# Gordian Unlinks

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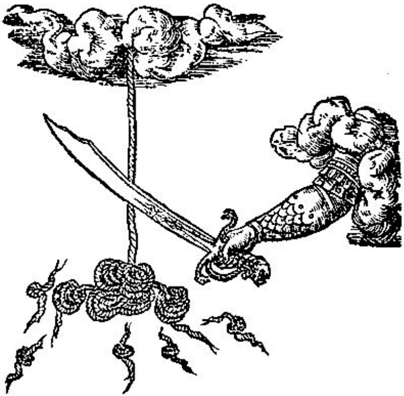
# Introduction

Given a sufficiently nice embedded space curve, we can thicken it into a physical rope. A pair of physical configurations is Gordian if there is no physical isotopy that takes one to the other. It is an (old) open problem to describe a Gordian unknot.

We will explore some special configurations of physical links by considering extrusion along with various packing constraints. This is a dual perspective to the topological sweep-out procedure Coward and Hass used to first describe a Gordian split link. There are some advantages that appear with our shifted view; we trade a general statement about knots and surfaces for tighter area bounds and rigidity, severely constraining the character of certain physical isotopies. In the end, we claim this is sufficient to describe a Gordian Unlink.

Related to work with Rob Kusner, Greg Buck

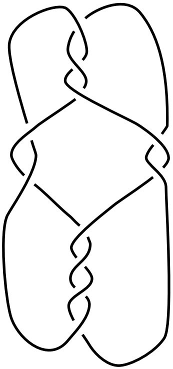
*Tanto monta, cortar como desatar.*

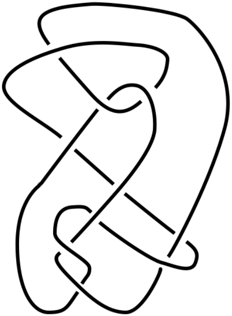


# Haken's "Gordian" Unknot

(Agol)

# Goeritz's "Gordian" Unknot and the Culprit





(Henrich and Kauffman, Kauffman and Lambropoulou)

# Smale Conjecture / Hatcher Theorem

$$\mathrm{Diff}(S^3) \simeq O(4)$$

One statement of Hatcher's theorem is:

**Theorem**

*The space of unknotted circles in* $\mathbb{R}^3$ *deformation retracts to the subspace of great circles in* $S^2$*.*

We can always "unknot a family of (topological) unknots.... There is no *topological* obstruction to finding a gradient flow taking any unknot to a round circle."

- Magic wand in 2D: Curve Shortening Flow: Move perpendicular to the curve proportional to curvature (Gage-Hamilton-Grayson).
- Put an electrostatic charge on the curve (O'hara, Freedman-He-Wang).
- Thicken the curve into "rope".
- Ricci Flow through singularities (Bamler and Kleiner).

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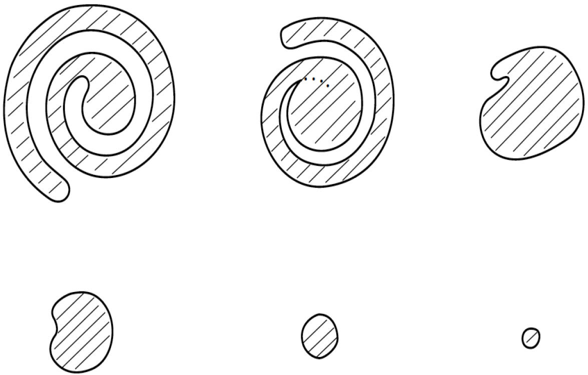
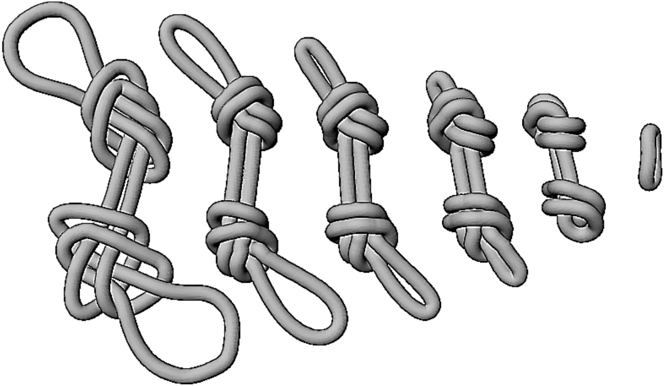
  **This works (in a very general sense.)**

