Configurations of Spheres

Wöden Kusner

Institute for Analysis and Number Theory
Graz University of Technology

AIM
September 2016

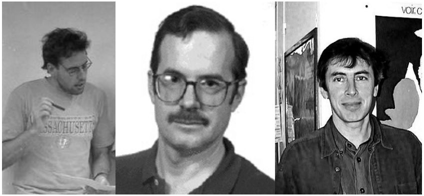
Abstract

In 1694, Newton and Gregory discussed how many non-overlapping unit spheres could be placed in contact with a central unit sphere: Is it 12 or possibly 13? This problem was unresolved until 1953, when Schütte and van der Waerden showed that 12 was the correct answer.

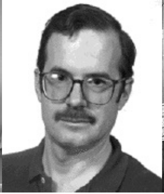
An alternate formulation, related to the Tammes best packing problem, is to consider the configuration space of 13 spheres touching a central sphere parametrized by radius and show that it is empty for radius 1.

This point of view opens up a variety of new configurational problems, namely, how does the geometry and topology of such a configuration space change as the radius is varied?

We will discuss some of the history and the current state of these problems.



Rob Kusner, Jeff Lagarias & Senya Shlosman.

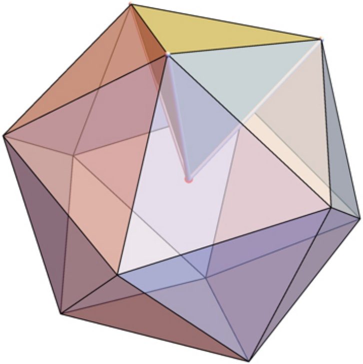




Rob Kusner, Jeff Lagarias & Senya Shlosman.
Tom Hales & Bob Connelly.


The Newton-Gregory Problem

Question

Is the regular icosahedron made of 20 regular tetrahedra?

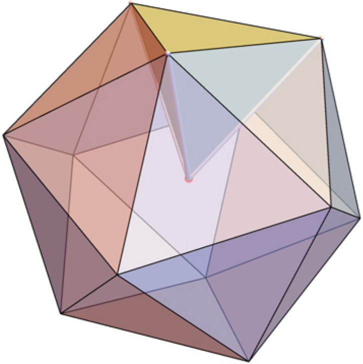


Question

Is the regular icosahedron made of 20 regular tetrahedra?

No! For circumradius 1, we can compute the edge length to be

$$\left(\frac{1}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})}\right)^{-1} = 1.0514\ldots.$$

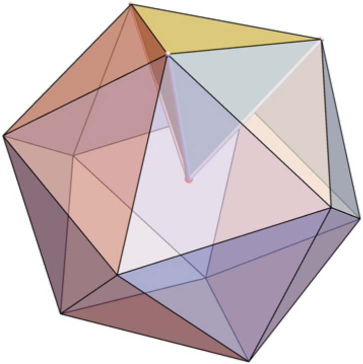


Question

Is the regular icosahedron made of 20 regular tetrahedra?

No! For circumradius 1, we can compute the edge length to be

$$\left(\frac{1}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})}\right)^{-1} = 1.0514\ldots.$$

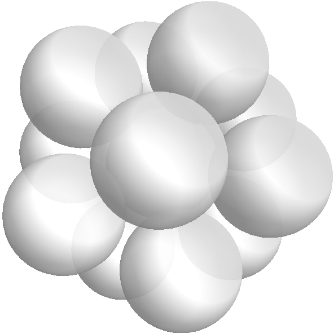


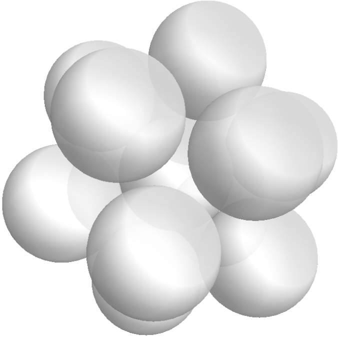
Riddle

Answer the question synthetically.

Remark

If we place unit spheres at the vertices of that regular icosahedron, there is a lot of space between them.

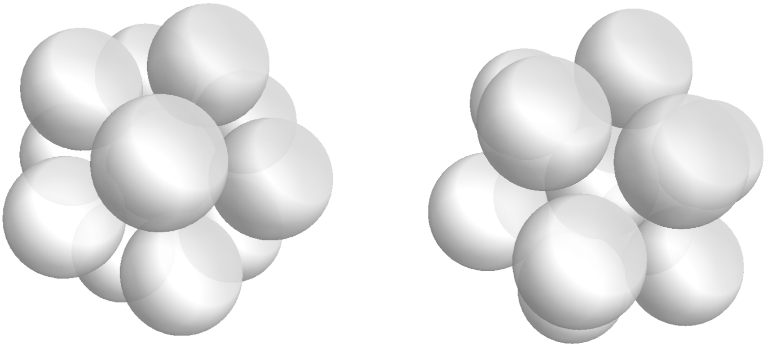




Aristotle: On The Heavens (c. 350 B.C.)

Remark

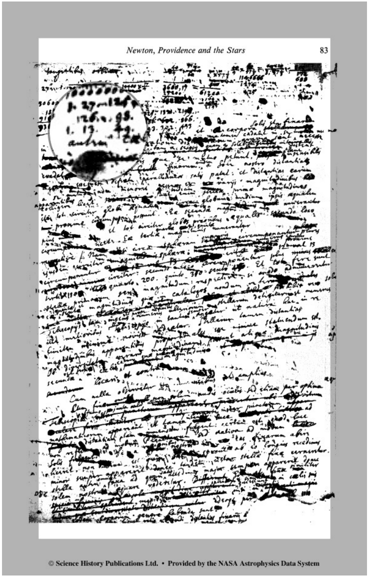
If we place unit spheres at the vertices of that regular icosahedron, there is a lot of space between them.



Question (Newton-Gregory)

Can we fit in a thirteenth sphere?

Newton and Gregory: Principia (revision c. 1694)



For Newton and Gregory, this was a problem of mechanics: Why the fixed stars don't all fall into the sun.

In a draft for the second edition of *Principia*, Newton considers stars of various magnitudes as modeled by arrangements of equal balls.

