Lecture 4) (and 5?) The classification of (crystallographic) root systems

4.1) We will examine root lattices (crystallographic):

One motivation is to find nice constructions, in low dimensions.

Another is to show that ~~there are not~~ such ~~[illegible]~~ simple constructions ~~that~~ don't really extend to high dimensions in a ~~[illegible]~~ good way.

The best known packings up through $\mathbb{R}^8$ are root lattices.

Consumes $\frac{1}{2}$ of the ~~[illegible]~~

Classification of (complex)(semi-)simple Lie algebras ($\sim$ complex Lie groups).

$\frac{1}{2}$ [illegible]

root systems, [illegible]

then [illegible] to get on [illegible]

A_1, A_2, A_3, E_8 optimal as sphere packings.

A Lattice is a discrete (+) subgroup of $\mathbb{R}^n$. For our purposes... it will be co-compact or have full rank, it will have a $\mathbb{Z}$ basis $\{a_1 \dots a_n\}$ that spans as an $\mathbb{R}$ basis.

e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

Can have different bases, but since such bases differ by an integer change of basis T (with integral inverse)

$$a_i' = \sum_{i=1}^{n} k_i a_i \qquad k_i \in \mathbb{Z}$$

$$\Rightarrow \det\{k_i\} = \pm 1$$

So the volume of $\mathbb{R}^n / \Lambda$

$= |\det \Lambda|$

$= \mathrm{Vol}\ \mathrm{span}\{a_1 \cdots a_n\}$

is a clear invariant of Λ.

So we can normalize (to SL_n, compact)

Then there is a clearly upper bound on the length of the shortest vector on all lattices of $|\det \Lambda| = 1$ so a bound on

The largest volume sphere that can be ~~placed~~ fact that at the Λ is.

$\varphi_i \in \Lambda$ for a packing

$\{\varphi_i\ B(\frac{\lambda_{min}}{2})\}$

$\frac{\text{length } \Lambda}{2}$

So... what is the optimal lattice

Optimal lattices... ~~Still open.~~ Blichfeldt 1935 ... up to $n = 8$.

$\{A_1\ A_2\ A_3\ D_4\ D_5\ E_6\ E_7\ E_8\}$

these are root lattices.

4.2) Good candidates in low dimensions

A root lattice is an integral lattice generated by roots.

$$\boxed{\text{integral lattice } \Lambda \iff \langle x, y \rangle \in \mathbb{Z} \quad \forall\, x, y \in \Lambda}$$

roots elements of a ~~roots~~ (root system)

$\Delta \subset \mathbb{R}^n \setminus \{0\}$ ~~with~~ (satisfying)

1) Δ finite, span $\mathbb{R}^n$

2) $\alpha \in \Delta \Rightarrow n\alpha \in \Delta \iff n = \pm 1$ (reduced)

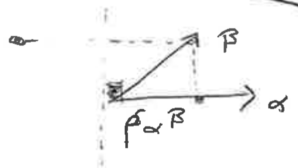
3) Δ invariant under $\perp$ reflection to any $\alpha \in \Delta$

$$\mathrm{proj}_\alpha \beta = \alpha \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

$$s_\alpha \beta := \beta - 2\,\mathrm{proj}_\alpha \beta \in \Delta$$

(spiegel)

$\Delta \ni s_\alpha \beta$



4) Crystalographic

$$n_{\beta\alpha} := \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}.$$

property 4 is the restriction that the projection of β onto α is an integer or $\frac{1}{2}$ integer multiple of α, but it really forces. $\triangle \rightsquigarrow \perp$
encodes
Lattice ...

Ex. $\boxed{\text{Try building up some other root systems, non-crystalographic.}}$

$\longrightarrow$

notice: 4) is strong...

