Critical packings, rigidity, and the radius function

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Abstract

There are a number of classical problems in geometric optimization that ask for the ”best” configuration of points with respect to some function. We are interested in the relationships between various notions of criticality for such functions on configuration spaces, in particular the injectivity or packing radius. This is not a Morse function, but it has been observed to be Morse-like, in that the topological notion of regularity can be defined in an analogous way. Furthermore, there is a geometric interpretation from rigidity theory that characterizes configurations as critical by the existence of a strut measure.

Best Packings

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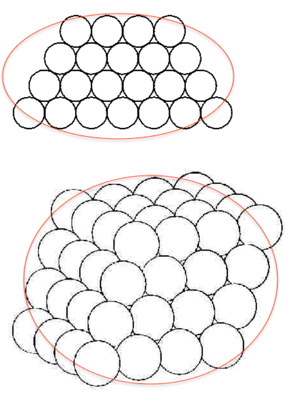
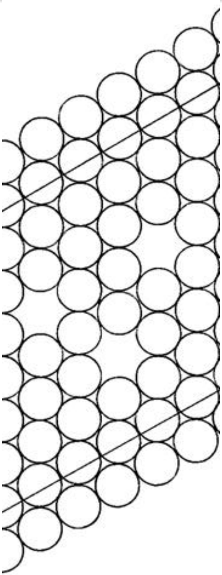


Definition

A *packing* $\mathcal{D}$ of a space X by D is a collection of discs $\{D_i\}$ congruent to D with pairwise disjoint interiors.



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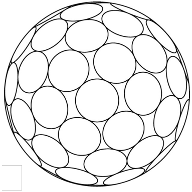
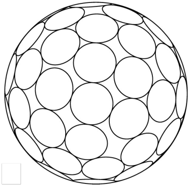
For many types of packings we can define a *density*. Today we will work in a nice compact space: $\mathbb{S}^2$. So consider the volume fraction $\frac{\mathrm{Vol}(\mathcal{D})}{\mathrm{Vol}(\mathbb{S}^2)}$.





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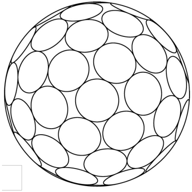
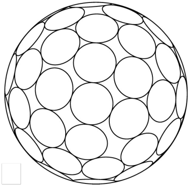




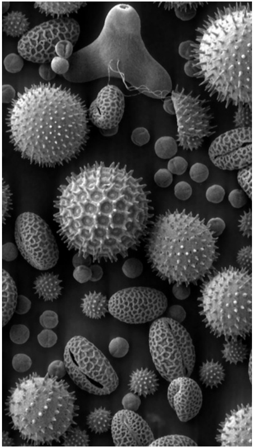
This is essentially the Tammes Problem.

Tammes Problem: P. M. L. Tammes (1930)

Question

What is the maximal radius possible for N equal spheres, all touching a central sphere of radius 1?

The original formulation of the *Tammes problem*: How many spherical caps of angular diameter θ that can be placed without overlap?



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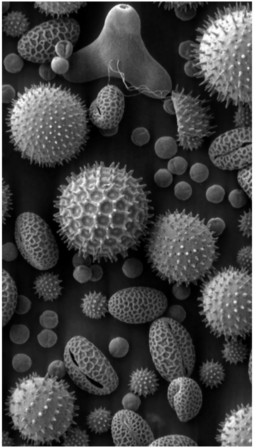

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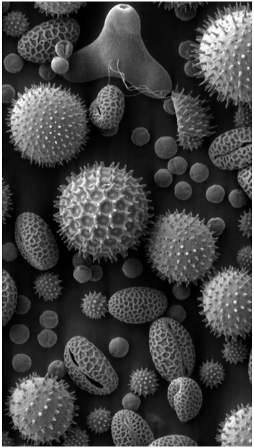
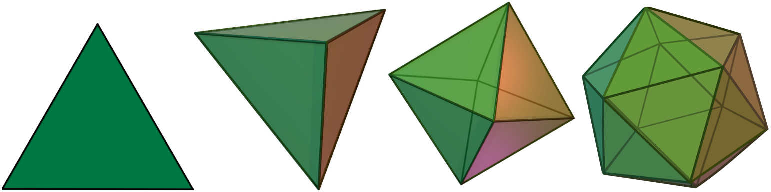


Remark

The maximizing configuration for 5 is not unique.