This method was abandoned, but history lends the names of Newton and Gregory to the problem.

Kepler: Epitome Astronomiae Copernicanae (c. 1620)


LIBER PRIMUS. 35

Si de fixis certius nihil conſtat, videtur illa regio infinita eſſe; nec Sol hic noſter aliud erit, quam una ex fixis, nobis major & clarior viſa, quia propior quam fixa: atq; ita circa quamlibet fixam poterit eſſe talis mundus, qualis circa nos eſt; Vel, quod eodem redit, inter innumerabiles locos in illa infinitâ fixarum congerie, Mundus hic noſter cum Sole ſuo erit unus, nulla re diverſus à locis alijs circa fixas ſingulares: Ut in ſubjecta figura litera M?

Ita quidem Brunus & veterum aliqui. At non ſequitur, ſi centra fixarum non ſunt in eadem ſuperficie Sphærica; propterea regionem per quam ſunt diſperſæ fixæ, eſſe vndiquaque ſibi ſimilem.

Habet enim illa omninò vacuum aliquem ſinum, cavumque ingens, à fixarum agmine, confertim circumfuſo,

C 2 ceu


36 EPITOMES ASTRONOMIÆ,

ceu à muro vel fornice quodam concluſum & circumſcriptum: & in hujus Cavi ingentis complexu, Tellus noſtra cum ſole & ſtellis mobilibus comprehenſa eſt.

Vide ſubjectam figuram.

Quod habes hujus rei Argumentum?

Si Regio fixarum vndique ſimiliter eſſet conſita ſtellis, etiam in vicinia noſtri mundi mobilis, ſic ut ſitus mundi ſolisque noſtri nullam haber ... circumſcriptionem præ ſitu fixæ alicujus ... nobis paucæ aliquæ fixæ ingentes, nec ul ... tot angulos habet Icoſaedron) poſſent ... dem à nobis diſtantiæ, & magnitudinis: ... haud multo plures, haberent jam diſtantiam duplicatam proximarum; aliæ ſuperiores triplicatam, & ſic conſequentes ſemper multipliciorem.

nc appare
duodecim
e omnes

Ac cùm omnium maximæ, tam appareant parvæ, ut vix inſtrumentis poſſint notari aut menſurari: quæ igitur duplo aut triplo &c. diſtarent longius, duplo & triplo apparerent minores, poſitis æqualibus ipſis veris magnitudinibus

Naive Bound

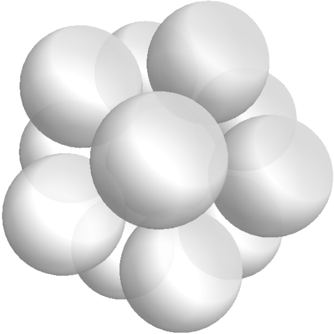
We know some good ways to arrange 12 spheres to touch a central one.

There is a bound given by the solid angle. Centrally projecting to the surface of the central sphere, this region has area

$$2\pi(1 - \cos\frac{\pi}{6}) = .26\ldots$$

giving a bound of

$$\frac{4\pi}{2\pi(1 - \cos\frac{\pi}{6})} = 14.9\ldots.$$



Naive Bound

We know some good ways to arrange 12 spheres to touch a central one.

There is a bound given by the solid angle. Centrally projecting to the surface of the central sphere, this region has area

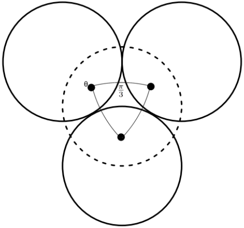
$$2\pi(1 - \cos\frac{\pi}{6}) = .26\ldots$$

giving a bound of

$$\frac{4\pi}{2\pi(1-\cos\frac{\pi}{6})} = 14.9\ldots.$$

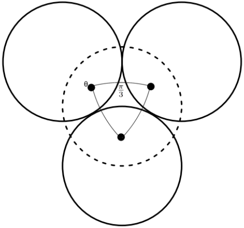
Less Naive Bound

Triangulate the contacts.



Less Naive Bound

Triangulate the contacts.

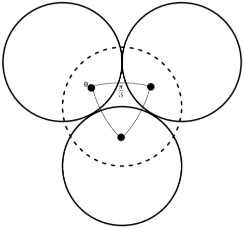


Assumption

Good triangulations exist and the area minimizer is the face of a regular tetrahedron.

Less Naive Bound

Triangulate the contacts.



Assumption

Good triangulations exist and the area minimizer is the face of a regular tetrahedron.

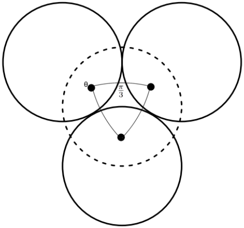
From Euler characteristic: $v - e + f = 2$ and $3f = 2e$ for triangulations of the sphere, giving $f = 2v - 4$.

The minimal face area is

$$T := 3\arccos\left(\frac{1}{3}\right) - \pi = 0.55\ldots.$$

Less Naive Bound

Triangulate the contacts.



Assumption

Good triangulations exist and the area minimizer is the face of a regular tetrahedron.

From Euler characteristic: $v - e + f = 2$ and $3f = 2e$ for triangulations of the sphere, giving $f = 2v - 4$.

The minimal face area is

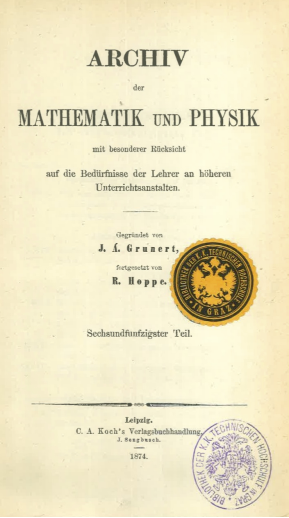
$$T := 3\arccos\left(\frac{1}{3}\right) - \pi = 0.55\ldots.$$

Then for $v = 13$

$$4\pi - 22 * T = 0.43...$$

but for $v = 14$

$$4\pi - 24 * T = -0.66\ldots.$$


ARCHIV

der

MATHEMATIK UND PHYSIK

mit besonderer Rücksicht

auf die Bedürfnisse der Lehrer an höheren Unterrichtsanstalten.