$$n_{\beta\alpha} = 2\frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle} = \frac{2|\beta|}{|\alpha|}\cos\theta$$

since $n_{\beta\alpha}$ and $n_{\alpha,\beta} \in \mathbb{Z}$

$$\Rightarrow \boxed{n_{\beta\alpha} n_{\alpha\beta} = 4\cos^2\theta \in \mathbb{Z}}$$

$$\Rightarrow 4\cos^2\theta \in \{0, 1, 2, 3, 4\}$$

and for

$$4\cos^2\theta = 4$$

$\Rightarrow$

$\alpha = \beta$ or

$\alpha = -\beta$

$4\cos^2\theta$	$n_{\beta\alpha}$	$n_{\alpha\beta}$	$\lvert\alpha\rvert / \lvert\beta\rvert$	$\cos\theta$	θ
3	+1	+3	$\sqrt{3}$	$+\sqrt{3}/2$	$\pi/6$
3	-1	-3	$\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$5\pi/6$
2	+1	+2	$\sqrt{2}$	$+\frac{\sqrt{2}}{2}$	$\pi/4$
2	-1	-2	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$3\pi/4$
1	+1	+1	1	$\frac{1}{2}$	$\pi/3$
1	-1	-1	1	$-\frac{1}{2}$	$2\pi/3$
0	0	0	(–)	0	$\frac{\pi}{2}$

$\lvert\alpha\rvert \geq \lvert\beta\rvert$

$\uparrow$

Table assum

A root system is (decomposable) reducible

if $\Delta = \Delta_1 \cup \Delta_2$ st

$\forall \; \alpha_1 \in \Delta_1 \, , \alpha_2 \in \Delta_2 \, ,$

$\langle \alpha_1 , \alpha_2 \rangle = 0$. ie

all components of 1 are

orthogonal to all compts of

another. else call

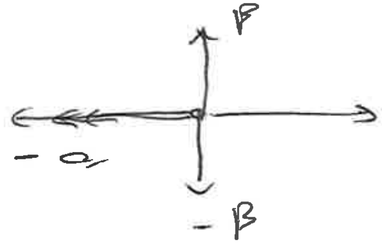
it irreducible.

(indecomposable.)

A_1

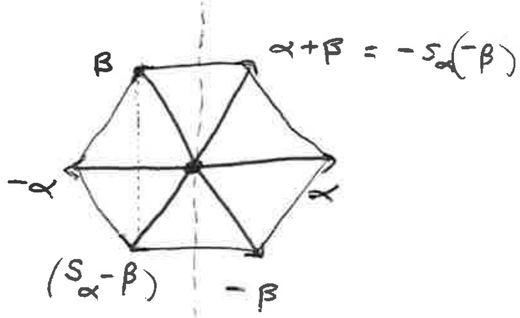
$n = 1$

$A_1 \times A_1$
(decomposable)



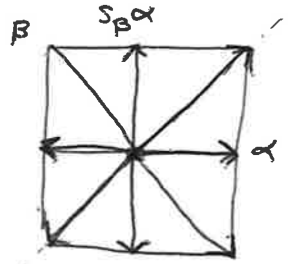
$\theta = \frac{\pi}{2}$

A_2



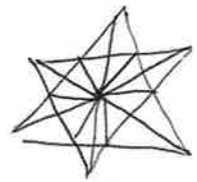
$\theta = \frac{\pi}{3}$

B_2



$\theta = \frac{\pi}{4}$

G_2 $\theta = \frac{\pi}{6}$



(note: Δ is not a basis !)

1.3) Classification of root systems

preliminaries

From a root system Δ we can choose a (non-unique) subset, the <u>simple roots</u>.

- For each $\alpha \in \Delta$, ~~there is a unique $\perp$ hyperplane its~~ consider $\alpha^{\perp}$, its (linear) orthogonal complement

since $|\Delta|$ is finite, $\Rightarrow \exists\, d$

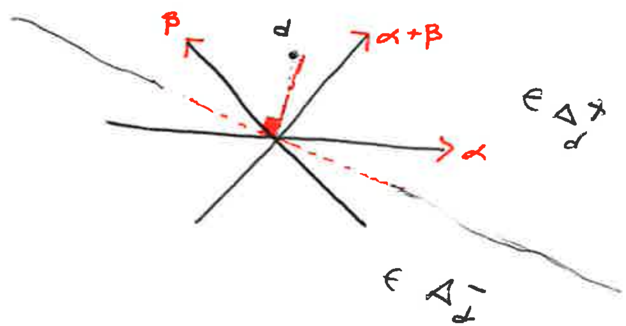
st $\forall\, \alpha \in \Delta$, $\langle \alpha, d \rangle \neq 0$.

