Critical packings and the radius function

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Leoben
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Abstract

In 1694, Newton and Gregory discussed how many non-overlapping unit spheres could be placed in contact with a central unit sphere: Is it 12 or possibly 13? This problem was unresolved until 1953, when Schütte and van der Waerden showed that 12 was the correct answer.

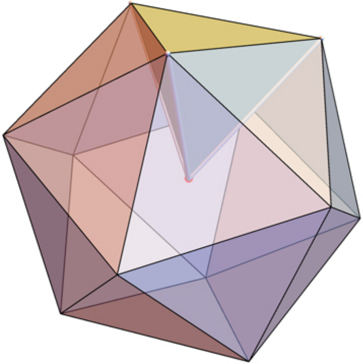
An alternate formulation, related to the Tammes best packing problem, is to consider the configuration space of 13 spheres touching a central sphere parametrized by radius and show that it is empty for radius 1.

This point of view leads us to wonder how the geometry and topology of such a configuration space changes as the radius is varied. Morse theory suggests we should be concerned with the “critical values” of the radius function.

The Newton-Gregory Problem

Question

Is the regular icosahedron made of 20 regular tetrahedra?



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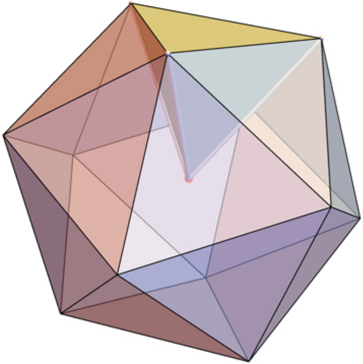
No! For circumradius 1, we can compute the edge length to be

$$\left(\frac{1}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})}\right)^{-1} = 1.0514\ldots.$$



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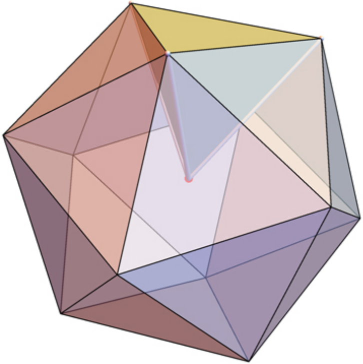
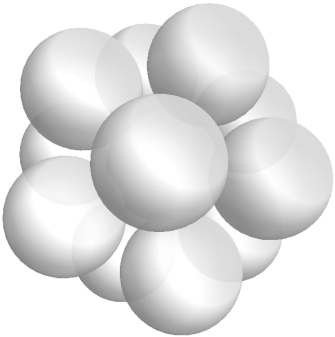


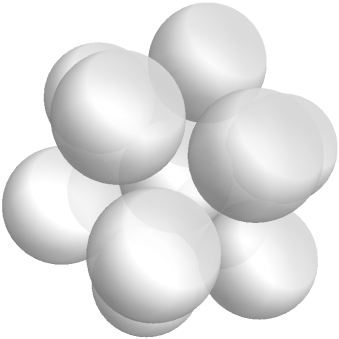
Riddle

Answer the question synthetically.

Remark

If we place unit spheres at the vertices of that regular icosahedron, there is a lot of space between them.





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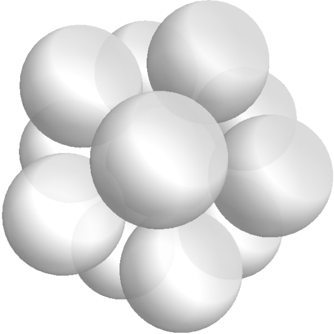
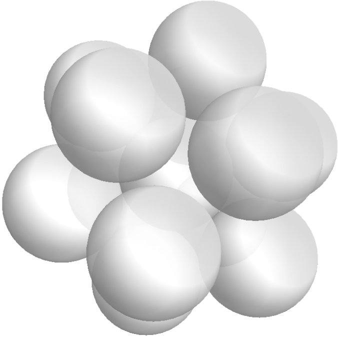
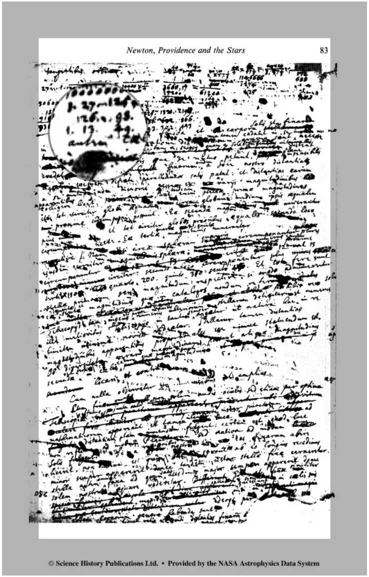




Question (Newton-Gregory)

Can we fit in a thirteenth sphere?

Newton and Gregory: Principia (revision c. 1694)


Newton, Providence and the Stars 83

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For Newton and Gregory, this was a problem of mechanics: Why the fixed stars don't all fall into the sun.

In a draft for the second edition of *Principia*, Newton considers stars of various magnitudes as modeled by arrangements of equal balls.

This method was abandoned, but history lends the names of Newton and Gregory to the problem.

Kepler: Epitome Astronomiae Copernicanae (c. 1620)

BSB

LIBER PRIMVS. 35

Si de fixis certius nihil conſtat, Videtur illa regio infinita eſſe; nec Sol hic noſter aliud erit, quam una ex fixis, nobis major & clarior viſa, quia propior quam fixa: atq; ita circa quamlibet fixam poterit eſſe talis mundus, qualis circa nos eſt; Vel, quod eodem redit, inter innumerabiles locos in illa infinitâ fixarum congerie, Mundus hic noſter cum Sole ſuo erit unus, nulla re diverſus à locis alijs circa fixas ſingulares: Vt in ſubjecta figura litera M:

M

Ita quidem Brunus & veterum aliqui. At non ſequitur, ſi centra fixarum non ſunt in eadem ſuperficie Sphærica; propterea regionem per quam ſunt diſperſæ fixæ, eſſe vndiquaque ſibi ſimilem.

Habet enim illa omninò vacuum aliquem ſinum, cavumque ingens, à fixarum agmine, confertim circumfuſo,

C 2 ceu

BSB

36 EPITOMES ASTRONOMIÆ,

ceu à muro vel fornice quodam concluſum & circumſcriptum: & in hujus Cavi ingentis complexu, Tellus noſtra cum ſole & ſtellis mobilibus comprehenſa eſt.

Vide ſubjectam figuram.

Quod habes hujus rei Argumentum?

Si Regio fixarum vndique ſimiliter eſſet conſita ſtellis, etiam in vicinia noſtri mundi [illegible] ſic ut ſitus mundi ſolisque noſtri nullam haber[illegible] circumſcriptionem præ ſitu fixæ alicujus; [illegible] nobis paucæ aliquæ fixæ ingentes, nec u[illegible] duodecim [illegible] ot angulos habet Icoſaedron) poſſent [illegible] omnes [illegible] dem à nobis diſtantiæ, & magnitudinis: [illegible] haud multo plures, haberent jam diſtantiam duplicatam proximarum; aliæ ſuperiores triplicatam, & ſic conſequentes ſemper multipliciorem.

