Overview

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November 25, 2013

Density

A *packing* of a region $X \subseteq \mathbb{R}^n$ by objects $C_i \subseteq X$ is a family $\mathscr{C} = \{C_i\}_{i \in I}$ with disjoint interiors.

The *upper density* ρ^+ of a packing $\mathscr{C}$ in $\mathbb{R}^n$ will be defined as

$$\rho^+(\mathscr{C}) = \limsup_{r \to \infty} \sum_{C_i \subseteq B(r)} \frac{\mathrm{Vol}(C_i)}{\mathrm{Vol}(B(r))},$$

where $B(r)$ is the ball of radius r centered at 0.

Example

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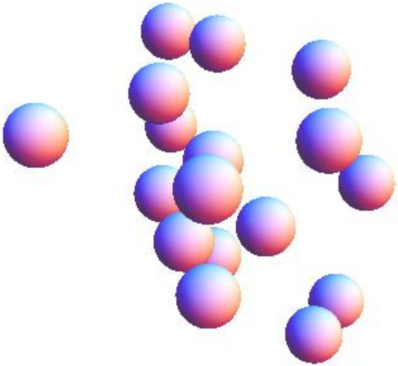

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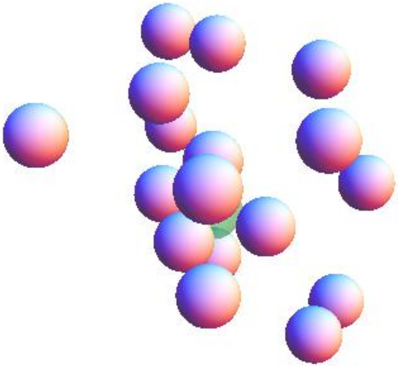

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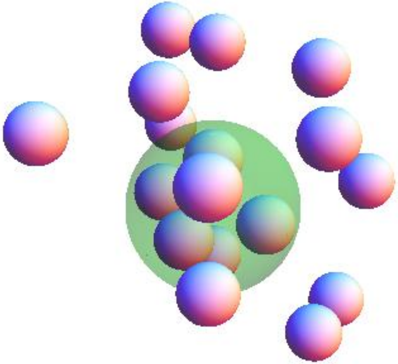

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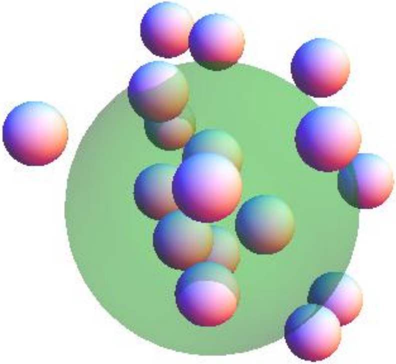

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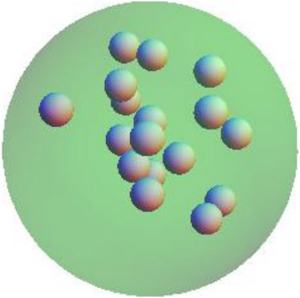

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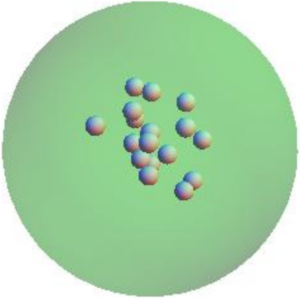

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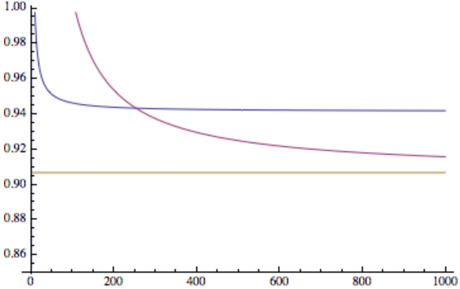

Upper bounds on density of unit radius cylinders relative to their length.
Blue: W. Kuperberg and G. Fejes Tóth.
Purple: New bound.
Yellow: Conjectured bound.

