(\alpha^2+\beta^2)}{(2r-\gamma)^2+\alpha^2+\beta^2}\Big). \end{aligned} \tag{5.12}$$

From the equation of S, we have

$$\alpha^2 + \beta^2 = 2r\gamma - \gamma^2. \tag{5.13}$$

The demoninators in (5.12) can be simplified to

$$(2r-\gamma)^2 + \alpha^2 + \beta^2 = 4r^2 - 4r\gamma + \gamma^2 + \alpha^2 + \beta^2 = 4r^2 - 2r\gamma.$$

The point in (5.12) can then be simplified to (α, β, γ). This proves that the composite function $P \circ Q$ is the identity function on the punctured sphere.

In the other direction, we want to show that

$$Q(P(x,y)) = Q\Big(\frac{4r^2x}{4r^2+x^2+y^2}, \frac{4r^2y}{4r^2+x^2+y^2}, \frac{2r(x^2+y^2)}{4r^2+x^2+y^2}\Big)$$

is equal to the point (x, y). To this end, we compute

$$\frac{2r\alpha}{2r-\gamma} = \frac{2r\frac{4r^2x}{4r^2+x^2+y^2}}{2r - \frac{2r(x^2+y^2)}{4r^2+x^2+y^2}} = x$$

and

$$\frac{2r\beta}{2r-\gamma} = \frac{2r\frac{4r^2y}{4r^2+x^2+y^2}}{2r - \frac{2r(x^2+y^2)}{4r^2+x^2+y^2}} = y.$$

This proves that the composite function $Q \circ P$ is the identity function on the complex plane.

Because $Q \circ P$ and $P \circ Q$ are identity functions, we conclude that P is a bijection. $\square$

The stereographic projection is not just a bijection, it has the feature of preserving circle and straight line. Suppose that a plane with equation

$$a\alpha + b\beta + c\gamma = d \tag{5.14}$$

for some constants a, b, c and d intersects with the sphere S. We know that the cross-section is circular in shape. We derive the equation that describes the stereographic projection of the cross-section as follows.

Consider a plane Π that contains the north pole of S. This plane intersects the sphere S at a circle as long as it is not a horizontal plane, i.e., when a and b are not both zero. Using the fact that the vector (a, b, c) is a normal vector of Π, we can write the equation of Π as

$$\begin{aligned}[(\alpha, \beta, \gamma) - (0, 0, 2r)] \cdot (a, b, c) &= 0 \\ a\alpha + b\beta + \gamma c &= 2rc.\end{aligned}$$

We note that when the plane Π passes through the north pole, the constant d in (5.14) is equal to $d = 2rc$. The intersection of Π and the horizontal plane is the straight line described by the equation $a\alpha + b\beta = 2rc$.

Next, suppose $a\alpha + b\beta + c\gamma = d$ is the equation of a plane Π that cut through the sphere S but does not pass through the north pole N. Let (x, y) be a point on the horizontal plane such that the point $(\alpha, \beta, \gamma) = P(x, y)$ lies on the plane Π. We obtain the equation

$$\begin{aligned}a\frac{4r^2x}{4r^2 + x^2 + y^2} + b\frac{4r^2y}{4r^2 + x^2 + y^2} + c\frac{2r(x^2 + y^2)}{4r^2 + x^2 + y^2} &= d \\ 4r^2ax + 4r^2by + 2rc(x^2 + y^2) &= d(4r^2 + x^2 + y^2) \\ (d - 2rc)(x^2 + y^2) - 4r^2ax - 4r^2by + 4r^2d &= 0.\end{aligned}$$

This equation defines a circle provided that $d \neq 2rc$, which happens if and only if the plane Π does not contain the north pole. By completing the squares we see that the center of the circle is located at

$$(x, y) = \frac{2r^2}{d - 2rc}(a, b)$$

and the radius is

$$\frac{2r}{|d - 2rc|}\sqrt{r^2(a^2 + b^2) - d(d - 2rc)}. \tag{5.15}$$

The number $r^2(a^2 + b^2) - d(d - 2rc)$ should be positive, because the radius is supposed to be positive. We can show this algebraically. We are assuming that the perpendicular distance between the plane and the center of S is less than r, otherwise the plane Π and the sphere S has no intersection, or intersect at a single point. We express this condition using the dot product

$$\left|(\alpha, \beta, \gamma - r) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}\right| < r.$$

This condition can be simplified to

$$(\alpha a + \beta b + \gamma c - rc)^2 < r^2(a^2 + b^2 + c^2)$$

which can be further simplified to

$$d^2 - 2drc < r^2(a^2 + b^2).$$

This is equivalent to the condition that $r^2(a^2 + b^2) - d(d - 2rc)$ is positive.

6 Linear fractional transformation

Summary:

- Automorphisms on Riemann sphere
- 3-transitivity of linear fractional transformation
- Cross ratio

In this lecture we study the mapping properties of linear fractional transformation, whose domain and range include the point at infinity.

6.1 Linear fractional transformations

We first treat the special case of affine transformations, which can be classified into three types.

A *translation* is a function $f : \mathbb{C} \to \mathbb{C}$ in the form

$$f_w(z) = z + w$$

for some complex constant w. Geometrically it is just moving horizontally by $\mathrm{Re}(w)$ and vertically by $\mathrm{Im}(w)$.

A *scaling function* is simply expanding by a constant real factor. It can be written as

$$g_a(z) = az$$

for some real constant a. This is also called a *dilation*.

A *rotation* by angle θ can be realized by multiplication by $(\cos\theta + i \sin\theta)$,

$$h_\theta(z) = (\cos\theta + i \sin\theta)z.$$

All three types of mappings preserve the shape of a geometric object in the domain. For example, if we draw a triangle on the complex plane and apply a translation or a dilation to it, we will get a triangle that is similar to the original one. In particular, straight line is mapped to a straight line. If we have two intersecting lines, the angle between them is invariant under translation, scaling and rotation.

If we apply a series of translation, dilation and rotation, the overall effect can be expressed in terms of an *affine function*

$$z \mapsto az + b$$

where a is a nonzero complex number and b is a complex number. This transformation is also called an *affine transformation*. We note that the composition of two affine functions is also affine.

If we mix affine functions with the inversion function, using function composition, we then obtain the class of linear fractional transformations.

Definition 6.1. A *linear fractional transformation* (or *Möbius* transformation, or *bilinear* transformation) is a function in the form

$$f(z) = \frac{az+b}{cz+d} \tag{6.1}$$

where a, b, c and d are complex numbers with $ad - bc \neq 0$.

If $b = 0$, $c = 0$ and $d = 1$, a linear fractional transformation reduces to a rotation followed by a dilation, i.e., $f(z) = az$.

If $a = 1$, $c = 0$ and $d = 1$, a linear fractional transformation is a translation $f(z) = z+b$.

If $a = 0$, $b = 1$, $c = 1$ and $d = 0$, a linear fractional transformation becomes the inversion function $f(z) = 1/z$.

We can verify that any linear fractional transformation can be expressed as a composition of translation, scaling, rotation, and inversion. If $c = 0$ and $d \neq 0$, then $(az + b)/d$ is just an affine transformation, and hence can be computed by first multiplying by a/d and then adding the constant b/d. When $c \neq 0$, by applying a partial fraction expansion, we can see that a linear fractional transformation can be written as:

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}. \tag{6.2}$$

The condition $ad - bc \neq 0$ in the definition of linear fractional transformation ensures that it is a non-constant function. In fact, we may scale all four parameters a, b, c and d by a nonzero complex constant such that $ad - bc = 1$.

Reciprocals can be taken for all complex numbers except zero. We may define $1/0$ as the point at infinity in the extended complex number system. In general, the behavior of a function $f(z)$ near the point at infinity can be seen by making a change of variable $w = 1/z$, and considering the function $g(w) = f(1/w)$. In the case of linear fractional transformation $f(z) = (az+b)/(cz+d)$, for $z \in \mathbb{C}$, we can study what happens in the vicinity of the point at infinity by making a change of variable $w = 1/z$, and consider the function

$$g(w) = \frac{a/w + b}{c/w + d}.$$

when w is close to 0. When w is close to zero but not equal to 0, we can write $g(w)$ as

$$g(w) = \frac{a + bw}{c + dw}.$$

Written in this form, we see that $g(w)$ approaches a/c as $w \to 0$. Hence, we can say that the original function $f(z)$ approaches a/c as z tends to infinity.

We can now refine the definition linear fractional transformation and define it as a function on the extended complex plane.

Definition 6.2. Given complex numbers a, b, c and d with $ad - bc \neq 0$, we extend the domain of function $f(z) = (az+b)/(cz+d)$ to the extended complex plane as follows. When $c = 0$, we define

$$f(z) = \begin{cases} (a/d)z + (b/d) & \text{if } z \neq \infty, \\ \infty & \text{if } z = \infty. \end{cases}$$

When $c \neq 0$, define

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c,\ z \neq \infty \\ \infty & \text{if } z = -d/c \\ a/c & \text{if } z = \infty. \end{cases}$$

Linear fractional mapping has the special property of transforming a circle to a circle or a straight line, and transforming a straight line to a circle or a straight line. It is obvious that a translation, dilation and rotation preserve shape, it remains to show that the inversion function transforms circle to circle. In fact, circles and straight lines on the complex plane are considered as the same type of objects on the Riemann sphere. Both of them corresponds to a circle on the Riemann sphere. We may think of a straight line on the complex plane as a circle that passes through the point at infinity.

Theorem 6.3. *A linear fractional transformation $f(z) = (az+b)/(cz+d)$ transforms circles and straight lines to circles and straight lines.*

6.2 Composition of linear fractional transformations

From (6.2), we know that a linear fractional transformation can be expressed as the composition of translation, dilation, rotation, and inversion.

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}. \tag{6.3}$$

We adopt the following convention for the computation with the point at infinity.

- For any extended complex number z, which may be a finite complex number or infinity, we define $z + \infty = \infty + z = \infty$.
- For any $b \in \bar{\mathbb{C}} \setminus \{0\}$, we define $b \cdot \infty = \infty \cdot b = \infty$
- For any $c \in \mathbb{C} \setminus \{0\}$, we define $c/\infty = 0$ and $c/0 = \infty$.
- $\infty - \infty$ and $0 \cdot \infty$ are undefined.

We can evaluate $f(z)$ at the point at infinity. When $c \neq 0$, we have

$$f(\infty) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{c\cdot\infty+d}.$$

With the above convention, we evaluate $c\dot{\infty} + d$ to ∞. The fraction $1/(c\cdot\infty+d)$ is equal to 0. As a result, $f(\infty)$ is equal to a/c when c is not zero.

When $z = -d/c$, we have $1/(c(-d/c)+d) = \infty$ because we define $1/0 = \infty$. We see that the above rules of calculations with ∞ is consistent with the definition $f(-d/c) = \infty$.

Theorem 6.4. *The composition of two linear fractional transformations is a linear fractional transformation.*

Proof. (sketch) In the followings, $f(z)$ is a linear fractional transformation $f(z) = (az + b)/(cz + d)$. It suffices to consider three cases.
(i) The function $f(z) + \alpha$ is a linear fractional transformation for any complex constant α.
(ii) The function $\beta f(z)$ is a linear fractional transformation for any nonzero complex constant β.
(iii) The function $1/f(z)$ is a linear fractional transformation. $\square$

We next consider the inverse of a linear fractional transformation. We can use the fact that a linear fractional transformation can be written as the composition of four types of functions — translation, rotation, dilation, and inversion. Because the inverse of translation (resp. rotation, dilation, inversion) is a translation (resp. rotation, dilation, inversion), we have the following theorem.

Theorem 6.5. *The inverse of a linear fractional function exists, and is a linear fractional transformation.*

We give a second, and more direct proof of this theorem below, by computing the inverse of a linear fractional transformation, and showing that the inverse is a linear fractional transformation.

Proof. Let $f(z)$ be a linear fractional transformation defined by $f(z) = (az + b)/(cz + d)$. We consider two cases.

Suppose that $c = 0$. In this case, we must have $a \neq 0$ and $d \neq 0$ (because we assume $ad - bc \neq 0$). The function $f(z)$ can be written in the form $az + b$, by writing a/d as a and b/d as b. The inverse of $f(z)$ is $g(w) = (w - b)/a$. We also have $f(\infty) = \infty$ in this case. The inverse of $f(z)$ is

$$g(w) = \begin{cases} (w-b)/a & \text{if } w \neq \infty, \\ \infty & \text{if } w = \infty. \end{cases}$$

We can check that $g \circ f$ and $f \circ g$ are identity functions.

Next suppose $c \neq 0$. From $w = (az + b)/(cz + d)$, we can apply the identity in (6.2) and write

$$\frac{cw - a}{bc - ad} = \frac{1}{cz + d}.$$

If $w \neq a/c$, we can solve for z in $\mathbb{C}$

$$z = \frac{1}{c}\Big(\frac{bc - ad}{cw - a} - d\Big) = \frac{b - dw}{cw - a}.$$

If $w = a/c$, we have $f(\infty) = w = a/c$. The inverse function of $f(z)$ is

$$g(w) = \begin{cases} \frac{b-dw}{cw-a} & \text{if } \infty \neq w \neq a/c \\ \infty & \text{if } w = a/c \\ -d/c & \text{if } w = \infty. \end{cases}$$

Since $g(f(z))$ and $f(g(w))$ are identity functions from Riemann sphere to itself, we conclude that f is a bijection. □

Example 6.1. Consider the function $f(z) = (z+1)/(z-i)$. When $z = i$, we define $f(i) = \infty$ and when $z = \infty$, we define $f(\infty) = 1$. To compute the inverse function of $f(z)$, we solve for z in the equation $w = f(z)$,

$$\begin{aligned} w &= \frac{z+1}{z-i} \\ wz - iw &= z + 1 \\ (w-1)z &= 1 + iw \\ z &= \frac{1 + iw}{w - 1}. \end{aligned}$$

The last line requires $w \neq 1$. The inverse of $f(z)$ is thus

$$g(w) = \begin{cases} \frac{1+iw}{w-1} & \text{if } w \neq 1,\ w \neq \infty, \\ \infty & \text{if } w = 1, \\ i & \text{if } w = \infty. \end{cases}$$

Because the inverse of a linear fractional transformation always exists, we can conclude that a linear fractional transformation is a bijection on the extended complex plane.

Theorem 6.6. *A linear fractional transformation defines a bijection on the Riemann sphere.*

Example 6.2. Consider the linear fractional transformations $f_1(z) = 1/z$ and $f_2(z) = 1 - z$. Both of them induce a permutation on the three points $0, 1, \infty$ on the Riemann sphere.

$$\begin{aligned} f_1(0) &= \infty, \qquad & f_1(1) &= 1, \qquad & f_1(\infty) &= 0, \\ f_2(0) &= 1, & f_2(1) &= 0, & f_2(\infty) &= \infty. \end{aligned}$$

In this example we want to exhaust all function that can be obtained by composing f_1 and f_2 recursively. For example, using f_1 and f_2, we can obtain functions in the form $f_1(f_2(z))$, $f_2(f_1(z))$, $f_1(f_1(f_2(z)))$, $f_2(f_1(f_2(z)))$, etc. How many functions can we get?

First, we see that we can get the identity function from

$$f_1(f_1(z)) = f_2(f_2(z)) = z.$$

We can discover two new functions from

$$\begin{aligned} f_1(f_2(z)) &= \frac{1}{1-z}, \\ f_2(f_1(z)) &= 1 - \frac{1}{z} = \frac{z-1}{z}. \end{aligned}$$

If we apply the inversion function to $f_2(f_1(z))$, we get

$$f_1(f_2(f_1(z))) = \frac{z}{z-1}.$$

We claim that these six functions

$$f_0(z) = z, \quad f_1(z) = 1/z, \quad f_2(z) = 1 - z, \quad f_3(z) = \frac{1}{1-z}, \quad f_4(z) = \frac{z-1}{z}, \quad f_5(z) = \frac{z}{z-1}$$

are all we can get. We can organize the above data in a composition table.

$\circ$	f_0	f_1	f_2	f_3	f_4	f_5
f_0						
f_1		f_0	f_3		f_5	
f_2		f_4	f_0			
f_3						
f_4						
f_5						

The non-empty cells correspond to the equations we know so far: $f_0 = f_1 \circ f_1 = f_2 \circ f_2$, $f_3 = f1 \circ f_2$, $f_4 = f_2 \circ f_1$, $f_5 = f_1 \circ f_4 = f_1 \circ f_2 \circ f_1$.

We complete the table, and note that the six functions are closed under composition.

$\circ$	$f_0(z) = z$	$f_1(z) = \frac{1}{z}$	$f_2(z) = 1-z$	$f_3(z) = \frac{1}{1-z}$	$f_4(z) = \frac{z-1}{z}$	$f_5(z) = \frac{z}{z-1}$
$f_0(z) = z$	z	$\frac{1}{z}$	$1-z$	$\frac{1}{1-z}$	$\frac{z-1}{z}$	$\frac{z}{z-1}$
$f_1(z) = \frac{1}{z}$	$\frac{1}{z}$	z	$\frac{1}{1-z}$	$1-z$	$\frac{z}{z-1}$	$\frac{z-1}{z}$
$f_2(z) = 1-z$	$1-z$	$\frac{z-1}{z}$	z	$\frac{z}{z-1}$	$\frac{1}{z}$	$\frac{1}{1-z}$
$f_3(z) = \frac{1}{1-z}$	$\frac{1}{1-z}$	$\frac{z}{z-1}$	$\frac{1}{z}$	$\frac{z-1}{z}$	z	$1-z$
$f_4(z) = \frac{z-1}{z}$	$\frac{z-1}{z}$	$1-z$	$\frac{z}{z-1}$	z	$\frac{1}{1-z}$	$\frac{1}{z}$
$f_5(z) = \frac{z}{z-1}$	$\frac{z}{z-1}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$	$\frac{1}{z}$	$1-z$	z

In every column and every row, each of the six functions appears once and only once.

As a matter of fact, these six functions correspond to the permutation of three points 0, 1 and ∞ on the Riemann sphere. Because there are six permutations in total, and the action on three points determines a linear fractional transformation, we obtain six functions accordingly.

6.3 Cross ratio

In this section we consider the question: Given two circles (or straight lines) on the complex plane, can we fine a linear fractional transformation that maps the first circle to the second circle. The answer is yes, and we can establish a stronger result. It is known that a circle is uniquely determined by three points. We prove in the next theorem that we can find a linear fractional transformation that maps three given points to three other given points.

Theorem 6.7. *Given any three distinct points z_1, z_2 and z_3 and any three distinct points w_1, w_2 and w_3, all in the extended complex plane $\bar{\mathbb{C}}$, we can find a linear fractional transformation that maps z_1 to w_1, z_2 to w_2, and z_3 to w_3.*

Proof. We first consider the special case $w_1 = 0$, $w_2 = 1$, and $w_3 = \infty$. We claim that we can find a linear fractional transformation that maps z_1 to 0, z_2 to 1 and z_3 to ∞, for any choice of z_1, z_2, and z_3. For example, when z_1, z_2 and z_3 are all in $\mathbb{C}$, the function

$$f(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \tag{6.4}$$

is a linear fractional transformation that satisfies the requirements.

There are three boundary cases to consider. When $z_1 = \infty$ and z_2 and z_3 are finite, we can map ∞ to 0, z_2 to 1, and z_3 to ∞ by

$$f(z) = \frac{z_2 - z_3}{z - z_3}.$$

When $z_2 = \infty$ and z_1 and z_3 are finite, we can map z_1 to 0, ∞ to 1, and z_3 to ∞ by

$$f(z) = \frac{z - z_1}{z - z_3}.$$

Finally, when $z_3 = \infty$ and z_1 and z_2 are finite, we can map z_1 to 0, z_2 to 1, and ∞ to ∞ by

$$f(z) = \frac{z - z_1}{z_2 - z_1}.$$

This completes the proof of the claim.

To prove the existence of linear fractional transformation in the theorem, we can consider a linear fractional transformation $f(z)$ that maps z_1 to 0, z_2 to 1, and z_3 to ∞, and another linear fractional transformation $g(w)$ that maps w_1 to 0, w_2 to 1, and w_3 to ∞. Then the composite function $g^{-1} \circ f$ is a linear fractional transformation that satisfies the required mapping properties.

For explicit calculations, we can set up an equation

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} \tag{6.5}$$

and write w as a function of z. The resulting function of z will be a linear fractional transformation that equals $g^{-1}(f(z))$. ☐

In view of the previous proof, we are motivated to give the following definition.

Definition 6.8. Given four complex numbers z_0, z_1, z_2 and z_3, we define the *cross ratio* as

$$[z_0, z_1, z_2, z_3] \triangleq \frac{z_0 - z_1}{z_0 - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

Example 6.3. Find a linear fractional transformation that maps 0 to 0, 1 to 1, and z_0 to ∞, where z_0 is a complex number different from 0 and 1.

Because we want to take z_0 to ∞, the linear fractional transformation has the form

$$f(z) = \frac{az + b}{z - z_0}$$

for some choice of complex constants a and b. Because we want $f(0) = 0$, the only choice of constant b is $b = 0$. From the requirement $f(1) = 1$, we derive that a is equal to $1 - z_0$. The answer is

$$f(z) = (1 - z_0)\frac{z}{z - z_0}.$$

Consider the case that z_0 is equal to i. The three points 0, 1 and i in the domain of the transformation lie on the circle with equation $|z - (1+i)/2|^2 = 1/2$. The three points 0, 1, ∞ in the range lie on the line $\mathrm{Im}(z) = 0$ (the real axis plus the point at infinity). In this case, the fractional linear transformation

$$z \mapsto \frac{(1-i)z}{z-i}$$

maps the circle with and radius $1/\sqrt{2}$ and center at $(1+i)/2$ to the real axis.

Example 6.4. In this example we construct a linear fractional transformation that maps three points $z_1 = 1$, $z_2 = 2$, and $z_3 = 3$ on the real axis to three points $w_1 = i$, $w_2 = 3i$, and $w_3 = 5i$ on the imaginary axis. We apply the equation in (6.5),

$$\frac{z-1}{z-3} \cdot \frac{2-3}{2-1} = \frac{w-i}{w-5i} \cdot \frac{3i-5i}{3i-i}.$$

After some algebraic calculations, we can simplify it to

$$w = 2iz - i.$$

The desired linear fractional transformation is

$$f(z) = 2iz - i = i(2z - 1).$$

An important property of the cross ratio is that it is invariant under linear fractional transformation.

Theorem 6.9. *Let $f(z)$ be a linear fractional transformation mapping z_k to z'_k, for $k = 0, 1, 2, 3$, then*

$$[z_0, z_1, z_2, z_3] = [z'_0, z'_1, z'_2, z'_3].$$

Proof. Since a linear fractional transformation can be obtained by composing an affine transformation and a complex inversion (see (6.3)), we can study the effect of applying a translation, a scaling and an inversion to the cross ratio.

Let z_0, z_1, z_2 and z_3 be four complex numbers. If we add a complex number ζ to all of them, the cross ratio of the resulting points is

$$\begin{aligned}
[z_0+\zeta, z_1+\zeta, z_2+\zeta, z_3+\zeta] &= \frac{z_0+\zeta-(z_1+\zeta)}{z_0+\zeta-(z_3+\zeta)} \cdot \frac{z_2+\zeta-(z_3+\zeta)}{z_2+\zeta-(z_1+\zeta)} \\
&= \frac{z_0-z_1}{z_0-z_3} \cdot \frac{z_2-z_3}{z_2-z_1} \\
&= [z_0, z_1, z_2, z_3].
\end{aligned}$$

By similar calculation, we can easily verify that, for any nonzero complex number α,

$$[\alpha z_0, \alpha z_1, \alpha z_2, \alpha z_3] = [z_0, z_1, z_2, z_3].$$

When z_0, z_1, z_2, and z_3 are all nonzero complex number in $\mathbb{C}$, we have

$$\begin{aligned}
[\frac{1}{z_0}, \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}] &= \frac{\frac{1}{z_0}-\frac{1}{z_1}}{\frac{1}{z_0}-\frac{1}{z_3}} \cdot \frac{\frac{1}{z_2}-\frac{1}{z_3}}{\frac{1}{z_2}-\frac{1}{z_1}} \\
&= \frac{z_1-z_0}{z_3-z_0} \cdot \frac{z_3-z_2}{z_1-z_2} \\
&= [z_0, z_1, z_2, z_3].
\end{aligned}$$

There are few more cases to check when z_0, z_1, z_2 and z_3 can take 0 or ∞ as the value. This is left as exercise. $\square$

An application in plane geometry. Because linear fractional transformation is bijective and maps circle to circle/straight line, the point z_0 lie on the circle that contains z_1, z_2 and z_3 if and only if $[z_0, z_1, z_2, z_3]$ lies on the real axis. We thus obtain a criterion that determine whether four distinct points lie on a circle. We say that a set of points are *concyclic* if they are on the same circle. We have the following test for concyclicity:

$$\text{Im}([z_0, z_1, z_2, z_3]) = 0 \text{ iff } z_0,\ z_1,\ z_2 \text{ and } z_3 \text{ are concyclic.} \tag{6.6}$$

This gives a convenient test to determine whether four points are located on a circle.

Example 6.5. Derive the equation of the circle that passes through $(0,0)$, $(1,0)$ and $(0,2)$ in the x-y plane.

We take $z_1 = 0$, $z_2 = 1$ and $z_3 = 2i$. Let $z = x + iy$ be a point that lie on the circle that contains z_1, z_2 and z_3. Applying the criterion in (6.6), we obtain the condition

$$\text{Im}\Big(\frac{x+iy}{x+iy-2i} \cdot \frac{1-2i}{1}\Big) = 0,$$

which can then be simplified to

$$x^2 + y^2 - x - 2y = 0,$$

or equivalently

$$(x - 1/2)^2 + (y - 1)^2 = \frac{5}{4}.$$

This is the equation of a circle with center at $(1/2, 1)$ and radius $\sqrt{5}/2$.

We implement a test of concyclicity in Python. This program takes for points with real coordinates as inputs, and returns 1 if the four points lie on a circle. Even though this geometrical question itself has nothing to do with complex numbers, we can use complex numbers and complex arithmetic to get the answer easily. The calculation is done in one line.

```
# Determine whether four distinct points lie on a circle
# Input four tuples:  pt1= (x1,y1), pt2= (x2,y2), pt3=
    (x3,y3), pt4=(x4,y4)
# return True if the four points pt1, pt2, pt3, pt4 are
    concyclic, False if not concyclic

def test_concyclic(pt0, pt1, pt2, pt3):
    z0 = complex(pt0[0],pt0[1])   # convert to complex numbers
    z1 = complex(pt1[0],pt1[1])
    z2 = complex(pt2[0],pt2[1])
    z3 = complex(pt3[0],pt3[1])
    return abs(((z0-z1)*(z2-z3)/((z0-z3)*(z2-z1))).imag)< 1e-8
```

For example, we can use this Python function to check that four points $(0, 0)$, $(1, 0)$, $(1, 2)$, and $(0, 2)$ lie on the circle in the previous example.

```
In [1]: test_concyclic((0,0), (1,2), (0,2), (1,0))
Out[1]: True
```

7 Complex differentiability

Summary

- Limit and continuity
- Approximation by linear function
- Cauchy-Riemann equations

In order to compute the derivative of a function at a point, we need to make a small difference in the domain and investigate the resulting change in the function value. In order to be able to have enough elbow room, we usually assume that the complex function is defined in an open set. In complex analysis, the term *domain/region* is reserved to mean an open and connected set in $\mathbb{C}$.

We can understand the term "connected" as "path-connected", i.e., any two points in the domain are connected by a path.

7.1 Limit and continuity of complex function

Definition 7.1. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *continuous at z_0* if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \;\Rightarrow\; |f(z) - f(z_0)| < \epsilon.$$

A function f is said to be *continuous* in a domain D if f is continuous at every point in D.

By consider the real and imaginary part separately, we can prove the following

Theorem 7.2. *A complex function f is continuous if and only if the real and imaginary parts are continuous.*

The proof of this theorem is straightforward and is omitted.

Example 7.1. Show that $f(z) = 1/z$ is continuous in the domain $\mathbb{C} \setminus \{0\}$.

Suppose $z = x + iy$ and $z \neq 0$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

The real part of $f(z)$ is $x/(x^2+y^2)$ and the imaginary part is $-y/(x^2+y^2)$. Both of them are continuous functions in the domain $\mathbb{C}\setminus\{0\}$. Hence $f(z)$ is continuous by the previous theorem.

Another way to understand continuity is by means of sequence. If a complex function f is continuous at z_0, and if $(z_k)_{k=1}^{\infty}$ is a sequence of complex numbers converging to z_0, then the image points $(f(z_k))_{k=1}^{\infty}$ should converge to $f(z_0)$. We can approach z_0 from any direction, and the limit of the sequence should equal $f(z_0)$.

The third way to define continuity of a function is by limit of functions. Given a complex function $g(z)$ and a complex number z_0, we say that the following limit

$$\lim_{z\to z_0} g(z)$$

and is equal to L if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |g(z) - L| < \epsilon.$$

We note that in this definition the function $g(z)$ need not be defined at z_0. With this notation, we have

Lemma 7.3. *A complex function $f(z)$ is continuous at z_0 iff*
(i) $f(z_0)$ is well defined,
(ii) the limit $\lim_{z\to z_0} f(z)$ exists, and
(iii) $\lim_{z\to z_0} f(z) = f(z_0)$.

The limit for complex function behaves in the same as the limit of vector function. Hence, complex limit has the same properties as in multi-variable calculus.

7.2 Complex differentiable function

By regarding a complex function $f(z)$ as a two-dimensional vector field,

$$f(x+iy) = u(x,y) + iv(x,y).$$

When we say that $f(x+iy)$ is *real differentiable*, if the vector-valued function $(u(x,y), v(x,y))$ is differentiable as in multivariable calculus. By definition, if $(u(x,y), v(x,y))$ is real differentiable at a point (x_0, y_0), we can approximate it by linear function,

$$\begin{bmatrix} u(x_0+\Delta x, y_0+\Delta y) \\ v(x_0+\Delta x, y_0+\Delta y) \end{bmatrix} \doteq \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{7.1}$$

when $(\Delta x, \Delta y)$ is small. The entries in the 2×2 matrix are the partial derivatives of u and v evaluated at (x_0, y_0). The symbol "$\doteq$" means that the difference between the left-hand side and the right-hand side of (7.1) is converging to zero faster than the linear term, i.e.,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\|\text{Difference between L.H.S and R.H.S. of (7.1)}\|}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

Example 7.2. The function $f(z)$ defined by $x^2 y + i(x+y)$ is real differentiable. Partial derivatives of the real part $u(x,y) = x^2 y$ and imaginary part $v(x,y) = x+y$ exist, and we have

$$\begin{bmatrix} (x_0+\Delta x)^2(y_0+\Delta y) \\ x_0+\Delta x+y_0+\Delta y \end{bmatrix} \doteq \begin{bmatrix} x_0^2 y_0 \\ x_0+y_0 \end{bmatrix} + \begin{bmatrix} 2x_0 y_0 & x_0^2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{7.2}$$

for any x_0 and y_0.

Definition 7.4. A complex function f is said to be *complex differentiable at z_0* if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \tag{7.3}$$

exists. This is equivalent to requiring that

$$f(z_0+\Delta z) \doteq f(z_0) + w_0 \cdot \Delta z, \tag{7.4}$$

were w_0 is the complex constant that equals to the limit in (7.3). The limit in (7.3) is denoted by $f'(z_0)$, if the limit exists.

The symbol $\doteq$ in (7.4) means that

$$f(z_0+\Delta z) = f(z_0) + w_0 \cdot \Delta z + o(\Delta z),$$

where $o(\Delta z)$ is a function such that $\lim_{\Delta z \to 0} |o(\Delta z)|/|\Delta z| = 0$.

Example 7.3. Consider the function

$$f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2 y - y^3).$$

The real and imaginary parts are $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2 y - y^3$, respectively.

The partial derivatives of u and v are

$$\begin{aligned} u_x(x,y) &= 3x^2 - 3y^2, \\ u_y(x,y) &= -6xy, \\ v_x(x,y) &= 6xy, \\ v_y(x,y) &= 3x^2 - 3y^2. \end{aligned}$$

Suppose we fix a base point $(x_0, y_0) = (2, 1)$. The linear approximation in (7.1) at $(2, 1)$ can be written as

$$\begin{bmatrix} u(2+\Delta x, 1+\Delta y) \\ v(2+\Delta x, 1+\Delta y) \end{bmatrix} \doteq \begin{bmatrix} 2 \\ 11 \end{bmatrix} + \begin{bmatrix} 9 & -12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

For general $z = x + iy$, the 2×2 matrix consisting of the partial derivatives is

$$\begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

We can use complex arithmetic to realize the linear approximation by

$$\begin{aligned} f(z + \Delta z) &= f(z) + (3x^2 - 3y^2 + i(6xy)) \cdot (\Delta x + i\Delta y) \\ &= f(z) + (3z^2)\Delta z. \end{aligned}$$

Hence, we can conclude that z^3 is complex differentiable and the complex derivative is $3z^2$.

As in the real case, a function that is complex differentiable at a point is continuous at that point (See Lemma 7.3).

Theorem 7.5. *If $f(z)$ is complex differentiable at z_0, then $f(z)$ is continuous at z_0.*

Proof. Suppose $f(z)$ is complex differentiable at z_0. By the definition of complex differentiability, we can find a complex number w_0 such that

$$f(z_0 + h) = f(z_0) + w_0 \cdot h + o(h),$$

where h is a complex variable and $o(h)$ is a function with the property that $|o(h)/h| \to 0$ as $h \to 0$. If we do not divide by h, the function $o(h)$ converges to zero as well, because

$$\lim_{h\to 0} o(h) = \lim_{h\to 0} \frac{o(h)}{h} h = \lim_{h\to 0} \frac{o(h)}{h} \cdot \lim_{h\to 0} h = 0 \cdot 0 = 0.$$

Therefore

$$\lim_{h\to 0} f(z_0+h) = f(z_0) + \lim_{h\to 0} w_0 \cdot h + \lim_{h\to 0} o(h) = f(z_0).$$

□

7.3 Cauchy-Riemann equations

We derive a necessary condition for complex differentiability.

Theorem 7.6 (Cauchy-Riemann equations). *Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at $z_0 = x_0 + iy_0$ (see Definition 7.4), where z_0 is in the domain of $f(z)$. Then the partial derivatives u_x, u_y, v_x and v_y at z_0 exist, and they satisfy*

$$u_x(x_0,y_0) = v_y(x_0,y_0), \quad \textit{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0).$$

Proof. Let f be a complex function that is complex differentiable at $z_0 = x_0 + iy_0$. The limit in (7.3) that defines the complex derivative does not depend on how we approach the point z_0. We can approach z_0 horizontally or vertically, and the results must be the same if the function is complex differentiable.

Let $\Delta z = \Delta x$ and take $\Delta x \to 0$.

$$\begin{aligned}
&\lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).
\end{aligned}$$

Next suppose $\Delta z = i\Delta y$ and take $\Delta y \to 0$.

$$\begin{aligned}
&\lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y) + iv(x_0, y_0+\Delta y) - u(x_0,y_0) - iv(x_0,y_0)}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y} \\
&= -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0).
\end{aligned}$$

By equating real and imaginary parts of the two limits, we get $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$. □

The function in Example 7.2 is real differentiable everywhere but not complex differentiable. As a matter of fact, the function in Example 7.2 fails the Cauchy-Riemann equation for all real numbers x_0 and y_0, and thus is not complex differentiable anywhere.

A necessary and sufficient condition for complex differentiability at a fixed point z_0 is given in Theorem 7.10 in the appendix of this lecture. It states that a function is complex differentiable at a point if and only if (i) it is real differentiable at this point and (ii) the Cauchy-Riemann equations are satisfied. Checking whether a real function is differentiable from definition is not an easy task. We can apply the following basic result from multivariable calculus (See Theorem 9.21 in [Rudin]).

Theorem 7.7. *Suppose $\vec{f}(x, y) = (u(x, y), v(x, y))$ is a two-dimensional vector field. A sufficient condition for $\vec{f}$ to be real differentiable at (x_0, y_0) is (i) partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0), and (ii) the partial derivatives u_x, u_y, v_x and v_y are continuous at (x_0, y_0).*

This theorem gives a sufficient condition for real differentiability. Combining this theorem with Theorem 7.10, we have the following sufficient condition for complex differentiability. The proof idea is, given conditions 1 and 2 in Theorem 7.7, the vector function $\vec{f}(x, y) = (u(x, y), v(x, y))$ is real differentiable at (x_0, y_0). Because the Cauchy-Riemann equations are satisfied at (x_0, y_0), the 2×2 matrix in (7.1) is in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We can thus write the linear approximation in (7.1) in terms of complex multiplication.

Theorem 7.8 (A sufficient condition for complex differentiability)**.** *A complex function f is complex differentiable at z_0 if*

1. *The partial derivatives u_x, u_y, v_x, v_y exists in a neighborhood of z_0;*
2. *u_x, u_y, v_x, v_y are continuous at z_0;*
3. *Cauchy-Riemann equations are satisfied at z_0.*

From the proof of Theorem (7.6), the complex derivative of $f(z)$ can be computed using the partial derivatives of the real and imaginary part of $f(z)$.

Corollary 7.9. *If a complex function $f(z) = u(x, y) + iv(x, y)$ is complex differentiable at $z_0 = x_0 + iy_0$, the complex derivative at z_0 can be computed by*

$$u_x(x_0, y_0) + iv_x(x_0, y_0),$$

or

$$v_y(x_0, y_0) - iu_y(x_0, y_0).$$

7.4 Appendix: A necessary and sufficient condition for complex differentiability at a point

Theorem 7.10. *Suppose $f(z) = u(x, y) + iv(x, y)$ is a complex function defined on a domain D which contains $z_0 = x_0 + iy_0$. Then $f(z)$ is complex differentiable at z_0 if and only if the real vector field $(u(x, y), v(x, y))$ is real-differentiable at (x_0, y_0) and the partial derivatives satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0).

Proof. ($\Rightarrow$) Suppose $f(z)$ is complex differentiable at z_0. This means that

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Let the limit be $a + ib$. When the magnitude of h is small, we can approximate $f(z_0 + h)$ by

$$f(z_0 + h) = f(z_0) + (a + ib) \cdot h + o(h), \tag{7.5}$$

where $o(h)$ represents a complex function of h that satisfies

$$\lim_{h\to 0} \frac{o(h)}{h} = 0.$$

By expressing $f(z)$ in terms of the real and imaginary parts and $h = \Delta x + i\Delta y$, we can write (7.5) as

$$u(x_0 + \Delta x, y_0 + \Delta y) = u(x_0, y_0) + a\Delta x - b\Delta y + \mathrm{Re}(o(\Delta x + i\Delta y)),$$
$$v(x_0 + \Delta x, y_0 + \Delta y) = v(x_0, y_0) + a\Delta y + b\Delta x + \mathrm{Im}(o(\Delta x + i\Delta y)).$$

In terms of matrix, we have

$$\begin{bmatrix} u(x_0 + \Delta x, y_0 + \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) \end{bmatrix} = \begin{bmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \begin{bmatrix} \mathrm{Re}(o(\Delta x + i\Delta y)) \\ \mathrm{Im}(o(\Delta x + i\Delta y)) \end{bmatrix}. \tag{7.6}$$

The length of the error term approaches to zero faster than $\sqrt{\Delta x^2 + \Delta y^2}$, because

$$\frac{\sqrt{\mathrm{Re}(o(\Delta x + i\Delta y))^2 + \mathrm{Im}(o(\Delta x + i\Delta y))^2}}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{|o(h)|}{|h|} = \left|\frac{o(h)}{h}\right|$$

converges to zero as $h \to 0$. This proves that the vector field $(u(x, y), v(x, y))$ is real differentiable at (x_0, y_0).

The fact that the 2×2 matrix in (7.6) is in the special form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ implies that the Cauchy-Riemann equations hold at (x_0, y_0).

($\Leftarrow$) In the reverse direction, we suppose that the vector field $(u(x, y), v(x, y))$ is real-differentiable at (x_0, y_0), and the Cauchy-Riemann equations are satisfied at (x_0, y_0). From the definition of real-differentiability, the quantity

$$\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}} \left\| \begin{bmatrix} u(x_0 + \Delta x, y_0 + \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) \end{bmatrix} - \begin{bmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{bmatrix} - \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \right\|$$

approaches zero as the vector $(\Delta x, \Delta y)$ approaches the origin. We thus have a vector function $\mathbf{g}(\Delta x, \Delta y)$ with variables Δx and Δy, satisfying equation

$$\begin{bmatrix} u(x_0 + \Delta x, y_0 + \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) \end{bmatrix} = \begin{bmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{bmatrix} + \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \mathbf{g}(\Delta x, \Delta y) \tag{7.7}$$

and

$$\frac{\|\mathbf{g}(\Delta x, \Delta y)\|}{\sqrt{\Delta x^2 + \Delta y^2}} \to 0$$

as $\sqrt{\Delta x^2 + \Delta y^2} \to 0$. $\square$

Moreover, from the Cauchy-Riemann equation, we can write (7.7) as

$$\begin{bmatrix} u(x_0 + \Delta x, y_0 + \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) \end{bmatrix} = \begin{bmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \mathbf{g}(\Delta x, \Delta y)$$

for some real numbers a and b.

If we identify the vector function $\mathbf{g}(\Delta x, \Delta y)$ with a complex-valued function $e(\Delta x + i\Delta y)$ whose real and imaginary parts equal the first and second component of $\mathbf{g}(\Delta x, \Delta y)$, we get

$$\mathbf{g}(\Delta x, \Delta y) = (\mathrm{Re}(e(\Delta x + i\Delta y), \mathrm{Im}(e(\Delta x + i\Delta y)),$$

and

$$f(z_0 + h) = f(z_0) + (a + ib) \cdot h + e(h).$$

Since $\lim_{h\to 0} |e(h)|/|h| = 0$, we can conclude that $f(z)$ is complex differentiable at $z_0 = x_0 + iy_0$.

8 Holomorphic functions

Summary:

- Example of holomorphic functions
- Differentiation rules

It turns out that being differentiable at an isolated point (as defined in Definition 7.4) is not a very useful property. Much more structure can be seen if we require a function to be complex differentiable in an open neighborhood of a point.

Definition 8.1. A complex function $f(z)$ is said to be *holomorphic* at a point z_0 if there is a neighborhood of z_0 such that f is complex differentiable at every point in the neighborhood. A function is said to be *entire* if it is complex differentiable at every point in $\mathbb{C}$. We denote the complex derivative of f by $\frac{df}{dz}$ when f is holomorphic.

If we can show that a function is complex differentiable at every point in the domain of definition, then it is holomorphic in the domain.

Remark. In this notes, we will use the term "analytic" to refer to complex functions that are locally representable by complex power series. We will show later that this is equivalent to the condition of complex differentiable at every point in the domain.

Using Theorem 7.8, we can check that a complex function is holomorphic if it satisfies the Cauchy-Riemann equations and has continuous first partial derivatives throughout the domain of definition.

8.1 Examples

The following examples illustrate how to determine whether a function is complex differentiable (or the contrary). We start with the simplest example.

Example 8.1. For any complex constant a, the function $f(z) = az$ is an entire function, i.e., it is complex differentiable at every point in the complex plane, and the derivative is equal to a. We can see this by checking the definition of complex derivative.

$$\left|\frac{(a(z_0+h)) - az_0}{h} - a\right| = 0.$$

Trivially, we have $\lim_{h\to 0} \frac{a(z_0+h)-az_0}{h} = a$.

Likewise, we can see that the derivative of a translation function $z \mapsto z + b$ is complex differentiable, and the derivative is equal to 1.

Example 8.2. The conjugate function $f(z) = \bar{z}$ is not complex differentiable anywhere. It is because $u_x = 1$ and $v_y = -1$, and $u_x \neq v_y$ at any point in $\mathbb{C}$. It fails the CR equations everywhere.

Example 8.3. The square function $f(z) = z^2$ is entire with derivative $2z$. The real and imaginary parts are $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, respectively. We check that the partial derivatives.

$$u_x = 2x,\ u_y = -2y,\ v_x = 2y,\ v_y = 2x$$

exist and are continuous at every point in $\mathbb{C}$. This check conditions 1 and 2 in Theorem 7.8. Furthermore, the Cauchy-Riemann equalities hold, because

$$u_x = v_y = 2x, \quad \text{and} \quad u_y = -v_x = -2y.$$

By Theorem 7.8, $f(z) = z^2$ is entire, and the complex derivative is $u_x + iv_x = 2z$.

Example 8.4. The function $f(z) = 1/z$ is defined in the domain $\mathbb{C} \setminus \{0\}$. It is complex differentiable everywhere in the domain because, for $z \neq 0$,

$$\begin{aligned}\frac{\frac{1}{z+h} - \frac{1}{z}}{h} &= \frac{1}{h}\Big(\frac{z-(z+h)}{(z+h)z}\Big) \\ &= -\frac{1}{z(z+h)}.\end{aligned}$$

When $h \to 0$, the limit of $-\frac{1}{z(z+h)}$ is $-1/z^2$. Therefore $f(z) = 1/z$ is complex differentiable for $z \in \mathbb{C} \setminus \{0\}$, and the complex derivative is $-1/z^2$.

Example 8.5. The function $f(z) = |z|^2 = x^2 + y^2$ has zero imaginary part. As a real-valued function it is real differentiable. However it is complex differentiable only at $z = 0$. We see this by computing the partial derivatives

$$\begin{aligned} u_x &= 2x, \quad & v_x &= 0, \\ u_y &= 2y, \quad & v_y &= 0. \end{aligned}$$

The Cauchy-Riemann equations are satisfied only at $z = 0$. Therefore it is not complex differentiable if $z \neq 0$. By Theorem 7.8, it is complex differentiable only at the origin $z = 0$. This function is not holomorphic at any point.

Example 8.6. We have seen in (3.5) that the complex exponential function can be written as

$$\exp(x+iy) = e^x \cos(y) + ie^x \sin(y).$$

This is an entire function because the Cauchy-Riemann equations

$$\begin{aligned} u_x &= e^x \cos(y) = v_y \\ u_y &= -e^x \sin(y) = -v_x \end{aligned}$$

are satisfied at all points, and the partial derivatives of u and v are continuous everywhere. By Theorem 7.8, the complex exponential function is complex differentiable everywhere, and the derivative is

$$\exp(z)' = u_x + iv_x = e^x \cos(y) + i(e^x \sin y) = \exp(z).$$

Example 8.7. Define a complex function

$$f(z) = \begin{cases} \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} & \text{if } x+iy \neq 0 \\ 0 & \text{if } x+iy = 0. \end{cases}$$

This function is continuous at $z=0$, satisfies the Cauchy-Riemann equations at $z=0$, but fail to be complex differentiable at $z=0$.

We first show that $f(z)$ is continuous at $z=0$. Take the absolute value of $f(z)$,

$$\begin{aligned} |f(z)| &= \frac{1}{x^2+y^2}|x^3-y^3+i(x^3+y^3)| \\ &= \frac{\sqrt{2(x^6+y^6)}}{x^2+y^2}. \end{aligned}$$

We simplify and upper bound x^6+y^6 as

$$(x^2+y^2)(x^4-x^2y^2+y^4) \leq (x^2+y^2)(x^4+2x^2y^2+y^4) = (x^2+y^2)^3.$$

We then get

$$|f(z)| \leq \sqrt{2(x^2+y^2)} = \sqrt{2}|z|$$

If we take $|z| \to 0$, we have $|f(z)| \to 0$. Therefore $f(z)$ approaches 0 as z approaches 0. The function $f(z)$ is a continuous function.

Table 1: Summary of the examples

Not differentiable	Real differentiable but not holomorphic	Holomorphic in a domain
Ex. 8.7	Ex. 8.2, 8.5	Ex. 8.1, 8.3, 8.4, 8.6

The real and imaginary parts of $f(z)$ are

$$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2}, \qquad v(x,y) = \frac{x^3 + y^3}{x^2 + y^2},$$

respectively. We next show that the Cauchy-Riemann equations hold at $z = 0$,

$$\begin{aligned} u_x(x,0) &= \frac{\partial x}{\partial x} = 1, & v_y(0,y) &= \frac{\partial y}{\partial y} = 1, \\ u_y(0,y) &= \frac{\partial(-y)}{\partial y} = -1, & v_x(x,0) &= \frac{\partial x}{\partial x} = 1. \end{aligned}$$

Therefore, we have $u_x(0,0) = v_y(0,0)$ and $u_y(0,0) = -v_x(0,0)$.

However, this function is not complex differentiable at $z = 0$, (and hence is not real differentiable by Theorem 7.10). If this function were complex differentiable, the complex derivative would be

$$u_x(0,0) + iv_x(0,0) = 1 + i$$

by Corollary 7.9. But if we approach $z = 0$ in a straight line from the point $1 + i$, we will get

$$\lim_{t\to 0} \frac{1}{t(1+i)}(f(t+it) - f(0)) = \lim_{t\to 0} \frac{1}{t(1+i)}\Big(u(t,t) + iv(t,t)\Big) = \frac{1}{t(1+i)} i\frac{2t^3}{2t^2} = \frac{1+i}{2}.$$

Obviously $1 + i$ is not equal to $(1 + i)/2$. Therefore $f(z)$ is not complex differentiable at $z = 0$.

8.2 Differentiation rules

The usual rules for differentiation continue to hold for holomorphic functions. We note that another common notation for complex derivative is $\frac{df}{dz}$. This is a complex function whose value at z_0 is $f'(z_0)$.

The followings are the basic differentiation formulae. Suppose $f(z)$ and $g(z)$ are holomorphic in a domain D.

$$\begin{aligned}(f(z)+g(z))' &= f'(z)+g'(z),\\ (ag(z))' &= ag'(z), \text{ where } a\in\mathbb{C},\\ (f(z)g(z))' &= f'(z)g(z)+f(z)g'(z),\\ (f(z)/g(z))' &= \frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2}, \quad \text{provided that } g(z)\neq 0.\end{aligned}$$

If the co-domain of $g(z)$ is inside the domain of f, and g, f are both holomorphic, we have the chain rule

$$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z).$$

The proof is the same as in calculus and omitted. We illustrate the differentiation rules by two examples.

Example 8.8. Show that $(z^n)' = nz^{n-1}$ for all $z\in\mathbb{C}$.

It is trivially true when $n=1$. We prove the rest by induction. Suppose that it is true for $n=k$. When $n=k+1$, we apply the product rule to see that

$$(z^k)' = (zz^{k-1})' = (1)z^{k-1} + z(k-1)z^{k-2} = kz^{k-1}.$$

Example 8.9. By combining the previous example with the sum rule, we see that polynomials are entire (complex differentiable at all point), and

$$\frac{d}{dz}\big(a_0+a_1z+a_2z^2+\cdots+a_nz^n\big) = a_1+2a_2z+\cdots+na_nz^{n-1}.$$

Example 8.10. Show that $(\sin z)' = \cos z$.

We differentiate both sides of the identity $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ to get

$$(\sin z)' = \Big(\frac{e^{iz}-e^{-iz}}{2i}\Big)' = \frac{(e^{iz}-e^{-iz})'}{2i} = \frac{ie^{iz}-(-i)e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z.$$

Example 8.11. We can show that a linear fractional function $f(z) = (az+b)/(cz+d)$ is complex differentiable and has nonzero derivative at any point in the domain of definition. Using the quotient rule for differentiation, we obtain

$$\Big(\frac{az+b}{cz+d}\Big)' = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

As part of the definition of LFR, the four constant a, b, c and d satisfies $ad - bc \neq 0$. The complex derivative of LFR is thus nonzero as long as z is not equal to $-d/c$.

Remark. We can consider the complex derivative of LFR at the point at infinity by making a change of variable $w = 1/z$ and investigate the function behavior near $w = 0$. Suppose $f(z) = (az + b)/(cz + d)$ is an LFR. Define $g(w) = f(1/w)$.

$$g(w) = \frac{a\frac{1}{w} + b}{c\frac{1}{w} + d} = \frac{wb + a}{wd + c}.$$

The derivative of $g(w)$ equals

$$g'(w) = \frac{b(wd + c) - d(wb + a)}{(wd + c)^2} = \frac{bc - da}{(wd + c)^2},$$

which is nonzero at $w = 0$.

