An upper bound on packing density for circular cylinders with high aspect ratio.

Wöden Kusner
University of Pittsburgh

March 17, 2013

Abstract

In the early nineties, A. Bezdek and W. Kuperberg used a relatively simple to argument to show a surprising result: The maximum packing density of circular cylinders of infinite length in three-dimensional Euclidian space is exactly $\pi/\sqrt{12}$, the planar packing density of the circle. We modify this result to find a bound on the packing density of finite length circular cylinders. In fact, the maximum density for a capped unit radius cylinder of length t is bounded above by $\pi/\sqrt{12} + 10/t$.

Circle Packing/Comments

Results in Circle Packing

- Toth

A packing of a region $R \subset \mathbb{R}^n$ by objects C_i is a family $\mathfrak{C} = \{C_i\}_{i \in I}$ $\mathbb{R}^n$ with disjoint interiors.
The *upper density* ρ^+ of a packing $\mathfrak{C}$ in $\mathbb{R}^n$ will be defined as

$$\rho^+(\mathfrak{C}) = \limsup_{r \to \infty} \frac{Vol(C_i \cap B_0(r))}{Vol(B_0(r))}$$

where $B_0(r)$ is the ball of radius r, centered at 0.

Circle Packing/Comments

Results in Circle Packing

- Voronoi Cells
- Toth

A packing of a region $R \subset \mathbb{R}^n$ by objects C_i is a family $\mathfrak{C} = \{C_i\}_{i \in I}$ $\mathbb{R}^n$ with disjoint interiors.
The *upper density* ρ^+ of a packing $\mathfrak{C}$ in $\mathbb{R}^n$ will be defined as

$$\rho^+(\mathfrak{C}) = \limsup_{r \to \infty} \frac{Vol(C_i \cap B_0(r))}{Vol(B_0(r))}$$

where $B_0(r)$ is the ball of radius r, centered at 0.

Circle Packing/Comments

Results in Circle Packing

- Voronoi Cells
- Volumes and Densitites
- Toth

A packing of a region $R \subset \mathbb{R}^n$ by objects C_i is a family $\mathfrak{C} = \{C_i\}_{i \in I}$ $\mathbb{R}^n$ with disjoint interiors.
The *upper density* ρ^+ of a packing $\mathfrak{C}$ in $\mathbb{R}^n$ will be defined as

$$\rho^+(\mathfrak{C}) = \limsup_{r \to \infty} \frac{Vol(C_i \cap B_0(r))}{Vol(B_0(r))}$$

where $B_0(r)$ is the ball of radius r, centered at 0.

Review of Bezdek and Kuperberg

Lower bound in density:

- Obvious construction.

Upper bound on density:

Review of Bezdek and Kuperberg

Lower bound in density:

- Obvious construction.

Upper bound on density:

- Fix a packing of Euclidian 3-space with bi-infinite cylinders.

Review of Bezdek and Kuperberg

Lower bound in density:

- Obvious construction.

Upper bound on density:

- Fix a packing of Euclidian 3-space with bi-infinite cylinders.
- Decompose 3-space into "Voronoi" cells associated with said cylinders. The Voronoi cell of a cylinder is the region closer to that particular cylinder.

Review of Bezdek and Kuperberg

Lower bound in density:

- Obvious construction.

Upper bound on density:

- Fix a packing of Euclidian 3-space with bi-infinite cylinders.
- Decompose 3-space into "Voronoi" cells associated with said cylinders. The Voronoi cell of a cylinder is the region closer to that particular cylinder.
- Slice the Voronoi cells perpendicular to the axis of associated cylinder.

Review of Bezdek and Kuperberg

Lower bound in density:

- Obvious construction.

Upper bound on density:

- Fix a packing of Euclidian 3-space with bi-infinite cylinders.
- Decompose 3-space into "Voronoi" cells associated with said cylinders. The Voronoi cell of a cylinder is the region closer to that particular cylinder.
- Slice the Voronoi cells perpendicular to the axis of associated cylinder.
- Show the area of each slice of a Voronoi cell is large.
- *Integrate*

Showing slices of Voronoi cells are large

- The slices are ”parabolic polygons.”