Ac cùm omnium maximæ, tam appareant parvæ, ut vix inſtrumentis poſſint notari aut menſurari: quæ igitur duplo aut triplo &c. diſtarent longius, duplo & triplo apparerent minores, poſitis æqualibus ipſis veris magnitudinibus

Gregory: Elements of Astronomy (c. 1715)

THE

ELEMENTS

OF

Aſtronomy,

PHYSICAL and GEOMETRICAL.

By DAVID GREGORY M. D. *Savilian* Profeſſor of ASTRONOMY at *Oxford*, and Fellow of the Royal-Society.

Done into *Engliſh*, with Additions and Corrections.

To which is annex'd,
Dr. HALLEY's Synopſis of the Aſtronomy of Comets.

In Two Volumes.

Vol. I.

LONDON:
Printed for JOHN MORPHEW near *Stationers-Hall*. M DCC XV.

Price of the Two Vols. 12*s*. Bound.

them in the ſame Spherical ſurface: Others following the Ancients, make their different diſtance to be the cauſe of their different Magnitude.

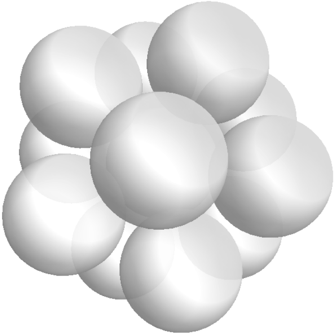
And this opinion is very much favoured by the number of the Fix'd Stars of the firſt and ſecond Magnitude. For if every Fix'd Star did the office of a Sun, to a portion of the Mundane ſpace nearly equal to this that our Sun commands, there will be as many Fix'd Stars of the firſt Magnitude, as there can be Syſtems of this ſort touching and ſurrounding ours; that is, as many equal Spheres as can touch an equal one in the middle of them. Now, 'tis certain from Geometry, that thirteen Spheres can touch and ſurround one in the middle equal to them, (for *Kepler* is wrong in aſſerting, in *B.* 1 of the *Epit.* that there may be twelve ſuch, according to the number of the Angles of an *Icoſaedrum*,) and juſt ſo many uncontroverted Stars of the firſt Magnitude are taken notice of by Obſervation. For Aſtronomers are not as yet agreed upon their number: *Hevelius* reckons the *bright Star in the Eagle's Shoulder* of the ſecond magnitude, whereas *Tycho* made it of the firſt; and on the contrary *Hevelius* makes the *little Dog* and *the right Shoulder of Orion* of the firſt magnitude, and *Tycho* of the ſecond. And there are others that *Hevelius* himſelf doubts of.

Again, if it be asked how many Spheres equal to the former can touch the firſt Order of Spheres, ſurrounding the Sphere placed in the beginning, (or rather a Sphere comprehending thoſe former thirteen together with a fourteenth in the Center;) the number of theſe will be found to be 52, or 4×13. For the centers of theſe Spheres of the ſecond Order are in a Spherical ſurface, which is quadruple of that in which

U the

Naive Bound

We know some good ways to arrange 12 spheres to touch a central one.



Naive Bound

We know some good ways to arrange 12 spheres to touch a central one.

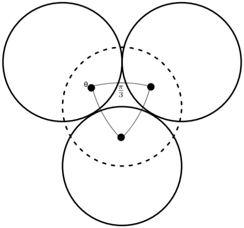
There is a bound given by the solid angle. Centrally projecting to the surface of the central sphere, this region has area

$$2\pi(1-\cos\frac{\pi}{6}) = .26\ldots$$

giving a bound of

$$\frac{4\pi}{2\pi(1-\cos\frac{\pi}{6})} = 14.9\ldots.$$

Triangulate the contacts.



Less Naive Bound

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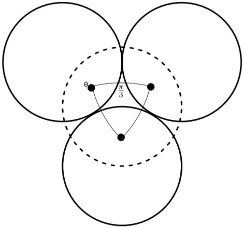

Assumption

Good triangulations exist such that the area minimizer is as above.

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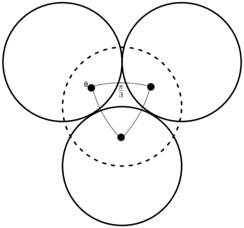

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From Euler characteristic: $v - e + f = 2$ and $3f = 2e$ for triangulations of the sphere, giving $f = 2v - 4$.

The minimal face area is

$$T := 3 \arccos\left(\frac{1}{3}\right) - \pi = 0.55 \ldots.$$

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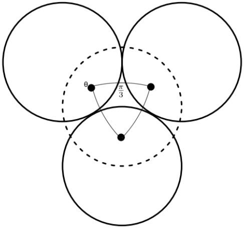

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$$T := 3\arccos\left(\frac{1}{3}\right) - \pi = 0.55\ldots.$$

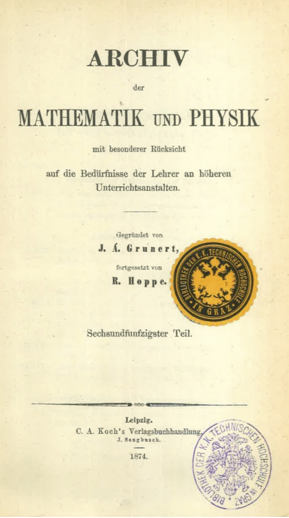
Then for $v = 13$

$$4\pi - 22 * T = 0.43...$$

but for $v = 14$

$$4\pi - 24 * T = -0.66\ldots.$$

Bender, Hoppe (c. 1874)


ARCHIV

der

MATHEMATIK UND PHYSIK

mit besonderer Rücksicht

auf die Bedürfnisse der Lehrer an höheren Unterrichtsanstalten.

Gegründet von

J. A. Grunert,

fortgesetzt von

R. Hoppe.

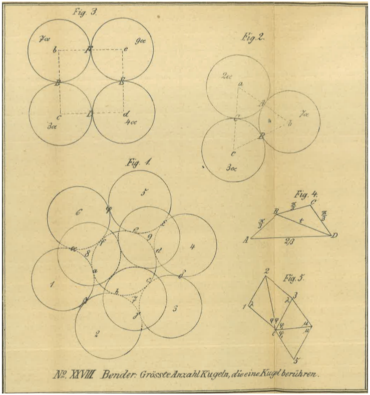
Sechsundfunfzigster Teil.

Leipzig.

C. A. Koch's Verlagsbuchhandlung.

J. Sengbusch.

1874.



Contact graphs: geometric graphs with a vertex for each kissing sphere and edges recording contact.

Schütte and van der Waerden, Leech (c. 1950)

Schütte and van der Waerden further analyzed such graphs and gave conditions on the contact graph showing that 13 unit spheres touching a central unit sphere would induce a graph that was not realizable.

