Lecture 11)

11.1) Critical Points and Jamming.

Last time we stated Thm.

ρ-regular value $\Rightarrow$ elt in tangent cone

no elt in tangent cone $\Leftrightarrow$ $\exists$ weights (non-negative, not all 0) ~~(positive)~~ st the <u>contact</u> graph is force balanced.

The contact graph is the ~~geometric~~ ~~structure of~~ set of ~~structs~~ ~~for~~ edges on S^2 between edges exactly distance θ apart for $\theta = 2\rho$

Thm $\mathrm{Conf}(N, 0)$

$\simeq \mathrm{Conf}(N, \frac{2\pi}{n})$

$=$

This also gives an algorithm for exploring critical / maximal configurations

$-$

ρ-evolution

update...

$=$

carefully...

11.2) Examples with pictures.

$N=3$, $N=4$ } full picture

$N = \dots$ max tomes.

$N = 10$ – local max?

$N = 12$ – local max?

higher critical points.

Jitterbug?

11.3) Local maxes and Edwards entropy.

Ansatz: ~~Idea~~: all jammed states have equal probability. of equal volume

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Related to the evolution of ρ flow. Also to model...

not always true...?

~~ex. cancelled~~

Boltzmann Measure

Discrete Setting:

N particles, each in one of k states.

Each state has an energy

$\mathcal{E}_i$, $i \in \{1, \dots, k\}$.

System closed so:

$$E = \sum_{i=1}^{k} E_i n_i \qquad n_i = \# \text{ particles in state } i$$

$$N = \sum_{i=1}^{k} n_i$$

W, the # of ways to place particles $\{n_1, \dots n_k\}$

is the multinomial coeff.

$$\frac{N!}{N_1!\, N_2! \dots N_k!}$$

LLN $\Rightarrow$

the optimal partition dominates for N large determine partition

that maximizes W.

replace with $\log W$ [log monotone ↗]

$$\log W = (N \ln N - N) - \sum_{i=1}^{K} (N_i \ln N_i - N_i) + \text{cr.}$$

L multiplier
$$f = \ln(W) + \alpha\Big(N - \sum_{i=1}^{K} N_i\Big) + \underbrace{\beta\Big(E - \sum_{i=1}^{K} E_i N_i\Big)}_{\text{csts}\ldots}$$

$$0 = \frac{\partial f}{\partial N_i} = - \ln N_i - (\alpha + \beta E_i)$$

$$\frac{\partial}{\partial N_i} \left(N_i \ln N_i + \bar{\alpha} N_i + \bar{\beta} E_i N_i \right)$$

$$\underset{-1}{-1 \bullet \ln N} - (\alpha + \beta \varepsilon_i]$$

$$- \ln N_i - (\alpha + \beta \varepsilon_i) = 0$$

$$N_i = e^{-\alpha - \beta E_i}$$

$$\frac{N_i}{N} = \frac{e^{-\beta E_i}}{\sum_{j=1}^{k} e^{-\beta E_j}}$$

$$Z = \sum_{j=1}^{k} e^{-\beta E_j}$$

$$\boxed{\beta = \frac{1}{T} \text{ or } \frac{1}{kT}}$$

$$\boxed{F = -T \ln Z \text{ - Free energy}}$$

$$Pr(\text{state } i) = \frac{e^{-\beta E_i}}{Z}$$

$$\bar{E} = \frac{E}{N} = \sum_{i=1}^{k} \frac{E_i N_i}{N}$$

$$= \sum_{i=1}^{k} \frac{E_i e^{\beta E_i}}{Z} = \frac{-\frac{\partial}{\partial\beta} Z}{Z} = -\frac{\partial}{\partial\beta} Z$$

Lemma: $-\ln Z$ convex:

Show. $\frac{-\partial}{\partial\beta} \ln Z$ monotone.

ie $\frac{\partial^2}{\partial\beta^2} \ln Z > 0$ Why?

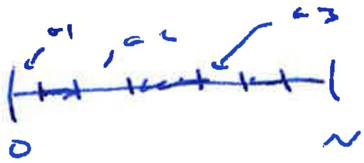
$\to$ write as variance of $E > 0$.

$\frac{1}{\beta} \ln Z :=$ Free energy.