The Tammes problem was initially solved for $N = 3, 4, 6$ and 12, with configurations of cap centers for $N = 3$ attained by vertices of an equatorial equilateral triangle and for $N = \{4, 6, 12\}$ by vertices of regular tetrahedron, octahedron and icosahedron.



Tammes Problem: L. Fejes-Tóth (1943)

Fejes-Tóth proved the following

Theorem

for N points on the sphere, there are 2 with angular distance

$$\theta \leq \arccos\left(\frac{(\cot(\omega)^2 - 1}{2}\right), \quad \omega = \left(\frac{N}{N-2}\right)\frac{\pi}{6}.$$

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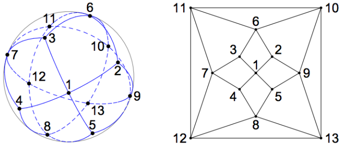
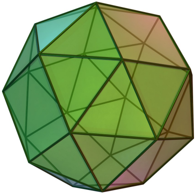
Remark

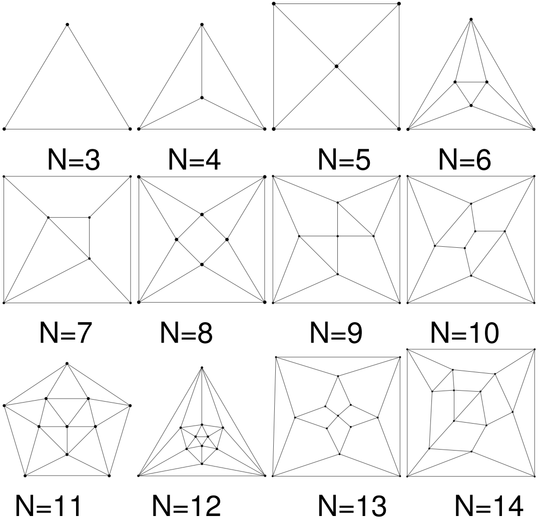
θ is the edge length of a equilateral spherical triangle with the expected area for an element of an $N-$vertex triangulation.

The Tammes problem has been solved exactly for only $3 \leq N \leq 14$ and $N = 24$. It was solved for $N = \{5, 7, 8, 9\}$ by Schütte and van der Waerden in 1951, $N = \{10, 11\}$ by Danzer in his 1963 Habilitationsschrift.



$N = 24$ was solved by Robinson in 1961 showing the configuration of centers were the vertices of a snub cube. Recently the Tammes problem was solved for the cases $N = \{13, 14\}$ by Musin and Tarasov, enumerating all plausible graphs by computer.




N=3
N=4
N=5
N=6
N=7
N=8
N=9
N=10
N=11
N=12
N=13
N=14

Tammes Problem: Approximate Values

N	$\theta(N)$	$r(N)$	Configuration	Source
3	$\delta_3 = \frac{2\pi}{3} = 120^\circ$	$3 + 2\sqrt{3} = 6.4641\ldots$	Equilateral Triangle	Fejes-Tóth (1943)
4	$\delta_4 = \arccos\left(-\frac{1}{3}\right) \approx 109.4712^\circ$	$2 + \sqrt{6} = 4.4494\ldots$	Regular Tetrahedron	Fejes-Tóth (1943)
5	$\delta_5 = \frac{2\pi}{4} = 90^\circ$	$1 + \sqrt{2} = 2.4142\ldots$	[Regular Octahedron minus a vertex]	Fejes-Tóth (1943) Fejes-Tóth (1943)
6	$\delta_6 = \delta_5 = 90^\circ$	$1 + \sqrt{2} = 2.4142\ldots$	Regular Octahedron	Fejes-Tóth (1943)
7	$\delta_7 \approx 77^\circ 52'$	$1.6913\ldots$	[NA]	Schutte and van der Waerden (1951)
8	$\delta_8 \approx 74^\circ 52'$	$1.5495\ldots$	Square Antiprism	Schutte and van der Waerden (1951)
9	$\delta_9 \approx 70^\circ 32'$	$1.3660\ldots$	[NA]	Schutte and van der Waerden (1951)
10	$\delta_{10} \approx 66^\circ 9'$	$1.2012\ldots$	[NA]	Danzer (1963)
11	$\delta_{11} \approx 63^\circ 26'$	$\frac{2}{\sqrt{5+\sqrt{5}-2}} \approx 1.1085085$	[Regular Icosahedron minus a vertex]	Danzer (1963)
12	$\delta_{12} = \delta_{11} \approx 63^\circ 26'$	$\frac{2}{\sqrt{5+\sqrt{5}-2}} = 1.1085\ldots$	Regular Icosahedron	Fejes-Tóth (1943)
13	$\delta_{13} \approx 57.1367^\circ$	–	[NA]	Musin and Tarasov (2013)
14	$\delta_{14} \approx 55.67057^\circ$	–	[NA]	Musin and Tarasov (2015)
24	$\delta_{24} \approx 43^\circ 41'$	$0.5926\ldots$	Snub Cube	Robinson (1961)

