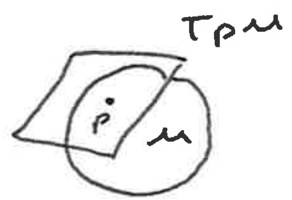
Lecture 9: More on functions and their critical points.

Last time, we defined a function (smooth) $f: M \to \mathbb{R}$ to be <u>Morse</u> if it had no non-degenerate critical points.

9.1. What does this mean?

we need to understand criticality so we introduce the Tangent space T_pM at a point $p \in M$.



We will consider velocities of curves

$$\gamma: [-1,1] \to M$$
$$\gamma(0) = p$$

and with respect to a chart $\varphi: \cancel{M} \to \mathbb{R}^n$

$U \subset M$
$p \in U$

$(\varphi \circ \gamma): [-1,1] \to \mathbb{R}^n$

differentiable at 0

The classes of curves identified by their tangents ~~...~~ at 0 define the tangent space at p

Given a smooth map.

$f: M \to N$, there is

an induced linear map.

$$f_*: T_pM \to T_pN$$

$$f_*(\gamma'(0)) = (f \circ \gamma)'(0).$$

(The differential or pushforward.)

$f: M \to \mathbb{R}$ smooth, p is a

critical point of f if

$$f_* \; T_pM \to T_{f(p)}\mathbb{R} = 0$$

$f(p)$ is a critical value

In local coordinates for T_pM

This is written.

$$\frac{\partial f}{\partial x_i}\Big|_p = 0 \quad \forall\, i = 1 \ldots n$$

Also, there is a bilinear form.

which can be written similarly.

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p = H_f(p)$$

or $f_{xx}(p)$

The Hessian of f at p.

This is non degenerate if

the matrix is non-singular.

The notion of index is well defined in this case.) # of negative eigenvalues.

6.2) Why do we care about non-degenerate critical points?

Morse Lemma.

If p is a non-degenerate critical point for f smooth, there is a local coordinate system about p st

$$f = \sum_{i=1}^{\lambda} - x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2$$

where λ is the index of f at p.

Proof (sketch)

We may shift s.t.

$$f(p) = 0$$

FTC and ~~FTC~~ Integral Remainder Taylor Thm. gives

$$f(x_1 \ldots x_n) = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1 \ldots tx_n)\, x_i \, dt$$

...

and in fact

$$f(x_1 \ldots x_n) = \sum_{i,j} h_{ij}(x_1 \ldots x_n) x_i x_j$$

for some functions h_{ij} smooth...

and in fact . Can symmetrise and can.

$$h_{ij} = \frac{1}{2} \frac{\partial f}{\partial x_i \partial x_j}(0)$$

in some coord chart.
for T_pM.

This can be inductively diagonalised in a nbhd of p, for example via the spectral thm + continuity argmt.

Ex in dim 2.

$$f(x,y) = x^2 h_{11} + 2xy h_{12} + y^2 h_{22}$$

$$\frac{\partial^2 f}{\partial x^2}\Big|_{0,0} = 2h_{11}(0,0)$$

etc...

by a coordinate change

$$h_{11}(0,0) \neq 0$$

cts $\Rightarrow$ $\exists$ nbhd st $h_{11} \neq 0$.

Then we can define a local change of coordinates.

$$\bar{x} = \sqrt{|h_{11}|}\left(x + \frac{h_{12}}{h_{11}} y\right)$$

$$\Rightarrow \bar{x}^2 = |h_{11}|\left(x^2 + 2\frac{h_{12}}{h_{11}} xy + \frac{h_{12}^2}{h_{11}^2} y^2\right)$$

Then for case $h_{11} > 0$

$$\bar{x}^2 = h_{11} x^2 + 2h_{12} xy + \frac{h_{12}^2}{h_{11}} y^2$$

Since

$$f = \underbrace{x^2 h_{11} + 2xy\, h_{12}} + y^2 h_{22}$$

$$\bar{x}^2 - \frac{h_{12}^2}{h_{11}} y^2 + h_{22} y^2$$

proves the same for other cases, higher dimensions.

Cor: non degenerate critical points are isolated.

