# Lecture 1: MAT670

Comment

: more breaks off

Definitions...

## 1.0) Admin

WKUSNER@gmail.com

Room ??? hopefully NT 020008

≡

Evaluation: • Short quizzes

• a project: $20^+$ minute talk

\+ written summary

I can update via email, or the TU Online...

— wkusner.github.io/MAT670/

We will take a 10 min break at ~ 15:50

If we end early / I run out of material...

we can stop.

=

## 1.1 Mathematics...

= Packings, Lattices and Configurations

Classical

Convex/

Discrete

Geometry

Applied Topology

Morse Theory

Geometry

of

numbers...

Rigidity

Stat Mech

Combinatorics

information theory

Various ideas from each, all ~~to~~ dealing with configurations...

{ Rigidity Theory, Configurations of linkages
{ Information Theory

Geometry of Numbers

Packing Problems ← my main motivation.

⋮

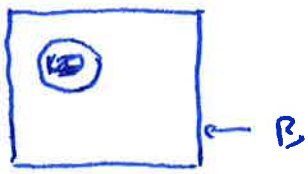
Easy to state, hard to solve.

≡ [ Packing, inequalities
  configuration spaces...

1.2 Packing problems.

Start with 2D

packing circles (Disks) in a box.




Want to maximize density (volume fraction)

$$\frac{\text{Volume}(K)}{\text{Volume}(B)}$$

Now, for $\bigcup_{i=1}^{N} K_i$

$K \sim K_i \sim K_j$ $\quad (K_i = \varphi K_j) : \varphi \in \mathbb{R}^2 \rtimes SO^2$

$K_i \subset B$

$\mathring{K}_i \cap \mathring{K}_j = \varphi$

Congruent, contained in Box, it is a packing.

e.g. consider the largest possible ~~radius~~ scale: $r$

st $\bigcup_{i=1}^{N} \varphi_i r K_i \subset B$, is a packing of B

{Minkowski notation}

or, find a ~~collection~~ configuration $\{X\}$ of

$N$ points in $B$ st

$d(x_i, x_j) > r$ for $i \neq j$

and $d(x_i, \partial B) \geq r \ \forall i$

---

These are hard problems.

infinite variety by changing

$N$, $K$ and $B$, dimension...

adding other constraints...

One case of interest is the $B = \mathbb{R}^2$ case...

The circle packing problem in the plane.

or... $B = \mathbb{R}^d$

or $B = S^2$

---

1.2) $B = \mathbb{R}^2$

We have an infinite collection of congruent Disks, now normalized to have unit radius ~~(closed disk)~~

$\mathring{K}_i \cap \mathring{K}_j = \varphi$

The (upper) density of such a packing $P$ is given by.

$$\delta^+(P) = \limsup_{r \to \infty} \frac{\mathrm{Vol}(P \cap r\mathbb{D}^2)}{\mathrm{Vol}(r\mathbb{D}^2)}$$

Similarly, Lower density...

These quantities may not be nice ~~well defined~~ in general...

- does not depend on $\vec{O}$
- does depend on $\mathbb{D}^2$ shape.

For general... $\delta^+(P) = \ldots \frac{P \cap D^n}{D^n}$

Does such a packing exist? ...

≡

We might look at this in more detail later ...

≡

How can we compute bounds on ~~the~~ density?

Clearly, $\delta^+(P) \leq 1$

≡ (bad...)

Also, we can construct lower bounds using highly structured packings.

For example.

$(2\mathbb{Z})^d$ is a packing.

Since this structure is periodic... we can work with the fundamental domain.

Easy Exercise: (why..?)

$$V_d(\text{Ball } S^d(r)) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d$$

easy in even dim.

$$V_{2d} = \frac{\pi^d}{d!} r^{2d}$$

density

$$\frac{\frac{\pi^d}{d!}}{2^{2d}} = \left( \frac{\pi^{\frac{d}{2}}}{(\frac{d}{2})!\, 2^d} \right)$$

---

We could improve this by considering better ~~lattice~~ periodic systems

[ Easy exercise: A ~~best~~ periodic system can approximate a densest packing. ]

but ...

[ Consider ~~E~~ A' $A_1 \overset{d}{\times} A_1 \ldots \quad A_d$ ?
root systems .. ]

1.3

## Better Bound (non-constructive)

---

Consider our packing to be saturated.

[ That is, $P$ st $P \cup_{\varphi} K$ is not a packing for any $\varphi$ in $\mathbb{R}^n \times SO(n)$

- No additional disks can be added.
- 1. they certainly exist (infinite algorithm.. !)
- 2. ~~They have~~ saturation does not decrease $\delta^+(P)$

---

Surprisingly useful idea...

$P$ Saturation $\Rightarrow d(x, K_i) < 1 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow$ ~~$\cup 2 K_i$~~ $\{2K_i\}$ is a covering.

~~b~~ double.

$\boxed{(2\varphi_i^{-1}(K))\varphi_i}$

Correct....

