Packing density bounds in higher dimensions.

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Abstract

We'll examine some of what is known about packing problems in higher dimensions, sketch an old argument of Blichfeldt and its generalizations to outer parallel bodies, and describe a new result for cylinders and polycylinders.

Who Cares?

Density and structure theorems have implications for:

- Materials Science
- Condensed Matter Physics
- Grain Growth, Annealing, Foams
- Liquid Crystals
- Coulomb/Minimal Energy Problems
- Constructing Efficient Codes
- Imaging Problems
- Approximation Theory
- ...

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Density

A *packing* of a region $X \subseteq \mathbb{R}^n$ by objects $C_i \subseteq X$ is a family $\mathscr{C} = \{C_i\}_{i \in I}$ with disjoint interiors.

The *upper density* ρ^+ of a packing $\mathscr{C}$ in $\mathbb{R}^n$ will be defined as

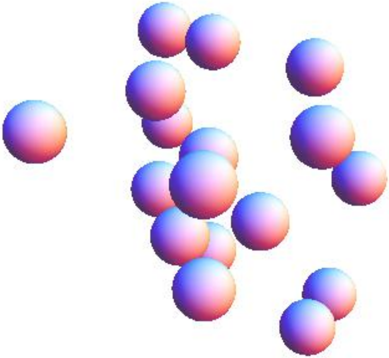
$$\rho^+(\mathscr{C}) = \limsup_{r \to \infty} \sum_{C_i \subseteq B(r)} \frac{\mathrm{Vol}(C_i)}{\mathrm{Vol}(B(r))},$$

where $B(r)$ is the ball of radius r centered at 0.

Density

Density

Example



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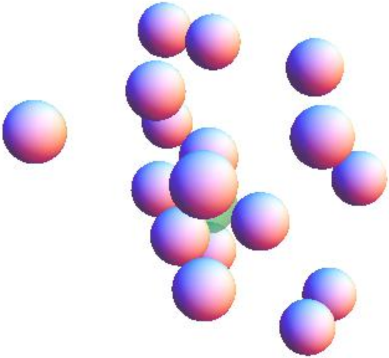

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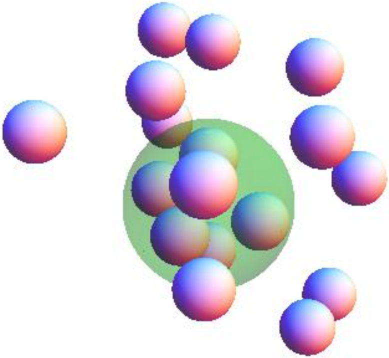

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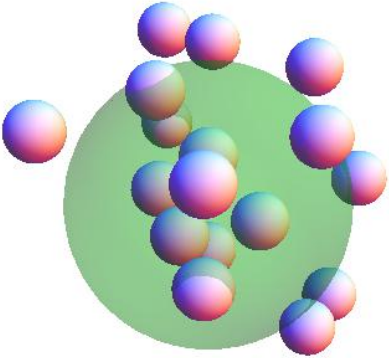

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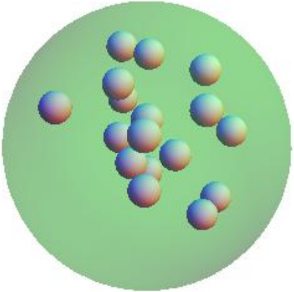

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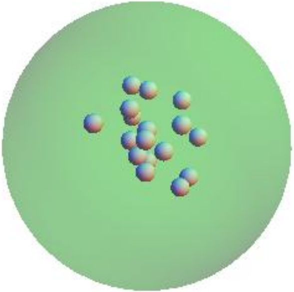

Packing in Higher Dimensions

- 1 Dimension: Easy for the most part, maybe number theory.
- 2 Dimensions : CCCB understood individually, other results.
- 3 Dimensions: Density for spheres known, new pathologies.
- Up to $\approx$ 24 Dimensions: Computer + miracles, sometimes.
- Higher Dimensions: Loose? bounds and conjectures.

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Blichfeldt Gauges

Given a packing of $\mathbb{R}^n$ by congruent objects $\mathscr{C} = \{C_i\}_{i \in I}$, there are a fixed body $C \subset \mathbb{R}^n$ and isometries $\{\phi_i\}_{i \in I}$ of $\mathbb{R}^n$ such that $C_i = \phi_i C$ for all i in I.