Gegründet von

J. A. Grunert,

fortgesetzt von

R. Hoppe.

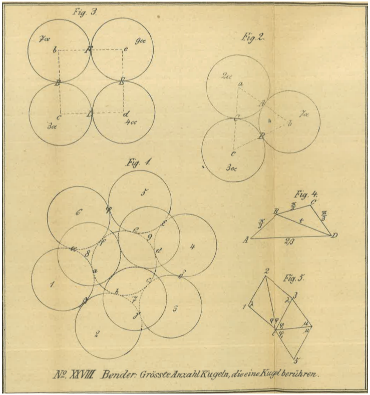
Sechsundfunfzigster Teil.

Leipzig.

C. A. Koch's Verlagsbuchhandlung,

J. Sengbusch.

1874.



Contact graphs: geometric graphs with a vertex for each kissing sphere and edges recording contact.

Schütte and van der Waerden, Leech (c. 1950)

Schütte and van der Waerden further analyzed such graphs and gave conditions on the contact graph showing that 13 unit spheres touching a central unit sphere would induce a graph that was not realizable.

Using similar techniques, Leech gave a proof consisting of only two pages. Much of this brevity seems to come from

> *certain details which are tedious rather than difficult being omitted.*

Schütte and van der Waerden, Leech (c. 1950)

Schütte and van der Waerden further analyzed such graphs and gave conditions on the contact graph showing that 13 unit spheres touching a central unit sphere would induce a graph that was not realizable.

Using similar techniques, Leech gave a proof consisting of only two pages. Much of this brevity seems to come from

certain details which are tedious rather than difficult being omitted.

Theorem (Schütte and van der Waerden)

We can only fit twelve spheres.

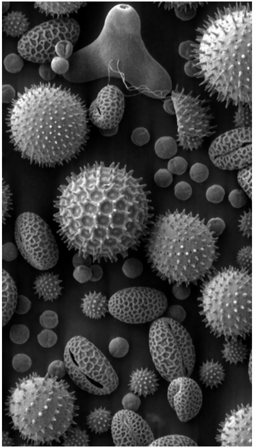
The Tammes Problem

P. M. L. Tammes (1930)

Question

What is the maximal radius possible for N equal spheres, all touching a central sphere of radius 1?

Another formulation of the *Tammes problem*: How many spherical caps of angular diameter θ that can be placed without overlap?



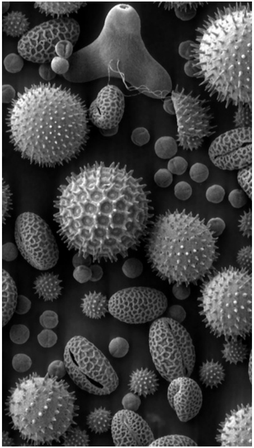
P. M. L. Tammes (1930)

Question

What is the maximal radius possible for N equal spheres, all touching a central sphere of radius 1?

Another formulation of the *Tammes problem*: How many spherical caps of angular diameter θ that can be placed without overlap?

Tammes was studying pollen grains and empirically determined 6 for $\theta = \frac{2\pi}{4}$ but no more than 4 for $\theta > \frac{2\pi}{4}$.

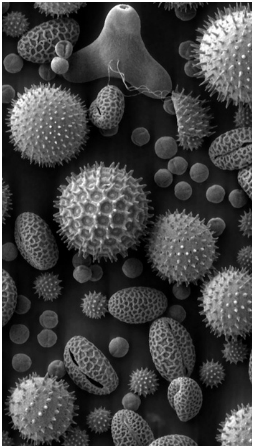


Question

What is the maximal radius possible for N equal spheres, all touching a central sphere of radius 1?

Another formulation of the *Tammes problem*: How many spherical caps of angular diameter θ that can be placed without overlap?

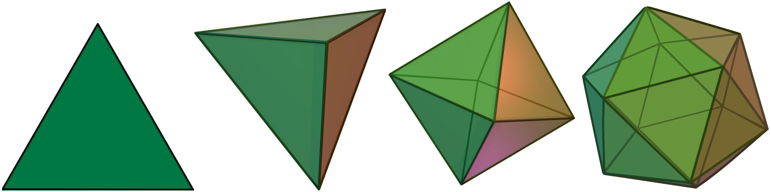
Tammes was studying pollen grains and empirically determined 6 for $\theta = \frac{2\pi}{4}$ but no more than 4 for $\theta > \frac{2\pi}{4}$.



Remark

The maximizing configuration for 5 is not unique.

The Tammes problem was solved for $N = 3, 4, 6$ and 12, with configurations of cap centers for $N = 3$ attained by vertices of an equatorial equilateral triangle and for $N = \{4, 6, 12\}$ by vertices of regular tetrahedron, octahedron and icosahedron.



L. Fejes-Tóth (1943)

Fejes-Tóth proved the inequality

Theorem

for N points on the sphere, there are 2 with angular distance

$$\theta \leq \arccos\left(\frac{(\cot(\omega)^2 - 1}{2}\right), \quad \omega = \left(\frac{N}{N-2}\right)\frac{\pi}{6}.$$

The inequality is sharp for $N = \{3, 4, 6, 12\}$.

L. Fejes-Tóth (1943)

Fejes-Tóth proved the inequality

Theorem

for N points on the sphere, there are 2 with angular distance

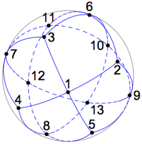
$$\theta \leq \arccos\left(\frac{(\cot(\omega)^2 - 1}{2}\right), \quad \omega = \left(\frac{N}{N-2}\right)\frac{\pi}{6}.$$

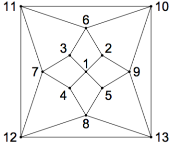
The inequality is sharp for $N = \{3, 4, 6, 12\}$.

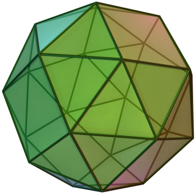
Remark

θ is the edge length of a equilateral spherical triangle with the expected area for an element of an $N-$vertex triangulation.

The Tammes problem has been solved exactly for only $3 \leq N \leq 14$ and $N = 24$. It was solved for $N = \{5, 7, 8, 9\}$ by Schütte and van der Waerden in 1951, $N = \{10, 11\}$ by Danzer in his 1963 Habilitationsschrift.