$\Rightarrow 2^d \cdot \delta^+(P) \geq 1$

$\Rightarrow \delta^+(P) \geq \frac{1}{2^d}$

We may go over some slight improvements in the future... but none seem satisfactory

Problem is that ~~the~~ ~~local structure~~ there is a lot more freedom in $d \geq 3$! even.

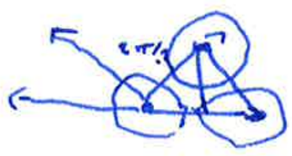
$\left[ (2\mathbb{Z})^4 \ldots \quad \Delta = 2\sqrt{4} = 4 \ldots \right.$

1.4) upper bounds in $d = 2$.

~~There is a~~

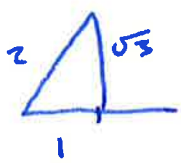
1) The Lower bound is constructed.

hex, $\Delta$ lattice




$$\delta = \frac{\pi}{\sqrt{12}}$$

equilateral $\Delta$. base 2




$$\frac{\pi}{2\sqrt{3}}$$

$$\frac{1}{2r} \geq \frac{\sqrt{3}}{2} 2r$$

$$r \leq \frac{\sqrt{\frac{2}{\sqrt{3}}}}{2}$$

$$\frac{\pi r^2}{1} = \text{density} \leq \frac{\pi}{\sqrt{12}} \quad //$$

For Lattices... ~~we may consider minimizing the det of the span for a unit lattice.~~

max minimum distance between 2 points... in a unit lattice..

can be small... []

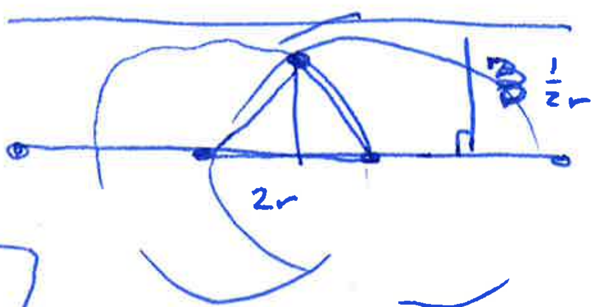
but is bounded above...

min $\frac{?}{2}$ dist. lin. also..

also on

note
not
only orthog
here

$\frac{1}{2r}$ is light of || here



$$\frac{1}{2r} \geq \frac{\sqrt{3}}{2} 2r$$

=>

$$r \leq \sqrt{\frac{2}{\sqrt{3}}}$$

maximum 2r => min $\frac{1}{2r}$

$$\frac{1}{2r} = \frac{\sqrt{3}}{2} 2r$$

~~$\frac{1}{2r} = \frac{\sqrt{3}}{2} 2r = \sqrt{3}\, r$~~

$$\frac{1}{2r} \geq \sqrt{3}\, r$$

r

1.5

In general, we need some analogous method to partition space.

≡

Classically: Dirichlet Voronoi diagrams are good...

~~$D_i$ & $P_i$ =~~ {x

$D_i$ associated with $P_i$

is

$$D_i = \{ x \in \mathbb{R}^2 : d(x, P_i) < d(x, P_j) \quad \forall j \neq i \}$$

when $P_i$ is a disk and $d$ = standard ~~metric~~ distance..

This is also the dist to the center of the circle..

Nice properties... half space decomposition wrt centers, so convex.

partition space... up to a 2 term.

... picture..

Dual notion: Delaunay Δ-ation.

– non unique ... □ ...

Difficult..

Characterization: circumcircles of Δ's are empty of other points/vertices..

[ Its circum center is the Voronoi vertex

Existence:

Consider a collection of N points in the plane

~~[illegible]~~.

Lift ~~to z = $x^2+y^2$ paraboloid~~...

$a, b, c, d$ in plane $x$-$y$

lift to $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ on $z = x^2 + y^2$

$d \subset$ circumcircle of $a, b, c$

$\Longleftrightarrow$

$\hat{d}$ is in lower $\frac{1}{2}$ space of $\langle \hat{a}, \hat{b}, \hat{c} \rangle$

A plane $\perp$ to paraboloid at $(p, q)$

has the form.

$$z = 2px + 2qy \mp (p^2 + q^2)$$

shifted ...

$$z = 2px + 2qy + (p^2 + q^2) + h^2$$

$$x^2 + y^2 = \quad ''$$

$$= \cancel{\text{than}} \; (x + p)^2 + (y + q)^2 = h^2.$$

IG

So place the $\perp$ to $(p, q, ~~\ldots~~\ p^2+q^2)$

at height $h^2$ is $\overleftrightarrow{p-j}$ to $x, y$

a circle radius $h$

$\Rightarrow$ lemma . . .   ; ~~Pfco~~

---

$\exists$ conn wall $\Rightarrow$ Existence of Realz $\triangle$

---

1.6

Thre

1.6

Lemma The Largest angle of

$\triangle ABC \in DT$ of a saturated packing.

satisfies:

$$\underbrace{\frac{\pi}{3} \leq \Theta}_{\text{obvious}} \leq \frac{2\pi}{3}$$

by largest angle of each $\triangle$.