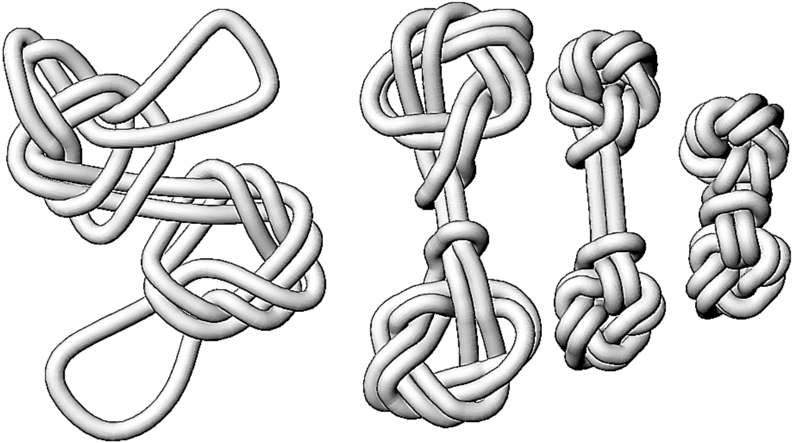


(Colding and Minicozzi)



(Pieranski, Przybyl, Stasiak)

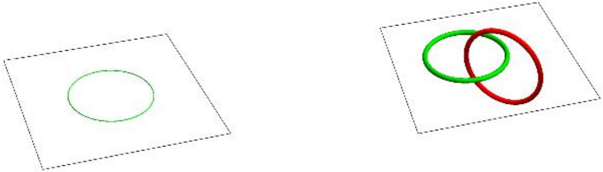
# Thickening/Shortening Rope



(Pieranski, Przybyl, Stasiak)

### Definition (Rope)

Given a (collection of) (closed, simple, $C^{1,1}$) curve(s) in $\mathbb{R}^3$, we can thicken it into a *rope* in the normal direction at least up to the normal injectivity radius. The original curve is its *core*. Typically, this is scaled to have thickness 1.



### Remark

*This is also the sweep-out by a unit normal disk* **or** *a unit ball.*

# Rope

In knot theory, equivalence is via (ambient) isotopy: we have a continuous family of embeddings of the knot between two configurations (that lifts to the whole space).

**Definition (Physical isotopy)**

An (ambient) isotopy of a knot or link that maintains this scaling (i.e. preserves length per component and injectivity radius at least 1) is a physical isotopy.

# Gordian: Topology vs Geometry

We can “Gordian” as a property of pairs:

**Definition (Gordian)**

A pair of physical configurations is *Gordian* if there is no physical isotopy that takes one to the other.

In this sense, a non-trivial knot is always Gordian if we compare it to an unknot.

**Question (Alexander (the Great))**

*Is there a Gordian Unknot? A Gordian Split Link?*

# A Thought Experiment

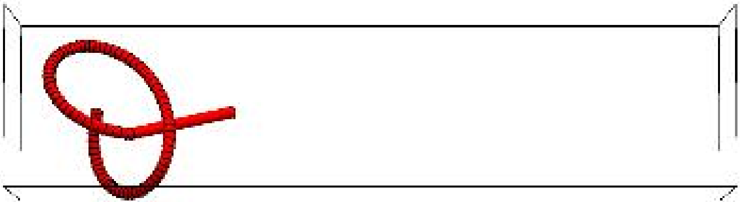
**Question**

*Is there a Gordian Strand?*

The core curve of a rope *could* be the image of an interval. To see if physical and topological theories are not distinct in this case is still not trivial.

If there is a free end, *reptation* will take rope to a standard form. In general we must consider a family of rescalings of the core curve, while reptating to preserve length.

# A Thought Experiment



# Gordian Links

For closed core curves, this trick does not work.

The existence of a Gordian unknot is still an open problem.

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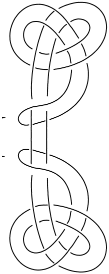
A Gordian Split Link has been exhibited by Coward and Hass.