6.3 A One parameter subgroup of diffeomorphisms:

of a manifold M is a smooth map

Φ $\varphi: \mathbb{R} \times M \longrightarrow M$

such that: for each $t \in \mathbb{R}$

Φ $\varphi_t : M \to M$ defined as $\varphi_t(p)$

$= \varphi_t(t, p)$ is a diffeomorphism

of M onto M <u>and</u>

for all $t, s \in \mathbb{R}$ $\varphi_{t+s} = \varphi_t \circ \varphi_s$

This defines a vector field X on M [for smooth $f: M \to \mathbb{R}$ by

$$X_q(f) = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h}$$

So this is an assignment of a function to each point q of M ... so it takes a directional derivative of f...

. X <u>generates</u> the map φ.

Lemma

A smooth vector field on M which vanishes outside of K compact $\subset M$ generates a unique 1p grp of diff. of M.

Proof: Consider smooth curve.

$\gamma : t \longmapsto \gamma(t) \in M$

with velocity. $\frac{\partial \gamma}{dt} \in T_{\gamma(t)} M$

~~defines~~ Characterized by.

$$\frac{d\gamma}{dt} f = \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}$$

Let $\overset{\Phi}{\varphi}$ be a 1p grp of d. generated by a vect field X as in the Lemma.

Then for fixed q

$$t \longmapsto \overset{\Phi}{\varphi}_t(q) \quad \text{satisfies.}$$

$$\frac{d \overset{\Phi}{\varphi}_t(q)}{dt} = X_{\cancel{\varphi_t(q)}}$$

$$\cancel{X_{\varphi(t)}}$$

$$X_{\varphi_t(q)}$$

$$\overset{\Phi}{\varphi}_0(q) = q$$

Since.

$$\frac{d\varphi^{\Phi}_t(q)}{dt}(f) = \lim_{h\to 0} \frac{f_{t+h}(q) - f(\varphi_t(q))}{h}$$

$$= \lim_{h\to 0} \frac{f(\varphi^{\Phi}_h(\varphi_t q)) - f(\varphi^{\Phi}_t(q))}{h} = X^{\Phi}_{\varphi_t(q)}(f)$$

Exist unique of solution.. (Picard–Lindelöf applied generally..).

has a unique solution

$\Rightarrow$ that smoothly depends on initial conditions..

$\Rightarrow$ for all $q \in M$, there is a neighborhood U

and $\varepsilon > 0$ st $\frac{d\varphi^{\Phi}_t(q)}{dt} = X^{\Phi}_{\varphi_t}(q)$ $\varphi^{\Phi}_0(q) = q$

has a unique smooth solution for $q \in U$

$|t| < \varepsilon$

Since X is compact

Since K is compact, there is a minimal $\varepsilon > 0$

for such neighborhoods with on K.

~~and for~~

for $q \notin K$, let $\varphi_t^{\Phi}(q) = q$

$\Rightarrow$ There is a unique solution $\varphi_t^{\Phi}(q)$ $t < \varepsilon$

smooth (in t and q)

for all q in M.

note. $\varphi^{\Phi}_{t+s} = \varphi^{\Phi}_t \circ \varphi^{\Phi}_s$ for $|t|, |s|, |t+s| < \varepsilon$

So φ^{Φ}_t is a diffeo for $|t| < \varepsilon$

but we can iterate this process by LNP...

for $t > \varepsilon$ by. $\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon + (t - \lfloor \frac{t}{\varepsilon} \rfloor \varepsilon)$

steps.. $\lfloor \frac{t}{\varepsilon} \rfloor +$

$\boxed{\varepsilon}$

6.4 ~~Homotopy~~ Theorem:
Diffeomorphism

For $f: M \to \mathbb{R}$, we have

Sublevelsets

$$M_a := f^{-1}(-\infty, a]$$

$$= \{p \in M : f(p) \leq a\}.$$

~~Given~~ If ~~...~~ f, smooth function

$M \to \mathbb{R}$, ~~...~~

$a < b$ and ~~...~~ $f^{-1}[a,b]$ is

Compact and contains no critical points of f

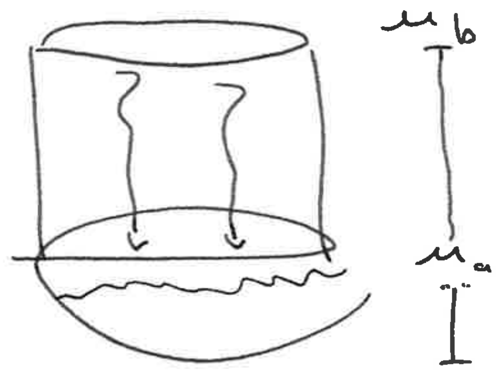
Then $M_a \cong M_b$

Idea:

$f^{-1}(b)$

$-\nabla f$ $\downarrow$

$f^{-1}(a)$



M admits a metric ...

need to define $\langle x, y \rangle$ the inner product of vectors tangent

So we can define

∇f, which is

Characterized by:

$$\langle X, \nabla f \rangle = Xf$$

The directional derivative of f along X. for any of X.

$\nabla f =$ (locally)

$\nabla f \cdot v$

Note ∇f vanishes at the critical points of f.

If $\gamma : \mathbb{R} \to \mathbb{R}$ is curve with velocity vector $\frac{d\gamma}{dt}$ then.

$$\left\langle \frac{d\gamma}{dt}, \nabla f \right\rangle = \frac{d f\circ\gamma}{dt}.$$

Define a smooth function.

$$\rho : M \to \mathbb{R}.$$

that is $\frac{1}{\langle \nabla f, \nabla f \rangle}$ on $f^{-1}[a,b]$ o ~~nbhd~~.

nbhd of a (cmpt) subset of ...

Then vf X defined as.

$$X_q = \rho(q)(\nabla f)_q$$

Satisfies the Lemma for 1p ~~set~~ grps.

(vanishes outside compact set)
smooth...

$\Rightarrow \exists$ 1p $\overline{\Phi}$ $\varphi_t : M \to M$
grp of diffeo.

For fixed q, consider

$$t \to f(\varphi_t(q))$$

If $\varphi_t(q) \in f^{-1}[a,b]$

Then.

$$\frac{d f(\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, \nabla f \right\rangle = \langle X, \operatorname{grad} f \rangle$$

$$= 1$$

$$\Rightarrow t \mapsto f(\varphi_t(q))$$

is lin with dh 1 for $f(\varphi_t(q))$ in $[a,b]$

$\varphi_{b-a} : M \to M$ is a diffeo.

$M^a \to M^b$.

Furthermore..

$r_t : M_b \to M_a$

$$r_t(q) \begin{cases} q & \text{if } f(q) \le a. \\ \varphi_{\underbrace{t(a-f(q))}}(q) & a \le f(q) \le b \end{cases}$$

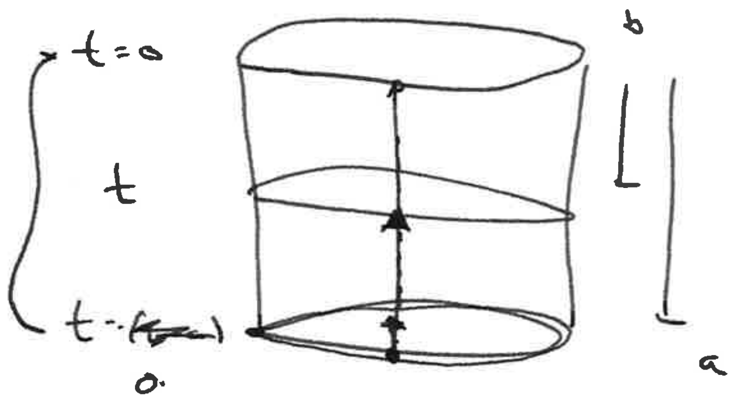
Then...

$$\begin{cases} r_0 = id. \\ r_1 \text{ is a retract of } M^b \to M^a. \end{cases}$$

retraction.

$\varphi_{(b-a)} : M \to M$ is diffeo.

$M^a \to M^b$



diffeos.

Lecture 10: Configuration Space and Morse-like results.

