Gordian Configurations

Woden Kusner

Vanderbilt University

33rd Summer Conference on Topology and its Applications
Western Kentucky University
July 17, 2018

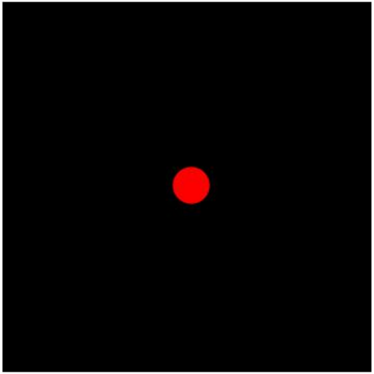
Introduction

While we are often interested in finding configurations that are “the best” – global optimizers of a nice function on configuration space, there is good reason to look at “the rest”. In particular, there are often configurations that are critical, which potentially witness a change in the topology of configuration space.

We will explore some cases appearing for spaces of points and curves where our function is “physical” and the configurations are “Gordian”.

Consider a ball in space.

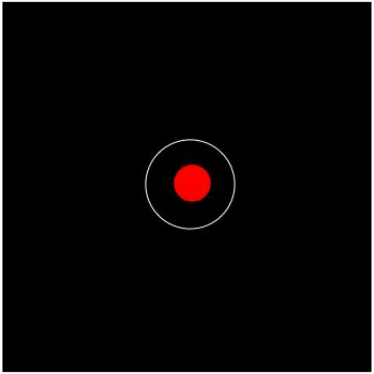
It is “point-like”, and can explore freely.



Consider a ball in space.

It is “point-like”, and can explore freely.

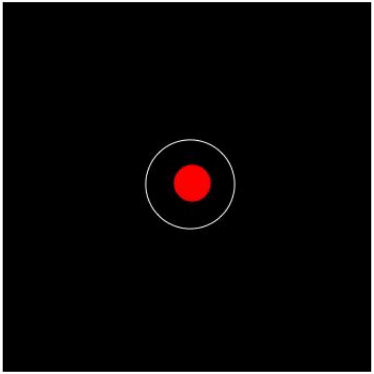
In a container, the ball is separated from the rest of space. When the container is deformed, the ball still remains “inside”.



Consider a ball in space.

It is “point-like”, and can explore freely.

In a container, the ball is separated from the rest of space. When the container is deformed, the ball still remains “inside”.



Where is the ball when the container is opened?

We can think of this a couple different ways.

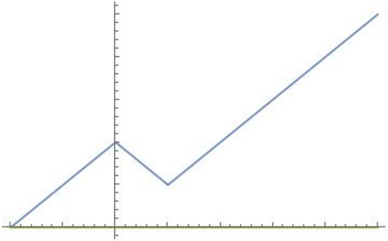
One way is to look at the connectedness of the configuration space of the ball.

This space can be described as part of the configuration space of a point with a function, the injectivity radius ρ, attached to it.

The parts of space that can be explored by a ball of a particular size are the *superlevel set* for ρ.

When the ball is small, this configuration space is connected.

As we consider larger and larger ρ, eventually there is a critical choice: go “inside” the container or “outside”.

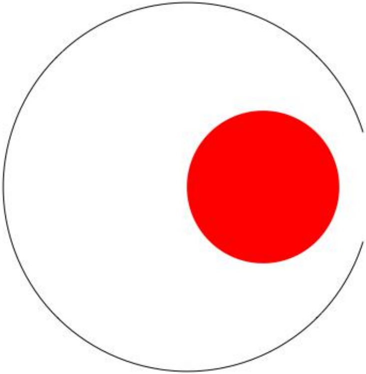


ρ as a function of of configuration space.

Another way is to think of the ball as passing through surfaces that span the mouth of the bottle.

The ball must then go “through” a surface separating the “inside” from the “outside”.

A surface might reach into the bottle. But (perhaps) there are natural choices to consider.



Gordian: Topology vs Geometry

Definition (physical configuration)

A configuration is *physical* (or thick) if it is constrained to remain in a superlevel set of the injectivity radius.

Definition (physical isotopy)

An isotopy is *physical* (or thick) if it is an isotopy through physical configurations.