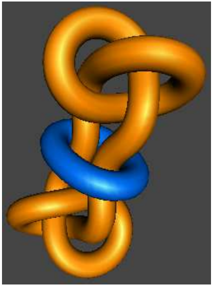
# Gordian Links

**Theorem (Coward-Hass)**

*"Physical knot theory is distinct from topological knot theory."*



? Freedman-He-Wang



! Coward-Hass

# Stoppers

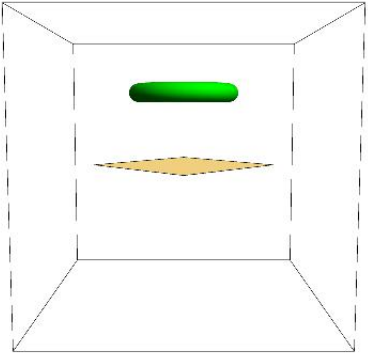
Toy Model: The existence of stopper knots.

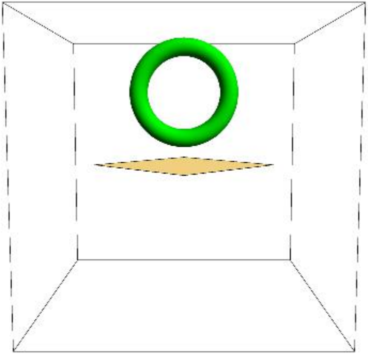
The model considers rope embedded in the upper halfspace and separated from the lower halfspace by a plane. The plane contains an *aperture* through which the rope must pass.

**Definition**

A *stopper knot* is a knot or link that cannot pass through a (planar) aperture of fixed area.

# Stoppers





Clearly, if the aperture is too small, no configuration of rope can pass. For example, it must have sufficient area to have a unit radius ball pass through. We can track area during an isotopy.

**Definition**

A rope is *accessible from above* (resp. *from below*) if its core curve has non-trivial intersection with the open upper (resp. lower) halfspace, and *accessible* if it is both accessible from above and below. A rope is *accessible critical* if its core curve has trivial intersection with either the upper or lower open half space, but non-trivial intersection with the aperture.

**Remark**

*If a rope is accessible, then the area of intersection with the aperture is greater than $\pi$. If a rope is accessible critical, then the area of intersection with the aperture is greater than $\pi$.*

In either case the core curve of the rope has non-trivial intersection with the aperture $A$. Consider a ball of radius 1 centered at a point in that intersection. This ball is contained completely inside the rope and occludes a circular unit disk.

# Stoppers

## Theorem

*There is no physical isotopy of a $(n, n)$-torus link that passes through an aperture with area less than $n\pi$.*

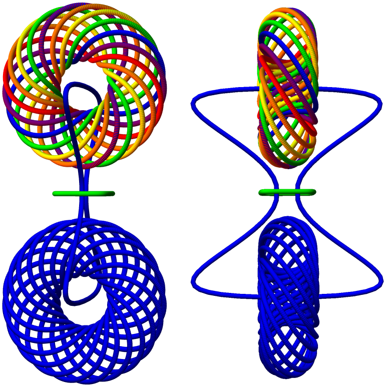
## Sketch

*Given an isotopy that passes through an aperture, consider the first time that a component of the $(n, n)$-torus link becomes inaccessible from above. That component is accessible critical. As the other $(n-1)$ components link with that component and this is the first time that any component has become inaccessible from above, any particular component must be either accessible or accessible critical. So there are at least $n$ disjoint occluding disks.*

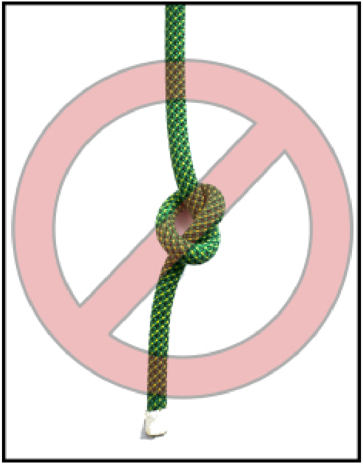
# Stoppers

**Theorem**

*Split links formed by a connect sum of two $(n, n)$-torus links cannot pass through a small planar aperture.*



# Stoppers In Real Life



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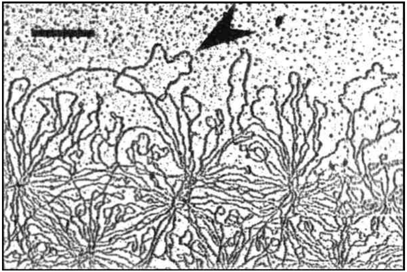
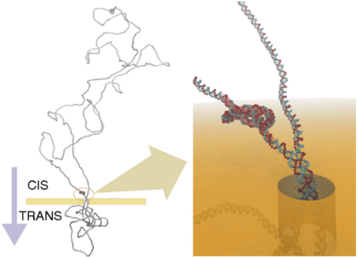


Lukes et al.