10.1) For smooth functions we have the diffeomorphism result that says:

$f: M \to \mathbb{R}$

- $a < b$
- $f^{-1}([a,b])$ cpt
- no critical values in $[a,b]$

$\Rightarrow M_a$ diffeo M_b

need to be rewritten out of order / missing pieces

There is also a handle attachment thm (not proved) for smooth

$f: M \to \mathbb{R}$

- p - nondegenerate critical point
- index γ, $f(p) = c$
- $\exists\ \varepsilon > 0$ st $f^{-1}([c-\varepsilon, c+\varepsilon])$ cpt, contains no critical pts except c

~~Then~~ $\Rightarrow M^{c+\varepsilon}$ has homotopy type $M^{c-\varepsilon} \bigcup_g \gamma$-cell

Morse type inequalities

c^r = # critical pts of index r

Then the homotopy type changes by attaching various cells of index r as we "climb" M via the Morse function.

$$\Rightarrow \sum (-1)^r c^r = \chi(M)$$

and in greatly considered

$$c^r \geq b_r(M)$$

the rth Betti numbers of M

[or the ranks of the rth homology grp.]

The key point for us is that the ~~Betti~~ Betti numbers can be computed for many spaces we like.

10.2. Betti numbers for $B\,\mathrm{Conf}(N)$

- Configuration space of pts on S^2

$$B\,\mathrm{Conf}(N) \longrightarrow \mathrm{Conf}(N-1, \mathbb{R}^2) / SO(2)$$

via Stereographic projection from

$N \geq 3$, ... we have a corresponding fact.

That allows us to ~~compute~~ ~~Betti numbers~~ from ~~the known~~ Betti numbers

as $P(\mathrm{Conf}\ N-1, \mathbb{R}^2) / P(SO(2))$

where.

$$P(\mathrm{Conf}(n-1, \mathbb{R}^2)) = (1+t)(1+2t)(\cdots)(1+(n-2)t)$$

$$P(SO2) = (1+t)$$

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pictures are punctured planes and the induction

up...

---

can use this idea to describe extensions

and induction again for configuration spaces in general..

for example the $K(\pi,1)$ type of config space

lemma for the fact ~~that the space config~~

— ~~follows from the other~~. that such spaces

really change nicely..

So . . .

	$r=0$	$r=1$	$r=2$ . . .	$r=3$
$n=3$	1	0	0	0
$n=4$	1	2	0	0
$n=5$	1	5	6	0
. . .	1			
	⋮			$(n-2)!$

[illegible]

~~N=2 fact~~ 1

10.2)

A min type function is a parametrized family functions.

$$f(x, p) : X \times M \longrightarrow \mathbb{R}$$

X ~~set~~ parameter space,

compact ....

For our purposes, finite discrete.

f smooth/smoothable for each parameter.

Our function, the injectivity radius

$= \cancel{\min_{i,j} d(p_i, p_j) \quad i \neq j}$

~~Conf(U) ⟶~~

$$\rho(\mathcal{U}) = \min_{\substack{i \neq j \\ u_i, u_j \in \mathcal{U}}} d(u_i, u_j)$$

$$\mathrm{Conf}(\mathcal{U}) \longrightarrow \mathbb{R}$$

is of this type.

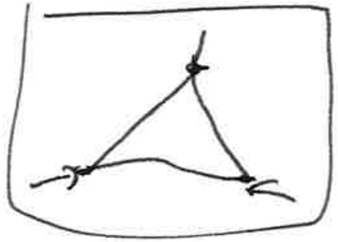
In general, topological regularity is characterized by the non-emptiness of a ~~...~~ intersection gradient cone of this type.

Cor: $\exists$ topology only changes at critical points x ~~on~~ ~~...~~ only occur when the gradient cone is empty, i.e.

$$\sum_x \omega_i \, df_{x_i}\Big|_{x_u} = 0 \qquad v_i > 0.$$

From our picture for injective vector, this is as giving some collection $\circ$ is not possible.

This ~~...~~

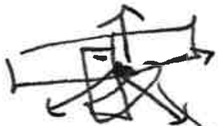


i.e. there is some other cone that intersects..

i.e. no elt in the positive span

For the case of injectivity radius

or ~~target~~ the convex can be empty

 definitions..

$\sum_i w_i \, df_{p_i}|_x = 0$

positive vertex for all

objects..

