MAT 118A Final Project

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1 Introduction

1.1 Historical Background

Sturm-Liouville problem is a class of PDE subject to some boundary conditions. The problem first arise back in the 1830s when French Mathematicians Charles-François Sturm and Joseph Liouville worked on the problem of heat conduction through a metal bar. Sturm was interested in specific differential equations that occurred in Poisson's theory of heat. Liouville was working on differential equations from the theory of heat. The papers published in 1836 by Sturm and Liouville on differential equations discussed the problem of expansions of functions in series. Today, the problem is well-known as the Sturm-Liouville problem (SLP), an eigenvalue problem in second order differential equations.

The simplest case of SLP that were developed back then takes the form $[p(x)y']' + [q(x) - \lambda r(x)]y = 0$ where y is some physical quantity and λ is the eigenvalue that constrains the equation so that y satisfies the boundary values. If the functions p, q, and r satisfy the constraint, the equation will have a family of solutions, called eigenfunction, with distinct eigenvalue. The 0 on the right hand side can be replaced with some source function f. In this paper we will derive and look at the case where the right hand side is zero.

1.2 Motivation

Suppose that we want to express an arbitrary function on a bounded interval as a linear combination of a set of orthogonal functions. We also want to compute the coefficients of the basis function. Then where do all these orthogonal functions come from? Is the set of orthogonal function complete in the sense that can the arbitrary function be expanded as the linear combination by some error approximation? When trying to all these questions, the Sturm-Liouville Theory was developed by Sturm and Liouville.

One thing to note is that when solving a PDE with separation of variable, the PDE will be reduced into several ODE that can be put into a Sturm-Liouville form. This is where Sturm-Liouville Theory becomes helpful to understand the eigenvalues and the corresponding eigenfunction.

1.3 Summary of full group project

In this project, we will provide the basic definitions and properties of functions on $L^2([a, b])$ space in the introduction. We will provide the derivation of the general form of Sturm-Liouville equations from the method of separation on a specific PDE. We will also show and prove that how the coefficients in the linear combination of a set of orthogonal function can be computed. We will demonstrate how regular SLP has real eigenvalues and the prove that the eigenfunctions are orthogonal to each other. I will focus on the derivation and proving orthogonality of eigenfunction of a SLP. I will also demonstrate an example where the orthogonality can be helpful to understand properties of Bessel function.

2 Background

Definition 1: The space of L^2 function $L^2([a,b])$ is given by all functions $f:[a,b] \to \mathbb{R}$ for which $\int_a^b |f(x)|^2 dx < \infty$

Definition 2: If $f, g \in L^2([a,b])$, then L^2 inner product is defined as $(f,g) = \int_a^b f(x)g(x)dx$

Definition 3: If $f, g \in L^2([a,b])$, then the norm of f is $||f|| = \sqrt{(f,f)} = (\int_a^b f(x)f(x)dx)^{\frac{1}{2}} \geq 0$

Definition 4: If $f, g \in L^2([a,b])$, $(f,g) = 0$, $(f,f) = ||f||^2 \neq 0$ and $(g,g) = ||g||^2 \neq 0$, then f and g are orthogonal to each other

Theorem: Suppose that $f \in L^2([a,b])$. If $\{f_1, f_2, \ldots\}$ forms an orthogonal system of function, then the series $\sum_{n=1}^{\infty} c_n f_n$ converges to f in the mean square sense if $\lim_{N \to \infty} e_N = 0$ where $e_N = \int_a^b |f(x) - \sum_{n=1}^{\infty} c_n f_n|^2 dx$.

Properties:

$$(f,g) = (g,f)$$
$$(f, g+h) = (f,g) + (f,h)$$
$$(f, \alpha g) = \alpha(f,g) \qquad \alpha \in \mathbb{R}$$

3 Main Result

Derivation: Suppose that we want to have separable solution of the form $u(x,t) = y(x)g(t)$ on bounded interval $[a,b]$ and for positive t. Suppose that we have the equation $u_t = \frac{d}{dx}(p(x)\frac{dy}{dx}) - q(x)u$. Then we have $u_x = y'(x)g(t)$ and $u_t = y(x)g'(t)$. Then by substitution

$$\begin{aligned} u_t &= \frac{d}{dx}(p(x)\frac{dy}{dx}) - q(x)u \\ \implies y(x)g'(t) &= \frac{d}{dx}(p(x)\frac{dy}{dx}) - q(x)y(x)g(t) \\ \implies \frac{y(x)g'(t)}{g(t)} &= \frac{d}{dx}(p(x)\frac{dy}{g(t)dx}) - q(x)y(x) \\ \implies \frac{y(x)g'(t)}{g(t)} &= -\frac{d}{dx}(p(x)y') + q(x)y(x) \\ \implies y(x)\lambda &= -(p(x)y')' + q(x)y(x) \end{aligned} \tag{1}$$

This ODE is often known as the Sturm-Liouville equation. If p, p' and q are continuous on $[a,b]$ and p is never zero on $[a,b]$, then the ODE boundary value problem is called a regular SLP. The boundary condition of the PDE will usually lead to the following two equations

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

where a_1 and a_2 cannot be both zero and b_1 and b_2 cannot be both zero.

