A brief analysis of packing regular pentagons in the plane

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ESI
October 2014

[1] Supported by an FWF special research program, NSF grant 1104102 and a fellowship from the University of Pittsburgh.

Abstract

It is a still an open problem to describe the densest packing of the plane by regular pentagons. Here, the general problem is described near a conjectured optimum by a nonlinear programming problem. Under certain assumptions, that optimum can be certified locally via a linear programming problem. This method can be used to computationally prove* the local optimality of the conjectured globally optimal pentagon packing.

Density

A *packing* of a region $X \subseteq \mathbb{R}^n$ by objects $\mathrm{P}_i \subseteq X$ is a countable family $\mathscr{P} = \{\mathrm{P}_i\}_{i \in I}$ with disjoint interiors.

The *upper density* ρ^+ of a packing $\mathscr{P}$ in $\mathbb{R}^n$ will be defined as

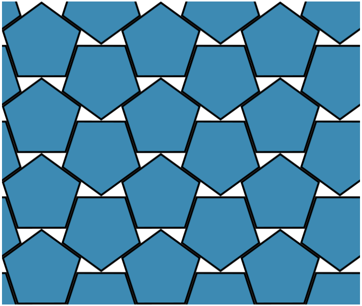
$$\rho^+(\mathscr{P}) = \limsup_{r \to \infty} \frac{\mathrm{Vol}(r\mathbb{B}^n \cap \mathscr{P})}{\mathrm{Vol}(r\mathbb{B}^n)},$$

where $\mathbb{B}^n$ is the unit n-ball centered at 0.

Remark

The packing density is bounded by the density of the densest “cell” in some “nice decomposition” of the packing.

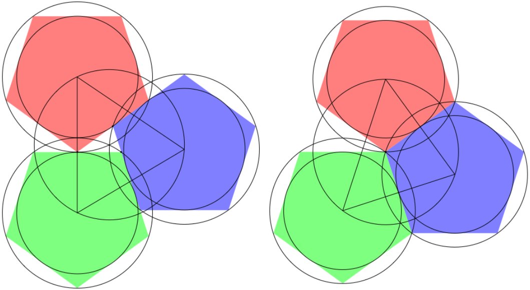
Pentagons



The best lower bound for the density of pentagon packings and the conjectured maximal density configuration is shown. This packing has a density of $(5-\sqrt{5})/3 = 0.92131\ldots$.

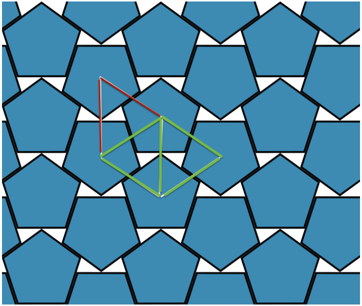
Pentagons

For circles, saturated packings and the existence of Delaunay triangulations can guide a proof of Thue's theorem.



This doesn't work for pentagons. The conjectured best packing is not a even critical point for density inside a Delaunay triangle.

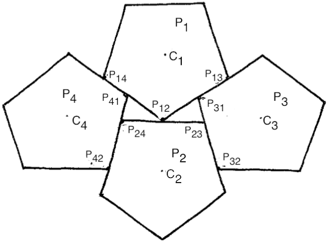
Pentagons

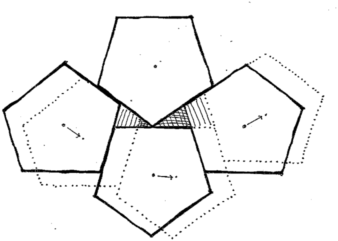


For four pentagons there is a pairing that gives a local result.

Nonlinear Program

The problem of finding the densest packing of pentagons can be phrased as the nonlinear program: *Maximize the density of a configuration of pentagons subject to the condition that the pentagons do not intersect.*





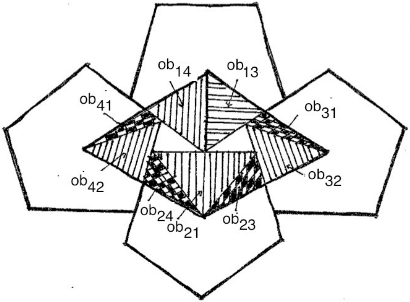
Parameterize the configuration space of 4 pentagons not with independent pentagons, but rather as a coupled system.