Dirichlet Cells

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- Decompose the packed region X.
- By definition, $C_i \subseteq D_i$, so if a generic Dirichlet cell D_i is large, we can get an upper bound on density.

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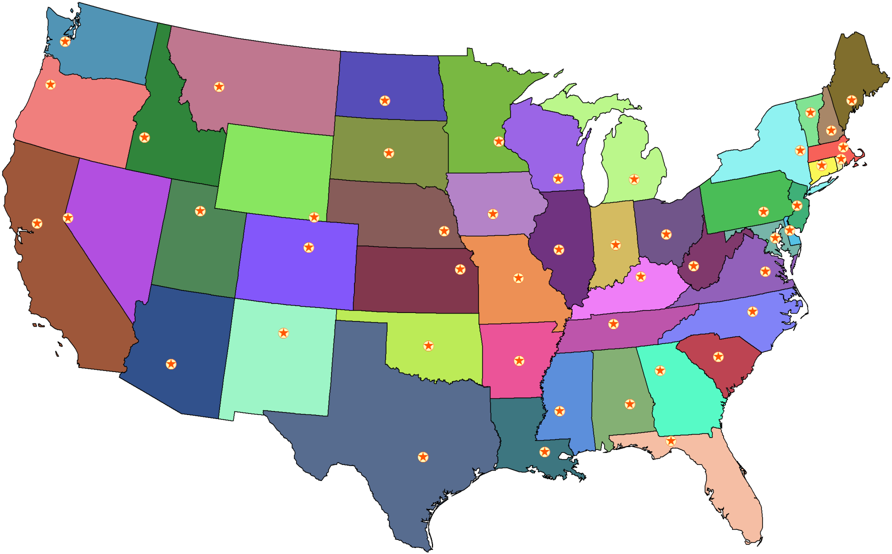

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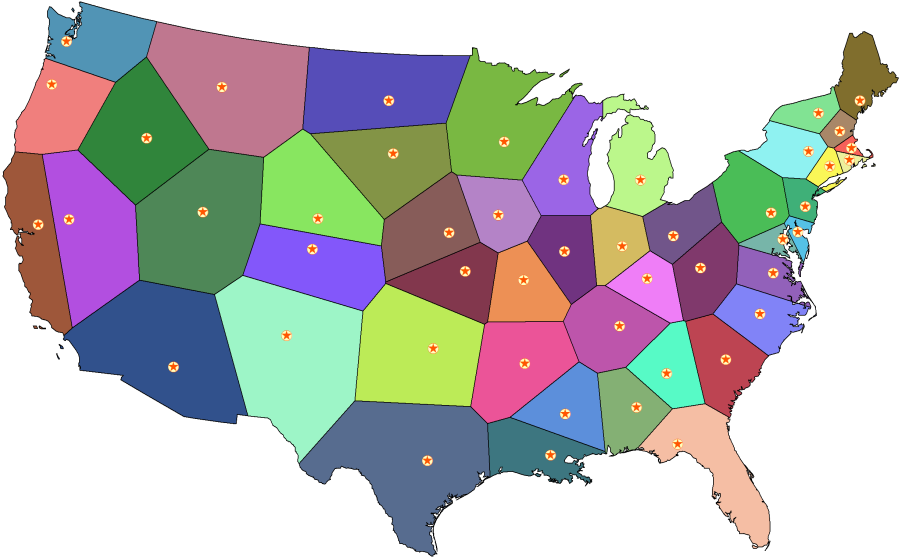


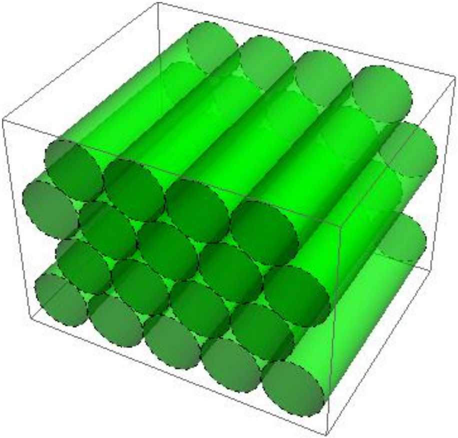
Bezdek and Kuperberg

A. Bezdek and W. Kuperberg show that the density of a packing by congruent infinite cylinders is at most the planar disk packing density. They show that the Dirichlet cells of infinite cylinders are large.