$N = 24$ was solved by Robinson in 1961 showing the configuration of centers were the vertices of a snub cube. Recently the Tammes problem was solved for the cases $N = \{13, 14\}$ by Musin and Tarasov by generating all candidate contact graphs.

Approximate Values

N	$\theta(N)$	$r_{max}(N)$	Configuration	Source
3	$\delta_3 = \frac{2\pi}{3} = 120^\circ$	$3 + 2\sqrt{3}$ ≈ 6.4641	Equilateral Triangle	Fejes-Tóth (1943)
4	$\delta_4 = \cos^{-1}(-\frac{1}{3})$ $\approx 109.4712^\circ$	$2 + \sqrt{6}$ ≈ 4.4495	Regular Tetrahedron	Fejes-Tóth (1943)
5	$\delta_5 = \frac{2\pi}{4} = 90^\circ$	$1 + \sqrt{2}$ ≈ 2.4142	Regular Octahedron minus one vertex	Fejes-Tóth (1943)
6	$\delta_6 = \frac{2\pi}{4} = 90^\circ$	$1 + \sqrt{2}$ ≈ 2.4142	Regular Octahedron	Fejes-Tóth (1943)
7	$\delta_7 \approx 77.8695^\circ$	≈ 1.6913	[No name]	Schutte and van der Waerden (1951)
8	$\delta_8 \approx 74.8585^\circ$	≈ 1.5496	Square Antiprism	Schutte and van der Waerden (1951)
9	$\delta_9 \approx 70.5288^\circ$	$\frac{1+\sqrt{3}}{2}$ ≈ 1.3660	[No name]	Schutte and van der Waerden (1951)
10	$\delta_{10} \approx 66.1468^\circ$	≈ 1.2013	[No name]	Danzer (1963)
11	$\delta_{11} \approx 63.4349^\circ$	$\frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}-1}$ ≈ 1.1085	Regular Icosahedron minus one vertex	Danzer (1963)
12	$\delta_{12} \approx 63.4349^\circ$	$\frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}-1}$ ≈ 1.1085	Regular Icosahedron	Fejes-Tóth (1943)
13	$\delta_{13} \approx 57.1367^\circ$	≈ 0.9165	[No name]	Musin and Tarasov (2013)
14	$\delta_{14} \approx 55.6706^\circ$	≈ 0.8759	[No name]	Musin and Tarasov (2015)
24	$\delta_{24} \approx 43.6908^\circ$	≈ 0.5926	Snub Cube	Robinson (1961)

Configuration Spaces

Definition

$\mathrm{Conf}(N, r)$ is the configuration space of N non-intersecting balls of radius r on the sphere. (Equivalently, caps of radius θ.)

- For r small, $\mathrm{Conf}(N, r) \simeq \mathrm{Conf}(N, 0)$.
- For r large, $\mathrm{Conf}(N, r) = \emptyset$.

Remark

The Tammes problem is equivalent to finding the maximal r such that $\mathrm{Conf}(N, r)$ is non-empty.

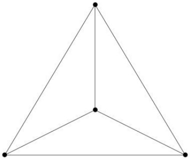
Remark

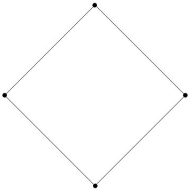
The result for $N = 13$ gives a solution to Newton-Gregory.

Critical radii

There are certain radii that are critical in the sense of (stratified) Morse Theory: The topology of the configuration space changes.

These radii also correspond to configurations of points that are force balanced: There exists a non-trivial strut measure on the contact graph that force balances all the vertices.

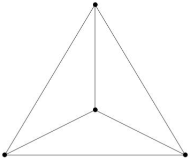


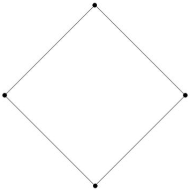


Critical radii

There are certain radii that are critical in the sense of (stratified) Morse Theory: The topology of the configuration space changes.

These radii also correspond to configurations of points that are force balanced: There exists a non-trivial strut measure on the contact graph that force balances all the vertices.





Remark

Such configurations obstruct the r-subgradient flow, which normally can be used to define a strong deformation retraction.

Critical radii

Theorem

To each critical value θ for the radius, there is a force balanced configuration.

Theorem

There are a finite number of critical values for the radius function.

Critical radii

Theorem

To each critical value θ for the radius, there is a force balanced configuration.

Theorem

There are a finite number of critical values for the radius function.

Remark

From this, it is pretty clear that the notion of criticality is one sided. And the Morse theoretic picture is not so nice. But for small N, we can really understand the configuration space.

Critical radii

Theorem

To each critical value θ for the radius, there is a force balanced configuration.

Theorem

There are a finite number of critical values for the radius function.

Remark

From this, it is pretty clear that the notion of criticality is one sided. And the Morse theoretic picture is not so nice. But for small N, we can really understand the configuration space.

Riddle

Draw a cartoon for $N = 3$.

Critical radii for 4 points

For 4 points, we have

- $0 \leq \theta < \frac{2\pi}{4}$:

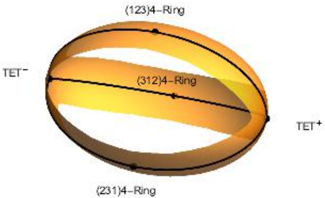
  $$\mathrm{Conf}(4, r(\theta)) \simeq \mathrm{Conf}(n, 0)$$

- $\frac{2\pi}{4} < \theta \leq \arccos\left(-\frac{1}{3}\right)$:

  $$\mathrm{Conf}(4, r(\theta)) \simeq \{0, 1\}$$

- $\theta > \arccos\left(-\frac{1}{3}\right)$:

  $$\mathrm{Conf}(4, r(\theta)) = \emptyset$$



Betti numbers for configuration space $\mathrm{Conf}(N,0)/SO(3)$ can be computed.