Remark. The complex number $-d/c$ is mapped to the point at infinity by the LFR $f(z) = (az + b)/(cz + d)$. We can study the behavior new the point $-d/c$ by considering the reciprocal $h(z) \triangleq 1/f(z)$. The function $h(z)$ is an LFR, and it maps $-d/c$ to the origin. We can compute the derivative of $h(z)$ as

$$h'(z) = \Big(\frac{cz + d}{az + b}\Big)' = \frac{c(az + b) - a(cz + d)}{(az + b)^2} = \frac{bc - ad}{(az + b)^2}$$

The value of $h'(z)$ at $z = -d/c$ is

$$h'(-d/c) = \frac{bc - ad}{(a(-d/c) + b)^2} = \frac{c^2(bc - ad)}{(-ad + bc)^2} = \frac{c^2}{bc - ad}$$

If the coefficient c is nonzero, we see that the function $f(z)$ has nonzero derivative at the point $-d/c$.

The remaining case $c = 0$ corresponds to affine function, which maps ∞ to ∞. The behavior of an affine function at the point at infinity is an exercise.

9 Properties of Holomorphic functions

Summary:

- Conformal property
- Phase portrait
- Harmonic functions

9.1 Conformal property

Suppose $f(z)$ is complex differentiable at a given point z_0 in $\mathbb{C}$, and suppose $f'(z_0)$ is nonzero. One can show that the function f is *conformal*, or *angle-reserving* at z_0.

Draw two parametric curves $\gamma_1(t)$ and $\gamma_2(t)$ through z_0. By "parametric curve" it means a smooth map from an interval in $\mathbb{R}$ to $\mathbb{C}$. Let the range of the parameter t in $\gamma_1(t)$ be $[a, b]$, where $a < b$, and $\gamma_1(t_1) = z_0$, for some t_1 in the range $[a, b]$. Likewise, let the range of t in $\gamma_2(t)$ be $[c, d]$, where $c < d$ and $\gamma_2(t_2) = z_0$, for some t_2 in $[c, d]$. In the domain of f, the angle between the tangent line of γ_1 at z_0 and the tangent line of γ_2 at z_0 is

$$\theta = \arg\Big(\frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}\Big).$$

Both $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ are nonzero, so that the argument is well defined. (See Fig. 2)

If we apply the function f to $\gamma_1(t)$ and $\gamma_2(t)$, we obtain a pair of curves intersecting at the point $w_0 = f(z_0)$,

$$w_0 = f(\gamma_1(t_1)) = f(\gamma_2(t_2)).$$

By applying the chain rule, we get the angle between the two tangent lines

$$\arg\Big(\frac{(f\circ\gamma_2)'(t_2)}{(f\circ\gamma_1)'(t_1)}\Big) = \arg\Big(\frac{f'(\gamma_2(t_2))\cdot\gamma_2'(t_2)}{f'(\gamma_1(t_1))\cdot\gamma_1'(t_1)}\Big) = \arg\Big(\frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}\Big) = \theta$$

In particular, if the two curves are perpendicular to each other in the domain, then it continues to be perpendicular to each other in the co-domain.

The conformal property explains why the conjugate function $f(z) = \bar{z}$ is not complex differentiable anywhere. The conjugate function is a reflection geometrically, and reflection reverses the orientation of angles.

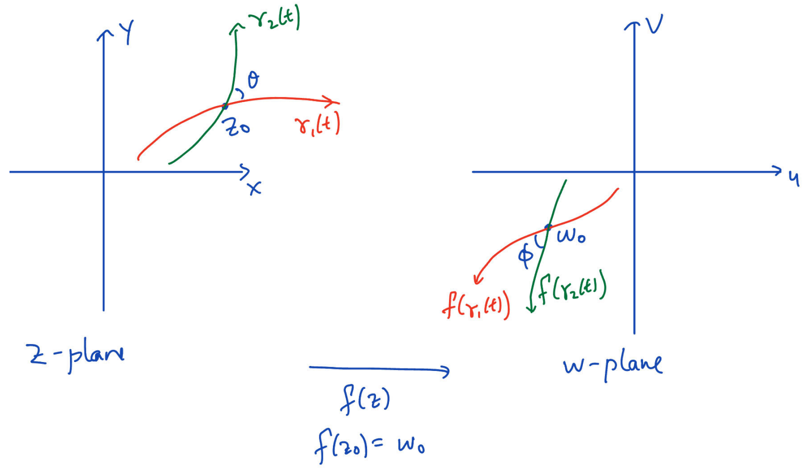


Figure 2: Angle-preserving property of holomorphic function

Definition 9.1. We say that two regions R_1 and R_2 in $\mathbb{C}$ are *conformally equivalent* if we can find a bijective holomorphic map $f(z)$ from R_1 to R_2.

Example 9.1. The upper half-plane is conformally equivalent to the unit disc. We can apply the linear fractional transformation $f(z) = (z-i)/(z+i)$. Since a linear fractional transformation maps a circle/straight line to circle/straight line, we can check the image of three specific points on the real axis.

$$\begin{aligned} f(1) &= \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{-2i}{2} = -i \\ f(0) &= \frac{0-i}{0+i} = \frac{-i}{i} = -1 \\ f(-1) &= \frac{-1-i}{-1+i} = \frac{(-1-i)^2}{2} = i \end{aligned}$$

The unique circle that passes through the three image points $-i$, -1 and i is the unit circle. Therefore, the function $f(z)$ maps the real axis to the unit circle. We further check that $f(i) = 0$, i.e., the point i in the upper half plane is mapped to the origin. The above argument shows that $f(z)$ maps the upper half-plane to the interior of the unit disc.

Since linear fractional transformation is always bijective and conformal, we can say that

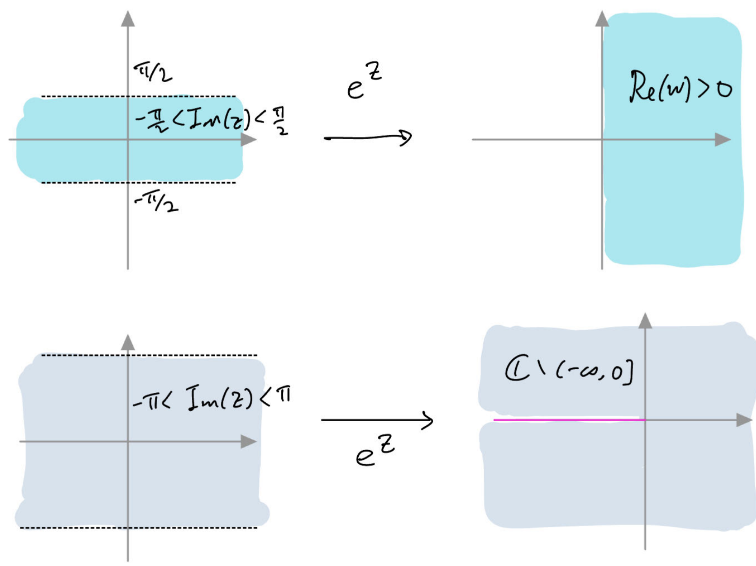


Figure 3: Mapping property of the complex exponential function.

the upper half-plane and the unit disc are conformally equivalent.

Example 9.2. The complex exponential function $f(z) = e^z$ maps the horizontal strip $\{z : -\pi/2 < \mathrm{Im}(z) < \pi/2\}$ conformally and bijectively to the right half plane $\{z : \mathrm{Re}(z) > 0\}$. Infinite horizontal strips are conformally equivalent to the unit open disk.

The complex exponential function also maps

$$\{z : -\pi < \mathrm{Im}(z) < \pi\} \to \mathbb{C} \setminus \{x + iy : y = 0 \text{ and } x \leq 0\}$$

conformally and bijectively (See Fig. 3).

9.2 Phase portrait

Phase portrait, which is also known as domain coloring, plots the argument of a complex function on a plane, by associating the numbers in the range $[0, 2\pi)$ with the color spectrum. The brightness corresponds to the magnitude of the complex function. A dark point means the function is zero. A bright color indicates the area where function magnitude is large. We take the phase portrait of the identity function as the reference (Fig. 4).

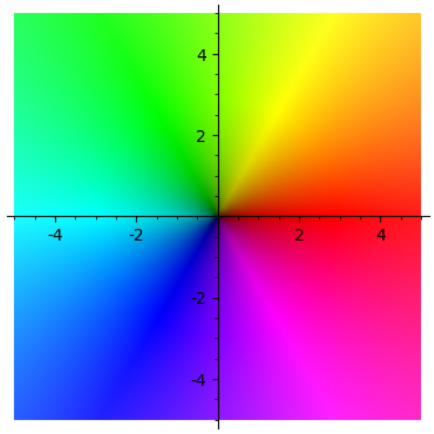


Figure 4: Phase portrait of $f(z) = z$

The phase portrait of some complex functions are shown in Fig. 5. The picture for the function $f(z) = 1/z$ has a bright point near the origin, because this function has large magnitude near the origin. We also note that as we go around the origin, we see the color in the reverse order. It is because the function $f(z) = 1/z$ multiply the argument by -1. In the phase portrait of $f(z) = z^2$, we see a zero of order 2 at the origin. As we go around the origin, we meet all the colors twice.

Example 9.3. The inverse function $f(z) = 1/z$ maps the open disc $|z-1| = 1$ to the half-plane $\mathrm{Re}(w) > 1/2$.

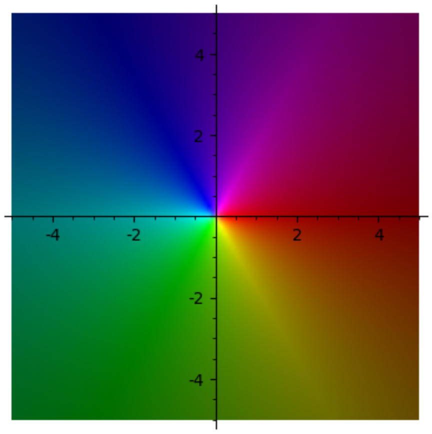
9.3 Harmonic functions

Another necessary condition for being holomorphic is that the real and imaginary parts are both harmonic functions. A twice-differentiable real-valued function $F(x, y)$ is said to be harmonic if it satisfies the Laplace equation

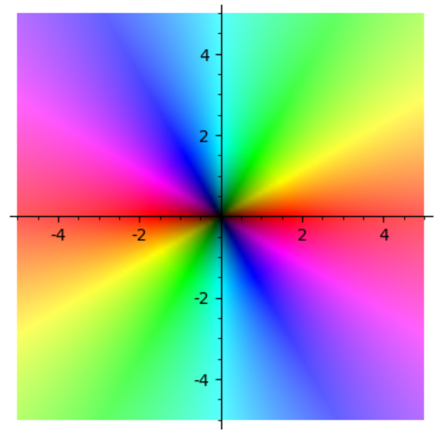
$$F_{xx} + F_{yy} = 0,$$

where F_{xx} and F_{yy} are the second partial derivatives of F with respect to x and y, respectively.

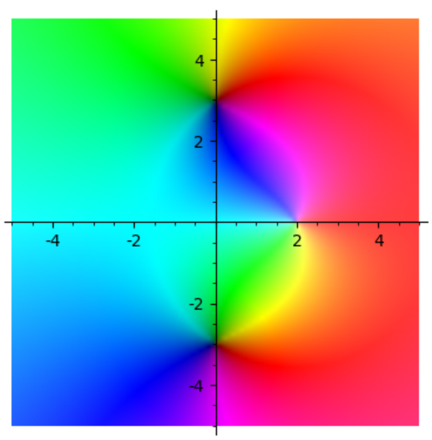
Suppose that $u(x, y)$ and $v(x, y)$ are twice differentiable, and the second-order derivatives are continuous functions, so that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. Under these assumptions,



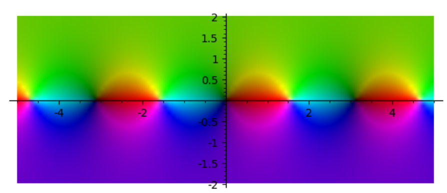
(a) Phase portrait of $f(z) = 1/z$



(b) Phase portrait of $f(z) = z^2$



(c) Phase portrait of
$f(z) = (z^2 + 9)/(z - 2)$



(d) Phase portrait of $f(z) = \tan(z)$

Figure 5: Phase portraits.

we can deduce that the real and imaginary part of a holomorphic function are both harmonic functions. (We will show in a later lecture that the additional smoothness requirements always hold.) In general, we say that a function is in C^2 if the second partial derivatives exist and are continuous.

Theorem 9.2. *Let $f(z)$ be a holomorphic function on a region R, and the real and imaginary parts are in C^2. Then*

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$$

in the region R.

Proof. The complex function $f(z)$ is holomorphic by assumption, hence must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. Take partial derivatives with respect to x and y, we obtain $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Because we can exchange the order of the partial derivatives, we get

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = v_{xy} - v_{xy} = 0.$$

Similarly, we can show that $v(x, y)$ is harmonic by first deriving $u_{xy} = v_{yy}$ and $u_{yx} = -v_{xx}$ from the Cauchy-Riemann equation, and then

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = -u_{xy} + u_{xy} = 0.$$

□

In the followings, we will assume that $u(x, y)$ and $v(x, y)$ are C^2 functions.

Motivated by the previous theorem we define the notion of harmonic conjugates

Definition 9.3. Given a harmonic function $u(x, y)$, we say that $v(x, y)$ is a *harmonic conjugate* of $u(x, y)$ if $u(x, y) + iv(x, y)$ is holomorphic.

Equivalently, we can define that $v(x, y)$ is a harmonic conjugate to u if u and v satisfy the Cauchy-Riemann equations. It is obvious that harmonic conjugate to a given harmonic functions, if exists, is not unique. It is because, if $v(x, y)$ is a harmonic conjugate to $u(x, y)$, then for any complex constant C, the function $v(x, y) + C$ is also a harmonic conjugate.

The existence of harmonic conjugate depends on the topology of the domain on which the function $u(x, y)$ is defined. It is known that if the domain D is simply connected, then a harmonic function $u(x, y)$ defined on D has a harmonic conjugate. We can formulate the problem in terms of differential form. We want to find a function $v(x, y)$ whose differential is given by

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy.$$

Since u_y and u_x are given, the problem of finding a harmonic conjugate to $u(x, y)$ reduces to solving the differential equation above. We prove in the appendix of this section that solution always exists if the region of definition D is simply connected (The proof relies an application of the Green's theorem).

Example 9.4. We consider the example

$$u(x, y) = \sin(x)\cosh(y)$$

. This is harmonic because

$$u_{xx} = -\sin(x)\cosh(y), \quad \text{and} \quad u_{yy} = \sin(x)\cosh(y).$$

We can set out to find its harmonic conjugate $v(x, y)$. From the condition $v_x = -u_y = -\sin(x)\sinh(y)$, we can obtain

$$v(x, y) = \int -\sin(x)\sinh(y)\, dx = \cos(x)\sinh(y) + C(y),$$

where $C(y)$ is a function of y. By applying the other Cauchy-Riemann equation $v_y = u_x$, we get

$$v_y = \cos(x)\cosh(y) + C'(y) = u_x = \cos(x)\cosh(y).$$

Hence the function $C(y)$ should be a constant. The harmonic conjugate of $u(x, y)$ can be written as

$$v(x, y) = \cos(x)\sinh(y) + C$$

for some constant C. The complex holomorphic function is

$$\begin{aligned}
\sin(x)\cosh(y) + i\cos(x)\sinh(y) &= \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^y + e^{-y}}{2} + i\Big(\frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^y - e^{-y}}{2}\Big) \\
&= \frac{1}{4i}[(e^{ix} - e^{-ix})(e^y + e^{-y}) - (e^{ix} + e^{-ix})(e^y - e^{-y})] \\
&= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \sin(z).
\end{aligned}$$

Example 9.5. Find a harmonic conjugate of the function

$$u(x, y) = -2x^2 + x^3 + 2y^2 - 3xy^2.$$

The function $u(x, y)$ is defined on the whole complex plane. We check that it is harmonic:

$$\begin{aligned} u_x &= -4x + 3x^2 - 3y^2 \\ u_{xx} &= -4 + 6x \\ u_y &= 4y - 6xy \\ u_{yy} &= 4 - 6x. \end{aligned}$$

Hence $u_{xx} + u_{yy}$ is identically equal to zero.

We now proceed to find a harmonic conjugate. We first integrate

$$v_x = -u_y = -4y + 6xy$$

with respect to x and get

$$v(x, y) = \int -4y + 6xy \, dx = -4xy + 3x^2 y + C(y),$$

where $C(y)$ is a constant that may involve y. Differentiate the above with respect to y,

$$v_y = -4x + 3x^2 + C'(y).$$

After comparing with u_x, we see that $C'(y) = -3y^2$, and hence $C(y) = -y^3 + C$ for some constant C. The answer is

$$v(x, y) = -4xy + 3x^2 y - y^3 + C,$$

where C is any constant.

When $u(x, y)$ is a *polynomial* in x and y, there is a slick method to directly obtain the complex function $f(z)$ whose real part is $u(x, y)$. We can write $z = x + iy$ and exploit the following equalities

$$u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = u(x, y) = \mathrm{Re}(f(z)) = \frac{f(z) + \overline{f(z)}}{2} = \frac{f(z) + f(\bar{z})}{2}.$$

In the last step we can replace $\overline{f(z)}$ by $f(\bar{z})$, because f is a polynomial in z.

We can re-do Example 9.5 by using this method.

$$\begin{aligned} u\Big(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\Big) &= -2\Big(\frac{z+\bar z}{2}\Big)^2 + \Big(\frac{z+\bar z}{2}\Big)^3 + 2\Big(\frac{z-\bar z}{2i}\Big)^2 - 3\Big(\frac{z+\bar z}{2}\Big)\Big(\frac{z-\bar z}{2i}\Big)^2 \\ &= -z^2 - \bar z^2 + z^3/2 + \bar z^3/2. \end{aligned}$$

By comparing it with $(f(z) + f(\bar z))/2$, we obtain $f(z) = -2z^2 + z^3$.

One can further streamline this method by directly computing

$$2u\Big(\frac{z}{2}, \frac{z}{2i}\Big) - u(0,0). \tag{9.1}$$

(We subtract $u(0,0)$ in order to preserve the constant term.) This will be a complex differentiable function f in variable z only and the real part is precisely $u(x,y)$. The complex function in Example 9.5 is thus.

$$\begin{aligned} &2u\Big(\frac{z}{2}, \frac{z}{2i}\Big) - u(0,0) \\ &= 2\Big(-2\big(\frac{z}{2}\big)^2 + \big(\frac{z}{2}\big)^3 + 2\big(\frac{z}{2i}\big)^2 - 3\big(\frac{z}{2}\big)\big(\frac{z}{2i}\big)^2\Big) \\ &= -2z^2 + z^3. \end{aligned}$$

9.4 Application to Dirichlet's problem

The Dirichlet problem involves solving the Laplace equation in a region with a given boundary condition. This equation holds significant importance in physics, as it can be used to determine the steady-state temperature distribution in a given region. However, solving the Dirichlet problem on a complex or irregularly shaped region can be quite challenging.

To simplify the problem, a technique called problem reduction can be employed. This technique involves transforming the original Dirichlet problem on a complex region into a Dirichlet problem on a standard and simpler shape, such as the unit circle. By utilizing a conformal map to transform the region to a circle, we can focus our attention on solving the Dirichlet problem specifically on the circle. Consequently, solving the Dirichlet problem on the original complex region boils down to finding an appropriate conformal map.

We note that Laplace equation and the physical application are not directly related to complex variables. However, it has been discovered that using the language of complex variables can greatly simplify matters. Let's consider a scenario where we have a region denoted as R with its boundary described by a closed curve γ_1. We identify a point (x,y)

with a complex number $z = x + iy$. Suppose we can transform this region to the unit disc in the w-plane using a holomorphic transformation

$$T(z) = u(x, y) + iv(x, y)$$

that has a non-vanishing derivative throughout the domain R. The boundary condition on R is transformed to a boundary condition on the unit circle.

We first show that the transformation T is locally one-to-one. Considering T as a multi-variable function with variables x and y, we can express its Jacobian matrix as:

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

Since we assume that the Cauchy-Riemann equations are satisfied, we can rewrite the determinant as

$$u_x v_y - v_x u_y = u_x(u_x) - v_x(-v_x) = u_x^2 + v_x^2 = |f'(z)|^2,$$

which is non-zero based on our assumption. This demonstrates that the Jacobian is not equal to zero.

Next, we will demonstrate that a solution to the Dirichlet problem on the unit circle corresponds to a solution in the original region R. Let $F(u, v)$ be a real-valued function that is twice differentiable and defined on the unit circle. We define the composite function G as

$$G(x, y) = F(u(x, y), v(x, y)).$$

We claim that

$$G_{xx} + G_{yy} = (F_{xx} + F_{yy})|T'(z)|^2 \tag{9.2}$$

Consequently, if $T'(z)$ is nonzero, we have $\Delta G = 0$ if and only if $\Delta F = 0$. The equation in (9.2), often referred to as the "transfer lemma," establishes a relationship between the Laplacian in the z-domain and the Laplacian in the w-domain.

The proof of equation (9.2) involves applying multivariable calculus while assuming the following conditions: (1) $u(x, y)$ is harmonic, (2) $v(x, y)$ is harmonic, and (3) the Cauchy-Riemann equations are satisfied.

Taking the derivative with respect to x again, and applying the product rule, we have

$$G_x = F_u u_x + F_v v_x.$$

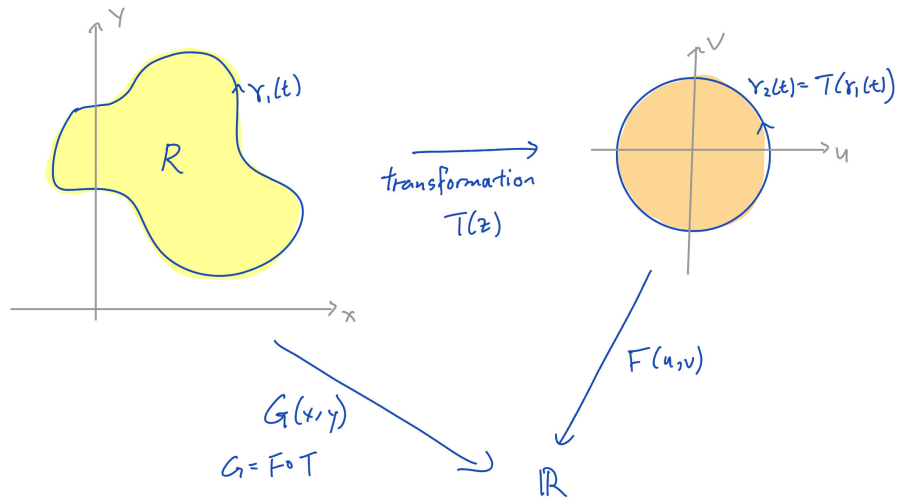


Figure 6: Transforming the domain of Dirichlet problem

Differentiating with respect to x again, and apply the product rule

$$G_{xx} = \big(F_u u_x + F_v v_x\big)_x = (F_u)_x u_x + F_u u_{xx} + (F_v)_x v_x + F_v v_{xx}.$$

Here, $(F_u)_x$ represents the partial derivative of F_u with respect to x, which can be computed as

$$(F_u)_x = F_{uu} u_x + F_{uv} v_x.$$

Likewise, the partial derivative of F_v with respect to x is

$$(F_v)_x = F_{vu} u_x + F_{vv} v_x.$$

By combining these results, we obtain

$$G_{xx} = F_{uu} u_x^2 + F_{uv} u_x v_x + F_u u_{xx} + F_{vu} u_x v_x + F_{vv} v_x^2 + F_v v_{xx}.$$

By repeating the same calculations with x replaced by y, we can derive the expression for G_{yy},

$$G_{yy} = F_{uu} u_y^2 + F_{uv} u_y v_y + F_u u_{yy} + F_{vu} u_y v_y + F_{vv} v_y^2 + F_v v_{yy}.$$

By substituting the expressions for G_{xx} and G_{yy} into the Laplacian ΔG, we obtain a lengthy expression:

$$\begin{aligned} G_{xx} + G_{yy} = F_{uu}(u_x^2 + u_y^2) + F_{vv}(v_x^2 + v_y^2) + (F_{uv} + F_{vu})(u_x v_x + u_y v_y) \\ + F_u(u_{xx} + u_{yy}) + F_v(v_{xx} + v_{yy}). \end{aligned}$$

Given that u and v are assumed to be harmonic, the terms in the second line of the expression vanish. Moreover, applying the Cauchy-Riemann equations allows us to simplify the term $(u_x v_x + u_y v_y)$ as follows,

$$u_x v_x + u_y v_y = u_x(-u_y) + u_y(u_x) = 0.$$

By Cauchy-Riemann again, we have $v_x^2 + v_y^2 = u_x^2 + u_y^2$. We can simplify the Laplacian of G to

$$G_{xx} + G_{yy} = (F_{uu} + F_{vv})(u_x^2 + u_y^2) = (F_{uu} + F_{vv})|f'(z)|^2.$$

This completes the derivation of (9.2).

9.5 Additional proof

In this optional section we prove that harmonic conjugate of a harmonic function exists if the domain of definition is simply connected. The main step is the Green's theorem from multi-variable calculus. In the followings, the functions u and v must be sufficiently smooth so that the Green's theorem can be applied.

Consider the differential form

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy.$$

We claim that the line integral

$$\int_{(x_0,y_0)}^{(x_1,y_1)} -u_y dx + u_x dy \tag{9.3}$$

does not depend on path. We can see it from Green's theorem, using the assumption that D is simply connected and $u(x, y)$ satisfies the Laplace equation. If C is a closed path enclosing a region S, we have

$$\oint_C -u_y dx + u_x dy = \iint_S \frac{\partial}{\partial x} u_x - \frac{\partial}{\partial y}(-u_y)\, ds = \iint_S u_{xx} + u_{yy}\, ds = 0.$$

We can thus define a function $v(x, y)$ by first fixing a point (x_0, y_0) in the domain D of $u(x, y)$, and define a function $v(x, y)$ by

$$v(x, y) \triangleq \int_{(x_0,y_0)}^{(x,y)} -u_y dx + u_x dy \tag{9.4}$$

for any point (x, y) in D. This is well-defined because the line integral is independent of path, and there is at least one path connecting (x_0, y_0) and (x, y).

We now check that the function $v(x, y)$ so defined is a harmonic conjugate of $u(x, y)$. Let (a, b) be any point in D. The partial derivative of v with respect to x at this point is

$$\begin{aligned} v_x(a, b) &= \lim_{\Delta x \to 0} \frac{v(a + \Delta x, b) - v(a, b)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(a,b)}^{(a+\Delta x,b)} -u_y dx + u_x dy \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(a,b)}^{(a+\Delta x,b)} -u_y dx \\ &= -u_y(a, b). \end{aligned}$$

In the last step we have used the assumption that u_y is continuous at (a, b).

Likewise, by consider the partial derivative of v with respect to y, we get

$$\begin{aligned} v_y(a, b) &= \lim_{\Delta y \to 0} \frac{v(a, b + \Delta y) - v(a, b)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(a,b)}^{(a,b+\Delta y)} -u_y dx + u_x dy \\ &= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(a,b)}^{(a,b+\Delta y)} u_x dy \\ &= u_x(a, b). \end{aligned}$$

We have used the continuity of u_x in the last step. This proves that the Cauchy-Riemann equations are satisfied at the point (a, b). By the sufficient condition in Theorem 7.8, we conclude that $u(x, y) + iv(x, y)$ is a holomorphic function in the domain D.

We can now conclude that a harmonic conjugate of $u(x, y)$ can be written as in (9.4) when D is simply connected. This provides the existence proof of harmonic conjugate, and is the theoretical basis for the computational examples in this lecture.

In general, the integration in (9.3) is not tractable. The purpose of the proof is to give a sufficient condition under which harmonic conjugate exists.

9.6 Appendix: Physical interpretation

Given a complex function $f(z) = u(x, y)+iv(x, y)$ that is complex differentiable in a domain D, we define a real vector field $(M(x, y), N(x, y))$ by

$$M(x, y) \triangleq u(x, y), \quad N(x, y) \triangleq -v(x, y).$$

This is the vector field corresponding to the conjugate of $f(z)$. This is sometime called the Polya's vector field of $f(z)$.

> **Theorem 9.4.** *If $f(z)$ is holomorphic in D, then the vector field $(M(x, y), N(x, y))$ has zero 2D curl and zero divergence in D.*

Proof. The 2-dimensional curl of $(M(x, y), N(x, y))$ is the usual 3-dimensional curl function applied to $(M(x, y), N(x, y), 0)$, with the third component set identically to zero. The curl of $(M(x, y), N(x, y), 0)$ is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M(x,y) & N(x,y) \end{vmatrix} = \vec{k}(-v_x - u_y).$$

By Cauchy-Riemann equation, $v_x + u_y$ is equal to $\vec{0}$ throughout the domain D.

On the other hand, the divergence of $(M(x, y), N(x, y))$ is

$$\frac{\partial}{\partial x} M(x, y) + \frac{\partial}{\partial y} N(x, y) = u_x + (-v)_y$$

which is also identically zero by Cauchy-Riemann equation. □

We thus see that a holomorphic function could be interpreted as the reflection of an irrotational and incompressible vector field. This is the basis of the application of complex analysis to solving some 2-dimensional partial differential equations.

9.7 Example: Joukowsky transformation

The Joukowsky map is defined by

$$J(z) \triangleq \frac{1}{2}\big(z + \frac{1}{z}\big)$$

for $z \in \mathbb{C} \setminus \{0\}$. We may extend the domain to the extendedcomplex plane by continuity by setting

$$J(0) \triangleq \infty, \ J(\infty) \triangleq \infty.$$

With this convention, the Joukowsky function has the special property that

$$J(z) = J(1/z)$$

for all $z \in \bar{\mathbb{C}}$.

The derivative of $J(z)$ is equal to

$$J'(z) = \frac{1}{2}\big(1 - \frac{1}{z^2}\big) = \frac{1}{2}\frac{(z-1)(z+2)}{z^2}.$$

for $z \neq 0$. The derivative of $1/J(z)$ at $z = 0$ is equal to

$$\frac{d}{dz}\frac{1+z^2}{2z}\Big|_{z=0} = 2$$

which is nonzero. Hence, $J(z)$ is conformal in the extended complex plane except ± 1.

10 Complex differential form

Summary:

- Tangent space
- Linear functional and real differential
- Complex differential
- $\bar{\partial}$ operator

The main reference for this lecture is [Spivak, Chapter 5] and [Miranda, Chapter IV].

In calculus, a differential dx is usually regarded as an infinitesimal quantity, representing a very small change in the x variable. The *total differential* of a bi-variate function $f(x, y)$ is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

However, the treatment of differential form in calculus is usually not very rigorous.

In this lecture we study complex differentiability by first considering multi-variable real functions and give a more rigorous definition of differential dx and dy. Then we extend it to the complex case and give a concrete meaning of the symbol dz. In the following sections we will borrow some notation from real manifolds. However, we do not need the full generality but only need the special case of dimension 2 and when the manifold is flat.

10.1 Tangent space and differential form

We first consider the Euclidean space $\mathbb{R}^2$ in the real case.

We define the tangent space at a given a point $\mathbf{p}$ in $\mathbb{R}^2$ as the set of all vectors that emanate from $\mathbf{p}$. The center point is represented by a position vector $\mathbf{p}$. To emphasize the starting point of a tangent vector, we write

$$(\mathbf{p}, \mathbf{v})$$

to represent a tangent vector emanating from the point $\mathbf{p}$.

The first component, $\mathbf{p}$, indicates that $\mathbf{p}$ is the initial point. We can add two vectors in $T_{\mathbf{p}}$ as follows,

$$(\mathbf{p}, \mathbf{v}_1) + (\mathbf{p}, \mathbf{v}_2) \triangleq (\mathbf{p}, \mathbf{v}_1 + \mathbf{v}_2)$$

and multiply by a scalar

$$a(\mathbf{p}, \mathbf{v}) \triangleq (\mathbf{p}, a\mathbf{v}).$$

However, if $\mathbf{p}$ and $\mathbf{q}$ are two distinct vector, the addition of $(\mathbf{p}, \mathbf{u})$ and $(\mathbf{q}, \mathbf{v})$ are not defined. (See Fig. 7)

Formally, given a point $\mathbf{p}$, the *tangent space at* $\mathbf{p}$ is defined as the set

$$T_{\mathbf{p}} \triangleq \{(\mathbf{p}, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^2\}.$$

For each point $\mathbf{p}$, the tangent space $T_{\mathbf{p}}$ is a two-dimensional vector space. We will use $\mathbf{e}_1$ and $\mathbf{e}_2$ to denote the standard basis. We will also use the shorter notation $\mathbf{v}_{\mathbf{p}}$ instead of $(\mathbf{p}, \mathbf{v})$. For concreteness, we can consider the example

$$2(4\mathbf{e_1} - \mathbf{e_2})_{\mathbf{p}} + (\mathbf{e_1} + 3\mathbf{e_2})_{\mathbf{p}} = (9\mathbf{e_1} + \mathbf{e_2})_{\mathbf{p}}.$$

Dot product of two vectors in $T_{\mathbf{p}}$ is done in the expected way

$$(\mathbf{p}, \mathbf{u}) \cdot (\mathbf{p}, \mathbf{v}) \triangleq (\mathbf{p}, \mathbf{u} \cdot \mathbf{v}). \tag{10.1}$$

A *vector field* is a mapping that associates, for each point $\mathbf{p}$, a tangent vector $\mathbf{v}_{\mathbf{p}}$, in a continuous and smooth manner. For instance, we can represent a vector in each tangent space using a formula, such as

$$((3x + y)\mathbf{e}_1 + (x - y)\mathbf{e}_2)_{(x,y)}$$

for $(x, y) \in \mathbb{R}^2$. In simple terms, the expression above says that at the point (x, y), we select the vector $(3x+y)\mathbf{e}_1+(x-y)\mathbf{e}_2$ from the associated tangent space $T_{(x,y)}$. To further simplify notation, very often we skip the subscript $_{(x,y)}$. In physics notation, the two unit vectors are commonly denoted as $\vec{\imath}$ and $\vec{\jmath}$. Therefore, if we write $(3x + y)\vec{\imath} + (x - y)\vec{\jmath}$, it represents a vector emanating from the point (x, y), with horizontal and vertical components equal to $3x + y$ and $x - y$, respectively. The graph of this vector field is depicted in Fig. 8.

The *cotangent space* at $\mathbf{p}$, denoted as $T^*_{\mathbf{p}}$, is the dual space of $T_{\mathbf{p}}$. It consists of all linear functionals, also known as *covectors*, that take a tangent vector at $\mathbf{p}$ as input and return a real number. A *differential form* is a continuous selection of a covector at each cotangent space.

An example of covector is a function that returns the x-coordinate of a vector.

$$\phi_1(x\mathbf{e}_1 + y\mathbf{e}_2) = x. \tag{10.2}$$

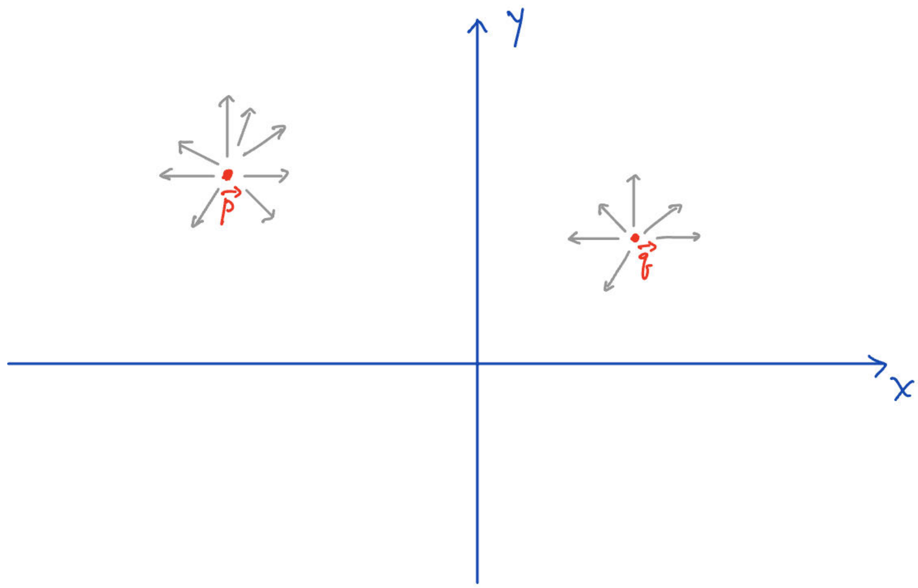


Figure 7: Tangent space at point **p** and tangent space at **q**

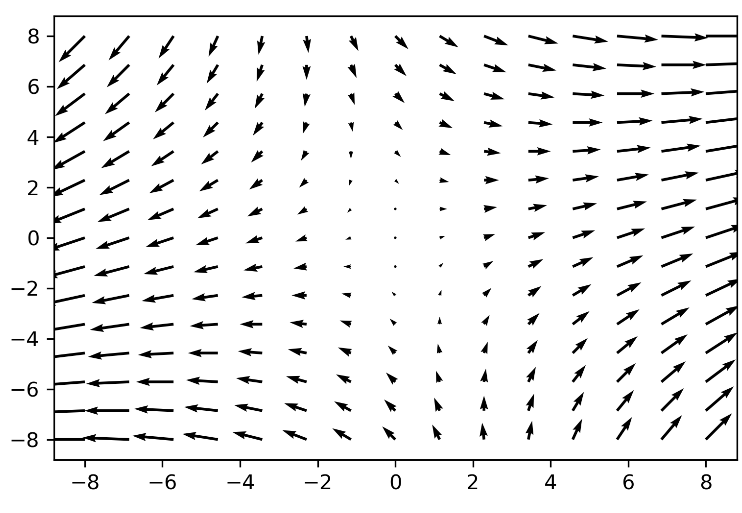


Figure 8: Plot of vector field $(3x + y)\vec{\imath} + (x - y)\vec{\jmath}$

Another simple example is a function that returns the y-coordinate

$$\phi_2(x\mathbf{e}_1 + y\mathbf{e}_2) = y. \tag{10.3}$$

In fact, a family of examples can be derived from directional derivatives. Given a differentiable function $f(x, y)$ defined at a specific point $\mathbf{p} = (x_0, y_0)$ and a vector $\mathbf{v}_\mathbf{p} = (r\mathbf{e}_1 + s\mathbf{e}_2)_\mathbf{p}$, the directional derivative of f at $\mathbf{p}$ in the direction of $\mathbf{v}_\mathbf{p}$ is given by

$$\lim_{h\to 0} \frac{f(x_0 + rh, y_0 + sh) - f(x_0, y_0)}{h} = \frac{\partial f(x_0, y_0)}{\partial x} r + \frac{\partial f(x_0, y_0)}{\partial y} s.$$

We denote the partial derivative with respect to the first coordinate by f_x, and the partial derivative with respect to the second coordinate by f_y. Using the dot product notation for tangent vectors in (10.1) and the gradient

$$(\nabla f)_\mathbf{p} = (f_x(x_0, y_0), f_y(x_0, y_0))_\mathbf{p},$$

we can express the directional derivative as $\nabla f_\mathbf{p} \cdot \mathbf{v}_\mathbf{p}$. This define a linear functional $\phi_f : T_\mathbf{p} \to \mathbb{R}$ by

$$\phi_f(\mathbf{v}_\mathbf{p}) = \nabla f_\mathbf{p} \cdot \mathbf{v}_\mathbf{p}.$$

Example 10.1. Let $f(x, y) = x^2y^3$. At the point $\mathbf{p} = (-1, 2)$, the function f gives rise to a linear functional

$$\begin{aligned}\phi_f(\mathbf{v}_\mathbf{p}) &= (2xy^3\mathbf{e}_1 + 3x^2y^2\mathbf{e}_2)_\mathbf{p} \cdot \mathbf{v}_\mathbf{p}\Big|_{x=-1,y=2} \\ &= (-16\mathbf{e}_1 + 12\mathbf{e}_2)_\mathbf{p} \cdot \mathbf{v}_\mathbf{p}\end{aligned}$$

If we change the point to $\mathbf{p}' = (2, 4)$, then the linear functional defined by f at $\mathbf{p}'$ is

$$\begin{aligned}\phi_f(\mathbf{v}_{\mathbf{p}'}) &= (2xy^3\mathbf{e}_1 + 3x^2y^2\mathbf{e}_2)_{\mathbf{p}'} \cdot \mathbf{v}_{\mathbf{p}'}\Big|_{x=2,y=4} \\ &= (2^8\mathbf{e}_1 + 3 \cdot 2^6 \cdot \mathbf{e}_2)_{\mathbf{p}'} \cdot \mathbf{v}_{\mathbf{p}'}\end{aligned}$$

This leads to the next definition.

Definition 10.1. Given a real differential function $f(x, y)$, we let *df* be the differential form that chooses the linear functional $(df)_\mathbf{p}$ at $\mathbf{p}$ that take a tangent vector $\mathbf{v}_\mathbf{p}$ as input and returns $\nabla f_\mathbf{p} \cdot \mathbf{v}_\mathbf{p}$.

We can express $(df)_{\mathbf{p}}$ as

$$(df)_{\mathbf{p}} = f_x\big|_{\mathbf{p}} \cdot \phi_1 + f_y\big|_{\mathbf{p}} \cdot \phi_2$$

If we consider $f(x, y) = x$, then $(df)_{\mathbf{p}}$ is the function that return the first coordinate of the input vector $\mathbf{v}_{\mathbf{p}}$. This is equivalent to the covector ϕ_1 defined in (10.2),

$$dx = \phi_1.$$

Similarly, if we take $f(x, y) = y$, then $(dy)_{\mathbf{p}}$ is the same as ϕ_2 in (10.3),

$$dy = \phi_2.$$

Hence, we can write df in the convenient form as

$$df = f_x dx + f_y dy,$$

but it is important to note that both sides of the equation are linear functionals. They become numbers only when evaluated at a specific tangent vector.

In calculus textbooks, it is often stated that we can assign any values to dx and dy, suggesting their linearly independence. This perspective makes sense when treating dx and dy as numbers. However, after defining differentials as functions, we are able to assert that dx and dy are linearly independent just like in linear algebra. More precisely, suppose there exists constant a and b such that $a \cdot dx + b \cdot dy$ is equal to the zero function. We apply this functional to the basis vector $\mathbf{e}_1$ to get

$$(a \cdot dx + b \cdot dy)(\mathbf{e}_1) = a \cdot 1 + b \cdot 0 = a,$$

which should be equal to zero as $a \cdot dx + b \cdot dy$ is a zero function. Likewise, by evaluating the functional $a \cdot dx + b \cdot dy$ to $\mathbf{e}_2$, we obtain $b = 0$. This proves the following theorem.

Theorem 10.2. *The two differential dx and dy are linearly independent, and hence form a basis of the cotangent space $T^*_{\mathbf{p}}$.*

Example 10.2. Let $f(x, y) = e^{x^2} \sin(y)$. The differential of f is

$$df = 2xe^{x^2} \sin(y)\, dx + e^{x^2} \cos(y)\, dy.$$

At the point $\mathbf{p} = (1, \pi)$, $(df)_{\mathbf{p}}$ is the linear functional

$$(df)_{(1,\pi)} = -e \cdot \phi_2.$$

This is a function that take a tangent vector with initial point $(1, \pi)$ and return $(-e)$ times the second coordinate of the tangent vector.

Since dx and dy form a basis of the cotangent space, we can write a real differential form as

$$a(x, y)dx + b(x, y)dy$$

for some real-valued functions $a(x, y)$ and $b(x, y)$. If a point $\mathbf{p} = (x_0, y_0)$ is given, the above differential form will pick up the differential

$$a(x_0, y_0)(dx)_{\mathbf{p}} + b(x_0, y_0)(dy)_{\mathbf{p}}$$

in the cotangent space $T^*_{(x_0,y_0)}$.

Remark. In this section, we have focused on defining the differential of a real-valued function. However, for vector-valued functions, such as $g : \mathbb{R}^m \to \mathbb{R}^n$, the differential of g can be represented by an $m \times n$ matrix. This matrix is constructed using the partial derivatives of the component functions of g. You can find more information on this topic in the book by Rudin [Rudin].

10.2 Complex differential form

We complexify the cotangent space by allowing complex numbers as the coefficients. This means that when forming a linear combination $a(dx)_{\mathbf{p}} + b(dy)_{\mathbf{p}}$, the value of a and b can be complex numbers[1].

As an example, consider the complex differential idy. It takes a vector $(r, s)_{\mathbf{p}}$ as input and returns the value $i \cdot s$. Here, i represents the imaginary unit. This shows that the complex differential idy assigns complex numbers to tangent vectors, allowing for a more general and flexible framework.

[1]This complexification process can be formally realized by tensoring $T^*_{\mathbf{p}}$ with $\mathbb{C}$ over $\mathbb{R}$. In mathematical symbols, the complex differential lives in the space $\mathbb{C} \otimes_{\mathbb{R}} T^*_{\mathbf{p}}$.

Definition 10.3. Define dz and $d\bar{z}$ to be the complex differential form whose component at a point $\mathbf{p}$ is

$$\begin{aligned}(dz)_{\mathbf{p}} &\triangleq (dx)_{\mathbf{p}} + i(dy)_{\mathbf{p}} \\ (d\bar{z})_{\mathbf{p}} &\triangleq (dx)_{\mathbf{p}} - i(dy)_{\mathbf{p}}.\end{aligned}$$

The symbol "$(dz)_{\mathbf{p}}$" represents a function, that takes a vector, say $(\alpha, \beta)_{\mathbf{p}}$ with an initial point $\mathbf{p}$, and return the complex number $\alpha + i\beta$. Similarly, the function $(d\bar{z})_{\mathbf{p}}$ accepts a vector $(\alpha, \beta)_{\mathbf{p}}$ as input and return the complex number $\alpha - i\beta$.

As in the real case, we will drop the subscript $\mathbf{p}$ in order to simplify notation.

The next example analyze a complex function $f(z)$ by treating it as a multi-variable real function.

Example 10.3. Find the differential of the real and imaginary parts of $f(z) = z^2$.

The real and imaginary parts are $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Their differentials are

$$\begin{aligned}du &= 2x\,dx - 2y\,dy \\ dv &= 2y\,dx + 2x\,dy.\end{aligned}$$

We can put du and dv together to form a complex differential,

$$\begin{aligned}du + idv &= (2x\,dx - 2y\,dy) + i(2y\,dx + 2x\,dy) \\ &= 2(x + iy)(dx + idy) \\ &= 2zdz.\end{aligned}$$

We will call it the differential of the complex-valued function f.

In general, for a complex function $f(z) = u(x, y) + iv(x, y)$, we define the complex differential df as

$$du + idv.$$

Despite the relationship between z and $\bar{z}$, it is important to note that the two complex differentials, dz and $d\bar{z}$, are linearly independent. This fact can be formulated in the following theorem.

Theorem 10.4. *Suppose $w_1 dz + w_2 d\bar{z} = 0$ for some complex numbers w_1 and w_2. Then $w_1 = w_2 = 0$.*

Proof. The condition in the theorem says that $w_1 dz + w_2 d\bar{z} = 0$ is a constant zero function. Thus, we can write it as

$$w_1(dx + idy) + w_2(dx - idy) = (w_1 + w_2)dx + i(w_1 - w_2)dy.$$

This expression should be equivalent to the constant zero function. We can evaluate it at the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, and the results should be zero. This leads to the following equations:

$$\begin{aligned} w_1 + w_2 &= 0 \\ i(w_1 - w_2) &= 0. \end{aligned}$$

Solving the system of linear equations, we easily get $w_1 = 0$ and $w_2 = 0$. □

The previous theorem establishes that dz and $d\bar{z}$ are linearly independent over $\mathbb{C}$. Therefore, the transformation from dx and dy to dz and $d\bar{z}$ corresponds to a change of basis in the complexified cotangent space.

The relationship between dx, dy, dz, and $d\bar{z}$ can be expressed as follows:

$$dx = \frac{dz + d\bar{z}}{2}, \tag{10.4}$$

$$dy = \frac{dz - d\bar{z}}{2i}. \tag{10.5}$$

This implies that dx and dy can be expressed as linear combinations of dz and $d\bar{z}$. This change of basis allows us to work with dz and $d\bar{z}$, which are more convenient when dealing with complex functions and complex analysis.

10.3 Partial differential operators

Suppose the real and imaginary parts of a complex function

$$f(x + iy) = u(x, y) + iv(x, y)$$

are real differentiable as functions of x and y. We want to express the complex differential df using the basis dz and $d\bar{z}$.

We start by writing the differentials of u and v as follows:

$$
\begin{aligned}
du &= u_x dx + u_y dy \\
dv &= v_x dx + v_y dy,
\end{aligned}
$$

where u_x, u_y, v_x and v_y are first-order partial derivatives of u and v. We can then express $du + idv$ as

$$
\begin{aligned}
du + idv &= (u_x dx + u_y dy) + i(v_x dx + v_y dy) \\
&= (u_x + iv_x)dx + (u_y + iv_y)dy.
\end{aligned}
$$

To simplify notation, we define

$$
\begin{aligned}
\frac{\partial f}{\partial x} &\triangleq u_x + iv_x \\
\frac{\partial f}{\partial y} &\triangleq u_y + iv_y.
\end{aligned}
$$

Continuing the derivation, we have

$$
\begin{aligned}
df = du + idv &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
&= \frac{\partial f}{\partial x}\Big(\frac{dz + d\bar{z}}{2}\Big) + \frac{\partial f}{\partial y}\Big(\frac{dz - d\bar{z}}{2i}\Big) \\
&= \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big)dz + \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)d\bar{z}.
\end{aligned}
$$

This expression shows that the differential df can be expressed in terms of dz and $d\bar{z}$ using the coefficients $\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$ and $\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$.

Definition 10.5. Define

$$
\begin{aligned}
\frac{\partial f}{\partial z} &\triangleq \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) \\
\frac{\partial f}{\partial \bar{z}} &\triangleq \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).
\end{aligned}
$$

The operator $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ is called the $\bar{\partial}$ *operator*.

We now relate the $\bar{\partial}$ operator to the Cauchy-Riemann equations.

Theorem 10.6. *For any complex function $f(z) = u(x, y)+iv(x, y)$, the two Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are equivalent to $\partial f/\partial \bar{z} = 0$. Moreover, when the Cauchy-Riemann equations are satisfied, we have*

$$\frac{\partial f}{\partial z} = f'(z),$$

where $\frac{\partial f}{\partial z}$ is defined in Definition 10.5.

Proof. To expand the partial derivative $\frac{\partial f}{\partial \bar{z}}$ with respect to $\bar{z}$ in terms of the partial derivatives of the real and imaginary parts of $f(z)$, we start with the expression

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} \triangleq \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big) &= \frac{1}{2}(u_x + iv_x + i(u_y + iv_y)) \\ &= \frac{1}{2}(u_x - v_y + i(v_x + u_y)).\end{aligned}$$

If $\partial f/\partial \bar{z}$ is zero, then the real and imaginary parts on the right-hand side of the equation must both be zero. This implies that the Cauchy-Riemann equations hold. Conversely, if the Cauchy-Riemann equations are satisfied, the real and imaginary parts of the expression above are both zero, and thus $\partial f/\partial \bar{z} = 0$.

Now, assuming that the Cauchy-Riemann equations hold, we can expand the expression for $\partial f/\partial z$ as follows:

$$\begin{aligned}\frac{\partial f}{\partial z} \triangleq \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) &= \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) \\ &= \frac{1}{2}(u_x + iv_x - i(-v_x + iu_x)) \\ &= (u_x + iv_x)\end{aligned}$$

By Corollary 7.9, we see that $\partial f/\partial z = f'(z)$. □

The $\bar{\partial}$ operator behaves like taking derivative with respect to $\bar{z}$. We can evaluate $\bar{\partial}\bar{z}$ and $\bar{\partial}z$ as follows,

$$\begin{aligned}\bar{\partial}\bar{z} &= \frac{\partial \bar{z}}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)(x - iy) = 1 \\ \bar{\partial}z &= \frac{\partial z}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)(x + iy) = 0.\end{aligned}$$

On the other hand, we have

$$\frac{\partial \bar{z}}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(x - iy) = 0$$
$$\frac{\partial z}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(x + iy) = 1.$$

Example 10.4. Apply the theory to the square function

$$f(z) = z^2 = (x^2 - y^2) + i(2xy).$$

The complex differential df is

$$\begin{aligned} df = d(x^2 - y^2) + id(2xy) &= 2xdx - 2ydy + 2i(ydx + xdy) \\ &= 2(x + iy)dx + 2i(x + iy)dy. \end{aligned}$$

Using (10.4) and (10.5), we can perform a change of basis and write df in terms of dz and $d\bar{z}$,

$$\begin{aligned} df &= 2z\frac{dz + d\bar{z}}{2} + 2iz\frac{dz - d\bar{z}}{2i} \\ &= z(dz + d\bar{z}) + z(dz - d\bar{z}) \\ &= 2zdz. \end{aligned}$$

The complex differential $d(z^2)$ is thus equal to $2zdz$. It only depends on z and dz, but does not depend on $d\bar{z}$.

