SHORT RESEARCH STATEMENT

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Introduction. My research interests are in geometric optimization and frustration, primarily dealing with discrete packing density and configuration problems. Because of the general nature of these problems, I call upon many different areas of mathematics: topology, soft and hard analysis, linear and non-linear programming, combinatorial and algebraic methods, and various sub-disciplines of geometry. I am interested in problems of discrete geometry and geometric optimization that are approachable by synthetic or analytic means and those where brute force computation is becoming tractable. The potential applications of optimal geometry are broad. For example, one encounters questions about geometric optimization and frustration in control theory, chemistry, condensed matter physics and materials science.

Motivation. The study of optimal configurations, resulting in minimized energy, density or other function, dates to antiquity. A modern motivation is found in Hilbert's 18th, from *Mathematische Probleme* [8], regarding *dense* configurations. Conway, Goodman-Strauss and Sloane [1] note that the definition of density in Hilbert's question is too malleable a notion to use in uniquely defining an optimal configuration, but is still natural to consider. Even then, there is are implicit assumptions in Hilbert's question, e.g. that the behavior of even planar configurations is understood. This is not the case.

Some Results. In [7], I extended the previous result of A. Bezdek and W. Kuperberg for bi-infinite cylinders to the case of finite height cylinders.

Theorem 1. (K). *The upper density* δ^+ *of a packing* $\mathscr{C}(t)$ *of* $\mathbb{R}^3$ *satisfies* $\delta^+(\mathscr{C}) \leq \frac{\pi}{\sqrt{12}} + \frac{10}{t}$.

This new result is one of very few non-trivial upper bounds for packings of bounded domains in $\mathbb{R}^3$ and the only known bound for cylinders that is asymptoticly sharp, improving a result of W. Kuperberg and G. Fejes Tóth [2].

Using results in affine algebra, the result for bi-infinite cylinders is also sufficient to prove higher dimensional packing density bounds for poly-cylinders [6]. This is the first non-trivial exact bound for higher dimensional objects.

Theorem 2. (K). $\delta^+(D^2 \times \mathbb{R}^n) = \delta^+(D^2)$ *for all natural numbers* n.

Work with Thomas Hales [3] proved that the best known lower bound for the density of regular pentagon packings is in fact the global maximizer for density.

Theorem 3 (Hales, K)**.** *The density of packings of pentagons in* $\mathbb{R}^2$ *is bounded by* $(5-\sqrt{5})/3$.

I had previous proved that the then conjectured optimal configuration for pentagons was locally optimal. This result was extended in work with Yoav Kallus [5] reformulating the local result to apply to a general/generic convex polygon.

Theorem 4 (Kallus, K)**.** *There is a open set in the configuration space of a generic polygon in the plane, in which the maximum density with respect to its finite Delaunay triangles is given by the optimal double lattice packing.*

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Current and future work. I propose to continue investigating optimal configurations, but in a broader sense: with different potentials and with a larger experimental component, both computational and physical. The long term goal is to generate a bestiary of interesting critical and optimal sets on manifolds with respect to various functions, e.g. discrepancy or injectivity radius, which would be of significant utility and interest in geometric optimization, but also important in fields such condensed matter physics, integration and approximation theory, and optimal control. This would reasonably subsume the following ongoing projects.

Spherical cap discrepancy. Given an geometric sphere $\mathbb{S}^d$ with radius 1 and normalized uniform measure σ and a spherical cap C embedded in $\mathbb{R}^{d+1}$, the *local spherical cap discrepancy* of a set X_N of N distinct points in the d-sphere, which describes the largest deviation of the point measure from the uniform measure with respect to spherical caps. I discovered that there is indeed a polynomial time algorithm to compute the discrepancy. By implementing this algorithm, it is possible to explicitly compute the discrepancy of point sets. The discrepancy is intimately connected to the sum of distances via an invariance principle.

Morse theory for configuration spaces. A configuration space of a collection of spheres in a container is a subspace of the configuration space of the centers of those spheres. As the the radius of the spheres changes, the topology and geometry of the configuration space changes exactly at the radius functions critical points. Using classical optimization theory and differential geometry, it is possible to equate that criticality to the existence of a strut measure and a balanced diagram. These critical points can then be classified computationally.

Sticky sphere clusters. Recent work in materials science [4] has led to interest in the enumeration of clusters of rigid sticky sphere clusters. These are essentially rigid bar frameworks with an integrality condition on the edge lengths. For clusters containing a small number of spheres, there are algorithmic approaches to generating and checking the allowed configurations. I am currently using a mechanical approach to construct examples of medium size clusters that are poorly behaved with respect to the rules used for current enumeration algorithms.

Hyperuniform structures. Consider points $(X_i)_i$ generated by a point process on a space, such that $\mu_n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$, converges weakly to the uniform measure on the space. For test sets B in this space, $\mu_n(B)$ is the random variable counting how many points the process at time n has placed in B. Low variance of $\mu_n(B)$ suggests that there is extra order in the point set. Torquato and Stillinger [9] identified this extra order in the context of thermodynamical ensembles in Euclidean space and called it *hyperuniformity.* I would like to continue to examine properties of such ensembles in the compact setting.

Interdisciplinary work. I would like to work with materials scientists and engineers to fabricate some of the special structures that appear in the literature. Known and conjectured critical domains and configurations present attractive experimental opportunities. The methods used to analyze configurations could be used to make metamaterials and mechanisms, leveraging a better understanding of the inherent geometric properties and defects. Such experimentation would also help by producing further mathematical conjectures and methods.

Further information may be found in my Research Statement (∼2016).

References

[1] J H Conway, C Goodman-Strauss, and N Sloane. Recent progress in sphere packing. *Current Developments in Mathematics*, pages 37–76, 1999.

[2] G Fejes Tóth and W Kuperberg. Blichfeldt's density bound revisited. *Mathematische Annalen*, 295(1):721–727, 1993.

[3] T Hales and W Kusner. Packings of regular pentagons in the plane. *arXiv preprint arXiv:1602.07220*, 2016.

[4] M C Holmes-Cerfon. Enumerating rigid sphere packings. *SIAM Review*, 58(2):229–244, 2016.

[5] Y Kallus and W Kusner. The local optimality of the double lattice packing. *Discrete & Computational Geometry*, pages 1–23, 2015.

[6] W Kusner. On the densest packing of polycylinders in any dimension. *arXiv preprint arXiv: submitted Fri. May 2 2014*, 2014.

[7] W Kusner. Upper bounds on packing density for circular cylinders with high aspect ratio. *Discrete & Computational Geometry*, 51(4):964–978, 2014.

[8] M W Newson. Mathematical Problems: Lecture delivered before the International Congress of Mathematicians at Paris in 1900 by Professor David Hilbert. *Bulletin of the American Mathematical Society*, 8:437–479, 1902.

[9] S Torquato and F H Stillinger. Local density fluctuations, hyperuniformity, and order metrics. *Physical Review E*, 68(4):041113, 2003.