Bezdek and Kuperberg

Lower bound for density of a packing of space by unit radius cylinders:

- Obvious construction.



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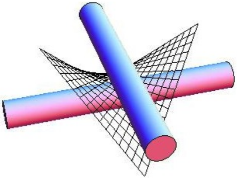

Two cylinders and associated Dirichlet cells.

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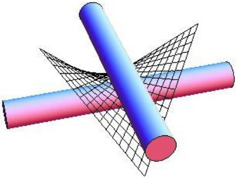

Two cylinders and associated Dirichlet cells.

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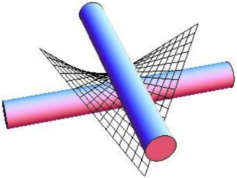
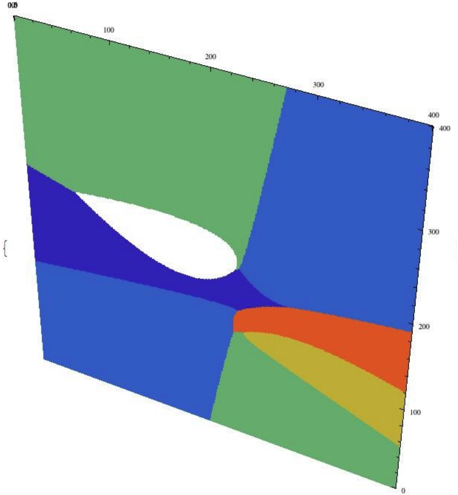


Two cylinders and associated Dirichlet cells.

- Slice the Dirichlet cells perpendicular to the axis of associated cylinder.
- Show that the area of each such Dirichlet slice is large. They are special "parabola-sided polygons."

Bezdek and Kuperberg



A slice of a random packing. Dirichlet Slice in white is a "parabola-sided polygon".

Modifications for Finite Cylinders

In the case of t-cylinders: unit radius cylinders with length t.

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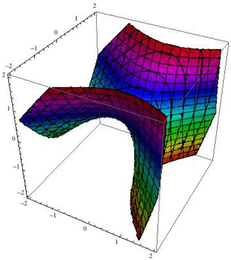
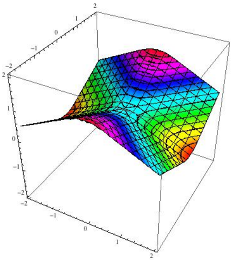
- Quandary: The Dirichlet cell of (the axis of) a finite t-cylinder need not contain the cylinder.
- Solution: Consider packings of *capped t-cylinders*: t-cylinders C_i^0 with hemispherical caps C_i^1 and C_i^2. Decompose the capped-cylinder Dirichlet cell D_i.

Modifications for Finite Cylinders

- Quandary: Dirichlet slices are not the same type of "parabola-sided polygons."





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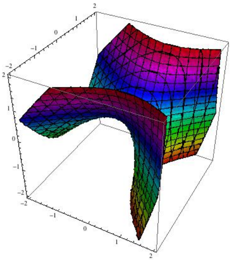

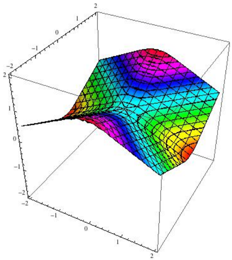

- Solution: We can characterize *some* Dirichlet slices.

Lemma

Slices sufficiently far away from the ends of any axis satisfy conditions that allow the area estimates of Bezdek and Kuperberg to apply.

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- Quandary: We don't know the volume of the regions D^1 and D^2.
- Solution: We know the volume of C^1 and C^2.

Packings Revisited

A *packing* of $X \subseteq \mathbb{R}^3$ by capped t-cylinders is a countable family $\mathscr{C} = \{C_i\}_{i \in I}$ of congruent capped t-cylinders C_i with mutually disjoint interiors and $C_i \subseteq X$.