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^+$ is a ***Blichfeldt gauge*** for a convex body $C \subset \mathbb{R}^n$ if for any collection of isometries $\Phi = \{\phi_i\}_{i \in I}$ of $\mathbb{R}^n$ where $\mathscr{C} = \{\phi_i C\}_{i \in I}$ is a packing and for all x in $\mathbb{R}^n$,

$$\sigma_\Phi(f)(x) := \sum_{i \in I} f(\phi_i^{-1} x) \leq 1.$$

Remark

Notice that the characteristic function $\mathbf{1}_C$ of C is a Blichfeldt gauge for C. It might be surprising there exist Blichfeldt gauges exist that are not dominated by $\mathbf{1}_C$.

Replacing $\mathbf{1}_C$ with a more general Blichfeldt gauge f lets one replace the characteristic function of the packing $\mathbf{1}_{\mathscr{C}}$ with a diffuse version $\sigma_\Phi(f)$. This new function $\sigma_\Phi(f)$ has the same general characteristics as $\mathbf{1}_{\mathscr{C}}$, is still bounded pointwise by 1 in the ambient space and is uniformly bounded independent of Φ in the moduli space of packings.

Remark

The function f may have greater mass than $\mathbf{1}_C$. This allows one to estimate the volume of the interstices of a packing and thereby bound the packing density.

Example

Blichfeldt used the radial function $2f_0$ where

$$f_0(r) = \begin{cases} \frac{1}{2}(2 - r^2) & : 0 \leq r \leq \sqrt{2} \\ 0 & : \sqrt{2} < r \end{cases}$$

and showed that f_0 is a Blichfeldt gauge. Then, for a packing $\mathscr{C} = \{\phi_i C\}_{i \in I}$ of a cube $t\mathbb{I}^n$ by spheres, the support of $\sigma_\Phi(f)$ is contained in a slightly larger cube $(t + 2\sqrt{2} - 2)\mathbb{I}^n$. A bound on the sphere packing density can then be extracted as follows. From the definition of the Blichfeldt gauge and integrating in spherical coordinates, one finds

$$(t + 2\sqrt{2} - 2)^n \geq |I| \int_{\mathbb{R}^n} f_0 \, \mathrm{dV} = \frac{|I| \operatorname{Vol}(\mathbb{B}^n) 2^{\frac{n+2}{2}}}{n+2}.$$

Blichfeldt Gauges

Example

When density is measured relative to a cube,

$$\delta^+(\mathscr{C}) = \frac{|I|\,\mathrm{Vol}(\mathbb{B}^n)}{t^n} \leq \frac{n+2}{2^{\frac{n+2}{2}}}\left(1 + \frac{2\sqrt{2}-2}{t}\right)^n.$$

By passing to the limit, the bound

$$\delta^+(\mathscr{C}) \leq \frac{n+2}{2^{\frac{n+2}{2}}}$$

holds for any sphere packing in $\mathbb{R}^n$.

Remark

This also holds when $t\mathbb{I}^n$ is replaced with $\mathbb{B}^n_{t/2}$.

Blichfeldt Gauges

Theorem (Blichfeldt)

If g is a Blichfeldt gauge for a body C, then $\delta^+(\mathscr{C}) \leq \mathrm{Vol}(C)/J(g)$ where

$$J(g) = \int_{\mathbb{R}^n} g \, \mathrm{dV}.$$

Theorem (Fejes Tóth–Kuperberg)

If $f(\alpha)$, $\alpha \geq 0$, is a real valued function such that $f(|x|)$ is a Blichfeldt gauge for the unit ball, and C is a convex body with inradius $r(C)$, then for any $\varrho \leq r(C)$

$$g(x) = f\left(\frac{d(x, C_{-\varrho})}{\varrho}\right)$$

is a Blichfeldt gauge for C, where $C_{-\varrho}$ is the inner parallel body of C at distance ϱ.

Blichfeldt Gauges

For better results, do not use f_0, but rather Blichfeldt's modified version f_1.

