Lecture 2.

A206 Mon's 15-17^{00}
Except 2 days + Last Day in Sem room
? NT02008

2.1 Last time: modern take on Thue's Result in $\mathbb{R}^2$: The density of a packing of the plane by congruent circles is bounded by the density of the regular hexagonal packing.

Idea: use saturation and Delaunay triangulation

Remark: Left out error analysis of limits ... etc.

[Remark: Rogers showed density bound from the regular simplex.

2.2 Fejes-Tóth inequality for S^2

Thm: For $n > 2$ points on $\mathbb{S}^2$

There exists a pair with spherical distance (angular...)

$$d \leq \arccos\left(\frac{\cot(\omega) - 1}{2}\right);$$

(*)

$$\omega = \frac{n}{n-2} \cdot \frac{\pi}{6}.$$

(Exercise)

Remark: can be rewritten as a density result for spherical caps of diameter d

$$\frac{n \cdot 2\pi (1 - \cos(\frac{d}{2}))}{4\pi} \rightsquigarrow \frac{\pi}{\sqrt{12}}.$$

Flat values also converge but not so nice to take limit.

Good lower bound? ie construct a sequence of sets of points that achieves this bound.

To prove F-T Thm, we need

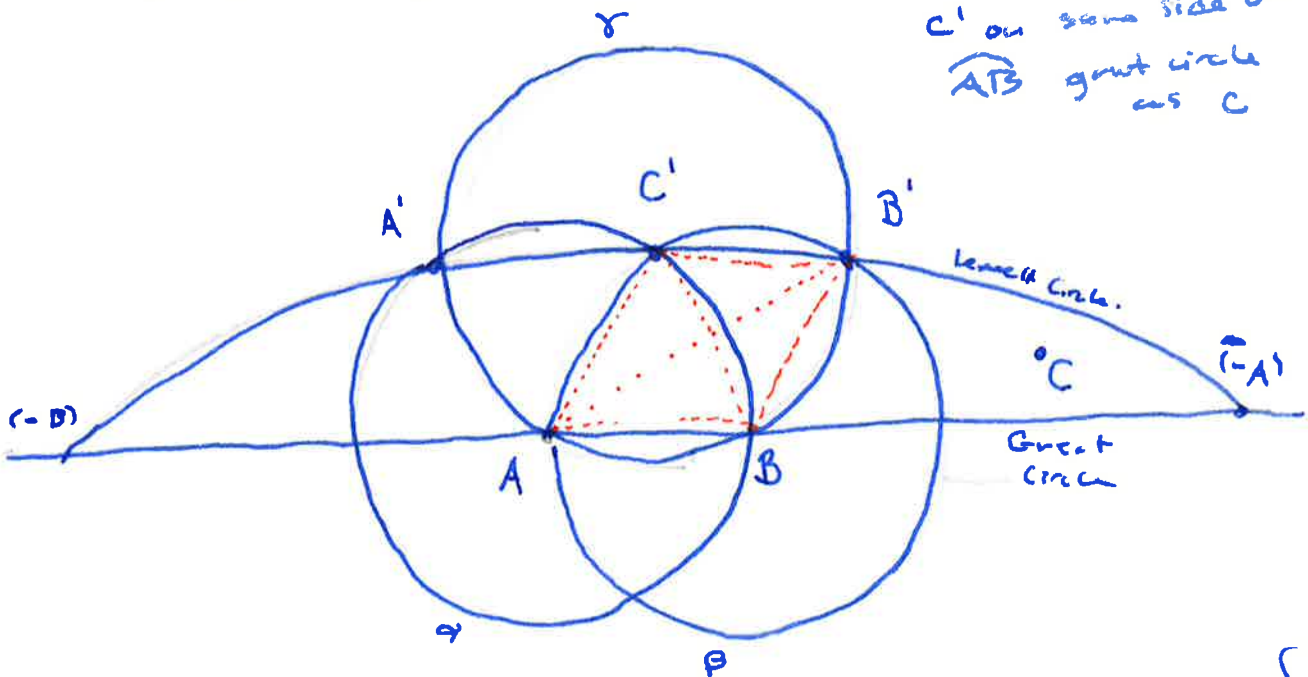
Lemma: given spherical Δ ABC

If: Area ΔABC < Area of equilateral $\Delta AB\hat{C}$, drawn on the shortest side $\overline{AB}$ of ΔABC

Then: The spherical radius of the (intersection) circumcircle of ABC is greater than length $\overline{AB}$.