Objective

The objective function for the nonlinear program is defined in terms of the areas and corresponds to the normalized density function on a neighborhood of the conjectured optimal configuration. The objective is made up of constituent functions which are the areas of various triangular regions of the pentagons and the area of the convex hull of $\{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4\}$. The objective function may be written as

$$\sum_{i=1}^{4} \frac{\mathrm{Area}(\mathrm{Hull}(\{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4\}) \cap \mathrm{P}_i)}{\mathrm{Area}(\mathrm{Hull}(\{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4\}))} - \frac{5 - \sqrt{5}}{3}.$$

Objective



Objective

Objective

Objective

[illegible]

Objective

$$\vdash \frac{1}{\dfrac{2\left(\left(-\dfrac{(1+\frac{1}{4}(1-\sqrt{5})+\frac{1}{4}(-1+\sqrt{5})-\frac{1}{4}\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(1+\sqrt{5}))(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})}{\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}}+x_2-x_5+\frac{1}{4}(-1+\sqrt{5})-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}\right)(\sin(\frac{\pi}{10}-x_6)+\frac{1}{4}(-1+\sqrt{5})+\frac{1}{4}(1-\sqrt{5}))-(x_1-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})-x_4-\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}})\cos(\frac{\pi}{10}-x_6)\right)}{\sqrt{(x_1-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})-x_4-\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}})^2+(-\dfrac{(1+\frac{1}{4}(1-\sqrt{5})+\frac{1}{4}(-1+\sqrt{5})-\frac{1}{4}\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(1+\sqrt{5}))(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})}{\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}}+x_2-x_5+\frac{1}{4}(-1+\sqrt{5})-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})})^2}\sqrt{\cos^2(\frac{\pi}{10}-x_6)+(\sin(\frac{\pi}{10}-x_6)+\frac{1}{4}(-1+\sqrt{5})+\frac{1}{4}(1-\sqrt{5}))^2}\sqrt{1-\dfrac{((-\frac{(1+\frac{1}{4}(1-\sqrt{5})+\frac{1}{4}(-1+\sqrt{5})-\frac{1}{4}\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(1+\sqrt{5}))(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})}{\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}}+x_2-x_5+\frac{1}{4}(-1+\sqrt{5})-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}}{((x_1-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})-x_4-\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}})^2+(-\frac{(1+\frac{1}{4}(1-\sqrt{5})+\frac{1}{4}(-1+\sqrt{5})-\frac{1}{4}\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(1+}{\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}}}}}}$$

$$\sqrt{5})(\tfrac{5}{8}-\tfrac{\sqrt{5}}{8})(3+\sqrt{5})+\tfrac{1}{8}\sqrt{(5-\sqrt{5})(3+\sqrt{5})})+\tfrac{1}{2}(-\tfrac{1}{4}\sqrt{\tfrac{5(3+\sqrt{5})}{5-\sqrt{5}}}x_1^2-\tfrac{1}{4}\sqrt{\tfrac{3+\sqrt{5}}{5-\sqrt{5}}}x_1^2+2\sqrt{\tfrac{2}{5-\sqrt{5}}}x_1^2+\tfrac{1}{2}\sqrt{\tfrac{1}{2}(3+\sqrt{5})}x_2x_1-x_8x_1-\tfrac{1}{2}\sqrt{\tfrac{1}{2}(1+\tfrac{2}{\sqrt{5}})(\tfrac{5}{8}-\tfrac{\sqrt{5}}{8})(3+\sqrt{5})}x_1+\tfrac{1}{8}\sqrt{\tfrac{5}{2}(3+\sqrt{5})}x_1-\tfrac{3}{8}\sqrt{\tfrac{1}{2}(3+\sqrt{5})}x_1+\tfrac{\sqrt{5}x_1}{4}-\tfrac{x_1}{4}-\tfrac{1}{2}\sqrt{\tfrac{1}{2}(5+\sqrt{5})}x_2-\tfrac{1}{8}\sqrt{(5-\sqrt{5})(3+$$