$\mathcal{U} \in \mathrm{Conf}(N)$

with $\rho(\mathcal{U}) \geq \Theta$

There is a contact graph.

vertices: $u_i \in \mathcal{U}$

edges $u_i u_j \iff d(u_i, u_j) = \Theta$

edges may be assigned a

positive weight w_i, and

if the associated graph

is force balanced..

$\iff$ ... is empty..

$\Rightarrow$ the only rel'd

for non-topologically ...

which are exactly the

free boundary ...

of pts..

Examples...

Thm: Suppose $N \geq 3$.

Then for ~~d(u_i, u_i)~~

$0 < \theta < \frac{\pi}{2}$.

$\mathrm{Conf}(N, \theta)$ is a def retract

of ... to $\mathrm{Conf}(N, 0)$

---

proof: first no bound ...

Each by pushing....

For $N > 4$ ...

$\mathrm{Conf}(N, \theta)$ small

$\Rightarrow$ pitch correct

Example $N = 4$

.

.

.

Strategy for tones maxima

Table .

.

.

.

Algorithm for searching ... by evolution of

force configurations....

Local Maxima..

rings...

rigidity?, ~~cost~~ control conditions

and weights..

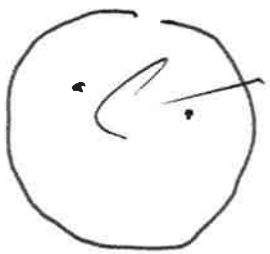
candidate graph sorting..

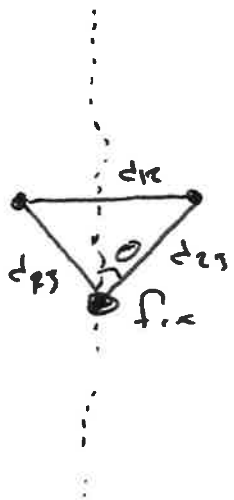
What is this!

this is a free hand [illegible].

I.e. for injectivity radius, it is ~~a variation in~~ the intersection of the convex cones of motion that increase extended dist between each pair of points.

Picture!: 3 pts.





6 dim.
4 dim space. fix 3
3 dim space.. fix θ
given by distance..

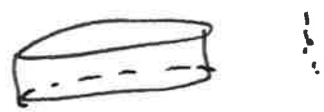
Cone for

d_{12} is upper 12 ...

etc..

So

topological regular value $\iff$ ~~diffeomorphic~~ ~~neighborhoods~~ ~~for the~~ neighborhoods of value have ~~diffeomorphic~~ sublevel sets that deformation retract to each other.



So critical ~~points~~ values are where the topology changes...

For the topological regular value, this deformation retraction is defined by the ~~evolution~~ flow of some vector field defined by f, the gradient flow as in the last lecture...

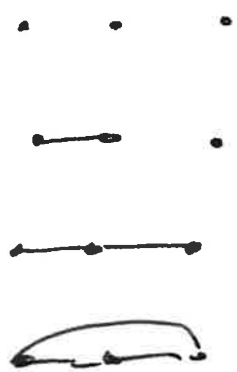
So the topology can only change at an obstruction to gradient flow.

For a min type function, this is a ~~...~~ given by a cone, ~~...~~ that is the ~~...~~ intersection of all vertical cones that increase f at the point $x \in X$.

1) ~~Category~~ Classify all distinct ~~geometrically~~ realizable, ~~configurat.~~ blowable configurations of n points on a sphere.

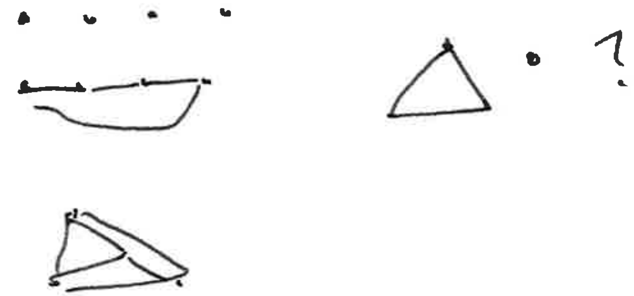
=

Finite graphs. 3 points.



Ex. to be balanced, fair.

need value = 2.



?

~~realizable~~ optimal

?

$\boxed{\text{g-t-circles}}$

geometry..

optimal