$N \backslash k$	0	1	2	3	4	5	6	7	8
3	1	0	0	0	0	0	0	0	0
4	1	2	0	0	0	0	0	0	0
5	1	5	6	0	0	0	0	0	0
6	1	9	26	24	0	0	0	0	0
7	1	14	71	154	120	0	0	0	0
8	1	20	155	580	1044	720	0	0	0
9	1	27	295	1665	5104	8028	5040	0	0
10	1	35	511	4025	18424	48860	69264	40320	0
11	1	44	826	8624	54649	214676	509004	663696	362880

Critical radii for 4 points

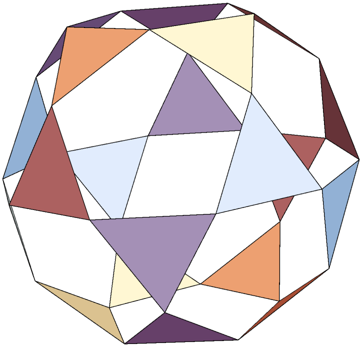
The Euler characteristic of our the configuration space is $\chi(\mathrm{Conf}(4,0))/SO(3)) = -1$.
Using the indexed sum of critical points of the function $\rho : \mathrm{Conf}(4,0)/SO(3) \to \mathbb{R}$ gives an alternative computation

$$\chi(\mathrm{Conf}(4,0)/SO(3)) = \sum_k (-1)^k \#(\text{ critical points of co-index } = k\,).$$

Since the 4-Ring has symmetry group D_4 of order 8 in $SO(3)$, there are $3 = |S_4/D_4|$ critical points of this type with co-index 1; and since TET has symmetry group A_4 of order 12 in $SO(3)$, there are really $2 = |S_4/A_4|$ critical points of this type with co-index 0

$$\chi(\mathrm{Conf}(4,0)/SO(3)) = 2 - 3 = -1$$

In general, the structure of the critical points are not nice.



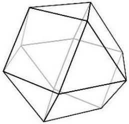
12 points: Frederick Charles Frank

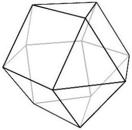
Frank argued that the supercooling of liquids can occur because the common arrangements of molecules in liquids assumes configurations far from what they would assume if frozen.

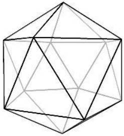
12 points: Frederick Charles Frank

Frank argued that the supercooling of liquids can occur because the common arrangements of molecules in liquids assumes configurations far from what they would assume if frozen.

He claimed that there are exactly 3 configurations of 12 unit spheres next to a given central sphere: The FCC configuration, the HCP configuration, and the dodecahedral configuration.



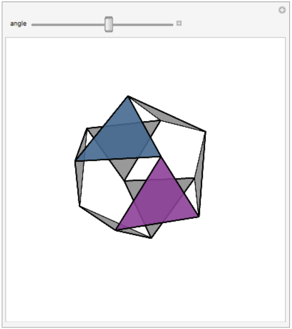


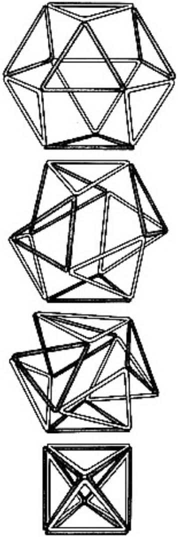


“ Consider the question of how many different ways one can put twelve billiard balls in simultaneous contact with another one, counting as different the arrangements which cannot be transformed into each other without breaking contact with the centre ball?’ The answer is *three*. Two which come to the mind of any crystallographer occur in the face-centred cubic and hexagonal close packed lattices. The third comes to the mind of any good schoolboy, and it is to put one at the center of each face of a regular dodecahedron. That body has five-fold axes, which are abhorrent to crystal symmetry: unlike the other two packings, this one cannot be continuously extended in three dimensions. You will find that the outer twelve in this packing do not touch each other.”

— Frederick Charles Frank

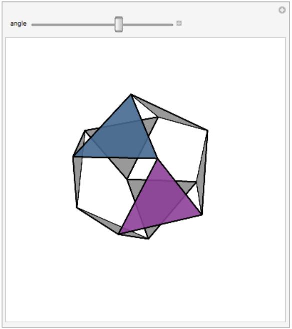
The Jitterbug

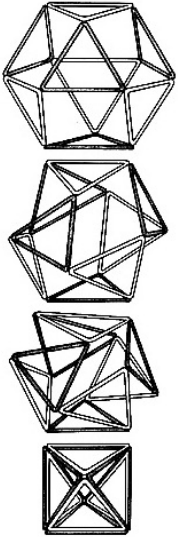




Connects icosahedron to FCC.

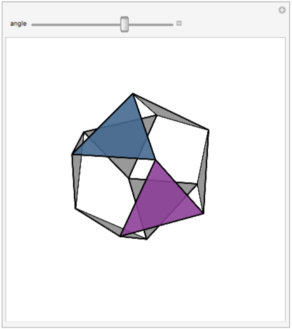
The Jitterbug

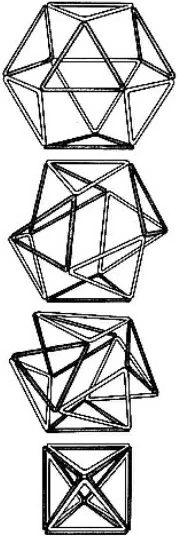




Connects icosahedron to FCC.

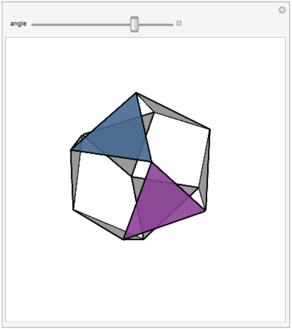
The Jitterbug



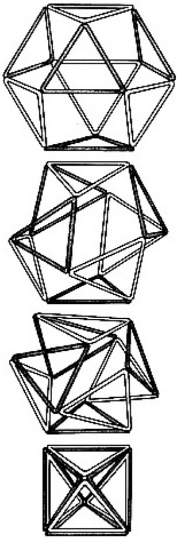


Connects icosahedron to FCC.