Example 10.5. The function $f(x+iy) = x+ix^2$ is continuous and smooth as a real function. We can express it in terms of z and $\bar{z}$ as

$$\begin{aligned} x + ix^2 &= \frac{z + \bar{z}}{2} + i\big(\frac{z + \bar{z}}{2}\big)^2 \\ &= \frac{1}{4}\big(2z + 2\bar{z} + iz^2 + 2iz\bar{z} + i\bar{z}^2\big). \end{aligned}$$

This is a function of both z and $\bar{z}$. We can check that $\partial f/\partial \bar{z}$ is equal to

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)(x + ix^2) \\ &= \frac{1}{2} + ix. \end{aligned}$$

The function $f(x+iy) = x + ix^2$ is not complex differentiable because $\partial f/\partial \bar{z}$ is not equal to zero at any point z.

Anyway we can still express $d(x+ix^2)$ in terms of the basis dz and $d\bar{z}$

$$d(x+ix^2) = (1+2ix)\,dx = (1+2ix)\frac{dz+d\bar{z}}{2} = \frac{1+2ix}{2}dz + \frac{1+2ix}{2}d\bar{z}.$$

For a complex function $f(z)$ that is complex differential in a domain D, we have the following results.

- If we apply the $\bar{\partial}$ operator to $f(z)$, the resulting function $\frac{\partial f}{\partial \bar{z}}$ is identically zero in D. We can interpret this as "f is a function of z only, but not a function of $\bar{z}$."
- Since $\partial f/\partial \bar{z}$ is zero, we can write the differential df in the form of

  $$df = \frac{\partial f}{\partial z}dz.$$

  We thus obtain a relationship between the complex differential, the partial derivative of f with respect to z, and the complex derivative $f'(z)$:

  $$df = \frac{\partial f}{\partial z}dz = f'(z)dz.$$

11 Multi-valued functions

Summary

- Parametric curve
- Multi-valued function
- The angle function

In the context of real-valued functions, the square root function $\sqrt{x}$ serves as a fundamental example of a multi-valued function. For any positive input value, there exist two possible choices of square roots. By convention, we typically select the positive root as the default choice.

Similarly, inverse trigonometric functions are also multi-valued. Take, for instance, the arc tangent function $\tan^{-1}(x)$. In this case, we usually choose the function value that falls within the range of $-\pi/2$ to $\pi/2$.

Formally, a *multi-valued function*, or simply a *multi-function*, is a function that maps a point in the domain to a set of values. For example, we can consider the square root function as a multi-function and assign the set $\{\sqrt{x}, -\sqrt{x}\}$ to a positive real number x. When viewing the arc tangent function $\tan^{-1}(x)$ as a multi-function, it maps a real number x to a set in the form of $\{\theta + \pi k : k \in \mathbb{Z}\}$, where θ is a number such that $\tan\theta = x$.

11.1 Parametric curves

We formally define a path as follows.

Definition 11.1. *parametric curve*, or a *path* is defined as a continuous function γ, that maps a closed interval $[a, b]$ to the complex numbers $\mathbb{C}$:

$$\gamma : [a, b] \to \mathbb{C}. \tag{11.1}$$

A path is *differentiable* if γ is a differentiable function, i.e., $\gamma'(t)$ exists. We say that a path is *piece-wise differentiable*) if it is a concatenation of finitely many differentiable paths.

The constants a and b represent the start and end times, respectively.

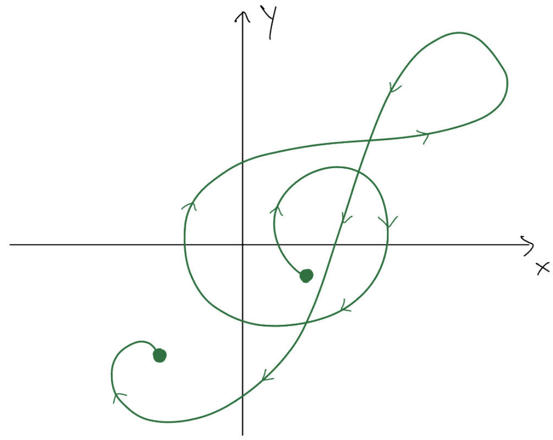


Figure 9: An example of parametric curve that has self-intersections.

For any t within the interval $[a, b]$, the complex number $\gamma(t)$ indicates the robot's location at time t. Self-intersection points are allowed in the parametric curve, and it possesses a sense of direction. An example of such a curve is depicted in Figure 9.

Definition 11.2. The parametric curve in (11.1) is said to be

- *simple* if there is no self-intersection, i.e., $\gamma(t_1) \neq \gamma(t_2)$ for any $a \leq t_1 < t_2 \leq b$, except when $t_1 = a$ and $t_2 = b$.
- *closed* if the start point is the same as the end point, i.e., $\gamma(a) = \gamma(b)$.

From multi-variable calculus, we know how to compute the length of the path using line integral. If the x and y coordinates of the point $\gamma(t)$ is $x(t)$ and $y(t)$, respectively, then we can compute the length of the path $\gamma(t)$ by

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t)^2)}\, dt,$$

provided that both $x'(t)$ and $y'(t)$ are continuous functions. In the followings we need to assume that the length of the path represented by $\gamma(t)$ is finite. For this purpose, we will assume that the real and imaginary parts of $\gamma(t)$ are both continuously differentiable.

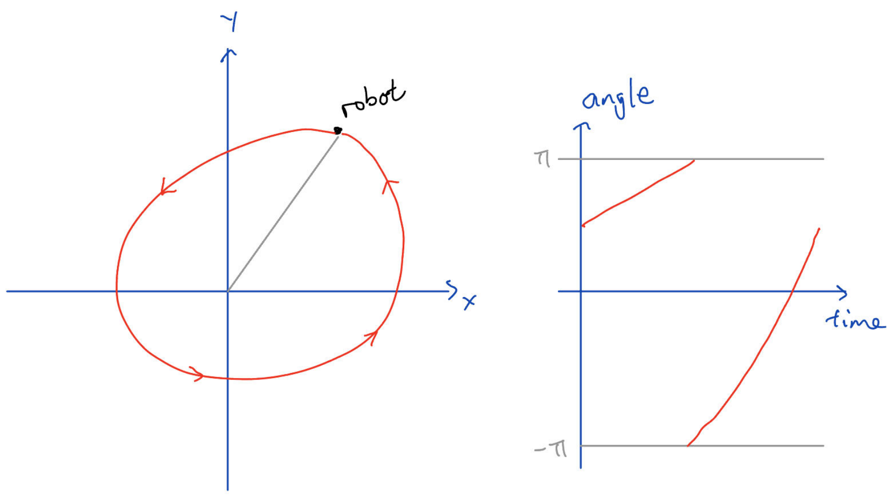


Figure 10: Going around the origin.

11.2 The discontinuity of the principal branch of the argument function

To illustrate the problem we aim to solve, we will consider a concrete scenario. In Python, we can calculate the principal argument of a complex number $x+iy$ using the atan2(y,x) function. This function computes the angle between the point $x+iy$ and the origin, with a range of $(-\pi,\pi]$.

For instance, let's compute the angle of the point $z=-1$ along the negative real axis.

```
from math import atan2

x , y = -1, 0
a = atan2(y,x)
print(f"The angle of {x}+i({y}) is {a}")
```

We can see that the angle is π.

```
The angle of -1+i(0) is 3.141592653589793
```

If we move the point slightly below the negative real axis, the angle is very close to $-\pi$. We can run

```python
from math import atan2

x , y = -1, -0.0000001
a = atan2(y,x)
print(f"The angle of {x}+i({y}) is {a}")
```

This will give

```
The angle of -1+i(-1e-07) is -3.1415925535897933
```

There is a sudden change of function value is due to the discontinuity of the function atan2(y,x) at the negative real axis.

Imagine a robot moving within a complex plane, where our objective is to measure the angle along its path. However, the robot can only compute the angle based on its location. It possesses GPS to determine its coordinates and has some computing capability. Nevertheless, the robot lacks intelligence and can only execute the instructions provided to it.

Suppose the location of the robot is given by $\gamma(t)$ for time t in the interval $[a, b]$. The value of $\gamma(t)$ is a complex number expressed as $x + iy$, where x and y represent the coordinates of the robot at time t.

If we compute the angle using the atan2 function on the robot, and the robot traverses a loop that includes the origin, we will observe a discontinuity jump as illustrated in Figure 10. This discontinuity occurs when the robot crosses the negative real axis. However, it is desirable to have a continuous measurement of the angle.

11.3 Computing the angle within a half-plane

One solution is to select a designated region in the complex plane as the domain and restrict the robot's movement within that domain. A suitable choice for the domain is a half-plane with its boundary passing through the origin. For instance, we can consider the right half-plane $H \triangleq \{x+iy \in \mathbb{C} : x > 0\}$. To construct a continuous angle function on the half-plane H, we can express a complex number $z = x + iy$ and define

$$A(z) \triangleq \tan^{-1}\left(\frac{y}{x}\right). \tag{11.2}$$

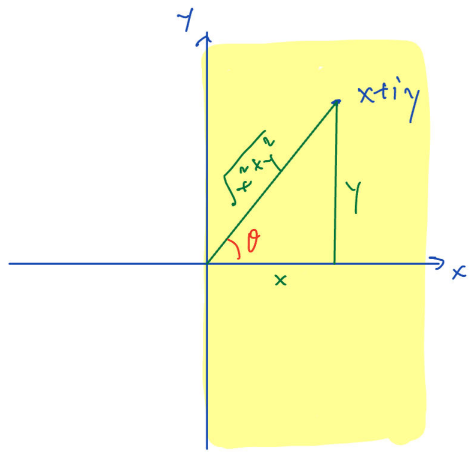


Figure 11: The right half-plane and the angle function.

The output range of this function lies within the open interval $(-\pi/2, \pi/2)$ (refer to Figure 11). Here, we leverage the property that the x-coordinate is strictly positive in the right half-plane. The mapping $(x, y) \mapsto \tan^{-1}(y/x)$ is well-defined and avoids division by zero errors. As long as the point $x + iy$ remains within the right half-plane, there will be no discontinuity in the angle measurement.

In reality, the robot has the capability to travel to any location. Therefore, we propose a solution for measuring the change of angle without restricting the robot to the right half-plane. The idea is to cover the entire complex plane (except the origin) using an infinite number of half-planes. For any given angle α, let H_α denote the half-plane with a boundary defined by a straight line passing through the origin, where the normal vector makes an angle of α radians with the positive real axis. An illustrative example is depicted in Fig. 12.

We can define an angle function on H_α by utilizing the existing A function as follows:

$$A_\alpha \triangleq A(e^{-i\alpha} z) + \alpha. \tag{11.3}$$

Geometrically, this process involves rotating the half-plane H_α to the right half-plane H_0, applying the function A, and then rotating it back to H_α. We record this construction in the following proposition.

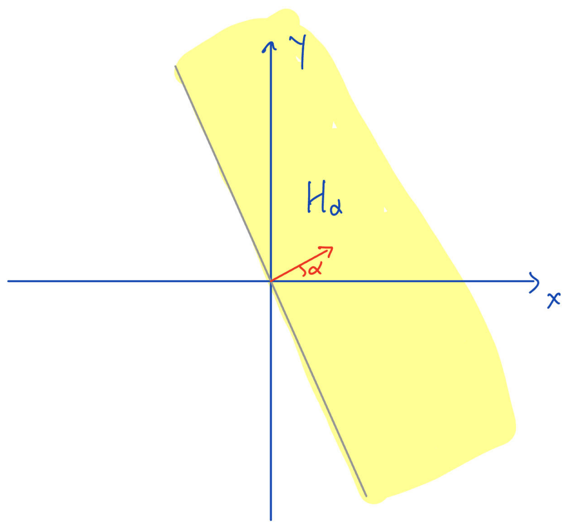


Figure 12: The half plane H_α.

Proposition 11.3. *We can define a continuous angle function in an open half plane in the complex plane, with the boundary being a straight line passing through the origin.*

When $|\alpha_1 - \alpha_2| \leq \pi/2$, i.e., when the difference between the two angles α_1 and α_2 is within $\pi/2$, the functions A_{α_1} and A_{α_2} are identical when restricted to the intersection of H_{α_1} and H_{α_2}.

We do not have to use the arc tangent function in computing the angle. We may use other trigonometric functions instead. For example, if the domain is the upper half-plane

$$H_{\pi/2} = \{x + iy \in \mathbb{C} :\ y > 0\},$$

we may use the arc cosine function

$$A_{\pi/2}(x+iy) = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

It define a continuous angle function on the upper half-plane, and its range is $(0, \pi)$

If the half-plane is

$$H_{\pi/4} = \{x + iy \in \mathbb{C} :\ x + y > 0\},$$

whose boundary is obtained by rotating counter-clockwise 45 degrees from the imaginary axis, we can compute the angle function as

$$A_{\pi/4}(x+iy) = \cos^{-1}\Big(\frac{(x-y)/\sqrt{2}}{\sqrt{x^2+y^2}}\Big).$$

The range of this function is $(-\pi/4, 3\pi/4)$.

11.4 The angle function

In this section we will describe how to select a branch of the argument function on a path (which may or may not be a simple path)

Since the robot's path may not be confined within a single half-plane, we can divide the path into small portions and cover each portion with a suitable half-plane. We apply the corresponding function A_α as long as the input point remains within the respective half-plane.

We make the following assumptions: firstly, the robot never reaches the origin where the angle function is not defined, and secondly, the robot is capable of measuring the distance traveled. Let $\gamma(t)$ represent the robot's movement on a path for $0 \le t \le b$. We can employ the following algorithm:

The robot starts at a designated point called the base point z_0. Since z_0 is non-zero, the distance to the origin $d_0 \triangleq |z_0|$ is positive. Let $\alpha_0 = \arg z_0$ denote the initial argument. At the beginning, we utilize the function A_{α_0} defined on the half-plane H_{α_0}. We know that the robot's location will remain within the half-plane H_{α_0} as long as it stays within a distance d_0 from the initial point. In the first segment of the trajectory with a length of d_0, we utilize the function A_{α_0} to measure the angle. Suppose that at time t_1, the robot has traveled a distance of d_0. For $0 \le t \le t_1$, we define

$$\phi(t) \triangleq A_{\alpha_0}(\gamma(t)), \quad \text{for } 0 \le t \le t_1.$$

We mark the point z_1 at time t_1. The angle at this point is $A_{\alpha_0}(z_1)$, and the distance to the origin is $d_1 \triangleq |z_1|$. At this moment, we change the base point from z_0 to z_1, and use the function A_{α_1}, which is defined on the half-plane H_{α_1}. Let t_2 be the time when the robot travel a distance of d_1 along the contour $\gamma(t)$ from the point z_1. For time t between t_1 and t_2, we continue the angle measure by

$$\phi(t) \triangleq A_{\alpha_1}(\gamma(t)) - A_{\alpha_1}(z_1) + \phi(t_1), \quad \text{for } t_1 \le t \le t_2.$$

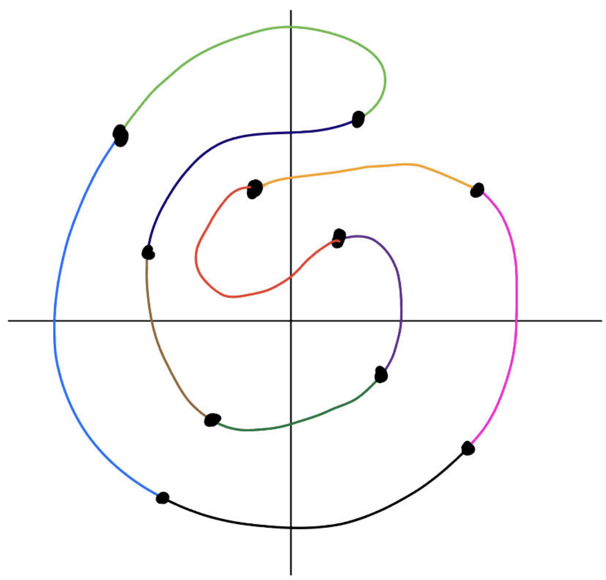


Figure 13: Computing the change of angle by dividing a curve into smaller parts

At time t_2, we update the base point to $z_2 \triangleq \gamma(t_2)$. The distance to the origin becomes $d_2 = |z_2|$, and the corresponding argument is $\phi(t_2)$. Let t_3 denote the time instance when the robot has traveled a distance of d_2 from the point z_2. For $t_2 \le t \le t_3$, we define

$$\phi(t) \triangleq A_{\alpha_2}(\gamma(t)) - A_{\alpha_2}(z_2) + \phi(t_2), \quad \text{for } t_2 \le t \le t_3.$$

This process can be repeated in a similar manner. The resulting function $\phi(t)$ is continuous and measures the angle between the robot and the origin. When the robot comes to a stop at time $t_n = b$, the total change in angle can be computed as

$$\sum_{k=0}^{n-1} A_{\alpha_k}(\gamma(t_{k+1})) - A_{\alpha_k}(\gamma(t_k)). \tag{11.4}$$

See Fig. 13 for a sample of division of path.

In conclusion, the procedure described in this section is able to obtain a continuously-varying angle function on any path, provided that the path does not pass through the origin, and the length of the curve is well-defined and finite.

12 Winding number, Riemann surface

Summary:

- Winding number
- Riemann surface of $\sqrt{z}$
- Derivative of inverse function

12.1 Winding number, branch point, branch cut

The previous section demonstrates that it is possible to define an angle function along a continuous path, even if the path is complex and may encircle the origin multiple times. We refer to such a function as a "choice" of the angle function. It is important to note that the choice is not unique, as we can add any integral multiple of 2π to the function.

However, regardless of the specific choice made, it is geometrically evident that the change in angle for any closed curve is always an integral multiple of 2π. With this in mind, we can now introduce the definition of the winding number.

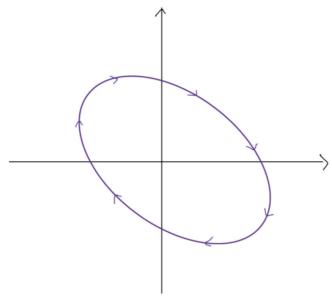
Definition 12.1. We define the *winding number* (also known as the *index*) of a closed parametric curve not passing through the origin by $(1/2\pi)$ times the change of angle.

The above definition of winding number is relative to the origin. If we want to study how many times a curve revolves around another point, we can translate the curve suitably and reduce the problem to the above definition.

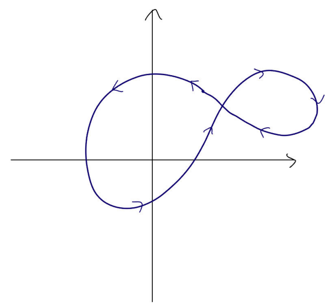
The winding number is an integer, and its sign represents the orientation of the closed path. For instance, the change in angle for the closed curve depicted in Figure 13 is 0. Additional examples are provided in Fig. 14, illustrating different winding numbers."

The need for a more involved process to compute the winding number arises due to the fact that the angle is a multi-function, and the origin serves as a *branch point* for this multi-function. We can provide a working definition of a branch point as follows:

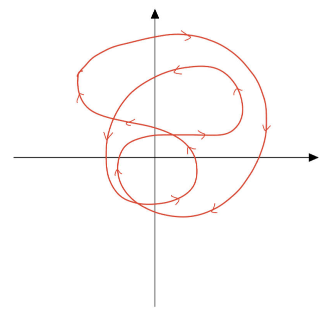
Consider a general multi-function F defined on the complex plane. For each z in the domain, $F(z)$ represents a set of possible values. Now, let's imagine traveling around a specific point z_0 within a small circle while continuously varying the function value. After



(a) Winding number $= -1$



(b) Winding number $= 1$



(c) Winding number $= 0$

Figure 14: Winding number of a closed curve relative to the origin.

completing one full cycle around z_0, we may observe that the function value does not return to its initial value. If this phenomenon persists for arbitrarily small circles centered at z_0, then we consider z_0 to be a branch point of F.

A *branch* of the multi-function F refers to an ordinary single-valued function that selects one value from the set $F(x)$ of possible values.

In order to make a continuous choice of value, one approach is to designate a *branch cut* and define a branch at every point in the complex plane except for the points located on the branch cut. The branch cut should be carefully selected in a manner that prevents any closed cycle from encircling a branch point

12.2 Riemann surface of function $\sqrt{z}$

Instead of introducing a branch cut, another approach is to extend the domain to a Riemann surface. Let's illustrate the construction of a Riemann surface for the square root function.

The typical domain used in the definition of the complex square root function is given by

$$D = \mathbb{C} \setminus \{x + iy \in \mathbb{C} : \ y = 0, x \leq 0\},$$

which involves removing the negative real axis from the complex plane. This removed half-line is referred to as a *branch cut.* To define a single-valued continuous square root function in the domain D, we can express $z \in D$ in polar coordinates as $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$. We then define

$$f(z) = \sqrt{r}e^{i\theta/2}.$$

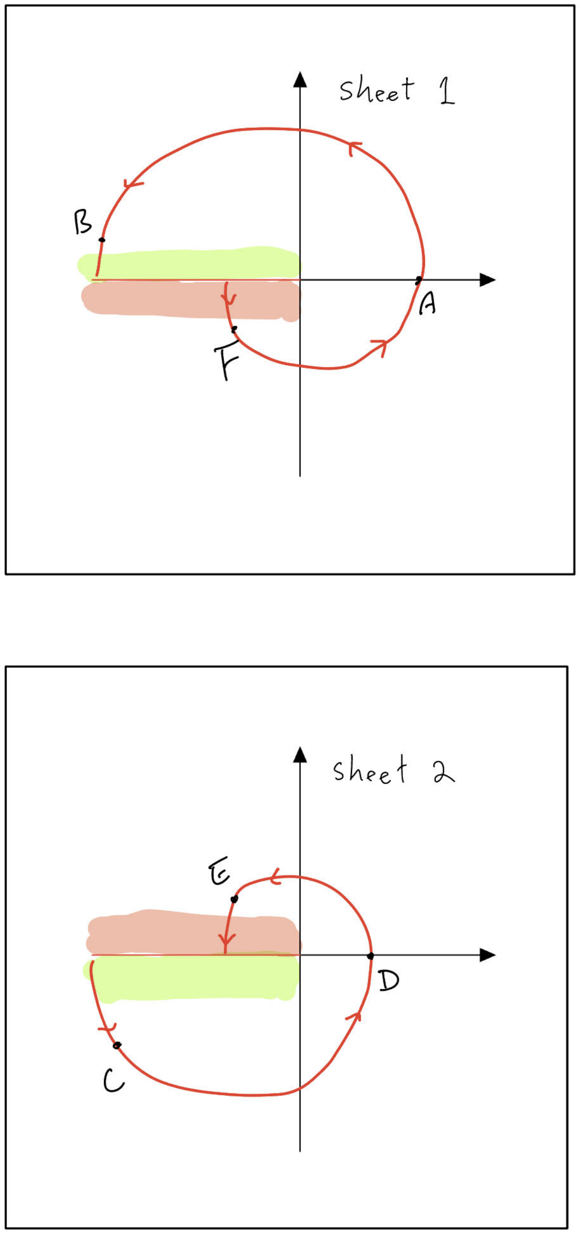


Figure 15: A model for the Riemann surface of the function $\sqrt{z}$.

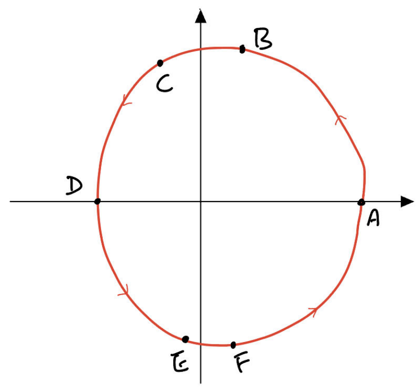


Figure 16: Image of the curve in Fig. 15 under the square root function.

Under this function, the image of D is the right half-plane. However, the drawback of this definition is that the square root of points on the negative real axis is not well-defined.

A better approach involves constructing a special surface and using it as the domain for the square root function. This surface is a two-sheeted covering of the complex plane. We take two copies of the complex plane and position one on top of the other, such that each complex number (except the origin) corresponds to two points—one in sheet 1 and one in sheet 2. The two origins are identified as the same point. We introduce a cut along the negative real axis and connect the two sheets as shown in Fig. 15. For any positive real number r, the point $-r$ in sheet 1 is connected to the point $-r - \epsilon i$ in sheet 2, where ϵ represents an infinitesimally small real number. Similarly, the point $-r$ in sheet 2 is connected to $-r - \epsilon i$ in sheet 1. In this model, we do not remove the negative real axis; instead, the negative real axis is replicated in both sheet 1 and sheet 2.

A double loop is illustrated in Fig. 15. This loop encircles the origin twice. After traveling along this curve once, the change in angle is 4π. Under the square root function, the corresponding image is a closed loop that winds around the origin once (see Fig. 16). It is important to note that both point A and point D lie on the positive part of the real axis. However, they belong to different sheets of the Riemann surface. We select the positive square root for point A in sheet 1, and the negative square root for point D in sheet 2. When we travel from point A to point D on the Riemann surface, the change in angle is 2π. After applying the square root function, the change in angle is divided by 2, resulting in an angle change of π.

12.3 Derivative of inverse function

We can study the inverse of a complex function by treating it as a vector field and reduce the problem to real analysis. We will identify a complex functions $f(z)$ and $g(z)$ as a vector-valued functions $\mathbf{f}(x, y)$ and $\mathbf{g}(x, y)$ whose domain and range are subsets of $\mathbb{R}^2$.

The following theorem from real analysis is called the inverse function theorem.

Theorem 12.2 ([Rudin] Thm 9.24)**.** *Let $\mathbf{g}$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, and suppose $\mathbf{g}$ is continuously differentiable in a domain D. If $\mathbf{g}'$ is invertible at $(x_0, y_0) \in D$, then there exist an open set $U \subseteq D$ that contains (x_0, y_0), and an open set $V \subseteq \mathbb{R}^2$ such that $\mathbf{g}$ is a bijection from U to V and the inverse of $\mathbf{g}$ is continuously differentiable in V.*

An equivalent way to state the conclusion is to say that the function $\mathbf{g}$ is *locally invertible* at the point (x_0, y_0).

We recall that when $\mathbf{g}(x, y)$ is a vector-valued function with components $u(x, y)$ and $v(x, y)$, then the derivative of $\mathbf{g}$ at a point (x_0, y_0), denoted by $\mathbf{g}'(x_0, y_0)$, is the 2×2 matrix

$$\mathbf{g}'(x_0, y_0) = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix},$$

which is also called the Jacobian of $\mathbf{g}$ at (x_0, y_0). Another common notation for this matrix is

$$\frac{\partial(u, v)}{\partial(x, y)}$$

Suppose that $g(z)$ is a holomorphic function in some domain D and the real and imaginary parts are both continuously differentiable. (This assumption can be relaxed to differentiable, using a corollary of the Cauchy-Gousat theorem.)

The following theorem is the complex analog of this theorem.

> **Theorem 12.3.** *Let $g(z)$ be a complex function that is continuously differentiable in D. If $g'(z_0) \neq 0$ at a point $z_0 \in D$, then*
>
> 1. *there exist open sets $U \subseteq D$ and $V \subseteq \mathbb{C}$, with $z_0 \in U$, such that g is a bijection from U to V,*
> 2. *the inverse g is continuous differentiable (as a real function) in V, and complex differentiable in V.*

Proof. Suppose the real and imaginary parts of the complex function $g(x+iy)$ are $u(x,y)$ and $v(x,y)$, respectively. Identify the complex function $g(z)$ with vector-valued function

$$\mathbf{g}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}.$$

Let z_0 be the complex number x_0+iy_0. The derivative $\mathbf{g}'$ at (x_0,y_0) is an invertible matrix, because its determinant

$$\begin{vmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{vmatrix} = \begin{vmatrix} u_x(x_0,y_0) & -v_x(x_0,y_0) \\ v_x(x_0,y_0) & u_x(x_0,y_0) \end{vmatrix} = u_x(x_0,y_0)^2 + v_x(x_0,y_0)^2 = |g'(z_0)|^2$$

is nonzero by the assumption $g'(z_0) \neq 0$.

By the inverse function theorem from real analysis, we know that $\mathbf{g}$ is real-differential and locally invertible at (x_0,y_0). Precisely, it means that there exists an open set U that contains the point (x_0,y_0), and an open set V in the image of $\mathbf{g}$ such that the image of U under the function $\mathbf{g}$ is exactly equal to V, and furthermore, $\mathbf{g}$ is injective on U. Hence, we know that there is a function $\mathbf{f}$ that maps from V to U and satisfies $\mathbf{g} \circ \mathbf{f} = id_V$ and $\mathbf{f} \circ \mathbf{g} = id_U$.

$$\begin{tikzcd} V \arrow[r, bend left, "\mathbf{f}"] & U \arrow[l, bend left, "\mathbf{g}"] \end{tikzcd}$$

It remains to show that the inverse of $\mathbf{g}$ is complex-differentiable at the point $g(x_0,y_0)$. Suppose the image of the point (x_0,y_0) under $\mathbf{g}$ is (r_0,s_0). We apply the chain rule to $\mathbf{g} \circ \mathbf{f} = id_V$, and get

$$\mathbf{g}'(x_0,y_0) \cdot \mathbf{f}'(r_0,s_0) = I_{2\times 2}$$

where $I_{2\times 2}$ denotes the 2×2 identity matrix. Hence, $\mathbf{f}'(r_0, s_0)$ is the inverse of $\mathbf{g}'(x_0, y_0)$

Because the matrix $\mathbf{g}'(x_0, y_0)$ is in the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for some constant a and b, the matrix $\mathbf{f}'(r_0, s_0)$ can be written as

$$\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

which is a matrix in the form

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

for some constants c and d, representing the complex number $g'(z_0)^{-1}$. □

Theorem 12.3 shows that if $g'(z)$ does not vanish at $z = z_0$, then the inverse of g is holomorphic, and the derivative of the inverse function at the point $g(z_0)$ is

$$(g'(z_0))^{-1}.$$

Example 12.1. The log function is the inverse function of exponential function. To differentiate the log function, we have to first pick a branch, so that $\log(z)$ is single-valued. Suppose we take the negative real axis as the branch cut (first part of Fig. 17). For

$$z \in \mathbb{C} \setminus \{x + iy : \ y = 0, \ x \le 0\},$$

we define $\log(z)$ as a complex number $u + iv$ such that $e^{u+iv} = z$ and $-\pi < v < \pi$. This defines a one-to-one onto mapping to the horizontal strip

$$\{u + iv : -\pi < v < \pi\}.$$

Differentiate both side of

$$\exp(\log(z)) = z,$$

we obtain

$$\exp(\log(z))' \cdot \log(z)' = 1.$$

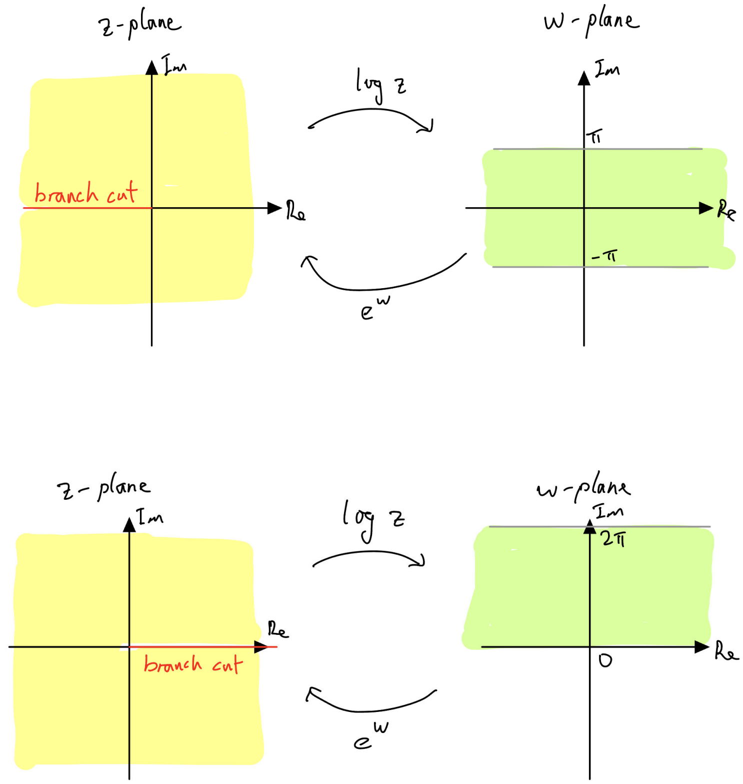


Figure 17: Two different branches of complex log function.

Using the fact that $\exp(w)' = \exp(w)$, we get

$$\log(z)' = \frac{1}{\exp(\log(z))} = \frac{1}{z}.$$

We have thus derive

$$\frac{d}{dz}\log(z) = \frac{1}{z}$$

for z in the chosen branch $\mathbb{C} \setminus \{x + iy : y = 0, x \le 0\}$.

We may take another choice of branch cut of the log function. If we pick the positive real axis as the branch cut, the corresponding area in the z-plane is shown in the second part of Fig. 17. We can repeat the same calculations and show that the derivative is also equal to $1/z$ for this choice of branch cut.

Example 12.2. In this example we compute the derivative of the square root function. Since the square root function is a multi-valued, we need to first make it single-valued by selecting a branch cut. Suppose we select the negative real axis as the branch cut. For each z not on the negative real axis, we define $\sqrt{z}$ as the unique complex number w on the right half plane such that $w^2 = z$. Hence, we impose the condition that $\sqrt{z}$ has positive real part. (Refer to the first half of Fig. 18.)

With this choice of branch cut, $\sqrt{z}$ is a bijection from $\mathbb{C} \setminus \{x + iy : y = 0, x \le 0\}$ to $\{u + iv : u > 0\}$. Let z be a complex number not lying on the negative real axis and $g(z) = z^2$ denote the squaring function. Differentiate both side of

$$g(\sqrt{z}) = z,$$

we obtain

$$g'(\sqrt{z}) \cdot (\sqrt{z})' = 1.$$

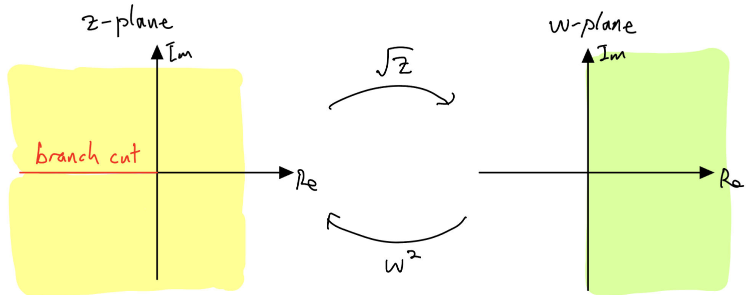
Since g is the squaring function, we know that $g'(w) = 2w$. Therefore

$$(\sqrt{z})' = \frac{1}{2\sqrt{z}}.$$

As a numerical example, we select the point $z = -i$, and compute the complex derivative

$$(\sqrt{z})'\big|_{z=-i} = \frac{1}{2e^{-i\pi/4}} = \frac{1+i}{2\sqrt{2}}.$$

Suppose we pick another branch cut for the square root function. Instead of the negative real axis, we may choose the positive real axis as the branch cut. The resulting single-valued square root function is mapping $\mathbb{C} \setminus \{x + iy : x \ge 0,\ y = 0\}$ to the upper half



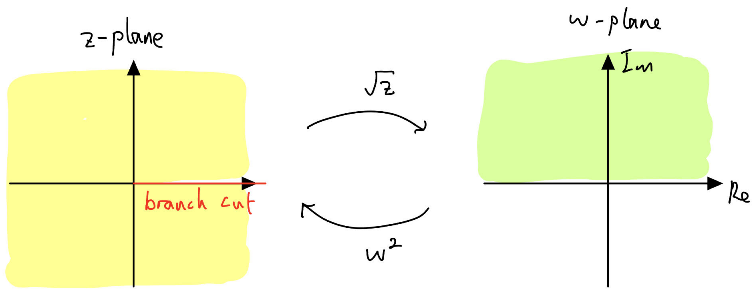


Figure 18: Two different branches of square root function.

plane $\{u + iv : v > 0\}$. Given a complex number z_0 not lying on the positive real axis, we now define $\sqrt{z}$ as the unique complex number $u + iv$ in the upper half plane such that $(u + iv)^2 = z$. With this choice of branch cut, we have $\sqrt{-i} = e^{3\pi i/4}$. So

$$(\sqrt{z})'\big|_{z=-i} = \frac{1}{2e^{3\pi i/4}} = \frac{-1-i}{2\sqrt{2}}.$$

We thus see that the derivative of the square root function has two possible values, and the function value depends on the choice of the branch cut.

12.4 Appendix: A more rigorous definition of branch point

We continue with the discussion of branch point.

Suppose we have a path $\gamma(t)$ defined as a continuous function on the interval $[0, b]$ in $\mathbb{C}$. At each point along the curve $\gamma(t)$, we select a value from the multi-function in such a way that the function values vary continuously as we travel along the curve. Let $v(t)$ be a continuous function of t for $0 \le t \le b$, such that $v(t) \in F(\gamma(t))$ for all t.

We are interested in understanding the change in the function value after traversing the curve. We define the difference in function values as

$$\delta(F, \gamma, v_0),$$

where F represents the multi-function, γ is the path, and v_0 denotes the initial function value, i.e., $v_0 = v(0)$ is an element in $F(\gamma(0))$.

It is desirable that the change in function value does not depend on the shape of the curve but only on the positions of the initial and final points, as well as the choice of the function value at the initial point. To capture this property, we introduce a name. We say that the multi-function F is *independent of path in a region* R if, for any closed path γ that lies entirely within the region R, and no matter how we select function values for each point on the curve such that the values vary continuously along the curve, the difference in function value after completing one loop around the path is zero.

Theorem 12.4. *If F is independent of path in a region R, then we can select a branch of F that is continuous on R (without any branch cut).*

Proof sketch. We want to construct a branch g of F on the region R such that $g(z) \in F(z)$ for all $z \in R$ and $g(z)$ is a continuous function on R.

Let's choose an arbitrary point z_0 from R. We select a single value v_0 from the set $F(z_0)$. For any other point z in R, we can draw a continuous path from z_0 to z. Along this path, we continuously vary the function value starting from the initial value v_0. (The assumption that R is path-connected guarantees the existence of such a path.)

We define $g(z)$ as the function value at the end of the path. To complete the proof, we need to show that $g(z)$ is well-defined, meaning it is independent of the specific shape of the path from z_0 to z, and that it is a continuous function. $\square$

Given this preliminary, we can give a formal definition of branch point.

Definition 12.5. A point z_0 is called a *regular point* of a multi-function F if there exists a small disc D centered at z_0 such that $F(z)$ is independent of path in the region D.

A point z_0 is called a *branch point* if, for every open disk D centered at z_0, there exists a closed path around z_0 such that the change in function value $\delta(F(z), \gamma, v_0)$ after traversing the closed path is nonzero, while every other point in $D \setminus \{z_0\}$ is a regular point.

13 Contour integral

Summary

- Integration on the real number line
- Definition of contour integral

In this lecture we first consider integrating a complex-valued function defined on a real interval, and then extend it to contour integral.

13.1 Review of line integral

In this section we recall the meaning of line integral for real-valued functions.

Let $\mathbf{F}(x, y)$ denote a vector-valued function

$$\mathbf{F}(x, y) = (M(x, y), N(x, y))$$

where $M(x, y)$ and $N(x, y)$ are continuous functions.

We represent a curve parametrically using a vector-valued function $\mathbf{r}(t) = (x(t), y(t))$ for $t \in [a, b]$. Typically, we assume that $x(t)$ and $y(t)$ are both continuously differentiable, and the tangent vector $(x'(t), y'(t))$ is nonzero for all $t \in [a, b]$.

Given these data, we can define the line integral of $\mathbf{F}$ over the curve C represented by

$$\int_C \mathbf{F} \cdot d\mathbf{r} \triangleq \int_a^b (M(x(t), y(t))x'(t) + N(x(t), y(t))y'(t))\, dt. \tag{13.1}$$

Physically, it represents the work done by the force field $\mathbf{F}$ on a particular moving along the curve C.

Mathematically, we interpret the integrand as the dot product of the vector field $(M(x(t), y(t)), N(x(t), y(t)))$ and the tangent vector $\mathbf{r}'(t) = (x'(t), y'(t))$. This explains the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$. To numerically approximate this integral, we can divide the interval $[a, b]$ into sub-intervals by creating a partition:

$$a = t_0 < t_1 < t_2 < \cdots t_{n-1} < t_n = b.$$

We then select a sample point t_k^* between t_k and t_{k+1} for $k = 0, 1, \dots, n-1$, and compute the sum

$$\sum_{k=0}^{n-1} \mathbf{F}(x(t_k^*), y(t_k^*)) \cdot \Big[\mathbf{r}'(t_k^*)(t_{k+1} - t_k)\Big].$$

Under the usual assumption of Riemann integrability, the above sum will converge as the width of the partition converges to 0. (The *width* is defined as the maximum of $t_{k+1} - t_k$ over all indices k.

13.2 Integration of complex-valued function defined on the real line

In this course integration is defined in the sense of Riemann integral. When $f(t)$ is complex-valued, we define the integral of $f(t)$ as a complex number whose real part is the integral of the real part of $f(t)$, and the imaginary is the integral of the imaginary part of $f(t)$.

Definition 13.1. Let f be a complex-valued function defined on the interval $[a, b]$ on the real number line, with $u(t)$ and $v(t)$ as the real and imaginary parts,

$$f(t) = u(t) + iv(t)$$

for $t \in [a, b]$. We define the integral of f as

$$\int_a^b f(t)\,dt \triangleq \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{13.2}$$

provided that both $u(t)$ and $v(t)$ are Riemann integrable over $[a, b]$.
When the function $f(t)$ is defined on the whole real number line, we define the improper integral as double limit,

$$\int_{-\infty}^{\infty} f(t)\,dt \triangleq \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(t)\,dt.$$

Example 13.1. Consider the complex function $f(x) = (x-1) + ix^2$, for $-1 \le x \le 1$. The integral of this function on the interval $[-1, 1]$ is

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (x-1)\,dx + i\int_{-1}^{1} x^2\,dx = -2 + \frac{2i}{3}.$$

Example 13.2. We can integrate the complex exponential function by considering the real and imaginary parts separately.

$$\int_0^{\pi} e^{it}\,dt = \int_0^{\pi} \cos(t)\,dt + i\int_0^{\pi} \sin(t)\,dt = 2i.$$

Integral transforms such as Fourier and Laplace transform are one of the major applications of complex analysis in science and engineering. For instance, the textbook [Stein] contains a whole chapter on Fourier transform.

Example 13.3. Given any function $f(t)$, the *Fourier transform* of $f(t)$ is defined as

$$F(\omega) \triangleq \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt,$$

where ω is a real number. If $f(t)$ is a probability distribution function, i.e., if $f(t) \geq 0$ for all $t \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(t)\,dt = 1$, then the *characteristic function* of the corresponding probability distribution is defined by

$$\phi(\omega) \triangleq \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt,$$

which is basically the same as Fourier transform, except that there is a sign change in the input variable ω.

Example 13.4. The (two-sided) *Laplace transform* of $f(t)$ can be viewed as a generalization of the Fourier transform,

$$\hat{f}(s) \triangleq \int_{-\infty}^{\infty} f(t)e^{-st}\,dt.$$

The variable s is complex number. When s is purely imaginary, it reduces to Fourier transform.

Example 13.5. The Mellin transform of a function $f(t)$ is

$$\int_{0}^{\infty} f(t)t^{s-1}\,dt,$$

where s is a complex number. The Mellin transform of exponential function is called the Gamma function

$$\Gamma(s) = \int_{0}^{\infty} e^{-t}t^{s-1}\,dt.$$

13.3 Contour integral

The notion of parametric curve in the complex plane is almost the same as in the real case.

Definition 13.2. A parametric curve C represented by

$$z : [a, b] \to \mathbb{C}$$
$$z(t) = x(t) + iy(t)$$

is said to be *smooth* if
(i) $x(t)$ and $y(t)$ are continuously differentiable on $[a, b]$, and
(ii) the vector $(x'(t), y'(t))$ is not equal to the zero vector for $t \in [a, b]$.

We note that the condition in (ii) means that the tangent vector is well-defined at all points on the curve. We write $z'(t) = x'(t) + iy'(t)$ as a tangent vector at t. To emphasize the direction/orientation of the curve from $t = a$ to $t = b$, we sometime use the word "contour" instead of "path". If a curve can be divided into finitely many parts and each part is smooth, then we call it a *piece-wise smooth* curve.

Remark. In the rest of the lecture notes, all curves are piece-wise smooth curve. Because a piece-wise smooth curve can be treated as a concatenation of smooth curves, we will describe the computation of a complex integral on a smooth curve.

One way to motivate contour integral of function $f(z)$ is to divide a smooth curve C in Definition 13.2 into many small parts, and consider the Riemann sum

$$\sum_{k=0}^{n-1} f(z_k^*) \cdot \Delta z_k$$

where z_k^* is a sample point in the k-th part, and Δz_k is the difference of the k-th part. If f is a continuous function, Riemann sum will converge to a constant when we take n approaching infinity. The limit is then called the contour integral of $f(z)$ over C. (See Fig. 19)

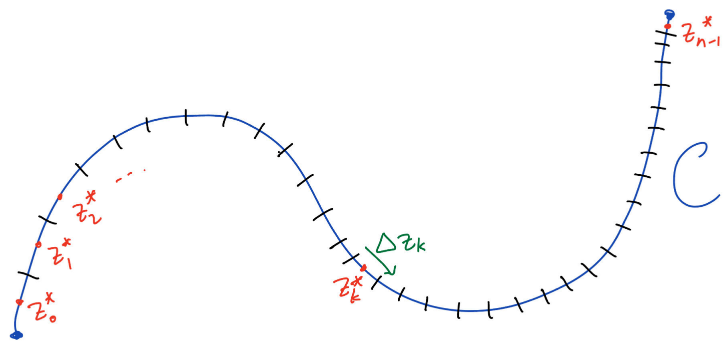


Figure 19: Division of a contour in constructing a Riemann sum

Definition 13.3. Given a continuous complex-valued function $f : \mathbb{C} \to \mathbb{C}$ and a smooth curve C, the *complex integral* (or *contour integral*) of f over C is defined as

$$\int_C f(z)\, dz \triangleq \int_a^b f(z(t)) \cdot z'(t)\, dt \tag{13.3}$$

with the right-hand side defined as in Definition 13.1.
If C can be decomposed into n parts $C_1, C_2, \ldots, C_n$, such that each part is a smooth contour, the integral of $f(z)$ over C is defined as the sum

$$\int_C f(z)\, dz \triangleq \sum_{k=1}^{n} \int_{C_k} f(z)\, dz.$$

Remark. Definition 13.3 is a generalization of Definition 13.1. The complex integral for a function $f : [a, b] \to \mathbb{C}$ can be regraded as a contour integral with the contour on the real axis from the complex point $z = a$ to $z = b$. If $a < b$, we can parametrize the curve by $z(t) = t + 0i$ for $t \in [a, b]$.

Notation: We write the integral as

$$\oint_C f(z)\, dz$$

if we want to emphasize that the curve C is closed.

Example 13.6. Integrate $f(z) = \bar{z}$ over the line segment from $z_1 = 0$ to $z_2 = 2i$. We parameterize the line segment using the y coordinate as the parameter,

$$C: \ z(y) = 0 + iy$$

for $0 \leq y \leq 2$. The tangent vector is constant i.

$$\int_C \bar{z}\, dz = \int_0^2 \overline{0 + iy} \cdot (i)\, dy = \int_0^2 (-iy)i\, dy = \left[\frac{y^2}{2}\right]_0^2 = 2.$$

Example 13.7. Consider the same complex function $f(z) = \bar{z}$ as in the previous example, but now we integrate over a semi-circle C' with center i and radius 1, from the point $z_1 = 0$ to the point $z_2 = 2i$. We represent the semi-circle by angle θ in the range $[-\pi/2, \pi/2]$,

$$z(\theta) = \cos\theta + i(1 + \sin\theta).$$

The tangent vector is

$$z'(\theta) = -\sin\theta + i\cos\theta.$$

Substitute the above data to the definition of contour integral, and compute

$$\begin{aligned}
\int_C f(z)\, dz &= \int_{-\pi/2}^{\pi/2} (z(\theta))^* \cdot z'(\theta)\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} (\cos\theta - i(1 + \sin\theta)) \cdot (-\sin\theta + i\cos\theta)\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \cos\theta + i(1 + \sin\theta)\, d\theta \\
&= 2 + i\pi.
\end{aligned}$$

The answer is not the same as in the previous example.

Examples 13.6 and 13.7 illustrate that if we change the contour, the contour integral will be different in general.

Example 13.8. Integrate $f(z) = z^2$ over the curve $C = \{x + iy \in \mathbb{C} : \ y = x^2\}$ from 0 to $1 + i$. We use the representation $z(x) = x + ix^2$ for $0 \leq x \leq 1$, with the variable x as the parameter. The derivative of $z(x)$ is $z'(x) = 1 + 2xi$.

$$\begin{aligned}\int_C z^2\,dz &= \int_0^1 (x + ix^2)^2 \cdot (1 + 2xi)\,dx \\ &= \int_0^1 (x^2 - 5x^4) + i(4x^3 - 2x^5)\,dx \\ &= \big[\frac{x^3}{3} - x^5\big]_0^1 + i\big[x^4 - \frac{x^6}{3}\big]_0^1 \\ &= -\frac{2}{3} + \frac{2}{3}i.\end{aligned}$$

We can interpret contour integral in terms of line integrals. The first one is through line integral for real vector field. Suppose

$$f(x + iy) = u(x, y) + iv(x, y)$$

and the curve C is parameterized by $z(t) = x(t) + iy(t)$ for $t \in [a, b]$. We can decompose the contour integral into two parts.

$$\begin{aligned}\int_C f(z)\,dz &= \int_a^b (u(x(t), y(t)) + iv(x(t), y(t))) \cdot (x'(t) + iy'(t))\,dt \\ &= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)\,dt \\ &\qquad + i\int_a^b u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)\,dt \\ &= \int_C udx - vdy + i\int_C udy + vdx.\end{aligned}$$

We are actually computing two line integral simultaneously. Indeed, some people take the above as the definition of contour integral.

Recall from Section 9.6 that there is a physical interpretation by considering the vector field $(M(x, y), N(x, y)) = (u(x, y), -v(x, y))$, which is called Polya's vector field of $f(z)$. If we make a substitution $u(x, y) = M(x, y)$ and $v(x, y) = -N(x, y)$, we can write the contour integral as

$$\int_C (M(x, y) - iN(x, y))\,dz = \int_C M dx + N dy + i\int_C M dy - N dx.$$

The real and imaginary parts are precisely the *work integral* and *flux integral* of the vector field $(M(x, y), N(x, y))$. This is a physical interpretation of the contour integral.

13.4 Appendix: Definition of line integral and contour integral using differential form

Another interpretation of line is through the differential form

$$\omega = M(x,y)dx + N(x,y)dy.$$

At the point (x_0, y_0), the linear functional $\omega_{(x_0,y_0)}$ maps a tangent vector $\mathbf{v} = (v_1, v_2)$ in $T_{(x_0,y_0)}$ to

$$\omega_{(x_0,y_0)}(\mathbf{v}) = M(x_0,y_0)v_1 + N(x_0,y_0)v_2.$$

(See Section 10.1) If the curve $\mathbf{r}(t)$ passes through the point (x_0, y_0) at time t_0, we can obtain the tangent vector at (x_0, y_0) as $(x'(t_0), y'(t_0))$. We apply the differential form $\omega = M(x,y)dx + N(x,y)dy$ to the tangent vector $(x'(t_0), y'(t_0))$ to obtain a scalar function

$$\omega_{(x(t),y(t))}(\mathbf{r}'(t)), \tag{13.4}$$

which is a function of parameter t.