Assume $\Theta \geq \frac{2\pi}{3}$. $\Rightarrow$ Circumradius $\triangle ABC > 2$.

if $A$ smallest Angle,

and $\Theta \geq \frac{2\pi}{3} \Rightarrow A \leq \frac{\pi}{6}$

$\Rightarrow \sin(A) \leq \frac{1}{2}$

Circumradius formula.

$$R = \frac{1}{2} \frac{\overline{BC}}{\sin A} \geq \frac{2}{\frac{1}{2}} \cdot \frac{1}{2} = 4$$

$\cdot\!\times\!\cdot$

Lemma

$\triangle$ density in a set $\triangle$ DT

$\leq \frac{\pi}{\sqrt{12}}$, $=$ sharp if equilat.

$B$ Largest Angle $\triangle ABC$.

$$\text{area} = \frac{1}{2}\, \overline{AB} \cdot \overline{CB} \sin B$$

$$\geq \frac{1}{2} \cdot 2 \cdot 2 \cdot \min_{[\frac{\pi}{3}, \frac{2\pi}{3})} \sin B$$

$$= \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2}$$ when $B = \frac{\pi}{3}$

So Area DT $\geq \sqrt{3}$

$\boxed{\text{Theorem}}$

$\Rightarrow$ density DT $\leq \frac{\pi}{2\sqrt{3}}$ ■

Density of P

$$= \sum_{\Delta \text{ in } DT} \frac{\text{area } D_t \times \text{Density of } DT}{\text{area } DT}$$

$$\leq \frac{\pi}{\sqrt{12}}$$

$\Rightarrow$ any finite union of DT has density $< \frac{\pi}{\sqrt{12}}$.

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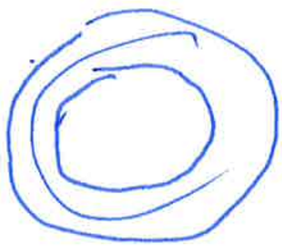
finite many of the geometry of the $\Delta$ is reached here.

so



error in the limit is $\partial$ here

$\Rightarrow$

$d < \frac{\pi}{\sqrt{12}}$.

# Lecture 2.

A206 Mon? 15-17$^{00}$
Except 2 days + Last Day in Sem room
? NTO2008

## 2.1 Last time: modern take on

Thue's Result in $\mathbb{R}^2$: The density of a packing of the plane by congruent circles is bounded by the density of the regular hexagonal packing.

Idea: use saturation and Delaunay triangulation

Remark: Left out error analysis of limits ... etc.

Remark: Rogers showed density bound from the regular simplex.

## 2.2 Fejes-Tóth inequality for $S^2$

Thm: For $n > 2$ points on $\mathbb{S}^2$

There exists a pair with spherical distance (angular...)

(*)
$$d \leq \arccos\left(\frac{\cot(\omega)-1}{2}\right);$$

$$\omega = \frac{n}{n-2} \cdot \frac{\pi}{6}.$$

(Exercise)
Remark: can be rewritten as a density result for spherical caps of diameter $d$

$$\frac{n \cdot 2\pi\,(1-\cos(\frac{d}{2}))}{4\pi} \rightsquigarrow \frac{\pi}{\sqrt{12}}.$$

Flat values also converge but not so nice to take limit.

Good lower bound? ie construct a sequence of sets of points that achieves this bound.

To prove F-T Thm, we need

Lemma: given spherical $\Delta$ ABC

If: Area $\Delta ABC$ < Area of equilateral $\Delta AB\hat{C}$, drawn on the shortest side $\overline{AB}$ of $\Delta ABC$

Then: The spherical radius of the (intersection) circumcircle of ABC is greater than length $\overline{AB}$.