Proof of Lemma:

1) Draw picture. C', st ABC' equilateral
C' on same side of $\widehat{AB}$ great circle as C



2) Show picture is "correct":

note: spherical triangles $\triangle ABA'$, $\triangle ABB'$, $\triangle ABC'$ have equal area.

$\Rightarrow$ the intersection locus of the plane $\{A', B', C'\}$ is the equal area $\triangle$ locus (Lexell Circle Thm)

By assumption, C is between the Lexell Circle and the Great circle $\widehat{AB}$.

As $\overline{AB}$ is the shortest edge of $\triangle ABC$ by assumptions

C does not lie in α or β.
$\Rightarrow C$ is not in γ.

But C is on the C' side of $\widehat{AB}$, $\Rightarrow$ circumcircle radius of

$\pi/2$ 95 $\triangle ABC > \overset{rad}{\gamma} =$ length $\overline{AB}$.

So to prove thm:

$n=3$ trivial, so $n\geq 4$.

WOLG $\text{Convex Hull}\{P_1,\dots,P_n\}\ni 0$

[else we could flow points $\xrightarrow{\text{towards}}$ pole equator and increase all distances.]

1) Triangulate convex hull.

Euler Char. $\Rightarrow$ $2n-4$ faces.

Exercise:

$$v-e+f=2$$

$$3f=2e \quad \text{for } \Delta \text{ lations}$$

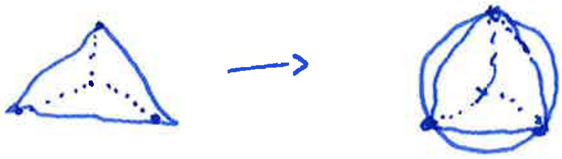
$$\Rightarrow 2n-2e+2f=4$$

$$\Rightarrow 2n-3f+2f=4$$

$$\Rightarrow f=2n-4 \quad ✓$$

Δation of Convexhull $\{p_1 \cdots p_n\}$

$\longrightarrow$ "Spherical net" by projection to the sphere, radially.



(edges are geodesics . . . .)

Now: for $\{p_1 \cdots p_n\}$,

Suppose length $\widehat{p_i p_j} > d$ $(*)$

for all $i \neq j$.

(Exercise) $(*)$

d is the length of the side of an equilateral spherical Δ of area

$$\frac{4\pi}{2n-4}.$$

(L'Huilier, analogue to Heron's formula)

Then $\Delta P_i P_j P_k$ of minimal area satisfies the Lemma:

Its area is small*, but edges are all longer than d.

* less than or equal to the equilateral Δ of side length d.

$\Rightarrow$ The circum circle $P_i P_j P_k$ has radius larger than d.

But the net construction $\Rightarrow$ The circum circle is empty.

So we may place a new point at the center...

This new collection satisfies the same inequalities for the same d!

This is absurd.

eg.: volume bound.

$$\mathrm{Vol}\, S^2 = 4\pi$$

and $\sum_{i=1}^{N} \text{area cap}\left(\frac{d}{2}\right) < 4\pi \quad \forall N > 0$

2.3) Higher Dimension Sphere packing.

Consider an early method of Blichfeldt (1929) to get density bounds in higher dimensions.

We start with a <u>convex body</u> C ~~(so think $C = \mathbb{B}^n$)~~ in $\mathbb{R}^n$: a compact, convex subset of $\mathbb{R}^n$ with non-empty interior.

Also, consider collections of isometries $\{\varphi_i\}_{i=1}^{\infty}$ such that $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing.

$\equiv$

For now $C = \mathbb{B}^n$

and a packing is of the form.

$$\{\varphi_i \mathbb{B}^n\}_{i=1}^{\infty}$$

Consider replacing the ball with its characteristic function:

$$\chi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

So it has mass.

$$I(\chi) = \int_{\mathbb{R}^n} \chi(x)\,dx$$

And a packing $\longrightarrow \sum_{i=1}^{\infty} \chi(\varphi_i^{-1} x)$

For general C, this works as well ...

Furthermore, we now can replace χ with some other function, f and in general, a packing with

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1} x)$$

And a density may be similarly defined to the packing density when each object has mass

$$I(f) = \int_{\mathbb{R}^n} f(x)\,dx$$

spread over $\mathrm{supp}(f)$.