Definition

The ***modified Blichfeldt gauge*** for $\mathbb{D}^n$ is the radial function

$$f_1(r) = \begin{cases} 1 & : 0 \leq r \leq 2 - \sqrt{2} \\ \frac{1}{2}(2-r)^2 & : 2 - \sqrt{2} \leq r \leq 1 \\ \frac{1}{2}(2-r^2) & : 1 \leq r \leq \sqrt{2} \\ 0 & : r > \sqrt{2} \end{cases}$$

Definition (Fejes Tóth–Kuperberg)

For the two-dimensional gauge f_1 defined above,

$$A_2 := J(f_1)/\operatorname{Vol}(\mathbb{D}^2) = (29 - 16\sqrt{2})/6 = 1.06209\ldots.$$

Blichfeldt Gauges

The results of Fejes Tóth and Kuperberg give an estimate for the density of ***infinite polycylinders*** as follows. Consider a polycylinder as a Minkowski sum $C(t) = \mathbb{D}^{n+2} + t\mathbb{I}^n$ in $\mathbb{R}^{n+2}$ and the gauge $g_t(x) = f_1(d(x, C(t)_{-1}))$, where f_1 is the modified Blichfeldt gauge and $C(t)_{-1}$ is the inner parallel body at distance 1, an n-cube of height t. An estimate of the integral $\int_{\mathbb{R}^{n+2}} g_t \,\mathrm{dV}$ gives a density bound. By integrating g_t over $C(t)_{-1} \times \mathbb{R}^2$ and noticing that contribution from the complement $\mathbb{R}^{n+2} \setminus (C(t)_{-1} \times \mathbb{R}^2)$ is of strictly lower order – it is bounded above by a constant times the $(n-1)$-Hausdorff measure of the boundary $\partial C(t)_{-1} \subset C(t)_{-1}$ – it follows that

$$\delta^+(C(t)) \leq \frac{\pi t^n}{\pi A_2 t^n + O(t^{n-1})}.$$

As t tends to infinity, this gives a bound of $1/A_2 = .941533\ldots$.

Dirichlet Cells

Remark

The packing density is bounded by the density of the densest "cell" in some "nice decomposition" of the packing. For example, Dirichlet (Voronoi) cells can give good results.

- In a packing $\mathscr{C}$, the *Dirichlet cell* D_i associated to an object C_i is the set of points no further from C_i than from any other object C_j, $i \neq j$.
- Dirichlet cells can be considered in $\mathbb{R}^n$ or restricted to $X \subseteq \mathbb{R}^n$..
- Partition the packed region X.
- By definition, $C_i \subseteq D_i$. If Dirichlet cells are generically large, we find a good upper bound on density.

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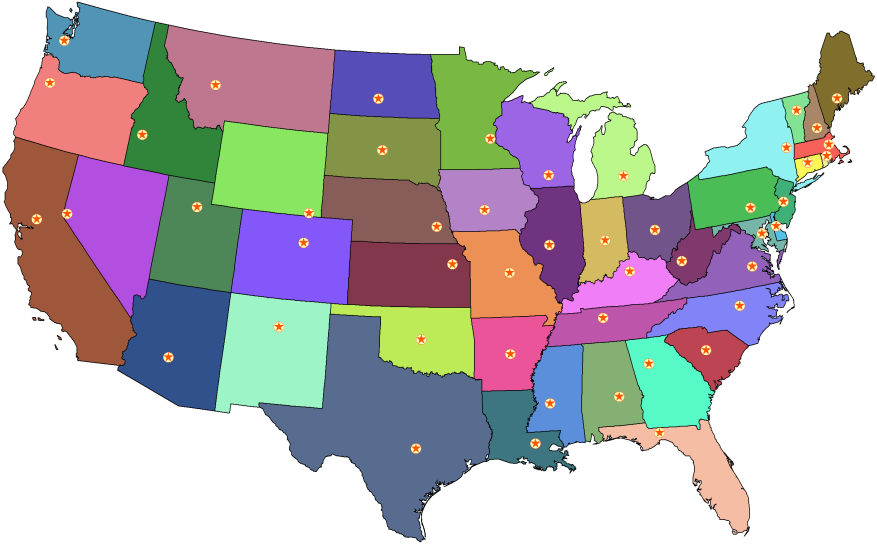