The integral of differential form $M(x,y)dx + N(x,y)dy$ along the curve $\mathbf{r}(t)$ is defined as the integral of the function in (13.4),

$$\int_C \omega \triangleq \int_a^b \omega_{(x(t),y(t))}(\mathbf{r}'(t))\, dt. \tag{13.5}$$

If we expand it in terms of the component functions M and N, we see that this is the same as the line integral in (13.1).

We can also define contour integral using complex differential form. Similar to (13.5), we can regard

$$\omega = f(z)dz = f(x+iy)(dx + idy)$$

as a mathematical object. It defines a (complex) linear functional $\omega_{x_0+iy_0}$ on the point $x_0 + iy_0$, mapping a tangent vector $\tau \triangleq \tau_1 + i\tau_2$ in the tangent space $T_{x_0+iy_0}$ to

$$\omega_{x_0+iy_0}(\tau) = f(x_0 + iy_0) \cdot (\tau_1 + i\tau_2)$$

Suppose the contour $\gamma(t)$ is represented by $z(t) = x(t) + iy(t)$ and it passes through the point $x_0 + iy_0$ at $t = t_0$, we can obtain the tangent vector at this point by

$$\gamma'(t_0) = x'(t_0) + iy'(t_0).$$

We obtain a complex-valued function

$$f(x(t) + iy(t)) \cdot \gamma'(t). \tag{13.6}$$

The integral of the complex-valued function in (13.6) is exactly the same as the contour integral

$$\int_a^b f(x(t) + iy(t)) \cdot (x'(t) + iy'(t))\, dt = \int_a^b f(z(t)) \cdot z'(t)\, dt.$$

It is now obvious that the line integral and contour integral belong to the same class of objects through differential form.

14 Calculus of contour integral

Summary:

- Independence of parameterization
- Path concatenation
- Primitive function / anti-derivative

14.1 Independence of parameterization

There are more than one way to represent a curve in parametric form, but the value of the contour does not depend on the parameterization.

Theorem 14.1. *The answer obtained from the definition of contour integral does not depend on the parameterization of the curve.*

Proof. Suppose a curve C is represented in two different ways. The first parameterization is $z(t)$ for $a \le t \le b$, and the second parameterization is $w(t)$ for $c \le t \le d$. We want to show that

$$\int_a^b f(z(t))z'(t)\,dt = \int_c^d f(w(\tau))w'(\tau)\,d\tau. \tag{14.1}$$

Because both $z(t)$ and $w(\tau)$ represent the same curve, there is a monotonically increasing function $\alpha : [c,d] \to [a,b]$ such that

$$w(\tau) = z(\alpha(\tau)).$$

We can illustrate the relationship among w, z and α by the diagram

$$\begin{tikzcd} {[c,d]} \arrow[r, "w"] \arrow[rr, bend right, "\alpha"'] & \mathbb{C} & {[a,b]} \arrow[l, "z"'] \end{tikzcd}$$

Apply chain rule and differentiate with respect to τ,

$$w'(\tau) = z'(\alpha(\tau))\alpha'(\tau).$$

We make a change of variable $t = \alpha(\tau)$, and substitute dt by $dt = \alpha'(\tau)d\tau$,

$$\begin{aligned}\int_a^b f(z(t))z'(t)\,dt &= \int_c^d f(z(\alpha(\tau)))z'(\alpha(\tau)) \cdot \alpha'(\tau)\,d\tau \\ &= \int_c^d f(w(\tau))\frac{w'(\tau)}{\alpha'(\tau)} \cdot \alpha'(\tau)\,d\tau \\ &= \int_c^d f(w(\tau))w'(\tau)\,d\tau.\end{aligned}$$

This proves the equality in (14.1). $\square$

The contour integral satisfies the usual operational rule as for line integral. For example, for a fixed curve C, we have linearity property

$$\int_C af(z) + bg(z)\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz,$$

where a and b are complex constants, and f and g are complex functions.

14.2 Varying the paths

Even though contour integral does not depend on parameterization, it does depend on the direction of the path.

Definition 14.2. The *negation* or the *reverse* of a curve C is the curve with the same locus with reverse direction. The negation of C is denoted by $-C$.

Without loss of generality, we may suppose the range of the parameter starts from 0. If a contour C is represented by $z(t)$, for t going from 0 to b, then $-C$ can be represented by $z(b - t)$. Similar to the proof of Theorem 14.1, we can apply the chain rule to obtain the following relationship between the integrals on a contour in opposite direction.

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz. \tag{14.2}$$

This is analogous to the formula

$$\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx.$$

in calculus.

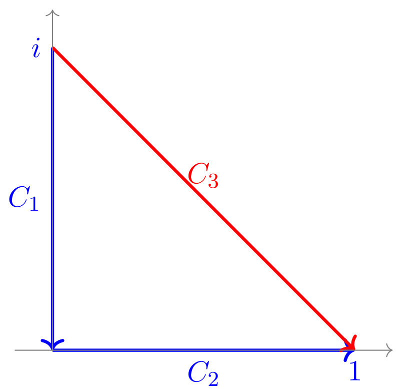


Figure 20: The three contour in Example 14.1.

For integer n and contour C, we use the notation nC to represent going through the curve C n times. This notation is particular convenient if C is a closed curve and we want to say "go around the curve C multiple times. For example, given a closed curve C, the symbol "2C" means going around C counter-clockwise 2 times. Thus, we have

$$\int_{2C} f(z)\,dz = 2\int_C f(z)\,dz.$$

Example 14.1. Consider the function $f(z) = xz = x^2 + ixy$. Compute the complex integral from i to 1 via (i) a vertical line segment from i to 0 and then a horizontal line segment from 0 to 1, (ii) a single line segment from i to 1.

Let C_1, C_2, and C_3 be paths as illustrated in Fig. 20.

The function $f(z)$ is zero on the path C_1. Therefore we can directly see that

$$\int_{C_1} f(z)\,dz = 0.$$

For contour C_2, we can represent it by C_2 using x as the parameter

$$z(x) = x, \qquad z'(x) = 1.$$

$$\int_{C_2} f(z)\,dz = \int_0^1 (x^2)\cdot(1)\,dx = 1/3.$$

Therefore, the contour integral along C_1 and C_2 is

$$\int_{C_1+C_2} f(z)\,dz = 0 + \frac{1}{3} = \frac{1}{3}.$$

If we go directly from i to 1 through C_3, we can parameterize the contour by the x coordinate,

$$z(x) = x + i(1-x)$$

for $0 \le x \le 1$. The tangent vector is constantly equal to $1-i$. We then compute the contour integral along C_3 by

$$\begin{aligned}\int_{C_3} f(z)\,dz &= \int_0^1 (x^2 + ix(1-x))(1-i)\,dx \\ &= (1-i)(\frac{1}{3} + \frac{i}{2} - \frac{i}{3}) \\ &= \frac{1}{6}(1-i)(2+i) \\ &= \frac{3-i}{6}.\end{aligned}$$

If we go around the triangle counter-clockwise, along the curves C_1, C_2 and $-C_3$, the integral of $f(z)$ is

$$\oint_{C_1+C_2+(-C_3)} f(z)\,dz = \frac{1}{3} - \frac{1}{2} + \frac{i}{6} = \frac{-1+i}{6}.$$

14.3 Fundamental theorem of calculus for complex integral

We define the notion of anti-derivative as in calculus.

Definition 14.3. A function $F(z)$ is called a *primitive* or *anti-derivative* of $f(z)$ if $F'(z) = f(z)$ for all z in the domain of definition of $f(z)$.

Implicit in the definition is that $F(z)$ is a holomorphic function. Because the derivative of a complex function is continuous, it also implicit in the definition that $f(z)$ is a continuous function.

Remark. Recall from the end of Section 10.3 that when $F(z)$ is holomorphic, the differential of F equals

$$dF = F'(z)dz = f(z)dz.$$

In general, if there exists a holomorphic function $F(z)$ such that $f(z)dz$, we say that $f(z)dz$ is an *exact* differential form. Hence, the condition in Definition 14.3 is the same as saying that the complex differential $f(z)dz$ is exact.

If $f(z)$ has a primitive, then the calculation of contour integral becomes much easier.

Theorem 14.4 (Fundamental theorem of calculus for complex integration)**.** *Suppose $f(z)$ is a continuous complex function defined on a domain D. If f has primitive $F(z)$ in D, then for any smooth contour C in D from z_1 to z_2, we have*

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

We note that the domain D in Theorem 14.4 needs not be simply-connected. The domain could be multiply connected. We can check that the proof below does not require that the domain is simply connected.

Proof. Suppose C is a smooth curve in D parameterized by $z(t)$ for $a \le t \le b$. We want to calculate

$$\begin{aligned}\int_C f(z)\,dz &\triangleq \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b \frac{d}{dt}F(z(t))\,dt.\end{aligned}$$

We have applied the chain rule of differentiation in the last step. Hence, we see that the integrand can be written as the derivative of some function. To facilitate notation, we may define $g(t) = F(z(t))$.

The remaining steps is to apply the fundamental theorem of calculus to the real and imaginary part. Suppose real and imaginary parts of $g(t)$ are $u(t)$ and $v(t)$, respectively, such that

$$g'(t) = u'(t) + iv'(t).$$

Integrate real and imaginary parts separately,

$$\begin{aligned}\int_C f(z)\,dz &= \int_a^b u'(t)\,dt + i\int_a^b v'(t) \\ &= u(b) - u(a) + i(v(b) - v(a)) \\ &= g(b) - g(a) \\ &= F(z(b)) - F(z(a)) \\ &= F(z_2) - F(z_1).\end{aligned}$$

This proves the fundamental theorem for contour integral. □

Theorem 14.4 can be extended easily to piece-wise smooth curve. Suppose a contour C is the concatenation of n smooth contours,

$$C = C_1 + C_2 + \cdots + C_n,$$

and for $j = 1, 2, \ldots, n$, C_j starts at z_{j-1} and ends at z_j. If $f(z)$ has a primitive function $F(z)$ in a domain that contains all C_j's, then

$$\begin{aligned}\int_C f(z)\,dz &= (\int_{C_1} + \int_{C_2} + \cdots + \int_{C_n})f(z)\,dz \\ &= (F(z_1) - F(z_0)) + (F(z_2) - F(z_1)) + \cdots + (F(z_n) - F(z_{n-1})) \\ &= F(z_n) - F(z_0).\end{aligned}$$

14.4 Independence of path

If $f(z)$ has a primitive, then the value of $\int_C f(z)\,dz$ only depends on the two end points of the contour and the anti-derivative $F(z)$, but does not depend on the shape of the contour.

Definition 14.5. Given a continuous complex function $f(z)$ on a domain D, we say that the integral $\int_C f(z)\,dz$ is *independent of path in domain D* if for any curve C, the contour integral $\int_C f(z)\,dz$ only depends on the start and end points, but does not depend on the shape of the curve C.

Theorem 14.4 says that the existence of primitive implies independence of path.

Example 14.2. We denote the circle with radius r, positive orientation, centered at the origin, by C_r. For nonegative integer n, the integral

$$\oint_{C_r} z^n \, dz$$

is equal to 0, because z^n has primitive $\frac{z^{n+1}}{n+1}$ throughout the complex plane.

When $n = -1$, we cannot apply Theorem 14.4, even though we have $(\log z)' = 1/z$ for all $z \neq 0$. The reason we cannot apply Theorem 14.4 is that the log function fails to be continuous, and hence is not holomorphic, in the punctured complex plane $\mathbb{C} \setminus \{0\}$. (We need a branch cut to make it holomorphic). Indeed the integral $\int_C 1/z \, dz$ is not equal to zero. We can compute the integral by parameterizing the circle with radius r by $\gamma(t) = re^{it}$, for $t \in [0, 2\pi]$. The derivative of $\gamma(t)$ with respect to t is $\gamma'(t) = rie^{it}$. Using this parameterization, we have

$$\int_{C_r} 1/z \, dz = \int_0^{2\pi} r^{-1} e^{-i\theta} (rie^{i\theta}) \, d\theta = \int_0^{2\pi} i \, d\theta = 2\pi i.$$

When $n \leq -2$, the function $f(z) = z^n$ is holomorphic in the punctured plane $\mathbb{C} \setminus \{0\}$, and have primitive function $\frac{z^{n+1}}{n+1}$ for $z \in \mathbb{C} \setminus \{0\}$. By Theorem 14.4, the integral $\oint_{C_r} z^n \, dz$ is zero.

We can double check the integral of z^n along C_r is

$$\begin{aligned} \int_{C_r} z^n \, dz &= \int_0^{2\pi} r^n e^{int} (rie^{it}) \, dt \\ &= \int_0^{2\pi} r^{n+1} i e^{i(n+1)t} \, dt. \end{aligned}$$

When the value of $n+1$ in the exponent is not equal to zero, we can evaluate the integral using the anti-derivative,

$$\int_{C_r} z^n \, dz = r^{n+1} i \Big[\frac{e^{i(n+1)t}}{i(n+1)} \Big]_0^{2\pi} = 0.$$

We summarize this example as a theorem.

Theorem 14.6. *Let C_r denote the circle centered at the origin, with radius r and counter-clockwise orientation.*

$$\oint_{C_r} z^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{-1\}. \end{cases}$$

As in multi-variable calculus, we have an equivalent condition for path independence.

Theorem 14.7. *Suppose $f(z)$ is a complex-valued continuous function defined on a domain D. The followings are equivalent:*

(a) For any piece-wise smooth curve C in domain D, the integral $\int_C f(z)\, dz$ only depends on the start and end point of C, but not on the shape of C.

(b) $\oint_C f(z)\, dz = 0$ for any closed contour C.

Proof. (a) $\Rightarrow$ (b). Suppose the integral $\int_C f(z)\, dz$ is independent of path. Let C be a closed contour. We pick two distinct points z_1 and z_2 on C, and divide C into two parts. The first part C_1 is the portion of C that goes from z_1 to z_2. The second part C_2 is the portion of C that goes from z_2 back to z_1.

$$\begin{aligned} \oint_C f(z)\, dz &= \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz \\ &= \int_{C_1} f(z)\, dz - \int_{-C_2} f(z)\, dz. \end{aligned}$$

Since C_1 and $-C_2$ have the same start point and end point, the two integral in the last line have the same value. Therefore, their difference must be zero.

(b) $\Rightarrow$ (a). Suppose $\oint_C f(z)\, dz$ is equal to 0 for any closed contour C. Let C_1 and C_2 be two contours with the same start point and end point. Concatenate C_1 with the reverse path $-C_2$. The resulting contour $C_1 + (-C_2)$ is a closed, and by assumption, the integral on $C_1 + (-C_2)$ is zero. We thus obtain

$$\begin{aligned}
\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz &= 0 \\
\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz &= 0 \\
\int_{C_1} f(z)\,dz &= \int_{C_2} f(z)\,dz.
\end{aligned}$$

□

15 Cauchy-Goursat theorem

Summary

- Arc length of a smooth curve.
- ML inequality
- Cauchy-Goursat theorem for triangle

15.1 Cauchy theorem version 1

If we assume that $f(z)$ is sufficiently smooth, then we can proof the Cauchy theorem easily as a consequence of Green theorem.

Theorem 15.1 (Cauchy theorem version 1)**.** *If C is a simple closed curve enclosing a simply connected region D and $f(z)$ is continuously differentiable in D, then*

$$\oint_C f(z)\,dz = 0.$$

Proof. Suppose u and v are the real and imaginary parts of f, respectively. The integral $\oint_C f(z)\,dz$ can be written as

$$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C u\,dy + v\,dx.$$

By Green's theorem, the first integral on the right-hand side equals

$$\iint \Big(- v_x - u_y \Big)\,dxdy$$

and the second integral equals

$$\iint \Big(u_x - v_y \Big)\,dxdy.$$

Because Riemann-Cauchy equations holds for functions u and v, both of them are equal to 0. $\square$

The assumption that the partial derivatives are continuous is redundant. We can prove the conclusion in Theorem 15.1 by assuming that $f(z)$ is complex differentiable in the domain D.

15.2 The length of a contour and ML inequality

We first prove a few useful results.

Theorem 15.2 (Triangle inequality for complex integral)**.** *Suppose $g(t)$ is a continuous complex function from $[a, b]$ to $\mathbb{C}$. Then*

$$\Big| \int_a^b g(t)\, dt \Big| \le \int_a^b |g(t)|\, dt. \tag{15.1}$$

Proof. Let α denote the complex integral $\alpha \triangleq \int_a^b g(t)\, dt$. If $\alpha = 0$, then (15.1) is obviously true, because the left-hand side is zero. Hence, we can suppose $\alpha \neq 0$. Write α in polar form $\alpha = re^{i\theta}$. (θ is defined because $\alpha \neq 0$.)

Then,

$$e^{-i\theta} \int_a^b g(t)\, dt = r$$

is a real number. We can re-write it as

$$\int_a^b e^{-i\theta} g(t)\, dt.$$

Let $u(t)$ and $v(t)$ be the real and imaginary parts of $e^{-i\theta} g(t)$, respectively. We get

$$\int_a^b u(t)\, dt = r \quad \text{and} \quad \int_a^b v(t)\, dt = 0.$$

But

$$u(t) \le \sqrt{u^2(t)} \le \sqrt{u^2(t) + v^2(t)} = |g(t)|.$$

By the monotonic property for real integration,

$$\Big| \int_a^b g(t)\, dt \Big| = r = \int_a^b u(t)\, dt \le \int_a^b |g(t)|\, dt.$$

□

To compute the arc length of a path that is parameterized by $z(t) = x(t) + iy(t)$, we can apply a result from multi-variable calculus:

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

We express the above integral using complex variable, by observing that the integrand can be written as

$$\sqrt{x'(t)^2 + y'(t)^2} = |x'(t) + iy'(t)| = |z'(t)|.$$

Hence, the arc length of a curve C can be formally defined as follows.

Definition 15.3. The *length* of a smooth curve C, represented by $z(t)$ for $a \le t \le b$, is defined as $\int_a^b |z'(t)|\, dt$.

Remark. We can also define the length of a smooth curve using complex differential $dz = dx + idy$:

$$\int_C |dz|.$$

The symbol $|dz|$ represents the differential $\sqrt{dx^2 + dy^2}$, which maps a tangent vector $\mathbf{v} = v_1 + iv_2$ to $\sqrt{v_1^2 + v_2^2}$. We now apply the integral in (13.5).

The ML inequality gives a bound on the absolute value of a contour integral in terms of the length of the path and the largest absolute value of the integrand on the path.

Theorem 15.4 (ML inequality)**.** *If $|f(z)| \le M$ for z on a smooth curve C and the length of C is equal to L, then*

$$\left| \int_C f(z)\, dz \right| \le ML.$$

Proof. Apply the triangle inequality (15.1) to get

$$\left| \int_C f(z)\, dz \right| = \left| \int_a^b f(z(t)) z'(t)\, dt \right| \le \int_a^b \big| f(z(t)) z'(t) \big|\, dt.$$

This yields

$$\left| \int_C f(z)\, dz \right| \le \int_a^b |f(z(t))| \cdot |z'(t)|\, dt \le M \int_a^b |z'(t)|\, dt = ML.$$

□

The next example illustrates an application of Theorem 15.4 in proving an inequality. The original inequality does not involve any integral, but the proof is facilitated by complex integral.

Example 15.1. For any complex numbers z_1 and z_2 with negative real part, prove that

$$|e^{z_2} - e^{z_1}| \le |z_2 - z_1|.$$

We note that $e^{z_2} - e^{z_1}$ is the same as $\int_C e^z \, dz$ for any path from z_1 to z_2. We take C to be the direct path from z_1 to z_2 in the form of a straight line. The integrand e^z has modulus less than 1, because z_1 and z_2 have negative real parts. The length of the line segment between z_1 and z_2 is precisely $|z_2 - z_1|$. Hence, by the ML inequality,

$$|e^{z_2} - e^{z_1}| = \left| \int_C e^z \, dz \right| \le 1 \cdot |z_2 - z_1|.$$

15.3 Cauchy-Goursat theorem for triangular domains

We need the following theorem, which is usually called "Cantor's intersection theorem" from point-set topology.

Theorem 15.5. *Suppose K_1, K_2, $K_3, \ldots$ are nonempty compact sets (in some metric space). If K_1, K_2, $K_3, \ldots$ is a decreasing sequence, i.e.,*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

then $\cap_{j=1}^{\infty} K_j$ is not empty.

We note that the conclusion of this theorem will fail without the compactness assumption. For example, consider a sequence of intervals

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

defined by $A_k \triangleq [k, \infty)$ for $k = 1, 2, 3, \ldots$. The sets are unbounded, and hence are not compact. Their intersection is empty.

If we let B_k be the open interval $(0, 1/k)$, for $k = 1, 2, \ldots$, we have a decreasing sequence of open intervals, and the intersection is empty.

The following version of Cauchy theorem is due to Goursat.

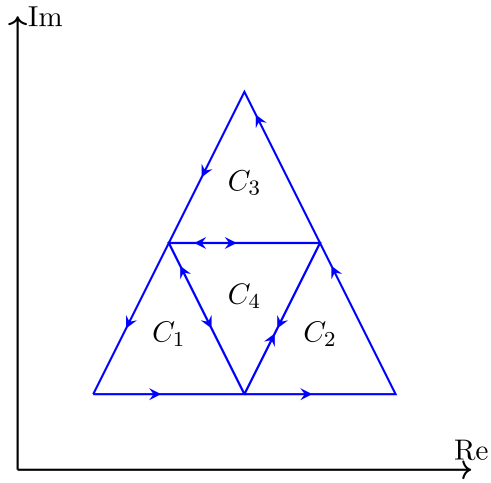


Figure 21: Division of a triangular regions into four equal parts.

Theorem 15.6 (Cauchy-Goursat for triangles)**.** *Suppose $f(z)$ is holomorphic in a domain D and T is a triangle contained inside D. Then*

$$\oint_{\partial T} f(z)\, dz = 0,$$

where ∂T denotes the boundary of triangle T.

Remark. It is important to note that the theorem only assumes that the derivative of $f(z)$ exists. We do not need to assume that $f'(z)$ is a continuous function of z.

Proof. Let C be the boundary of triangle T with counter-clockwise orientation. Let L be the perimeter of triangle T.

Suppose $\int_C f(z)\, dz = I$. We want to show that $|I| = 0$. The following figure depicts the triangle T on which $f(z)$ is holomorphic.

Divide the triangle T into four equal parts. The four sub-triangles are similar to each other, and have the same area and perimeter. Let C_i be the boundary of the i-th sub-triangle, for $i = 1, 2, 3, 4$, all in the counter-clockwise direction (See Fig. 21).

The integrals on the internal lines cancel. Hence,

$$I = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz.$$

By triangle inequality,

$$|I| \leq \Big|\int_{C_1} f(z)\,dz\Big| + \Big|\int_{C_2} f(z)\,dz\Big| + \Big|\int_{C_3} f(z)\,dz\Big| + \Big|\int_{C_4} f(z)\,dz\Big|.$$

Suppose $T^{(1)}$ is the sub-triangle that has the largest integral in magnitude,

$$\Big|\int_{\partial T^{(1)}} f(z)\,dz\Big| = \max_{j=1,2,3,4}\Big|\int_{C_j} f(z)\,dz\Big|$$

(Here $\partial T^{(1)}$ means the boundary of $T^{(1)}$.)

This implies

$$\frac{|I|}{4} \leq \Big|\int_{\partial T^{(1)}} f(z)\,dz\Big|$$

by the choice of $T^{(1)}$.

Recursively, for $k \geq 1$, divide $T^{(k)}$ into four equal parts, and let $T^{(k+1)}$ be the sub-triangle that has the largest integral in absolute value. Thus,

$$T \supseteq T^{(1)} \supseteq T^{(2)} \supseteq T^{(3)} \supseteq T^{(4)} \supseteq \cdots$$

and

$$\frac{|I|}{4^k} \leq \Big|\int_{\partial T^{(k)}} f(z)\,dz\Big|$$

for $k = 1, 2, 3, \ldots$ The perimeter of $T^{(k)}$ is $L/2^k$, for $k \geq 1$.

By Cantor intersection theorem (Theorem 15.5), there is a point z_0 that lies inside $T^{(k)}$ for all k. Since f is assumed to be holomorphic in T, it is complex differentiable at the point z_0. Suppose that the derivative is $f'(z_0)$. By the definition of complex differentiability, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0) \tag{15.2}$$

where $\epsilon(z)$ is a function such that $|\epsilon(z)| \to 0$ as $z \to z_0$.

For any k, we can integrate both sides of (15.2) on the boundary of $T^{(k)}$.

$$\int_{\partial T^{(k)}} f(z)\,dz = \int_{\partial T^{(k)}} f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0)\,dz.$$

Because $f(z_0) + f'(z_0)(z - z_0)$ is a linear function in z, it has an anti-derivative. By Theorem 14.4, $\int_{\partial T^{(k)}} f(z_0) + f'(z_0)(z - z_0)\,dz$ is zero. So,

$$\int_{\partial T^{(k)}} f(z)\,dz = \int_{\partial T^{(k)}} \epsilon(z)(z - z_0)\,dz$$

for any $k = 1, 2, 3, \ldots$

We upper bound $|z - z_0|$ by the diameter of $T^{(k)}$. In general, the diameter of a set S in $\mathbb{C}$ is the supremum of the Euclidean distance between two points in S,

$$\operatorname{diam}(S) \triangleq \sup_{p,p' \in S} \{\text{distance between } p \text{ and } p'\}.$$

For example, the diameter of a circle as defined above is the diameter in the usual sense. The diameter of an ellipse is the length of the major axis.

Let d denote the diameter of original triangle T. We know that the diameter of $d^{(k)}$ equals $d/2^k$, for $k = 1, 2, 3, \ldots$, because in each subdivision step we scale down the triangle by a factor of 2. By the very definition of diameter, we have $|z - z_0| \leq d^{(k)}$ for all $z \in T^{(k)}$.

We now fix a small and positive real number δ, and choose a sufficiently large k such that $|\epsilon(z)| \leq \delta$ for all $z \in T^{(k)}$. This is possible as $\epsilon(z) \to 0$ as $z \to z_0$. For each k, we can bound $|\epsilon(z)(z - z_0)|$ by $\delta d/2^k$. Apply ML inequality (Theorem 15.4),

$$\begin{aligned}\frac{1}{4^k}|I| &\leq \Big| \int_{\partial T^{(k)}} \epsilon(z)(z - z_0)\,dz \Big| \\ &\leq \delta \frac{d}{2^k} \cdot \frac{L}{2^k}.\end{aligned}$$

Hence

$$|I| \leq \delta d L.$$

Since d and L are constant and δ can be arbitrarily small, we must have $I = 0$. $\square$

We can immediately extend Cauchy theorem to domain with polygonal shape.

Theorem 15.7 (Cauchy theorem for rectangle)**.** *Suppose $f(z)$ is holomorphic in an open rectangle R, then*

$$\oint_{\partial R} f(z)\,dz = 0.$$

Proof. Pick one of the diagonal of the rectangle R, and consider the two triangles obtained by dividing R along this diagonal. The integral along the boundary of R is the sum of the two integrals along the two triangles. Since the integrals on the triangles are both 0, the sum is also 0. $\square$

With the same argument as in the proof of Theorem 15.7, we can show that Cauchy-Goursat theorem is valid for polygons, such as parallelogram, trapezoid, etc.

16 Local primitive function and Cauchy theorem for regions of various shapes

Summary

- Construction of local primitive function
- Cauchy theorem for convex and star-shape regions
- Application to the definition of complex log function

If a primitive function exists, Theorem 14.4 guarantees that the integral of $f(z)$ depends only on the initial and final points of the contour, regardless of the shape of the contour. However, it is not always the case that a primitive function exists for every function $f(z)$. As we observed in Example 14.2, the integral of the function $1/z$ over a circle centered at the origin is not zero. However, if we introduce a branch cut and prohibit closed paths from encircling the origin, we can conclude that $\int_C 1/z\,dz = 0$ for any closed curve C within the domain $\mathbb{C} \setminus \{x + iy :, y = 0, x \leq 0\}$. This motivates us to consider a more restricted notion of a primitive function, as the existence of such a function may be possible within a smaller domain.

16.1 Application to the evaluation of real integral

We illustrate an application of Cauchy theorem to the evaluation of Fourier transform. Consider the probability distribution function of the standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Our objective is to demonstrate that for any $t \in \mathbb{R}$, the Fourier transform of $f(x)$ is given by

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-itx}\,dx = e^{-t^2/2}.$$

It is sufficient to consider the case where $t > 0$. We examine the complex function $f(z) = e^{-z^2/2}$ and integrate it along the rectangular contour depicted in Fig. 22.

Since $e^{-z^2/2}$ is an entire function, we can apply Cauchy theorem (Theorem 15.7) to see that the integrals along C_1, C_2, C_3 and C_4 are all zero,

$$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0.$$

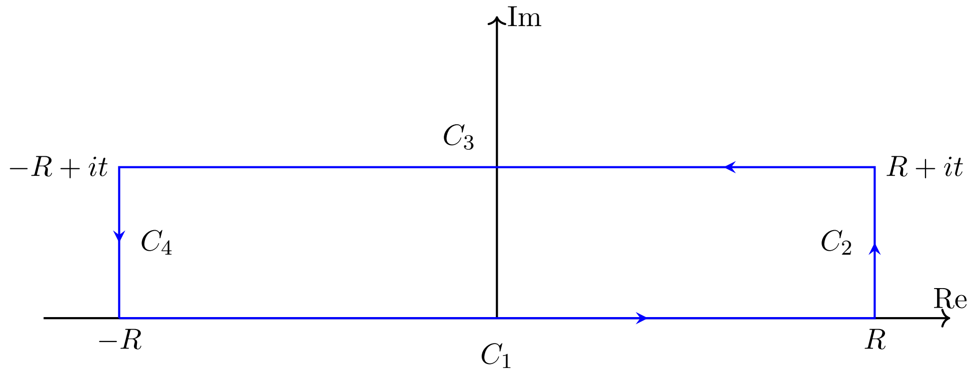


Figure 22: Closed contour used in the derivation of the Fourier transform of the Gaussian probability distribution function.

The integral along C_1 from $-R$ to R on the real axis equals

$$\int_{C_1} e^{-z^2/2}\,dz = \int_{-R}^{R} e^{-x^2/2}\,dx.$$

This integral tends to $\sqrt{2\pi}$ as $R \to \infty$.

The contour C_2 is a vertical straight line segment from R to $R+it$. We parameterize C_2 by $z(y) = R+iy$ for $0 \le y \le t$. The derivative is constant $z'(y) = i$. Apply the triangle inequality (Theorem 15.2) to obtain an upper bound the integral over C_2,

$$\begin{aligned}\Big|\int_{C_2} f(z)\,dz\Big| &= \Big|\int_0^t e^{-(R+iy)^2/2}\cdot i\,dy\Big|\\ &\le \int_0^t e^{-(R^2-y^2)/2}\,dy\\ &\le e^{-(R^2-t^2)/2}\cdot t.\end{aligned}$$

For any fixed t, the upper bound approaches 0 as R approaches infinity. Therefore the integral of f over contour C_2 tends to 0 as R tends to ∞.

Likewise, we can show that the integral over contour C_4 approaches 0 as $R \to \infty$.

Finally we consider the contour C_3. We represent the reversed contour $-C_3$ using the parameterization $z(x) = x + it$, where x ranges from $-R$ down to R. The derivative is

$z'(x) = 1$. Using this parameterization, we have

$$\int_{-C_3} f(z)\,dz = \int_{-R}^{R} e^{-(x+it)^2/2}\,dx$$
$$= e^{t^2/2}\int_{-R}^{R} e^{-x^2/2}e^{-ixt}\,dx.$$

By Cauchy theorem for rectangle, for each $R > 0$, we obtain the equation

$$-\int_{C_3} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_4} f(z)\,dz.$$

Taking limit as $R \to \infty$, we see that the integrals over C_2 and C_4 approaches zero, and the integral over C_1 approaches $\sqrt{2\pi}$. Thus, we have

$$e^{t^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ixt}\,dx = \sqrt{2\pi}.$$

After performing some algebraic operations, we obtain the desired formula for $t > 0$:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ixt}\,dx = e^{-t^2/2}.$$

The proof for $t < 0$ follows the exact pattern, except that the contour is modified to lies below the real axis.

16.2 Existence of local primitive function

Definition 16.1. We say that a function $f(z)$ *has a primitive at a given domain D* if we can find a holomorpic function $F(z)$ defined on D such that $F'(z) = f(z)$ for all $z \in D$. We will sometime call $F(z)$ a *local primitive* of $f(z)$ in the domain D.

We note the domain D in the above definition may not necessarily be the entire domain of definition of $f(z)$. Instead, it can be a small open set that contains a specific point or a restricted region within the overall domain of $f(z)$.

There are several sufficient conditions that can guarantee the existence of local primitive function. The key part of the argument is contained in the following theorem.

Consider a continuous complex function $f(z)$ defined on a domain D (which is assumed to be open and connected, but may not be simply connected). We pick a base point z_0

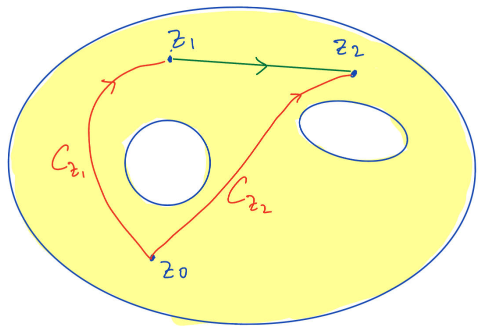


Figure 23: Constructing local primitive function by integral.

in D. Since D is connected, there is at least one path from z_0 to z that lies completely inside D. For each point $z \in D$, we select a path from z_0 to z, and denote it by C_z. We then define a candidate function for the primitive function as

$$F(z) \triangleq \int_{C_z} f(z)\, dz. \tag{16.1}$$

Furthermore, we assume that the paths are selected in a way that if z_1 and z_2 are two points connected by a line segment lying inside the domain D, then the following equation holds:

$$F(z_2) = F(z_1) + \int_{z_1 \to z_2} f(z)\, dz. \tag{16.2}$$

Here, we introduce a non-standard notation

$$\int_{z_1 \to z_2} f(z)\, dz,$$

which denote the contour integral of $f(z)$ on the line segment from z_1 to z_2. (See Fig. 23) If the line segment from z_1 and z_2 does not lie entirely inside D, then there is nothing to check in (16.2). The integral $\int_{z_1 \to z_2} f(z)\, dz$ makes no sense in this case anyway.

Theorem 16.2. *Assuming that $f(z)$ is a continuous function defined on a domain D, and that $F(z)$ defined in (16.1) satisfies the condition in (16.2), then $F(z)$ is holomorphic in D with derivative $f(z)$.*

We note that the axiom of choice is used in defining $F(z)$.

Proof. Let z be a point in domain D. Since D is an open set (as required by the definition of a domain), we can draw an open disc $D(z, r)$ centered at z with radius r such that $D(z, r)$ is contained inside D. A point in $D(z, r)$ can be written as $z + h$, where h is a complex number with modulus less than r.

Let h be a nonzero complex number with modulus less than r. We want to compute the difference $F(z + h) - F(z)$ and divide it by h, and show that the result tends to $f(z)$ as $h \to 0$. According to the assumption on how the function $F(z)$ is defined, the difference $F(z+h) - F(z)$ equals the integral obtained by integrating $f(z)$ from z to $z+h$ along the line segment connecting z and $z + h$. This is valid because the line between z and $z+h$ is contained within the open disc $D(z, r)$ and, therefore, also within the domain D.

By assumption, we have

$$F(z + h) = F(z) + \int_{z \to z+h} f(w)\, dw.$$

We re-write it as

$$\frac{F(z + h) - F(z)}{h} = \frac{1}{h} \int_{z \to z+h} f(w)\, dw.$$

On the other hand, if we integrate the constant function $f(z)$ along the same line segment, we get

$$\int_{z \to z+h} f(z)\, dw = hf(z),$$

This gives

$$\begin{aligned} \frac{F(z + h) - F(z)}{h} - f(z) &= \frac{1}{h} \int_{z \to z+h} f(w)\, dw - \frac{1}{h} \int_{z \to z+h} f(z)\, dw \\ &= \frac{1}{h} \int_{z \to z+h} (f(w) - f(z))\, dw. \end{aligned}$$

Since $f(z)$ is continuous at z by assumption, for any small positive real number $\epsilon 0$, we can find a sufficiently small positive real number δ such that $|f(w) - f(z)| < \epsilon$ for all w satisfying $|w - z| < \delta$. Then, for a complex number h with $|h| < \min(r, \delta)$, we can apply the ML inequality to obtain

$$\Big|\frac{F(z + h) - F(z)}{h} - f(z)\Big| \le \frac{1}{|h|}\epsilon \cdot (\text{length of line segment from } z \text{ to } z + h) = \epsilon$$

because the length of the line segment from z to $z+h$ is exactly equal to $|h|$. Since ϵ can be arbitrarily chosen, we have proved that for any given real number $\epsilon > 0$, we can guarantee that

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|\leq\epsilon$$

for all complex numbers h with $|h|\leq\min(\delta,r)$, where δ is determined from ϵ through the continuity of $f(z)$. This is equivalent to saying that

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$$

This completes the proof that $f(z)$ has a primitive function $F(z)$ in the domain D. $\square$

Using this technical result, we can now prove that the existence of local primitive function is logically equivalent to independent of path.

Theorem 16.3. *Let f be a continuous complex-valued function defined in a region region D. Then integral $\int_C f\,dz$ is independent of path in region D if and only if $f(z)$ has a primitive function $F(z)$ in the domain D.*

Proof. ($\Leftarrow$) This is Theorem 14.4.

($\Rightarrow$) We apply Theorem 16.2 to prove that (b) $\Rightarrow$ (c). Let us arbitrarily choose a base point z_0 in D. Since it is assumed that the integral $\int_C f(z)\,dz$ is independent of path, we can define $F(z)$ by integrating $f(z)$ along any path from z_0 to z. The condition in (16.2) automatically holds due to the assumption of path independence. Thus, the function $F(z)$ serves as the primitive of $f(z)$ in the region D. $\square$

We summarize what we know about path independence and existence of primitive. Let f be a continuous complex-valued function defined in a region D. The followings are equivalent:

(a) The contour integral $\oint_C f(z), dz$ around any closed contour C in the region D is zero.

(b) The integral $\int_C f\,dz$ is path independent for any piece-wise smooth curve C in the region D.

(c) Function $f(z)$ has a primitive function $F(z)$ in the domain D.

(d) the differential form $f(z)\,dz$ is exact, i.e., there exists function $F(z)$ defined on D such that $dF=f\,dz$.

We now apply Theorem 16.2 to the case where D is a convex set and $f(z)$ is holomorphic.

Theorem 16.4 (Existence of primitive in a convex domain)**.** *If $f(z)$ is complex differentiable at every point inside a convex domain D, then f has a primitive in D.*

Proof. We arbitrarily choose a point z_0 in the domain D. Since D is convex, the straight line from z_0 to z lies entirely inside D for any point $z \in D$. We define a function $F(z)$ as

$$F(z) \triangleq \int_{z_0 \to z} f(z)\, dz.$$

The condition in (16.2) holds by appealing to the Cauchy-Goursat theorem for triangles. Consider two points z_1 and z_2 in the convex domain D. The three points z_0, z_1 and z_2 form a triangle that lies inside D. By applying the Cauchy-Goursat theorem for triangles (Theorem 15.6), we know that the integral along the boundary of the triangle is zero

$$\int_{z_0 \to z_1} f(z)\, dz + \int_{z_1 \to z_2} f(z)\, dz + \int_{z_2 \to z_0} f(z)\, dz = 0$$

See Fig. 24. In terms of the function $F(z)$, this can be expressed as

$$F(z_2) = F(z_1) + \int_{z_1 \to z_2} f(z)\, dz.$$

Hence, all the conditions in Theorem 16.2 are satisfied. We can now apply Theorem 16.2 to conclude that $F(z)$ is a primitive of $f(z)$ in the domain D. $\square$

Because the complex plane is convex, we can apply Theorem 16.4 to the whole complex plane to get

Corollary 16.5. *If $f(z)$ is entire, then $f(z)$ has a primitive defined on the whole complex plane.*

Example 16.1. Using this theorem, we now know that the following functions have primitives in the complex plane:

$$\sin(\cos(e^z)), \qquad \cos^{32}(z)\sin^{53}(z),$$

because they are entire function.

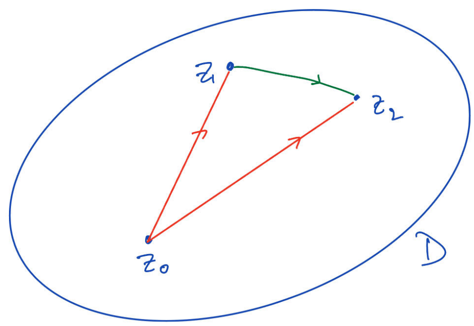


Figure 24: Proof of Theorem 16.4 about construction of primitive function for convex region.

Example 16.2. Whether a function has primitive depends on the domain. The inverse function $1/z$ has primitive in the open disc

$$D(3,3) \triangleq \{z \in \mathbb{C} : |z-3| < 3\},$$

because $1/z$ is defined and differentiable in this open disc. The primitive is given by a branch of the complex log function. For example, we can take the principal branch $\mathrm{Log}(z)$.

However, if we consider the punctured disc

$$\{z \in \mathbb{C} : 0 < |z| < 3\},$$

we cannot apply Theorem 16.4, because the punctured disc is not convex. In this case, it is true that no primitive function exists for $1/z$ in the punctured disc $D(0,3)$. It is because the integral of $1/z$ on a circle centered at 0 with radius $r < 3$ is $2\pi i$, which is nonzero. By Theorem 16.3, no such primitive function exists.

Example 16.3. Consider the inverse square function $1/z^2$ in the punctured complex plane $\mathbb{C} \setminus \{0\}$. The punctured complex plane is not convex, and hence we cannot apply Theorem 16.4. Nevertheless, we do have a primitive function for $1/z^2$, namely $-1/z$, which is well defined and differentiable in the punctured complex plane.

We have proved that we can always construct a primitive for a given holomorphic function defined in a convex domain. Even though a lot of common shapes, such as circle, semicircle, rectangle, and parallelogram, etc., are convex, we want to extend it further to include nonconvex sets.

In fact, the proof of Theorem 16.4 does not require the full force of convexity. It only requires that the domain D has a *star shape*, which means that we can find a point z_0 in D such that the line segment between z_0 and any point in D lies entirely inside D (See Fig. 25). Given a star-shaped domain S and the special point z_0, we can define a primitive function $F(z)$ as in the proof of Theorem 16.4.

Theorem 16.6 (Existence of primitive in a star-shaped domain)**.** *If $f(z)$ is complex differentiable at every point inside a domain D that has a star shape, then f has a primitive in D.*

Proof. The proof is exactly the same as in the proof of Theorem 16.4. □

A more general form of the Cauchy theorem is stated in terms of simply connected region

Theorem 16.7 (Cauchy theorem for simply connected regions)**.** *Suppose D is a simply connected region. Then any holomorphic function $f(z)$ defined in the domain D has a primitive, and therefore*

$$\oint_C f(z)\, dz = 0$$

for any closed curve C in the domain D.

A typical proof relies on the fact that any simple closed curve in a simply connected domain can be continuously shrunk to a point. However, providing a rigorous proof requires a deeper understanding of topology and is beyond the scope of this notes.

16.3 Defining a branch of log function by integral

We provide an alternate definition of the complex log function by contour integral. This definition is analogous to the definition of the real-valued log function through integration in some calculus books,

$$\ln(x) = \int_1^x \frac{1}{u}\, du,$$

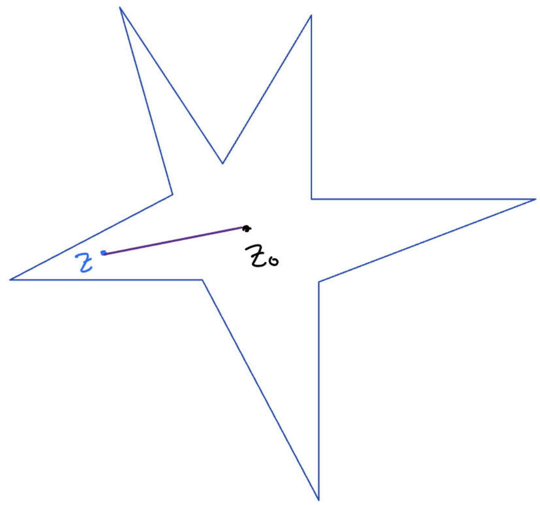


Figure 25: Star-shaped region

for $x > 0$.

Let D be the set defined as

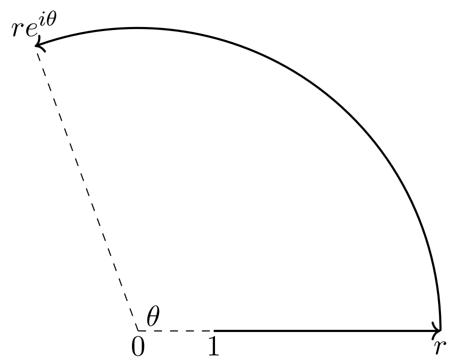
$$D \triangleq \mathbb{C} \setminus \{x + iy \in \mathbb{C} : \ x \leq 0, y = 0\},$$

which consists of all complex numbers except for the negative real axis. This set represents a star-shaped domain. We choose the point $z_0 = 1$ as the base point. Every point z in D can be reached through a straight line from 1 to z. Because $1/z$ is holomorphic in the domain D, we can define $\mathrm{Log}(z)$ using the same argument as in the proof of Theorem 16.4:

$$\mathrm{Log}(z) \triangleq \int_{1 \to z} \frac{1}{z} \, dz.$$

Here, the contour in the integral is the line segment from 1 to z. The function $\mathrm{Log}(z)$ in this way serves as a primitive of $1/z$ in the domain D.

Now, consider a point z not on the negative real axis. We can express it as $z = re^{i\theta}$, where $-\pi < \theta < \pi$. Since a primitive exists, the integral of $1/z$ is independent of path (as long as the path does not cross the branch cut). We can choose a more convenient path for calculation purposes. Consider a contour C from 1 to z that first goes horizontally from 1 to r, and then travels along an arc from r to $re^{i\theta}$.



We can compute the integral of $1/z$ from 1 to z in two parts. First, consider the integral from 1 to r:

$$\int_1^r \frac{1}{z}\,dz = \int_1^r \frac{1}{x}\,dx = \ln r.$$

This integral simplifies to $\ln r$.

The integral from r to $re^{i\theta}$ along the circular arc is

$$\begin{aligned}\int_r^{re^{i\theta}} \frac{1}{z}\,dz &= \int_0^\theta r^{-1}e^{-i\theta}(ire^{i\theta})\,d\theta \\ &= i\theta.\end{aligned}$$

The second integral simplifies to $i\theta$.

Combining the results, we can express this branch of the complex log function as

$$\mathrm{Log}(z) = \ln(r) + i\theta.$$

This matches the definition of principal log function in Definition 4.7.

17 Cauchy integral formula

Summary

- Cauchy integral formula
- Deformation theorem
- Liouville theorem

17.1 Cauchy integral formula

The Cauchy integral formula is a landmark in complex analysis.

Theorem 17.1 (Cauchy integral formula)**.** *Consider a domain D and a circular path C with counter-clockwise orientation inside D. If function $f(z)$ is holomorphic in D, then for any z in the interior of C,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} \, dw. \tag{17.1}$$

The Cauchy integral formula provides an integral representation of a holomorphic function. The integral on the right-hand side is taken over a circle with an arbitrary radius, provided that the function $f(z)$ is holomorphic within the circle. However, the integrand fails to be holomorphic at only one point, namely, at the point z.

A remarkable feature about the Cauchy integral formula is that, in the calculation of the contour integral, we only require the value of the function on the circle C. The fraction $1/(w-z)$ acts as a weighting factor. By using different weighting factors, we can determine the value of function f at any point z inside the circle. Hence, the value of f within the circle can be completely determined by its values on the boundary.

The proof of Theorem 17.1 relies on a theorem commonly known as the "deformation theorem." The following statement represents a particular case of the general deformation theorem, which is adequate for deriving the Cauchy integral formula for circle.

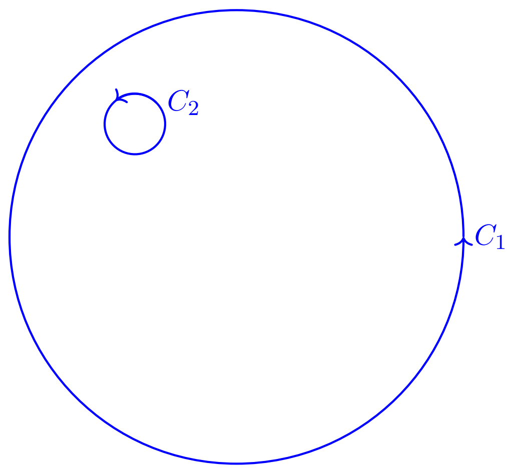


Figure 26: The two circular paths in Theorem 17.2.

Theorem 17.2 (Deformation theorem for circles)**.** *Let C_1 and C_2 be two circular contours with the same orientation, such that C_2 is contained inside C_1 (See Fig. 26). Suppose $f(z)$ is holomorphic at all points that are inside C_1 and on the curve C_1, with the possible exception at the center of circle C_2, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Proof. Since $f(z)$ is holomorphic at every point on the outer circle C_1, there exists a disc slightly larger than C_1 in which $f(z)$ is holomorphic, except possibly at the center of C_2.

We divide the region between the circles into four parts by drawing a horizontal and a vertical line passing through the centers of the inner circle C_2 (See Fig. 27). Each closed path is contained within a convex set where the function $f(z)$ is holomorphic. Consequently, the integral of $f(z)$ over each closed path is equal to zero.

On the other hand, the sum of the integrals over the four closed paths is equivalent to the difference between the integrals over C_1 and C_2. This demonstrates that the integrals of $f(z)$ over C_1 and C_2 are equal. $\square$

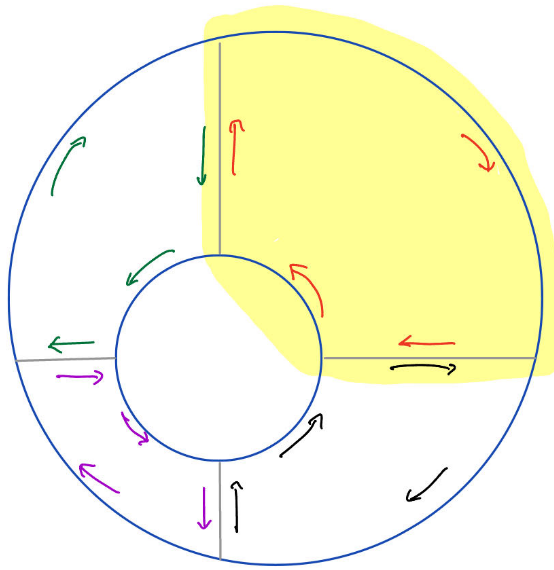


Figure 27: Proof of Theorem 17.2.

Proof of Cauchy integral formula. Consider an arbitrary point z within the interior of the circle C. Draw another circle, denoted as C', centered at z with a radius ρ. Choose the radius ρ such that C' lies entirely inside C (see Fig. 28). The circular contour C' is traversed in a counterclockwise direction.

