Gordian Configurations

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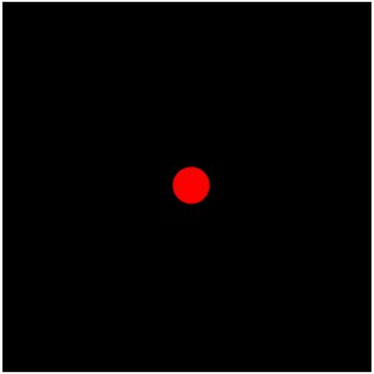
Abstract

While we are often interested in finding configurations that are ”the best”, those global optimizers of a nice function on configuration space, there is good reason to look at ”the rest”. In particular, there are often configurations that are critical, which implies a potential change the topology of configuration space for a threshold value of our function.

We will look at some cases appearing for spaces of points and curves; where our function is ”physical”, and our configuration is ”Gordian”, i.e. a hard shell potential yielding spheres and rope, and the connectedness of our configuration space changes.

Consider a ball in space.

It is “point-like”, and can explore freely.

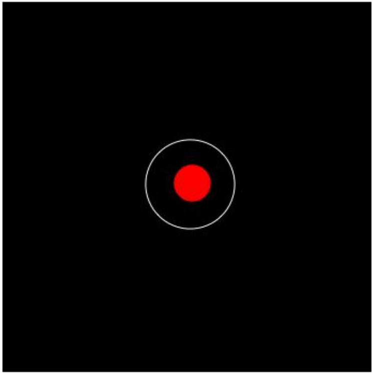


Gordian: Topology vs Geometry

Consider a ball in space.

It is “point-like”, and can explore freely.

In a container, the ball is separated from the rest of space. When the container is deformed, the ball still remains “inside”.



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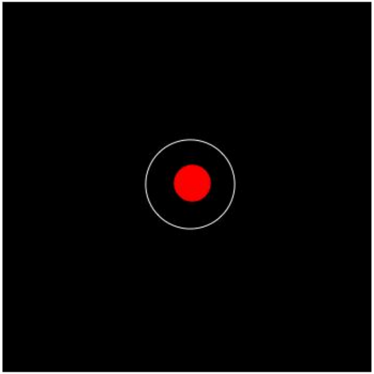


Where is the ball when the container is opened?

We can think of this a couple different ways.

One way is to look at the connectedness of the configuration space of the ball.

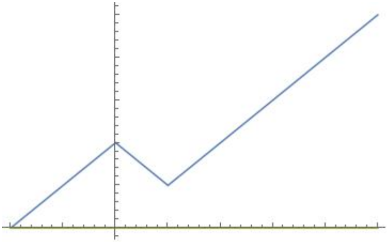
This space can be described as part of the configuration space of a point with a function, the injectivity radius ρ, attached to it.

Gordian: Topology vs Geometry

The parts of space that can be explored by a ball of a particular size are the *superlevel set* for ρ.

When the ball is small, this configuration space is connected.

As we consider larger and larger ρ, eventually there is a critical choice: go “inside” the bottle or “outside”.

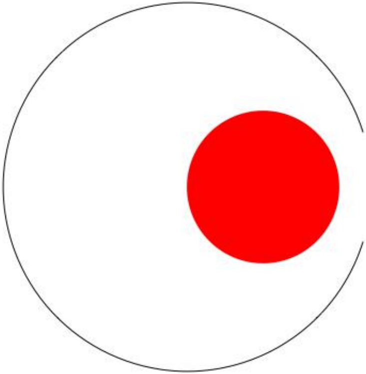


ρ as a function of of configuration space.

Another way is to think of the ball as passing through surfaces that span the mouth of the bottle.

The ball must then go "through" a surface separating the "inside" from the "outside".

This seems a bit softer: a surface might reach into the bottle. But maybe there are some natural choices of surfaces to consider.



Gordian: Topology vs Geometry

Definition (physical configuration)

A configuration is *physical* (or thick) if it is constrained to remain in a superlevel set of the injectivity radius.