$T \to \infty \quad Pr(i) \to \frac{1}{k}$

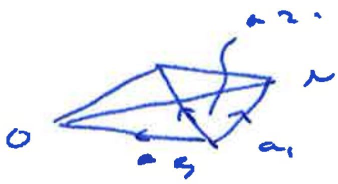
Hard disc model.

intuel.



$$\Omega_{N,3} = \{(a_1, a_2, a_3)$$

$$0 \le a_1, a_2, a_3, \quad a_1 + a_2 + a_3 \le N-3.$$



$$\text{Volume } \Omega_{(N,3)} = \frac{(N-3)^3}{3!}$$

ingenl.

$$\text{vol}(\Omega_{N,k}) = \frac{(N-k)^k}{k!}$$

Give energy E_0 per particle.

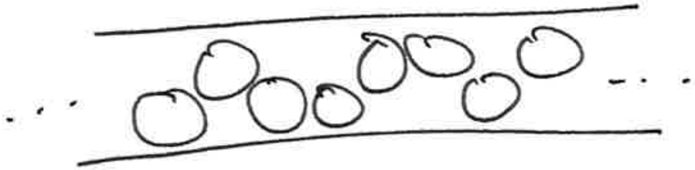
$z = e^{-\beta E_0}$ fugacity. $\longrightarrow$ define

$$Z_N(z) = \sum_{k=0}^{N} z^k \, \mathrm{vol}(\Omega_{N,k})$$

$$= \sum_{k=0}^{N} \left(\frac{(N-k)^k}{k!}\right) z^k$$

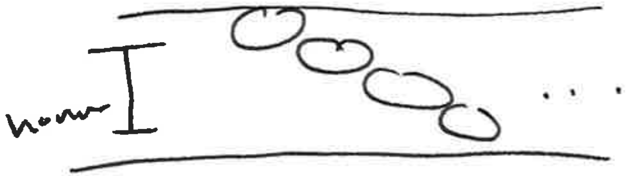
11.4) Quasi-1D model with open ends.

~~Model~~. narrow channel : Combinatorial analysis.



← Jammed.
local maximal density.

Critically Saddle



length is t.

disks $r=1$ channel $2 < \text{diam}_{\text{channel}} < 2+\sqrt{3}$.

Combinatorics : homology maxima for t ?

The length is determined by # of adjacent
at t opposite balls.

(1) (0)

Counting is # of binary words with no
adjacent 1's

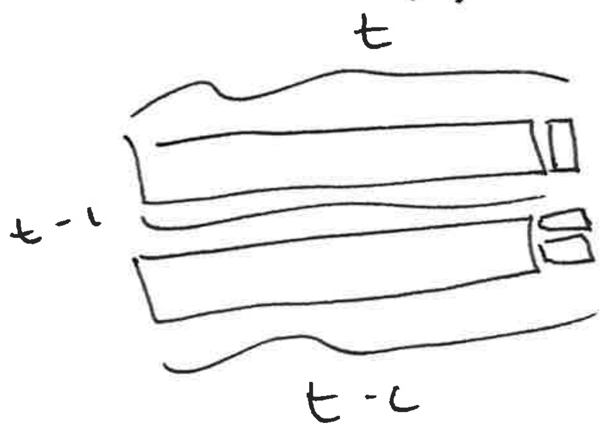
This is eq to a tiling domino problem. or stars and bars.

sequences formed of... 0 01

This satisfies the fibbonacci

recursion. (r not up/down symmetric.)

~~Words ending 0 of length ℓ~~



and

1, 2 ...

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given exactly k 1s in a word of length ℓ,

$\Rightarrow$ inserting k 1s into a sequence of $\ell - k$ 0's. , 0, 0, 0, ... 0,

So into $\ell - k + 1$ slots.

So for each $k \Rightarrow \binom{\ell - k + 1}{k}$ choices of order

So if length of a "1" is 2

and a "0" is L

we have

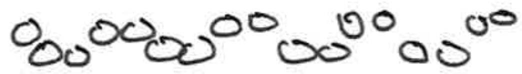
the $t = \ell = k \cdot 2 + (\ell - k)L + \underbrace{2}_{\text{stop.}}$

We can also compute the max* of rectangle

$$\frac{\binom{n-k+1}{k}}{} \longrightarrow \frac{k}{n} \rightarrow \frac{5}{10} - \frac{\sqrt{5}}{10}$$

And then is a "unique" decent party Case.

~ on last line as well...



Ex. lower off line ★ appears
even.

⟶ typical configuration looks like

$\frac{1}{2} + \frac{\sqrt{5}}{10}$ (1s) (∞)

(compute configs ...) ⟶ ($\frac{1}{2\epsilon}$)

pairing values of central pts

flow to joined attractor ...

probably minimal size central

sudden change to near 0 pairings

$\Longrightarrow$

? compute these values!

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also.. for $\underline{\underline{\text{large}}}$ N

so studying low modes easy.