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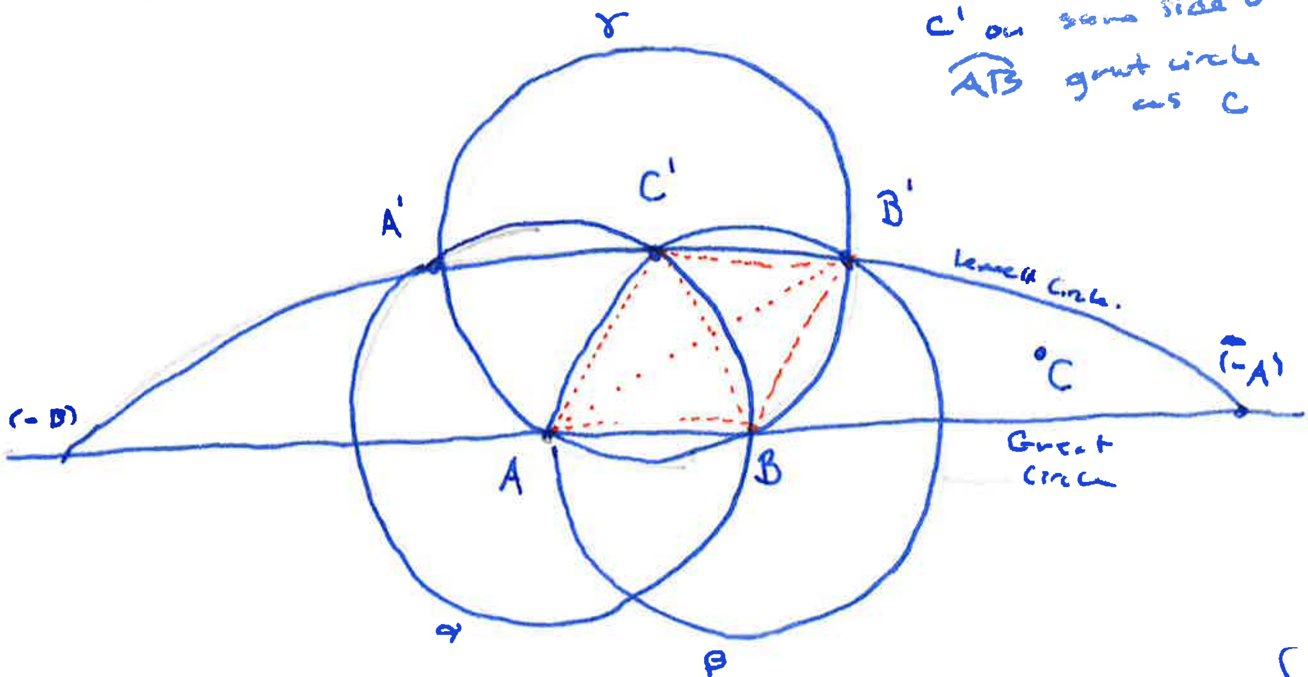
Proof of Lemma:

1) Draw picture. C', st ABC' equilateral
C' on same side of $\widehat{AB}$ great circle as C



2) Show picture is "correct":

note: spherical triangles $\Delta ABA'$, $\Delta ABB'$, $\Delta ABC'$ have equal area.

$\Rightarrow$ the intersection locus of the plane $\{A', B', C'\}$ is the equal area $\Delta$ locus (Lexell Circle Thm)

By assumption, $C$ is between the Lexell Circle and the Great circle $\widehat{AB}$.

As $\overline{AB}$ is the shortest edge of $\Delta ABC$ by assumptions $C$ does not lie in $\alpha$ or $\beta$.
$\Rightarrow C$ is not in $\gamma$.

But $C$ is on the $C'$ side of $\widehat{AB}$, $\Rightarrow$ circumcircle radius of

$\pi r/2$
$95$ $\Delta ABC > \overset{rad}{\gamma}$ = length $\overline{AB}$.

20

So to prove thm:

$n=3$ trivial, so $n \geq 4$.

WOLG Convex Hull $\{P_1, \dots, P_n\} \ni 0$

[ else we could flow points $\xrightarrow{\text{towards}}$ pole equator and increase all distances. ]

1) Triangulate convex hull.

Euler Char. $\Rightarrow$ $2n-4$ faces.

Exercise:

$$v - e + f = 2$$

$3f = 2e$ for $\Delta$ lations

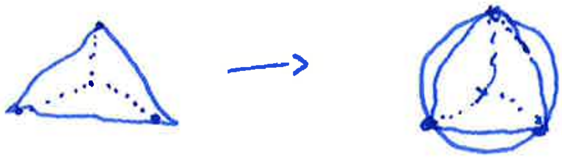
$\Rightarrow 2n - 2e + 2f = 4$

$\Rightarrow 2n - 3f + 2f = 4$

$\Rightarrow f = 2n - 4$ ✓

$\Delta$ation of Convexhull $\{p_1 \cdots p_n\}$

$\longrightarrow$ "Spherical net" by projection to the sphere, radially.



(edges are geodesics . . . .)

Now: for $\{p_1 \cdots p_n\}$,

Suppose length $\widehat{p_i p_j} > d^{(*)}$

for all $i \neq j$.

(Exercise) $(*)$

$d$ is the length of the side of an equilateral spherical $\Delta$ of area

$\frac{4\pi}{2n-4}$. (L'Huilier, analogous to Heron's formula)

Then $\Delta P_i P_j P_k$ of minimal area satisfies the Lemma:

Its area is small*, but edges are all longer than d.

* less than or equal to the equilateral $\Delta$ of side length d.

$\Rightarrow$ The circum circle $P_i P_j P_k$ has radius larger than d.

But the net construction $\Rightarrow$ The circum circle is empty.

So we may place a new point at the center...

This new collection satisfies the same inequalities for the <u>same d</u>!

This is absurd.

e.g.: volume bound.

$$\mathrm{Vol}\, S^2 = 4\pi$$

and $\sum_{i=1}^{N} \text{area } \mathrm{cap}(\frac{d}{2}) < 4\pi \quad \forall N > 0$

2.3) Higher Dimension Sphere packing.

Consider an early method of Blichfeldt (1929) to get density bounds in higher dimensions.