Objective

$$2((-\frac{(1+\frac{1}{4}(1-\sqrt{5})+\frac{1}{4}(-1+\sqrt{5})-\frac{1}{4}\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})}(1+\sqrt{5}))(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})}{\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}}+x_2-x_5+\frac{1}{4}(-1+\sqrt{5})-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}$$

$$)(1+\sqrt{5}))(x_1+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}})+x_2-x_5+\frac{1}{4}(-1+\sqrt{5})-\sqrt{(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})})^2\sqrt{\cos^2(\frac{\pi}{10}-x_6)+(\sin(\frac{\pi}{10}-x_6)+\frac{1}{4}(-1+\sqrt{5})+\frac{1}{4}(1-\sqrt{5}))^2}$$

$$x_1^2+2\sqrt{\frac{2}{5-\sqrt{5}}}x_1^2+\frac{1}{2}\sqrt{\frac{1}{2}(3+\sqrt{5})}x_2x_1-x_8x_1-\frac{1}{2}\sqrt{\frac{1}{2}(1+\frac{2}{\sqrt{5}})(\frac{5}{8}-\frac{\sqrt{5}}{8})(3+\sqrt{5})}x_1+\frac{1}{8}\sqrt{}$$

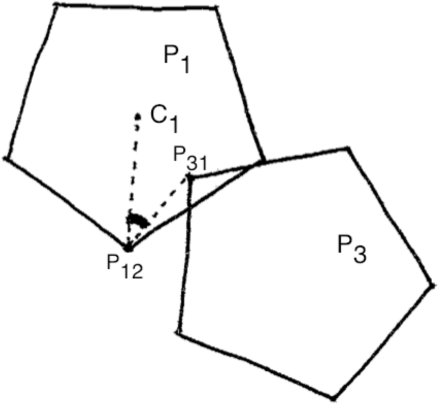
Constraints

The constraint functions for the nonlinear program are conditions forcing non-intersection of the pentagons. Locally, it is sufficient to require that no vertex of a pentagon be in the interior of another. Let p_{ij} be the vertex of P_i that is in contact with P_j. The constraint that a vertex of P_k does not lie in P_j may be considered as an angle constraint, written as

$$\mathrm{Angle}\{\mathrm{C}_i - \mathrm{p}_{ij}, \mathrm{p}_{ki} - \mathrm{p}_{ij}\} - \frac{3\pi}{10} \geq 0$$

for appropriate choices of i, j, k in $\{1, 2, 3, 4\}$.

Constraints



Violation of an angle constraint.

Numerical Check

Trying to solve the nonlinear program destroyed a computer. A linear approximation of the program can be found and easily solved numerically, indicating a (possibly degenerate) critical point at 0 .

A restricted bordered Hessian test indicates 0 is a local max. There are some directions where the geometry forces the objective function be constant or linear.

Careful error analysis should show 0 is a local maximum.

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Conical Programs

The nonlinear programming problem for pentagons satisfies certain assumptions that allow it to be sliced and analyzed. In general,

Proposition

a nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \ \textit{subject to} \ g_r(x) \geq 0, r \in I$$

satisfying the following assumptions, has an isolated local maximum at 0 with f(0) = 0. □

Conical Programs

Definitions

- *Let* e_1 *be the standard unit vector* $\{1, 0, \ldots, 0\}$ *in* $\mathbb{R}^n$.
- *Let* $F(t) = \nabla f(te_1)$.
- *Let* $G_r(t) = \nabla g_r(te_1)$.
- *Let* $E = \{te_1 : t \in \mathbb{R}\}$.
- *Let* H *be the orthogonal complement of* E *so that* $\mathbb{R}^n = E \oplus H$.

Assumptions

- *The index set* I *is a finite set.*
- *For* r *in* I, *the objective and constraint functions* f *and* g_r *are analytic functions on a neighborhood of* 0.

Conical Programs

Assumptions

- *Assume* $f(0) = g_r(0) = 0$ *for all* r *in* I.
- *There is a bounded solution at* 0 *to the linear program*

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \ \mathit{subject\ to}\ G_r(0) \cdot x \geq 0, r \in I$$

- E *is the set of solutions in* $\mathbb{R}^n$ *to the program*

$$F(0) \cdot x = 0 \ \mathit{subject\ to}\ G_r(0) \cdot x \geq 0, r \in I.$$

- *There is an* $\epsilon > 0$ *so the functions* $g_r(te_1) = 0$ *for all* $t \in (-\epsilon, \epsilon)$, *for all* r *in* I.
- *Assume* $\frac{\partial}{\partial t} f(te_1)|_{t=0} = 0$, $\frac{\partial^2}{\partial t^2} f(te_1)|_{t=0} < 0$.

Sketch of Proof

Lemma

Given the assumptions, the linear program

$$\max_{x \in H} F(0) \cdot x \ \textit{subject to} \ G_r(0) \cdot x \geq 0, r \in I$$

has a unique maximum at $x = 0$

Sketch of Proof

Proof.