$\Rightarrow \exists$ partition Δ into

positive $\Delta_d^{+} = \{\alpha \in \Delta : (\alpha, d) > 0$

negative $\Delta_d^{-} = \{\alpha \in \Delta : \langle \alpha, d \rangle < 0$

A root $\alpha \in \Delta_d^{+}$ is <u>simple</u> if it is not the sum of 2 other positive roots.



A set of simple roots is a fundamental system of Δ.

- The choice of δ affects the fundamental system. but such systems are equivalent under the actions of reflections ~~through~~ group $\langle S_\alpha \rangle_{\alpha \in \Delta}$.

=

a fundamental system of Δ is an $\mathbb{R}$-basis for $\mathbb{R}^n$

Sketch: (Lin ind)

$$\sum_{i=1} c_i \alpha_i = 0, \quad \alpha_i \in fs$$

partition into $c_i > 0 = (+)$

are zero $c_i \leq 0 = (-)$

$$\Rightarrow (+)^2 = (+)(-)$$

$$= |c_+||c_-| \underbrace{\langle \alpha \cdot \beta \rangle}_{} \leq 0$$

(span)

$\gamma \perp$ to all simple roots

Then by orbitary.

by $\langle S \rangle$

$\gamma \perp$ to all roots.

$\triangle$ can be reconstructed from a fundamental system via reflection $\boxed{\text{picture}}$.

Sketch. ~~orbit d. to ...~~

Consider <u>Weyl chamber</u> of f_s ← (finite spanning set.)

$$\{ x \in \mathbb{R}^n \text{ st } \langle x, f_s \rangle > 0 \}$$

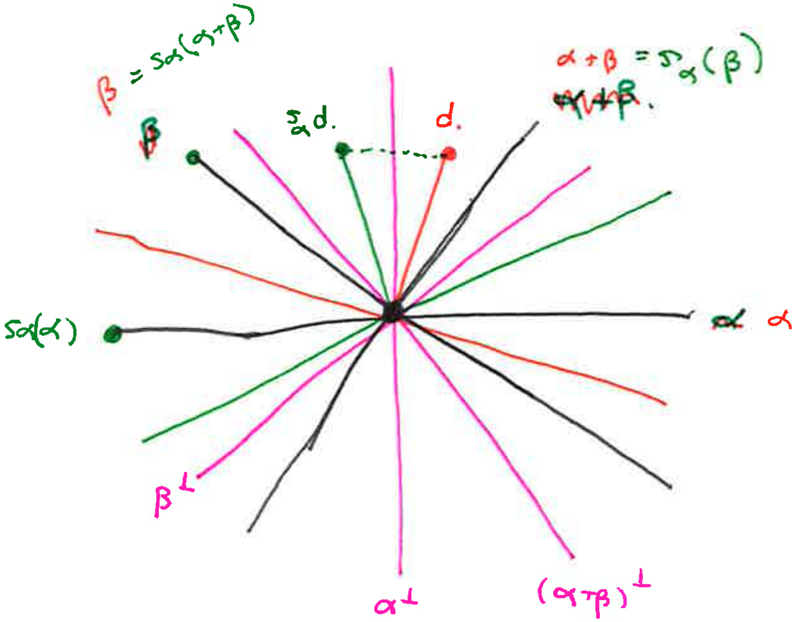
~~... to get to ... ...~~

Then $d \in$ weyl cham doesn't change the f_s.

any other f_s is based on d' in another chamber. (finite set of chamber)

path $d \to d'$ thru faces exists

$\to$ s_{faces} $d \to d'$ (chamber).



$\beta = s_\alpha(\alpha+\beta)$
$\alpha+\beta = s_\alpha(\beta)$
β
$s_\alpha d.$
$d.$
$s_\alpha(\alpha)$
α
α
$\beta^\perp$
$\alpha^\perp$
$(\alpha+\beta)^\perp$

If α, β not $\parallel$, $(\alpha, \beta) > 0 \Rightarrow$

$\alpha - \beta \in \Delta.$

$=$

$$n_{\alpha\beta} := \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} > 0$$