So :

$$\Delta_f = \delta_{\substack{\text{packing} \\ C}} \frac{I(f)}{\mathrm{Vol}(C)}$$

Can think of this as
spreading out and renormalizing
the characteristic function
in some (nice) way.
hopefully)

" Cut up the sphere and spread it out. "

Insight: There are functions f
where $\sum_{i=1}^{\infty} f(\varphi_i^{-1} x)$ are

~~pointwise and unifo~~

Uniformly pointwise bounded
over $\mathbb{R}^n$ and collections
of isometries $\{\varphi_i\}$ st

$\{\varphi_i C\}$ is a packing!

2.41) Given a convex body C in $\mathbb{R}^n$

f is a <u>Blichfeldt gauge</u> for C

if for any $\{\varphi_i\}_{i=1}^{\infty}$ of

isometries of $\mathbb{R}^d$ st $\{\varphi_i C\}_{i=1}^{\infty}$

is a packing:

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1} x) \leq 1$$

for all $x \in \mathbb{R}^n$.

/// add non negative condition...

Lemma: If f is a Blichfeldt gauge.

$$(*) \qquad \delta(C) < \frac{\mathrm{Vol}(C)}{I(f)}$$

proof sketch. Def $\Rightarrow \Delta \overset{\leq}{\bullet} 1$

$\Rightarrow (*)$

2.5) Back to $C = \mathbb{B}^n$

we can consider radially symmetric functions by an averaging argument

(no benefit to anisotropy, since the $SO(n)$ kills it...)

$\equiv$

So distance functions!

Rankin does a complicated study to get good function

We consider

$$f_0(x) = \begin{cases} 1 - \frac{1}{2}|x|^2 & |x| \le \sqrt{2} \\ 0 & |x| > \sqrt{2} \end{cases}$$

Can show the following for a packing of unit radius Balls.

$\boxed{\text{Lemma:}}$

Consider m unit Balls with centers

$\{a_i, \dots, n_i\}$.

Then clearly, for all pairs we have.

(*) $(a_i - a_j)^2 + \dots + (n_i - n_j)^2 \geq 4$.

Then by considering

$\Rightarrow \sum_{1 \leq i < j \leq m} (*)$,

we have. (extra cross terms cancel...)

$$m \sum_{i=1}^{m} (a_i^2 + \dots + n_i^2) - \left(\sum_{i=1}^{m} a_i\right)^2 - \left(\sum_{i=1}^{m} b_i\right)^2 \dots - \left(\sum_{i=1}^{m} n_i\right)^2$$

$$\geq 2m(m-1)$$

$$\Rightarrow \sum (a_i^2 + \dots + n_i^2) \geq 2(m-1)$$

This is an arbitrary arrangement, so we consider a collection of m spheres within $\sqrt{2}$ of $\vec{0}$ with masses defined by f_0 centered at $(a_i \dots n_i)$.

we see that

$$\sum_{i=1}^{m} \frac{(2 - r_i^2)}{2} \leq \frac{2m - 2(m-1)}{2} = 1$$

and ~~this~~ $f_0(x)$ is a Blichfeldt gauge.

2.6) So we can integrate this

$$I(f) = \int_0^{\sqrt{2}} \frac{2-r^2}{2} \, dr^n = \frac{4 \cdot 2^{n/2}}{n+2} \cdot \mathrm{Vol}(B^n) \quad \frac{2^{\frac{n+2}{2}}}{n+2}$$

Exercise (spherical shells)

$\boxed{\text{Limit argument for a box}}$

For a large box with k spheres in it.

Then $(t + 2\sqrt{2} - 2)^n$ box contains the gauges

$$(t + 2\sqrt{2} - 2)^n \geq |I| \int_{\mathbb{R}^n} f_0 \, dx$$

$$= \frac{|I| \, \mathrm{Vol}(B^n) \, 4 \cdot 2^{n/2}}{n+2}$$

$$\Rightarrow \delta\left(\begin{array}{c}\text{of these } k \text{ balls in}\\ \text{the box } t^n\end{array}\right)$$

$$= \frac{|I| \operatorname{Vol}(B^n)}{t^n}$$

$$\leq \frac{n+2}{4 \cdot 2^{n/2}} \left(1 + \frac{2\sqrt{2} - 2}{t}\right)^n$$

and for $t \to \infty$

$$\delta(\cancel{\quad}) \leq \frac{n+2}{4 \cdot 2^{n/2}}$$

$$\frac{2^{\frac{n}{2}+1}}{n+2}$$

for any packing of $\mathbb{R}^n$ by $\mathbb{B}^n$

$\square$

Can do this explicity for Balls too, also still missing Boundary term.