Using similar techniques, Leech gave a proof consisting of only two pages. Much of this brevity seems to come from

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Theorem (Schütte and van der Waerden)

We can only fit twelve spheres.

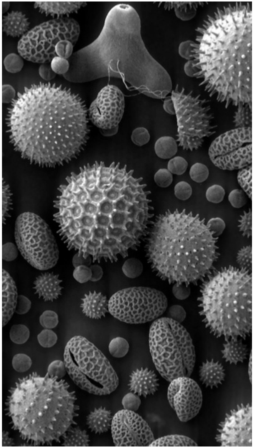
The Tammes Problem

P. M. L. Tammes (1930)

Question

What is the maximal radius possible for N equal spheres, all touching a central sphere of radius 1?

Another formulation of the *Tammes problem*: How many spherical caps of angular diameter θ that can be placed without overlap?



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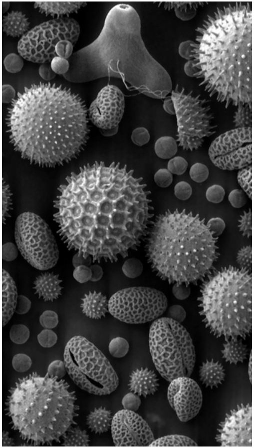
Tammes was studying pollen grains and empirically determined 6 for $\theta = \frac{2\pi}{4}$ but no more than 4 for $\theta > \frac{2\pi}{4}$.



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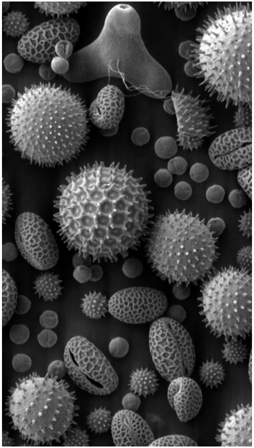
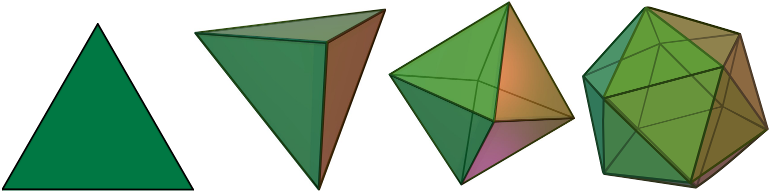


Remark

The maximizing configuration for 5 is not unique.

The Tammes problem was solved for $N = 3, 4, 6$ and 12, with configurations of cap centers for $N = 3$ attained by vertices of an equatorial equilateral triangle and for $N = \{4, 6, 12\}$ by vertices of regular tetrahedron, octahedron and icosahedron.



Fejes-Tóth proved the inequality

Theorem

for N points on the sphere, there are 2 with angular distance

$$\theta \leq \arccos\left(\frac{(\cot(\omega)^2 - 1}{2}\right), \quad \omega = \left(\frac{N}{N-2}\right)\frac{\pi}{6}.$$

The inequality is sharp for $N = \{3, 4, 6, 12\}$.

L. Fejes-Tóth (1943)

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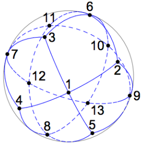
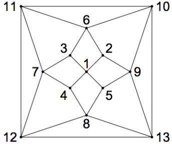
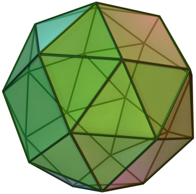
Remark

θ is the edge length of a equilateral spherical triangle with the expected area for an element of an $N-$vertex triangulation.

The Tammes problem has been solved exactly for only $3 \leq N \leq 14$ and $N = 24$. It was solved for $N = \{5, 7, 8, 9\}$ by Schütte and van der Waerden in 1951, $N = \{10, 11\}$ by Danzer in his 1963 Habilitationsschrift.







$N = 24$ was solved by Robinson in 1961 showing the configuration of centers were the vertices of a snub cube. Recently the Tammes problem was solved for the cases $N = \{13, 14\}$ by Musin and Tarasov by generating all candidate contact graphs.

Configuration Spaces

Critical Packings

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We have some solutions for the global maxima for the Tammes Problem, but there could be other interesting configurations. What about other critical points? Are there local maxima?

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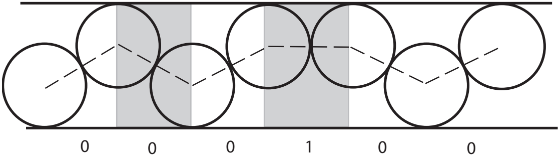

Remark

One "similar" model that can be analyzed completely is the quasi-1D packing problem. Such packings have lots of maxima.

Key Players

Definition

The classical *configuration space* $\mathrm{Conf}(N) := \mathrm{Conf}(\mathbb{S}^2, N)$ of N distinct labeled points on the unit 2-sphere $\mathbb{S}^2$.

Remark

There also is a reduced configuration space *to consider:*

$$\mathrm{Conf}(N)/SO(3).$$

Also assume $N \geq 3$ *to avoid degenerate cases.*

Definition

Configurations are $\mathbf{U} := (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N)$, where the $\mathbf{u}_j \in \mathbb{S}^2$ are distinct points.

Key Players

Definition

The *injectivity radius function* $\rho : \operatorname{Conf}(N) \to \mathbb{R}^+$ assigns a configuration $\mathbf{U} := (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N) \in (\mathbb{S}^2)^N$ the value

$$\rho(\mathbf{U}) := \frac{1}{2}\left(\min_{i \neq j} \theta(\mathbf{u}_i, \mathbf{u}_j)\right),$$

where $\theta(\mathbf{u}_i, \mathbf{u}_j)$ is the angular distance between $\mathbf{u}_i$ and $\mathbf{u}_j$.

Remark

Since ρ is invariant under the action of $SO(3)$, it descends to a well defined function on the reduced space.

Definition

$$\operatorname{Conf}(N;\theta) := \{\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_N) : \rho(\mathbf{U}) \geq \frac{\theta}{2}\}.$$

Variable Radii

Definition

Equivalently, $\mathrm{Conf}(N, \theta)$ is the configuration space of N non-intersecting caps of radius θ.

- For θ small, $\mathrm{Conf}(N, \theta) \simeq \mathrm{Conf}(N, 0)$.
- For θ large, $\mathrm{Conf}(N, \theta) = \emptyset$.

Remark

The Tammes problem is equivalent to finding the maximal r such that $\mathrm{Conf}(N, r)$ is non-empty.

Remark

The result for $N = 13$ gives a solution to Newton-Gregory.

Morse Theory

Morse theory concerns how topology changes for the super level sets of a smooth real-valued function on a manifold.

Definition (super level set)

For $f : M \to \mathbb{R}$, $M^a := \{x \in M : f(x) \geq a\}$ is a superlevel set.

Theorem

Given a smooth function $f : M \to \mathbb{R}$ and an interval $[a, b]$ with compact preimage, and $[a, b]$ contains no critical values. Then M^a is diffeomorphic to M^b.