The fraction $f(w)/(w-z)$ is holomorphic at every point on the curves C and C' as well as the points enclosed between them. We can apply Theorem 17.2 to obtain the following equality,

$$\int_C \frac{f(w)}{w-z}\,dw = \int_{C'} \frac{f(w)}{w-z}\,dw.$$

Hence, it is sufficient to prove

$$\int_{C'} \frac{f(w)}{w-z}\,dw = 2\pi i f(z). \tag{17.2}$$

We start by writing

$$f(w) = f(z) + [f(w) - f(z)]$$

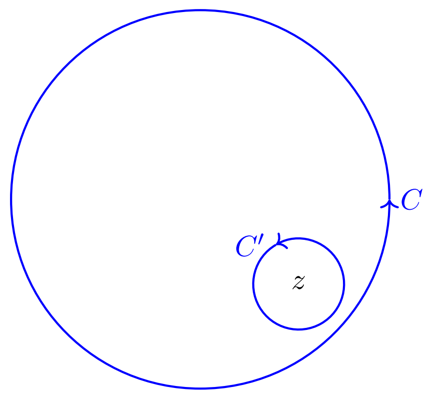


Figure 28: Proof of Cauchy integral formula.

and then decompose the integral in (17.2) into two parts:

$$\int_{C'} \frac{f(w)}{w-z}\,dw = \int_{C'} \frac{f(z)}{w-z}\,dw + \int_{C'} \frac{f(w)-f(z)}{w-z}\,dw. \tag{17.3}$$

To evaluate the first integral on the right-hand side of (17.3), we can employ the same calculations as in Example 14.2. We parameterize C' as $z+\rho e^{i\theta}$, where θ ranges from 0 to 2π. Then we have

$$\int_{C'} \frac{f(z)}{w-z}\,dw = f(z)\int_{C'} \frac{1}{w-z}\,dw = f(z)\int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\,d\theta = 2\pi i f(z).$$

To simplify notation, we introduce $g(w) \triangleq f(w) - f(z)$. On the circle C', the modulus $|g(w)|$ can be expressed as

$$|g(z+\rho e^{i\theta})|$$

for θ in the interval $[0, 2\pi]$. Since $[0, 2\pi]$ is a closed and bounded interval, and $|g(z+\rho e^{i\theta})|$ is a continuous function of θ, we can conclude that a maximum exists and is finite. We define

$$M_\rho \triangleq \sup_{\theta\in[0,2\pi]} |g(z+\rho e^{i\theta})| = \max_{w\in C'} |g(w)|.$$

By ML inequality (Theorem 15.4), we can upper bound the modulus of the integral

$$\int_{C'} \frac{g(w)}{w-z}\,dw.$$

by

$$\Big|\int_{C'} \frac{g(w)}{w-z}\,dw\Big| \le \frac{M_\rho}{\rho} 2\pi\rho = 2\pi M_\rho.$$

Since the function g is continuous, M_ρ approaches 0 as ρ approaches 0.The integral in (17.1) is thus equal to 0. This proves that the right-hand side of (17.3) is equal to $2\pi i f(z)$. $\square$

Example 17.1. Calculate $\int_C \frac{e^z}{z^2+1}\,dz$ when C is the contour (i) $|z-i|=1$, (ii) $|z+i|=1$, (iii) $|z-2|=1$, (iv) $|z|=4$, all with positive orientation.

The denominator can be factorized as

$$z^2+1=(z+i)(z-i).$$

The function $e^z/(z^2+1)$ is not holomorophic at the points i and $-i$.

(i) Let the circle centered at i with radius 1 and positive orientation be denoted by C_1. It encloses the point $z=i$ but not $z=-i$.

$$\int_{C_1} \frac{e^z}{z^2+1}\,dz = \int_{C_1} \frac{\frac{e^z}{z+i}}{z-i}\,dz = 2\pi i\Big(\frac{e^z}{z+i}\Big)\Big|_{z=i} = \pi e^i.$$

(ii) Let C_2 be the circular contour centered at $-i$ with radius 1 and positive orientation. It encloses the point $z=-i$ but not $z=i$.

$$\int_{C_2} \frac{e^z}{z^2+1}\,dz = \int_{C_2} \frac{\frac{e^z}{z-i}}{z+i}\,dz = 2\pi i\Big(\frac{e^z}{z-i}\Big)\Big|_{z=-i} = -\pi e^{-i}.$$

(iii) The circle centered at $z=2$ with radius 1 does not contain the singular points $\pm i$, and hence is holomorphic inside and on the boundary of the circle. By Cauchy theorem, the integral is equal to 0.

(iv) If we integrate along the circle centered at the origin with radius 4 counter-clockwise, the answer is the sum of the integrals in parts (i) and (ii),

$$\pi e^i - \pi e^{-i} = \pi(e^i - e^{-i}) = 2\pi i \sin(1).$$

17.2 Liouville theorem

Using the Cauchy estimate, we can establish the Liouville theorem. Recall that an *entire* function is a complex function that is holomorphic at every point in the complex plane. We can motivate the Liouville theorem by first looking at some examples.

(1) The functions e^z, $\sin(z)$ and $\cos(z)$ are entire but not bounded. It is important to note that the function $\sin(z)$ and $\cos(z)$ are *not* bounded, because $|\sin(z)|$ approaches infinity when we move along the vertical line $x + iy$, with a constant x and $y \to \infty$.

(2) The function $e^{-|z|}$ defined for all $z \in \mathbb{C}$ is bounded, but is not entire.

(3) The inverse function $1/z$ is unbounded, and is not defined at the origin.

Theorem 17.3 (Liouville theorem)**.** *An entire and bounded function must be a constant function.*

Proof. Suppose $f(z)$ is entire and there exists a real number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We want to show that $f(z_1) = f(z_2)$ for any complex numbers z_1 and z_2.

The idea of proof is to draw a large circle centered at the origin. The radius R is selected such that

$$R \geq 2 \max(|z_1|, |z_2|).$$

(The exact value of the factor 2 is not very important. We can replace it by any real number larger than 1.)

For any complex number w on the circle with radius R and centered at the origin, we have

$$R = |w| = |w - z_k + z_k| \leq |w - z_k| + |z_k| \leq |w - z_k| + R/2.$$

for $k = 1, 2$. Hence,

$$|w - z_1| \geq \frac{R}{2} \text{ and } |w - z_2| \geq \frac{R}{2} \tag{17.4}$$

for all w on the circle $|w| = R$. Apply Cauchy integral formula to express $f(z_1)$ and $f(z_2)$ as integrals over the circle, and write the difference as

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w - z_1} - \frac{f(w)}{w - z_2} \, dw.$$

Simplify the integrand on the right to

$$f(z_1) - f(z_2) = \frac{z_1 - z_2}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w - z_1)(w - z_2)} \, dw.$$

By the ML inequality, and the two assumptions in (17.4), we get

$$|f(z_1) - f(z_2)| \leq \frac{|z_1 - z_2|}{2\pi} \frac{M}{(R/2)^2} \cdot 2\pi R = O(\frac{1}{R}).$$

Because $f(z)$ is entire, we can enlarge the circle $|w| = R$ by increasing the radius R, and the above argument is still valid. Because R can be arbitrarily large, we see that $|f(z_1) - f(z_2)|$ must be equal to 0. $\square$

17.3 Fundamental theorem of algebra

We have a quick proof of the fundamental theorem of algebra using Liouville theorem.

Theorem 17.4 (Fundamental theorem of algebra)**.** *A polynomial with complex coefficient of degree at least 1 has a root in* $\mathbb{C}$.

Proof. Suppose $p(z)$ is a polynomial

$$c_0 + c_1 z + c_2 z^2 + \cdots c_d z^d$$

with degree $d \geq 1$. Without loss of generality, we can assume that $c_d = 1$.

We will prove this by contradiction. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ is a well-defined and holomorphic for all $z \in \mathbb{C}$, i.e., $1/p(z)$ is an entire function.

Consider the ratio

$$\frac{p(z)}{z^d} = \frac{c_0}{z^d} + \frac{c_1}{z^{d-1}} + \cdots + 1.$$

As $|z| \to \infty$, this ratio approaches 1. Hence, there exists a sufficiently large real number R such that

$$\Big|\frac{p(z)}{z^d}\Big| > \frac{1}{2}$$

for all z with modulus larger than R. In other words, we have

$$\Big|\frac{1}{p(z)}\Big| < \frac{2}{|z^d|} < \frac{2}{R^d}$$

for all $|z| > R$.

On the other hand, for $|z| \leq R$, the modulus $|1/p(z)|$ is bounded, because $1/|p(z)|$ is a continuous function of z and $|z| \leq R$ is a compact set.

As a result, $|1/p(z)|$ is bounded by a constant for all $z \in \mathbb{C}$. By Liouville theorem, $1/p(z)$ is a constant, contradicting that the assumption that the degree of $p(z)$ is at least 1. $\square$

17.4 Appendix: Cauchy-Green formula / Pompeiu formula

There is a version of Green theorem that only assume that the real $u(x, y)$ and imaginary parts $v(x, y)$ of the complex function $f(z) = u(x, y)+iv(x, y)$ are continuously differentiable, without assuming that $f(z)$ is complex differentiable. This theorem is also called *Pompeiu formula*.

Theorem 17.5. *Let R be a bounded region whose boundary is a smooth closed curve C. Let $f(z) = u(x, y) + iv(x, y)$ be a complex function, with $u(x, y)$ and $v(x, y)$ both in C^1. We have the following formula*

$$\begin{aligned} 2\pi i f(z) &= \oint_C \frac{f(w)}{w-z}\,dw - \iint_R \frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}\,d\bar{z} \wedge dz \\ &= \oint_C \frac{f(w)}{w-z}\,dw - 2i \iint_R \frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}\,dA. \end{aligned}$$

The first term is the Cauchy integral formula. The double integral in the first line is written using the differential 2-form $d\bar{z} \wedge dz$. The partial derivative respective to $\bar{z}$ is the del-bar operator (See Definition 10.5). We can transform the double integral with respect to the differential 2-form $dx \wedge dy$, and obtain the alternate form in the second line in the theorem.

This formula describes what would happen if we do not assume that the function $f(z)$ is not holomorphic in the region R. The second term vanishes if $\bar{\partial} f$ is identically zero, which happens precisely when f is a holomorphic function. We note that this formula can be proved using multivariable calculus, without invoking Cauchy-Goursat theorem. A proof of this formula can be found in Chapter 2 of [Gong].

18 Taylor series expansion

Summary

- Power series, radius of convergence, and term-wise differentiation
- Power series expansion of holomorphic function
- Cauchy integral formula for higher-order derivatives
- Uniqueness of power series expansion

18.1 Convergence of power series

We study power series in more details in the remainder of this lecture. For ease of notation, we assume that the center of the power series is the origin. All results below hold in the general case when the center is not the origin.

Proposition 18.1. *Suppose $\sum_{k=0}^{\infty} a_k z^k$ converges at $z_0 \neq 0$, then it converges absolutely for all z with $|z| < |z_0|$.*

Proof. By the n-th term test (Prop. 3.10) we know that $|a_k z_0^k| \to 0$ as $k \to \infty$. Fix $\epsilon > 0$. there is a sufficiently large N such that

$$|a_k z_0^k| < \epsilon, \qquad \forall k \geq N.$$

Hence, for all $k \geq 0$, we have

$$|a_k z_0^k| \leq \max(\epsilon, |a_0|, |a_1 z_0|, |a_2 z_0^2|, \ldots, |a_{N-1} z_0^{N-1}|) \triangleq C.$$

Let z be a complex number with $|z| < |z_0|$. Let r be the ratio $|z|/|z_0|$. For each k, re-write $|a_k z^k|$ as

$$|a_k z^k| = |a_k z_0^k| \frac{|z|^k}{|z_0|^k} \leq C r^k,$$

Because $r < 1$ and $\sum_k C r^k$ is a convergent geometric series, we can apply comparison test and conclude that $\sum_k a_k z^k$ is convergent absolutely for $|z| < |z_0|$ (See Prop. 3.11). $\square$

By taking the contrapositive, we can re-write Prop. 18.1 as

Proposition 18.2. *If $\sum_k a_k z^k$ diverges at z_1, then $\sum_k a_k z^k$ diverges whenever $|z| > |z_1|$.*

Proof. We prove by contradiction. Suppose $\sum_k a_k z^k$ converges for some point $z = z_2$ with modulus $|z_2|$ strictly larger than $|z_1|$. By the previous proposition, the power series must converge (absolutely) at $z = z_1$. This contradicts the assumption that $\sum_k a_k z^k$ diverges at z_1. $\square$

In view of the previous two proposition, we can make the following definition.

Definition 18.3. Given a power series $\sum_{k=0}^{\infty} a_k z^k$, let

$$R \triangleq \sup\Big\{|z| : \sum_k a_k z^k \text{ converges}\Big\}. \tag{18.1}$$

The value of R in (18.1) is called the *radius of convergence* of $\sum_{k=0}^{\infty} a_k z^k$. The region $|z| < R$ is called the *region of convergence*. If $\sum_k a_k z^k$ converges for all z, then we write $R = \infty$, and say that the radius of convergence is infinite. If $\sum_k a_k z^k$ converges only at $z = 0$, then $R = 0$.

Example 18.1. The power series that define the complex functions $\exp(z)$, $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$ have radius of convergence $R = \infty$.

Example 18.2. If we expand the function $1/(1+z^2)$ as a power series, the resulting power series is

$$1 - z^2 + z^4 - z^6 + \cdots$$

The radius of convergence is 1.

An explicit expression for calculating the radius of convergence is given by the Hadamard formula.

Theorem 18.4 (Hadamard formula for radius of convergence)**.** *Given a complex power series* $\sum_{k=0}^{\infty} a_k z^k$*, the radius of convergence can be computed by*

$$R = \frac{1}{\limsup_k |a_k|^{1/k}}.$$

Proof. Suppose z is a complex number with $|z| < R$. Pick a real number R_1 between $|z|$ and R, i.e., $|z| < R_1 < R$. By the definition of R in the theorem,

$$\frac{1}{R_1} > \limsup |a_k|^{1/k}.$$

Using the property of limsup, there exists a sufficiently large N such that for all $k \geq N$,

$$|a_k|^{1/k} < 1/R_1 \qquad \Rightarrow \qquad |a_k| < 1/R_1^k.$$

This yields

$$|a_k z^k| < \frac{|z|^k}{R_1^k}.$$

Since $|z|^k/R_1^k < 1$, the geometric series $\sum_k |z|^k/R_1^k$ is convergent. By comparison test, $\sum_k a_k z^k$ is convergent.

Now suppose that $|z| > R$. Pick a real number R_2 such that $|z| > R_2 > R$. Taking reciprocal,

$$\frac{1}{R_2} < \limsup |a_k|^{1/k}.$$

By the defining property of limsup,

$$\frac{1}{R_2} < |a_k|^{1/k} \quad \text{for infinitely many } k.$$

Hence $|a_k z^k| > \left|\frac{z}{R_2}\right|^k > 1$ for infinitely many k. By the nth-term test (Prop. 3.10)), the power series diverges whenever $|z| > R$. □

We recall that by *uniform convergence*, we mean that for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ that depends only on ϵ but not on the value of z, such that

$$\left|\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{\infty} a_k z^k\right| < \epsilon$$

for all $n \geq N(\epsilon)$ and for all $z \in S$. The number $N(\epsilon)$ should be a function of ϵ only independent of z. The following theorem gives a sufficient condition for uniform convergence. The proof idea is basically Weierstrass's M-test.

Theorem 18.5. *Suppose a complex power series $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely at $z = z_0$, then $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly in the closed disc $S = \{z : |z| \leq |z_0|\}$.*

Proof. We first note that the series $\sum_k a_k z^k$ converges absolutely for all $z \in S$. It is because

$$\sum_{k=0}^{\infty} |a_k||z|^k \leq \sum_{k=0}^{\infty} |a_k||z_0|^k,$$

and the series on the right converges by assumption.

For $k \geq 0$, let $M_k \triangleq |a_k||z_0|^k$. By assumption, $\sum_{k=0}^{\infty} M_k$ converges. Given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\sum_{k=n+1}^{\infty} M_k \leq \epsilon, \qquad \text{for all } n \geq N(\epsilon).$$

So, for this choice of $N(\epsilon)$, we have

$$\Big|\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{\infty} a_k z^k\Big| = \Big|\sum_{k=n+1}^{\infty} a_k z^k\Big| \leq \sum_{k=n+1}^{\infty} |a_k||z|^k \leq \sum_{k=n+1}^{\infty} M_k \leq \epsilon$$

for all $z \in S$. This proves that $\sum_k a_k z^k$ is uniformly convergent to $\sum_{k=0}^{\infty} a_k z^k$ for $z \in S$. $\square$

18.2 Differentiation of power series

Using the property of uniform convergence, we can prove that the function $\sum_{k=0}^{\infty} a_k z^k$ is complex differentiable within the region of convergence.

Theorem 18.6 (Term-wise differentiation of power series)**.** *Suppose power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$.*

1. *The series $\sum_{k=1}^{\infty} k a_k z^{k-1}$ obtained by term-wise differentiation has the same radius of convergence R.*

2. *The function defined by $\sum_{k=0}^{\infty} a_k z^k$ is complex differentiable at every z with $|z| < R$, and the complex derivative is given by $\sum_{k=1}^{\infty} k a_k z^{k-1}$.*

Proof. We first use the Hadamard formula in Theorem 18.4 to show that $\sum_k k a_k z^{k-1}$ has radius of convergence equal to R. We note that the radius of convergence of $\sum_k k a_k z^{k-1}$ is the same as the radius of convergence of $\sum_k k a_k z^k$, and the radius of convergence of the latter series can be computed by

$$\frac{1}{\limsup_k |k a_k|^{1/k}} = \frac{1}{\limsup_k \sqrt[k]{k} |a_k|^{1/k}} = \frac{1}{\limsup_k |a_k|^{1/k}} = R.$$

In the second equality, we have used the fact that $\lim_{k\to\infty} \sqrt[k]{k} = 1$.

To prove the second part, we pick a complex number z_0 inside the region of convergence, i.e., $|z_0| < R$. Let δ be a real number such that $|z_0| + \delta < R$. The choice of δ is made to ensure that the open disc $\{z : |z - z_0| < \delta\}$ lies inside the region of convergence.

We set up some notation. For $n \geq 0$, let $f_n(z)$ denote the n-th partial sum

$$f_n(z) \triangleq \sum_{k=0}^{n} a_k z^k,$$

and let $f(z)$ denote the limit

$$f(z) \triangleq \sum_{k=0}^{\infty} a_k z^k.$$

By Theorem 18.5, $f_n(z)$ converges to $f(z)$ uniformly in the closed disk $S = \{z : |z| \leq |z_0| + \delta\}$.

By part 1 and Theorem 18.5, the series $\sum_{k=1}^{\infty} k a_k z^{k-1}$ converges absolutely and uniformly in the closed disk S.

For each $n \geq 0$, define

$$\varphi_n(h; z_0) \triangleq \frac{f_n(z_0 + h) - f_n(z_0)}{h} = \sum_{k=0}^{n} a_k \frac{(z_0 + h)^k - z_0^k}{h} \tag{18.2}$$

for nonzero h with $|h| \leq \delta$. When $n \to \infty$, $\varphi_n(h; z_0)$ converges to

$$\frac{f(z_0 + h) - f(z_0)}{h}.$$

The convergence is uniform for $|h| \leq \delta$. Indeed, we can use the identity

$$\alpha^k - \beta^k = (\alpha - \beta)(\alpha^{k-1} + \alpha^{k-2}\beta + \alpha^{k-3}\beta^2 + \cdots + \alpha\beta^{k-2} + \beta^{k-1}),$$

to upper bound the k-th term in the summation in (18.2),

$$\begin{aligned}
\left| a_k \frac{(z_0 + h)^k - z_0^k}{h} \right| &= |a_k| \frac{|(z_0 + h)^k - z_0^k|}{|h|} \\
&= |a_k| \Big(\sum_{j=0}^{k-1} |z_0 + h|^j |z_0|^{k-1-j} \Big) \\
&\leq |a_k| k (|z_0| + \delta)^{k-1}.
\end{aligned}$$

Because the series $\sum_{k=1}^{\infty} |a_k| k (|z_0| + \delta)^{k-1}$ is convergent and does not depend on the value of h, we can see that the convergence $\varphi(h; z_0)$ is uniform over h.

Consider the double limit

$$\lim_{n \to \infty} \Big(\lim_{h \to 0} \varphi_n(h; z_0) \Big).$$

For each n, $f_n(z)$ is a polynomial. Hence $\lim_{h \to 0} \varphi_n(h; z_0) = \sum_{k=1}^{n} k a_k z_0^{k-1}$. This gives

$$\lim_{n \to \infty} \Big(\lim_{h \to 0} \varphi_n(h; z_0) \Big) = \lim_{n \to \infty} \sum_{k=1}^{n} k a_k z_0^{k-1} = \sum_{k=1}^{\infty} k a_k z_0^{k-1}. \tag{18.3}$$

(Convergence of the last summation is justified by part 1.)

On the other hand, since φ_n converges uniformly for $|h| \leq \delta$, we can exchange the order of the two limits to get

$$\lim_{h \to 0} \Big(\lim_{n \to \infty} \varphi_n(h; z_0) \Big) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{18.4}$$

Since (18.3) and (18.4) are equal, the limit in (18.4) exists and is equal to the power series in (18.3). We thus obtain

$$f'(z_0) = \sum_{k=1}^{\infty} k a_k z_0^{k-1}.$$

This proves that a power series is complex differentiable, and we can differentiate it termwise. $\square$

By repeatedly applying Theorem 18.6, we can differentiate a power series arbitrarily many times.

Theorem 18.7. *Suppose the radius of convergence of power series $\sum_{k=0}^{\infty} a_k z^k$ is positive. For any positive integer j, the function $f(z)$ defined by a power series $\sum_{k=0}^{\infty} a_k z^k$ can be differentiated j-th times at any point in the interior of the region of convergence, and the j-th derivative can be computed by the power series*

$$f^{(j)}(z) = \sum_{k=j}^{\infty} k(k-1)(k-2)\cdots(k-j+1) a_k z^{k-j}.$$

In real analysis, we have example of infinitely differentiable functions that does not have power series expansion. The following real continuous function

$$f(x) \triangleq \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x^4} & \text{if } x > 0 \end{cases}$$

is infinitely smooth. However, the derivatives of all orders at $x = 0$ are all equal to zero. The power series expansion of $f(x)$ at $x = 0$ is the constant function 0, which is not the same as the original function $f(x)$. This function fails to have power series expansion at $x = 0$.

The main result in the next section says that we do not have this kind of example in the complex case.

18.3 Existence of power series expansion for holomorphic functions

We have shown in the previous section that a power series is holomorphic. It turns out that the converse is also true, and it is the next objective.

Definition 18.8. A complex function is called *analytic at a point z_0* if we can draw a small circle centered at z_0 such that the function in this circle is the same as a power series centered at z_0. We say that a function is *analytic* if it is analytic at any point in the domain of definition.

The next theorem guarantees that if a complex function $f(z)$ is holomorphic at z_0, then $f(z)$ is analytic at z_0.

Theorem 18.9 (Taylor expansion)**.** *Suppose $f(z)$ is a complex-valued function that is complex differentiable in a domain that contains the closed disc $|z - z_0| \le r$. Then $f(z)$ has Taylor series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in the open disc $|z - z_0| < r$. Moreover, for $n = 0, 1, 2, 3, \ldots$, the coefficient a_n is given by

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(w)}{(w - z_0)^{n+1}}\, dw. \tag{18.5}$$

Recall that when we say that a function $f(z)$ is holomorphic at a point z_0, it means that $f(z)$ is complex differentiable in an open disc $|z - z_0| < \rho$, for some positive radius ρ. Thus, the above theorem applies to every point at which the function is holomorphic.

A heuristic argument is as follows. Consider a contour C defined as $|z - z_0| = r$ with a counter-clockwise orientation. By Cauchy integral formula (Theorem 17.1), for any complex number z inside the contour C, we can express $f(z)$ as

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw.$$

The proof idea of this theorem is to expand $1/(w - z)$ using a geometric series,

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w - z_0 - (z - z_0)} \\ &= \frac{1}{w - z_0}\Big(\frac{1}{1 - \frac{z - z_0}{w - z_0}}\Big) \\ &= \frac{1}{w - z_0} + \frac{z - z_0}{(w - z_0)^2} + \frac{(z - z_0)^2}{(w - z_0)^3} + \frac{(z - z_0)^3}{(w - z_0)^4} + \cdots \end{aligned} \tag{18.6}$$

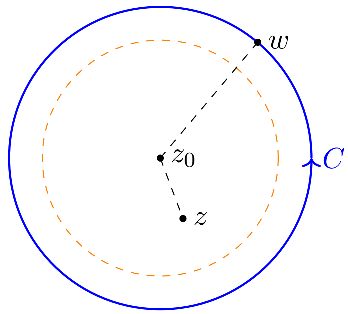


Figure 29: Proof of Taylor expansion of holomorphic function.

This geometric series converges because $|z - z_0| < |w - z_0|$. If we can justify the exchange of infinite summation and contour integral, then we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} f(w) \frac{(z-z_0)^n}{(w-z_0)^{n+1}}\, dw \\ &= \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \oint_C \frac{f(w)\, dw}{(w-z_0)^{n+1}} \Big) (z-z_0)^n. \end{aligned}$$

This is a Taylor series centered at z_0.

Proof. To make the above argument rigorous, we can utilize the property of uniform convergence. Let ρ be a real number in the range $0 < \rho < r$. Consider a complex number z that lies inside the circle $|z - z_0| < \rho$ (See Fig. 29).

For each positive integer n, consider the partial sum

$$\sum_{k=0}^{n} a_k (z-z_0)^k,$$

where a_k is defined in (18.5). We want to show that it converges to $f(z)$ as $n \to \infty$, for $|z - z_0| < \rho$.

Because $|(z - z_0)/(w - z_0)| < 1$, we can use geometric series to simply the infinite sum

$$\sum_{k=n+1}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}} = \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+2}} \frac{1}{1 - \frac{z-z_0}{w-z_0}} = \frac{1}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}.$$

We can thus write $1/(w - z)$ as

$$\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \cdots + \frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \frac{1}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}.$$

Multiply each term by $f(w)$ and integrate over the curve C. (Since this is a finite sum, there is no problem in exchanging finite summation and integration.)

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = \sum_{k=0}^{n}\Big(\frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z_0)^{k+1}}\Big)(z-z_0)^k + \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)}\frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\,dw.$$

Using the definition of coefficients a_n in the theorem, we can write the right-hand side of the above equation as

$$\sum_{k=0}^{n} a_k(z-z_0)^k + \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)}\frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\,dw.$$

This holds for any $n \geq 0$. The summation is a partial sum of the desired Taylor series expansion. The integral on the right is the corresponding error term.

To bound the error term, we observe that

$$\begin{aligned} |z-z_0| &< \rho \\ |w-z| &\geq r - |z-z_0| \\ |w-z_0| &= r. \end{aligned}$$

Because $|f(w)|$ is a continuous function, the value of $|f(w)|$ on the circle C is bounded by some constant M. We now apply the ML inequality (Theorem 15.4),

$$\Big|\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)}\frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\,dw\Big| \leq \frac{1}{2\pi}\frac{M}{r-|z-z_0|}\frac{\rho^{n+1}}{r^{n+1}}\cdot 2\pi r = \frac{M\rho}{r-|z-z_0|}\Big(\frac{\rho}{r}\Big)^n.$$

Take $n \to \infty$, the modulus of the remainder term approach 0. (This step depends on the assumption that $\rho < r$). Consequently, the Taylor series is convergent at z, and converges to the intended value $f(z)$. □

Because a convergent power series is differentiable, we have proved a key feature of holomorphic functions

Theorem 18.10. *A complex function is holomorphic in a domain if and only if it is analytic. Moreover, a holomorphic is infinitely differentiable, i.e., the n-th order derivative exists for all positive integer n.*

Because of this connection, in the remainder of this lecture notes we will use the two terms "holomorphic" and "analytic" interchangeably.

We summarize what we know about holomorphic function below. Let $f(z)$ denote a continuous complex-valued function in a domain D. Then the followings are equivalent:

(a) $f(z)$ is holomorphic in D, i.e., at each point $z_0 \in D$, the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists.

(b) $f(z)$ is real-differentiable as a multi-variable function, and satisfies the Cauchy-Riemann equations

(c) $f(z)$ is analytic in D, i.e., at each point $z_0 \in D$, $f(z)$ can be expanded locally as a power series centered at z_0.

Theorem 18.5 also sheds light on the radius of convergence. The radius of convergence of a function $f(z)$ at the center point z_0 is the largest radius r such that there is no singularity within the open disc $|z - z_0| < r$.

Example 18.3. Consider the function $f(z) = \frac{1}{z(z-2)}$. It is complex differentiable everywhere except two points, namely 0 and 2. Suppose we want to find the Taylor series function at the point i. We can draw a circle with radius 1, centered at i, so that $f(z)$ is holomorphic inside the circle. This is the largest circle that does not contain any of the two problematic points 0 and 2. By Theorem 18.5, the power series expansion of $f(z)$ with center i has radius convergence equal to 1.

Now, suppose we change the center to $1 + i$. The distance from $1 + i$ to the two points 0 and 2 are both $\sqrt{2}$. Hence, the radius of convergence of the power series with center $1 + i$ is $\sqrt{2}$.

18.4 Uniqueness of Taylor series

The identity theorem is another surprising result in complex analysis. It says that if a holomorphic function is identically zero in a small open disk, then the function must be identically zero throughout the domain. This is related to the uniqueness of the coefficients of Taylor series. We first recall the notion of accumulation point.

Definition 18.11. An *accumulation point* (a.k.a. *cluster point* or *limit point*), of a set S of points is a point p such that every neighborhood of p contains at least one point of S not equal to p.

It is known that in a point p in a metric space is an accumulation point of S if and only if every neighborhood of p contains infinitely many points of S.

Theorem 18.12 (Uniqueness of coefficients in Taylor series)**.** *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

are functions defined by power series, both centered at z_0, and both of them have positive radii of convergence. If we can find a set S of points in the common region of convergence, such that $f(w) = g(w)$ for all $w \in S$ and z_0 is an accumulation point of S, then we must have $a_k = b_k$ for all k.

Proof. From the hypothesis in the theorem, we can find a sequence $(z_j)_{j=1}^{\infty}$ in S that converges to z_0. For example, we can let z_j to be a point in S with $0 < |z_j - z_0| < 1/j$. By construction, this sequence converges to z_0, and $z_j \neq z_0$ for all j.

Let $h(z)$ be the difference of the two functions $f(z)$ and $g(z)$. By assumption we have $h(z_j) = 0$ for all $j = 1, 2, 3, \ldots$. Since $h(z)$ is holomorphic at z_0, we can write $h(z)$ has a power series centered at z_0,

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n,$$

where $c_n = a_n - b_n$ is the coefficient of the n-th term.

We claim that $c_n = 0$ for all n.

Suppose the claim is false and suppose there is a nonzero coefficient. Let m be the smallest integer such that $c_m \neq 0$. If $m = 0$, i.e., $c_0 \neq 0$, then by the continuity of power series, the value of $h(z)$ is nonzero in a small neighborhood of z_0, contradicting the hypothesis that there exists a sequence of points z_j converging to z_0 and $h(z_j) = 0$ for all j.

Suppose $m \geq 1$. We can write $h(z)$ as

$$h(z) = (z-z_0)^m \big[c_m + c_{m+1}(z-z_0) + c_{m+2}(z-z_0)^2 + \cdots\big].$$

Because $c_m \neq 0$ by assumption, the power series inside the square bracket is nonzero for z in a sufficiently small neighborhood of z_0. The first factor $(z - z_0)^m$ is nonzero whenever $z \neq z_0$. As a result, we see that $h(z)$ is nonzero in a punctured disc centered at z_0. This yields the same contradiction as in the previous paragraph.

Therefore the claim is true, and we must have $c_n = 0$ for all n. This proves that $a_n = b_n$ for all n. □

18.5 Application to generating functions

The method of generating function is a powerful way to solve difference equation and linear recurrence relation. This is a topic in combinatorics, computer science and discrete mathematics. In general, a linear recurrence relation of order k is a recursive definition in the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, \ldots, c_k$ are constants. If k initial values a_0, $a_1, \ldots, a_{k-1}$ are given, we can use the above equation to define an infinite sequence. The question is to study this sequence, e.g. by giving some asymptotic estimates of the numbers when n is large. In signal processing, it is the same as z-transform.

The method of generating function is to consider a power series, whose coefficients are the numbers in the sequence $(a_k)_{k=1}^{\infty}$,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Provided that the sequence does not increase too fast (say no faster than exponential growth), then we can study the associated complex-valued function $f(z)$ and try to obtain information about the sequence $(a_k)_{k=0}^{\infty}$ from the function.

In the following we consider the standard example: Fibonacci sequence. This example contains all the ingredient in the method of generating function. The Fibonacci numbers are recursively defined by $F_0 = F_1 = 1$, and

$$F_k = F_{k-1} + F_{k-2}$$

for $k \geq 2$. The first few numbers in this sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Let

$$f(z) = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + F_5 z^5 + \cdots$$
$$= 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \cdots$$

The k-th coefficient (for $k = 0, 1, 2, \ldots$) is the k-th Fibonacci number. The power series has positive radius of convergence, because the k-th Fibonacci number is upper bounded by 2^k. (This can be proved by mathematical induction) Hence, $f(z)$ has a non-trivial region of convergence with radius at least 1/2.

By comparing $f(z)$, $zf(z)$ and $z^2 f(z)$, we can see that the function $f(z)$ satisfies the following equation

$$f(z) - zf(z) - z^2 f(z) = 1.$$

In the region of convergence, $f(z)$ is equal to a rational function in z:

$$f(z) = \frac{1}{1 - z - z^2}.$$

We will write the function $f(z)$ in a more convenient form,

$$f(z) = \frac{1}{(1 - \alpha z)(1 - \beta z)}$$

where α and β are the reciprocal root of $1 - z - z^2$. To this end, we consider the *reciprocal polynomial*, which is commonly known as the *characteristic polynomial*:

$$P(u) = u^2 - u - 1,$$

whose coefficients are the same as in $f(z)$ but in reverse order. In this example, one of the roots is the famous Golden ratio. We denote the two roots by

$$\varphi_1 = \frac{1 + \sqrt{5}}{2}, \qquad \varphi_2 = \frac{1 - \sqrt{5}}{2}.$$

Hence, we have

$$f(z) = \frac{1}{(1 - \varphi_1 z)(1 - \varphi_2 z)}.$$

We can apply partial fraction expansion, and expand the two resulting fraction by geometric series,

$$f(z) = \frac{\varphi_1/\sqrt{5}}{1 - \varphi_1 z} - \frac{\varphi_2/\sqrt{5}}{1 - \varphi_2 z}$$
$$= \sum_{k=0}^{\infty} \Big(\frac{\varphi_1^{k+1}}{\sqrt{5}} - \frac{\varphi_2^{k+1}}{\sqrt{5}} \Big) z^k.$$

By the uniqueness of power series expansion, the k-th Fibonacci number is equal to

$$F_k = \frac{\varphi_1^{k+1}}{\sqrt{5}} - \frac{\varphi_2^{k+1}}{\sqrt{5}}.$$

Because the Fibonacci numbers are integers, and the absolute value of φ_2 is less than 1, we can further simplify the expression to

$$F_k = round\Big(\frac{\varphi_1^{k+1}}{\sqrt{5}}\Big)$$

where $round()$ is the function that rounds a real number to the nearest integer.

We thus see that Fibonacci grows exponentially fast, and asymptotically the ratio between two consecutive numbers is the Golden ratio.

In general, the asymptotic growth rate of a sequence defined by linear recurrence relation is controlled by the largest root of the characteristic function.

19 Identity theorem

Summary

- Cauchy integral formula for higher-order derivatives
- Cauchy estimate
- Identity theorem

The fact that a holomorphic function has power series expansion has many surprising consequences. The scenario is very different from real-differentiable function.

19.1 Cauchy integral formula for higher-order derivatives

At this point, we know that the derivative of a holomorphic function of any order exists. There is a formula for the higher-order derivatives that is analogous to the original Cauchy integral formula.

Theorem 19.1. *If a function f is holomorphic in a domain D, then f is infinitely differentiable. The n-th derivative at $z = z_0$ can be evaluated by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw \tag{19.1}$$

where C is a circle centered at z_0 such that $f(z)$ is holomorphic at every point inside and on the circle C. In particular, we have

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^2}\, dw.$$

Remark. In some books, the formula in (19.1) is proved directly from Cauchy theorem, without mentioning power series. In this notes, we derive it as an easy corollary of Taylor series expansion. The difficult part has already been done in the proof of Theorem 18.9.

Proof. From Theorem 18.9 we know that the Taylor series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \tag{19.2}$$

converges in a neighborhood of $z = z_0$, with coefficient a_n equal to

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}} \, dw.$$

If we substituting $z = z_0$ into (19.2), we get

$$f(z_0) = a_0 = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z_0} \, dw,$$

which is the same as Cauchy's integral formula in Theorem 17.1. Since Taylor series can be differentiated term-wise, we can differentiate both sides of (19.2) and substitute $z = z_0$ to get

$$f'(z_0) = a_1 = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2} \, dw.$$

If we differentiate n times and substitute $z = z_0$, we obtain (19.1). □

Based on (19.1) one can estimate the modulus of the derivative of a holomorphic function. This is commonly known as the Cauchy estimate for derivatives

Corollary 19.2 (Cauchy estimate)**.** *If $|f(z)| \le M$ for z on the circle*

$$\{z : \, |z - z_0| = r\}$$

and f is holomorphic in a neighborhood of the disc $|z - z_0| \le r$, then

$$|f^{(n)}(z_0)| \le M \frac{n!}{r^n}$$

for $n = 0, 1, 2, 3, \ldots$

Proof. Take absolute value on both sides of (19.1) and apply ML inequality (Theorem 15.4)

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} \, dw \right| \le \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n! M}{r^n}.$$

□

An application of the formula in (19.1) is to evaluate integral in the form

$$\oint_C \frac{f(z)}{(z - z_0)^m} \, dz$$

where m is an integer larger than or equal to 2, and z_0 is a complex number contained in the interior of contour C. Here, it is assumed that C has simple shape, e.g. a circle or a square, such that $f(z)$ is holomorphic in the region bounded by C and the Cauchy integral formula applies.

Example 19.1. Suppose we want to integrate $e^{i\omega z}/(z-z_0)^2$, for some real constant ω and complex constant z_0,

For simplicity, we take $z_0 = 1$, and C is a circle that contains 1 in the interior. We can evaluate the integral

$$\oint_C \frac{e^{i\omega z}}{(z-1)^2}\,dz$$

by applying the equation in (19.1),

$$\oint_C \frac{e^{i\omega z}}{(z-1)^2}\,dz = 2\pi i \frac{d}{dz} e^{i\omega z}\Big|_{z=1} = 2\pi i (i\omega) e^{i\omega z}\Big|_{z=1} = -2\pi\omega e^{i\omega}.$$

Example 19.2. We apply the Cauchy integral formula for higher derivatives to evaluate a trigonometric integral:

$$\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta \tag{19.3}$$

where a and b are positive constants satisfying $a > b$. The assumption $a > b$ ensures that the integrand never vanishes for any θ. Integrals of this type can be transformed to a contour integral on the unit circle by making a substitution $z = e^{i\theta}$, and use the formula

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

We transform the trigonometric integral in (19.3) to

$$\oint_C \frac{1}{(a + b(z+z^{-1})/2)^2} \cdot \frac{1}{iz}\,dz$$

where C is the unit circle with counter-clockwise orientation. We single out the integrand and simplify it as follows:

$$\begin{aligned}\frac{1}{(a + b(z+z^{-1})/2)^2} \cdot \frac{1}{iz} &= \frac{4z}{i(bz^2 + 2az + b)^2} \\ &= \frac{4z}{ib^2(z^2 + 2(a/b)z + 1)^2}.\end{aligned}$$

Denote the root of the quadratic polynomial $z^2 + 2(a/b)z + 1$ in the demoninator as

$$\alpha \triangleq \frac{-a + \sqrt{a^2 - b^2}}{b}, \qquad \beta \triangleq \frac{-a - \sqrt{a^2 - b^2}}{b}.$$

We need to determine whether α and β are inside the unit circle. We first use the fact that the product $\alpha\beta = 1$ to see that exactly one of the roots α and β is inside the unit circle. Since β is negative and less than -1, and hence is outside the unit circle. we know that the first root α must be inside the unit circle. We express the contour integral in the form of the Cauchy integral formula,

$$\oint_C \frac{1}{(a + b(z + z^{-1})/2)^2} \cdot \frac{1}{iz}\, dz = \frac{4}{ib^2} \oint_C \frac{z}{(z-\alpha)^2 (z-\beta)^2}\, dz.$$

To apply Cauchy integral formula, we consider the function

$$f(z) = \frac{z}{(z-\beta)^2}.$$

We note that the function $f(z)$ is holomorphic inside the unit circle C. Compute the derivative of $f(z)$

$$f'(z) = \frac{1 \cdot (z-\beta)^2 - z(2(z-\beta))}{(z-\beta)^4} = -\frac{\beta + z}{(z-\beta)^3},$$

and evaluate it at $z = \alpha$. This gives

$$f'(\alpha) = -\frac{\beta + \alpha}{(\alpha - \beta)^3} = -\frac{-2a/b}{8(a^2 - b^2)^{3/2}/b^3} = \frac{ab^2}{4(a^2 - b^2)^{3/2}}.$$

The final answer is

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta &= \frac{4}{ib^2} 2\pi i f'(\alpha) \\
&= \frac{4}{ib^2} 2\pi i \frac{ab^2}{4(a^2 - b^2)^{3/2}} \\
&= 2\pi \frac{a}{(a^2 - b^2)^{3/2}}.
\end{aligned}$$

We can double check the answer by noting that the answer should be a real and positive number. In the degenerate case of $b = 0$, the answer should be equal to $2\pi/a^2$. Another sanity check is: if increase both a and b by a factor of k, then the integral should decrease by a factor of k^2. We see that the answer passes all checks.

19.2 Identity theorem

Another peculiar property of holomorphic function: if we know the values of a holomorphic function in a small disk inside the domain, then the function values in the whole domain are uniquely determined. As long as it has a nonempty interior so that we can find a sequence of points converging to an accumulation point, then there is only one way to extend the function to a larger domain. This property is content of the identity theorem.

In the proof below, we need a result about connected set. In point-set topology, a set is said to be *disconnected* if it can be decomposed as the union of two nonempty disjoint open sets. A set is *connected* if it is not disconnected, i.e., if it cannot be decomposed as the union of two nonempty disjoint open subsets. A set that is open and closed is often called a *clopen set*. It is easy to see that empty set is a clopen set. The complement of the empty set, which is the whole space, is also clopen.

Theorem 19.3. *The subsets of a path-connected open set S in an Euclidean space (e.g. $\mathbb{R}^2$) that are clopen (in the relative topology) are precisely the empty set and S.*

Sketch of proof. One can prove that a path-connected set is always connected[2]. In the other direction, we know that an open and connected set in an Euclidean space is path-connected[3].

It remains to prove that the empty set and the whole set are the only clopen sets in a connected set. This can be easily done, because if A is a clopen set in S that is not the empty set and not the entire set S, then A and A^c are nonempty and disjoint subsets, whose union is the whole space. The complement A^c is open (in the relative topology of S) because A is assumed to be closed. We can thus write S as a union of two nonempty disjoint open sets, violating that assumption that S is connected. $\square$

[2] A proof that a path-connected set is connected

[3] A proof from mathonline.com

Theorem 19.4 (Identity Theorem). *Suppose f and g are holomorphic in a domain D. The followings are equivalent:*

1. *$f(z) = g(z)$ for all $z \in D$.*
2. *$f(z) = g(z)$ on a sequence of distinct points in D that has a limit point in D.*
3. *There exists a point w in D such that $f^{(n)}(w) = g^{(n)}(w)$ for all $n = 0, 1, 2, 3, \ldots$.*

Proof. The two directions "(1) implies (2)", and "(1) implies (3)" are trivial. To show that the two reverse directions are also valid, it is sufficient to consider the difference $h(z) \triangleq f(z) - g(z)$, and show that $h(z) = 0$ for all $z \in D$.

(2) $\Rightarrow$ (1). Suppose $h(z) = 0$ on a sequence of distinct points $(z_n)_{n=1}^{\infty}$, which is a sequence of distinct points with a limit point in D. Denote the limit point by z_0. We have $h(z_n) = 0$ for all $n = 1, 2, 3, \ldots$

Since z_0 is in the interior of the domain D and $h(z)$ is holomorphic at z_0, we can find a radius $r > 0$ such that h is complex differentiable in the open disc $|z - z_0| < r$. By Theorem 18.9, we can expand h as a power series at the point z_0,

$$h(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k, \tag{19.4}$$

where c_k are complex numbers, and the equality in (19.4) holds for all z in the open disc $|z - z_0| < r$.

Because the sequence $(z_n)_{n=1}^{\infty}$ converges to z_0, it will eventually fall inside this open disc $|z - z_0| < r$. So, we can find an integer N such that $|z_n - z_0| < r$ for all $n \geq N$. Obviously this subsequence also converges to z_0. By Theorem 18.12, there is a unique choice of coefficients c_k's such that $h(z_n) = 0$ for all $n \geq N$. But in fact, the power series with all zero coefficients is one such possibility. Therefore, we can conclude that the coefficients c_k in (19.4) must be all equal to zero. This proves that $h(z)$ is identically equal to zero in the open disc $|z - z_0| < r$.

We now consider the subset in D in which $h(z)$ is equal to 0,

$$\{z \in D : h(z) = 0\}.$$

We know that it has nonempty interior. Let

$$U \triangleq \{z \in D : h(z) = 0\}^\circ,$$

where $^\circ$ denote the interior of a set. By the argument in the previous paragraph, U is nonempty, as it contains the open disc $|z - z_0| < r$.

By the meaning of "interior", U is an open set by construction. We claim that U is also a closed set. We can see why it is closed by checking that any accumulation point of U is in U. Let w_0 be such an accumulation point. There is a sequence of points $w_1, w_2, w_3, \ldots$ in U converging to w_0. By the continuity of function $h(z)$, we have

$$h(w_0) = h(\lim_k w_k) = \lim_k h(w_k) = 0.$$

The point w_0 is in the domain D, and by the hypothesis in the theorem, h is holomorphic in a small open disc centered at w_0. Using the same argument as in the first part of the proof, $h(z)$ is identically zero in this disc. Hence w_0 is in the interior of $\{z \in D : h(z) = 0\}$, and thus is in U.

Since D is connected, by the fact stated in Theorem 19.3, any open and closed set in D is either the empty set or the set D. But U is non-empty. The only possibility is that $U = D$. Therefore, all points in D is in the set U, and hence satisfy $h(z) = 0$. This proves that $h(z)$ is identically equal to 0 in domain D.

(3) $\Rightarrow$ (1). There is an open disc with positive radius, centered at w, such that the function h is holomorphic inside this open disc. By hypothesis, the n-th derivative $h^{(n)}(w)$ is equal to 0 for all positive integer n. Therefore, using the same idea in the proof of "(2) implies (1)", we can say that $h(z)$ is equal to zero for every point z within this open disc, and therefore $h(z) = 0$ for all points z in D. □

The identity theorem gives a constraint on the vanishing locations of holomorphic function.

Corollary 19.5. *Suppose $f(z)$ is an analytic function defined on a domain D that is not constantly equal to 0. Then the set*

$$\{z \in D : f(z) = 0\}$$

has no accumulation point in domain D (See Definition 18.11 for the meaning of accumulation point).

From the proof of the identity theorem, we have proved that a nonzero holomorphic function cannot vanish on an open disc. This gives another way to think about the identity theorem. Suppose we are given a holomorhpic function f and we want to change the function values $f(z)$ for some points z inside a small set (perhaps we would like to increase the modulus of the function value), without modifying the function value outside this set. The identity theorem says that this kind of local operations are impossible, if we wish to maintain the analyticity of the function. We say that a holomorphic function is *rigid* precisely when we are referring to this property.

We record this fact below.

Corollary 19.6. *If $f(z)$ is analytic in a domain D and the values of $f(z)$ is zero within an open disc in D, then f is necessarily the zero function in D.*

Example 19.3. There is only one way to extend the real exponential function e^x, for real value x, to an analytic function that is defined on the whole complex plane. Suppose $g(z)$ is an entire function with the property that $g(x + i0) = e^x$ for all real number x. we can find a sequence of points $(x_k)_{k=1}^{\infty}$ on the real axis that approaches the origin $x = 0$. By the identity theorem, there is only one analytic function such that the value of x_k is e^{x_k} for all k, and the complex exponential function defined by (3.18) is one such choice. Therefore, the complex exponential function defined in (3.18) is the unique analytic function that agrees with the real exponential function.

Likewise, we can see that there is only one way to extend the sine, cosine, and tangent functions to the complex domain.

Remark. A common proof technique in differential geometry is partition of unity. This method involves a family of "bump functions", each of which is constantly equal to 1 within a compact set, and drops rapidly to zero outside this compact set, so that the sum of the functions in this family is the constant function that identically equal to 1. In view of the identity theorem, thee is no partition of unity in complex analysis if we want each bump function to be an analytic function.

20 Maximum modulus principle, zero of a function

Summary

- Maximum modulus principle
- Zeros of a function
- Classification of isolated singularity

20.1 Maximum modulus principle

Another special feature of holomorphic function is the maximum modulus principle.

Theorem 20.1 (Maximum modulus principle)**.** *Let $f(z)$ be a holomorphic function defined on a domain D. If $|f(z)|$ attains a maximum value at an interior point in D, then $f(z)$ is a constant function.*

Remark. Note that the maximum value mentioned in the theorem should be located in the interior of D, because D is assumed to be open and does not include any boundary point. An equivalent form of the maximum modulus principle is: if f is a nonconstant analytic function, then the maximum value of $|f(z)|$ over a domain D occurs on the boundary of D.

Proof. Suppose the modulus of $f(z)$ attains maximum at an interior point z_0 of D. We can find a sufficiently small $\epsilon > 0$ such that all points z in the open disc $|z - z_0| < \epsilon$ lie inside the domain D. By assumption, we have

$$|f(z)| \leq |f(z_0)| \quad \text{whenever } |z - z_0| < \epsilon.$$

Pick any real number r that is between 0 and ϵ. By Cauchy integral formula (Theorem 17.1),

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz.$$

Represent the circle $|z - z_0| = r$ by $z = z_0 + re^{i\theta}$, for $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \end{aligned}$$

The above equation says that the value $f(z_0)$ at the center of the circle equals the mean of the function values on the circle.

Taking modulus on both sides, we get

$$\begin{aligned}|f(z_0)| &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta \\ &\leq |f(z_0)|.\end{aligned}$$

By the sandwich argument, all of the above equalities hold, and we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta,$$

or equivalently

$$\int_0^{2\pi} \Big[|f(z_0)| - |f(z_0+re^{i\theta})|\Big]\,d\theta = 0.$$

Since $|f(z_0)| \geq |f(z_0+re^{i\theta})|$ by hypothesis, by the monotone property of integral, we obtain

$$|f(z_0)| = |f(z_0+re^{i\theta})|$$

for all $0 \leq \theta \leq 2\pi$.

Because the radius r can be any positive real number less than ϵ, we obtain

$$|f(z_0)| = |f(z_0+re^{i\theta})|$$

for all $r \in (0,\epsilon)$ and $\theta \in [0,2\pi]$. Thus, the modulus $|f(z)|$ is equal to a constant on the open disc $|z-z_0| < \epsilon$. By considering the Cauchy-Riemann equations of $f(z)$, we can see that $f(z)$ must be a constant function in the open disc $|z-z_0| < \epsilon$, and the constant is $f(z_0)$. Finally, by the Identity Theorem (Theorem 19.4), $f(z)$ is constant throughout D. □

Example 20.1. Find the maximum value of $|z^2 - z|$ in $|z| \leq 1$.

By the maximum modulus principle, we only need to consider the boundary $|z| = 1$. On the boundary the function to be maximized is

$$|z^2 - z| = |z| \cdot |z-1| = |z-1|.$$

The maximum value occurs at the point on the unit circle that is farthest away from $z = 1$. The maximum value is thus $|(-1)^2 - (-1)| = 2$.

Example 20.2. The maximal modulus principle implies the corresponding statement for harmonic function: if $u(x, y)$ is harmonic inside a region and if D is a closed disc within the region, then the maximum value of $u(x, y)$ over the closed disc D is attained somewhere on the boundary of D.

We can see this by considering a harmonic conjugate $v(x, y)$ of $u(x, y)$, and the complex function

$$f(z) = e^{u(x,y)+iv(x,y)}.$$

Because $u(x, y) + iv(x, y)$ and exp are both holomorphic, so is their composite function. The modulus of f is $e^{u(x,y)}$. By the monotonicity of the real exponential function, the modulus $|f(z)|$ attains the maximum value exactly at the location where $u(x, y)$ attains the maximum value. Hence, by maximum modulus principle, the maximum value of $u(x, y)$ in the disc D can only occur at the boundary of D.

20.2 Zeros of a function

Definition 20.2. A complex number z_0 is called a *zero* of a function f if $f(z_0) = 0$.

Notation: We say that "f has a zero at z_0" if z_0 is a zero of f.