Question

We have some solutions for the global maxima for the Tammes Problem, but there could be other interesting configurations. What about other critical points? Are there local maxima?

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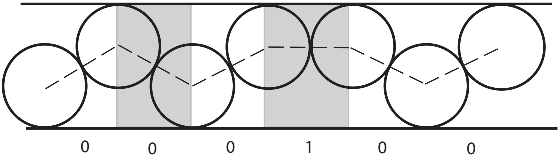

Remark

One model that can be analyzed completely is the quasi-1D packing problem. Such packings have lots of local maxima.

Key Players

Definition

The classical *configuration space* $\mathrm{Conf}(N) := \mathrm{Conf}(\mathbb{S}^2, N)$ of N distinct labeled points on the unit 2-sphere $\mathbb{S}^2$.

Definition

The *reduced configuration space*

$$\mathrm{BConf}(N) := \mathrm{Conf}(N)/SO(3).$$

Assume $N \geq 3$ to avoid degenerate cases.

Definition

Configurations are $\mathbf{U} := (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N)$, where the $\mathbf{u}_j \in \mathbb{S}^2$ are distinct points.

Key Players

Definition

The *injectivity radius function* $\rho : \operatorname{Conf}(\mathbb{S}^2, N) \to \mathbb{R}^+$ assigns a configuration $\mathbf{U} := (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N) \in (\mathbb{S}^2)^N$ the value

$$\rho(\mathbf{U}) := \frac{1}{2}\big(\min_{i \neq j} \theta(\mathbf{u}_i, \mathbf{u}_j)\big),$$

where $\theta(\mathbf{u}_i, \mathbf{u}_j)$ is the angular distance between $\mathbf{u}_i$ and $\mathbf{u}_j$.

Remark

Since ρ is invariant under the action of $SO(3)$, it decends to a well-defined function on $\operatorname{BConf}(N; \theta)$, which we also denote ρ.

Remark

To study critical packings, study the injectivity radius function.

Morse Theory

Morse theory concerns how topology changes for the super level sets of a smooth real-valued function on a manifold.

Definition (super level set)

For $f : M \to \mathbb{R}$, $M^a := \{x \in M : f(x) \geq a\}$ is a superlevel set.

Theorem

Given a smooth function $f : M \to \mathbb{R}$ and an interval $[a, b]$ with compact preimage, and $[a, b]$ contains no critical values. Then M^a is diffeomorphic to M^b.

It is at the *critical values* of the function that the topology of the super level changes.

Morse Theory

The topological change is described by adding up contributions of the *critical points* at critical values.

Definition

A *Morse function* is a smooth function with isolated critical points, non-degenerate, with only one critical point at each critical level.

Definition

Non-degenerate means that the function is C^2 and its Hessian $[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}]$ is nonsingular at the critical point.

Superlevel sets topology changes at a critical level by adding a cell of dimension equal to the *co-index* of the critical point: the number of positive eigenvalues of the Hessian.