It is only at the critical values of the function that the topology of the super level *might* change.

Remark

The injectivity radius function is not Morse.

The injectivity radius function ρ on $\mathrm{Conf}(N)$ is not smooth: it is a min-function for a finite number of smooth functions. But we may still be inspired by Morse theory to pass between the topological, analytic and geometric notions of “critical”.

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To vary a configuration $\mathbf{U} = (\mathbf{u}_1, ..., \mathbf{u}_N) \in \mathrm{Conf}(N) \subset (\mathbb{S}^2)^N$ along a tangent vector $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$ to $\mathrm{Conf}(N)$ at $\mathbf{U}$, use an ersatz exponential map.

For sufficiently small $\mathbf{V}$, define a nearby configuration

$$\mathbf{U}\#\mathbf{V} = \left(\frac{\mathbf{u}_1 + \mathbf{v}_1}{|\mathbf{u}_1 + \mathbf{v}_1|}, ..., \frac{\mathbf{u}_N + \mathbf{v}_N}{|\mathbf{u}_N + \mathbf{v}_N|}\right) \in \mathrm{Conf}(N) \subset (\mathbb{S}^2)^N$$

by summing and projecting each factor back to $\mathbb{S}^2$.
In particular, the $\mathbf{V}$-directional derivative of a smooth function f on $\mathrm{Conf}(N)$ at $\mathbf{U}$ is $\frac{d}{dt}|_{t=0} f(\mathbf{U}\#t\mathbf{V})$.

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Definition

$\mathbf{U}$ is a critical point for smooth f provided all its $\mathbf{V}$-derivatives vanish at $\mathbf{U}$. That is, the increment $f(\mathbf{U}\#\mathbf{V}) - f(\mathbf{U}) = o(\mathbf{V})$.

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In particular, the $\mathbf{V}$-directional derivative of a smooth function f on $\mathrm{Conf}(N)$ at $\mathbf{U}$ is $\frac{d}{dt}|_{t=0} f(\mathbf{U}\#t\mathbf{V})$.

Definition

A configuration $\mathbf{U} = (\mathbf{u}_1, ..., \mathbf{u}_N) \in \mathrm{Conf}(N)$ is *critical for maximizing* ρ provided for every sufficiently small $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$, we have $\max[\rho(\mathbf{U}\#\mathbf{V}) - \rho(\mathbf{U}), 0] = o(\mathbf{V})$.
That is, a configuration $\mathbf{U}$ is *critical for maximizing* if there is no variation $\mathbf{V}$ that can increase ρ to first order.

Morse Theory

If we are not critical for maximizing, there exists a variation $\mathbf{V}$ which increases ρ to first order. By the definition of ρ as a min-function, the distance between all pairs $(\mathbf{u}_i, \mathbf{u}_j)$ realizing the minimal angular distance $\theta(\mathbf{u}_i, \mathbf{u}_j) = \theta_o$ increases to first order along $\mathbf{V}$. So there is also a notion of regular value analogous to the smooth case.

If we are not critical for maximizing, there exists a variation $\mathbf{V}$ which increases ρ to first order. By the definition of ρ as a min-function, the distance between all pairs $(\mathbf{u}_i, \mathbf{u}_j)$ realizing the minimal angular distance $\theta(\mathbf{u}_i, \mathbf{u}_j) = \theta_o$ increases to first order along $\mathbf{V}$. So there is also a notion of regular value analogous to the smooth case.

Theorem (Topological Regularity)

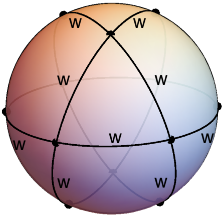
If such a variation exists for all configurations in this $\rho = \theta_o$-level set, then this level is topologically regular: *that is, the variations provide a deformation retraction from $\mathrm{Conf}(N; \theta_o - \varepsilon)$ to $\mathrm{Conf}(N; \theta_o + \varepsilon)$ for some $\varepsilon > 0$.*

Definition

For $\mathbf{U} \in \mathrm{Conf}(N;\theta)$, the *contact graph* of $\mathbf{U}$ is the graph embedded in $\mathbb{S}^2$ with vertices given by points $\mathbf{u}_i$ in $\mathbf{U}$ and edges given by the geodesic segments $[\mathbf{u}_i, \mathbf{u}_j]$ when $d(\mathbf{u}_i, \mathbf{u}_j) = \theta$.

Definition

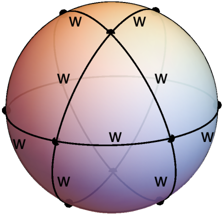
A *stress graph* for $\mathbf{U} \in \mathrm{Conf}(N;\theta)$ is a contact graph with nonnegative weights w_e on each geodesic edge $e = [\mathbf{u}_i, \mathbf{u}_j]$.



A stress graph associates a system of *tangential forces* to edges $e = [\mathbf{u}_i, \mathbf{u}_j]$ of the contact graph. The forces have magnitude w_e, are tangent to $\mathbb{S}^2$ at each point $\mathbf{u}_i$ of $\mathbf{U}$, and point outward at the ends of each edge.

Definition

A stress graph is *balanced* if the sum of the forces in $T_{\mathbf{u}_i}\mathbb{S}^2$ is zero for all points of $\mathbf{U}$. A configuration $\mathbf{U}$ is *balanced* if it has a balanced stress graph for some choice of non-negative, not everywhere zero weights on its edges.



Balanced Graphs

Theorem

For each critical value θ for the injectivity radius ρ, there exists a balanced configuration $\mathbf{U}$. The vertices of the contact graph are a subset of the points in $\mathbf{U}$ and the geodesic edges of the contact graph all have length θ.

Theorem

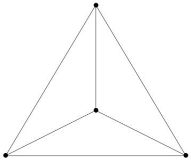
If a configuration $\mathbf{U}$ on $\mathbb{S}^2$ is balanced, then $\mathbf{U}$ is critical for maximizing the injectivity radius ρ.

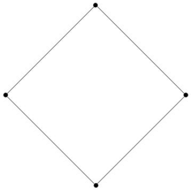
Summary

There are certain radii that are critical: The topology of the configuration space changes*.

These radii also correspond to configurations of points that are force balanced: There exists a non-trivial strut measure on the contact graph that force balances all the vertices.

Such configurations obstruct the ρ-subgradient flow, which would give a deformation retraction.





Critical radii for 4 points

For 4 points, we have

- $0 \leq \theta < \frac{2\pi}{4}$:

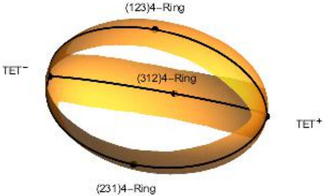
  $$\mathrm{Conf}(4, r(\theta)) \simeq \mathrm{Conf}(n, 0)$$

- $\frac{2\pi}{4} < \theta \leq \arccos\left(-\frac{1}{3}\right)$:

  $$\mathrm{Conf}(4, r(\theta)) \simeq \{0, 1\}$$

- $\theta > \arccos\left(-\frac{1}{3}\right)$:

  $$\mathrm{Conf}(4, r(\theta)) = \emptyset$$



Betti Numbers

Betti numbers for configuration space $\mathrm{Conf}(N, 0)/SO(3)$ can be computed.

$N \setminus k$	0	1	2	3	4	5	6	7	8
3	1	0	0	0	0	0	0	0	0
4	1	2	0	0	0	0	0	0	0
5	1	5	6	0	0	0	0	0	0
6	1	9	26	24	0	0	0	0	0
7	1	14	71	154	120	0	0	0	0
8	1	20	155	580	1044	720	0	0	0
9	1	27	295	1665	5104	8028	5040	0	0
10	1	35	511	4025	18424	48860	69264	40320	0
11	1	44	826	8624	54649	214676	509004	663696	362880

Critical radii for 4 points

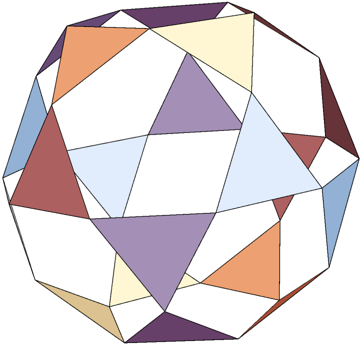
The Euler characteristic of our the configuration space is $\chi(\mathrm{Conf}(4,0))/SO(3)) = -1$.
Using the indexed sum of critical points of the function $\rho : \mathrm{Conf}(4,0)/SO(3) \to \mathbb{R}$ gives an alternative computation

$$\chi(\mathrm{Conf}(4,0)/SO(3)) = \sum_k (-1)^k \#(\text{ critical points of co-index } = k\,).$$

Since the 4-Ring has symmetry group D_4 of order 8 in $SO(3)$, there are $3 = |S_4/D_4|$ critical points of this type with co-index 1; and since TET has symmetry group A_4 of order 12 in $SO(3)$, there are really $2 = |S_4/A_4|$ critical points of this type with co-index 0

$$\chi(\mathrm{Conf}(4,0)/SO(3)) = 2 - 3 = -1$$

In general, the structure of the critical points are not nice.



Exploring Configuration Space

Remark

The configuration space of points on the sphere is a nice space to experiment with. It inherits a nice metric, the tangent space is easy to work with, it is trivial to sample, the symmetries are natural.

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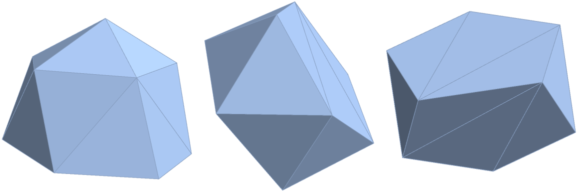
We can easily hack together an approximation of a sub-gradient descent algorithm for the injectivity radius function. With that, it is a quick step to make some conjectures about local maxima, global maxima, and the distribution of maxima just by exploring the basins of attraction randomly.



Thank you for your attention!

wkusner.github.io

Supported by Austrian Science Fund (FWF) Project 5503

Extras

12 points: Frederick Charles Frank

Frank argued that the supercooling of liquids can occur because the common arrangements of molecules in liquids assumes configurations far from what they would assume if frozen.

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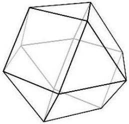
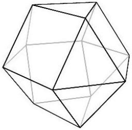
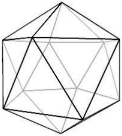
He claimed that there are exactly 3 configurations of 12 unit spheres next to a given central sphere: The FCC configuration, the HCP configuration, and the dodecahedral configuration.



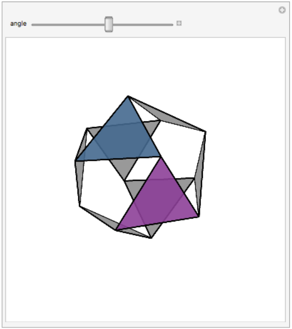


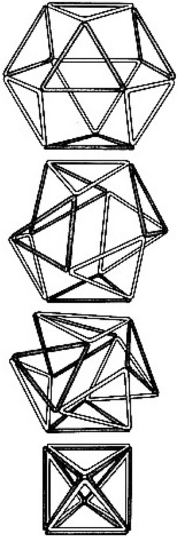


“ Consider the question of how many different ways one can put twelve billiard balls in simultaneous contact with another one, counting as different the arrangements which cannot be transformed into each other without breaking contact with the centre ball?’ The answer is *three*. Two which come to the mind of any crystallographer occur in the face-centred cubic and hexagonal close packed lattices. The third comes to the mind of any good schoolboy, and it is to put one at the center of each face of a regular dodecahedron. That body has five-fold axes, which are abhorrent to crystal symmetry: unlike the other two packings, this one cannot be continuously extended in three dimensions. You will find that the outer twelve in this packing do not touch each other.”

— Frederick Charles Frank

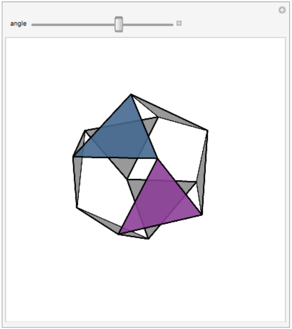
The Jitterbug

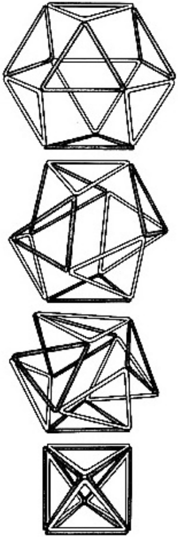




Connects icosahedron to FCC.

The Jitterbug





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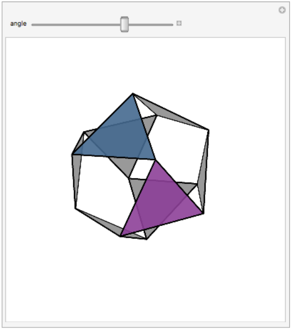

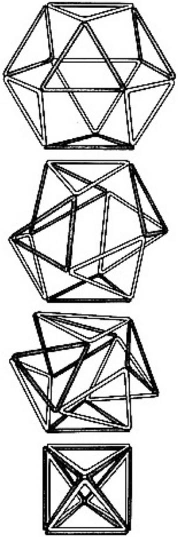

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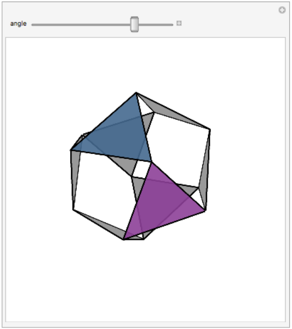

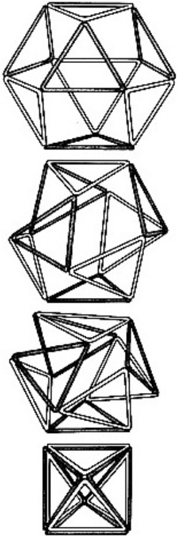

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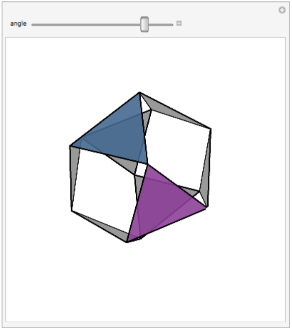

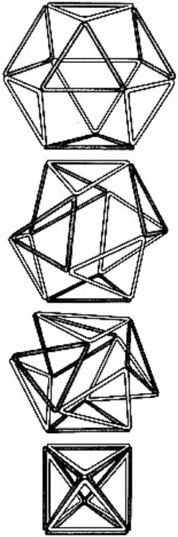

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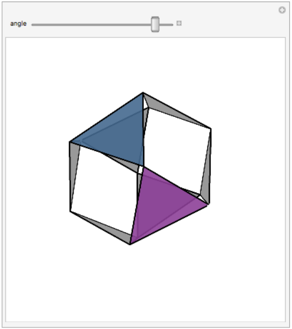

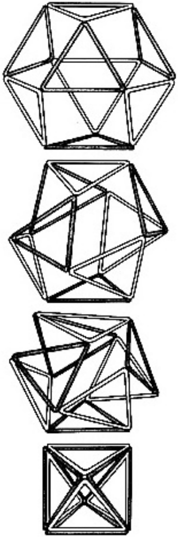

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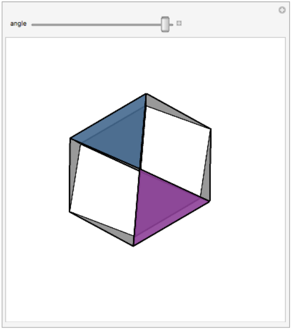

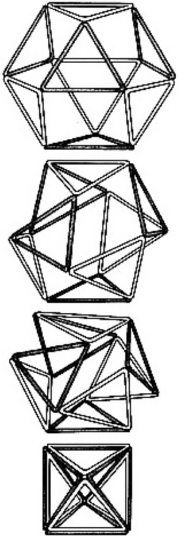

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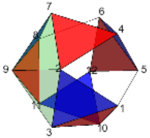
angle

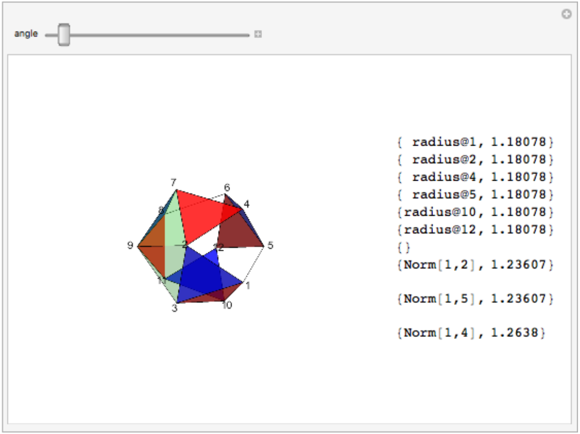



```
{ radius@1, 1.17557}
{ radius@2, 1.17557}
{ radius@4, 1.17557}
{ radius@5, 1.17557}
{radius@10, 1.17557}
{radius@12, 1.17557}
{}
{Norm[1,2], 1.23607}

{Norm[1,5], 1.23607}

{Norm[1,4], 1.23607}
```

Connects icosahedron to HCP.



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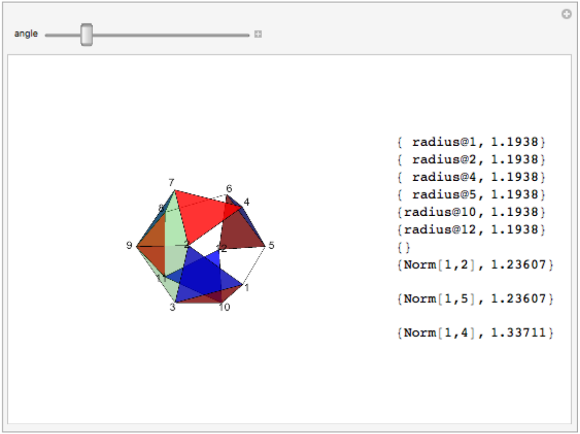

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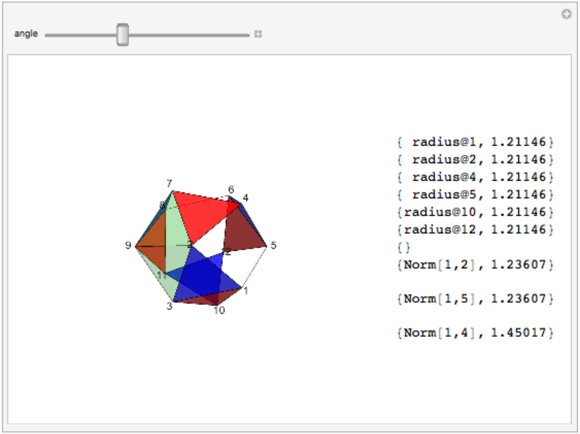

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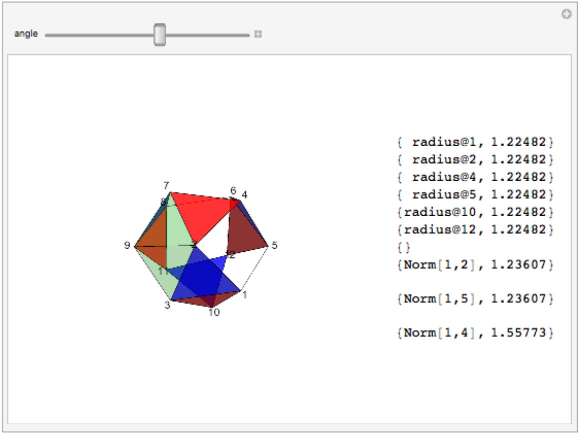

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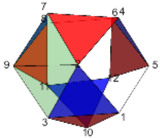
angle



```
{ radius@1, 1.23305}
{ radius@2, 1.23305}
{ radius@4, 1.23305}
{ radius@5, 1.23305}
{radius@10, 1.23305}
{radius@12, 1.23305}
{}
{Norm[1,2], 1.23607}

{Norm[1,5], 1.23607}

{Norm[1,4], 1.65533}
```