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For a packing $\mathscr{C}$ of $\mathbb{R}^3$, the *restriction* of $\mathscr{C}$ to $X \subseteq \mathbb{R}^3$ is defined to be a packing of $\mathbb{R}^3$ by capped t-cylinders $\{C_i : C_i \subseteq X\}$.

Density Revisited

The *density* $\rho(\mathscr{C}, R, R')$ of a packing $\mathscr{C}$ of $\mathbb{R}^3$ by capped t-cylinders with $R \leq R'$ is defined as

$$\rho(\mathscr{C}, R, R') = \sum_{C_i \subseteq B(R)} \frac{\mathrm{Vol}(C_i)}{\mathrm{Vol}(B(R'))}.$$

Then the upper density ρ^+ of a packing $\mathscr{C}$ of $\mathbb{R}^3$ by capped t-cylinders may be written as

$$\rho^+(\mathscr{C}) = \limsup_{R \to \infty} \rho(\mathscr{C}, R, R).$$

Main Theorem

Fix $t_0 = \frac{4}{3}(\frac{4}{\sqrt{3}} + 1)^3 = 48.3266786\ldots$.

Theorem

Fix $t \geq 2t_0$. Fix $R \geq 2/\sqrt{3}$. Fix a packing $\mathscr{C}$ of $\mathbb{R}^3$ by capped t-cylinders. Then

$$\rho(\mathscr{C}, R - 2/\sqrt{3}, R) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}.$$

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This implies the bound for the upper density of a packing of $\mathbb{R}^3$.

Main Theorem gives the general bound

Corollary

Fix $t \geq 2t_0$. The upper density of a packing $\mathscr{C}$ of $\mathbb{R}^3$ by capped t-cylinders satisfies the inequality

$$\rho^+(\mathscr{C}) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}.$$

Let V_R and W_R be subsets of the index set I, with $V_R = \{i : C_i \subseteq B(R)\}$ and $W_R = \{i : C_i \subseteq B(R - 2/\sqrt{3})\}$. By definition,

$$\rho^+(\mathscr{C}) = \limsup_{R \to \infty} \left(\sum_{W_R} \frac{\mathrm{Vol}(C_i)}{\mathrm{Vol}(B(R))} + \sum_{V_R \setminus W_R} \frac{\mathrm{Vol}(C_i)}{\mathrm{Vol}(B(R))} \right).$$

Main Theorem gives the general bound

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As R grows, the term $\sum_{V_R \setminus W_R} \operatorname{Vol}(C_i)/\operatorname{Vol}(B(R))$ tends to 0. Further analysis of the right-hand side yields

$$\rho^+(\mathscr{C}) = \limsup_{R\to\infty} \rho(\mathscr{C}, R - 2/\sqrt{3}, R).$$

By the Main Theorem, the stated inequality holds.

Density Computation

Fix a packing $\mathscr{C}$. Fix $R \geq 2/\sqrt{3}$ and restrict to $\mathscr{C}^*$.

- A is the union of the axes a_i over I^*.
- μ is the 1-dimensional Hausdorff measure on A.
- X is the subset of qualified points of A.
- Y is the subset of A given by $\{x \in A : B_x(\frac{4}{\sqrt{3}}) \text{ contains no ends}\}$.
- Z is the subset of A given by $\{x \in A : B_x(\frac{4}{\sqrt{3}}) \text{ contains an end}\}$.

$Y \subseteq X \subseteq A$ from our Proposition and $Z = A - Y$ by definition.
The sets are A, X, Y, and Z are measurable. The set A is just a finite disjoint union of lines in $\mathbb{R}^3$. The area of the Dirichlet slice d_x is piecewise continuous on A, so X is a Borel subset of A.The conditions defining Y and Z make them Borel subsets of A. The ball $B(R)$ is finite volume, so I^* has some finite cardinality n.