Example 20.3. For a polynomial $f(z) = a_0 + a_1 z + \cdots a_d z^d$, there are at most d zeros. The set of zeros is a finite set.

Example 20.4. The complex exponential function has no zero, because the equation $e^z = 0$ has no solution. However, if c is a nonzero complex constant, the equation $e^z = c$ has infinitely many solutions. The zero of $e^z - c$ are precisely the complex log of c.

Theorem 20.3. *The zeros of a nonzero function are isolated.*

More precisely, the theorem says that if $f(z)$ is a nonzero holomorphic function and z_0 is a zero of f, then we can find a small radius r such that the function $f(z)$ is nonzero in the punctured disk $0 < |z - z_0| < r$.

Example 20.5. The function $\sin(\pi/z)$ is equal to zero for z in the set

$$\{\frac{1}{n} : n = \pm 1, \pm 2, \pm 3, \ldots\}.$$

The origin is an accumulation point of this set. However, the function $\sin(\pi/z)$ is not zero. It does not violate the Theorem 20.3 because the domain of this function does not include the origin, and the origin is not a zero of the function.

The proof follows directly from Corollary 19.5, which states that the set of zeros of a nonzero function cannot have an accumulation point in the domain of the function. Another way to express the same idea is that there are finitely many zeros in a closed and bounded region.

Corollary 20.4. *Suppose $f(z)$ is a nonzero holomorphic function defined in a domain D. Then in any closed and bounded region in D, the function $f(z)$ has finitely many zeros.*

Proof. We prove by contradiction. Suppose B is a closed and bounded region in D such that $f(z)$ has infinitely many zeros. Because B is closed and bounded, we can apply Bolzano-Weierstrass theorem and conclude that the set of zeros of $f(z)$ in B has an accumulation point. The accumulation point is a point in D. Hence, by the identity theorem, the function $f(z)$ is a constant zero function, violating the assumption that $f(z)$ is a nonzero function. □

Definition 20.5. If f is not identically equal to zero, the *order* of a zero is the exponent of the first nonzero term in the power series expansion of f at z_0. In other words, $f(z)$ has a zero of order m at z_0 if the power series of $f(z)$ at z_0 is

$$a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + a_{m+2}(z-z_0)^{m+2} + \cdots$$

with $a_m \neq 0$. A zero of order 1 is called a *simple* zero. A zero order 2 is called a *double* zero.

The order of zero is also referred to as the *order of vanishing*. We know that a holomorphic function can always be expanded as a power series at any point in its domain, and the coefficients are uniquely determined (Theorem 18.12). This ensures that the order of a zero is well defined.

Because power series can be differentiated term-wise, the order of zero is the smallest integer m such that $f'(z_0)$, $f''(z_0), \ldots, f^{(m-1)}(z_0)$ are all equal to zero but $f^{(m)}(z_0)$ is nonzero.

The behavior of function $f(z)$ near the point z_0 can be approximated by the power function $(z - z_0)^m$. If z_0 is a zero of order m and we travel around the point z_0 on a small circle centered at z_0, the image through the function f will cycle m times around the point $f(z_0)$.

20.3 Classification of singular points

Definition 20.6. A point z_0 in $\mathbb{C}$ is called a *singular point* of $f(z)$ if $f(z)$ is not complex differentiable at z_0. It is called an *isolated singular point* if $f(z)$ is not holomorphic at z_0 but is holomorphic in a punctured disc $0 < |z - z_0| < r$ for some radius r.

Example 20.6. The function $f(z) = 1/(z^2 + 1)$ has two isolated singular points located at $z = i$ and $z = -i$.

Example 20.7. The function $\frac{1}{\sin(\pi/z)}$ has a non-isolated singular point at $z = 0$. It is because the sequence $\{1/n\}_{n=1}^{\infty}$ converges to $z = 0$, and this function is singular at $z = 1/n$, for $n = 1, 2, 3, 4, \ldots$.

Example 20.8. The principal log function with domain

$$\mathbb{C} \setminus \{x + iy : \, y = 0, x \leq 0\}$$

has a non-isolated singularity at the origin.

Isolated singularity can classified into three categories.

Definition 20.7. An isolated singular point z_0 of $f(z)$ is called

(i) a *removable singularity* if $|f(z)|$ is bounded in a neighborhood of z_0;

(ii) a *pole* if $|f(z)| \to \infty$ as $z \to z_0$;

(iii) an *essential singularity* otherwise.

We note that the three cases exhaust all possibilities; an isolated singular point that is not removable and is not a pole is by definition an essential singularity. Because essential singularity is more complicated, we will mainly consider singularity of the first two types.

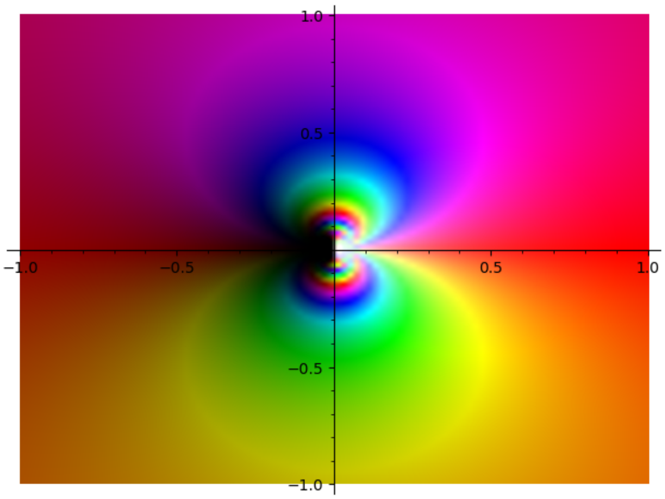


Figure 30: Phase portrait of $e^{1/z}$ near the origin.

Definition 20.8. A complex function is said to be *meromorphic* if the singular points are isolated, and all of them are removable or poles, i.e., there is no essential singularity within the domain.

Example 20.9. The function $f(z) = \frac{\sin(z)}{z}$ has a removable singularity at $z = 0$. This function is basically the same as the function defined by the power series

$$\frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

The power series has infinite radius convergence, and hence is an entire function.

Example 20.10. For any positive integer n, the function $\frac{z^n-1}{z-1}$ has a removable singularity at $z = 1$. The function is not defined at $z = 1$. But if we take limit as z approach 1, the function value has limit equal to n. The behavior of this function is essentially the same as the polynomial $z^{n-1} + z^{n-2} + \cdots + z + 1$.

Example 20.11. The function $f(z) = \frac{1}{z(z+1)}$ has two poles located at $z = 0$ and $z = -1$.

Example 20.12. The complex trigonometric functions $\tan(z)$ and $\cot(z)$ have infinitely many poles. The poles of $\tan(z)$ are the same as the zeros of $\cos(z)$, and they are located at $\pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2, \ldots$

Example 20.13. The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. We can see that it is an essential singular point by approaching $z = 0$ from the right and from the left. For real variable x, if we take $x \to 0$ from the right, the value of $f(z)$ tends to positive infinity. Hence it is not bounded near $z = 0$. On the other hand if we take x approaching 0 from the negative real axis, then $f(z)$ tends to 0. The modulus is not approaching infinity. See Fig. 30 for the phase portrait of this function. (Compare with the other phase portraits in Fig. 5.)

The complex function in 20.7 is not meromophic, because it has non-isolated singularity. The function $e^{1/z}$ in Example 20.13 are not meromorphic because it contains an essential singularity.

20.4 Appendix: Casorati-Weierstrass theorem

The behavior near an essential singularity is complicated. The Casorati-Weierstrass theorem captures part of the complications.

Theorem 20.9 (Casorati-Weierstrass)**.** *Suppose $f(z)$ is homophobic in $U \setminus \{z_0\}$, where U is an open set and z_0 is an essential singularity. For any positive real numbers ϵ and δ, and for any choice of complex number w, there exists a complex number z U such that*

$$0 < |z - z_0| < \delta, \text{ and } |f(z) - w| < \epsilon.$$

An equivalent but more compact way to express the conclusion in the theorem is: If N is any neighborhood of z_0 contained in U, then the image of $N \setminus \{z_0\}$ under the function f is dense in $\mathbb{C}$.

We can formulate this theorem in terms of a game. In this game a complex function $f(z)$ and a point z_0 in the domain of $f(z)$ are given. Player A first selects a small neighborhood of z_0 and a complex number w. Then, Player B tries to find a sequence of points in the neighborhood such that the image of the sequence under the function f is convergent with limit w. Player B wins if he/she can find such a sequence, otherwise Player A wins.

If z_0 is an essential singular point, then Player B will always win.

The proof of Casorati-Weierstrass theorem is skipped in this note. We will focus more on the other two types of singularity.

21 Laurent series

Summary:

- Riemann's theorem on removable singularity
- Laurent series
- Residue

The following theorem is an extension of Theorem 17.2 to multiple holes.

Theorem 21.1 (Cauchy theorem for function with isolated singularity)**.** *Suppose complex function $f(z)$ defined on a domain D has isolated singularity, i.e., $f(z)$ is holomorphic in D except for some isolated points. Let C be a simple closed curve in D with counter-clockwise orientation, so that no singular point of $f(z)$ lie on C. Let z_1, $z_2, \dots, z_m$ denote the singular points inside C. For $k = 1, 2, \dots, m$, let C_k be a positively-oriented circular contour centered at z_k, with the radius r_k is chosen such that $f(z)$ is holomorphic in the punctured disc $0 < |z - z_k| < r_k$. Then*

$$\oint_C f(z)\, dz = \sum_{k=1}^{m} \oint_{C_k} f(z)\, dz.$$

We note that the assumption of isolated singularity is used in justifying why we can choose a sufficiently small radius r_k so that there is only one singular point in the open disc $D(z_k, r_k)$. Fig. 31 illustrates the notation in the theorem.

We will use a property of simple closed curve: the contour C divides the complex plane into two parts: interior and exterior. Moreover, for any point α in the interior, the winding number of C relative to α is 1, and for any point β on in the exterior, the winding number of C relative to β is equal to 0. (Why it is true is the content of the Jordan curve theorem.)

Proof. We first illustrate the proof when C is a circle and there are two isolated singular points inside C. We divide the ares bounded between the outer circle and the inner circles into six parts. We first draw a straight line that joins the two singular points z_1 and z_2. Then we draw perpendicular lines at z_1 and at z_2. We obtain six closed paths. The sum

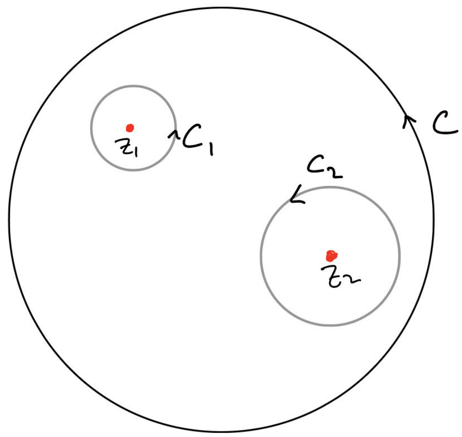


Figure 31: Cauchy theorem with isolated singularity.

of these six closed paths is the sum of the three circles C, $-C_1$ and $-C_2$, because all the internal line segments cancel.

Note that we take we take the negation of C_1 and C_2. See Fig. 32.

For each cycle, the area bounded within a closed path is not convex. However, each closed path is contained in a convex region, that avoids the two singular points z_1 and z_2. Since $f(z)$ is holomorphic in the convex region, the function $f(z)$ has a primitive, and hence the integral along the closed path is equal to 0.

Each of the six closed paths is contained inside a convex region, in which $f(z)$ is holomorphic. Therefore the integral of all six closed paths equal 0.

The proof idea can be modified to incorporate more than two inner circles. The shape of circle is not crucial. The theorem is valid for other simple shapes such as ellipse, square, rectangle, etc. The crucial point is that the winding number of the curve C relative to each interior point is 1. $\square$

21.1 Removable singularity

The next theorem justifies the name "removable singularity".

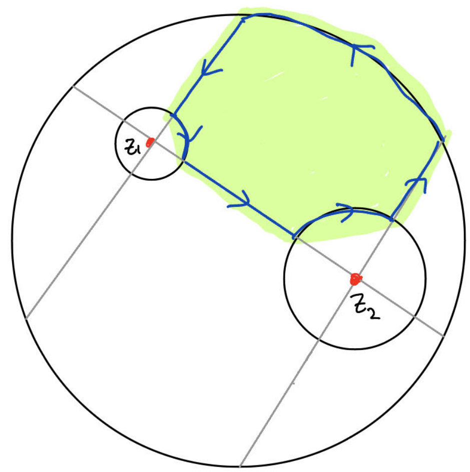


Figure 32: Dividing the area into six parts. Each part is contained in a convex set in which the function to be integrated is holomorphic.

Theorem 21.2 (Riemann's theorem on removable singularity)**.** *Suppose f is holomorphic in a domain that contains a punctured disc $\{z : 0 < |z - z_0| < \epsilon\}$. If $f(z)$ is bounded in this punctured disc, then we can re-defined f at z_0 so that f becomes holomorphic at z_0.*

More precisely, the theorem is saying that there exists a function $\tilde{f}(z)$ that is holomorphic for $|z - z_0| < \epsilon$ such that $\tilde{f}(z) = f(z)$ whenever $0 < |z - z_0| < \epsilon$.

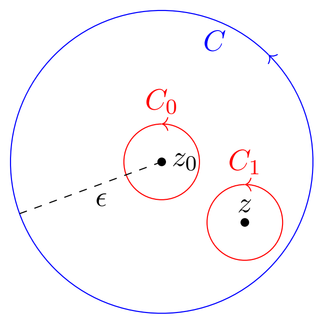
Proof. Suppose f is holomorphic and bounded in a punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$. We want to prove that we can define $f(z_0)$ appropriately to make $f(z)$ continuous at z_0.

Let $D(z_0, \epsilon)$ denote the open disc $|z - z_0| < \epsilon$ and let $D(z_0, \epsilon) \setminus \{z_0\}$ denote the punctured disc. By the hypothesis in the theorem, there is a real number M such that $|f(z)| \le M$ for z in the punctured disc $D(z_0, \epsilon) \setminus \{z_0\}$. Define

$$\tilde{f}(z) \triangleq \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw,$$

where C is the circle $|z - z_0| = \epsilon$, with counter-clockwise orientation, and z is any point inside the circle. We want to show that (i) $\tilde{f}(z) = f(z)$ for all $z \in D(z_0, \epsilon) \setminus \{z_0\}$, and (ii)

$\tilde{f}$ is analytic in $D(z_0, \epsilon)$.



Let z be a point in the puncture disc $0 < |z - z_0| < \epsilon$. Draw a small circle C_0 of radius r_0 with center at z_0, such that C_0 is inside the circle C but does not contain z. Draw another small circle C_1 of radius r_1 with center at z such that C_1 is inside the circle C but does not contain z_0.

By Theorem 21.1, we have

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw = \frac{1}{2\pi i} \int_{C_0} \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw. \tag{21.1}$$

Since $f(z)$ is analytic inside the circle C_1, we can apply Cauchy integral formula (Theorem 17.1) to obtain

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw = f(z).$$

We next prove that $|\int_{C_0} \frac{f(w)}{w-z}\, dw|$ can be made arbitrarily small. For complex number w on C_0, the distance $|w - z|$ is lower bounded by $|z - z_0| - r_0$. By ML inequality (Theorem 15.4)

$$\Big| \int_{C_0} \frac{f(w)}{w - z}\, dw \Big| \leq \frac{M}{|z - z_0| - r_0} 2\pi r_0,$$

which approaches 0 as $r_0 \to 0$. Because (21.1) holds for all sufficiently small radius r_1 of C_1, we can conclude that $\tilde{f}(z) = f(z)$. Since this holds for all $z \in D(z_0, \epsilon) \setminus \{z_0\}$, this proves the part (i).

For part (ii), we just need to show that $\tilde{f}(z)$ is holomorphic at the center z_0, because $\tilde{f}(z)$ is equal to $f(z)$, which is assumed to be holomorphic in $D(z_0; \epsilon) \setminus \{z_0\}$. We first use the definition of $\tilde{f}$ at z_0 to write

$$\begin{aligned}\frac{\tilde{f}(z_0+h)-\tilde{f}(z_0)}{h} &= \frac{1}{2\pi i h}\oint_C \frac{f(w)}{w-z_0-h} - \frac{f(w)}{w-z_0}\,dw \\ &= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0-h)(w-z_0)}\,dw\end{aligned}$$

where C is the circle $|z-z_0|=\epsilon$, with counter-clockwise orientation. We then show that it converges as $h\to 0$. Because we already has a target limit, we calculate

$$\begin{aligned}\Big|\frac{\tilde{f}(z_0+h)-\tilde{f}(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}dw\Big| &= \frac{1}{2\pi}\Big|\int_C \frac{f(w)h}{(w-z_0)^2(w-z_0-h)}\,dw\Big| \\ &\leq \frac{1}{2\pi}\frac{M|h|}{\epsilon^2(\epsilon-|h|)}2\pi\epsilon.\end{aligned}$$

The last inequality is the ML inequality. For fixed ϵ, it converges to 0 as $|h|$ approaches 0. Therefore $f'(z_0)$ exists and is equal to

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}\,dw.$$

This proves part (ii). □

In view of Theorem 21.2, we can ignore any removable singularity, because we can always re-define the function appropriately so that it is no longer a singularity. We note that Theorem 21.2 fails for real functions. The real-valued function $f(x)=|x|$ is not differentiable at $x=0$ and is bounded near 0. It is continuous at $x=0$, but is not analytic at $x=0$.

21.2 Pole order

Notation: We sometime say that "$f(z)$ has a pole at z_0" if z_0 is a pole of $f(z)$.

Suppose $f(z)$ has a pole at z_0. By definition we have $|f(z)|\to\infty$ as $z\to z_0$, and $f(z)$ is nonzero in a small neighborhood of z_0. Consider the reciprocal function $1/f(z)$. It is bounded in a neighborhood of z_0. Indeed, by the meaning of "diverging to infinity", it means that for any positive positive M, we can find a sufficiently small $\epsilon>0$, such that $|f(z)|>M$ for all $0<|z-z_0|<\epsilon$. In other words, we have $|1/f(z)|<1/M$ for all $|z-z_0|<\epsilon$. We can take M to be an arbitrary positive real number, and see that $|1/f(z)|$ is bounded by $1/M$ in a punctured disc.

The singular point z_0 is a removable singularity of $1/f(z)$. Since $|f(z)| \to \infty$ as $z \to z_0$, the domain of function $1/f(z)$ can be extended to an open disc $|z - z_0| < \epsilon$ by defining

$$g(z) \triangleq \begin{cases} 1/f(z) & \text{if } 0 < |z - z_0| < \epsilon \\ 0 & \text{if } z = z_0. \end{cases}$$

By the theorem of removable singularity, the new function $g(z)$ is holomorphic at z_0, can thus be expressed as a power series centered at z_0. The constant term must be equal to zero because $g(z_0) = 0$. However, the coefficients of the power series expansion of $g(z)$ cannot be all zero, because $g(z)$ has an isolated zero at z_0. Let m be the order of the zero z_0 of $g(z)$. (The order m must be finite by Corollary 20.4.) We can write

$$g(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + a_{m+2}(z - z_m)^{m+2} + \dots$$

where a_m is a nonzero coefficient. We can write $f(z)$ as

$$f(z) = \frac{1}{(z - z_0)^m} \cdot \Big(\frac{1}{a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \cdots} \Big) \tag{21.2}$$

Because a_m is a nonzero constant, the denominator is nonzero within a small neighborhood of z_0. Hence, the fraction enclosed in the parentheses is holomorphic at z_0. We can thus write the fraction in the parentheses as a power series (Theorem 18.9),

$$\begin{aligned} f(z) &= (z - z_0)^{-m}(c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots \\ &= c_0(z - z_0)^{-m} + c_1(z - z_0)^{-m+1} + c_2(z - z_0)^{-m+2} + \cdots \end{aligned}$$

We note that the first coefficient c_0 is nonzero, because a_m is nonzero. Indeed, we have $c_0 = 1/a_m$.

This leads to define the order of a pole as the zero order of the reciprocal function.

Definition 21.3. The *order* of a pole z_0 of $f(z)$ is the smallest positive integer m such that $\lim_{z \to z_0}(z - z_0)^m f(z)$ is a nonzero constant. A pole of order 1 is called a *simple* pole. A pole of order 2 is called a *double* pole.

Example 21.1. Consider a rational function $f(z) = \frac{(z-1)(z-2)}{(z-3)(z+i)^2}$. This function $f(z)$ has two zeros of order 1 at $z = 1$ and $z = 2$. There is a simple pole at $z = 3$, because

$$f(z) = \frac{1}{z - 3} \Big[\frac{(z - 1)(z - 2)}{(z + i)^2} \Big]$$

and $\frac{(z-1)(z-2)}{(z+i)^2}$ is nonzero when evaluated at $z=3$. We can formally check this by calculating

$$\lim_{z\to 3}(z-3)f(z)=\lim_{z\to 3}\frac{(z-1)(z-2)}{(z+i)^2}=\frac{(3-1)(3-2)}{(3+i)^2}\neq 0.$$

Another pole of $f(z)$ is located at $z=-i$. This is a double pole because

$$\lim_{z\to -i}(z+i)f(z)=\lim_{z\to -i}\frac{(z-1)(z-2)}{(z-3)(z+i)}=\infty,$$

but

$$\lim_{z\to -i}(z+i)^2f(z)=\lim_{z\to -i}\frac{(z-1)(z-2)}{z-3}=\frac{(-i-1)(-i-2)}{-i-3}\neq 0.$$

Example 21.2. The function $f(z)=\frac{\sin(z)}{z^n}$ has a pole of order $n-1$ at the origin. It is because $\sin(z)/z$ has a removable singularity at $z=0$. If we multiply $f(z)$ by z^{n-1}, the product $f(z)z^{n-1}$ has a finite limit as z approaches 0. On the other hand, the reciprocal $1/f(z)$ has poles at $z=\pm k\pi$, for $k=1,2,3,\dots$

21.3 Laurent series and residue

We have seen that at a pole z_0 of order m, we can express the function as an infinite sum

$$\sum_{k=-m}^{\infty}c_k(z-z_0)^k,$$

where c_{-m} is a nonzero coefficient, in a neighborhood of z_0.

In the case of essential singularity, we can also expand the function as an infinite series. For example, we can expand the function $e^{1/z}$ at $z=0$ as

$$e^{1/z}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots$$

which converges whenever $z\neq 0$ (see Example 20.13).

This motivates the definition of Laurent series.

Definition 21.4. A *Laurent series* is an infinite sum in the form

$$\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n. \tag{21.3}$$

The first summation $\sum_{n=0}^{\infty}c_n(z-z_0)^n$ is called the *analytic part*, or the *regular part*. The second summation $\sum_{n=1}^{\infty}c_{-n}(z-z_0)^{-n}$ is called the *principal part*. We say that the Laurent series converges if both the analytic and principal parts converge.

If the principal part is zero, a Laurent series reduces to a Taylor series. Sometime we write a Laurent series in the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$$

to emphasize the analytic part and the principal part.

Remark. Laurent series is closely related to Fourier series. Suppose the coefficients a_n's in (21.3) are conjugate symmetric, i.e., $c_{-n} = c_n^*$, and the Laurent series converges on the unit circle. If we substitute z by $e^{2\pi it}$, for some real number t, then

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$

is a real function with period 1. In fact we can re-write it as

$$\begin{aligned} &c_0 + \sum_{n=1}^{\infty}(c_n e^{2\pi int} + c_{-n}e^{-2\pi int}) \\ &= c_0 + \sum_{n=1}^{\infty} \mathrm{Re}(c_n)\cos(2\pi nt) - \mathrm{Im}(c_n)\sin(2\pi nt), \end{aligned}$$

which is in the form of a Fourier series. When the domain of the Laurent series is restricted to the unit circle, it is the same as complex Fourier series.

In order to simplify notation, we consider $z_0 = 0$ below. The first structural result is that the region of convergence has annulus shape.

Theorem 21.5. *The region of convergence of a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ has the shape of an annulus $R_1 < |z| < R_2$. (If $R_1 > R_2$, the region of convergence is empty.)*

Proof. By definition, Laurent series converges if and only if the analytic part and the principal parts converge. The analytic part is a power series, and hence is convergent inside an open disc, say with radius R_2. The value of R_2 is given by the Hadamard formula (Theorem 18.4).

For the principal part, we make a substitution $u = 1/z$,

$$\sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} b_n u^n.$$

This is a power series with variable u, and hence will converge in an open disc, say with radius $1/R_1$.

Combining the two parts, we see that this Laurent series converges if $|z| < R_2$ and $1/|z| = |u| < 1/R_1$. Therefore, the region of convergence is in the form

$$R_1 < |z| < R_2.$$

□

The next example illustrates the dependency on the region of convergence.

Example 21.3. Let $f(z)$ denote the function $\frac{1}{(1-z)(3-z)}$. Find the Laurent series expansion at three regions (i) $|z| < 1$, (ii) $1 < |z| < 3$, and (iii) $3 < |z|$.

We first obtain the partial fraction expansion of $f(z)$,

$$f(z) = \frac{1}{2}\frac{1}{(1-z)} - \frac{1}{2}\frac{1}{(3-z)}.$$

(i) When $|z| < 1$, we can use geometric series to expand both terms in the partial fraction expansion,

$$\begin{aligned}\frac{1}{(1-z)(3-z)} &= \frac{1}{2}(1+z+z^2+z^3+\cdots) - \frac{1}{2}\cdot\frac{1}{3}(1+(z/3)+(z/3)^2+\cdots \\ &= \frac{1}{2}\sum_{k=0}^{\infty}(1-(1/3)^{k+1})z^k.\end{aligned}$$

This Laurent series converges for $|z| < 1$, and is indeed a power series.

(ii) If $1 < |z| < 3$, we write the first fraction in the partial fraction expansion as

$$\frac{-1}{2}\frac{1}{1-(1/z)}$$

We can then expand $f(z)$ in the annulus $1 < |z| < 3$ as

$$\begin{aligned}\frac{1}{(1-z)(3-z)} &= \frac{-1}{2}\frac{1}{1-(1/z)} - \frac{1}{2\cdot 3}\frac{1}{1-(z/3)} \\ &= -\frac{1}{2}(1+\frac{1}{z}+\frac{1}{z^2}+\cdots) - \frac{1}{6}(1+\frac{z}{3}+\frac{z^2}{3^2}+\cdots).\end{aligned}$$

The principal part of this Laurent series is non-zero and contains infinitely many terms.

(iii) When $3 < |z|$, the Laurent series expansion has zero analytic part.

$$\begin{aligned}\frac{1}{(1-z)(3-z)} &= \frac{-1}{2}\frac{1}{1-(1/z)} + \frac{1}{2z}\frac{1}{1-3/z} \\ &= -\frac{1}{2}(1+\frac{1}{z}+\frac{1}{z^2}+\cdots) + \frac{1}{2z}(1+\frac{3}{z}+\frac{3^2}{z^2}+\cdots)\end{aligned}$$

Remark. In contrast to power series, it is not true in general that $c_n = f^{(n)}(0)/n!$ for Laurent series. Nevertheless, the coefficients can be computed using contour integrals that is similar to the case of power series.

Theorem 21.6 (Existence and uniqueness of Laurent series)**.** *Suppose $f(z)$ analytic in an annulus $R_1 < |z - z_0| < R_2$. where R_1 and R_2 are positive real constants. There exist coefficients $(a_k)_{k=-\infty}^{\infty}$ such that*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

for all z in the annulus $R_1 < |z - z_0| < R_2$.
The coefficients a_k is uniquely determined by contour integral

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

for $k \in \mathbb{Z}$, where C is a simple contour in the annulus $R_1 < |z - z_0| < R_2$ going around z_0 with positive orientation.

The proof idea is similar to the analogous theorem for power series (Theorem 18.9). The existence part of the theorem is relegated to the appendix at the end of this section. We note that the contour C in the theorem must lie within the annulus $R_1 < |z - z_0| < R_2$. The computation of the coefficients a_k's depend on the region of convergence.

We summarize the relationship between principal part of Laurent series and pole order: Suppose z_0 is an isolated singularity of $f(z)$. Expand $f(z)$ using a Laurent series with center at z_0.

1. If z_0 is a removable singularity, then the principal part is zero.

2. If z_0 is a pole of $f(z)$, then the principal part has finitely many terms. The pole order

is the integer m such that $a_{-m}(z-z_0)^{-m}$ is the smallest-degree term with a nonzero coefficient.

3. If z_0 is an essential singularity, then the principal part has infinitely many terms.

We define the notion of residue in terms of the coefficients of a Laurent series.

Definition 21.7. Consider a complex function $f(z)$ that is analytic in a domain except some isolated singular points. The *residue* of $f(z)$ at a point z_0 is defined as the coefficient b_1 in the Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n},$$

with convergence in a small punctured disc $0 < |z-z_0| < \epsilon$ centered at z_0, for some $\epsilon > 0$.

It is important to specify the region of convergence $0 < |z-z_0| < \epsilon$ in this definition, because the coefficients depend on the choice of the region of convergence.

Notation: There are several notation for residue, e.g. $\mathrm{Res}(f; z_0)$, $\mathrm{Res}_{z_0}(f)$, and $\operatorname*{Res}_{z=z_0}(f)$.

When $f(z)$ is analytic at z_0, the residue at z_0 is zero.

From the formula for computing the coefficients of a Laurent series in Theorem 21.6, we have the following equation

$$\mathrm{Res}(f; z_0) = \frac{1}{2\pi i}\int_C f(z)\,dz, \tag{21.4}$$

where C is a small contour around z_0 that contains no singular point except z_0. In fact, we can take (21.4) as an alternate definition of of residue.

Example 21.4. The residue of $e^{1/z}$ at $z = 0$ is 1, because the coefficient of $1/z$ in

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

is equal to 1.

Theorem 21.1 can be expressed in terms of residues.

Theorem 21.8 (Residue theorem)**.** *Suppose f is analytic in a domain D except for some isolated singularities. If C is a simple closed curve enclosing singular points z_1, $z_2, \ldots, z_n$ in the interior, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

Proof. For $k = 1, 2, \ldots, n$, we draw a small circle C_k centered at z_k so that the circle does not contain the other singular points. By Theorem 21.1,

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

Since the residue of f at z_k is equal to the integral $\frac{1}{2\pi i}\int_{C_k} f(z)\,dz$, the integral around the contour C_k can be replaced by the corresponding residue.

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

□

21.4 Appendix: Existence of Laurent series expansion

For notation convenience, we assume that the center z_0 is the origin.
We set up the proof by picking two finite real numbers r_1 and r_2 such that

$$R_1 < r_1 < r_2 < R_2.$$

In general, R_2 may equal to infinity. In this case r_2 can be any real number larger than r_2. We consider the region with modulus less than r_2 and larger than r_1, by drawing a circle C_1 with radius r_1 centered at origin, and a circle C_2 with radius r_2 centered at the origin, with counter-clockwise orientation.

Let z be a complex number in the region between C_1 and C_2. Our setup ensures that function $f(z)$ is well-defined on C_1 and C_2, and the input value z stays away from the two boundaries.

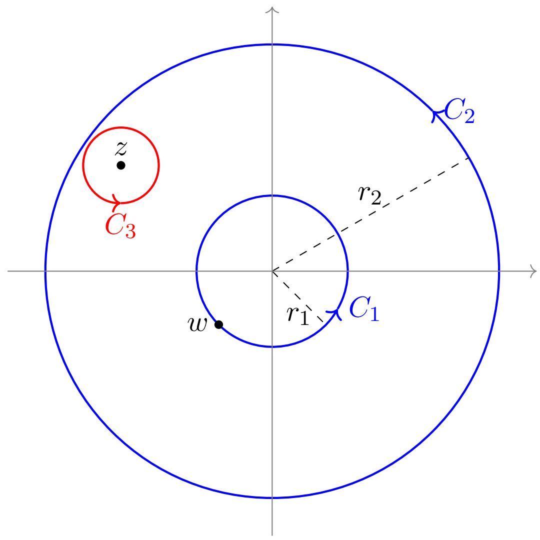


Figure 33: Proof of the existence of Laurent series expansion

Draw a positively-oriented circle C_3 centered at z with radius r_3 such that C_3 is within the area between C_1 and C_2. The notation is illustrated in Figure 33.

We can extend the argument in the deformation theorem (Theorem 17.2) to more than one circles. The integral over the outer circle C_2 is the same as the sum of integrals over the two inner circle C_1 and C_3,

$$\int_{C_2} \frac{f(w)}{w-z}\, dw = \int_{C_1} \frac{f(w)}{w-z}\, dw + \int_{C_3} \frac{f(w)}{w-z}\, dw.$$

By Cauchy integral formula, $f(z)$ can be expressed as a contour integral along the circle C_3. Therefore, we can write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_3} \frac{f(w)}{w-z}\, dw && (21.5)\\ &= \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z}\, dw. && (21.6) \end{aligned}$$

Because f is continuous, and hence bounded on C_1 and C_2, the contour integrals

$$\int_{C_1} \frac{f(w)}{w-z}\, dz \quad \text{and} \quad \int_{C_2} \frac{f(w)}{w-z}\, dz$$

are well-defined and finite. We will not have $\infty - \infty$.

We claim that the first integral can be expanded as a convergent Taylor series

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} a_n z^n, \tag{21.7}$$

and the second integral can be represented as

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw = \sum_{n=1}^{\infty} b_n z^{-n}. \tag{21.8}$$

The proof of (21.7) is similar to the proof of Theorem 18.9 and is skipped. We note that the derivation only requires that f is holomorphic on the contour C_2, so that we can apply the ML inequality in one of the steps. We do not require that $f(z)$ is analytic throughout the circular region enclosed by C_2.

In the remainder of the proof we consider (21.8). We write

$$\frac{1}{z-w} = \frac{1}{z(1-w/z)} = \frac{1}{z}\big(1 + \frac{w}{z} + \frac{w^2}{z^2} + \frac{w^3}{z^3} + \cdots\big)$$

and observe that it converges for $|w| < |z|$. Since the location of z is outside C_1, we have convergence for all $w \in C_1$,

$$\frac{1}{z}\sum_{n=0}^{\infty} \frac{w^n}{z^n} = \sum_{n=1}^{N} \frac{w^{n-1}}{z^n} + \sum_{n=N+1}^{\infty} \frac{w^{n-1}}{z^n} = \sum_{n=1}^{N} \frac{w^{n-1}}{z^n} + \frac{w^N}{z^N(z-w)}.$$

Thus, when $|w| < |z|$, we have

$$-\frac{f(w)}{w-z} = \sum_{n=1}^{N} \frac{f(w)w^{n-1}}{z^n} + \frac{f(w)w^N}{z^N(z-w)}.$$

Integrate the above along the curve C_1, and exchange the order of summation and integration to obtain

$$-\int_{C_1} \frac{f(w)}{w-z}\,dw = \sum_{n=1}^{N}\Big(\int_{C_1} f(w)w^{n-1}\Big)z^{-n} + \int_{C_1} \frac{f(w)}{z-w}\frac{w^N}{z^N}\,dw.$$

The summation is the partial sum of the principal part in the Laurent series. We next show that remainder term approach zeros as N approaches infinity. This is another standard application of the ML inequality. For $w \in C_1$ and z outside C_1, we have

$$|w| = r_1, \quad \text{and } |z-w| \geq |z| - r_1.$$

Furthermore, $|f(w)|$ is bounded by some constant M for $w \in C_1$, because C_1 is a compact set and $|f(w)|$ is a continuous function on C_1. By ML inequality (Theorem 15.4),

$$\Big| \int_{C_1} \frac{f(w)}{z-w} \frac{w^N}{z^N} \, dw \Big| \leq \frac{M}{(|z|-r_1)} \frac{r_1^N}{|z|^N} 2\pi r_1.$$

Because $\frac{r_1}{|z|} < 1$, we have $\frac{r_1^N}{|z|^N} \to 0$ as $N \to \infty$. This proves the equality in (21.8) with

$$b_n = \frac{1}{2\pi i} \int_{C_1} f(w) w^{n-1} \, dw.$$

We have thus proved that, for $r_1 < |z| < r_2$, the complex function $f(z)$ can be expanded as

$$\frac{1}{2\pi i} \sum_{n=0}^{\infty} \Big(\int_{C_2} \frac{f(w)}{w^{n+1}} \, dw \Big) z^n + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \Big(\int_{C_1} f(w) w^{n-1} \, dw \Big) z^{-n}.$$

By the deformation theorem, we can replace the contour C_1 and C_2 in the above integrals by a simple contour C with positive orientation in the annulus $r_1 < |z| < r_2$, going around the origin once. We can then simplify it to

$$f(z) = \sum_{n=-\infty}^{\infty} \Big(\frac{1}{2\pi i} \int_C \frac{f(w)}{w^{n+1}} \, dw \Big) z^n.$$

Finally, since r_1 can be any number larger than R_1, and r_2 can be any number less than R_2 (satisfying $r_1 < r_2$), we have a convergent Laurent series for any z in the annulus $R_1 < |z| < R_2$.

22 Residues

Summary:

- Uniqueness of Laurent series
- Computation of residues

The residue of a function $f(z)$ at a given point is defined as the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of $f(z)$ in a punctured disc $0 < |z - z_0| < r$. We first show that it is well defined, but proving that the coefficients of Laurent series expansion are uniquely determined by the function values. We then derive formula for the computations of residues.

22.1 Uniqueness of Laurent series coefficients

Suppose that a function $f(z)$ is represented by a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$$

in an annulus $R_1 < |z - z_0| < R_2$, for some nonzero positive constants R_1 and R_2. We can uniquely recover the coefficients from the function values. For notational convenience we assume that the center z_0 is 0.

The idea is to choose a contour C that lie within the annulus with winding number 1 (respect to the center of the circle). We first show that the integral

$$\oint_C f(z)\, dz$$

is independent of the choice of C. Suppose there are two closed curves C_1 and C_2 in the annulus, each of them revolves around the center once counter-clockwise (Fig. 34). The integral of $f(z)$ over the two curves must be the same. We can see this by dividing them into several parts, by drawing line segments from the center of the circles. In each part, we connect a portion of C_1 and a portion by C_2 by two line segments. The resulting closed curve is included inside a convex set (highlighted in as a shaded aea in Fig. 35), in which the function $f(z)$ is holomorphic. Therefore, by Theorem 16.4, the integral of $f(z)$ over every closed curve inside this domain is zero. We note that the closed curve may have

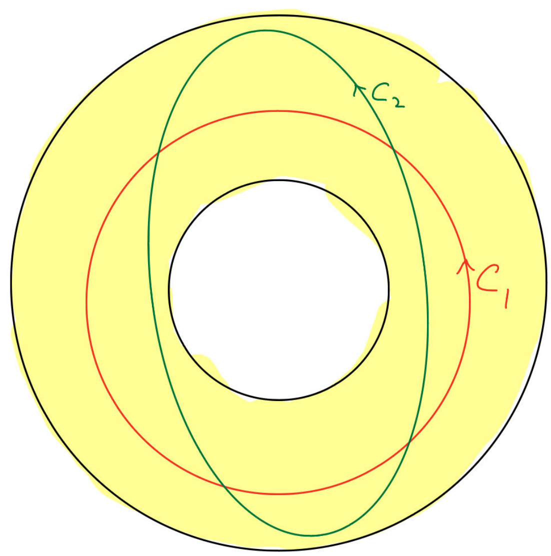


Figure 34: Two closed curves in an annulus

self-intersection, but it is allowed in Theorem 16.4. The sum of the integral over all parts is equal to the integral over C_1 minus the integal over C_2, and the difference is equal to 0. This proves that we can take any closed curve in the annulus with winding number when we compute the contour integral.

A non-rigorous proof of uniqueness may proceed a follows: Assume that we can exchange the order of contour integral and infinite sum. By using the basic integrals in Theorem 14.6, we obtain

$$\oint_C z^k\,dz = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise.} \end{cases}$$

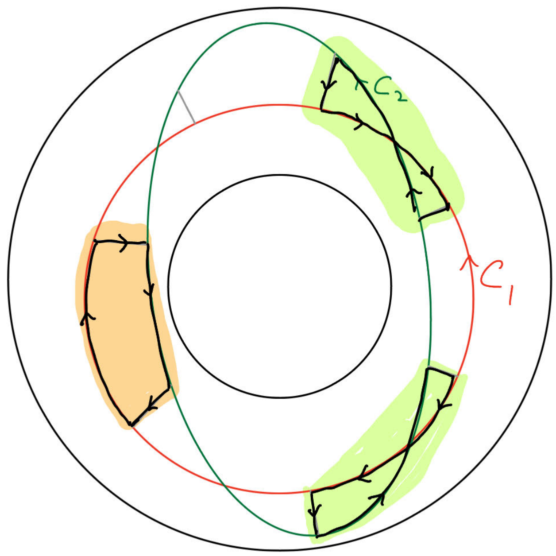


Figure 35: Division of the two contours into parts

This gives

$$\begin{aligned}
\oint_C f(z)\,dz &= \oint_C \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k\,dz \\
&= \sum_{k=-\infty}^{\infty} \oint_C c_k (z-z_0)^k\,dz \\
&= 2\pi i c_{-1}.
\end{aligned}$$

Therefore c_{-1} is equal to $(2\pi i)^{-1} \oint_C f(z)\,dz$. The value of other coefficients c_n, for $n \neq -1$, can be obtained by shifting the Laurent series so that c_n is the coefficient of the term with power -1. This can be done by multiplying $f(z)$ by z^{-n-1} and repeating the above argument,

$$\oint_C f(z) z^{-n-1}\,dz = \oint_C \sum_{k=-\infty}^{\infty} c_k (z-z_0)^{k-n-1} = 2\pi i c_n.$$

To make this argument rigorous, we divide the Laurent series into three parts,

$$f(z) = \sum_{k=2}^{\infty} c_{-k} z^{-k} + \frac{c_{-1}}{z} + \sum_{k=0}^{\infty} c_k z^k \tag{22.1}$$

Interchanging the order of integral and summation that consists of finitely many terms is always justified. We only need to assume that the integral of each term exists. Our task is to show that the integral of the first term and the third term on the right-hand side of (22.1) are both zero.

We recall that, when we say that the Laurent series converges, it means that the analytic part and the principal part converge. The third term is a power series $\sum_{k=0}^{\infty} c_k z^k$, and by assumption, it is convergent in the annulus $R_1 < |z| < R_2$. Because the region of convergence is an open disc, the power series indeed converges at all point in the disc $|z| < R_2$, and is a holomorphic function of z for $|z| < R_2$. Because an open disc is convex, by Theorem 16.4, the power series has a primitive function in the open disc, and hence the integral of this power series is zero if the contour is a closed curve inside the region of convergence. Therefore

$$\oint_C \Big(\sum_{k=0}^{\infty} c_k z^k \Big)\, dz = 0.$$

We now consider the first summation in (22.1). We want to show that it has anti-derivative in the region $|z| > R_1$. We first make a substitute $w = 1/z$. Let $g(w)$ denote the Taylor series

$$g(w) \triangleq \sum_{k=2}^{\infty} c_{-k} w^k.$$

Since $f(z)$ converges in the annulus $R_1 < |z| < R_2$, we know that the power series $g(w)$ is convergent in the region $R_2^{-1} < |w| < R_1^{-1}$. However, if $g(w)$ converges at any point w in this annulus, it also converges for all point with modulus less than $|w|$ (using the fact that power series is absolute convergent). We thus see that the power series $g(w)$ indeed converges in the open disc $|w| < R_1^{-1}$.

Consider the function $h(w)$ represented by the series

$$h(w) \triangleq \sum_{k=2}^{\infty} \frac{c_{-k}}{k-1} w^{k-1}.$$

This has the same radius of convergence as $g(w)$ by the Hadamard formula. We differentiate with respect to variable w,

$$\frac{d}{dw}h(w) = \sum_{k=2}^{\infty} c_{-k} w^{k-2}.$$

Here, exchange of derivative and infinite sum is permitted because it is a power series (Theorem 18.6). We get the relationship between power series $h(w)$ and power series $g(w)$,

$$\frac{d}{dw}h(w) = w^{-2} g(w). \tag{22.2}$$

We substitute w by $1/z$ and apply the chain rule of differentiation,

$$\begin{aligned}\frac{d}{dz}h(1/z) &= \frac{d}{dw}h(w)\Big|_{w=1/z} \cdot \Big(\frac{1}{z}\Big)' \\ &= (1/z)^{-2} g(1/z) \cdot (-1/z^2) \\ &= -g(1/z).\end{aligned}$$

This proves that $\sum_{k=2}^{\infty} c_{-k} z^{-k}$ has anti-derivative $-h(1/z)$ in the region $|z| > R_1$. Because it has anit-derivative, if we integrate it over any closed contour in the region $|z| > R_1$, the result is equal to 0.

Finally, by Cauchy integral formula (Theorem 17.1), the integral of $\oint_C c_{-1}/z\, dz$ is the same as $2\pi i c_{-1}$.

With all these preparation, we now integrate both sides of (22.1),

$$\begin{aligned}\oint_C f(z)\, dz &= \oint_C \Big(\sum_{k=2}^{\infty} c_{-k} z^{-k} \Big)\, dz + \oint_C \frac{c_{-1}}{z}\, dz + \oint_C \Big(\sum_{k=0}^{\infty} c_k z^k \Big)\, dz \\ &= 0 + 2\pi i c_{-1} + 0 = 2\pi i c_{-1}.\end{aligned}$$

We thus obtain

$$c_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz, \tag{22.3}$$

The other coefficients in the Laurent series can be obtained by dividing $f(z)$ by z^{n+1}, for some integer n, and integrate the result along the curve C. The coefficient that is associated with the term with degree -1 in $f(z)/z^{n+1}$ is c_n. Therefore, by repeating the above argument, we have

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz.$$

for all integers n.

In summary, the coefficients in the Laurent series expansion is uniquely determined by the function value within the annulus of convergence. This means that if we apply two different methods to derive Laurent series of a given function, the two answers are the same. In particular, we see that the residue of a function at given point is uniquely determined.

22.2 Calculation of residues

For pole with small order, the residue can be computed efficiently. If z_0 is a pole of $f(z)$ with order m, then

$$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$
$$(z-z_0)^m f(z) = b_m + \cdots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots .$$

We can extract the coefficient b_1 by differentiating it $m-1$ times, taking limit as $z_{\rightarrow} z_0$, and dividing by $(m-1)!$. In particular, for pole with order 1,

$$\mathrm{Res}(f;z_0) = \lim_{z\to z_0} (z-z_0)f(z),$$

and for pole with order 2,

$$\mathrm{Res}(f;z_0) = \lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 f(z).$$

We summarize in the following theorem.

Theorem 22.1. *Suppose $f(z)$ has a pole of order m at the point z_0. The residue can be computed by*

$$\mathrm{Res}(f;z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big).$$

There is a useful formula for computing the residue of function in the form of $P(z)/Q(z)$.

Proposition 22.2. *If $f(z) = \frac{P(z)}{Q(z)}$, where $Q(z)$ has a simple zero at z_0 and $P(z)$ is holomorphic at z_0, then*

$$\mathrm{Res}(f;z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Proof. Suppose $Q(z)$ has a simple zero at z_0. By definition, it means that the power series expansion of $Q(z)$ at z_0 has the form

$$Q(z) = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$$

where a_1 is a nonzero constant. We factor out the common factor $(z - z_0)$,

$$Q(z) = (z - z_0)(a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + \cdots).$$

Denote the power series in the parenthesis by $g(z)$, so that

$$Q(z) = (z - z_0)g(z).$$

The function $g(z)$ is holomorphic in a neighborhood of z_0 (Theorem 18.6), and has the property that $g(z_0) = a_1 \neq 0$.

The residue of $f(z)$ at z_0 equals

$$\begin{aligned}\lim_{z \to z_0}(z - z_0)f(z) &= \lim_{z \to z_0}(z - z_0)\frac{P(z)}{(z - z_0)g(z)} \\ &= \lim_{z \to z_0}\frac{P(z)}{g(z)} \\ &= \frac{P(z_0)}{g(z_0)}.\end{aligned}$$

By product rule for differentiation,

$$Q'(z) = g(z) + (z - z_0)g'(z).$$

We see that $g(z_0)$ is the same as $Q'(z_0)$. We obtain

$$\text{Res}(f; z_0) = \frac{P(z_0)}{g(z_0)} = \frac{P(z_0)}{Q'(z_0)}.$$

□

Example 22.1. The function $f(z) = (z^2 + 1)/(z^3 - 1)$ has simple pole at the cube roots of unity. Let $\omega = e^{2\pi i/3}$. We compute the residue of $f(z)$ at 1, ω and ω^2 using the formula in Prop. 22.2. In the calculations below, we use the relation $1 + \omega + \omega^2 = 0$.

$$\begin{aligned}\text{Res}(f; 1) &= \frac{1^2 + 1}{3(1)^2} = \frac{2}{3} \\ \text{Res}(f; \omega) &= \frac{\omega^2 + 1}{3(\omega)^2} = \frac{-\omega}{3\omega^2} = \frac{\omega}{3} \\ \text{Res}(f; \omega^2) &= \frac{(\omega^2)^2 + 1}{3(\omega^2)^2} = \frac{\omega + 1}{3\omega} = \frac{-\omega^2}{3\omega} = -\frac{\omega}{3}.\end{aligned}$$

Example 22.2. Compute

$$\int_C \frac{dz}{z(z-1)(z-2)}$$

with C being the contour $|z| = 1.5$ with counter-clockwise orientation.

The contour C contains two poles at $z = 0$ and $z = 1$. The residues at these two poles are

$$\begin{aligned}
\operatorname{Res}\Big(\frac{1}{z(z-1)(z-2)}; 0\Big) &= \lim_{z\to 0} z \frac{1}{z(z-1)(z-2)} = \frac{1}{2} \\
\operatorname{Res}\Big(\frac{1}{z(z-1)(z-2)}; 1\Big) &= \lim_{z\to 1} (z-1) \frac{1}{z(z-1)(z-2)} = -1.
\end{aligned}$$

Apply residue theorem (Theorem 21.8),

$$\int_{|z|=1.5} \frac{dz}{z(z-1)(z-2)} = 2\pi i(\frac{1}{2} - 1) = -\pi i.$$

Example 22.3. Repeat Example 22.2 with contour C replaced a circle of radius 3 centered at the origin (counter-clockwise).

The contour now contains all three poles at 0, 1 and 2. The residue at $z = 2$ is

$$\operatorname{Res}\Big(\frac{1}{z(z-1)(z-2)}; 2\Big) = \lim_{z\to 2} (z-2) \frac{1}{z(z-1)(z-2)} = \frac{1}{2}.$$

By residue theorem (Theorem 21.8), the answer is

$$\int_{|z|=3} \frac{1}{z(z-1)(z-2)}\, dz = 2\pi i[\frac{1}{2} - 1 + \frac{1}{2}] = 0.$$

Example 22.4. Evaluate

$$\int_C \frac{dz}{z(z-1)^2}$$

over the contour $C : |z| = 2$ with counter-clockwise orientation.

The contour C encloses the simple pole at $z = 0$ and the double pole at $z = 1$.

$$\begin{aligned}
\operatorname{Res}\Big(\frac{1}{z(z-1)^2}; 0\Big) &= \lim_{z\to 0} z \frac{1}{z(z-1)^2} = 1 \\
\operatorname{Res}\Big(\frac{1}{z(z-1)^2}; 1\Big) &= \lim_{z\to 1} \frac{d}{dz}\Big[(z-1)^2 \frac{1}{z(z-1)^2}\Big] = -1.
\end{aligned}$$

By residue theorem, the integral is equal to $2\pi i(1 + (-1)) = 0$.

23 Evaluation of real integral

Summary

- Semi-circular contour
- Jordan lemma

23.1 Semi-circular contour

Example 23.1. Derive

$$\int_{-\infty}^{\infty} \frac{\cos(bx)}{1+x^2}\,dx = \frac{\pi}{e^b} \quad \text{for } b > 0.$$

We complexify the integral into a contour integral

$$\int_{-\infty}^{\infty} \frac{e^{ibz}}{1+z^2}\,dz.$$

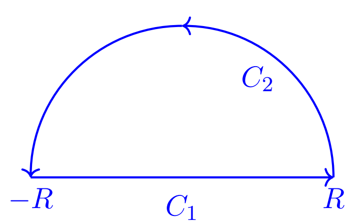
In this integral we integrate along the real axis. The original integral is the real part of this complex integral. The imaginary part is zero, because $\sin(x)/(1+x^2)$ is an odd function. To be more accurate, we emphasize that what we are computing is the principal value

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ibz}}{1+z^2}\,dz,$$

so that the imaginary part $\lim_{R\to\infty} \int_{-R}^{R} \sin(x)/(1+x^2)\,dx$ is indeed equal to zero. The problem reduces to showing

$$\int_{-\infty}^{\infty} \frac{e^{ibz}}{1+z^2}\,dz = \frac{\pi}{e^b} \quad \text{for } b > 0.$$

Let C_1 and C_2 be the contours in the figure below.