We start with a <u>convex body</u> $C$ ~~(so think $C = \mathbb{B}^n$)~~

in $\mathbb{R}^n$: a compact, convex subset of $\mathbb{R}^n$ with non-empty interior.

Also, consider collections of isometries $\{\varphi_i\}_{i=1}^{\infty}$ such that $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing.

$\equiv$

For now $C = \mathbb{B}^n$

and a packing is of the form:

$$\{\varphi_i \, \mathbb{B}^n\}_{i=1}^{\infty}$$

Consider replacing the ball with its characteristic function.

$$\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

So it has mass.

$$I(\chi) = \int_{\mathbb{R}^n} \chi(x)\,dx$$

And a packing $\longrightarrow \sum_{i=1}^{\infty} \chi(\varphi_i^{-1} x)$

For general $C$, this works as well...

Furthermore, we now can replace $\chi$ with some other function, $f$ and in general, a packing with

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1} x)$$

And a density may be similarly defined to the packing density

when each object has mass

$$I(f) = \int_{\mathbb{R}^n} f(x)\,dx$$

spread over supp$(f)$.

So :

$$\Delta_f = \sup_{\substack{\text{packing} \\ C}} \frac{I(f)}{Vol(C)}$$

$\equiv$

Can think of this as spreading out and renormalizing the characteristic function in some (nice) way.
hopefully)

" Cut up the sphere and spread it out. "

Insight: There are functions $f$ where $\sum_{i=1}^{\infty} f(\varphi_i^{-1} x)$ are

~~pointwise and unifo~~

Uniformly pointwise bounded over $\mathbb{R}^n$ and collections of isometries $\{\varphi_i\}$ st

$\{\varphi_i C\}$ is a packing!

2.41) Given a convex body $C$ in $\mathbb{R}^n$

$f$ is a <u>Blichfeldt gauge</u> for $C$

if for any $\{\varphi_i\}_{i=1}^{\infty}$ of

isometries of $\mathbb{R}^d$ st $\{\varphi_i C\}_{i=1}^{\infty}$

is a packing,

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1} x) \leq 1$$

for all $x \in \mathbb{R}^n$.

add non negative condition...

Lemma: If $f$ is a Blichfeldt gauge.

$$(*) \qquad \delta(C) \leq \frac{\mathrm{Vol}(C)}{I(f)}$$

proof sketch. Def $\Rightarrow \Delta \overset{\leq}{\bullet} 1$

$\Rightarrow (*)$

2.5) Back to $C = \mathbb{B}^n$

we can consider radially symmetric functions by an averaging argument

(no benefit to anisotropy, since the $SO(n)$ kills it... )

$\equiv$

So distance functions!

Rankin does a complicated study to get good function

We consider

$$f_0(x) = \begin{cases} 1 - \frac{1}{2}|x|^2 & |x| \le \sqrt{2} \\ 0 & |x| > \sqrt{2} \end{cases}$$

---

Can show the following for a packing of unit radius Balls.

$\boxed{\text{Lemma:}}$

Consider $m$ unit Balls with centers

$\{a_i, \dots, n_i\}$.

Then clearly, for all pairs we have.

$$(*) \quad (a_i - a_j)^2 + \dots + (n_i - n_j)^2 \geq 4 .$$

Then by considering.

$$\Rightarrow \sum_{1 \leq i < j \leq m} (*) ,$$

we have. (extra cross terms cancel...)

$$m \sum_{i=1}^{m} (a_i^2 + \dots + n_i^2) - \left(\sum_{i=1}^{m} a_i\right)^2 - \left(\sum_{i=1}^{m} b_i\right)^2 \dots - \left(\sum_{i=1}^{m} n_i\right)^2$$

$$\geq 2m(m-1)$$

$$\Rightarrow \sum (a_i^2 + \dots + n_i^2) \geq 2(m-1)$$

This is an arbitrary arrangement, so we consider a collection of $m$ spheres within $\sqrt{2}$ of $\vec{0}$ with masses defined by $f_0$ centered at $(a_i \dots n_i)$.

we see that

$$\sum_{i=1}^{m} \frac{(2 - r_i^2)}{2} \le \frac{2m - 2(m-1)}{2} = 1$$

and ~~this~~ $f_0(x)$ is a Blichfeldt gauge.

2.6) So we can integrate this

$$I(f) = \int_0^{\sqrt{2}} \frac{2-r^2}{2}\, dr^n = \frac{4 \cdot 2^{n/2}}{n+2} \cdot \mathrm{Vol}(B^n) \qquad \frac{2^{\frac{n+2}{2}}}{n+2}$$

Exercise (spherical shells)

$\boxed{\text{Limit argument for a box}}$

For a large box with $k$ spheres in it.