Morse Theory

To vary a configuration $\mathbf{U} = (\mathbf{u}_1, ..., \mathbf{u}_N) \in \mathrm{Conf}(N) \subset (\mathbb{S}^2)^N$ along a tangent vector $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$ to $\mathrm{Conf}(N)$ at $\mathbf{U}$. The component $\mathbf{v}_i$ is tangent vector to $\mathbb{S}^2$ at $\mathbf{u}_i$, for $i = 1, 2, ..., N$. For sufficiently small $\mathbf{V}$, we can define a nearby configuration

$$\mathbf{U}\#\mathbf{V} = \left(\frac{\mathbf{u}_1 + \mathbf{v}_1}{|\mathbf{u}_1 + \mathbf{v}_1|}, ..., \frac{\mathbf{u}_N + \mathbf{v}_N}{|\mathbf{u}_N + \mathbf{v}_N|}\right) \in \mathrm{Conf}(N) \subset (\mathbb{S}^2)^N$$

by summing and projecting each factor back to $\mathbb{S}^2$.
In particular, the $\mathbf{V}$-directional derivative of a smooth function f on $\mathrm{Conf}(N)$ at $\mathbf{U}$ is $\frac{d}{dt}|_{t=0} f(\mathbf{U}\#t\mathbf{V})$.

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Definition

$\mathbf{U}$ is a critical point for smooth f provided all its $\mathbf{V}$-directional derivatives vanish at $\mathbf{U}$. The increment $f(\mathbf{U}\#\mathbf{V}) - f(\mathbf{U}) = o(\mathbf{V})$, $o(\mathbf{V})$ is a function on $T_{\mathbf{U}}\,\mathrm{Conf}(N)$ tending to 0 faster than linearly in $\mathbf{V}$.

Remark

The injectivity radius function is not Morse.

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Definition

A configuration $\mathbf{U} = (\mathbf{u}_1, ..., \mathbf{u}_N) \in \mathrm{Conf}(N)$ is *critical for maximizing* ρ provided for every sufficiently small $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$, we have $\max[\rho(\mathbf{U}\#\mathbf{V}) - \rho(\mathbf{U}), 0] = o(\mathbf{V})$. That is, a configuration $\mathbf{U}$ is critical if no variation $\mathbf{V}$ can increase ρ to first order.

Negating *critical for maximizing* means there exists a variation $\mathbf{V}$ which does increase ρ to first order. By the definition of ρ as a min-function, the distance between pairs $(\mathbf{u}_i, \mathbf{u}_j)$ realizing the minimal angular distance $\theta(\mathbf{u}_i, \mathbf{u}_j) = \theta_o$ increases to first order along $\mathbf{V}$.

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There is also a notion of regular value analogous to the smooth case.

Theorem (Topological Regularity)

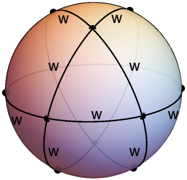
If such a variation exists for all configurations in this $\rho = \theta_o$-level set, then this level is topologically regular: *that is, the variation provides a deformation retraction from $\mathrm{Conf}(N; \theta_o - \varepsilon)$ to $\mathrm{Conf}(N; \theta_o + \varepsilon)$ for some $\varepsilon > 0$.*

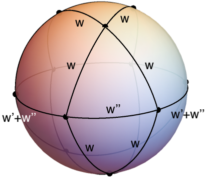
Definition

For $\mathbf{U} \in \mathrm{Conf}(N)[r] = \mathrm{Conf}(N;\theta)$, the *contact graph* of $\mathbf{U}$ is the graph embedded in $\mathbb{S}^2$ with vertices given by points $\mathbf{u}_i$ in $\mathbf{U}$ and edges given by the geodesic segments $[\mathbf{u}_i, \mathbf{u}_j]$ when $d(\mathbf{u}_i, \mathbf{u}_j) = \theta$.

Definition

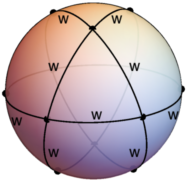
A *stress graph* for $\mathbf{U} \in \mathrm{Conf}(N)[r] = \mathrm{Conf}(N;\theta)$ is a contact graph with nonnegative weights w_e on each geodesic edge $e = [\mathbf{u}_i, \mathbf{u}_j]$.