Definition (physical isotopy)

An isotopy is *physical* (or thick) if it is an isotopy through physical configurations.

There may be a difference between topological and physical theories.

If we have a standard configuration:

Definition (Gordian)

A physical configuration is *Gordian* if it is in a different component of configuration space than a standard configuration.

Equivalently

Definition (Gordian)

A physical configuration is *Gordian* if there is no physical isotopy taking it to a standard configuration.

Gordian: Topology vs Geometry

Otherwise, we consider a relative notion:

Definition (Gordian)

A pair of physical configurations is *Gordian* if they are in distinct components of configuration space.

Equivalently

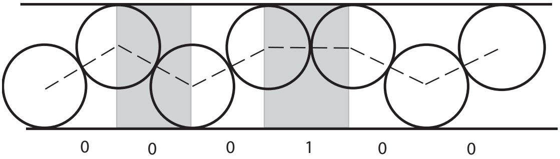
Definition (Gordian)

A pair of physical configurations is *Gordian* if there is no physical isotopy that takes one to the other.

Disk Packings

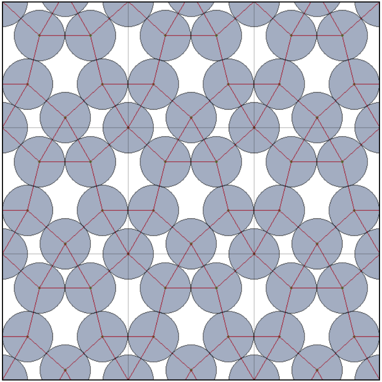
Question

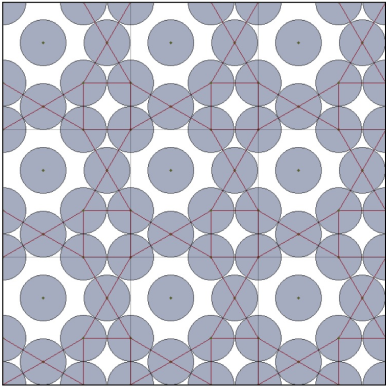
Are there different configurations of points that are isolated local maxima for injectivity radius? These would be Gordian.



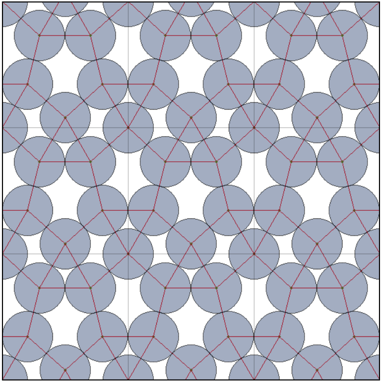
One model [Bowles, Ashwin] that can be analyzed completely is a quasi-1D packing problem. Such packings have lots of maxima.

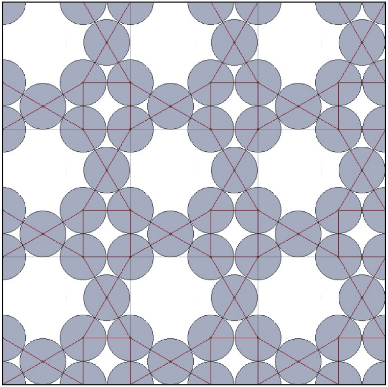
Local maxima on the square torus [Musin].





Local maxima on the square torus [Musin]. A Gordian pair.





Morse theory considers topology changes of superlevel sets of smooth real-valued functions on a manifold.

Definition (superlevel set)

For $f : M \to \mathbb{R}$, $M^a := \{x \in M : f(x) \geq a\}$ is a superlevel set.