$t$

Then $(t + 2\sqrt{2} - 2)^n$ box contains the gauges

$$(t + 2\sqrt{2} - 2)^n \ge |I| \int_{\mathbb{R}^n} f_0\, dx$$

$$= \frac{|I|\, \mathrm{Vol}(B^n)\, 4 \cdot 2^{n/2}}{n+2}$$

$\Rightarrow \delta\left(\begin{array}{c}\text{of these } k \text{ balls in} \\ \text{the box } t^n\end{array}\right)$

$$= \frac{|I| \operatorname{Vol}(B^n)}{t^n}$$

$$\leq \frac{n+2}{4 \cdot 2^{n/2}} \left(1 + \frac{2\sqrt{2} - 2}{t}\right)^n$$

and for $t \to \infty$

$$\delta(\mathcal{B}^n) \leq \frac{n+2}{4 \cdot 2^{n/2}}$$

$$\frac{2^{\frac{n}{2}+1}}{n+2}$$

for any packing of $\mathbb{R}^n$ by $\mathbb{B}^n$

$\square$

Can do this explicity for Balls too, also still missing Boundary term.

Lecture 3 18.4.2016

Last time: Fejes Tóth inequality
Blichfeldt sphere packing Bound.

## 3.1) Some remarks

- Where is the Fejes Tóth inequality sharp? For $n$ points

$$d \leq \arccos\left(\frac{\cot(\omega)-1}{2}\right);$$

$$\omega = \frac{n}{n-2}\cdot\frac{\pi}{6},$$

$d$ angular edge length of the equilateral triangle with area $\frac{4\pi}{2n-4}$ on unit sphere.

So, this should be sharp for an equilateral $\triangle$-ulation ...

$n = \underline{3}, 4, 6, 12.$, and perhaps $\xrightarrow{n} \infty$

- There are other proofs, Longer.

~~Again, There exist e.g. a trivial Blichfeldt gauge for $C$. is given by $C$.~~

3.2 aside] We saw last time.

$$f_0(x) = \begin{cases} 1 - \frac{1}{2}|x|^2 & x \leq \sqrt{2} \\ 0 & x > \sqrt{2} \end{cases}$$

is a Blichfeldt gauge for $\mathbb{B}^n$.

It is non-trivial for our purposes.

that is:

$$\int_{\mathbb{R}^n} f_0(x) \neq \int_{\mathbb{R}^n} \chi_{\mathbb{B}^n}$$

(at least for $n \neq 2$)

$$\int_{\mathbb{R}^n} f_0(x)\, dV = \frac{2^{1+\frac{n}{2}}}{2+n} \operatorname{vol}(\mathbb{B}^n)$$

$\Rightarrow$

$$\cancel{\limsup_{t\to\infty}} \quad \frac{\mathrm{Vol}(B^n \cap \mathrm{Box}(t))}{\mathrm{Vol}(\mathrm{Box}(t))} \underset{\underline{\text{always}}}{\leq}$$

$$\frac{n+2}{2^{1+\frac{n}{2}}}\left(1 + \frac{\sqrt{2}-2}{t}\right)^n$$

$$+ \; \frac{C\, t^{n-1}}{t^n}$$

$$\longrightarrow \quad \limsup_{t\to\infty}\left(\frac{\mathrm{Vol}\, B^n \cap \mathrm{Box}(t)}{\mathrm{Vol}(\mathrm{Box}(t))}\right) \leq \boxed{\frac{n+2}{2^{1+\frac{n}{2}}}}$$

---

note, we could use.

$$\delta = \limsup_{t\to\infty} \frac{\mathrm{Vol}\{B^n : B^n \subseteq \mathrm{Box}(t)\}}{\mathrm{Vol}\ \mathrm{Box}(t)}$$

3.4) What does this look like?

$f_0$ : Security bounds.

| | | |
|---|---|---|
| $n=1$ | 1.06 | |
| | 1 | |
| | .88 | |
| | .75 | |
| | .61 | $\dfrac{n+2}{2^{1+\frac{n}{2}}}$ |
| | .5 | |
| | .39 | |
| | .31 | |
| | .24 | |
| $n=10$ | .19 | |

Lower Bounds:

$\dfrac{1}{2^n}$ (Saturation)

⋮

Ball (1992?)

$$\frac{2n}{2^n}$$

Vance $n \equiv 0 \bmod 4$

$$\frac{\frac{6}{e} n}{2^n}$$

!!!

Venkatesh (Space) $\frac{e^{-\gamma}}{2} n \log\log(n) / 2^n$

Upper

$\dfrac{n+2}{2^{\frac{n}{2}+1}}$

⋮

⋮

$2^{-.599 n}$ Kabatjanskiĭ ~~Kabatjansky~~ Levenštein

(Cohn, Elkies, Kumar)

• Known good constructions in Low dimensions / Special ones...

need to talk about this at some point

• applications

• ...

Exact Results. (upper)

$d=1$ trivial

$d=2$ we proved it. (1890... ???) Thue.

$d=3$ — Hales, twice → complicated geometric analysis
2000 ish..

$d=8$, $d=24$ } 1 month ago: LP duality
constructed a special function..

Remember...

$$\mathbb{Z}^n \sim \frac{\pi^{n/2}}{2^n\left(\frac{n}{2}\right)!}$$

3.4) Kabatjanskiĭ + Levenštein.