We compute the complex integral

$$\int_{C_1+C_2} \frac{e^{ibz}}{1+z^2}\,dz$$

along the closed path formed by C_1 and C_2. When R is large enough, the contour includes the simple pole at $z = i$. By residue theorem (Theorem 21.8), the complex integral is equal to

$$\int_{C_1+C_2} \frac{e^{ibz}}{1+z^2}\,dz = 2\pi i \operatorname{Res}(\frac{e^{ibz}}{1+z^2}; i) = 2\pi i \frac{e^{-b}}{2i} = \frac{\pi}{e^b}.$$

We next show that the integral long the semi-circular path C_2 tends to zero as the radius increases. When z is on C_2, we can write z as $Re^{i\theta}$, for $0 \le \theta \le \pi$. The modulus of e^{iz} on C_2 is upper bounded by 1, because

$$|e^{ibz}| = |e^{ibR(\cos\theta + i\sin\theta)}| = e^{-bR\sin\theta},$$

which is less than 1 when $0 \le \theta \le \pi$. Let R be a sufficiently larger number so that $|1+z^2| > R^2 - 1$ for all z on the circle $|z| = R$. By ML inequality (Theorem 15.4),

$$\Big| \int_{C_2} \frac{e^{ibz}}{1+z^2}\,dz \Big| \le \frac{1}{R^2-1}(\pi R).$$

Putting all the calculations together, we obtain

$$\int_{C_1+C_2} \frac{e^{ibz}}{1+z^2}\,dz - \int_{C_1} \frac{e^{ibz}}{1+z^2}\,dz = \int_{C_2} \frac{e^{ibz}}{1+z^2}\,dz$$

$$\Big| \frac{\pi}{e^b} - \int_{-R}^{R} \frac{\cos(bx)}{1+x^2}\,dx \Big| \le \frac{\pi R}{R^2-1}.$$

This holds whenever $R > 1$. Since $R/(R^2-1)$ has limit zero as $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(bx)}{1+x^2}\,dx = \frac{\pi}{e^b}.$$

We check that when $b = 0$, the integral is simplified to a standard result in calculus:

$$\int_{\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi.$$

We mentioned in Example 13.4 that Fourier transform and Laplace transform are important examples of complex integrals. In this section we illustrate how to evaluate some real integrals that are related to Fourier transform. The Jordan lemma is a useful tool for this purpose.

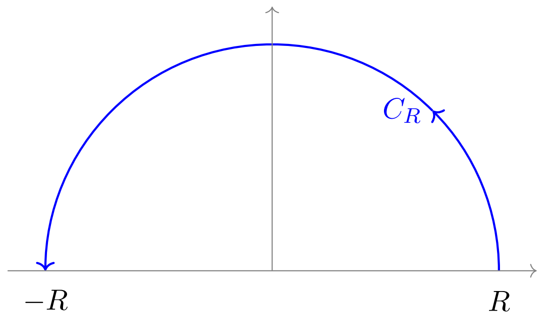


Figure 36: The semi-circular contour in Jordan lemma.

23.2 Jordan lemma

The Jordan lemma provides a convenient way to check that the integral on the semi-circular arc tends to zero as the radius tends to infinity.

Lemma 23.1 (Jordan lemma)**.** *Consider the semi-circular contour C_R in Fig. 36. Suppose that for all sufficiently large R, $f(z)$ is holomorphic on C_R, and we can find a function M_R of R that approaches 0 as R approaches ∞, such that $|f(z)| \leq M_R$ for all z on C_R. Then for any positive real constant a,*

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\, dz = 0.$$

Proof. The idea of proof is to represent the circular path by $z = Re^{i\theta}$, for $\theta \in [0, \pi]$. Substituting z by $Re^{i\theta}$ in e^{iaz}, we obtain

$$e^{iaR(\cos\theta + i\sin\theta)} = e^{-aR\sin\theta + iaR\cos\theta}.$$

The magnitude is $e^{-aR\sin\theta}$. Hence it is of interest to give a good estimate to the integral

$$\int_0^\pi e^{-aR\sin\theta}\, d\theta.$$

We prove the following claim. For any positive real number a and radius R, we have the inequality

$$\int_0^\pi e^{-aR\sin\theta}\, d\theta < \frac{\pi}{aR}. \tag{23.1}$$

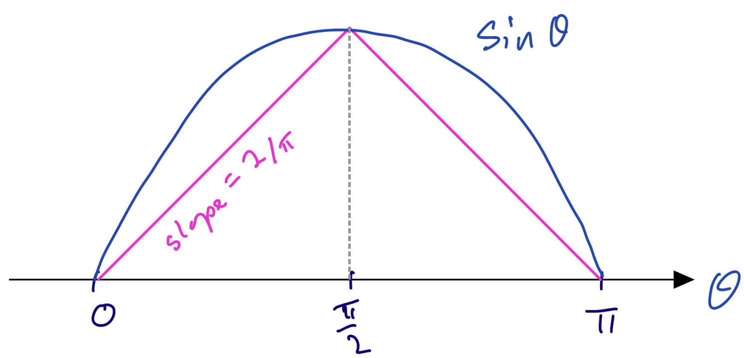


Figure 37: Proof of Jordan inequality

For θ in the range $0 \le \theta \le \pi/2$, the value $\sin\theta$ is larger than or equal to $2\theta/\pi$. We can see this by drawing a line segment from the origin to the point $(\pi/2, 1)$, and see that the graph of $\sin\theta$ is below the graph of $\sin\theta$ in this range. (See Fig. 37) We thus get an upper bound

$$\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta \le \int_0^{\pi/2} e^{-aR(2\theta/\pi)}\,d\theta = \left[\frac{-\pi}{2aR}e^{-2b\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{2aR}(1-e^{-aR}) < \frac{\pi}{2aR}.$$

By using the symmetry of the graph of sine function, the same argument apply to the second part of the interval from $\pi/2$ to π,

$$\int_{\pi/2}^{\pi} e^{-aR\sin\theta}\,d\theta < \frac{\pi}{2aR}.$$

This proves (23.1) for any positive real numbers a and R.

We apply this inequality to derive an upper bound

$$\begin{aligned}\left|\int_{C_R} f(z)e^{iaz}\,dz\right| &\le \int_0^{\pi} \left|f(Re^{i\theta})\right|e^{-aR\sin\theta}\cdot|Rie^{i\theta}|\,d\theta \\ &\le RM_R\int_0^{\pi} e^{-aR\sin\theta}\,d\theta \\ &= M_R\frac{\pi}{a}.\end{aligned}$$

Since a is a constant and M_R approaches 0 as $R\to\infty$, the upper bound approaches 0 as R approaches ∞. $\square$

Jordan lemma is useful in evaluating integral of type

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x)\, dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x)\, dx$$

where $P(x)$ and $Q(x)$ are polynomials and $\deg Q - \deg P \geq 1$.

Example 23.2. Evalaute the integral

$$\int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx,$$

where a and b are positive constants with $a > b$.

Because the function to be integrated is an even function, we can transform the problem to the evaluation of

$$\int_{-R}^{R} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx,$$

and take limit as R approaching infinity. This integral above is the same as the contour integral

$$\int_{L_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\, dz$$

on the line segment L_R from $-R$ to R. Let C_R denote the semi-circular path in Fig. 36 from R to $-R$ along the circle $|z| = R$ on the upper-half plane. By residue theorem (Theorem 21.8), the integral

$$\int_{L_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} + \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

is equal to $2\pi i$ time the sum of residue of the integrand inside the semi-circle bounded by L_R and C_R.

The integrand has two simple poles in the upper half plane at $z = ai$ and $z = bi$. The residue at the pole ai equals

$$\operatorname{Res}\left(\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}; ai\right) = \frac{e^{iz}}{(z+ai)(z^2+b^2)}\Big|_{z=ai} = \frac{e^{-a}}{2ai(b^2-a^2)}.$$

The other residue at bi is

$$\operatorname{Res}\left(\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}; bi\right) = \frac{e^{iz}}{(z+bi)(z^2+a^2)}\Big|_{z=bi} = \frac{e^{-b}}{2bi(a^2-b^2)}.$$

On the other hand, because

$$\Big|\frac{1}{(z^2+a^2)(z^2+b^2)}\Big| = \frac{1}{(|z|^2+a^2)(|z|^2+b^2)}$$

approaches 0 as $|z| \to \infty$, by applying Jordan lemma, we can see that the integral on C_R approaches 0 as R approaches infinity. Therefore

$$\begin{aligned}\lim_{R\to\infty} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz &= 2\pi\Big(\frac{e^{-a}}{2ai(b^2-a^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}\Big)\\ &= \frac{\pi}{2(a^2-b^2)}\Big(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\Big).\end{aligned}$$

The final answer is

$$\int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{4(a^2-b^2)}\Big(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\Big)$$

Example 23.3. Show that

$$\int_0^\infty \frac{\cos x}{x^4+x^2+1}\,dx = \frac{\pi}{\sqrt{3}}e^{-\sqrt{3}/2}\sin\big(\frac{1}{2}+\frac{\pi}{6}\big)$$

We first note that the integrand is an even function. It is sufficient to show that

$$\int_{-\infty}^\infty \frac{\cos x}{x^4+x^2+1}\,dx = 2\frac{\pi}{\sqrt{3}}e^{-\sqrt{3}/2}\sin\big(\frac{1}{2}+\frac{\pi}{6}\big).$$

Since $\frac{\cos x}{x^4+x^2+1}$ is an odd function. We can show this by evaluating the principal value

$$\int_{-\infty}^\infty \frac{e^{iz}}{z^4+z^2+1}\,dz.$$

The roots of z^4+z^2+1 are sixth roots of unity, because

$$(z^2-1)(z^4+z^2+1) = z^6-1.$$

There are two roots that lie on the upper-half plane, namely

$$\alpha \triangleq e^{i\pi/6} = \frac{1}{2}+\frac{\sqrt{3}i}{2}, \text{ and } \beta \triangleq e^{i\pi/3} = -\frac{1}{2}+\frac{\sqrt{3}i}{2}.$$

We let

$$f(z) = \frac{e^{iz}}{z^4+z^2+1}$$

Let L_R denote the line segment on the real axis from $-R$ to R, and C_R be the semi-circle in the upper half plane from R to $-R$, centered at the origin. By residue theorem, we know that for all $R > 1$,

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i(\mathrm{Res}(f,\alpha) + \mathrm{Res}(f,\beta)). \tag{23.2}$$

The first integral is equal to the real integral

$$\int_{-R}^{R} \frac{\cos x}{x^4 + x^2 + 1}\,dx.$$

By Jordan lemma, the second integral tends to 0 as R tends to ∞.

Using Theorem 22.2, we calculate the two residues below:

$$\begin{aligned}
\mathrm{Res}(f;\alpha) &= \frac{e^{i\alpha}}{4\alpha^3 + 2\alpha} = \frac{e^{i/2-\sqrt{3}/2}}{-3+\sqrt{3}i} \\
\mathrm{Res}(f;\beta) &= \frac{e^{i\alpha}}{4\beta^3 + 2\beta} = \frac{e^{-i/2-\sqrt{3}/2}}{3+\sqrt{3}i}
\end{aligned}$$

The sum of residues at α and β equal

$$\begin{aligned}
e^{-\sqrt{3}/2}\Big[\frac{e^{i/2}}{-3+\sqrt{3}i} + \frac{e^{-i/2}}{3+\sqrt{3}i}\Big] &= e^{-\sqrt{3}/2}\frac{3(e^{i/2} - e^{-i/2}) + \sqrt{3}i(e^{i/2} + e^{-i/2})}{3i^2 - 9} \\
&= e^{-\sqrt{3}/2}\frac{6i\sin(1/2) + 2\sqrt{3}i\cos(1/2)}{-12} \\
&= -\frac{ie^{-\sqrt{3}/2}}{\sqrt{3}}\big[\frac{\sqrt{3}}{2}\sin(1/2) + \frac{1}{2}\cos(1/2)\big] \\
&= -\frac{ie^{-\sqrt{3}/2}}{\sqrt{3}}\sin(1/2 + \pi/6).
\end{aligned}$$

Substituting it back to (23.2), and taking limit as R approaching infinity, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1}\,dx = \frac{2\pi e^{-\sqrt{3}/2}}{\sqrt{3}}\sin(1/2 + \pi/6).$$

24 Indented contour and keyhole contour

Summary

- Indented contour
- Keyhole contour

24.1 Indented contour

In some examples the contour contains one or more poles of the integrand. We can approximate the contour integral by adding some dents on the contour.

A common contour is shown in Fig. 38. The indented contour is contained in a star-shaped domain. For example, we can consider the whole complex plane with the negative imaginary axis removed.

$$\mathbb{C} \setminus \{x+iy : \, y \leq 0, x = 0\}.$$

This is a star-shaped domain. We may take the point i as the center of the star shape. If a complex function has a pole at the origin, and is holomophic everywhere else, by invoking the Cauchy theorem for star-shaped region (Theorem 16.6), we can see that any closed contour within this domain has zero integral. So in particular, the integral of this function over the indented circular path is zero.

Example 24.1. In this example we derive the Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$

The function $\sin(x)/x$ has a removable singularity at $x = 0$, as we can see from the power series expansion:

$$\frac{1}{x}(x - x^3/3! + x^5/5! - \cdots) = 1 - x^2/3! + x^4/5! - \cdots$$

The function to be integrated is a continuous function.

We want to replace $\sin x$ by e^{iz} in the integral, so that $\sin x$ is the imaginary part of e^{ix} when x is real number. We note that the function e^{iz}/z has a simple pole at the origin.

Since $\frac{\sin x}{x}$ is an even function and $\frac{\cos x}{x}$ is odd, it is sufficient to prove

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \Big(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \Big) \frac{e^{iz}}{z} \, dz = \pi i.$$

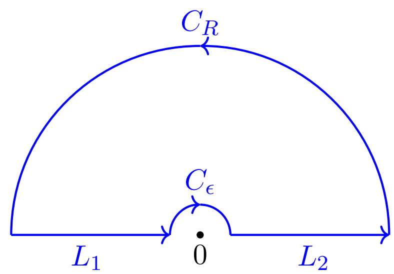


Figure 38: Indented semi-circular path. The outer semi-circle has radius R and the inner semi-circle has radius ϵ.

Note that we avoid the pole of $\frac{e^{iz}}{z}$ at the origin by not including the interval $[-\epsilon, \epsilon]$. We consider the indented contour in Fig. 38.

We have

$$\int_{L_1+L_2+C_R+C_\epsilon} \frac{e^{iz}}{z}\, dz = 0.$$

By Jordan lemma (Lemma 23.1), the integral of $\frac{e^{iz}}{z}$ along C_R decreases to 0 as $R \to \infty$. The integrals on the real axis approaches $\int_{-\infty}^{\infty} e^{iz}/z\, dz$ as $R \to \infty$ and $\epsilon \to 0$. The problem reduces to proving

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = -\pi i.$$

The integrand can be represented by Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2} - \frac{iz^2}{6} + \cdots$$

For small enough $\epsilon > 0$, the analytic part $i - \frac{z}{2} - \frac{iz^2}{6} + \cdots$ is bounded (because it converges and is continuous at $z = 0$). (Another way to see this is by observing $(e^{iz} - 1)/z$ has a removable singularity at $z = 0$.)

By ML inequality (Theorem 15.4),

$$\Big| \int_{C_\epsilon} i - \frac{z}{2} - \frac{iz^2}{6} + \cdots\, dz \Big| \to 0$$

as $\epsilon \to 0$. The integral of $1/z$ on C_ϵ is equal to

$$\int_{C_\epsilon} \frac{1}{z}\, dz = \int_{\pi}^{0} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\, d\theta = -\pi i.$$

This proves that

$$\int_{L_1+L_2} \frac{e^{iz}}{z}\,dz + \int_{C_\epsilon} \frac{1}{z}\,dz = -\int_{C_R} \frac{e^{iz}}{z}\,dz - \int_{C_\epsilon} i - \frac{z}{2} - \frac{iz^2}{6} + \cdots\,dz.$$

Take absolute value of both sides and take limit as $R \to \infty$ and $\epsilon \to 0$,

$$\Big|\int_{L_1+L_2} \frac{e^{iz}}{z}\,dz - \pi i\Big| \leq \Big|\int_{C_R} \frac{e^{iz}}{z}\,dz\Big| + \Big|\int_{C_\epsilon} i - \frac{z}{2} - \frac{iz^2}{6} + \cdots\,dz\Big|.$$

As the two terms on the right-hand side tends to zero as $R \to \infty$ and $\epsilon \to 0$, we obtain

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_{L_1+L_2} \frac{e^{iz}}{z}\,dz = \pi i.$$

This completes the derivation of the Dirichlet integral.

We extract a useful result that is used in the previous example.

Theorem 24.1. *Suppose $f(z)$ has a simple pole at z_0 with residue*

$$\mathrm{Res}(f; z_0) = \lim_{z\to z_0} (z - z_0) f(z).$$

Let C_r denote a part of a circle with radius r and positive orientation, and with angle θ_1 to θ_2, relative to the center z_0. That is, it is parameterized by $z_0 + re^{i\theta}$for $\theta_1 \leq \theta \leq \theta_2$. Then

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = i(\theta_2 - \theta_1)\,\mathrm{Res}(f; z_0).$$

Proof. See the notes of Tutorial 10. ☐

Example 24.2. In this example we compute the real integral

$$\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$$

using complex analysis.

The complex cube root function $\sqrt[3]{z}$ is a multi-valued function. We put the branch cut a the negative imaginary axis in the computation of $\sqrt[3]{z}$. More precisely, given a complex number that does not lie on the negative imaginary axis, we write it in polar form as

$$z = re^{i\theta}$$

for $r > 0$ and $-\pi/2 < \theta < 3\pi/2$, we define the branch of $\sqrt[3]{z}$ by $\sqrt[3]{r}e^{i\theta/3}$.

With the branch of $\sqrt[3]{z}$ so defined, we integrate the function

$$f(z) = \frac{\sqrt[3]{z}}{z^2+1}$$

on the indented contour in Fig. 38. Note that we avoid the branch point at the origin. By the residue theorem (Theorem 21.8), the integral is equal to the residue at the pole at i multiplied by $2\pi i$. We calculate the pole at i by the formula in Prop. 22.2.

$$\text{Res}(\frac{\sqrt[3]{z}}{z^2+1}; i) = \frac{\sqrt[3]{z}}{2z}\Big|_{z=i} = \frac{e^{i\pi/6}}{2i}.$$

By decomposing the indented contour into four parts, we obtain

$$\int_{C_R} \frac{\sqrt[3]{z}\,dz}{z^2+1} + \int_{L_1} \frac{\sqrt[3]{z}\,dz}{z^2+1} + \int_{C_\epsilon} \frac{\sqrt[3]{z}\,dz}{z^2+1} + \int_{L_2} \frac{\sqrt[3]{z}\,dz}{z^2+1} = \pi e^{i\pi/6}.$$

The integral of $f(z)$ on L_1 equals

$$\int_{L_1} \frac{\sqrt[3]{z}dz}{z^2+1} = \int_{\epsilon}^{R} \frac{\sqrt[3]{x}}{x^2+1}\,dx.$$

We represents the line segment L_2 by parameter x for x from $-R$ to $-\epsilon$. The integral on L_2 is

$$\int_{L_2} \frac{\sqrt[3]{z}dz}{z^2+1} = \int_{-R}^{-\epsilon} \frac{\sqrt[3]{|x|}e^{i\pi/3}}{x^2+1}\,dx = e^{i\pi/3}\int_{\epsilon}^{R} \frac{\sqrt[3]{x}}{x^2+1}\,dx.$$

The integral on C_R approaches 0 as $R \to 0$, because

$$\Big|\int_{C_R} \frac{\sqrt[3]{z}dz}{z^2+1}\Big| \le \frac{\sqrt[3]{R}}{|R^2-1|}(\pi R)$$

by ML inequality. When $R > 1$, the ratio $\frac{R^{1+1/3}}{R^2-1}$ has limit 0 as $R \to \infty$.

Lastly, we consider the integral over C_ϵ for $0 < \epsilon < 1$,

$$\Big|\int_{C_\epsilon} \frac{\sqrt[3]{z}dz}{z^2+1}\Big| \le \frac{\sqrt[3]{\epsilon}}{1-\epsilon^2}(\pi\epsilon).$$

The ratio $\frac{\epsilon^{1+1/3}}{1-\epsilon^2}$ tends to 0 as ϵ tends to 0.

By taking limits as $\epsilon \to 0$ and $R \to \infty$, we obtain

$$(1+e^{i\pi/3})\int_0^\infty \frac{\sqrt[3]{x}}{x^2+1} = \pi e^{i\pi/6}.$$

Divide both side by $(1+e^{i\pi/3})$,

$$\begin{aligned}\int_0^\infty \frac{\sqrt[3]{x}}{x^2+1} &= \pi\frac{e^{i\pi/6}}{1+e^{i\pi/3}} \\ &= \frac{\pi}{e^{i\pi/6}+e^{-i\pi/6}} \\ &= \frac{\pi}{2\cos(\pi/6)} = \frac{\pi}{\sqrt{3}}.\end{aligned}$$

24.2 Keyhole contour

A keyhole contour has shape as shown in Fig. 39. We can consider the keyhole contour when the integrand is a multi-value function. We can pick a branch of the function so that the branch cut lie outside the keyhole contour. The keyhole contour consists of four parts. The outer and inner circular part has radius R and ϵ, respectively. The line segments L_1 and L_2 lie above and below the positive real axis. Let 2δ denote the distance between L_1 and L_2. The methodology is to first evaluate the contour on this contour, and then take limit as $\epsilon \to 0$, $\delta \to 0$, and $R \to \infty$.

The region

$$\mathbb{C} \setminus \{x+iy:\ y=0, x \geq 0\}$$

is a star-shaped region. We can thus apply the Cauchy theorem and the residue theorem to the keyhole contour.

Example 24.3. Suppose a is a constant between 0 and 1. We can prove that

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin(a\pi)}$$

by converting it to a complex integral on the keyhole contour.

Consider the complex function

$$f(z) = \frac{z^{a-1}}{1+z}.$$

The numerator is a multi-valued function. We take the branch with branch cut located on the positive real axis. For complex number z that is not on the positive real axis, we write $z = re^{i\theta}$, where $r > 0$ and $0 < \theta < 2\pi$, and calculate z^{a-1} by

$$z^{a-1} = (re^{i\theta})^{a-1} = r^{a-1}e^{i\theta(a-1)}. \tag{24.1}$$

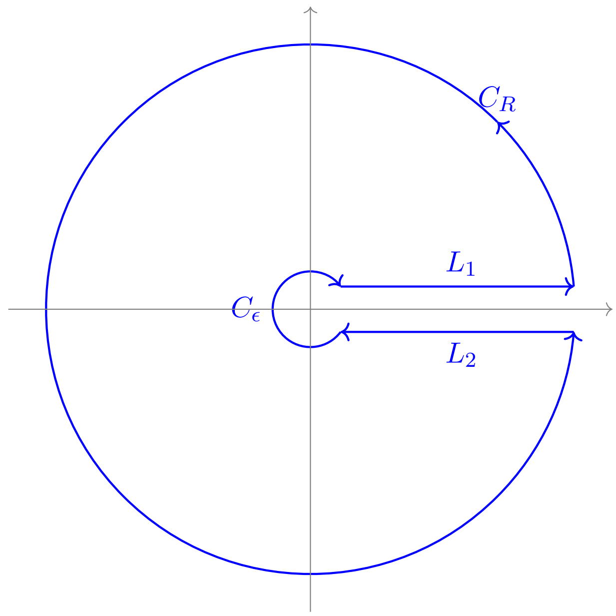


Figure 39: Keyhole contour

Because there is a simple pole inside the keyhole contour at $z = -1$, the integral of $f(z)$ is equal to

$$\left(\int_{C_R} + \int_{L_2} + \int_{C_\epsilon} + \int_{L_1}\right) f(z)\, dz = 2\pi i \operatorname{Res}(f(z); -1).$$

The residue at $z = -1$ is

$$\begin{aligned} \operatorname{Res}(f(z); -1) &= \lim_{z \to -1} (1+z) f(z) \\ &= \lim_{z \to -1} z^{a-1}. \end{aligned}$$

Using the branch of z^{a-1} in (24.1), we obtain

$$\operatorname{Res}(f(z); -1) = e^{i\pi(a-1)}.$$

The integral of $f(z)$ on the horizontal line L_1 has limit

$$\lim_{\substack{R \to \infty \\ \epsilon \to 0 \\ \delta \to 0}} \int_{L_1} f(z)\, dz = \int_0^\infty \frac{x^{a-1}}{1+x}\, dx$$

Because L_2 is on the other side of the branch cut, the variable of integration x is multiplied by a factor of $e^{2\pi i}$,

$$\begin{aligned}\lim_{\substack{R\to\infty\\ \epsilon\to 0\\ \delta\to 0}}\int_{L_2} f(z)\,dz &= -\int_0^\infty \frac{x^{a-1}e^{2\pi i(a-1)}}{1+xe^{2\pi i}}\,dx\\ &= -e^{2\pi(a-1)i}\int_0^\infty \frac{x^{a-1}}{1+x}\,dx.\end{aligned}$$

Using similar argument as in Example 24.2, we can show that the integral of $f(z)$ on C_R and C_ϵ approaches 0.

Putting all the data together, we have

$$\begin{aligned}\int_0^\infty \frac{x^{a-1}}{1+x}\,dx - e^{2\pi(a-1)i}\int_0^\infty \frac{x^{a-1}}{1+x}\,dx &= 2\pi i e^{i\pi(a-1)}\\ \int_0^\infty \frac{x^{a-1}}{1+x}\,dx &= \frac{2\pi i e^{i\pi(a-1)}}{1-e^{2\pi(a-1)i}}\\ &= \frac{2\pi i}{e^{-i\pi(a-1)}-e^{i\pi(a-1)}}\\ &= -\frac{\pi}{\sin((a-1)\pi)}\\ &= \frac{\pi}{\sin(a\pi)}.\end{aligned}$$

This completes the proof in this example.

Remark. The integral formula in the previous example in fact is valid if a is replaced by a complex number with real part between 0 and 1, i.e., if

$$0 < \mathrm{Re}(a) < 1.$$

This condition is required to show that the integrals over the two circular paths approach 0.

We can use the keyhole contour to evaluate real integral of the form

$$\int_0^\infty \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials with $\deg Q \geq \deg P + 2$ and $Q(x) \neq 0$ for $x \geq 0$. The assumption that $Q(x) \neq 0$ for $x \geq 0$ ensures that there is no division by 0 in the integral. We demonstrate the procedure using the following example.

Example 24.4. Evaluate

$$\int_0^\infty \frac{1}{x^3+1}\,dx.$$

Note that the integrand is no an even function. We cannot evaluate it through the integral $\int_{-\infty}^{\infty} 1/(x^3+1)\,dx$.

The trick to multiply the function to be integrated by a log function, and consider the complex integral

$$\int_C \frac{\log z}{z^3+1}\,dz,$$

over the keyhole contour C as shown in Fig. 39. For the complex log function we take the nonnegative real axis as the branch cut, i.e., for complex number in polar form $re^{i\theta}$, with $0<\theta<2\pi$, the log function is evaluated as

$$\log r + i\theta \quad \text{for } 0<\theta<2\pi.$$

The contour C consists of four parts. The outer circle C_R has radius R and positive orientation. The inner circle C_ϵ has radius ϵ and negative orientation. The distance between L_1 and L_2 is 2δ. When $R\to\infty$, $\epsilon\to 0$ and $\epsilon\to 0$, the integrals along L_1 and L_2 have limits

$$\int_{L_1} \frac{\log z}{z^3+1}\,dz \to \int_0^\infty \frac{\log x}{x^3+1}\,dx$$
$$\int_{L_2} \frac{\log z}{z^3+1}\,dz \to -\int_0^\infty \frac{\log x + 2\pi i}{x^3+1}\,dx.$$

The integral over C_ϵ has modulus upper bounded by

$$\Big|\int_{C_\epsilon} \frac{\log z}{z^3+1}\,dz\Big| \le 2\pi\epsilon M_\epsilon \max_{|z|=\epsilon} |\log z|$$

where M_ϵ denotes the maximum of $|1/(z^3+1)|$ on the circle $|z|=\epsilon$. Since it is assumed that $Q(z)$ is defined at $z=0$, M_ϵ can be upper bounded by another constant independent of ϵ. In this example M_ϵ is approach 1 as ϵ approaches 0, and hence we can say that $M_\epsilon<2$ for all sufficiently small ϵ. The modulus of $\log(z)$ is no more than the modulus of $\log(\epsilon)+i2\pi$. Hence, as $\epsilon\to 0$, the modulus of the integral of C_ϵ is upper bounded by a constant times $\epsilon|\log\epsilon + 2\pi i|$, which decreases to zero as $\epsilon\to 0$.

For complex number z on C_R, the modulus $|\log(z)/(z^3+1)|$ is upper bounded by a constant times $\frac{|\log R + 2\pi i|}{R^3-1}$. The integral over C_R approaches 0 as R approaches infinity.

Therefore,

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0\\ R\to\infty}} \int_{L_1+L_2+C_R+C_\epsilon} \frac{\log(z)}{z^3+1}\,dz = -2\pi i \int_0^\infty \frac{1}{x^3+1}\,dx.$$

We can re-write the above equation as

$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\lim_{\substack{\epsilon\to 0\\ \delta\to 0\\ R\to\infty}} \frac{1}{2\pi i}\int_{L_1+L_2+C_R+C_\epsilon} \frac{\log(z)}{z^3+1}\,dz.$$

The polynomial z^3+1 has three roots, namely, -1, $e^{\pi i/3}$ and $e^{5\pi i/3}$. By residue theorem (Theorem 21.8), we can compute the integral by

$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\Big[\operatorname{Res}\big(\frac{\log(z)}{z^3+1};-1\big) + \operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{\pi i/3}\big) + \operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{5\pi i/3}\big)\Big].$$

Since the pole of $\log(z)/(z^3+1)$ are all simple roots, we can use the formula in Prop. 22.2. Evaluate the residues at -1, $e^{\pi i/3}$ and $e^{5\pi i/3}$ by

$$\begin{aligned}
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};-1\big) &= \frac{\log(z)}{3z^2}\Big|_{-1} = \pi i\frac{1}{3}\\
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{\pi i/3}\big) &= \frac{\log(z)}{3z^2}\Big|_{e^{\pi i/3}} = \frac{\pi i}{3}\frac{e^{-2\pi i/3}}{3}\\
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{5\pi i/3}\big) &= \frac{\log(z)}{3z^2}\Big|_{e^{5\pi i/3}} = \frac{5\pi i}{3}\frac{e^{-10\pi/3}}{3}.
\end{aligned}$$

Adding the three residues, we get

$$\frac{i\pi}{3}\big[1+\frac{-1-\sqrt{3}i}{6}+5\frac{-1+\sqrt{3}i}{6}\big] = -\frac{2\sqrt{3}}{9}\pi,$$

and

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\sqrt{3}}{9}\pi.$$

Example 24.5. We give another derivation of the integral in the previous example, using a different contour shown in Fig 40. The function $f(z)$ has a simple pole at $e^{\pi i/3}$ when the radius R is larger than 1. By Residue Theorem, for all sufficiently larger R, we have

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), e^{\pi i/3})$$

We compute the residue of $f(z)$ at the sixth root of unity $e^{\pi i/3}$ by using the formula in Proposition 22.2,

$$\operatorname{Res}(f(z), e^{\pi i/3}) = \frac{1}{3x^2}\Big|_{z=e^{\pi i/3}} = -\frac{1}{3}e^{\pi i/3}.$$

The integral of $f(z)$ along the line segment L_1 is

$$\int_{L_1} f(z)\,dz = \int_0^R \frac{1}{x^3}\,dx.$$

On the other hand, the integral of $f(z)$ from 0 to $Re^{2\pi i/3}$, along the opposite of L_2, is

$$\int_{-L_2} f(z)\,dz = \int_0^R \frac{1}{1+(e^{2\pi i/3}t)^3} e^{2\pi i/3}\,dt = \int_0^R \frac{1}{1+t^3} e^{2\pi i/3}\,dt.$$

The integral on C_R approaches 0 as $R \to \infty$, by using ML inequality.

Therefore, by taking limit as $R \to \infty$, we obtain

$$(1 - e^{2\pi i/3}) \int_0^\infty \frac{1}{1+x^3}\,dx = 2\pi i \frac{1}{3}(-e^{\pi i/3})$$

The answer is equal to

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^3}\,dx &= -\frac{2\pi i e^{\pi/3}}{3(1-e^{2\pi i/3})} \\
&= \pi \frac{2i}{3(e^{\pi/3} - e^{-\pi i/3})} \\
&= \frac{\pi}{3\sin(\pi/3)} \\
&= \frac{2\pi}{3\sqrt{3}} = \frac{2\sqrt{3}\pi}{9}.
\end{aligned}$$

24.3 Appendix: Differentiation and residue at the point of infinity

Definition 24.2. Given a complex function $f(z)$, we say that it is *continuous at ∞* if the function $g(w) \triangleq f(1/w)$ is continuous at $w = 0$. Likewise, the function $f(z)$ is said to be *complex differentiable at ∞* if $g(w) = f(1/w)$ is complex differentiable at $w = 0$, and is *holomorphic at ∞* if $g(w)$ is holomorphic at $w = 0$.

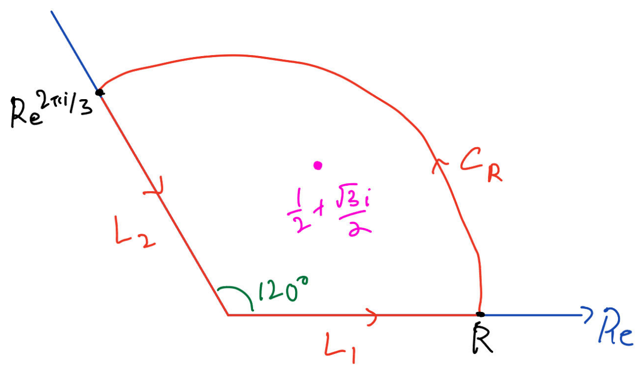


Figure 40: A contour with the shape of a fan.

Example 24.6. Consider the function $f(z) = (z+1)/(z-1)$. It is differentiable at ∞. The function $g(w) = f(1/w)$ is

$$g(w) = \frac{\frac{1}{w}+1}{\frac{1}{w}-1} = \frac{1+w}{1-w}.$$

As $w \to 0$, we have $g(w) \to 1$. The derivative of $g(w)$ is

$$g'(w) = \frac{2}{(1-w)^2}$$

At the point $w = 0$, the derivative of $g(w)$ is $g'(0) = 2$.

To understand the behavior of a function $f(z)$ at the point at infinity, we make a change of variable $w = 1/z$. The variable $w = 1/z$ is called the *local parameter*/*uniformizer* at ∞.

Definition 24.3. Given a complex function $f(z)$, make a change of variable and define a new function $g(w) = f(1/w)$. We say that the function $f(z)$ is analytic at $z = \infty$ if $g(w)$ is analytic at $w = 0$. The point at infinity is said to be a removable singularity (resp. pole, or essential singularity) if $g(w)$ has a removable regularity (resp. pole, or essential singularity) at $w = 0$.

Example 24.7. The function $f(z) = z + 8$ has a simple pole at $z = \infty$, because $g(w) = f(1/w) = \frac{1}{w} + 8$ has a simple pole at $w = 0$.

Example 24.8. The function $f(z) = (z+1)/z^3$ has a doube zero at $z = \infty$, because $g(w) = f(1/w) = w^2 + w^3$ has a double zero at $w = 0$.

Example 24.9. The function $f(z) = e^z$ has an essential singularity at $z = \infty$, because

$$g(w) = \exp(1/w) = 1 + \frac{1}{w} + \frac{1}{2w^2} + \cdots$$

has an essential singularity at $w = 0$.

The residue at the point at infinity is a little bit more complicated. We take the definition by contour integral as the definition of residue at the point at infinity.

Definition 24.4. Suppose $f(z)$ has finitely many singular points in the complex plane, so that $f(z)$ is holomorphic in the domain $|z| > R$ for some R. The *residue at ∞* of $f(z)$ is defined as

$$\operatorname{Res}(f;\infty) \triangleq \frac{1}{2\pi i}\oint_{C'} f(z)\,dz$$

where C' is a circle containing all singular points in the interior, with *clockwise orientation.*

The assumption that $f(z)$ has finitely many singular points is the same as assuming that the point at infinity is an isolated singular point.

By making a change of variable $w = 1/z$, $dw = -1/z^2\,dz$, we get

$$\frac{1}{2\pi i}\int_{C'} f(z)\,dz = \frac{1}{2\pi i}\int_{C} \frac{-1}{w^2} f\big(\frac{1}{w}\big)\,dw = \operatorname{Res}\Big(\frac{-1}{w^2} f\big(\frac{1}{w}\big);0\Big)$$

where C is the image of C' under the transformation $w = 1/z$. In the w-plane, the function $\frac{-1}{w^2} f\big(\frac{1}{w}\big)$ has a isolated singularity at $w = 0$. This proves the following theorem.

Theorem 24.5. *Suppose f has finitely many singular points and C is a contour with counter-clockwise orientation, containing all singular points in the interior. Then*

$$\frac{1}{2\pi i}\oint_{C} f(z)\,dz = \operatorname{Res}\big(\frac{1}{z^2} f\big(\frac{1}{z}\big);0\big).$$

We can compare with (22.2), where we also have a factor of $1/w^2$. The explanation for the appearance of this factor is that the notion residue is actually defined for differential form $f(z)\,d(z-z_0)$, where $z-z_0$ is the local parameter at z_0. In the finite complex plane, the differential $d(z-z_0)$ is the same for all z_0. So, for simplicity we usually say "residue of a function $f(z)$ at z_0". However, for the point at infinity, the local parameter is $1/z$, and the relationship between $d(1/z)$ and dz is given by

$$d(1/z) = -z^{-2}\,dz.$$

So, the differential $f(w)dw$ with local parameter $w = 1/z$ is converted to a usual differential by

$$f(w)\,dw = f(1/z)\frac{dw}{dz}\,dz = f(1/z)(-z^{-2})\,dz.$$

Example 24.10. We can use residue at infinity to do Example 22.3. To find the residue of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

at the point at infinity, we make a substitute $w = 1/z$ and calculate

$$f(1/w)(1/w^2) = \frac{1}{\frac{1}{w}\left(\frac{1}{w}-1\right)\left(\frac{1}{w}-2\right)}\frac{1}{w^2} = \frac{w}{(1-w)(1-2w)}.$$

This function is holomorphic at $w = 0$. Therefore, the residue of $f(z)$ at the point at infinity is 0, and hence the integral $\int_C f(z)\,dz$ is zero. This conforms with the answer in Example 22.3.

Example 24.11. Evaluate

$$\oint_{|z|=2} \frac{4z+1}{z(z-1)}\,dz.$$

There are two simple poles at $z = 0$ and $z = 1$. Both of them are inside the contour. We compute the answer by three methods.

<u>Method 1.</u> Using the residue at infinity, we can calculate the complex integral by

$$\begin{aligned}\oint_{|z|=2} \frac{4z+1}{z(z-1)}\,dz &= 2\pi i \operatorname{Res}\left(\frac{1}{w^2}\frac{4(1/w)+1}{(1/w)((1/w)-1)}; 0\right)\\ &= 2\pi i \operatorname{Res}\left(\frac{4+w}{w(1-w)}; 0\right)\\ &= 2\pi i \cdot 4 = 8\pi i.\end{aligned}$$

Method 2. By the residue theorem (Theorem 21.8), The integral is equal to $2\pi i$ times the sum of the residues of the integrand at $z = 0$ and $z = 1$. Let $f(z) = (4z+1)/(z(z-1))$. The residues of $f(z)$ at $z = 0$ and $z = 1$ are, respectively,

$$\begin{aligned}\operatorname{Res}(f;0) &= \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{4z+1}{z-1} = -1 \\ \operatorname{Res}(f;1) &= \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1} \frac{4z+1}{z} = 5.\end{aligned}$$

The integral is thus equal to $2\pi i(5-1) = 8\pi i$.

Method 3. Expand the function $f(z)$ using partial fraction expansion. Write $f(z)$ as

$$\frac{4z+1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

for some constants A and B. The constants A and B are solutions to

$$\begin{aligned}A + B &= 4 \\ -A &= 1.\end{aligned}$$

So, we have $A = -1$ and $B = 5$, and

$$\oint_{|z|=2} f(z)\,dz = -\oint_{|z|=2} \frac{1}{z}\,dz + 5\oint_{|z|=2} \frac{1}{z-1}\,dz = (-1)(2\pi i) + 5(2\pi i) = 8\pi i.$$

Example 24.12. Evaluating a contour integral from the perspective from the point of infinity can be useful when most of the singular points are inside the circle. Consider the example of contour integral

$$\oint_C \frac{1}{(z-1)(z-2)(z-3)\cdots(z-10)}\,dz$$

over the contour $C : |z| = 9$. The integrand has ten poles. Nine of them lie inside C and one of them outside C. A direct application of the Residue Theorem requires the evaluation of the residues at the nine singular point inside the contour.

An alternate method is to use the residue at the point at infinity. However, we cannot apply Theorem 24.5, but we can adopt the proof idea. Make a change of variable $z = 1/w$, and rewrite the integral as

$$\oint_{C'} \frac{1}{(w^{-1}-1)(w^{-1}-2)(w^{-1}-3)\cdots(w^{-1}-10)}(-w^{-2})\,dw$$

over a contour $C' : |w| = 1/9$ with clockwise orientation. The integral can be simplified as

$$\oint_{C'} \frac{-w^8}{(1-w)(1-2w)(1-3w)\cdots(1-10w)}\, dw.$$

There are two poles inside the contour C'. Let

$$g(w) = \frac{w^8}{(1-w)(1-2w)(1-3w)\cdots(1-10w)}.$$

The residue of $g(w)$ at $w = 0$ and $w = 0.1$ are

$$\begin{aligned} \operatorname{Res}(g; 0) &= 0 \\ \operatorname{Res}(g; 0.1) &= \left.\frac{w^8}{(1-w)(1-2w)(1-3w)\cdots(1-9w)}\right|_{w=0.1} \\ &= \frac{0.1^8}{(1-0.1)(1-0.2)\cdots(1-0.9)} = \frac{10}{9!} \end{aligned}$$

respectively. The answer is equal to $20\pi i/9!$

25 Argument principle

Summary:

- Winding number
- Argument principle
- Rouché theorem

25.1 Analytic definition of winding number

The technical part of winding number is contained in the following theorem. This is about a closed but not necessarily simple smooth curve.

Theorem 25.1. *If C is a closed piece-wise smooth curve not passing through a point z_0, then*

$$\frac{1}{2\pi i}\oint_C \frac{1}{z-z_0}\,dz$$

is an integer.

Proof. Represent the curve by

$$\gamma : [0,1] \to \mathbb{C} \setminus \{z_0\}.$$

Write $\gamma(t) - z_0$ in polar form

$$\gamma(t) - z_0 = r(t)e^{i\theta(t)},$$

for t between 0 and 1.

We denote the distance between $\gamma(t)$ and z_0 by $r(t)$, and the angle with respect to the given point z_0 by θ. Since it is assumed that $\gamma(t)$ does not pass through z_0, we have $r(t) > 0$ for all t.

Differentiate $\gamma(t)$ with respect to t to get

$$\gamma'(t) = [r'(t) + ir(t)\theta'(t)]e^{i\theta(t)}.$$

Using the definition of complex integral, we obtain

$$\begin{aligned}
\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{[r'(t)+ir(t)\theta'(t)]e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{r'(t)}{r(t)}\,dt + \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt \\
&= \frac{1}{2\pi i}[\log(r(1)) - \log(r(0))] + \frac{1}{2\pi}[\theta(1)-\theta(0)].
\end{aligned}$$

Since the curve is closed, we have $r(0) = r(1)$, and $\theta(1) - \theta(0) = 2\pi k$ for some integer k. The first term above is thus equal to zero, and the second term is an integer, which is the number of times the curve γ goes around the point z_0. □

In view of the previous theorem, we make the following definition.

Definition 25.2. The *winding number* of a closed curve C around a point z_0 is defined as

$$n(C; z_0) \triangleq \frac{1}{2\pi i}\oint_C \frac{1}{z-z_0}\,dz.$$

Other notation for the winding number includes: $w(C, z_0)$, $ind(C, z_0)$. When the curve is parameterized by $\gamma(t)$, we also write it as $n(\gamma; z_0)$.

If γ is not closed, we can use the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$$

to compute the change of angle relative to the point z_0. (cf. Section 11.4).

25.2 Argument principle

The argument principle is about the integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$ when C is simple closed curve. This integral is always equal to an integer, because it can be interpreted as the winding number of some curve around the origin.

Theorem 25.3. *Suppose C is a simple closed curve parameterized by $\gamma(t)$, for $0 \le t \le 1$, and $f(z)$ is a complex function that does not vanish on the curve C. Then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n(f(\gamma(t));0).$$

It the same as the number of times the curve $f(\gamma(t))$ goes around the origin (in the counter-clockwise direction).

The non-vanishing of the function $f(z)$ on C ensures that the integral is well defined. Hence, the curve $f(\gamma(t))$ does not pass through the origin. We note that if we set the function $f(z)$ to $f(z) = z - z_0$, it reduces to Theorem 25.1.

Proof. Consider the curve C' that is parameterized by $f(\gamma(t))$, for t in $[0,1]$, and let $g(t) = f(\gamma(t))$ for $0 \le t \le 1$. The winding number of C' around 0 is

$$\begin{aligned} n(C';0) &= \frac{1}{2\pi i}\oint_{C'} \frac{1}{z}\,dz \\ &= \frac{1}{2\pi i}\int_0^1 \frac{g'(t)}{g(t)}\,dt \\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\,dt \\ &= \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz. \end{aligned}$$

□

The previous proof shows that $(2\pi i)^{-1}\int_C f'/f\,dz$ is always an integer. In fact, it is counting the number of zeros minus the number of poles of $f(z)$ (counted with multiplicity) inside C.

Theorem 25.4 (Argument principle)**.** *Suppose C is the boundary of a simply connected region and f is a nonzero meromorphic function defined in the interior of C. Then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz$$

is equal to the number of zeros minus the number of pole of f (counted with multiplicity) inside C.

Proof. Suppose z_1, $z_2, \dots, z_n$ are the zeros or poles of f in the interior of C. There could be only finitely many such points. Otherwise, either the zeros will have a limit point, or the poles will have a limit point, which contradicts that f is a nonzero function (Theorem 19.4).

For each $k = 1, 2, \dots, n$, we write $f(z)$ as

$$f(z) = a_{m_k}(z - z_k)^{m_k} + a_{m_k+1}(z - z_k)^{m_k+1} + a_{m_k+2}(z - z_k)^{m_k+2} + \cdots$$

The integer m_k is the smallest exponent such that the coefficient is nonzero. It is positive if z_k is a zero, and is negative if z_k is a pole. (Note that we use the assumption that there is no essential singularity here.) (m_k cannot be zero because z_k is either a pole or zero.) By construction, the coefficient a_{m_k} is nonzero.

The derivative is

$$f'(z) = m_k a_{m_k}(z - z_k)^{m_k-1} + (m_k + 1)a_{m_k+1}(z - z_k)^{m_k} + (m_k + 2)a_{m_k+2}(z - z_k)^{m_k+1} + \cdots .$$

The ratio f'/f can be written as

$$\frac{m_k a_{m_k}(z - z_k)^{m_k-1}}{a_{m_k}(z - z_k)^{m_k}} \Big[\frac{1 + c_1(z - z_k) + c_2(z - z_k)^2 + \cdots}{1 + d_1(z - z_k) + d_2(z - z_k)^2 + \cdots}\Big]$$

where c_j and d_j are some complex numbers. The fraction inside the square bracket is a holomorphic function in a neighborhood of z_k and the value at z_k is 1. Therefore, we can represent it as a power series with constant term 1. As a result, for each $k = 1, 2, \dots, n$, we can write f'/f as

$$\frac{f'(z)}{f(z)} = \frac{m_k}{z - z_k} + \text{ an analytic function of } z.$$

By consider the residues of $f'(z)/f(z)$ at $z_1, z_2, \dots, z_n$ together, we obtain

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = m_1 + m_2 + \cdots + m_n.$$

For $k = 1, 2, \dots, n$, the integer m_k is either positive or negative. It is positive (resp. negative) when z_k is a zero (resp. pole) of $f(z)$, and is equal to the order of zero (resp. pole) of z_k. The sum $m_1 + m_2 + \cdots + m_n$ is thus the number of zeros minus the number of poles in the interior of C, counted with multiplicity. □

We can combine Theorem 25.3 and Theorem 25.4. Suppose the contour C and complex function$f(z)$ satisfy the conditions in Theorem 25.4. The change of argument of $f(z)$, after traveling the contour C once, is equal to 2π times the difference between the number of zeros and the number of poles inside the contour C. This is the explanation why Theorem 25.4 is called the argument principle. In some textbooks, such as [BrownChurchill], the argument principle is formulated as

$$Z - P = \frac{1}{2\pi}(\Delta_C \arg f(z)),$$

where Z denotes the number of zeros of $f(z)$ inside C, P denotes the number of poles of $f(z)$ inside C, and $\Delta_C \arg f(z)$ is the change of the argument of $f(z)$ as we travel along the contour C. (cf. Equation (11.4).) Using the formulation in this notes, the connection is

$$n(f \circ \gamma; 0) = \frac{1}{2\pi}(\Delta_C \arg f(z)) = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = Z - P.$$

Example 25.1. Consider the function $f(z) = (z-1)(z-i)(z-3)$, and a circle C centered at the origin with radius 2. Suppose that the orientation is counter-clockwise. The function is holomorphic. The number of zeros inside the circle C can be obtained using the argument principle by

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz &= \frac{1}{2\pi i}\int_C \frac{(z-i)(z-3)+(z-1)(z-3)+(z-1)(z-i)}{(z-1)(z-i)(z-3)}\, dz \\
&= \frac{1}{2\pi i}\Big[\int_C \frac{1}{z-1}\, dz + \int_C \frac{1}{z-i}\, dz + \int_C \frac{1}{z-3}\, dz\Big] \\
&= \frac{1}{2\pi i}[2\pi i + 2\pi i + 0] = 2.
\end{aligned}$$

25.3 Rouché theorem

From the fundamental theorem of algebra we know that a polynomial of degree d has d roots, counted with multiplicities. Rouché theorem provides a method to estimate the locations of the roots. Rouché theorem applies to holomorphic functions in general, not just to polynomials.

Theorem 25.5 (Rouché theorem)**.** *Suppose C is a simple closed curve with positive orientation. If f and g are functions that are holomorphic in a neighborhood containing C and its interior, and*

$$|f(z)| > |g(z)|$$

for all $z \in C$, then the number of zeros of $f + g$ inside C is the same as the number of zeros of f inside C. (A zero of order m is counted as m zeros.)

We note that the assumption $|f(z)| > |g(z)|$ implies (i) $f(z) \neq 0$ for all $z \in C$, and (ii) $f(z) + g(z) \neq 0$ for all $z \in C$.

Proof. Parameterize the curve C by $\gamma(t)$, for $t \in [0, 1]$. The main idea of proof is that the curve $1 + g(\gamma(t))/f(\gamma(t))$, for $t \in [0, 1]$ lies strictly inside the circle $|z - 1| = 1$ with radius 1 and center $z = 1$. The winding number of the curve $\big((1 + \frac{g}{f}) \circ \gamma\big)(t)$ around the origin is thus zero. By Theorem. 25.3, this is saying that

$$\frac{1}{2\pi i} \int_\gamma \frac{\big(1 + \frac{g}{f}\big)'}{1 + \frac{g}{f}} \, dz = 0. \tag{25.1}$$

Figure 41 shows a closed curve in the circle $|z - 1| = 1$. No matter what the shape is, the winding number around the origin is 0.