Theorem 1: Suppose that $\{f_1, f_2, \ldots\}$ is a set of orthogonal functions of L^2 on $[a,b]$. If $f \in [a,b]$, then $f(x) = \sum_{n=1}^{\infty} a_n f_n$ where $a_n = \frac{<f, f_n>}{||f_n||^2} = \frac{\int_a^b f(x)f_n(x)dx}{\int_a^b f_n^2(x)dx}$

Proof: Let $f_m \in \{f_n\}$ for some fixed $m \in \mathbb{N}$. Then

$$\begin{aligned}
\int_a^b f(x)f_m(x)dx &= \int_a^b \sum_{n=1}^{\infty} c_n f_n(x) f_m(x)dx \\
&= \sum_{n=1}^{\infty} \int_a^b f_n(x)f_m(x)dx \\
&= \sum_{n=1}^{\infty} c_n < f_n, f_m > \\
&= c_m < f_m, f_m > \\
&= c_m ||f_m||^2 \\
\implies c_m &= \frac{< f, f_m >}{||f_m||^2}
\end{aligned} \tag{2}$$

which is the projection of f onto f_m

Theorem 2: (Logan, page 116) The regular SLP has infinitely many eigenvalues λ_n where $n \in \mathbb{N}$. The eigenvalues are real and $\lim_{n\to\infty} |\lambda_n| = \infty$. The eigenfunction corresponding to distinct eigenvalues are orthogonal, and the orthogonal set is complete, meaning that every square integrable function f on $[a, b]$ can be expanded $f(x) = \sum_{n=1}^{\infty} a_n f_n$ in the mean square sense.

Proof: Suppose that λ_1 and λ_2 are distinct eigenvalues with the corresponding eigenfunction y_1 and y_2. Then by the last equation from the derivation we have

$$y_1\lambda_1 = -(p(x)y_1')' + q(x)y_1$$

$$y_2\lambda_2 = -(p(x)y_2')' + q(x)y_2$$

Then by multiplying the first equation with y_2 and the second equation by y_1 and subtracting and integrating over $[a, b]$ we have

$$(\lambda_1 - \lambda_2)\int_a^b y_1y_2dx = \int_a^b [-y_2(py_1')' + y_1(py_2')']dx$$

Note that we can calculate the right hand side by

$$\frac{d}{dx}[p(y_1y_2' - y_2y_1')] = -y_2(py_1')' + y_1(py_2')'$$

Then by the fundamental theorem of calculus we have

$$(\lambda_1 - \lambda_2)\int_a^b y_1y_2dx = -y_2(py_1')' + y_1(py_2')'|_a^b$$

Then by the boundary condition that we derived in derivation we get

$$(\lambda_1 - \lambda_2)\int_a^b y_1y_2dx = 0$$

which implies that

$$\int_a^b y_1y_2dx = 0$$

since λ_1 and λ_2 are distinct, showing that y_1 and y_2 are orthogonal.

The eigenvalues of a SLP can be shown to be real by considering the complex conjugate of the eigenfunctions. By similar procedure from above we will get

$$(\lambda - \bar{\lambda})\int_a^b y\bar{y}dx = 0$$

Since $y\bar{y} > 0$, we must have $\lambda = \bar{\lambda}$, which implies that λ must be real.

Theorem 3: If v_j and v_k are eigenfunctions corresponding to distinct eigenvalues λ_j and λ_k, then v_j and v_k are orthogonal on $[a, b]$, meaning that $< v_j, v_k >= \int_a^b v_j(x)v_k(x)dx = 0$. If $j = k$, then $< v_j, v_k >= \int_a^b v_j(x)v_k(x)dx = 1$.

Example: We can define J_m to be the Bessel function of first kind of order m and let α_{mn} denote its nth positive zero. Then the SLP with $m \geq 0$

$$x^2y + xy' + (\lambda^2x^2 - m^2)y = 0 \qquad 0 < x < a$$

$$y(a) = 0, \quad y(0) \text{ is finite}$$

The eigenfunctions of this regular SLP is $y_n = J_m(\frac{\alpha_{mn}x}{a})$ Then by Theorem 3, the orthogonality of Bessel function is 0 shows that

$$\int_0^a J_m(\frac{\alpha_{ml}x}{a})J_m(\frac{\alpha_{mk}x}{a}) = 0$$

for $k \neq l$ and when $k = l$ we have

$$\int_0^a J_m(\frac{\alpha_{mk}x}{a})J_m(\frac{\alpha_{mk}x}{a}) = 1$$

4 Discussion

Since all second-order ordinary differential functions can be reduced into Sturm-Liouville form, the Sturm-Liouville theory reveals information about the eigenvalue and eigenfunction problem. Theorem 2 allows us to express an arbitrary function equipped with L^2 on a bounded interval as a linear combination of a set of orthogonal functions. Theorem 1 allows us to compute the coefficients of the basis function and check how good is our approximation. Theorem 3 allows us to compute $< v_j, v_k >$ without doing any direct computations.

Most of the special functions (Bessel function, hyperbolic functions) appear as solution to particular SLP. Knowing the eigenvalues and eigenfunction of these SLP helps us understand properties of some of these special function. The example above showed the property of Bessel function, which is difficult to obtain with direct computation.

There are limitations to the theorem and example of regular SLP that we did above. One limitation is that the theorems and examples that we used are limited to a bounded domain. These theorems may not hold on an infinite domain. In fact, a lot of PDE problems restrict x on the whole plane or half plane. Then the ODEs that we get from the method of separation may not be SLP on a bounded interval. The example that we did above also has bounded interval as domain.

Another limitation is that in our SLP we assume the function p, p' and q are continuous on $[a, b]$ and p is never zero on $[a, b]$. If the given p and q are not smooth function and if p achieved zero on $[a, b]$, these theorems and derivations that we have above may not work.

5 References

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