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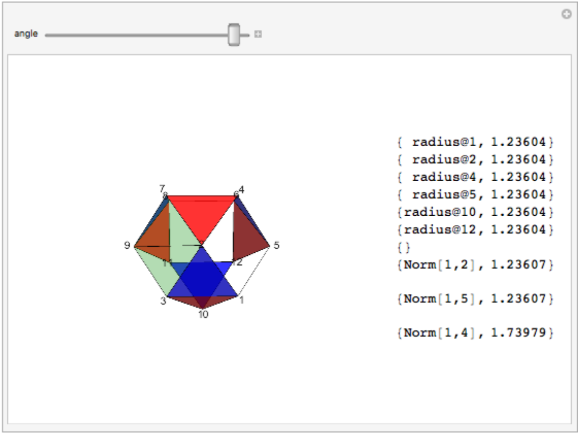

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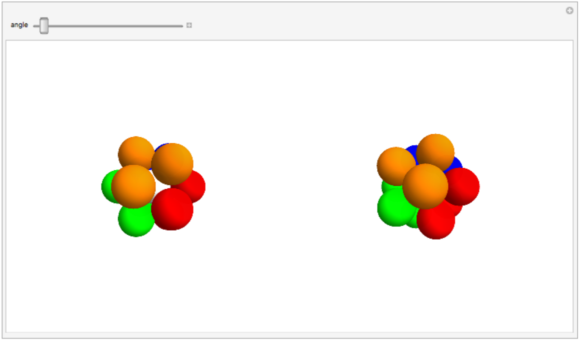
Motions of spheres



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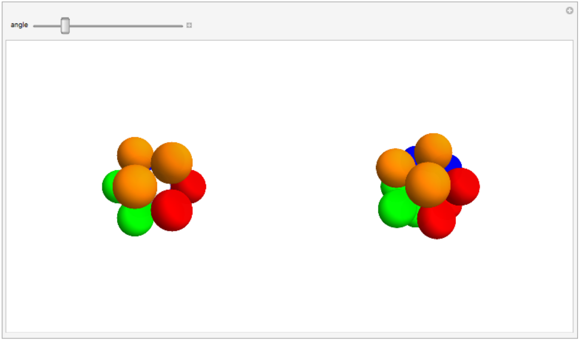

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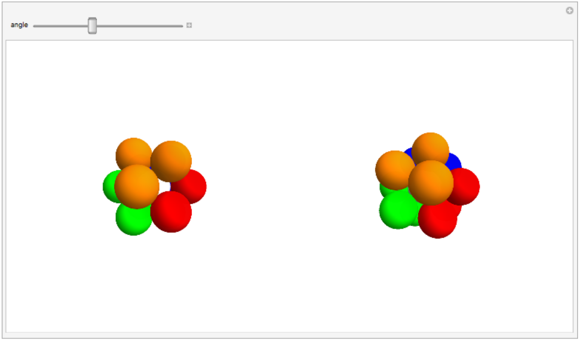

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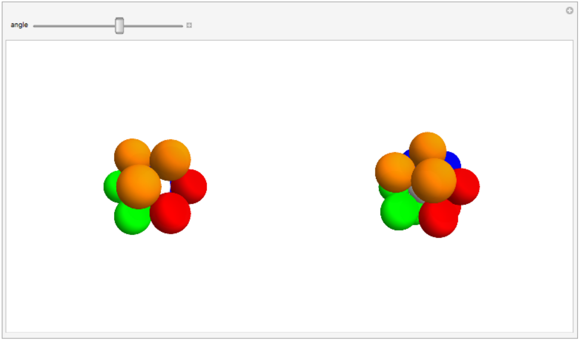

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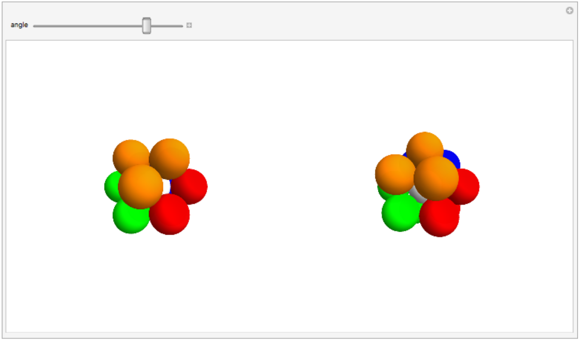

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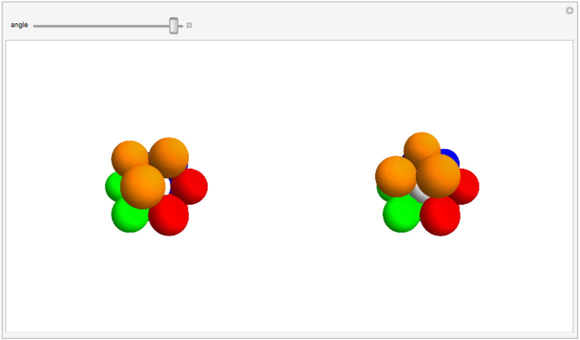

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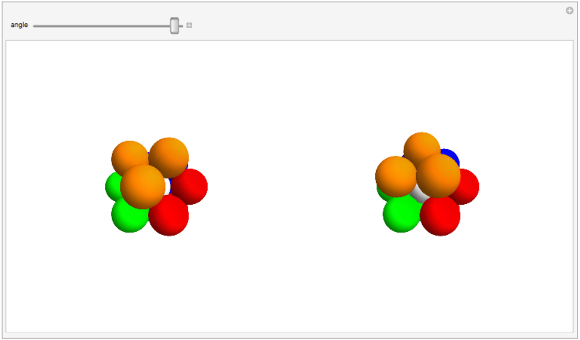

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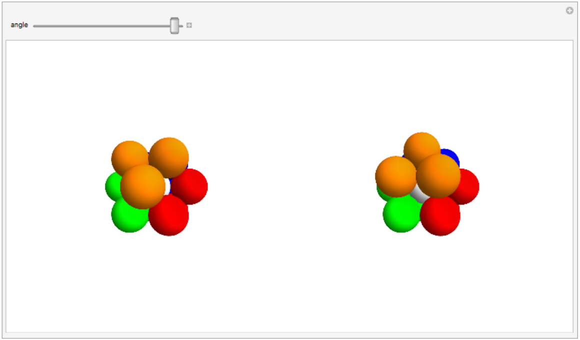

Motions of spheres

The assertion that there are exactly three arrangements is *false*. There *is* an underlying kernel of truth in this statement: each of three the arrangements above is “remarkable”.

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Question (Conway and Sloane)

What rearrangements of 12 unit spheres are possible via motions maintaining contact with the central unit sphere?

Motions of spheres

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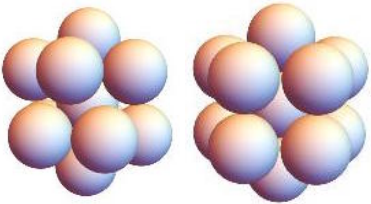
Question (Conway and Sloane)

What rearrangements of 12 unit spheres are possible via motions maintaining contact with the central unit sphere?

They demonstrate that within the component of configuration space of 12 unit spheres connected to the icosahedron, arbitrary permutations of all 12 touching spheres are possible.

The equatorial spheres can be moved towards poles and can be rotated to form half-geodesic graphs, like the bars of a birdcage.