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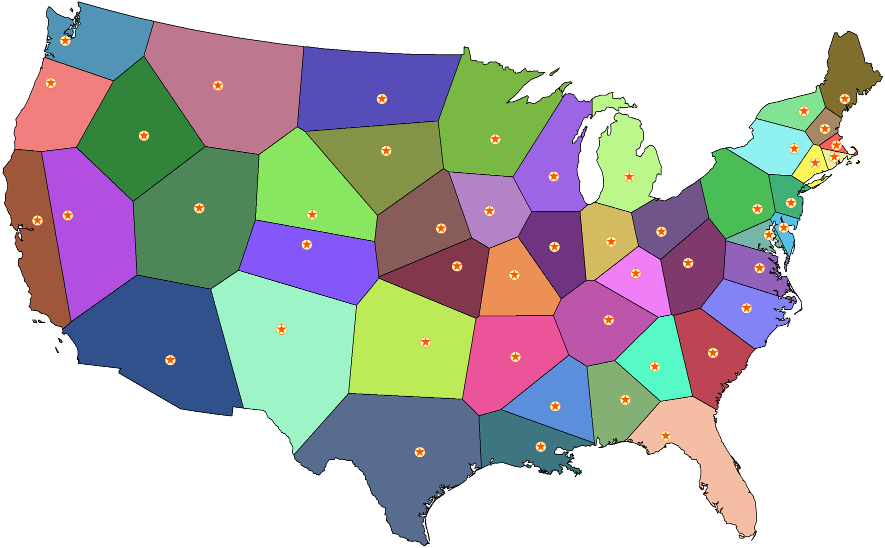
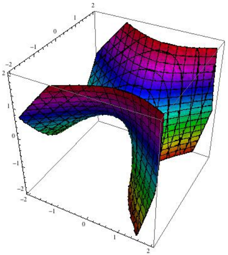
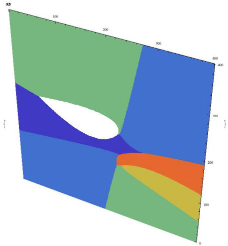


Dirichlet Cells

The Dirichlet cells of a sufficiently nice collection of points (or balls) will be convex polytopes, since they are defined by the intersections of half spaces. In general, they can be much more complex.





Polycylinders

Definition

A ***polycylinder*** is a set isometric to $\mathbb{D}^2 \times \mathbb{R}^n$ in $\mathbb{R}^{n+2}$ (or the Minkowski sum $\mathbb{D}^{n+2} + \mathbb{R}^n$ in $\mathbb{R}^{n+2}$).

Definition

A ***d-flat*** is a d-dimensional affine subspace of $\mathbb{R}^n$.

Definition

The ***parallel dimension*** $dim_{||}\{F, \ldots, G\}$ of a collection of flats $\{F, \ldots, G\}$ is the dimension of their maximal parallel sub-flats.

The notion of parallel dimension can be interpreted in several ways, allowing a modest abuse of notation.

Polycylinders

- For a collection of flats $\{F, \ldots, G\}$, consider their tangent cones at infinity $\{F_\infty, \ldots, G_\infty\}$. The parallel dimension of $\{F, \ldots, G\}$ is the dimension of the intersection of these tangent cones. This may be viewed as the limit of a rescaling process $\mathbb{R}^n \to r\mathbb{R}^n$ as r tends to 0, leaving only the scale-invariant information.
- For a collection of flats $\{F, \ldots, G\}$, consider each flat as a system of linear equations. The corresponding homogeneous equations determine a collection of linear subspaces $\{F_\infty, \ldots, G_\infty\}$. The parallel dimension is the dimension of their intersection $F_\infty \cap \cdots \cap G_\infty$.

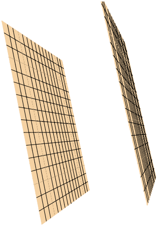
Polycylinders

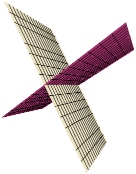
Definition

Two disjoint d-flats are ***parallel*** if their parallel dimension is d, that is, if every line in one is parallel to a line in the other.

Definition

Two disjoint d-flats are ***skew*** if their parallel dimension is less than d.





Polycylinders

Lemma

A pair of disjoint n-flats in $\mathbb{R}^{n+k}$ with $n \geq k$ has parallel dimension strictly greater than $n-k$.

Proof.