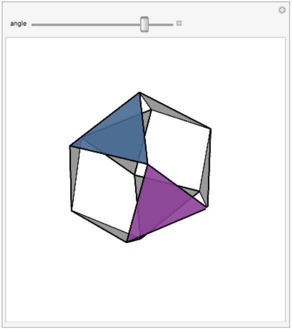
The Jitterbug

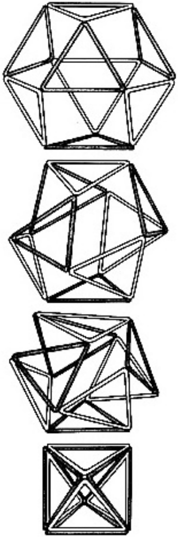


Connects icosahedron to FCC.



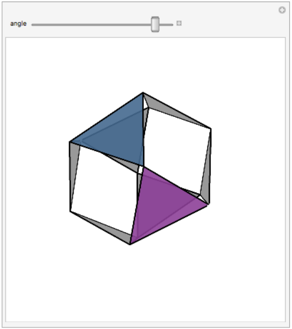
The Jitterbug

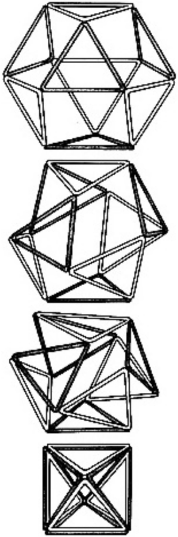




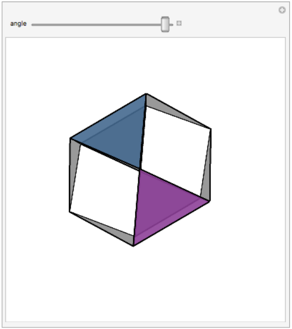
Connects icosahedron to FCC.

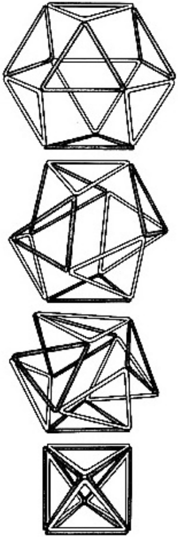
The Jitterbug



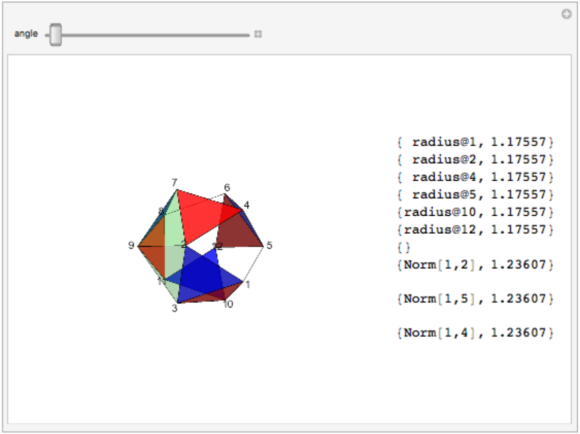


Connects icosahedron to FCC.

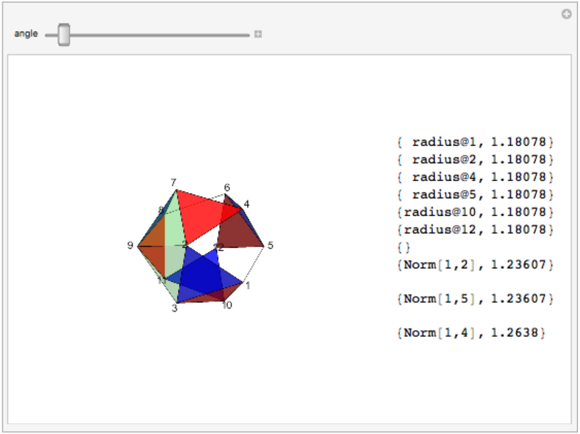




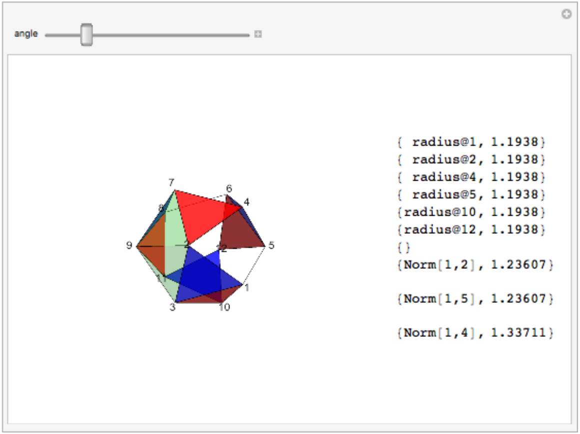
Connects icosahedron to FCC.



Connects icosahedron to HCP.

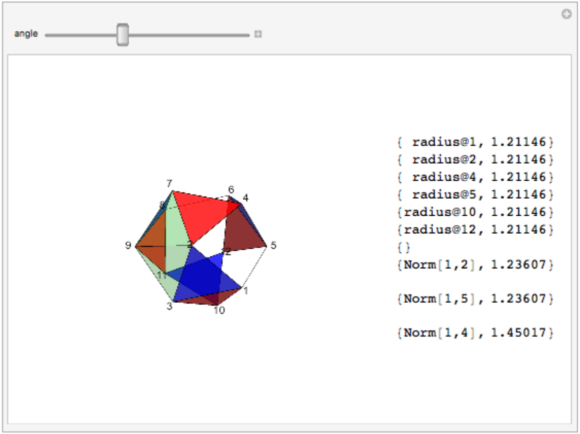


Connects icosahedron to HCP.

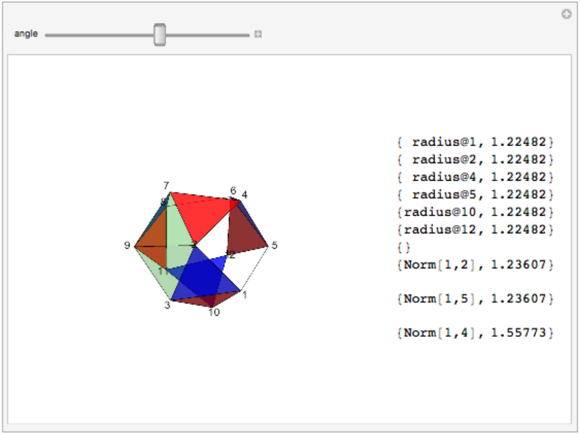


Connects icosahedron to HCP.