$\Rightarrow n_{\alpha\beta}$ or $n_{\beta\alpha} = 1$

$$n_{\alpha\beta} = 1 \Rightarrow \boxed{\begin{aligned} s_\beta(\alpha) &= \alpha - 2\beta\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \\ &= \alpha - \beta n_{\alpha\beta} = \alpha - \beta \in \Delta. \end{aligned}}$$

Similarly.

$$m_{\beta\alpha} = 1 \Rightarrow s_{\cancel{\alpha}}(-\beta) = \alpha - \beta$$

If α, β distinct simple roots

$\langle \alpha, \beta \rangle \leq 0$

If $\langle \alpha, \beta \rangle > 0$

Then $\alpha - \beta = \gamma \in \Delta$

$\Rightarrow \gamma$ or $-\gamma \in \Delta^+$

and

$\alpha = \beta + \gamma$ or

$\beta = \gamma + (-\gamma)$

(fs)
simple roots decomposable $\iff$ root system decomposable.

— similar argument —

4.4) Classification via Coxeter graphs

Simple roots $\leftrightarrow$ root systems

classify

elements of a fundamental system

α, β are $\perp$ or obtuse angle.

$\Rightarrow \theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}.$

That is.

$n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2\theta$

$\in \{0, 1, 2, 3\}$

The Coxeter graph of

Δ has a vertex for

each simple root and

an edge ~~of~~ of

weight

$n_{\alpha\beta} \cdot n_{\beta\alpha}$ between

α and β //

Well defined from the Weyl chamber up to, since simple roots → simple roots

$f_C \longrightarrow f_{C'}$

by reflections...

$\langle s_\alpha \rangle$

The Dynkin diagram is the Coxeter graph with arrows on o=o and o≡o edges pointing to the shorter vector

(possible. Lengths are defined by angles of irreducible system.)

Ex.

$A_1 \longrightarrow$ o

$A_1 \times A_1 \longrightarrow$ o o

can rescale one other.

$A_2 \longrightarrow$ o—o

$B_2 \longrightarrow$ o⇒o

$G_2 \longrightarrow$ o⇛o

End Lecture 4)

Lecture 5) Classification of Coxeter graphs.

(ignore lengths)

an independent ~~set~~ set of n unit vectors. $\{v_1 \cdots v_n\}$ spanning $\mathbb{R}^n$ is admissible if $\forall$ $i \neq j$, $\langle v_i, v_j \rangle \leq 0$

and

$$4 \langle v_i, v_j \rangle^2 = 4 \cos^2 \theta_{ij} \in \{0, 1, 2, 3\}$$

$$\boxed{\text{A normalized set of simple roots is admissible.}}$$

(note, angles are the defining feature)

An admissible diagram is the Coxeter graph of an admissible set.

The simple roots of an irreducible set Δ are not all orthogonal

$\Rightarrow$

Coxeter graphs connected.

so only need to classify

Connected admissible diagrams

Thm: The Dynkin Diagram of an irreducible root system is of the type.

n vertices.

A_n 

$n \geq 1$

B_n 

$n \geq 2$

C_n 

$n \geq 3$

D_n 

$(n \geq 4$

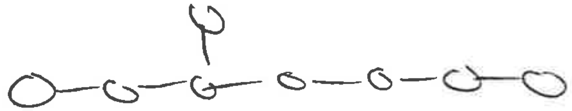
(each also unique)

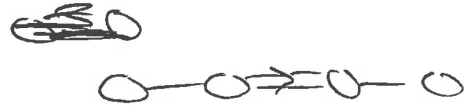
infinite families

Exceptional of type

E_6 

E_7 

E_8 

F_4 

G_2 

It will suffice to classify admissible diagrams.

B definitions subsets of vectors.

satisfy the admissibility condition inside their span (could be disconnected)

Lemma

A connected admissible diagram is a tree.

Consider $v = \sum_{i=1}^{n} v_i$, $\{v_i\}$ admissible.

v_i lin indp $\Rightarrow$ $v \neq 0$

$0 <$

$\Rightarrow \langle v, v \rangle = \sum \langle v_i, v_i \rangle$

$+ \sum_{i<j} 2\langle v_i, v_j \rangle$

$= n + \sum_{i<j} 2\langle v_i, v_j \rangle$

if v_i is connected to v_j in the Coxeter graph, then the table

$\Rightarrow$

$2\langle v_i, v_j \rangle \in \{-1, -\sqrt{2}, -\sqrt{3}\}$

$\Rightarrow \sum_{i<j} 2\langle v_i, v_j\rangle$ has

at most $n-1$ terms.

not equal to 0

$\Rightarrow$ at most $n-1$ connected vertices.

but by assumption, it is connected $\Rightarrow$ exactly $n-1$ pairs of connected vertices $\Rightarrow$ tree.