There may be a difference between topological and physical theories.

Gordian: Topology vs Geometry

If we have a standard configuration:

Definition (Gordian)

A physical configuration is *Gordian* if it is in a different component of configuration space than a standard configuration.

Equivalently

Definition (Gordian)

A physical configuration is *Gordian* if there is no physical isotopy taking it to a standard configuration.

Gordian: Topology vs Geometry

Otherwise, we consider a relative notion:

Definition (Gordian)

A pair of physical configurations is *Gordian* if they are in distinct components of configuration space.

Equivalently

Definition (Gordian)

A pair of physical configurations is *Gordian* if there is no physical isotopy that takes one to the other.

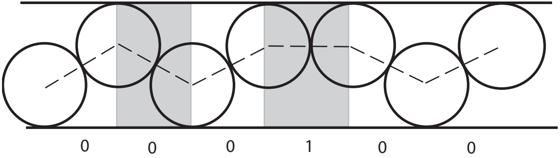
Disk Packings

Question

Are there different configurations of points that are isolated local maxima for injectivity radius? These would be Gordian.

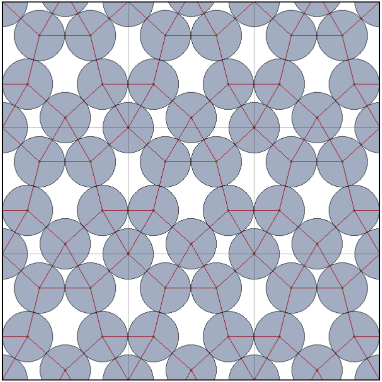
Question

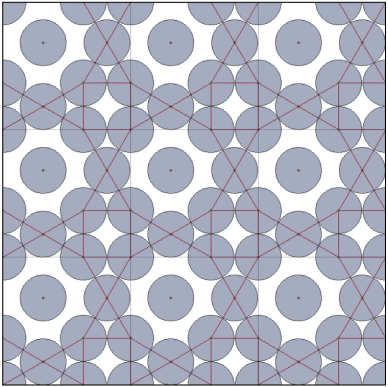
Are there different configurations of points that are isolated local maxima for injectivity radius? These would be Gordian.



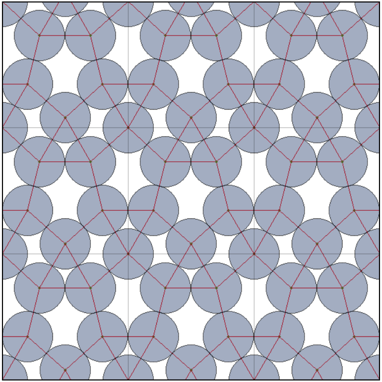
One model [Bowles, Ashwin] that can be analyzed completely is a quasi-1D packing problem. Such packings have lots of maxima.

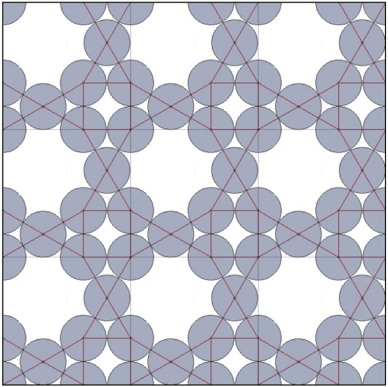
Local maxima on the square torus [Musin].





Local maxima on the square torus [Musin]. A Gordian pair.

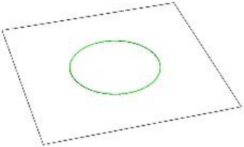


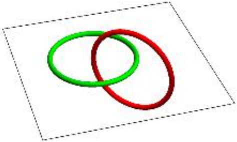


Rope

Definition (Rope)

Given a (closed, $C^{1,1}$) curve with bounded curvature and fixed length in $\mathbb{R}^3$, we can thicken it into a *rope* in the normal direction, up to the normal injectivity radius. The original curve is its *core*. Typically, this is scaled to have injectivity radius 1.





Definition (Physical isotopy)

An isotopy of a knot or link that maintains this scaling (i.e. preserves length per component and injectivity radius at least 1) is a physical isotopy.

