Quick guide to Strum-Liouville Problems

Group Member: Chia-Yuan Chang, Yanbing Wu

Motivation

We want to express an arbitrary function on a bounded interval as a linear combination of a set of orthogonal functions. We also want to compute the coefficients of the basis function and check how good is our approximation. The questions arise: where do all these orthogonal functions come from? Is the set of orthogonal functions complete in the sense that can the arbitrary function be expanded as the linear combination in the mean square sense? All these problems can be answered and explained with the Strum-Liouville Theory.

One discovery is that when solving a PDE with the separation of variables, the PDE will be reduced into several ODE that can be put into a Sturm-Liouville form. This is where Strum-Liouville Theory becomes helpful to understand this process.

Background

A Sturm-Liouville equation is a second-order linear differential equation in the form

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y = -\lambda w(x)y \qquad a < x < b$$

where $p(x), q(x), w(x)$ are some arbitrary function and $w(x)$ is the weight function. We will refer to this equation equipped with boundary conditions as a Sturm-Liouville problem (SLP). If p, p', and q are continuous on $[a, b]$ and p is never zero on $[a, b]$, then the ODE boundary value problem is called a regular SLP.

Suppose that $w(x)$ is non-negative on $[a, b]$ and f, g are functions on $[a, b]$. Then we define the inner product of f, g with respect to the weighted function $w(x)$ as $< f, g >= \int_a^b w(x)f(x)g(x)dx$

Theorem 1: (Trinity, page 4) Suppose that $\{f_1, f_2, \ldots\}$ is a set of orthogonal functions on $[a, b]$ with some weight function $w(x)$. If $f \in [a, b]$, then $f(x) = \sum_{n=1}^{\infty} a_n f_n$ where $a_n = \frac{<f,f_n>}{||f_n||^2} = \frac{\int_a^b f(x)f_n(x)w(x)dx}{\int_a^b w(x)f_n^2(x)dx}$

Theorem 2: (Logan, page 116) The regular SLP has infinitely many eigenvalues λ_n where $n \in \mathbb{N}$. The eigenvalues are real and $\lim_{n\to\infty} |\lambda_n| = \infty$. The eigenfunctions corresponding to distinct eigenvalues is orthogonal, and the orthogonal set is complete, meaning that every square-integrable function f on $[a, b]$ can be expanded $f(x) = \sum_{n=1}^{\infty} a_n f_n$ in the mean square sense.

Theorem 3: (Trinity, page 19) If v_j and v_k are eigenfunctions corresponding to distinct eigenvalues λ_j and λ_k, then v_j and v_k are orthogonal on $[a, b]$ with respect to $w(x)$, meaning that $< v_j, v_k >= \int_a^b v_j(x)v_k(x)w(x)dx = 0$. If $j = k$, then $< v_j, v_k >= \int_a^b v_j(x)v_k(x)w(x)dx = 1$.

Results

Consider the following regular SLP.

$$y'' + \lambda y = 0 \qquad 0 < x < 1$$
$$y(0) + y'(0) = 0$$
$$y(1) + 3y'(1) = 0$$

Consider the scenario in which we apply the method of separation to some PDE and that PDE will result in the above SLP. If $\lambda = w^2 > 0$, then we have $y = Acos(wx) + Bsin(wx)$. Then we can plug in the boundary condition and get

$$A + wB = 0$$
$$A(cos(w) - 3wsin(w)) + B(sin(w) + 3wcos(w)) = 0$$

We can calculate the Wronskian determinant to get

$$D(w) = \begin{vmatrix} 1 & w \\ cos(w) - 3wsin(w) & sin(w) + 3wcos(w) \end{vmatrix}$$

We know that y has a non-trivial solution when D is equal to zero. Then we get

$$tan(w) = \frac{-2w}{1 + 3w^2}$$

Then by numerical approximation and looking at the intersection of the functions on Figure 1, we see that

$$w_n \approx n\pi \qquad n = 1, 2, 3, \ldots$$
$$\lambda_n = w_n^2 \approx n^2\pi^2$$

Then we have the eigenfunction

$$y_n = cos(wx) - sin(wx)$$

In this example we have verified Theorem 2 that there are infinitely real eigenvalues $\lambda_n \approx n^2\pi^2$ and it's obvious that λ_n tends to infinity as n goes to infinity. Then by theorem 2, the solution y can be expanded as the linear combination of orthogonal functions y_n in the mean square sense.