From the assumptions, the linear program

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on E. The restricted program feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I \text{ and } x \in H\}$ is a subset of the feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I\}$ for the full program. Thus, the program

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

attains its maximum exactly on the (non-empty) intersection

$$E \cap \{x : G_r(0) \cdot x \geq 0, r \in I\} \cap H = 0.$$

□

Sketch of Proof

Definition

A ***finitely generated cone*** is a subset of $\mathbb{R}^n$ which is the non-negative span of a finite set of non-zero vectors $\{v_1, \dots v_m\}$ in $\mathbb{R}^n$, which are called the ***generators*** of the cone.

Definition

A ***conical linear program*** is a linear program with a constraint set that is a finitely generated cone.

The linear programs described throughout this section are always constrained to be on the intersection of half-spaces with 0 on the boundary. These are conical programs.

Definition

For a cone C, the set $C^p := \{x \in \mathbb{R}^n : v \cdot x \leq 0 \text{ for all } v \in C\}$ is the ***polar cone*** of C.

Sketch of Proof

Lemma

A conical linear program with $F \neq 0$ given by

$$\max_{x \in \mathbb{R}^n} F \cdot x \ \mathit{subject\ to}\ G_r \cdot x \geq 0, r \in I$$

(a) has a unique maximum at $x = 0$ iff F is in the interior of the polar cone C^p of $C = \{x : G_r \cdot x \geq 0, r \in I\}$ (b) has a bounded solution iff F is in the polar cone C^p of $C = \{x : G_r \cdot x \geq 0, r \in I\}$ and attains its maximum exactly on the non-negative span of the generators v_i satisfying $F \cdot v_i = 0$.

Sketch of Proof

Proof.

If F is in the interior of the polar cone C^p, then $F \cdot v_i < 0$ for all generators v_i. Therefore $F \cdot x$ is uniquely maximized in C at the vertex. If F is on the boundary of the polar cone, then $F \cdot x$ is maximized in C exactly on the span of the generators v_i for which $F \cdot v_i = 0$ as $F \cdot v_j < 0$ otherwise. If F is outside the polar cone, then $F \cdot v_i > 0$ for some generator v_i. Then $F \cdot x$ is unbounded in C. $\square$

Sketch of Proof

Lemma

Given the assumptions, there exists $\epsilon > 0$ *such that for all* t *in* $(-\epsilon, \epsilon)$*, the linear program*

$$\max_{y_t \in H} F(t) \cdot y_t$$

subject to

$$G_r(t) \cdot y_t \geq 0, r \in I$$

has a unique maximum at $y_t = 0$.

Sketch of Proof

Proof.

The program for $t \in (-\epsilon, \epsilon)$, for y_t in H, for each t in $(-\epsilon, \epsilon)$, for some $\epsilon > 0$, is a conical program on all of $\mathbb{R}^n$ with a cone C_t in $\mathbb{R}^n$ of co-dimension ≥ 1 with constraints $e_1 \cdot y_t \geq 0$ and $-e_1 \cdot y_t \geq 0$. By previous lemmas, $F(0)$ is in the polar cone of

$$C_0 = \{y_0 : G_r(0) \cdot y_0 \geq 0, e_1 \cdot y_0 \geq 0, -e_1 \cdot y_0 \geq 0\}.$$

As $f, g_r \in C^\omega$, the condition of $F(t)$ being in the interior of the polar cone C_t^p is open and the feasible set

$$C_t = \{y_t : G_r(t) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0\}$$

is conical. By a previous lemma, the program has a unique maximum at $y_t = 0$ for each t in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. □

Sketch of Proof

Lemma

Given the assumptions and ϵ *as in the previous lemmas, for all* $t \in (-\epsilon, \epsilon)$ *there exists* $\delta(t) > 0$ *and a cube* $Q(t) \subset \mathbb{R}^n$ *of side length* $2\delta(t)$ *such that*

$$\{(F(t) + Q(t)) \cap (\partial(C_t^p) + Q(t))\} = \emptyset.$$

Proof.

This follows from a previous lemma, which shows $F(t)$ is in the interior of the polar cone C_t^p. Then $F(t)$ and the boundary of C_t^p can be separated and the existence of Q is trivial. □

Sketch of Proof

Corollary

Given the assumptions and ϵ *as in previous lemmas, for all* $t \in (-\epsilon, \epsilon)$,

$$(F(t) + \Delta) \cdot y_t \leq 0$$

whenever y_t *satisfies*

$$(G_r(t) + \Delta_r) \cdot y_t \geq 0, r \in I \text{ and } e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0$$

where Δ *and* Δ_r *are any points in the* $2\delta(t)$*-cube* $Q(t)$ *and* y_t *is in* H.