(maybe with multiple edges ~~[illegible]~~ but not too many)

Lemma ~~[illegible]~~ of each vertex degree is at most 3, with multiplicity.

Fix vertex v, connected to vertices $\{v_1 \ldots v_k\}$.

Tree $\Rightarrow \langle v_i, v_j\rangle = 0$ for $i \neq j$

$\Rightarrow \{v_1 \ldots v_k\}$ orthonormal.

also, $\{\{v_1 \ldots v_k\}, v\}$ are lin ind (simple roots)

Lemma

$$\Rightarrow \frac{\operatorname{proj}_{\langle v_1 \ldots v_k\rangle^\perp}(v)}{\quad '' \quad '' \quad ''} = v_0 \neq 0$$

and

$\{v_0, \ldots v_k\}$

orthonormal set.

$\Rightarrow$

$$v = \sum_{i=0}^{k} \langle v, v_i \rangle v_i$$

<u>and</u>

$$\langle v, v \rangle = \sum_{i=0}^{k} \langle v, v_i \rangle^2 = 1$$

since

$\langle v, v_0 \rangle^2 \neq 0 \Rightarrow$

$$\sum_{i=1}^{k} 4 \langle v, v_i \rangle^2 < 4$$

but

$4 \cos^2\theta$

$= 4 \langle v, v_i \rangle^2 =$ # edges.

$=$ [illegible]

from

v on

to v_i

$\Rightarrow$ degree of v ~~# of edges~~ w/

multiplicity is at

most 3.

Corr. OEO is the only admissible diagram with a triple edge.

Consider only double and single edges from now on.

~~Lemma (Simple ~~Trees~~)~~ (Chains Collapse.)

Ref: A simple chain is a collection of vertices connected by single edges.

Lemma (Simple Chain collapse.)

A simple chain representing $\{v_1 \ldots v_k\}$ can be replaced by $V = \sum_{i=1}^{k} v_i$ yielding an admissible diagram.

WTS... v unit vector and the collapsed diagram is admissible.

$$\langle v, v \rangle = k + \sum_{i<j} 2\langle v_i, v_j \rangle$$

and $\langle v_i, v_j \rangle = 0 \quad \forall i \neq j$

except $j = i+1$

(recall) simple edge $\Rightarrow$ $2\langle v_i, v_{i+1} \rangle = -1$

$$\Rightarrow \cancel{k} \; \langle v, v \rangle = k - (k-1)$$

$$= 1$$

recall ~~admissible~~

Consider u not represented in the chain.

it connects to only 1 vertex in the chain (by tree), say v_j.

$$\langle u, v \rangle = \sum_{i=1}^{k} \langle u, v_i \rangle$$

$$= \langle u, v_j \rangle$$

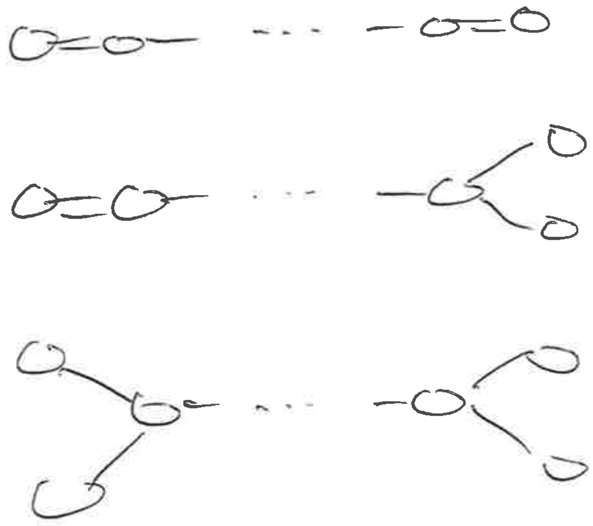
$\Rightarrow$ the angle ~~with~~ $\theta_{u,v}$

and θ_{u,v_j} ~~are~~

~~the~~...

are the same $\Rightarrow$ admissible.