Question (Alexander (the Great))

Is there a Gordian Unknot? A Gordian Split Link?

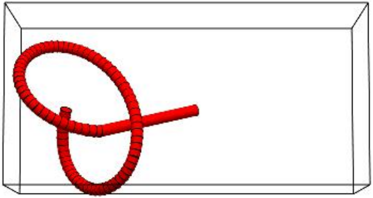
Rope

Question

Is there a Gordian Strand?

The core curve of a rope *could* be the image of an interval. To see if physical and topological theories are not distinct in this case is still not trivial.

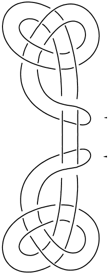
If there is a free end, *reptation* will take rope to a standard form. In general we must consider a family of rescalings of the core curve, while reptating to preserve length.



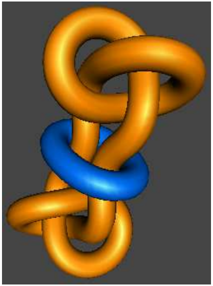
Rope

Theorem (Coward-Hass)

"Physical knot theory is distinct from topological knot theory."



? Freedman-He-Wang

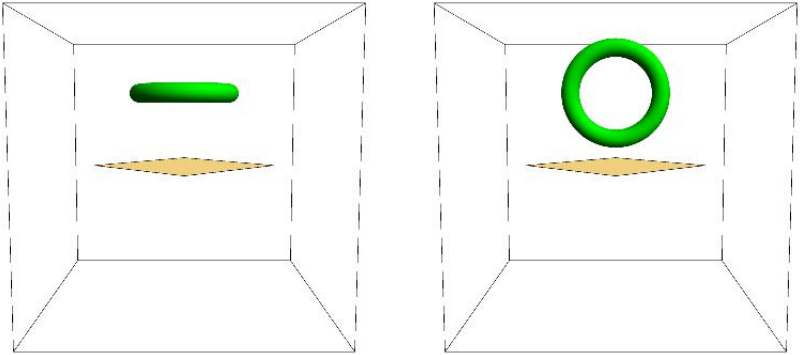


! Coward-Hass

Alternative: the existence of stopper knots. A *stopper knot* is a knot or link that cannot pass through a (planar) aperture of fixed area.

The model is rope embedded into the upper halfspace and separated from the lower halfspace by a plane. The plane contains an *aperture* through which the rope must pass.

Gordian: Topology vs Geometry



Rope

Clearly, if the aperture is too small, no configuration of rope can pass. For example, it must have sufficient area to have a unit radius ball pass through. We can track this during an isotopy.

Definition

A rope is *accessible from above* (resp. *from below*) if its core curve has non-trivial intersection with the open upper (resp. lower) halfspace, and *accessible* if it is both accessible from above and below. A rope is *accessible critical* if its core curve has trivial intersection with either the upper or lower open half space, but non-trivial intersection with the aperture.

Remark

If a rope is accessible, then the area of intersection with the aperture is greater than π*. If a rope is accessible critical, then the area of intersection with the aperture is greater than* π*.*

In either case the core curve of the rope has non-trivial intersection with the aperture A. Consider a ball of radius 1 centered at a point in that intersection. This ball is contained completely inside the rope and occludes a circular unit disk.

Theorem

There is no physical isotopy of a (n, n)-torus link that passes through an aperture with area less than $n\pi$.

Proof.

Given an isotopy that passes through an aperture, consider the *first* time that a component of the (n, n)-torus link becomes inaccessible from above. That component is accessible critical. As the other $(n-1)$ components link with that component and this is the first time that any component has become inaccessible from above, any particular component must be either accessible or accessible critical. So there are at least n disjoint occluding disks. $\square$

Rope

By application of an isoparametric inequality for space curves, the result extends to non-planar apertures giving:

Theorem

Split links formed by a connect sum of two (n, n)*-torus links and a sufficiently short surround curve are Gordian.*

We can also arrive at an analogous result by tightening the bound on the occlusion area (via Schur's Theorem), which can then be used to show that any rope passing through a small aperture or a sufficiently short space curve must be an unknot.

Rope