Proof.

By a previous lemma, $F(t) + \Delta$ is in the interior of the polar cone $C^p_{t,\Delta}$, where
$C_{t,\Delta} = \{y_t : (G_r(t) + \Delta_r) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0, r \in I\}$. $\square$

Sketch of Proof

Lemma

Given the assumptions and ϵ as in the previous lemmas, for all $t \in (-\epsilon, \epsilon)$, let $y_t = x - te_1 \in H$. Choose $\Delta = \Delta(y_t)$ and $\Delta_r = \Delta_r(y_t)$ in the $2\delta(t)$-cube $Q(t)$ to be the corner given by the sign of $x - te_1 = y_t$. Then there is an ϵ_t for which

$$(F(t) + \Delta(y_t)) \cdot y_t \leq 0 \implies f(x) - f(te_1) \leq 0$$

and

$$(G_r(t) + \Delta_r(y_t)) \cdot y_t \leq 0 \implies g_r(x) - g_r(te_1) = g_r(x) \leq 0$$

for all y_t satisfying $\|y_t\| \leq \epsilon_t$.

Sketch of Proof

Proof.

This follows from the local expansions of the nonlinear program. By this choice of $\Delta(y_t)$ and $\Delta_r(y_t)$,

$$\begin{aligned} f(x) - f(te_1) &= F(t) \cdot (x - te_1) + O(t^2) = F(t) \cdot y_t + O(t^2) \\ &\leq F(t) \cdot y_t + \delta(t)\|y_t\|_1 = (F(t) + \Delta(y_t)) \cdot y_t \end{aligned}$$

and

$$\begin{aligned} g_r(x) = g_r(x) - g_r(te_1) &= G_r(t) \cdot (x - te_1) + O(t^2) = G_r(t) \cdot y_t + O(t^2) \\ &\leq G_r(t) \cdot y_t + \delta(t)\|y_t\|_1 = (G_r(t) + \Delta_r(y_t)) \cdot y_t. \end{aligned}$$

□

Sketch of Proof

By the previous lemmas, for t in $(-\epsilon, \epsilon)$, the program

$$\max_{y_t \in H} (F(t) + \Delta) \cdot y_t \text{ subject to } (G_r + \Delta_r) \cdot y_t$$

is uniquely maximized at $y_t = 0$ for any choice of Δ, Δ_r in the $2\delta(t)$ cube $Q(t)$. Then there is an ϵ_t neighborhood of 0 where $f(y_t + te_1)$ is less than $f(te_1)$ on

$$\cup_{\Delta_r \in Q(t)} \{y_t : (G_r + \Delta_r) \cdot y_t \geq 0, r \in I, y_t \in H\},$$

which contains the feasible set $\{y_t : g_r(y_t + te_1) \geq 0, r \in I, y_t \in H\}$. Therefore the nonlinear programs $f(y_t + te_1)$ subject to $g_r(y_t + te_1) \geq 0$, $y_t \in H$, parameterized by t in $(-\epsilon, \epsilon)$, have local maxima at $y_t = 0$. This gives the following:

Sketch of Proof

Theorem

Given the assumptions, a fixed t in $(-\epsilon, \epsilon)$ and choosing Δ and Δ_r as in the previous lemmas, for x satisfying $g_r(x) \geq 0$ for all r in I and $y_t = x - te_1$ in H, there exist linear programs

$$\max_{y_t \in H} (F(t) + \Delta(y_t)) \cdot y_t \ \mathit{subject\ to}\ (G_r(t) + \Delta_r(y_t)) \cdot y_t \geq 0$$

that give solutions to the nonlinear programs

$$\max_{x \in H + te_1} f(x) \ \mathit{subject\ to}\ g_r(x) \geq 0$$

in an ϵ_t neighborhood of te_1 in $H + te_1$. □

Sketch of Proof

By choice of a sufficiently small ϵ and a minimal non-zero ϵ_t, the previous theorem gives an open neighborhood of 0 in which the maximum value of the original nonlinear program occurs on E. The assumptions for the first and second t-derivatives at 0 shows 0 to be a local maximum for the nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0.$$

Theorem

A nonlinear program satisfying the assumptions has an isolated local maximum at 0 with f(0) = 0. □


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