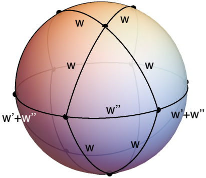
Balanced Graphs

A stress graph associates a system of *tangential forces* to edges $e = [\mathbf{u}_i, \mathbf{u}_j]$ of the contact graph. The forces have magnitude w_e, are tangent to $\mathbb{S}^2$ at each point $\mathbf{u}_i$ of $\mathbf{U}$, and point outward at the ends of each edge.



Definition

A stress graph is *balanced* if the sum of the forces in $T_{\mathbf{u}_i}\mathbb{S}^2$ is zero for all points of $\mathbf{U}$. A configuration $\mathbf{U}$ is *balanced* if it has a balanced stress graph for some choice of non-negative, not everywhere zero weights on its edges.



Remark

Critical for maximizing means the subdifferential cone is empty. For injectivity radius this means the configuration is balanced.

Balanced Graphs

Theorem

Suppose $N \geq 3$. Then for angular diameter $0 \leq \theta < \frac{\pi}{N}$,the space $\mathrm{Conf}(N;\theta)$ is a strong deformation retract of the full configuration space $\mathrm{Conf}(N) = \mathrm{Conf}(N;0)$.

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Proof.

The balanced contact graph on $\mathbb{S}^2$ of a $\frac{\theta}{2}$-critical N-configuration has geodesic edges with angular length θ. In order to balance, its total angular length must be at least 2π, the length of a complete great circle. Thus if $\theta < \frac{2\pi}{N}$, then the total length $N\theta < 2\pi$ and there is no balanced N-configuration in $\mathrm{Conf}(N;\theta)$ and $\frac{\theta}{2}$ is not a critical value for ρ. In this case, a weighted ρ-gradient-flow provides the strong deformation retraction of $\mathrm{Conf}(N)$ to $\mathrm{Conf}(N;\theta)$. □

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Proof.

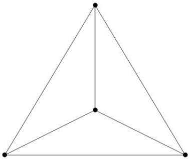
For every critical value $\frac{\theta}{2}$ for ρ, there is a critical stratum defined by the balanced contact graphs achieving edge length θ. The set of critical strata is a semianalytic set. It is a finite Boolean combination of one-sided analytic constraints characterizing the balance criteria over a finite collection of valid contact graphs on the sphere $\mathbb{S}^2$.
Connected components of a semianalytic set are semianalytic and form a locally finite family. The connected components of semianalytic sets are connected via a concatenation of semianalytic arcs. This implies that the injectivity radius ρ is constant on components.
Were it not, there would exist a configuration $\mathbf{U}$ somewhere along the interior of a semi-analytic arc contained in a connected component of the critical stratum, and a variation $\mathbf{V}$ tangent to the arc at $\mathbf{U}$ that increased ρ to first order, contradicting $\mathbf{U}$ is critical for maximizing ρ.
The above implies the set of critical values for ρ on $\mathrm{Conf}(N)$ is a locally finite subset of the half-open interval $(0, \frac{\pi}{2}]$. However, the smallest critical value is $\frac{\pi}{N}$, and so the set of critical values is in fact finite. □

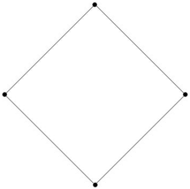
Summary

So there are certain radii that are critical: The topology of the configuration space changes.

These radii also correspond to configurations of points that are force balanced: There exists a non-trivial strut measure on the contact graph that force balances all the vertices.

Such configurations obstruct the r-subgradient flow, which normally can be used to define a strong deformation retraction.





Critical Radii For Small N

Riddle

Draw a cartoon for $N = 3$*. Start with* $\mathbb{S}^1$*.*

Riddle

Draw a cartoon for $N = 3$. Start with $\mathbb{S}^1$.

For 4 points, we have

- $0 \leq \theta < \frac{2\pi}{4}$:

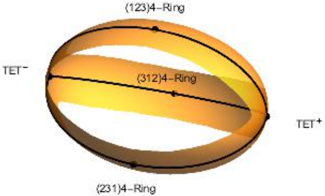
$$\mathrm{Conf}(4, \theta) \simeq \mathrm{Conf}(4, 0)$$

- $\frac{2\pi}{4} < \theta \leq \arccos\left(-\frac{1}{3}\right)$:

$$\mathrm{Conf}(4, \theta) \simeq \{0, 1\}$$

- $\theta > \arccos\left(-\frac{1}{3}\right)$:

$$\mathrm{Conf}(4, \theta) = \emptyset$$



Betti numbers for configuration space $\mathrm{Conf}(N,0)/SO(3)$ can be computed.

