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August 2020

My research is motivated by the study of optimal configurations: those resulting in a minimized energy, density, or other function. The isoperimetric problem--maximize the area surrounded by a fixed-length curve--and Plateau's problem--find a minimal surface with a given boundary--are classical examples. A more modern motivation, closest to some of my current work, is found in Hilbert's 18th problem regarding *dense* configurations. One version reads: "What is the densest arrangement of congruent shapes in space?" Conway, Goodman-Strauss, and Sloane note that density is too malleable a notion to use in uniquely defining an optimal configuration, but this can be fixed. Even then, there are implicit assumptions in Hilbert's question, e.g. that planar configurations, or functions besides density are already understood. This is not the case. In fact, it is almost a disgrace how little is known about such problems.

Some key results. In previous work, I extended results of Bezdek and Kuperberg for bi-infinite cylinders to the case of finite height cylinders, giving one of very few non-trivial upper bounds for packings of bounded domains, and the only known bound for cylinders that is asymptotically sharp. Using results in affine algebra, the result for bi-infinite cylinders is also sufficient to prove higher-dimensional packing density bounds for poly-cylinders, the first non-trivial exact bound for higher-dimensional objects.

With Hales, I proved that the best known lower bound for the density of regular pentagon packings is, in fact, the global maximizer for density. This was an extension of previous work showing local optimality. This result was extended in other directions with Kallus, reformulating the local result to apply to a generic convex polygon. With Grabner and Brauchart, I defined and exhibited hyperuniform structures in the compact setting. This was the first mathematically rigorous construction of hyperuniformity in a compact setting, extended in work with Grabner, Brauchart, and Ziefle. With Kusner, I showed that the space of thick, length trading links has local minimizers. This improves a result of Coward and Hass by exhibiting Gordian links with respect to the weaker constraints.

Current and future *work*. I propose to continue investigating optimal configurations in a broader sense: with different potentials and with a larger experimental component, both computational and physical. The long-term goal is to generate a bestiary of interesting critical and optimal sets on manifolds with respect to various functions, e.g. discrepancy or injectivity radius, which would be of significant utility and interest in geometric optimization, but also important in fields such as condensed matter physics, integration and approximation theory, and optimal control. This would reasonably subsume the following ongoing projects, aspects of which are appropriate for all levels of research:

Spherical cap discrepancy and other energies. Given a geometric sphere with normalized uniform measure, a local spherical cap discrepancy can be associated with a point set. This describes the largest deviation of the point measure from the uniform measure with respect to spherical caps. I discovered that there is a polynomial-time algorithm to compute that function. Implementing this algorithm, it is possible to explicitly compute the discrepancy of point sets. The discrepancy is intimately connected to the sum of distances via an invariance principle. Other energy computations related to sums of distances, e.g. Reisz energies can also be computed with some (usually greater) ease. I would like to computationally explore the statistics and geometry of configuration spaces under these potentials using implementations of these algorithms.

Morse theory for configuration spaces. A configuration space of a collection of spheres in a container is a subspace of the configuration space of the centers of those spheres. As the radius of the spheres changes, the topology and geometry of the configuration space change exactly at the radius function's critical points. Using classical optimization theory and differential geometry, it is possible to equate that criticality to the existence of a strut measure and a balanced diagram. These critical points can then be classified and the configuration space understood in this context.

Sticky sphere clusters. Recent work in materials science and condensed matter has led to an interest in the enumeration of rigid sticky sphere clusters. These are essentially rigid bar frameworks with an integrality condition on the edge lengths. For clusters containing a small number of spheres, there are algorithmic approaches to generating and checking the allowed configurations. I am currently using a mechanical approach to construct examples of medium size clusters that are poorly behaved with respect to the rules used for current enumeration algorithms; cf. Holmes-Cerfon.

Hyperuniform structures. Consider points generated by a point process on a space, with the empirical measure converging weakly to the uniform measure on the space. For test sets in this space, there is the obvious random variable counting how many points the process has placed in that test set. Low variance across a family of such test sets suggests that there is extra order in the point set. Torquato and Stillinger identified this extra order in the context of thermodynamical ensembles in Euclidean space and called it hyperuniformity. I would like to continue to examine the properties of such ensembles as related to these other projects, e.g. in the compact setting.

Gordian configurations. Examples of Gordian configurations can be found in describing the geometry and topology of spaces of kissing spheres and in attempts to construct a Gordian (un)knot. The first example of kissing spheres is an extension of several previously mentioned projects, but focused on metastable states in the discrete setting. The (seemingly more ancient) second example of Gordian knot construction is additionally inspired by a geometric approach to the Smale conjecture: while it is known (due to Hatcher) that unknotted round circles are isotopic, it is not believed that a physical potential will "unknot any unknot" without hitting singularities.

Interdisciplinary work. I would like to work with materials scientists, biologists, and engineers, at other schools and in industry, to fabricate some of the special structures that appear in the literature and my own work. Known and conjectured critical domains and configurations present some attractive experimental opportunities. The methods used to analyze configurations should be used to make metamaterials and mechanisms, leveraging a better understanding of the inherent geometric properties and defects. Experimentation would also help by producing further mathematical conjectures and methods.