The rings of five freely rotate relative to each other. Conway and Sloane note the conjugation action gives all 5-cycles.

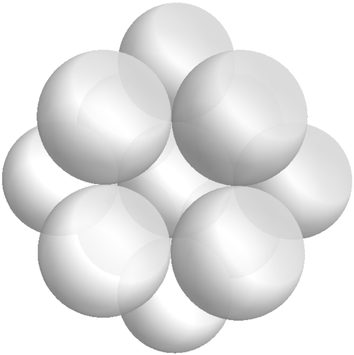
Remark

The fact that all 5-cycles can be produced this way is nontrivial. All 5-cycles generate A_{12}.

Hidden Symmetry

The Jitterbug gives a smooth motion from the icosahedron to the FCC configuration. This has an axis of 4-fold symmetry and 3 layers. Therefore conjugating a rotation with the Jitterbug describes an odd permutation.

With the icosahedral Rubik's cube, this generates all of S_{12}.

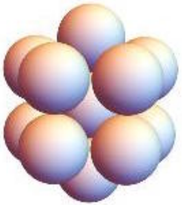


Theorem (Conway and Sloane)

Spheres at the vertices of a regular icosahedron can be arbitrarily permuted.

For $1 + \epsilon > r > 1$,it is possible to get at least A_{12}.

In the “Rubik’s cube”, one can perturb 4 vertical pairs to so the radius can be increased slightly and 1 pair passes. This yields a different type of configuration, but the same idea applies.



Higher critical radii for 12 points

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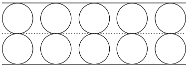

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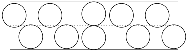

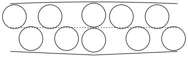

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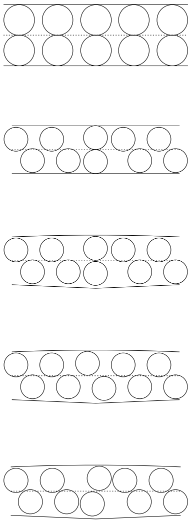

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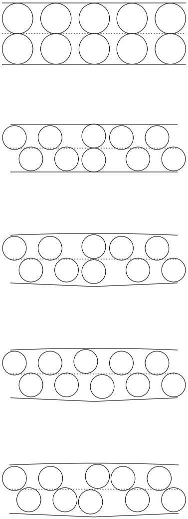

Remark

$\mathrm{Conf}(12, r)$ *near maximal* r *is a set with* $12!$ *components and* $\mathrm{Conf}(12, r)$ *just above radius* 1 *connects them. There must be a critical value above* $r = 1$.

Structure of the critical sets

Proposition

The set of balanced configurations on the sphere is a semialgebraic set.

We may write the collection of balanced configurations of N points on the sphere $\mathbb{S}^2$ as follows: Work initially in $\mathbb{R}^{3N} \times \mathbb{R}^{N(N-1)} \times \mathbb{R}^N$: the first, second and third factors will describe (1) realizations of balanced graphs, (2) edge weights and (3) normal slack variables, respectively. Partition the first factor into N triples of coordinates labeled $\{u_i\}_{i=1,\ldots N}$. Label the coordinates of the second factor w_{ij}. Label the coordinates of the third factor k_i.

Consider all labeled subgraphs G of the labeled complete graph K_N with $|V[G]| = N$. We will use each such graph G to describe a particular system of polynomials and take their **disjunction**. This is a finite collection of systems.

Configuration

:
To be a configuration, the following conditions must be satisfied.

- *points lie on the unit sphere:*

$$\forall i \in V[G]; ||u_i||^2 - 1 = 0.$$

- *points are distinct:*

$$\forall i, j \in V[G], i \neq j; ||u_i - u_j||^2 > 0.$$

Realization

To be realized as a contact graph the following must be satisfied:

- *pairs of points with edges in G have equal distance:*

$$\forall [ij], [mn] \in E[G]; ||u_i - u_j||^2 - ||u_m - u_n||^2 = 0.$$

- *all pairs have distance at least the edge distance:*

$$\forall [ij] \in E[G], \forall [mn] \in E[K_N]; ||u_m - u_n||^2 - ||u_i - u_j||^2 \geq 0.$$

Balance

Because we may have some zero weights, but they are necessarily not all zero, the balance condition may be described as a **disjunction** over $i \in V[G]$ of the following **conjunction** of conditions.

Remark

This will be empty for indices i where u_i that is only balanced with zero stresses. This will encode the full set of weights and corresponding normal slacks that exist for the entire configuration for indices i where u_i is force balanced with some incident positive stresses. Since a balanced configuration must have at least one vertex incident to an edge with a positive stress, a disjunction over the index i of vertices u_i gives all the balanced configurations, weights and normal slacks.

Balance

- if there are non-zero stresses that balance on the sphere at u_i; there is a positive normal slack force of magnitude k_i at u_i and parallel to u_i:

$$\sum_{j:[ij]\in E[G]} w_{ij}(u_i - u_j) - k_i u_i = 0$$

$$k_i > 0$$

- the stresses must extend to a system of non-negative stresses that balance on the sphere at every point u_m of a configuration; there is a non-negative normal slack force of magnitude k_m at u_m and parallel to u_m:

$$\forall m \in V[G]; \sum_{n:[mn]\in E[G]} w_{mn}(u_m - u_n) - k_m u_m = 0$$

$$\forall m \in V[G]; k_m \geq 0$$

Balance

- additional constraints on the weights from the definition of stress graph:

$$\forall [mn] \in E[G]; w_{mn} \geq 0$$

$$\forall [mn] \in E[K_N]; w_{mn} - w_{nm} = 0$$

$$\forall [mn] \in E[K_N] \setminus E[G]; w_{mn} = 0$$

Corollary

The set of critical values of ρ is finite.

Proof.

The previous section describes a semialgebraic set in $\mathbb{R}^{3N} \times \mathbb{R}^{N(N-1)} \times \mathbb{R}^{N}$. By Tarski-Seidenberg, we may project to the first factor, $\mathbb{R}^{3N}$, forgetting weights and normal slacks. This describes the set of balanced configurations on the sphere as a semialgebraic set. This holds similarly for the weights and normal slacks.

The square of the injectivity radius function ρ is a semialgebraic function, so Tarski-Seidenberg also implies that the ρ-image of the set of balanced configurations on the sphere is a semialgebraic subset of $\mathbb{R}$.

$\square$

Corollary

The straight line distances between points realizing critical configurations are algebraic numbers.

Proof.

The previous construction also defines the ρ-image as a semialgebraic set over $\mathbb{Q}$ or the field of real algebraic numbers $\bar{\mathbb{Q}} \cap \mathbb{R}$. The Taski-Seidenberg quantifier elimination algorithm works over these fields. That is, the set of critical values of ρ are in fact a finite subset of $\mathbb{R}$ defined by polynomials with real algebraic coefficients. □

Remark

We work on the sphere only for convenience... similar conditions can be defined for a variety of varieties, once the necessary changes have been made.