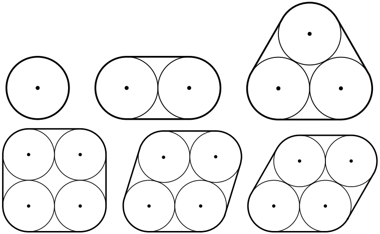



Suma-Micheletti

There are ways to replace a planar aperture with a spanning disk. A cone works.

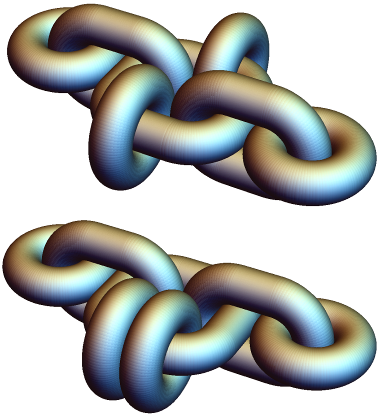
(Daina Taimina)

If the aperture curves are sufficiently short surround curves of the other component(s) strands, they end up being planar.



(Canterella, R. Kusner, Sullivan)

# A Pair of Gordian Links



# Schur's Lemma

Schur Comparison Lemma according to Chern:

> *If an arc is "stretched", the distance between its endpoints becomes longer.*

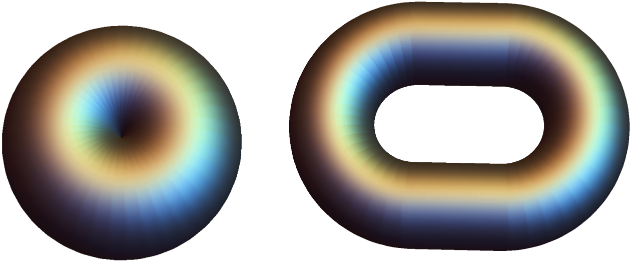
Two spheres centered on the core curve separated by arclength at least $\pi$ are disjoint: The distance between the ends of a curve with curvature bounded by $1$ of length $\pi$ is at least 2.

The minimizing chord distance for such curves of length less than $\pi$ is achieved along a planar circular arc of curvature 1.

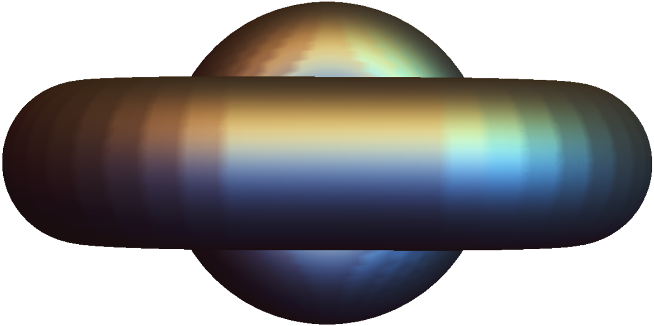
This version of Schur's Lemma, together with some geometric constraints forces canonical shapes for certain isotopies:

An isotopy that splits a link through a ”surround 2 strands” curve (as in the Coward-Hass counterexample) must have a relative component isotopic to a half-circle core curve of length $\pi$. To pass through, both relative components are isotopic to half-circle core curve of length $\pi$

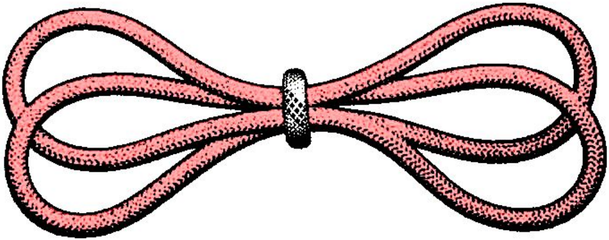
Analogous result holds for the ”surround 4 strands” curve.



# Relative Isotopy



# A Gordian Unlink



# The End? Thank You!