Complex l’hopital rule

Theorem.

Let $f, g : \mathbb{C} \to \mathbb{C}$ be complex functions differentiable at a point z_0. Suppose $f(z_0) = 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$. Then:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof.

Since f and g are differentiable at z_0, their derivatives are defined by the limits of their difference quotients:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{and} \quad g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}.$$

Using the hypotheses $f(z_0) = 0$ and $g(z_0) = 0$, these expressions simplify to:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)}{z - z_0} \quad \text{and} \quad g'(z_0) = \lim_{z \to z_0} \frac{g(z)}{z - z_0}.$$

For any $z \neq z_0$, the term $z - z_0$ is non-zero. Therefore, we can algebraically rewrite the fraction $\frac{f(z)}{g(z)}$ by dividing both the numerator and the denominator by $z - z_0$:

$$\frac{f(z)}{g(z)} = \frac{\frac{f(z)}{z-z_0}}{\frac{g(z)}{z-z_0}} \quad \text{for } z \neq z_0.$$

We now take the limit as $z \to z_0$. Since the limit of the denominator exists and is equal to $g'(z_0) \neq 0$, we can apply the quotient rule for limits:

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{\frac{f(z)}{z-z_0}}{\frac{g(z)}{z-z_0}} \\
&= \frac{\lim_{z \to z_0} \frac{f(z)}{z-z_0}}{\lim_{z \to z_0} \frac{g(z)}{z-z_0}} \\
&= \frac{f'(z_0)}{g'(z_0)}.
\end{aligned}$$

\qed