Showing slices of Voronoi cells are large

- The slices are "parabolic polygons."
- Satisfy certain conditions that allow a truncation and replacement to a nicer object.

Showing slices of Voronoi cells are large

- The slices are "parabolic polygons."
- Satisfy certain conditions that allow a truncation and replacement to a nicer object.
- Explicit computation of the area bound of such objects.

Modifications for Finite Cylinders

- The Voronoi cell of a finite cylinder doesn't need to contain the cylinder.

- Solution: We do know that the area is less than one. (was problem 1.) We can integrate areas of slices of
  Also need to work in a bounded area/
  So we have a series of inequalities

Modifications for Finite Cylinders

- The Voronoi cell of a finite cylinder doesn't need to contain the cylinder.
- Solution: Cap the ends of the cylinder.

- Solution: We do know that the area is less than one. (was problem 1.) We can integrate areas of slices of
  Also need to work in a bounded area/
  So we have a series of inequalities

Modifications for Finite Cylinders

- The Voronoi cell of a finite cylinder doesn't need to contain the cylinder.
- Solution: Cap the ends of the cylinder.
- Voronoi slices are no longer parabolic polygons

- Solution: We do know that the area is less than one. (was problem 1.) We can integrate areas of slices of
  Also need to work in a bounded area/
  So we have a series of inequalities

Modifications for Finite Cylinders

- The Voronoi cell of a finite cylinder doesn't need to contain the cylinder.
- Solution: Cap the ends of the cylinder.
- Voronoi slices are no longer parabolic polygons
- Theorem: Slices that are far away from the ends of ANY CYLINDER satisfy conditions that allow the same truncation and replacements done by Bezdek and Kuperberg.

- Solution: We do know that the area is less than one. (was problem 1.) We can integrate areas of slices of
  Also need to work in a bounded area/
  So we have a series of inequalities

Modifications for Finite Cylinders

- The Voronoi cell of a finite cylinder doesn't need to contain the cylinder.
- Solution: Cap the ends of the cylinder.
- Voronoi slices are no longer parabolic polygons
- Theorem: Slices that are far away from the ends of ANY CYLINDER satisfy conditions that allow the same truncation and replacements done by Bezdek and Kuperberg.
- We now don't know the area of some cells, also, the caps contribute volume.
- Solution: We do know that the area is less than one. (was problem 1.) We can integrate areas of slices of
  Also need to work in a bounded area/
  So we have a series of inequalities

Modifications for Finite Cylinders

- The Voronoi cell of a finite cylinder doesn't need to contain the cylinder.
- Solution: Cap the ends of the cylinder.
- Voronoi slices are no longer parabolic polygons
- Theorem: Slices that are far away from the ends of ANY CYLINDER satisfy conditions that allow the same truncation and replacements done by Bezdek and Kuperberg.
- We now don't know the area of some cells, also, the caps contribute volume.
- Solution: We do know that the area is less than one. (was problem 1.) We can integrate areas of slices of
  Also need to work in a bounded area/
  So we have a series of inequalities

Main Theorem

Fix $t_0 = \frac{4}{3}(\frac{4}{\sqrt{3}} + 1)^3 = 48.3266786\ldots.$

Theorem

Fix $t \geq 2t_0$. Fix $B_0(R)$, a ball of radius R centered at 0. Fix a packing $\mathfrak{C}$ of $B_0(R)$ by capped t-cylinders completely contained in $B_0(R - 2/\sqrt{3})$. Then

$$\rho(\mathfrak{C}) \leq \left(\frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}\right).$$

Huristic

We can expand and find a dominating hyperbola, giving

$$\rho(\mathfrak{C}) \leq \pi/\sqrt{12} + 10/t$$

for a packing $\mathfrak{C}$ given by congruent capped cylinders.

Quick Corollaries

- Upper bound on the packing density cylinders/spherocylinders.

Quick Corollaries

- Upper bound on the packing density cylinders/spherocylinders.
- Strict result for ray cylinders.

Quick Corollaries

- Upper bound on the packing density cylinders/spherocylinders.
- Strict result for ray cylinders.
- Bounds for packings of non-congruent unit radius cylinders.