By the argument principle (Theorem 25.4), the number of zeros of $f + g$ inside C can be computed by

$$\frac{1}{2\pi i} \int_C \frac{f' + g'}{f + g} \, dz,$$

while the number of zeros of f inside C can be computed by

$$\frac{1}{2\pi i} \int_C \frac{f'}{f} \, dz.$$

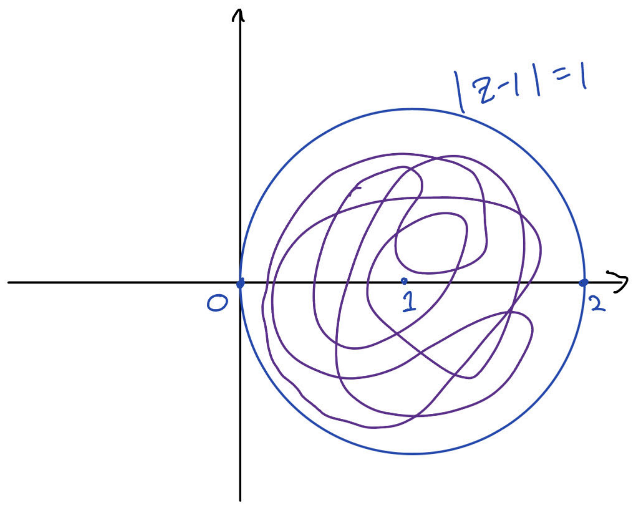


Figure 41: A closed curve inside the circle $|z-1|=1$.

We need to find a relationship among $\frac{(1+g/f)'}{1+g/f}$, $\frac{f'+g'}{f+g}$ and $\frac{f'}{f}$. Indeed, one can verify that

$$\begin{aligned}\frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}+\frac{f'}{f} &= \frac{g'f-gf'}{f(f+g)}+\frac{f'}{f}\\ &= \frac{g'f-gf'+f'f+f'g}{f(f+g)}\\ &= \frac{f'+g'}{f+g}.\end{aligned}$$

Integrating both sides and divide by $2\pi i$,

$$\frac{1}{2\pi i}\int_C \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}\,dz+\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz=\frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}\,dz.$$

We have shown in (25.1)) that the first integral is zero. Therefore

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz=\frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}\,dz.$$

This proves that the number of zeros of $f+g$ inside C is the same as the number of zeros of f inside C (counted with multiplicity). $\square$

Example 25.2. Using Rouché theorem, we can show that the polynomial $z^{10} + 3z^3 + 1$ has exactly 3 complex roots inside the unit circle. For $|z| = 1$, we check that

$$|3z^3| = 3, \quad \text{but } |z^{10} + 1| < 2.$$

Apply Rouché theorem with $f(z) = 3z^3$ and $g(z) = z^{10}+1$. The number roots of $z^{10}+3z^3+1$ inside the unit circle is the same as the number of roots of $3z^3$ inside the unit circle. Since $3z^3$ has a triple root at $z = 0$, $z^{10} + 3z^3 + 1$ has three roots inside the unit circle.

Besides root counting, Rouché also has other applications. We give two examples of different sorts below.

Example 25.3. Evaluate the integral

$$\int_{|z|=1} \frac{6z^5 + 3z^2 - 8z}{z^6 + z^3 - 4z^2}\, dz$$

over the unit circle, with counter-clockwise orientation.

We note that the numerator of the integrand is the derivative of the denominator. By the argument principle (Theorem 25.4), the integral is equal to $2\pi i$ times the number of zeros of $z^6 + z^3 - 4z^2$ inside the unit circle. By Rouché theorem, $z^6 + z^3 - 4z^2$ and $-4z^2$ has the same number of zeros inside the unit circle. The answer is thus $2\pi i(2) = 4\pi i$.

Example 25.4. Consider a polynomial of degree n with leading coefficient equal to 1,

$$h(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n.$$

Show that there is some point z on the unit circle such that $|h(z)| \geq 1$.

Suppose on the contrary that $|h(z)| < 1$ for all z on the unit circle. Apply the Rouché theorem with $f(z) = z^n$ and $g(z) = -h(z)$. We can check that the condition $|f(z)| > |g(z)|$ is satisfied on the unit circle, i.e.,

$$|f(z)| = |z^n| = 1 > |-h(z)| = |g(z)| \quad \text{for all } z \text{ with } |z| = 1.$$

By Rouché theorem, the function f and $f + g$ have the same number of zeros inside the unit circle. On one hand, $f(z) = z^n$ has exactly n zeros, namely, n repeated roots at $z = 0$, inside the unit circle. However

$$f(z) + g(z) = c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$$

has at most $n - 1$ zeros. The polynomial $f + g$ cannot have n zeros inside the unit circle. This contradiction shows that there must be some point z with $|z| = 1$ such that $|h(z)| \geq 1$.

We give a proof of the fundamental theorem of algebra as an application of Rouché theorem.

Theorem 25.6. *A polynomial over $\mathbb{C}$ of degree $d \geq 1$ has d roots in $\mathbb{C}$ (counted with multiplicity).*

Proof. Consider a polynomial in the form

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + + \cdots a_1 z + z_0$$

with $a_d \neq 0$. The idea of proof is to compare the polynomial with a simpler polynomial $a_d z^d$. This is a nonzero polynomial as a_d is nonzero.

Let $f(z)$ be the polynomial consisting of a single term $a_d z^d$, and let

$$g(z) \triangleq a_{d-1} z^{d-1} + + \cdots a_1 z + z_0$$

be the difference of polynomial $p(z)$ and $f(z)$.

We choose a real number R that is strictly larger than

$$\frac{|a_{d-1}| + |a_{d-2}| + \cdots + |a_0|}{|a_d|}.$$

On the circle $|z| = R$, we have

$$|f(z)| = |a_d| R^d.$$

The magnitude of $g(z)$ on the circle $|z| = R$ is strictly less than the magnitude of $f(z)$, because, when $|z| = R$,

$$\begin{aligned} g(z) &\leq |a_{d-1}|R^{d-1} + |a_{d-2}R^{d-2} + \cdots + |a_0| \\ &\leq R^{d-1}(a_{d-1} + a_{d-2} + \cdots + |a_0|) \\ &< |a_d|R^d = |f(z)|. \end{aligned}$$

We can thus apply Rouché theorem to conclude that $p(z)$ and $a_d z^d$ has the same number of roots. Because $a_d z^d$ has d roots, counted with multiplicity, so does $p(z)$. $\square$

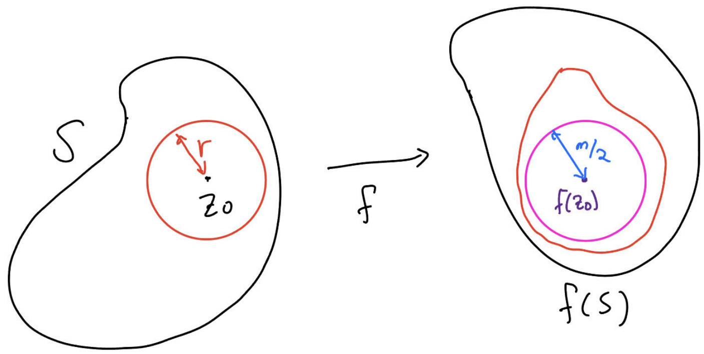


Figure 42: Proof of open mapping theorem

25.4 Appendix: Open mapping theorem

We give a proof of the open mapping theorem as an application of Rouché theorem. In topology, a continuous function is defined as a function such that the inverse image of any open set is open. The open mapping theorem in complex analysis is a statement in the other direction.

Theorem 25.7 (Open mapping theorem)**.** *The image of an open set under a non-constant holomorphic map is open.*

Proof. Let $f(z)$ be a nonconstant holomorphic function, and let S be an open subset inside the domain of $f(z)$. We want to show that the image of S under f is an open set.

Let z_0 be a point in S. The function $f(z) - f(z_0)$ has a zero at z_0. Since $f(z)$ is nonconstant, the zeros of the function $f(z)-f(z_0)$ are isolated. We can choose a sufficiently small radius r so that the open disc $D(z_0, r)$ belongs to S, and $f(z) - f(z_0)$ is equal to 0 only at $z = z_0$ for all $|z - z_0| \leq r$ (See Fig. 42).

Let $g(z)$ denote the function $f(z) - f(z_0)$. Draw a circle $C : |z - z_0| = r$ centered at z_0 with radius r. Since C is compact and $|g(z)|$ is continuous, we can take the minimum

$$m \triangleq \min_{z \in C} |g(z)| = \min_{z \in C} |f(z) - f(z_0)|$$

of $|g(z)|$ over the circle C, and the minimum value m is positive.

We claim that the open disc $|w - f(z_0)| < m/2$ is contained in the image of $f(z)$. That is, we want to show that any point in the open disc $D(f(z_0), m/2)$ has a pre-image in S under the function f.

Let w be any complex number satisfying $|w - f(z_0)| < m/2$. For any z on the circle C, we have

$$\begin{aligned} |f(z) - w| &= |f(z) - f(z_0) - (w - f(z_0))| \\ &\geq |f(z) - f(z_0)| - |w - f(z_0)| \\ &> m - m/2 = m/2. \end{aligned}$$

Therefore, $|f(z) - w|$ is strictly larger than $|w - f(z_0)|$ for all z on the circle $|z - z_0| = r$. By Rouché theorem, $f(z) - z_0$ and $f(z) - f(z_0) - (w - f(z_0)) = f(z) - w$ has the same number of zeros for z in the open disc $D(z_0, r)$. But by our choice of r, $f(z) - f(z_0)$ has exactly one zero in this open disc. Therefore, there exists exactly one point z in the open disk $D(z_0, r)$ such that $f(z) = w$. This point z is in S, because $D(z_0, r)$ is in S. This proves that every point in the open disc $D(f(z_0), m/2)$ is in the image of S.

We thus show that for every point z_0 in S, we can find a small radius r_{z_0}, such that $D(f(z_0), r_{z_0})$ is contained inside the image of S. The image of S can thus be written as a union

$$\bigcup_{z_0 \in S} D(f(z_0), r_{z_0}).$$

Since arbitrary union of open set is open, we prove that S is an open set. $\square$

The open mapping theorem has the same strength as the maximal modulus principle. In fact, we can prove the maximal modulus principle from the open mapping theorem. Suppose z_0 is a maximum point of a non-constant function $f(z)$ such that z_0 lies in the interior of the domain. From the proof of Theorem 25.7, the image $f(z_0)$ of z_0 is contained in a disc $D(f(z_0), r_{z_0})$ that is completely inside the range of f. The complex numbers $f(z_0) + \epsilon(\pm 1 \pm i)$ are in the range of f for a small but nonzero real number ϵ. One of them will have modulus larger than $f(z_0)$. This contradicts the assumption that maximum is attained at $z = z_0$.

26 Gamma function and Riemann zeta function

Summary:

- Local uniform convergence of sequence of functions
- Analytic continuation by power series
- Gamma function and Riemann zeta function

The Gamma function and Riemann's zeta functions are originally defined as a function with real numbers as the input value.

For positive real number x, the Gamma function $\Gamma(x)$ is defined by an integral

$$\Gamma(x) \triangleq \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{26.1}$$

It is easy to verify that $\Gamma(1) = 1$, and $\Gamma(n+1) = n\Gamma(n)$ for positive integer n. Hence, for positive integer n, $\Gamma(n) = (n-1)!$. With more effort, one can calculate that $\Gamma(1/2) = \sqrt{\pi}$. A graph of the Gamma function is shown in Fig. 43.

The Riemann's zeta function is defined for real number x that is strictly larger than 1,

$$\zeta(x) \triangleq \sum_{n=1}^\infty \frac{1}{n^x}. \tag{26.2}$$

The convergence can be verified by the p-test. When $x = 1$, the series is the harmonic series and hence is divergent. Special values of ζ function include $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. A plot of the Riemann zeta function for real input x is shown in Fig. 44.

The main objective of this lecture is to show that the domains of Gamma function and zeta function can be extended to the complex plane.

26.1 Morera's theorem

The basic Cauchy theorem says that given a function $f(z)$ that is holomorphic in a convex or star-shaped domain, then the integral of $f(z)$ over any closed curved in D is zero. This statement remains valid if the domain is simply connected. However, if the domain D is multiply connected, the conclusion of Cauchy theorem may fail.

Morera theorem provide a sufficient condition under which a function $f(z)$ is holomorphic. In the next theorem, the domain D is connected but may be multiply connected.

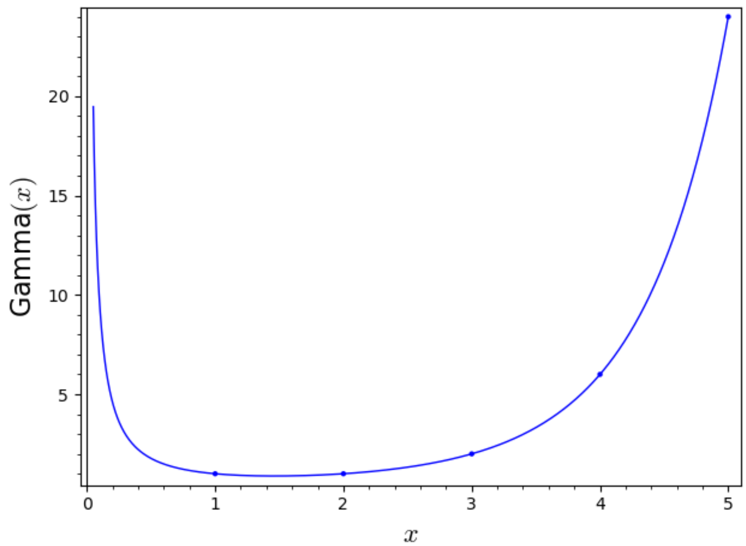


Figure 43: Plot of the gamma function with real input.

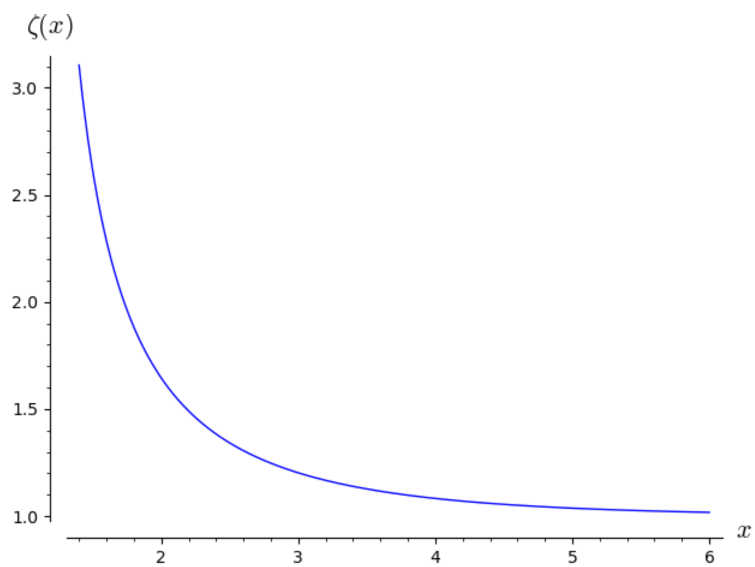


Figure 44: Plot of the zeta function with real input.

Theorem 26.1 (Morera theorem)**.** *Consider a continuous complex function f in a domain D. If*

$$\oint_C f(z)\,dz = 0$$

for all closed curves in D, then f is holomorphic in D.

Proof. The theorem is proved by exhibiting an anti-derivative of f. Choose and fix a point z_0 in D. For each other point $z \in D$, we can find a path from z_0 to z, because the domain D is assumed to be connected. Since $\int_C f\,dz$ is zero for all closed path C, the integral from z_0 to z is independent of path. We can thus define a function F by

$$F(z) \triangleq \int_{z_0}^{z} f(w)\,dw,$$

with the integral taken over any smooth curve from z_0 to z. The function $F(z)$ so defined is differentiable and derivative $F'(z)$ is equal to $f(z)$ for all $z \in D$. (See the proof of Theorem 16.2) Therefore $F(z)$ is holomorphic in D.

In general, the derivative of a holomorphic function is differentiable (see Theorem 18.10), i.e., $F'(z)$ is complex differentiable at every point $z \in D$. Since $F'(z) = f(z)$, we conclude that $f(z)$ is complex differentiable at every point $z \in D$. $\square$

We note that the condition in Morera theorem is sufficient but not necessary. It is possible that a complex function $f(z)$ is holomorphic in a region but fails to have zero integral over all closed curve. The complex log function defined on the punctured complex plane $\mathbb{C} \setminus \{0\}$ is such an example.

When we have more information about the shape of the domain D, we have a stronger result that only checks the contours with triangular shape.

Theorem 26.2. *Suppose f is a complex function that is continuous on a convex set D. Then f is holomorphic in D if and only if $\int_T f(z)\,dz = 0$ for all triangular contours T.*

Proof. The forward direction in the theorem is the Cauchy-Goursat theorem for triangle (Theorem 15.6).

To prove the backward direction, we choose an arbitrary base point z_0 in the open disc D. Define a function $F(z)$ by the integral

$$\int_{z_0 \to z} f(w)\, dw.$$

Here, the notation $\int_{z_0\to z} f(w)\, dw$ stands for the integral of f on the line segment from z_0 to z. The line segment lies within the domain D because it is assumed that D is convex. By the assumption that the integral of $f(z)$ over any triangle is zero, we have

$$F(z_2) = F(z_1) + \int_{z_1 \to z_2} f(w)\, dw$$

for any z_1 and z_2 in the domain. We can invoke Theorem 16.2 and conclude that $F(z)$ is a holomorphic function with derivative $f(z)$. Since $F(z)$ is infinitely differentiable (Theorem 18.10), the derivative of $f(z)$ exists for all $z \in D$. □

26.2 Weierstrass's theorem on convergence of sequence of functions

One way to construct holomorphic function is by taking the limit of a sequence of functions, such as power series. We list below a few notion of convergence for sequence of functions.

Definition 26.3. Let $(f_k)_{k=1}^\infty$ denote a sequence of functions defined in a common domain D.

- The sequence $(f_k)_{k=1}^\infty$ *converges* to f, if for any point z_0 in the domain D and $\epsilon > 0$, there is an integer N (which may depend on z_0) such that

$$|f_k(z_0) - f(z_0)| < \epsilon \quad \text{for all } k \geq N.$$

- The sequence $(f_k)_{k=1}^\infty$ *converges uniformly* to f, if for any $\epsilon > 0$, there exists a sufficiently large integer N (which only depends on ϵ) such that

$$|f_k(z) - f(z)| < \epsilon \quad \text{for all } k \geq N \text{ and } z \in D.$$

- The sequence $(f_k)_{k=1}^\infty$ *converges locally uniformly* to f if at any point z_0 in the domain D, there is a neighborhood U containing z_0 such that the sequence $(f_k)_{k=1}^\infty$ converges to f uniformly in U.

The logical relationship among these three mode of convergence is:

$$\text{Converge uniformly} \Rightarrow \text{Converge locally uniformly} \Rightarrow \text{Converge}$$

"Locally uniform convergence" is somewhere between "convergence" and "uniform convergence".

Remark. We note that locally uniform convergence is equivalent to compact convergence. This means that a sequence of functions $(f_k)_{k=1}^{\infty}$ converges locally uniformly in a domain D if and only if $(f_k)_{k=1}^{\infty}$ converges on any compact subset of D. In [Stein], the authors state the theorems using compact convergence in place of local uniform convergence.

We quote two properties about convergence of functions from real analysis.

Proposition 26.4. *Suppose $(f_k)_{k=1}^{\infty}$ is a sequence of continuous functions that converge to f locally uniformly. Then the function f is continuous.*

Proposition 26.5 (Weierstrass M test)**.** *Suppose $(f_k(z))_{k=1}^{\infty}$ is a sequence of complex-valued function defined on a domain D. If we can find positive real numbers M_k, for $k \geq 1$, such that*
(i) $|f_k(z)| \leq M_k$ for all $z \in D$, and
(ii) $\sum_{k=1}^{\infty} M_k$ converges,
then the infinite sum $\sum_{k=1}^{\infty} f_k(z)$ converges absolutely and uniformly on the domain D.

The proofs of these two propositions can be found in a real analysis book.

We can exchange the order of contour integration and limit if we have locally uniform convergence.

Theorem 26.6. *Suppose $f_k(z)$ is a continuous complex-valued function for $k = 1, 2, 3, \dots$ If $(f_k(z))_{k=1}^{\infty}$ converges locally uniformly to $f(z)$ in domain D, then*

$$\lim_{k\to\infty} \int_C f_k(z)\, dz = \int_C f(z)\, dz$$

for any piece-wise smooth curve C in the domain D.

Proof. We first not that the limit function $f(z)$ is continuous, by Prop. 26.4.

The proof relies on the fact that the points in C form a compact set. Suppose the curve C is represented by $\gamma(t)$, for t in the interval $[a,b]$. Because the interval $[a,b]$ is closed and bounded, the image under a continuous function $\gamma(t)$ is also closed and bounded.[4] Because we assume that the derivative $\gamma'(t)$ is continuous, except finitely many points, the modulus of $g'(t)$ is finite for all $t \in [a,b]$. Hence the length of $C = \int_a^b |\gamma'(t)|\,dt$ is finite.

Because C is a compact set, we can apply the hypothesis that $f_k(z)$ converges to $f(z)$ for z on the curve C. Let ϵ be an arbitrarily small positive real numbers. There exists integer N such that $|f_k(z) - f(z)| < \epsilon$ for all $z \in C$ and for all $k \geq N$. We now apply the ML estimate (Theorem 15.4)

$$\Big| \int_C f_k(z)\,dz - \int_C f(z)\,dz \Big| = \Big| \int_C f_k(z) - f(z)\,dz \Big| \leq \text{length of } C \cdot \epsilon.$$

This is the same as saying that $\int_C f_k(z)\,dz$ is converging to $\int_C f(z)\,dz$ as $k \to \infty$. □

As a consequence of Morera theorem, we can prove

Theorem 26.7. *If $f_k(z)$, for $k = 1, 2, \ldots$, are holomorphic functions defined on a region R, converging to locally uniformly to $f(z)$, then $f(z)$ is holomorphic in R.*

Proof. Since it is assumed that the function f_k is holomorphic for any k, the function f_k is continuous in the domain R for any k. The basic property of uniform convergence in Prop. 26.4 says that the pointwise limit $f(z)$ is a continuous function.

Let z_0 denote a point in R, and D be an open disc centered at z_0. We choose the radius to be sufficiently small so that the open disc lies completely inside region R. We want to show that $f(z)$ is holomorphic in open disc D.

Consider a triangle T in D. By Theorem 26.6, we have

$$\int_T f_k(z)\,dz \to \int_T f(z)\,dz.$$

By appealing to the Cauchy-Goursat theorem for triangle (Theorem 15.6), the integral $\int_T f_k(z)\,dz$ is equal to zero for each k, because f_k is holomorphic. Therefore, $\int_T f(z)\,dz = 0$. Since this holds for any triangular contour T in D, by Theorem 26.2, the limit function f is holomorphic in D.

[4] https://proofwiki.org/wiki/Continuous_Image_of_Compact_Space_is_Compact

The argument in the previous paragraph holds for any point in the region R. This proves that f is holomorphic. $\square$

Remark. The analogous statement of Theorem 26.7 in real analysis is false. It is possible to construct a sequence of smooth functions that converges pointwise to a function that is not differentiable at some point. The Weierstrass approximation theorem asserts that given any continuous real-valued function on a compact set, we can approximate it by a sequence of polynomials, and the convergence is uniform. If we take the function $f(x) = |x|$ to start with, then by Weiestrass approximation theorem, we can find a sequence of polynomial functions, which are all infinitely smooth, that converge to a function that is not differentiable at $x = 0$.

The next theorem is due to Weierstrass.

Theorem 26.8 (Weierstrass)**.** *Suppose $(f_k)_{k=1}^{\infty}$ is a sequence of holomorphic functions that converges to a limit function f locally uniformly. Then f is holomorphic and the derivatives f_k' converges to f' as $k \to \infty$.*

Proof. Let C be a simple closed curve in D. It encloses a compact set in the domain. By Heine-Borel theorem and locally uniform convergence, the functions f_k converge uniformly on the area enclosed by C. By Theorem 26.7, the limit function f is holomorphic in D.

It remains to prove that f_k' converges to f' pointwise as $k \to \infty$.

Let z_0 be a point in D. Select a radius r such that the open disc $|z - z_0| < r$ is contained in D. Let C be the boundary $|z - z_0| = r$ with counter-clockwise orientation. We use the Cauchy integral formula for the first derivative,

$$f_k'(z) = \frac{1}{2\pi i} \int_C \frac{f_k(w)}{(w-z)^2}\, dw,$$

which holds for integers $k = 1, 2, 3, \ldots$ From the assumption that $f_k(w)$'s converge locally uniformly to $f(w)$ on C, we have $f_k(w)/(w-z)^2$ converges to $f(w)/(w-z)^2$ on C as well. Therefore

$$\frac{1}{2\pi i} \int_C \frac{f_k(w)}{(w-z)^2}\, dw \to \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\, dw \quad \text{as } k \to \infty.$$

This proves that $f_k'(z) \to f'(z)$ as $k \to \infty$. $\square$

Theorem 26.9. *We can extend the Riemann zeta function, which is originally defined for real $x > 1$, to a holomorphic function in the half plane* $\mathrm{Re}(s) > 1$*, by*

$$\zeta(s) \triangleq \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{26.3}$$

Proof. In the followings we write the real and imaginary parts of complex number s as σ and t, respectively, so that $s = \sigma + it$. Recall from Definition 4.8 that, for complex s, the meaning of n^s is

$$n^s \triangleq e^{(\ln n)s}.$$

The modulus of n^s is equal to

$$|n^s| = |e^{(\ln n)(\sigma+it)}| = |e^{(\ln n)\sigma}| = n^\sigma.$$

We choose a real number a that is strictly larger than 1, and consider the half plane $\mathrm{Re}(s) > a$. For complex number s in this region, we have the bound

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma} \leq \frac{1}{n^a}$$

for all integer $n \geq 1$.

Because it is known that the summation $\sum_{n=1}^{\infty} n^{-a}$ converges for $a > 1$, the series in (26.3) converges absolutely by Weierstrass M test. More precisely, this means for each complex number s in the half plane $\mathrm{Re}(s) > a$, the partial sum

$$\sum_{n=1}^{m} \frac{1}{n^s}$$

is convergent as $m \to \infty$, and the convergent is uniform (because absolute convergence implies uniform convergence). Since we can take the constant a to be any real number larger than 1, we can now apply Theorem 26.7 to conclude that the infinite summation in (26.3) is a holomorphic function in the half plane $\mathrm{Re}(a) > 1$. $\square$

It requires more effort to extend the domain of the Gamma function. Because the integral in the definition of real gamma function in (26.1) is an imporper integral.

Theorem 26.10. *We can extend the domain of the real Gamma function to the right-half plane* $\mathrm{Re}(s) > 0$*. The extended function is given by the integral*

$$\Gamma(s) \triangleq \int_0^\infty x^{s-1} e^{-x}\, dx$$

and is holomorphic in the domain $\mathrm{Re}(s) > 0$*.*

Proof. The proof follows [Stein, Chapter 6, Prop. 1.1]. Let ϵ and M denote two real numbers such that $\epsilon < M$. We first show that the integral in the definition of $\Gamma(s)$ is absolutely convergent for s in the vertical strip $\delta < \mathrm{Re}(s) < M$.

We continue to use the customary notation in the literature and denote the real part of s by σ. For complex number s with real part σ, we have

$$|x^{s-1} e^{-x}| = x^{\sigma-1} e^{-x}$$

for all positive real numbers x. We treat the improper integral in the theorem as the limit

$$\int_0^\infty x^{s-1} e^{-x}\, dx = \lim_{k \to 0} \int_{1/k}^k x^{s-1} e^{-x}\, dx.$$

We introduce the notation

$$f_k(s) \triangleq \int_{1/k}^k x^{s-1} e^{-x}\, dx.$$

We first show that for each complex number s with real part σ between δ and M, the functions $f_k(s)$ converge uniformly as $k \to \infty$. Indeed, we can show that it converges absolutely. We divide the integral into two parts.

$$\int_{1/k}^1 x^{s-1} e^{-x}\, dx, \text{ and } \int_1^k x^{s-1} e^{-x}\, dx.$$

In the first integral, if we take the absolute value of the integrand, we can upper bound it by

$$\begin{aligned}\int_{1/k}^{1} |x^{s-1}e^{-x}|\,dx &\leq \int_{1/k}^{1} x^{\sigma-1}\,dx\\ &\leq \Big[\frac{x^{\sigma}}{\sigma}\Big]_{1/k}^{1}\\ &= \frac{1}{\sigma}\big(1-k^{-\sigma}\big)\\ &\leq \frac{1}{\sigma} \leq \frac{1}{\delta}.\end{aligned}$$

To bound the second integral, we choose a constant C_σ, for $\delta < \sigma < M$, such that

$$x^{\sigma-1} \leq C_\sigma e^{x/2}$$

for x in the range $[1, \infty)$. We choose the constants C_σ such that $C_\sigma \leq C_\tau$ whenever $\sigma \leq \tau$. For example, we can take C_σ to be the maximum of $x^{\sigma-1}e^{-x/2}$ over $x \in [1, \infty)$.

Then

$$\begin{aligned}\int_{1}^{k} x^{\sigma-1}e^{-x}\,dx &\leq \int_{1}^{k} C_\sigma e^{x/2}e^{-x}\,dx\\ &= C_\sigma\big[-2x^{-x/2}\big]_1^k\\ &= 2C_\sigma(e^{-1/2} - e^{-1/k})\\ &\leq 2C_M e^{-1/2}.\end{aligned}$$

Therefore, we obtain an upper bound

$$\int_{1/k}^{k} |x^{s-1}e^{-x}|\,dx \leq \frac{1}{\delta} + 2C_M e^{-1/2},$$

which is independent of k and s. This shows that the integral

$$\int_{1/k}^{k} x^{s-1}e^{-x}\,dx$$

converges absolutely, for all complex number s in the vertical strip $\delta < \mathrm{Re}(s) < M$, as $k \to \infty$. This finish the proof of the claim that $f_k(s)$ converges uniformly in any vertical strip $\delta < \mathrm{Re}(s) < M$.

Because δ and M can be any real numbers, subject to the constraint $\delta < M$, the convergence of the sequence $(f_k(s))_{k=1}^{\infty}$ is locally uniform. By Theorem 26.7, the limit function $\Gamma(s)$ is holomorphic. $\square$

Theorem 26.11. *For* $\mathrm{Re}(s) > 0$*, the complex Gamma function satisfies*

$$\Gamma(s+1) = s\Gamma(s).$$

Proof. The proof is done by integration by parts for complex integral. Let a and b to be real numbers with $a < b$. If we integrate from a to b, we can write

$$\int_a^b x^s e^{-x}\, dx = \big[-x^s e^{-x}\big]_a^b + \int_a^b s x^{s-1} e^{-x}\, dx.$$

We then take limit as $b \to \infty$ and $a \to 0$. □

26.3 Analytic continuation by power series and open discs

A method of extending the domain of an analytic function is by power series expansion. Suppose we have a converging power series with center z_0 and radius of convergence r_0. It defines an analytic function

$$f_0(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

in the open disc $D(z_0, r_0)$. Let z_1 be a point inside this open disc. Expand the function $f_0(z)$ with center z_1,

$$f_1(z) = \sum_{k=0}^{\infty} b_k (z - z_1)^k$$

and let the radius of convergence is r_1. If the open disc $D(z_1, r_1)$ is not completely inside $D(z_0, r_0)$ (see Fig. 45), we can extend the domain of $f_0(z)$ to the union of the two open disc,

$$g(z) = \begin{cases} f_0(z) & \text{if } z \in D(z_0, r_0), \\ f_1(z) & \text{if } z \in D(z_1, r_1) \setminus D(z_0, r_0). \end{cases}$$

This definition makes sense because the two functions $f_0(z)$ and $f_1(z)$ have the same value for z in the intersection. The function with the extended domain is analytic, because it has power series representation at every point in open discs $D(z_0, r_0)$ and $D(z_1, r_1)$. By the Identity Theorem (Theorem 19.4), this is the only way to extend $f_0(z)$ to $D(z_0, r_0) \cup D(z_1, r_1)$.

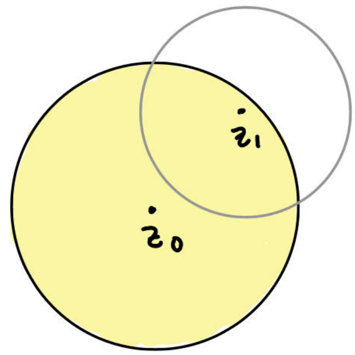


Figure 45: Extending the domain of a function by power series.

Example 26.1. As a toy example we consider the power series expansion of $\frac{1}{z-1}$ centered at the origin. This defines a function

$$f_0(z) = -1 - z - z^2 - z^3 - z^4 - \dots$$

This defines a function with the open disc $D(0,1)$ as the domain.

We can extend the domain beyond the open disc $D(0,1)$ by taking one point z_1 in $D(0,1)$ and consider the power series of $f_0(z)$ at z_1. For instance suppose we take $z_1 = i/2$ and expand the function $f_0(z)$ at z_1. This process is possible by Theorem 18.9. Since we know that the function has closed-form expression $1/(z-1)$, we can leverage on this knowledge and compute the power series centered at $i/2$ by

$$\begin{aligned}
\frac{1}{z-1} &= \frac{1}{-1+i/2+z-i/2} \\
&= \frac{1}{-1+i/2} \cdot \frac{1}{1+\frac{z-i/2}{-1+i/2}} \\
&= \frac{1}{-1+i/2}\Big[1 - \frac{z-i/2}{-1+i/2} + (\frac{z-i/2}{-1+i/2})^2 - (\frac{z-i/2}{-1+i/2})^3 + \cdots\Big]
\end{aligned}$$

This power series converges when z is in the open disc $D(i/2, \sqrt{5}/2)$. We can thus define another function by this power series

$$f_1(z) = \frac{1}{-1+i/2}\Big[1 - \frac{z-i/2}{-1+i/2} + (\frac{z-i/2}{-1+i/2})^2 - (\frac{z-i/2}{-1+i/2})^3 + \cdots\Big]$$

for $|z - i/2| < \sqrt{5}/2$. We can combine $f_0(z)$ and $f_1(z)$ and define a function $g(z)$ on the union of $D(0,1)$ and $D(i/2, \sqrt{5}/2)$ by

$$g(z) \triangleq \begin{cases} f_0(z) & \text{if } z \in D(0,1), \\ f_1(z) & \text{if } z \in D(i/2, \sqrt{5}/2). \end{cases}$$

If z is in the intersection of the two open disc, we can define $g(z)$ by either $f_0(z)$ or $f_1(z)$.

Take another point in the new domain and consider the power series expansion on this point. If this power series converges in some region outside the old domain, we can further extend the domain. We can continue this process indefinitely. This will recover the original function $1/(z-1)$ on the complex plane.

We remark that this method will fail in the strange case of natural boundary as in Example 26.2. It is not possible to analytically continue the function in Example 26.2 to a larger domain.

Example 26.2. (natural boundary) Consider the lacunary power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + z^{16} + \cdots.$$

("Lacunary" means there are a lot gaps in the series.) This series converges absolutely for $|z| < 1$ by comparing with a simple geometric series,

$$|z| + |z|^2 + |z|^4 + |z|^8 + |z|^{16} + \cdots \leq \sum_{k=1}^{\infty} |z|^k.$$

However, this series diverges at all (2^n)-th roots of unity, for all positive integer n. For example,

$$\begin{aligned} f(-1) &= -1 + 1 + 1 + 1 + \cdots = \infty, \\ f(i) &= i - 1 + 1 + 1 + \cdots = \infty, \\ f(-i) &= -i - 1 + 1 + 1 + \cdots = \infty, \\ f(e^{2\pi i/8}) &= e^{2\pi i/8} + e^{2\pi i/4} + e^{2\pi i/2} + 1 + 1 + \cdots = \infty. \end{aligned}$$

The set of singular points of $f(z)$ on the unit circle is countable and dense, and hence are not isolated. It is not possible to analytically extend the domain to a larger open set. The unit circle is a natural boundary of the domain of definition.

The singular points on the unit circle are plotted in Fig. 33.

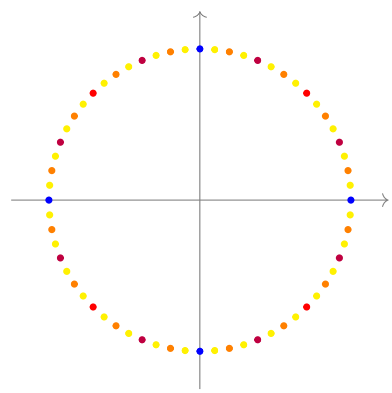


Figure 46: The singular points of a lacunary series are dense on the unit circle.

Using the recursive formula, $\Gamma(s+1) = s\Gamma(s)$, which also holds for complex number s in the right-half plane $\operatorname{Re} s > 0$, we can define the Gamma function for α in the vertical strip $-1 < \operatorname{Re}(s) \le 0$, except $s = 0$, by the same formula,

$$\Gamma(s) \triangleq \frac{\Gamma(s+1)}{s}, \quad \text{if } -1 < \operatorname{Re}(s) \le 0, s \ne 0.$$

We can further extend the domain to complex numbers in the vertical strip $-2 < \operatorname{Re}(s) \le -1$, except $s = -1$, by the formula

$$\Gamma(s) \triangleq \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)}, \quad \text{if } -2 \le \operatorname{Re}(s) \le -1, \text{ except } s = -1.$$

Repeating this process, we can define $\Gamma(s)$ for s in the complex plane except $0, -1, -2, \ldots$ (See Fig. 47).

Using the method of analytic continuation using power series, in theory, we can extend the domain to all other complex numbers. A more systematic way is to consider the auxiliary function

$$\xi(s) \triangleq \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

and establishing the functional equation

$$\xi(1-s) = \xi(s).$$

We refer the reader to other books in analytic number theory for the derivation.

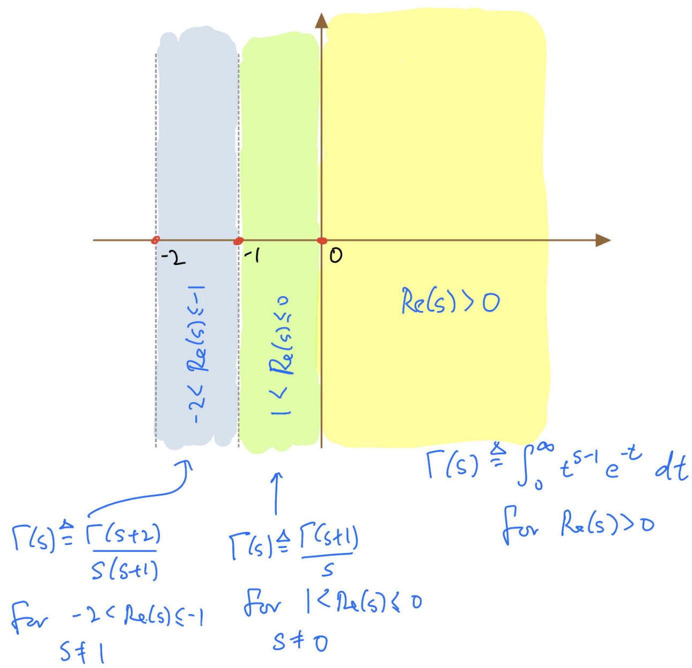


Figure 47: Analytic continuation in defining Gamma function for $\mathrm{Re}(s) \leq 0$.

It turns out that we can extend the domain of the zeta function to the whole complex plane except the point $s = 1$. Because of the uniqueness of analytic continuation, the values of the zeta function on the other part of the complex plane is uniquely defined by the values on the right-half plane $\{x + iy : x > 1\}$. It turns out that $\zeta(-1)$ is equal to $-1/12$, and we can thus write it as the following sum

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

The value of $\zeta(z)$ at -2, -4, -6, etc. are zeros. These are called the trivial zeros of the Riemann zeta function.

It was Riemann's vision that the values of the zeta function on the complex plane contain important information about the distribution of primes. For example, the fact that $s = 1$ is a pole is equivalent to the infinitude of primes. The prime number theorem can be deduced from the property that the zeta function does not vanish on the vertical line $\operatorname{Re} s = 1$.

The famous Riemann hypothesis is the conjecture about the location of the zeros in the critical strip $0 < \operatorname{Re}(s) < 1$.

Conjecture 26.12 (Riemann hypothesis)**.** *Let $\zeta(s)$ denote the complex function obtained by analytic continuation of the zeta function defined for positive real numbers larger than 1. All zeros in the critical strip $0 < \operatorname{Re}(s) < 1$ lie on the vertical line $\operatorname{Re}(s) = 1/2$.*

We remark that, by the Identity Theorem, the values of $\zeta(s)$ in the strip $0 < \operatorname{Re}(s) < 1$ (and other part of the complex plane), are uniquely determined by $\zeta(x)$ for real $x > 1$. There is no ambiguity in determining the locations of the zeros of $\zeta(s)$.

26.4 Appendix: Schwarz reflection principle

To motivate the idea of reflection principle, we consider a symmetric region D about the real axis, i.e., $D = \{\bar{z} : z \in D\}$, and suppose f is a complex function that is known to be analytic on D. As D is a connected domain, the domain D must intersect the real axis in at least one point. Moreover, since any domain is open by assumption, the intersection with the real axis must contain at least one open interval with positive length (See Fig. 48). We note that the domain D need not be convex.

Let $f(z)$ be an analytic function on D. By checking with the Cauchy-Riemann equations, one can show that the function $\overline{f(\bar{z})}$ is analytic in the same domain.

Theorem 26.13. *If $f(z)$ is analytic in a symmetric domain D along the real axis, then the function $\overline{f(\bar{z})}$ is analytic in D.*

Proof. We first note that because of the symmetry assumption, $f(\bar{z})$ is well-defined, i.e., $\bar{z}$ is in D whenever z is in D. Suppose $f(z) = u(x,y) + iv(x,y)$. We want to show that

$$g(z) = \overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$$

is analytic in D as well.

Let $p(x,y)$ and $q(x,y)$ denote the real and imaginary parts of $g(z)$, respectively. We check that

$$\begin{aligned}
p_x(x,y) &= \frac{\partial}{\partial x} u(x,-y) = u_x(x,-y) \\
q_y(x,y) &= \frac{\partial}{\partial y}(-v(x,-y)) = v_y(x,-y) \\
p_y(x,y) &= \frac{\partial}{\partial y} u(x,-y) = -u_y(x,-y) \\
q_x(x,y) &= \frac{\partial}{\partial x}(-v(x,-y)) = -v_x(x,-y).
\end{aligned}$$

Since $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations, $p(x,y)$ and $q(x,y)$ also satisfy the Cauchy-Riemann equations. Because $f(z)$ is assumed to be analytic, the real-valued functions $p(x,y)$ and $q(x,y)$ are continuously differentiable. By the sufficient condition for complex differentiability (Theorem 7.7), $\overline{f(\bar{z})}$ is analytic in D. □

Furthermore, suppose that $f(z)$ is real-valued on the intersection of D and the real axis. For examples, $f(x)$ could be a convergent power series with real coefficients. Then the difference $f(z) - \overline{f(\bar{z})}$ is analytic, and is equal to zero for z on the real axis. By the identity theorem (Theorem 19.4), $f(z) - \overline{f(\bar{z})}$ must be identically equal to zero on the whole domain. This proves that $f(z) = \overline{f(\bar{z})}$ throughout the domain D.

The reflection principle is a kind of converse to this result.

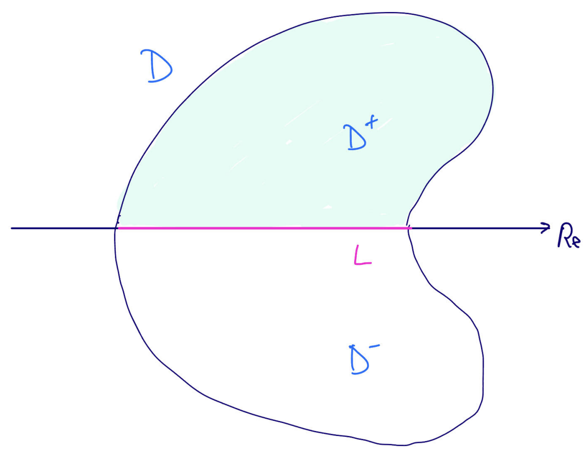


Figure 48: Symmetric domain

Theorem 26.14 (Reflection principle)**.** *Suppose D is a nonempty domain symmetric along the real axis. Let D^+ and L denote the intersection of D with the upper half plane and real axis, respectively. If f is a complex function defined on $D^+ \cup L$ such that (i) f is analytic in D^+, (ii) continuous in $D^+ \cup L$, and (iii) real-valued on L, then $f(z)$ has a unique extension to D, and the extended function satisfies $f(z) = \overline{f(\bar{z})}$ for $z \in D$.*

Proof. Let D^- be the intersection of D and the lower half plane. We define a new function $g(z)$ on domain D by

$$g(z) = \begin{cases} f(z) & \text{if } z \in D^+ \cup L \\ \overline{f(\bar{z})} & \text{if } z \in D^-. \end{cases}$$

By the same proof as in the proof of Prop. 26.13, $g(z)$ is analytic within D^-. Moreover, $g(z)$ is continuous in the symmetric domain D.

We show that $g(z)$ is analytic throughout the domain D using Morera theorem (Theorem 26.2). Suppose T is a triangular contour. If T completely lies inside D^+, or completely lies inside D^-, then the integral $\int_T g(z)\,dz = 0$. So, it remains to consider triangle T that overlaps with the real axis.

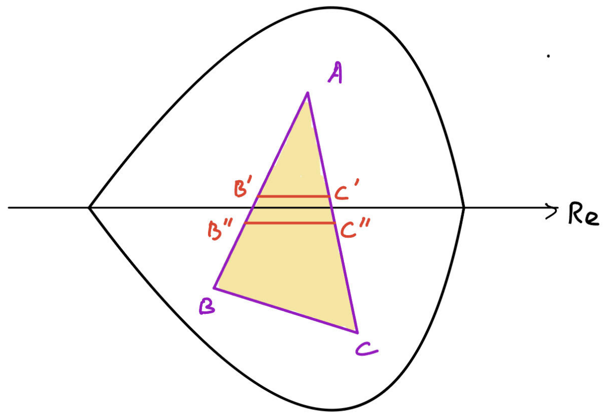


Figure 49: Triangular contour in a symmetric region.

Denote the vertices of T by points A, B and C, and suppose A is in D^+, and B and C are in D^- (See Fig. 49). Let B' and C' be two points on line AB and AC, respectively, so that $B'C'$ is parallel and is above the real axis. Similarly let B'' and C'' be two points on AB and BC so that $B''C''$ is below and parallel to the real axis. Select the points such that the parallel distance between $B'C'$ and $B''C''$ is ϵ. The contour integral of $g(z)$ on the triangle T can be decompose into three parts, namely, $AB'C'$, $B'B''C''C'$, and $B''BCC''$. The two contours that are inside D^+ or D^- have zero integral. The contour along $B'B''C''C'$ approaches 0 as ϵ approaches 0 by the continuity of $g(z)$. This shows that the integral of $g(z)$ along the triangle T is zero.

The same argument applies to other triangles in domain D. Hence, we have

$$\int_T g(z)\,dz = 0$$

for any triangle T in D. By Morera theorem for triangular contour, we conclude that $g(z)$ is analytic inside D. $\square$

Actually we have established a more general principle in the previous theorem. The assumption that the line of reflection has zero slope is not crucial. The function can also be complex-valued on the line of reflection. We single out the key part in the following theorem.

Theorem 26.15 (symmetry principle)**.** *Let D^+ and D^- be two regions that are symmetric along the common line L with positive length. We have a holomorphic function f^+ defined on D^+, and a holomorphic function f^- defined on D^-. If f^+ and f^- can be extended to the common line of intersection and $f^+(z) = f^-(z)$ for all $z \in L$, then the function $g(z)$ defined by*

$$g(z) \triangleq \begin{cases} f^+(z) & \text{if } z \in D^+ \\ f^-(z) & \text{if } z \in D^- \\ f^+(z) \text{ or } f^-(z) & \text{if } z \in L \end{cases}$$

is holomorphic in the combined region $D^+ \cup L \cup D^-$.

The proof is the same as in the proof of Theorem 26.14.

26.5 Appendix: Hurwitz theorem

The theorem in this section depicts another special property of holomorphic function. It is about the zeros of a sequence of functions that locally uniformly, and the zeros of the limiting function. The proof is based on Rouché theorem.

Theorem 26.16 (Hurwitz theorem)**.** *Let $(f_k)_{k=1}^{\infty}$ be a sequence of holomorphic function defined on a domain D. Suppose that the functions $f_k(z)$ do not vanish in D, and converges locally uniformly to $f(z)$. Then, unless $f(z)$ is identically zero, $f(z)$ has no zero in D.*

Proof. By Theorem 26.8, we know that the pointwise limit $f(z)$ is holomorphic in D. Suppose $f(z)$ is not identically zero, but there is a point z_0 in the domain D such that $f(z_0) = 0$. We will derive a contradiction.

Consider a small circle $|z - z_0| = \epsilon$ with radius ϵ so that $f(z) \neq 0$ for all z on this circle. This can be done because the zeros of a nonzero holomorphic function are isolated. Let m be the minimum value of $f(z)$ on this circle. Because $f(z)$ is not zero on this circle, we have $m > 0$. The minimum exists and is positive because $f(z)$ is continuous on the circle and the circle is a compact set.

Because the sequence of functions $f_k(z)$ converges locally uniformly, together with an application of Heine-Borel theorem, for any $\epsilon > 0$, we can find a sufficiently large integer N such that $|f_N(z) - f(z)| < \epsilon$ for all z on this circle. We choose an ϵ that is strictly less than m.

Given this data, we apply Rouché theorem (Theorem 25.5) and compare the values of the functions $f(z)$ and $f_N(z) - f(z)$ on the circle $|z - z_0| = \epsilon$. Because the modulus of $f_N(z) - f(z)$ is less than the modulus of $f(z)$ for all z on the circle, Rouché theorem says that $f(z)$ and $f(z) + (f_N(z) - f(z))$ has the same number of zeros inside the open disc $|z - z_0| < \epsilon$. However, it is assumed that $f_N(z)$ does not vanish anywhere in the domain. This contradicts that z_0 is a zero of $f(z)$. □

Hurwitz theorem is false for real functions. We can consider a sequence of real functions $f_n(x) = x^2 + 1/n$, for $n = 1, 2, 3, \ldots$, for x in the interval $[-1, 1]$. When n approaches infinity, this sequence converges uniformly to $f(x) = x^2$, which as a zero at $x = 0$.

Using the same type of argument as in the proof of Theorem 26.16, uniform limit of one-to-one holomorphic functions is also one-to-one.

Theorem 26.17. *Let $(f_k)_{k=1}^{\infty}$ be a sequence of complex functions that is holomorphic and injective in a domain D. If this sequence converges locally uniformly, then the limit function is either injective in D or a constant function.*

Proof. The steps in this proof are in parallel to the proof of Theorem 26.16. We prove by contradiction and suppose that $f(z)$ is not injective. Let z_0 and z_1 be two points in the domain of $f(z)$ such that $f(z_0) = f(z_1) = w_0$. Pick a sufficiently small radius ϵ such that $f(z) - w_0$ does not vanish on the two circles $|z - z_0| = \epsilon$ and $|z - z_1| = \epsilon$. Let m be the minimum of $f(z)$ over all z on these two circles. We have $m > 0$.

Because f_k converges locally uniformly, we can find a large enough integer N such that $|f_N(z) - f(z)| < m$ on both circles. By Rouché theorem, the two functions $f(z) - w_0$ and $f(z) - w_0 + (f_N(z) - f(z)) = f_n(z) - w_0$ has the same number of zeros inside the circles $|z - z_0| = \epsilon$ and $|z - z_1| = \epsilon$. Since $f_N(z)$ is assumed to be an injective function, if we subtract w_0 from $f_N(z)$, the function $f_N(z) - w_0$ has at most one zero inside the two circles. It contradictss that $f(z) - w_0$ has two zeros. □

In Theorem 26.17, it is possible that a sequence of injective functions converges uniformly to a constant function. For example, the functions $f_k(z) = z/k$, defined for z in a compact set, converge uniformly to the zero function, which is not injective.