So Coxeter graphs of type



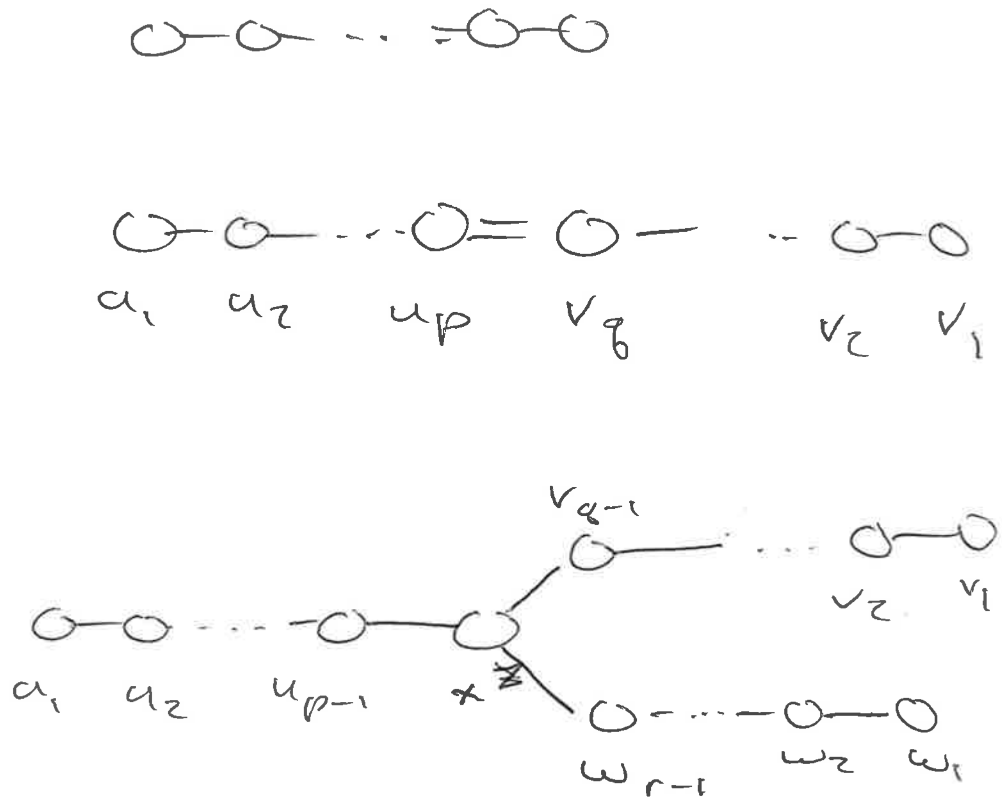
forbidden.

Can contain at most

1 branch

XOR

1 double edge



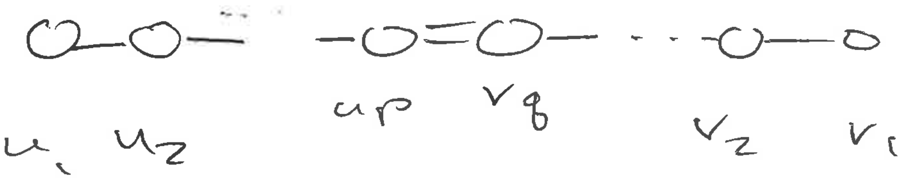
The simple chains are

or $\longrightarrow$ A_n.

B_n

C_n

F_4



• $u = \sum_{i=1}^{p} i u_i$

• $2\langle u_i, u_{i+1}\rangle = -1 \quad \forall\ 1 \le i \le p-1$

$$\langle u, u\rangle = \sum_{i=1}^{p} i^2 \langle u_i, u_i\rangle + \sum_{i<j} 2ij \langle u_i, u_j\rangle$$

$$= \sum_{i=1}^{p} i^2 + \sum_{i=1}^{p-1} (i(i+1)) \cdot \underbrace{2\langle u_i, u_{i+1}\rangle}_{-1}$$