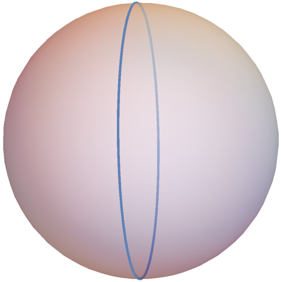
Theorem (Morse)

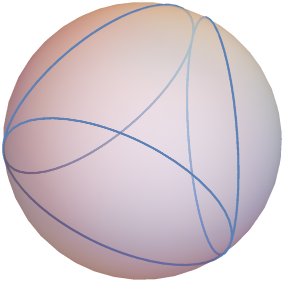
Given a smooth function $f : M \to \mathbb{R}$ and an interval $[a, b]$ with compact preimage, and $[a, b]$ contains no critical values. Then M^a is diffeomorphic to M^b.

It is at the *critical values* of the function that the topology of the superlevel set changes.

Critical Packings

For two and three points on a sphere, the configuration spaces of disks become more and more constrained as the radii grow, but they are still connected. The only critical points are the global maxima.





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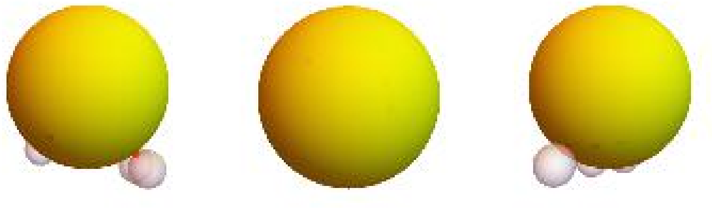
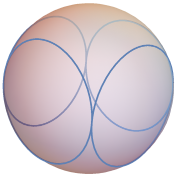


Critical Packings

Four disks around the equator form a bottleneck, and the superlevel sets for higher values of ρ are disconnected. There are two neighborhoods around the global two possible maxima. So any configuration of disks with (angular) radii greater than $\pi/4$ is Gordian. The disks are trapped in half of their possible configurations, described by an orientation.



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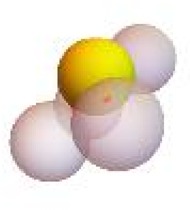
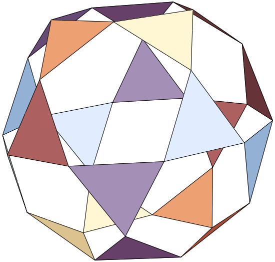


We might expect this to be a general principle: Maximal configurations (and those close to a local max for ρ) should be Gordian.

But the superlevel sets of the configuration space of 5 points are always connected.



Critical Packings

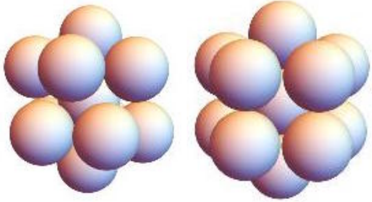
Question (Conway and Sloane)

What rearrangements of 12 unit spheres are possible via motions maintaining contact with the central unit sphere?

They demonstrate that within the component of configuration space of 12 unit spheres connected to the icosahedron, arbitrary permutations of all 12 touching spheres are possible.

Critical Packings

The equatorial spheres can be moved towards poles and can be rotated to form half-meridian contact graphs, like the bars of a birdcage.



The rings of five freely rotate relative to each other, like a "Rubik's Cube". Conway and Sloane note that this in fact gives all 5-cycles.

Remark

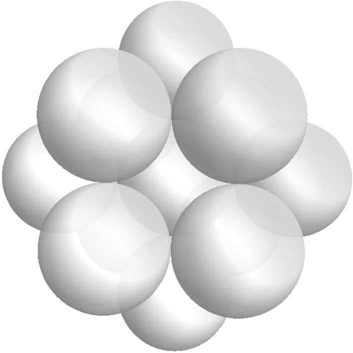
The fact that all 5-cycles can be produced this way is nontrivial. All 5-cycles generate A_{12}.

See also : http://www.puzzleforge.com/wp/

Critical Packings

The Jitterbug gives a smooth motion from the icosahedron to the FCC configuration. This has an axis of 4-fold symmetry and 3 layers. Therefore conjugating a rotation with the Jitterbug describes an odd permutation.