By homogeneity of $\mathbb{R}^{n+k}$, let $F = F_\infty$. As F_∞ and G are disjoint, G contains a non-trivial vector $\mathbf{v}$ such that $G = G_\infty + \mathbf{v}$ and $\mathbf{v}$ is not in $F_\infty + G_\infty$. It follows that

$$dim(\mathbb{R}^{n+k}) \geq dim(F_\infty + G_\infty + \text{span}(\mathbf{v})) > dim(F_\infty + G_\infty)$$

$$= dim(F_\infty) + dim(G_\infty) - dim(F_\infty \cap G_\infty).$$

Count dimensions to find $n + k > n + n - dim_{\|}(F_\infty, G_\infty)$. □

Polycylinders

Corollary

A pair of disjoint n-flats in $\mathbb{R}^{n+2}$ has parallel dimension at least $n-1$.

Definition

The ***core*** a_i of a polycylinder C_i isometric to $\mathbb{D}^2 \times \mathbb{R}^n$ in $\mathbb{R}^{n+2}$ v the distinguished n-flat defining C_i as the set of points at most distance 1 from a_i.

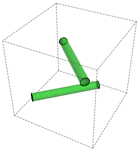
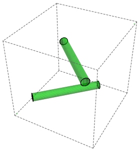
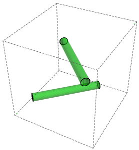
In a packing $\mathscr{C}$ of $\mathbb{R}^{n+2}$ by polycylinders, for every pair of polycylinders C_i and C_j, there are parallel $(n-1)$-dimensional subflats $b_i \subset a_i$ and $b_j \subset a_j$ which define a product foliation

$$\mathscr{F}^{b_i,b_j} : \mathbb{R}^{n+2} \to \mathbb{R}^{n-1} \times \mathbb{R}^3$$

with $\mathbb{R}^3$ leaves that are orthogonal to b_i and to b_j.

Polycylinders

Given a point x in a_i, there is a distinguished $\mathbb{R}^3$ leaf $F_x^{b_i,b_j}$ that contains the point x. The foliation $\mathscr{F}^{b_i,b_j}$ restricts to foliations of C_i and C_j with right-circular-cylinder leaves.

Polycylinders

Definition

In a packing $\mathscr{C}$ of $\mathbb{R}^{n+2}$ by polycylinders, the ***Dirichlet cell*** D_i associated to a polycylinder C_i is the set of points in $\mathbb{R}^{n+2}$ no further from C_i than from any other polycylinder in $\mathscr{C}$.

The Dirichlet cells of a packing partition $\mathbb{R}^{n+2}$, because $C_i \subset D_i$ for all polycylinders C_i. To bound the density $\delta^+(\mathscr{C})$, it is enough to fix an i in I and consider the density of C_i in D_i. For the Dirichlet cell D_i, there is a slicing as follows.

Definition

Given a fixed a polycylinder C_i in a packing $\mathscr{C}$ of $\mathbb{R}^{n+2}$ by polycylinders and a point x on the core a_i, the plane p_x is the 2-flat orthogonal to a_i and containing the point x. The ***Dirichlet slice*** d_x is the intersection of D_i and p_x.

Note that p_x is a sub-flat of $F_x^{b_i,b_j}$ for all j in I.

Polycylinders

For any point x on the core a_i of a polycylinder C_i, the results of Bezdek and Kuperberg for $\mathbb{R}^3$ apply to the Dirichlet slice d_x.

Lemma

A Dirichlet slice is convex and, if bounded, a parabola-sided polygon.

Lemma

Let $S_x(r)$ be the circle of radius r in p_x centered at x. The vertices of d_x are not closer to $S_x(1)$ than the vertices of a regular hexagon circumscribed about $S_x(1)$.

Lemma

Let y and z be points on the circle $S_x(2/\sqrt{3})$. If each of y and z is equidistant from C_i and C_j, then the angle yxz is smaller than or equal to $2\arccos(\sqrt{3}-1) = 85.8828\ldots^\circ$.

Polycylinders

Planar objects satisfying these lemmas have area no less than $\sqrt{12}$. As the bound holds for all Dirichlet slices, it follows that $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ in $\mathbb{R}^{n+2}$. The product of the dense disk packing in the plane with $\mathbb{R}^n$ gives a polycylinder packing in $\mathbb{R}^{n+2}$ that achieves this density. With known results for $n = 0$ and $n = 1$,

Theorem

$\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ *for all natural numbers n.*