$$= \cancel{p^2 - (p-1)(i+1)}$$

$$\sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1)$$

$$= p^2 - \sum_{i=1}^{p-1} i$$

$$= p(p+1)/2$$

similarly

$$\langle v, v \rangle = \frac{q(q+1)}{2}$$

$$\langle u, v \rangle = pq \langle u_p, v_q \rangle$$

also

~~by cases~~ ~~not~~

~~since only $U=0$ is not~~

~~orthogonal~~ relation

since $U=0$ is the only

non orthogonal relation,

and $\forall \langle u_p, v_q \rangle^2 = 2$

$$\Rightarrow \langle u, v \rangle^2 = \frac{p^2 q^2}{2}$$

and

since u, v not $\parallel$

$$\langle u, v \rangle^2 \lneq \langle u, u \rangle \langle v, v \rangle$$

$$\Rightarrow \frac{p^2 q^2}{2} < \frac{p(p+1)}{2} \frac{q(q+1)}{2}$$

$$p, q \in \mathbb{Z}$$

So

$$2pq < (p+1)(q+1)$$

$$\Rightarrow \boxed{(p-1)(q-1) < 2}$$

$$\Rightarrow \quad p = q = 2$$

$\boxed{F_4}$

or

$$p = 1 \quad q = (-)$$

$\boxed{B_n, C_n}$

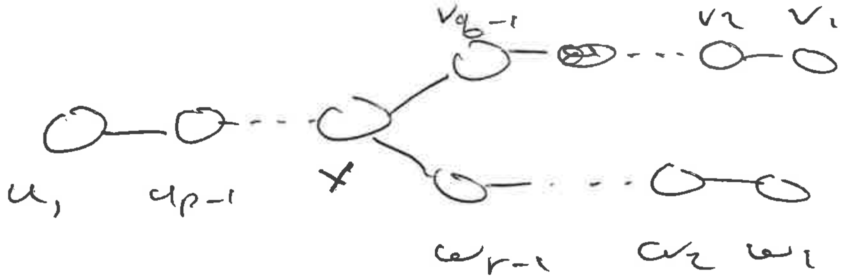
Finally

D_n

E_6

E_7

E_8



Same as previous

$$u = \sum_{i=1}^{p-1} i u_i$$

note. u, v, w ~~[illegible]~~ [illegible]

mutually orthogonal vectors.

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x is also not in

$\langle u, v, w \rangle$

$$1 = \langle x, x \rangle \geq \underbrace{\langle e, \tfrac{u}{\|u\|} \rangle^2 + \langle x, \tfrac{v}{\|v\|} \rangle^2 + \langle x, \tfrac{w}{\|w\|} \rangle^2}_{\text{projection length in } \langle u, v, w \rangle.}$$

also

$\langle x, u_i \rangle^2 = 0$ unless

$i = p-1$ and

$4 \langle x, u_{p-1} \rangle^2 = 1$

(single edge)

Then. $\langle x, u \rangle^2 = \sum_{i=1}^{p-1} i^2 \langle e, u_i \rangle$

$= (p-1)^2 \langle x, u_{p-1} \rangle^2 = \frac{(p-1)^2}{4}$

and $\langle u, u \rangle = \frac{p(p-1)}{2}$ ($p \to p-1$ gone or lost)

$\Rightarrow$

$$\langle x, \frac{u}{\|u\|} \rangle^2 = \frac{(p-1)^2}{4} \cdot \frac{2}{p(p-1)}$$

$$= \left(1 - \frac{1}{p}\right)/2$$

Then

$$2 > \left(1-\frac{1}{p}\right) + \left(1-\frac{1}{q}\right) + \left(1-\frac{1}{r}\right)$$

$$\Rightarrow \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$ and we assume $p, q, r \geq 2$

---

assume. $p \geq q \geq r \geq 1$, integers.

$\Rightarrow r = 2$

Then $q = 2 \Rightarrow p = (\sim)$

$q = 3 \Rightarrow \frac{1}{q} + \frac{1}{r} = \frac{5}{6}$

$\Rightarrow 3 \leq p < 6$

$q = 4 \Rightarrow \emptyset$

Exercises of

A_u

B_n

C_n

D_u

$E_{6\ 7\ 8}$

F_4

G_2

$\boxed{\text{Construct}}$

next

time

...

...