Density Computation

By Definition:

$$\rho(\mathscr{C}, R-2/\sqrt{3}, R) = \frac{\sum_{I^*} Vol(C_i)}{\sum_{I^*} Vol(D_i)} = \frac{\sum_{I^*} Vol(C_i^0) + \sum_{I^*} Vol(C_i^{1,2})}{\sum_{I^*} Vol(D_i^0) + \sum_{I^*} Vol(D_i^{1,2})}.$$

$$Vol(C_i^0) = t\pi.$$

$$Vol(C_i^{1,2}) = \frac{4}{3}\pi.$$

$$C_i^j \subseteq D_i^j.$$

Therefore:

$$\rho(\mathscr{C}, R-2/\sqrt{3}, R) \leq \frac{nt\pi + n\frac{4}{3}\pi}{\sum_{I^*} Vol(D_i^0) + n\frac{4}{3}\pi}.$$

Bound Lemma

The main theorem follows from a bound:

Lemma

For $t \geq 2t_0$,

$$\sum_{I^*} Vol(D_i^0) \geq n(\sqrt{12}(t - 2t_0) + \pi(2t_0)).$$

The sum $\sum_{I^*} \mathrm{Vol}(D_i^0)$ may be written as an integral of the area of the Dirichlet slices d_x over A

$$\sum_{I^*} \mathrm{Vol}(D_i^0) = \int_A \mathrm{Area}(d_x)\, \mathrm{d}\mu.$$

Bound Lemma

Using the area estimates from the main proposition, there is an inequality

$$\int_A \mathrm{Area}(d_x)\,\mathrm{d}\mu \geq \int_X \sqrt{12}\,\mathrm{d}\mu + \int_{A-X} \pi\,\mathrm{d}\mu.$$

As $\sqrt{12} > \pi$ and the integration is over a region A with $\mu(A) < \infty$, passing to the subset $Y \subseteq X$ gives

$$\int_X \sqrt{12}\,\mathrm{d}\mu + \int_{A-X} \pi\,\mathrm{d}\mu \geq \int_Y \sqrt{12}\,\mathrm{d}\mu + \int_{A-Y} \pi\,\mathrm{d}\mu = \int_{A-Z} \sqrt{12}\,\mathrm{d}\mu + \int_Z \pi\,\mathrm{d}\mu.$$

Bound Lemma

The measure of Z is the measure of the subset of A that is contained in all the balls of radius $4/\sqrt{3}$ about all the ends of all the cylinders in the packing. This is bounded from above by considering the volume of cylinders contained in balls of radius $4/\sqrt{3}+1$. If the cylinders completely filled the ball, they would contain at most axis length $\frac{4}{3}(\frac{4}{\sqrt{3}}+1)^3 = t_0$. As each cylinder has two ends, there are at worst $2n$ disjoint balls to consider. Therefore $2nt_0 \geq \mu(Z)$.
Provided $t \geq 2t_0$, we have the inequality

$$\int\limits_{A-Z} \sqrt{12}\,\mathrm{d}\mu + \int\limits_{Z} \pi\,\mathrm{d}\mu \geq (nt - 2nt_0)\sqrt{12} + 2n(t_0)\pi.$$

Corollaries

Corollary

(Main Theorem) Fix $t \geq 2t_0$. If the density of a packing of $\mathbb{R}^3$ by capped t-cylinders exists then it does not exceed

$$\left(\frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}\right).$$

Corollary

Fix $t \geq 2t_0 + 2$. If the density of a packing of $\mathbb{R}^3$ by t-cylinders exists then it does not exceed

$$\left(\frac{t}{\frac{\sqrt{12}}{\pi}(t - 2 - 2t_0) + (2t_0) + \frac{4}{3}}\right).$$

Other Implications

- We can leverage the packing result for finite capped cylinders to get bounds for other objects. For example:
  - The maximum packing density of infinite circular cylinders with ray axes is exactly $\frac{\pi}{\sqrt{12}}$.
  - Any objects we can pack densely with capped cylinders. For example, tubes with low but strictly positive curvature.
  - Non-congruent cylinders of sufficiently length.

- We can find a dominating hyperbola numerically, giving

$$\rho^{+}(\mathscr{C}) \leq \pi/\sqrt{12} + 10/t$$

for a packing $\mathscr{C}$ given by congruent capped t-cylinders when $t \geq 0$.