With the icosahedral Rubik's cube, these generates all of S_{12}.

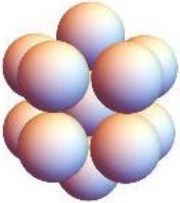


Theorem (Conway and Sloane)

Spheres at the vertices of a regular icosahedron can be arbitrarily permuted.

For $1 + \epsilon > r > 1$,it is possible to get at least A_{12}.

In the “Rubik’s cube”, one can perturb the vertical pairs slightly, which allows the radii to be increased while still allowing the rings to rotate.



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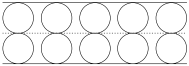

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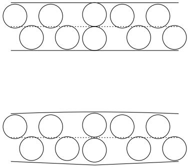

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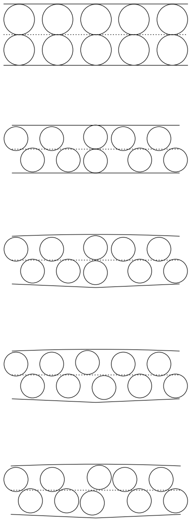

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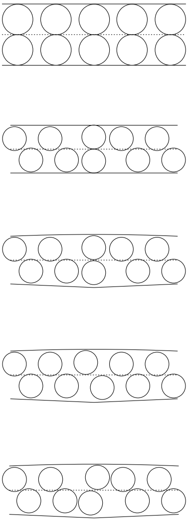


Remark

The configuration space of such spheres is disconnected at the global max for injectivity radius. There must be there critical values above radius 1.

Critical Packings

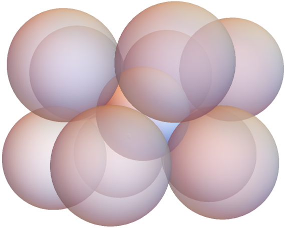
But what about local *maxima*? *Likely* in the previous regime.

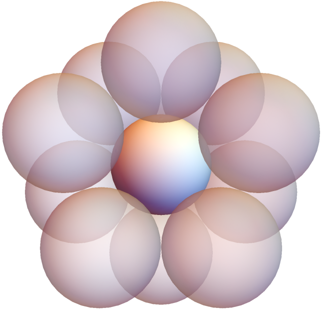
But what about local *maxima*? *Likely* in the previous regime.

Theorem ($2N$ Double Ring)

There is a local max configuration for all even N, corresponding to the vertices of a regular antiprism.

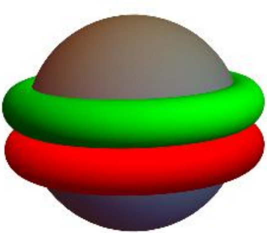
This provides a framework for constructing “unlikely” examples.





Theorem (Continuum Double Ring)

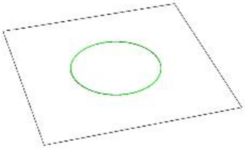
The per-component length of a pair of thickened tropical bands cannot be increased.

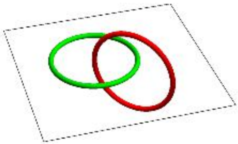


Rope

Definition (Rope)

Given a (closed, $C^{1,1}$) curve with bounded curvature and fixed length in $\mathbb{R}^{\not\models}$, we can thicken it into a *rope* in the normal direction, out to its (normal) injectivity radius. The original curve is its *core*. Typically, this is scaled to have injectivity radius 1.





Rope

Definition (Physical isotopy)

An isotopy of a knot or link that maintains this scaling (i.e. preserves length per component and injectivity radius at least 1) is a physical isotopy.

Question (Alexander (the Great))

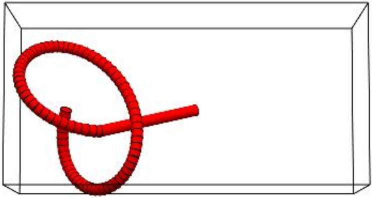
Is there a Gordian Unknot? A Gordian Split Link?

Question

Is there a Gordian Strand?