N \ k	0	1	2	3	4	5	6	7	8
3	1	0	0	0	0	0	0	0	0
4	1	2	0	0	0	0	0	0	0
5	1	5	6	0	0	0	0	0	0
6	1	9	26	24	0	0	0	0	0
7	1	14	71	154	120	0	0	0	0
8	1	20	155	580	1044	720	0	0	0
9	1	27	295	1665	5104	8028	5040	0	0
10	1	35	511	4025	18424	48860	69264	40320	0
11	1	44	826	8624	54649	214676	509004	663696	362880

Critical Radii For Small N

Remark

The Euler characteristic of the configuration space is $\chi(\mathrm{Conf}(4,0))/SO(3)) = -1$.

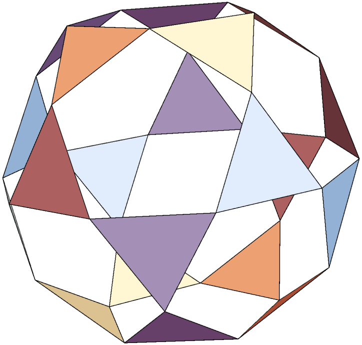
We can pretend that the Morse inequalities work: the indexed sum of critical points of the function $\rho : \mathrm{Conf}(4,0)/SO(3) \to \mathbb{R}$ gives an alternative computation

$$\chi(\mathrm{Conf}(4,0)/SO(3)) = \sum_k (-1)^k \#(\text{ critical points of co-index } = k\,).$$

- 4Ring has symmetry group D_4 of order 8 in $SO(3)$, there are $3 = |S_4/D_4|$ critical points with co-index 1.
- TET has symmetry group A_4 of order 12 in $SO(3)$, there are $2 = |S_4/A_4|$ critical points with co-index 0.

$$\chi(\mathrm{Conf}(4,0)/SO(3)) = 2 - 3 = -1$$

In general, the structure of the critical points and the configuration spaces are not nice.



The highest stratum

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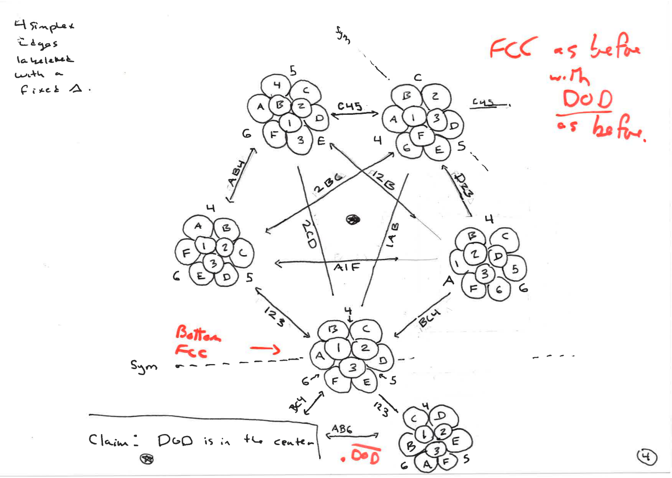

Near FCC

Exploring Configuration Space

Remark

BUT: The configuration space of points on the sphere is a nice space to experiment with. It inherits a nice metric, the tangent space is easy to work with, it is trivial to sample, the symmetries are natural.

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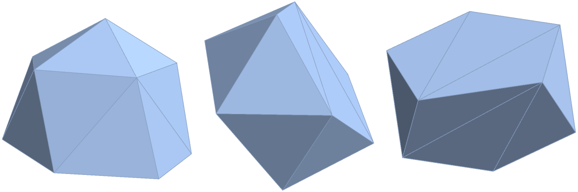
Even I can hack together a sub-gradient descent algorithm for the injectivity radius function. With that, it is a quick step to make some conjectures about local maxima, global maxima, and the distribution of maxima just by exploring the basins of attraction.



Thank you for your attention!

wkusner.github.io

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