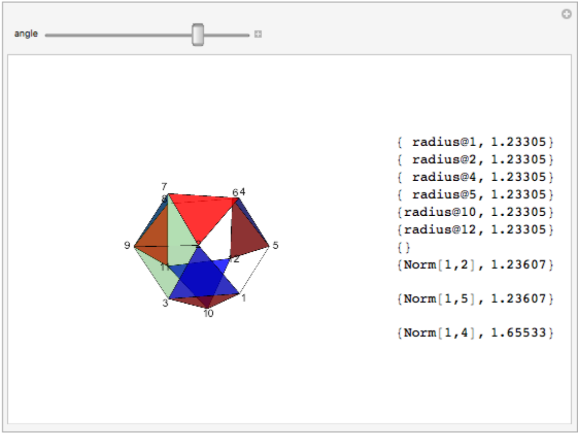
The Kagome Jitterbug



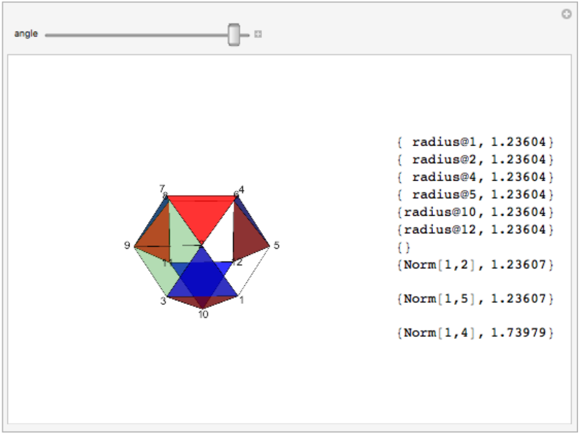
Connects icosahedron to HCP.



Connects icosahedron to HCP.

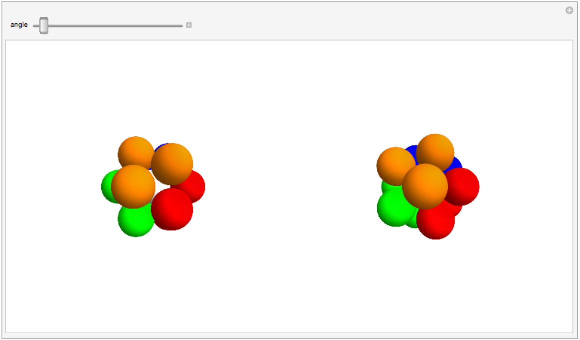


Connects icosahedron to HCP.

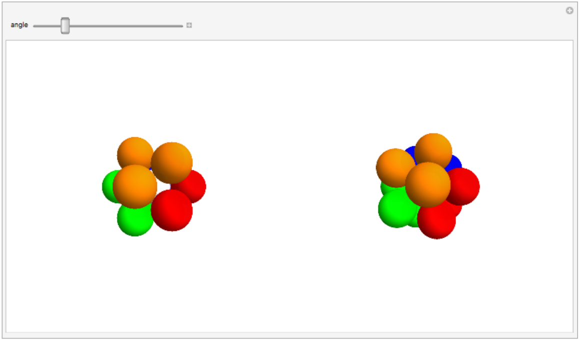


Connects icosahedron to HCP.