Thm: In high dimensions. for

$1 < r < 2$

$$f_d(x) = \begin{cases} M(d, \arccos(1 - 2/r^2))^{-1} & |x| < r \\ 0 & |x| \geq r \end{cases}$$

is a Blichfeldt gauge for $\mathbb{R}^d$

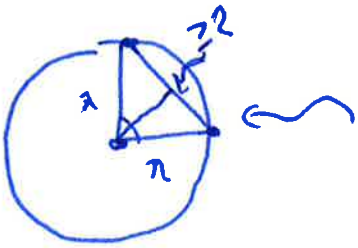
where $M(d, \varphi)$ is the maximum number of points on ~~the (d-1)~~ $S^{d-1}$ with pairwise distance $\geq \varphi$.

or... # pts in $\mathbb{R}^d$ st the angle spanned by any 2 pts is at least $\varphi$. (proof to....)

proof: for any packing of unit balls in $\mathbb{R}^d$, for $r < 2$

There are at most $M(d, \arccos(1 - 2/r^2))$ centers at dist less than $r$. and can show to

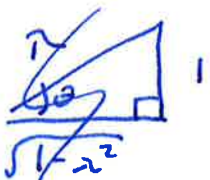
angle at $x$ spanned by segment of length $\geq 2$, ends within $r$ from $x$ is at least $\arccos(1 - 2/r^2)$




chord length is: $2 - 2(a \cdot b)$


$\cos\theta = \frac{1}{r}$

$2 \arccos\left(\frac{1}{r}\right)$




$\cos\theta = \frac{\sqrt{1-r^2}}{r}$

$A \cdot B = \|A\| \, \|B\| \cos\theta$

$r^2 \cos\theta$

$2 - 2 a \cdot b$

$2 - 2 r^2 \cos\theta$

$$\left|\frac{z}{r}\right|^2 \le \left|\frac{d}{r}\right|^2 = 2 - 2(u \cdot v)$$

$$= 2 - 2\cos\theta$$

$$\left|\frac{z}{r}\right|^2$$

$$\frac{4}{r^2} \le 2 - 2\cos\theta$$

$$\frac{2}{r^2} \le 1 - \cos\theta$$

$$\boxed{\cos\theta \le 1 - \frac{2}{r^2}}$$

Thm.

$$\delta(B^d) \le \lambda^{-d} \mu(d, \arccos(1 - 2/\lambda^2))$$

$$1 \le \lambda \le 2.$$

equiv. $\Rightarrow$

$$\rightsquigarrow \delta(B^d) \le \sin(\tfrac{\varphi}{2})^d \mu(d, \varphi)$$

$$\tfrac{\pi}{3} < \varphi \le \pi$$

for $d$ large,

$\varphi$ ~~small~~... $\le 63°$.

$$\mu(d,\varphi) \le \sin(\varphi/2)^{-d}\, 2^{-d \cdot .599 + \dots}$$

$$\Rightarrow \delta(B) \le 2^{-d \cdot .599 + o(d)}.$$

3.5)

So:

We defined a Blichfeldt gauge for $\mathbb{R}^n$. Can we extend this?

Abstractly:

Given a (convex) body $C$ in $\mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ is a <u>Blichfeldt gauge for $C$</u> if,

for any collection of isometries $\{\varphi_i\}_{i=1}^{\infty}$ of $\mathbb{R}^n$ s.t. $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing,

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1} x) \leq 1 \quad \forall \; x \in \mathbb{R}^n$$

- Remark... could add non-negativity condition...
  really want mass of $f$ $I(f) \geq I(\chi_C)$
- Remark. the index set could also be finite...
- Remark... $f$ could be much stranger than we allow... occurs first..
- Exercise *** (Think about such functions distributionally?)

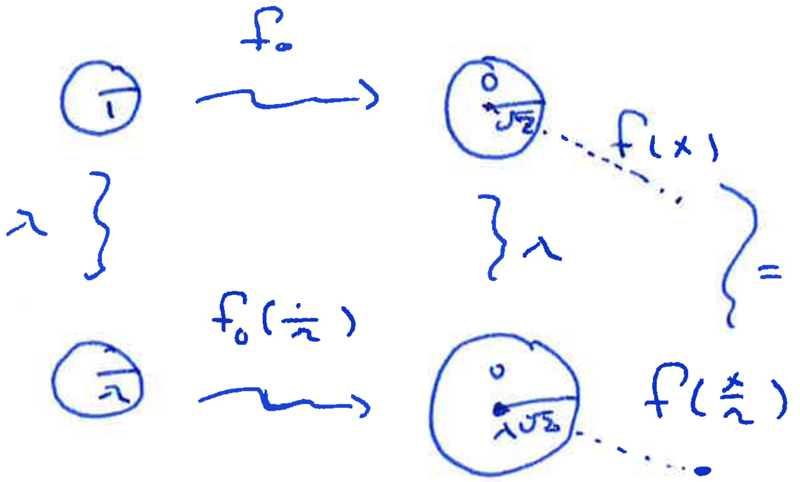
There exists a Blichfeldt gauge for $C$.

eg. $\chi_C$.

