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1.

contraction mapping theorem

$$|g(x) - g(y)| \leq \lambda |x - y|$$

$$g'(c) = \varepsilon \cos(c)$$

$$\varepsilon > |\varepsilon \cos(c)|$$

mean value theorem $\dfrac{|g(x) - g(y)|}{|x - y|} \leq g'(c)$, assume

$$\lambda = |\varepsilon \cos(c)|$$

$$\lambda \leq \varepsilon$$

since $e \in [0, 1)$

$$\boxed{|g(x) - g(y)| \leq \varepsilon |x - y| \text{ true for } x, y \in \mathbb{R}}$$

Therefore, Kepler's equation has a unique fixed point, by contraction mapping theorem.

2.

a.) $LU = A$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 3 & 7 & 5 \end{bmatrix} -\tfrac{1}{2}R_1 + R_2 \qquad \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} 5R_2 + R_3$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} -\tfrac{3}{2}R_1 + R_3$$

$$\text{check} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$$LY = B, \quad B = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} \qquad \begin{aligned} y_1 &= 0 \\ \tfrac{1}{2}y_1 + y_2 &= -5 \\ 3/2\, y_1 + (-5)y_2 + y_3 &= 7 \end{aligned} \qquad Y = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$$

$$UX = Y, \quad \begin{aligned} 2x_1 - 2x_2 + 4x_3 &= 0 \\ -2x_2 - x_3 &= -5 \\ -6x_3 &= -18 \end{aligned} \qquad \begin{aligned} x_1 &= -5 \\ x_2 &= 1 \\ x_3 &= 3 \end{aligned} \qquad \boxed{X = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}}$$

b.) multiply diagonals to get det(LU)

$\det(LU) = (1)\cdot(2\cdot -2 \cdot -6) = \boxed{24 = def(A)}$

c.) $\det(A) \neq 0$, then the solution is unique

d.) did this above, I assumed solving for X is always part of LU decomposition, now I know it is just solving for the lower & upper triangular matrices

e.) A^{-1} does exist because $\det(A) \neq 0$

f.) there is a 1e-15 absolute error due to a bunch of infinite repeating decimals in A^{-1}, leading to an tiny round-off error

3.

a.) $x = \begin{bmatrix} -\sqrt{3} \\ -6 \\ 4 \\ 2 \end{bmatrix}$

$\|x\|_1 = \sum_{i=1}^{n} |x_i| = |-\sqrt{3}| + |-6| + 4 + 2$

$\boxed{\|x\|_1 = 12 + \sqrt{3}}$

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{(-\sqrt{3})^2 + (-6)^2 + 4^2 + 2^2}$

$\boxed{\|x\|_2 = \sqrt{59}}$

$\boxed{\|x\|_\infty = \max(|x_i|) = 6}$

b.) $M = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -6 & 4 \\ -1 & 2 & 8 \end{bmatrix}$ $\|M\|_1 = \max_{1 \le j \le m} \left(\sum_{i=1}^{n} |a_{ij}|\right) = (6, 13, 19) = 19$

$\|M\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{m} |a_{ij}|\right) = (15, 12, 11) = 15$

c.) (i) $A = \begin{bmatrix} 9 & 9 & 9 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\|A\|_1 = 11$, $\|A\|_\infty = 27$ $\|A\|_1 < \|A\|_\infty$, $11 < 27$ ✓

(ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ $\|A\|_1 = 1$, $\|A\|_\infty = 1$ $\|A\|_1 = \|A\|_\infty$, $1 = 1$ ✓

(iii) $\begin{bmatrix} 9 & 1 & 1 \\ 9 & 1 & 1 \\ 9 & 1 & 1 \end{bmatrix}$ $\|A\|_1 = 27$, $\|A\|_\infty = 11$ $\|A\|_1 > \|A\|_\infty$, $27 > 11$ ✓

d. $\|M\|_2 = \sigma_{max}(M) = \sqrt{\lambda_{max}(MM^T)}$ $M = M^{-1}$

$MM^T = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -6 & 4 \\ -1 & 2 & 8 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 5 & -6 & 2 \\ 7 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 83 & 4 & 63 \\ 4 & 56 & 18 \\ 63 & 18 & 69 \end{bmatrix}$

$\det(MM^T - \lambda I) = \det \begin{bmatrix} 83-\lambda & 4 & 63 \\ 4 & 56-\lambda & 18 \\ 63 & 18 & 69-\lambda \end{bmatrix}$

$83-\lambda[(56-\lambda)(69-\lambda) - 18^2] - 4[(4)(69-\lambda) - 18(63)]$
$+ 63[4(18) - (56-\lambda)63] = 0$

$-18^2(83-\lambda) + (83-\lambda)(56-\lambda)(69-\lambda) - 16(69-\lambda)$ etc...

$\lambda_{max} \approx 142.025$ $\|M\|_2 = 11.917$

Orthogonal Matrix means $M^T = M^{-1}$

$\|M\|_2 = \sqrt{\lambda_{max}(I)} = 1$ $K_2(M) = 1$

$\|M\|_F = \sqrt{\sum \sum |a_{ij}|^2} = c$ $K_F(M) = c^2$