The core curve of a rope *could* be the image of an interval. To see that the physical and topological theories are not distinct in this case is still not trivial.

If there is a free end, *reptation* will take rope to a standard form. In general we must consider a family of homothetic transformations of the core curve, while reptating to preserve length.



Rope

For closed core curves, this trick does not work.

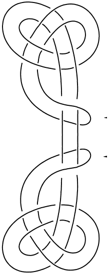
An equivalent statement to Hatcher's theorem is that the space of *smooth* unknotted simple closed loops deformation retracts to the space of standard loops. That is, we can always "unknot a family of topological unknots.... and there is no topological obstruction to finding a gradient flow that takes any unknot to the standard knot."

The existence of a Gordian unknot is still an open problem.

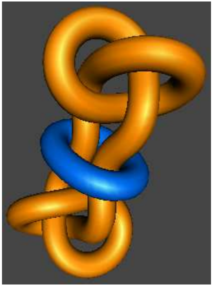
A Gordian Split Link was exhibited by Coward and Hass.

Theorem (Coward-Hass)

"Physical knot theory is distinct from topological knot theory."



? Freedman-He-Wang



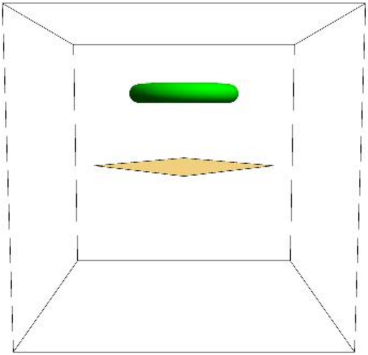
! Coward-Hass

Rope

Alternative: the existence of stopper knots. A *stopper knot* is a knot or link that cannot pass through a (planar) aperture of fixed area.

The model is rope embedded into the upper halfspace and separated from the lower halfspace by a plane. The plane contains an *aperture* through which the rope must pass.

Gordian: Topology vs Geometry



Rope

Clearly, if the aperture is too small, no configuration of rope can pass. For example, it must have sufficient area to have a unit radius ball pass through. We can track this during an isotopy.

Definition

A rope is *accessible from above* (resp. *from below*) if its core curve has non-trivial intersection with the open upper (resp. lower) halfspace, and *accessible* if it is both accessible from above and below. A rope is *accessible critical* if its core curve has trivial intersection with either the upper or lower open half space, but non-trivial intersection with the aperture.

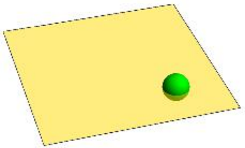
Lemma

If a rope is accessible, then the area of intersection with the aperture is greater than π. If a rope is accessible critical, then the area of intersection with the aperture is greater than π.

Proof.

In either case, the core curve of the rope has non-trivial intersection with the aperture A. Consider a ball of radius 1 centered at a point in that intersection. This ball is contained completely inside the rope and occludes a circular unit disk. $\square$

Gordian: Topology vs Geometry



It is not entirely clear that rope, as defined, is the outer parallel body/sweep out by unit radius balls of the core curve.

Theorem

There is no physical isotopy of a (n, n)*-torus link that passes through an aperture with area less than* $n\pi$.

Proof.

Given an isotopy that passes through an aperture, consider the *first* time that a component of the (n, n)-torus link becomes inaccessible from above. That component is accessible critical. As the other $(n-1)$ components of the (n, n)-torus link are all linked with that component and this is the first time that a component has become inaccessible from above, any particular component must be either accessible or accessible critical. So there are at least n disjoint occluding disks. $\square$

By application of an isoperametric inequality for space curves, the result extends to non-planar apertures giving:

Theorem

Split links formed by a connect sum of two(n, n)*-torus links and a sufficiently short surround curve are Gordian.*

We can also arrive at an analogous result by tightening the bound on the occlusion area (via Schur's Theorem), which can then be used to show that any rope passing through a small aperture or a sufficiently short space curve must be an unknot.

Rope