---

- If $f(x)$ is a Blichfeldt gauge for $C$,

  $f(x/\lambda)$ is a Blichfeldt gauge for $\lambda C$, the $\lambda$ scaling of $C$.



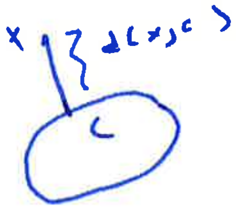

For a body $C$, define its insphere radius to be $r(C)$, the radius of the largest sphere contained in $C$

For $0 \leq \gamma \leq r(C)$ define $C_\gamma$ to be the <u>inner parallel body</u> at <u>depth</u> $\gamma$

The set of points

$$C_\gamma := \{ x \in C \text{ st } x + \gamma B^d \subset C \}$$


We can define the straight line distance to a body by. $d(x, C)$




Then... if $f$ is a radial Blichfeldt gauge for $B^n$,

ie $f(x) = F(|x|)$

Then $g(x) = F\left(\dfrac{d(x, C_\gamma)}{\gamma}\right)$

define.

Claim... $g(x)$ is a Blichfeldt gauge for $C$.

This is fairly easy to see by picture...

~~Consider a point $x \in \mathbb{R}^n$.~~

We would like to show that

$$\sum_{i=1}^{\infty} g(\varphi_i^{-1} x) \le 1 \quad \text{for all packings of } C \quad \{\varphi_i C\}_{i=1}^{\infty}.$$

$$\forall x \in \mathbb{R}^n$$

Consider a point $x \in \mathbb{R}^n$

and a pt $x_i$ in $\varphi_i C_\gamma$

~~x~~ at distance

$d(x, \varphi_i C_\gamma)$ from $x$.


Then $\gamma B^n_{x_i}$ is in $C$

~~and~~ $\Rightarrow$ $\{\varphi_i \, \gamma B^n_{x_i}$ is a $\}$.

packing

( Since $C$ is convex, then ... call )
well defined..

Since.

$\Longrightarrow$ $F(\frac{x}{\gamma})$ is a Blackwell [illegible] fn

$\gamma B^n$

$\Longrightarrow$

and $C$ is ... we $x$ is

and defined.. $d(x, C\gamma)$

$$\Rightarrow \sum B\, F\left(\frac{d(x, C\gamma)}{\gamma}\right)$$

$$\le 1 .$$

As with Blichfeldt guys and other approaches... is there [illegible] ?

if we have a convex body like an

$A + B^n_1$, it could be ..., provided there is enough high dimensional mass ...

$$f(x) = \begin{cases} f_0 & |x| > 1 \\ 1 - f_0(2 - x] & |x| \leq 1 \end{cases}$$

$$\rightsquigarrow \quad \delta \leq \left( \frac{2}{d+2} \sqrt{2}^{\,d} (1 + b_d) \right)^{-1}$$

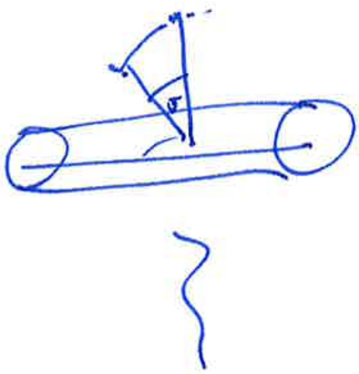
$$b_d = \frac{1}{(\sqrt{2})^d (d+1)}$$

$$\cdot - (\sqrt{2} - 1)^{d+1} \left(1 + \frac{\sqrt{2}}{d+1}\right)$$

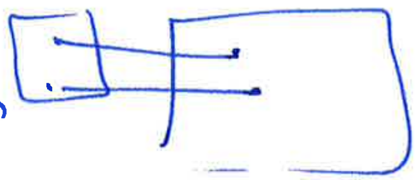
$\delta(f \ldots)$
$f$ Bd...

| | $f_0$ |
|---|---|
| 1 | 1.06 |
| .94 | 1 |
| .84 | .88 |
| .72 | .75 |
| .60 | .61 |
| .49 | .5 |
| .39 | .39 |
| .31 | .31 |
| .24 | .24 |
| .18 | .19 |

So this works for things with a lot of "2d" mass...
or large



vs



.94 + Error

1 + Error...

~~two~~

newer data
as it is now

Id blackbody
Eng...

47

$l$ cylinder.

$$\underline{\quad l \quad} \times B^n$$

$$I(g) = l \cdot \frac{I_{d-1} f}{V}$$

$$l \cdot \frac{\int_{d-1} (f)}{V(B^{d-1})} \cdot VB^{d-1} + \frac{\int_d f}{VB^d} V_d b.$$

$V$( $l$ cylinder )

$$= l V(B^{d-1}) + VB^d.$$

$= )$.

$$\delta(l \text{ cyl}) \leq \frac{l(VB^{d-1}) + VB^d)}{l\, I_{d-1} f + I_d f}$$

end errors for cylinder..