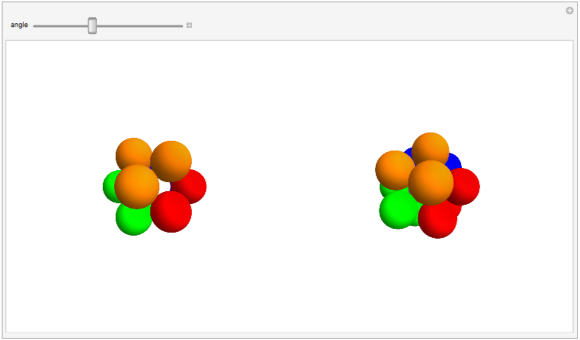
Motions of spheres



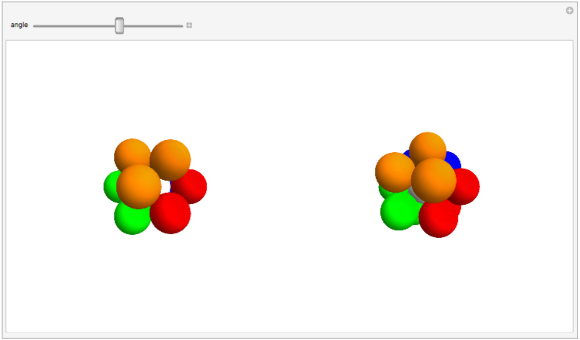
Motions of spheres



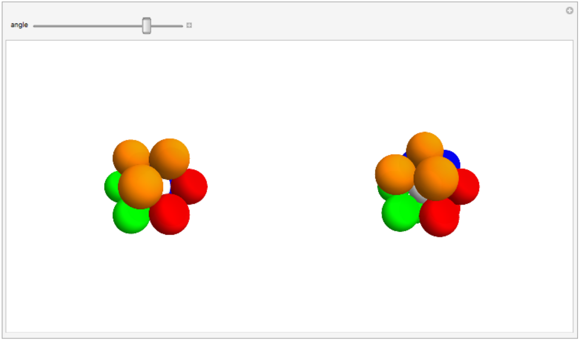
Motions of spheres



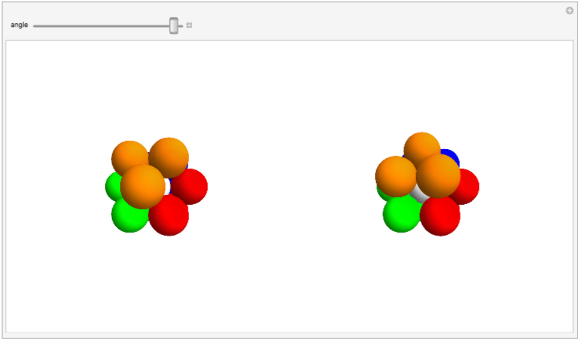
Motions of spheres



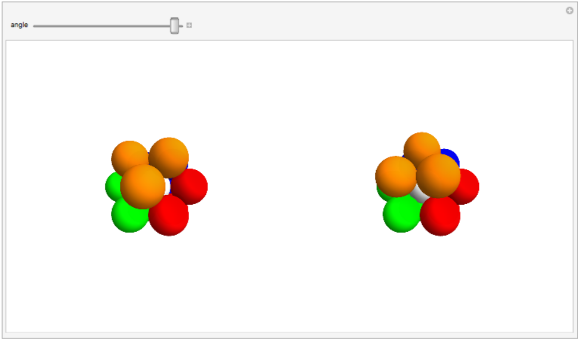
Motions of spheres



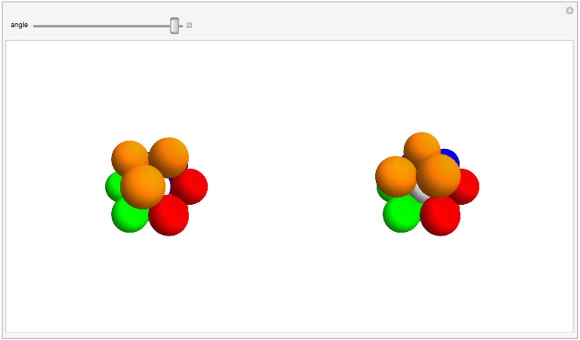
Motions of spheres



Motions of spheres



Motions of spheres



The assertion that there are exactly three arrangements is *false*. There *is* an underlying kernel of truth in this statement: each of three the arrangements above is “remarkable”.

The assertion that there are exactly three arrangements is *false*. There *is* an underlying kernel of truth in this statement: each of three the arrangements above is “remarkable”.

Question (Conway and Sloane)

What rearrangements of 12 *unit spheres are possible via motions maintaining contact with the central unit sphere?*

Motions of spheres

The assertion that there are exactly three arrangements is *false*. There *is* an underlying kernel of truth in this statement: each of three the arrangements above is “remarkable”.

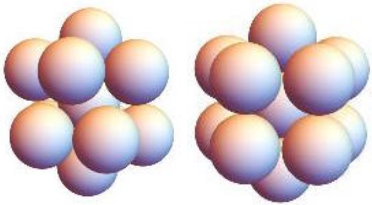
Question (Conway and Sloane)

What rearrangements of 12 unit spheres are possible via motions maintaining contact with the central unit sphere?

They demonstrate that within the component of configuration space of 12 unit spheres connected to the icosahedron, arbitrary permutations of all 12 touching spheres are possible.

An icosahedral “Rubik’s cube”

The equatorial spheres can be moved towards poles and can be rotated to form half-geodesic graphs, like the bars of a birdcage.



The rings of five freely rotate relative to each other. Conway and Sloane note the conjugation action gives all 5-cycles.

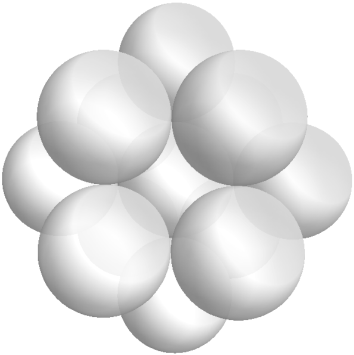
Remark

The fact that all 5-cycles can be produced this way is nontrivial. All 5-cycles generate A_{12}.

Hidden Symmetry

The Jitterbug gives a smooth motion from the icosahedron to the FCC configuration. This has an axis of 4-fold symmetry and 3 layers. Therefore conjugating a rotation with the Jitterbug describes an odd permutation.

With the icosahedral Rubik's cube, this generates all of S_{12}.

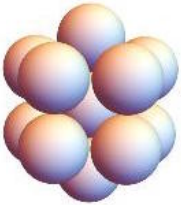


Theorem (Conway and Sloane)

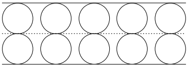
Spheres at the vertices of a regular icosahedron can be arbitrarily permuted.

For $1 + \epsilon > r > 1$,it is possible to get at least A_{12}.

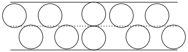
In the “Rubik’s cube”, one can perturb 4 vertical pairs to so the radius can be increased slightly and 1 pair passes. This yields a different type of configuration, but the same idea applies.

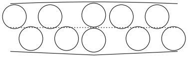


For $1 + \epsilon > r > 1$,it is possible to get at least A_{12}.



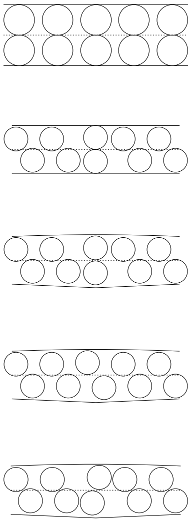
In the “Rubik’s cube”, one can perturb 4 vertical pairs to so the radius can be increased slightly and 1 pair passes. This yields a different type of configuration, but the same idea applies.





For $1+\epsilon > r > 1$,it is possible to get at least A_{12}.

In the “Rubik’s cube”, one can perturb 4 vertical pairs to so the radius can be increased slightly and 1 pair passes. This yields a different type of configuration, but the same idea applies.



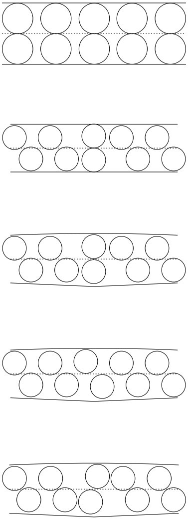
Higher critical radii for 12 points

For $1 + \epsilon > r > 1$,it is possible to get at least A_{12}.

In the "Rubik's cube", one can perturb 4 vertical pairs to so the radius can be increased slightly and 1 pair passes. This yields a different type of configuration, but the same idea applies.

Remark

$\mathrm{Conf}(12, r)$ *near maximal* r *is a set with* $12!$ *components and* $\mathrm{Conf}(12, r)$ *just above radius* 1 *connects them. There must be a critical value above* $r = 1$.



A remark on random jammed packings

:(

Thank you for your attention!

wkusner.github.io

Supported by Austrian Science Fund (FWF) Project 5503