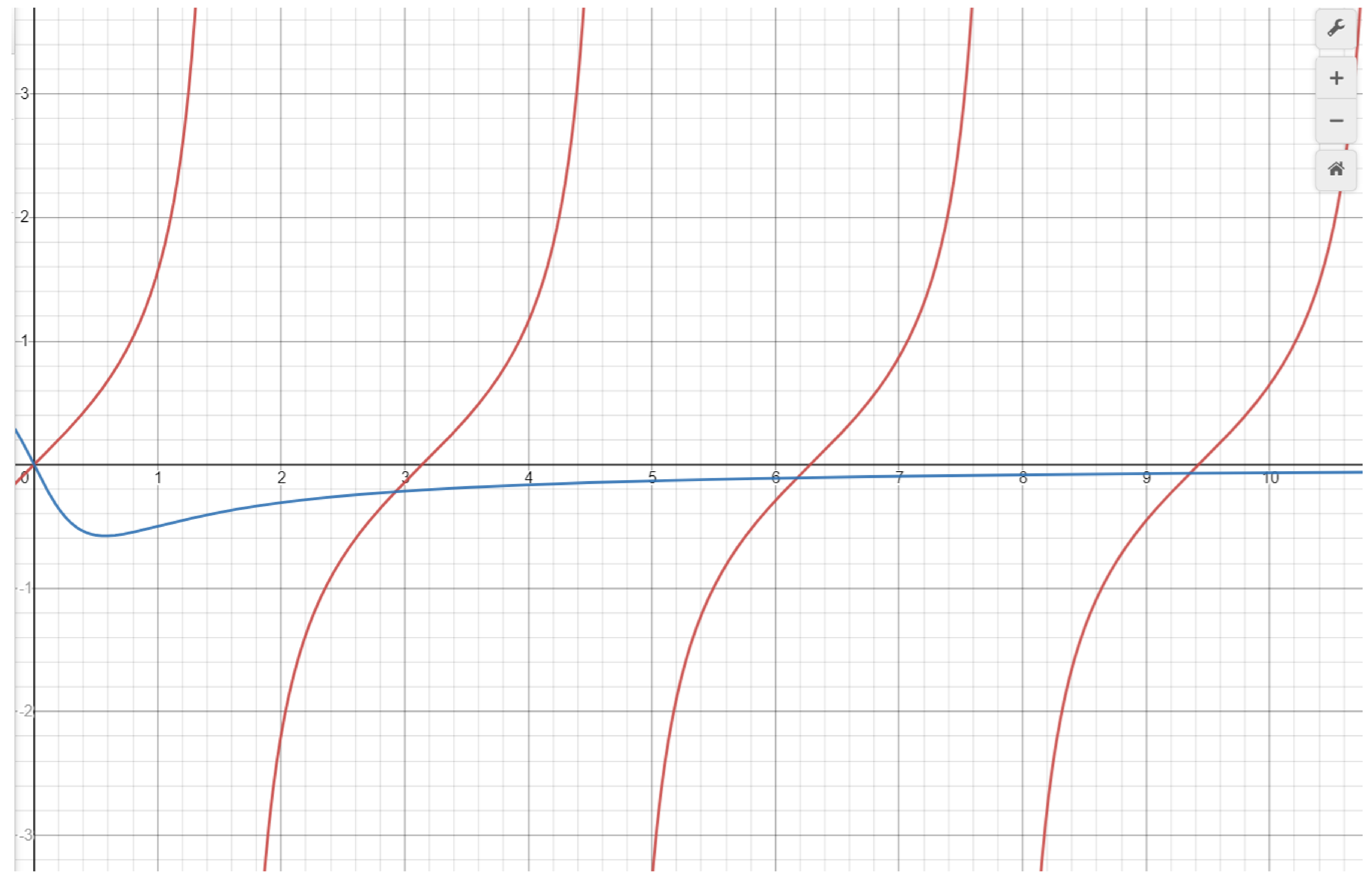

Fig. 1: Graph of the SLP: $tan(k)$ versus $\frac{-2k}{1+3k^2}$

Discussion

Overall, the Strum-Liouvile Theory is important as SLP appears frequently from the method of separation. We have the following procedure:

1. Use method of separation on a PDE problem
2. Reduce the PDE problem into several ODEs
3. The ODEs become SLP and can be solved using standard techniques
4. Use Theorem 1,2, and 3 to calculate coefficients in the orthogonal expansion

The theory of Sturm-Liouville helps us to understand eigenvalues about functions on $L^2([a, b])$. Theorem 3 tells us that distinct eigenfunctions on a bounded interval are orthogonal to each other, which allows us to directly get the result of $< v_j, v_k >$ without doing any direct computations.

Most special functions (Airy function, Bessel function, hyperbolic functions) appear as solutions to particular SLP. Knowing the eigenvalues and eigenfunctions of these SLP helps us understand the properties of some of these special functions.

There are certain limitations to the example of regular SLP that we just did. One limitation is that the theorem and model we used are limited to a bounded domain. Therefore even though the method of separation may still work, these three Theorems may not hold on an infinite domain.

References

[1] Ryan C Daileda. *Introduction to Sturm-Liouville Theory*. URL: http://ramanujan.math.trinity.edu/rdaileda/teach/s12/m3357/lectures/lecture_4_10_short.pdf.

[2] J. David Logan. *Applied Partial Differential Equations*. Springer, 2015, p. 167.

[3] Trinity University. *13.2: Sturm-Liouville Problems*. URL: https://math.libretexts.org/@go/page/9475.

[4] Wikimedia. *Sturm–Liouville theory*. URL: https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory.