MAT 185A Take Home Final

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Part A

Theorem 1. (Cauchy Integral Formula) If f is analytic in a simply connected region $D \subseteq \mathbb{C}$ where D is an open subset of $\mathbb{C}$, then $f(z) = \frac{1}{2\pi i} \oint\limits_C \frac{f(w)}{w-z} dw$ for any simple closed curve C in D and for any $z \in C$.

Corollary 2. If f is analytic in a simply connected region $D \subseteq \mathbb{C}$ where D is an open subset of $\mathbb{C}$, then $f(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(z + re^{it}) dt$ is the average of its values on the circle of radius r center z.

Proof: Assuming f is analytic in a simply connected domain containing any closed ball center at some point z with radius r. We parametrize the complex variable w with $w = z + re^{it}$ such that $0 \le t \le 2\pi$, which implies $dw = ire^{it}dt$. Then by Theorem 1, it follows that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint\limits_C \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{f(z+re^{it})}{re^{it}} ire^{it} dt \\ &= \frac{1}{2\pi} \int\limits_0^{2\pi} f(z+re^{it}) dt \end{aligned} \tag{1}$$

With the parameterization, $w = z + re^{it}$ is now a circle with radius r and center at z in the complex plane. We divide such circle into N evenly spaced points $t_0 < t_1 < \ldots < t_N$ where the endpoints are $t_0 = 0$ and $t_N = 2\pi$. We define $\Delta t = \frac{2\pi}{N}$ with $t_k = k\Delta t = \frac{2\pi k}{N}$, $t_0 = \frac{2\pi 0}{N} = 0$, $t_N = \frac{2\pi N}{N} = 2\pi$. Then we substitute t_k to get $\hat{z}_0 = z + re^{it_0}$, $\hat{z}_k = z + re^{it_k}$, and $\hat{z}_N = z + re^{it_N}$ for $k \in \{0, \ldots, N\}$. Equation (1) can be written in terms of Riemman sum as the following

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int\limits_0^{2\pi} f(z+re^{it}) dt \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} f(\hat{z}_k) \Delta t \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} f(\hat{z}_k) \frac{2\pi}{N} \qquad (\Delta t = \frac{2\pi}{N}) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\hat{z}_k) \end{aligned} \tag{2}$$

Thus $f(z)$ is the average of $f(\hat{z}_1), \ldots, f(\hat{z}_N)$ as N tends to ∞.

Corollary 3. If f is analytic in a simply connected region $D \subseteq \mathbb{C}$ where D is an open subset of $\mathbb{C}$, then $u(x,y) = \frac{1}{2\pi} \int\limits_0^{2\pi} u((x,y) + re^{it})dt$ corresponds the real value and $v(x,y) = \frac{1}{2\pi} \int\limits_0^{2\pi} v((x,y) + re^{it})dt$ corresponds the imaginary value of the average of its values on the circle of radius r center z where $z = x + iy$.

Proof: Let $f(x,y) = u + iv = u(x,y) + iv(x,y)$. Let $\hat{z}_k$ be the points on the circle. Then combined with Corollary 2 we have

$$\begin{aligned} f(x,y) = u + iv &= \frac{1}{2\pi} \int\limits_0^{2\pi} (u + iv)dt \\ &= \frac{1}{2\pi} \int\limits_0^{2\pi} u((x,y) + re^{it})dt + \frac{i}{2\pi} \int\limits_0^{2\pi} v((x,y) + re^{it})dt \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} u(\hat{z}_k)\Delta t + \frac{i}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} v(\hat{z}_k)\Delta t \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} u(\hat{z}_k)\frac{2\pi}{N} + \frac{i}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} v(\hat{z}_k)\frac{2\pi}{N} \qquad (\Delta t = \frac{2\pi}{N}) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} u(\hat{z}_k) + i \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} v(\hat{z}_k) \end{aligned} \tag{3}$$

Thus $\frac{1}{2\pi} \int\limits_0^{2\pi} u((x,y) + re^{it})dt$ corresponds the real part and $\frac{1}{2\pi} \int\limits_0^{2\pi} v((x,y) + re^{it})dt$ corresponds to the imaginary part of the average of its values on the circle of radius r center z.

Theorem 4. If f is analytic in a simply connected region $D \subseteq \mathbb{C}$ where D is an open subset of $\mathbb{C}$, then u and v are harmonic.

Proof: If $f(z) = u(x,y) + iv(x,y)$ is analytic, then by Cauchy-Riemann Equation $u_x = v_y$ and $u_y = -v_x \implies u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$, $v_{yy} = u_{xy}$, $v_{xx} = -u_{yx}$. Thus $\Delta u = u_{xx} + u_{yy} = 0$ and $\Delta v = v_{xx} + v_{yy} = 0$. By definition u and v are harmonic.

Lemma 5. If $x_k \in \mathbb{R}$ where $k \in \{0, \ldots, N\}$, then $\frac{1}{N} \sum\limits_{k=0}^{N} x_k \leq \max\limits_{0 \leq k \leq N} x_k$.

Proof: We observe that $x_k \leq \max\limits_{0 \leq k \leq N} x_k$ for any $k \in \{0, \ldots, N\}$. Then $\sum\limits_{k=0}^{N} x_k \leq N \max\limits_{0 \leq k \leq N} x_k \implies \frac{1}{N} \sum\limits_{k=0}^{N} x_k \leq \max\limits_{0 \leq k \leq N} x_k$

Theorem 6. The real part of a non-constant harmonic function can never take a strict maximum at an interior point $z_0 = (x_0, y_0)$.

Proof: Assuming U is a neighborhood of z_0 and assuming that there is a strict maximum at an arbitrary interior point $z_0 = (x_0, y_0)$ for contradiction, then we have $|u(x,y)| < u(x_0, y_0)$ for all (x,y) in $U \setminus \{z_0\}$ where f is analytic. We denote $B_r(z_0) \subseteq U$ to be a circle with radius r center z_0. By Corollary 3,

$$u(x_0, y_0) = \frac{1}{2\pi} \int\limits_0^{2\pi} u((x_0, y_0) + re^{it})dt = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} u(\hat{z}_k)$$

is the average of the real part of f on $B_r(z_0)$, where $\hat{z}_k$ are boundary points on the circles that divide the circle $B_r(z_0)$ evenly into N pieces for $k \in \{0, \ldots, N\}$ and $0 \leq t \leq 2\pi$. Then by Lemma 5, it follows that

$u(x_0, y_0) \leq \max\limits_{1 \leq k \leq N} u(\hat{z}_k)$ as N goes to infinity. Notice that the points $\hat{z}_k$ are contained in the deleted neighborhood $U \setminus \{z_0\} \implies |u(\hat{z}_k)| < u(x_0, y_0)$ from our assumption. This is a contradiction because we assume that $u(x_0, y_0)$ is the strict maximum in the delete neighborhood $U \setminus \{z_0\}$. However we have found at least one of the element $u(\hat{z}_k) \in U \setminus \{z_0\}$ larger than or equal to $u(x_0, y_0)$. Thus our assumption that there is a strict maximum at an arbitrary interior point is incorrect. Thus the real part of a non-constant harmonic function can never take a strict maximum at an interior point.

Theorem 7. Every harmonic function u has a harmonic conjugate v such that $f(z) = u + iv$ is analytic.

Proof: Assuming $\Delta u = 0$, we want to find v such that $f = u + iv$ satisfies the Cauchy-Riemann equation, meaning that $u_x = v_y$ and $u_y = -v_x$. If $f(z) = u + iv$, then by taking $\Delta z = \Delta x$ it follows that

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= u_x - iu_y \end{aligned} \tag{4}$$

We define $g(z) = u_x - iu_y$. To show that $g(z)$ is analytic, we need Cauchy-Riemann equation to hold for $g(z)$, meaning that we need $u_{xx} = -u_{yy}$ and $u_{xy} = u_{yx}$. If $g(z)$ is analytic, then g has has an antiderivative. Let $f(z) = \hat{u} + i\hat{v}$ be the antiderivative of g, meaning that $f'(z) = g(z) = u_x - iu_y$. By taking $\Delta z = \Delta x$, it follows that $f'(z) = \hat{u}_x + i\hat{v}_x$. Then combining both equation, we get $u_x = \hat{u}_x$ and $-u_y = \hat{v}_x$. Since f satisfies Cauchy Riemann equation, it follows that $\hat{u}_y = -\hat{v}_x = u_y$. Thus $\nabla \hat{u} = \nabla u \implies \hat{u} = u + C$ for some constant C. Thus $f(u) = \hat{u} + i\hat{v} = (u + C) + i\hat{v}$ is analytic.

Part B

Theorem 8. (Laurent Expansion Theorem) If f is analytic in an annulus $r < |z - z_0| < R$, then

$$f(z) = \sum_{k=1}^{\infty} c_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} c_k(z - z_0)^k = \sum_{k=-\infty}^{\infty} c_k(z - z_0)^k$$

converges for all $r < |z - z_0| < R$ and converges uniformly in any subannalus $r + \epsilon < |z - z_0| < R - \epsilon$ for any ϵ satisfying $0 < \epsilon < R$, where

$$c_{-k} = \frac{1}{2\pi i} \oint_C f(w)(w - z)^{k-1} dw \quad \text{for } k \geq 1$$

and

$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{k+1}} dw \quad \text{for } k \geq 0$$

.

Note that when $z_0 = 0$ we have

$$f(z) = \sum_{k=1}^{\infty} c_{-k} z^{-k} + \sum_{k=0}^{\infty} c_k z^k = \sum_{k=-\infty}^{\infty} c_k z^k$$

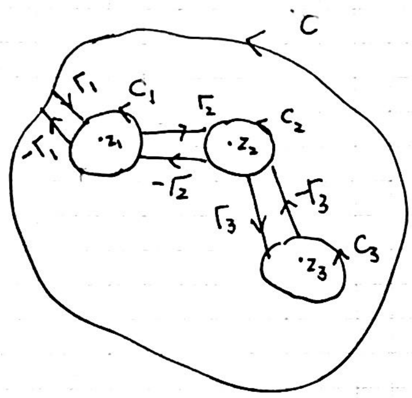
converges for all $r < |z| < R$ and converges uniformly in any subannalus $r < r + \epsilon < |z| < R - \epsilon < R$ for any ϵ satisfying $0 < \epsilon < R$

Theorem 9. If $f(z) = \sum\limits_{k=0}^{\infty} g_k(z)$ is finite for all $z \in \mathbb{C}$ and the series converges absolutely and uniformly on a simply closed curve C, then

$$\int_C f(z)dz = \int_C (\sum_{k=0}^{\infty} g_k(z))dz = \sum_{k=0}^{\infty} \int_C g_k(z)dz$$

Theorem 10. (Residue Theorem) If f is analytic inside C except at a finite number of point singularities $z_1, \ldots, z_n$, then f has a unique Laurent expansion that converges in $\epsilon \leq |z - z_i| \leq R$ for some ϵ satisfying $0 < \epsilon < R$ and being arbitrarily small and $\oint\limits_C f(z)dz = 2\pi i \sum\limits_{i=1}^{n} R(f; z_i)$ where $R(f; z_i)$ is the coefficient of $\frac{1}{z - z_i}$ in the Laurent expansion of f at z_i.

Proof: Let C be a simply closed curve that encloses $z_1, \ldots, z_n$ so that z_i are inside C. Let $C_1, \ldots, C_n$ be simply closed curve such that only z_i is contained inside C_i. We connect C and C_1 with Γ_1, C_i and C_{i+1} with Γ_{i+1}.



Then we traverse through C_i and Γ_i to get

$$\oint\limits_{C+\sum\limits_{i=1}^{n}\Gamma_i-\sum\limits_{i=1}^{n}\Gamma_i+\sum\limits_{i=1}^{n}C_i} f(z)dz = 0 \implies \oint\limits_{C} f(z)dz = \sum_{i=1}^{n}\oint\limits_{C_i} f(z)dz$$

Hence to evaluate $\oint\limits_{C} f(z)dz$ it is sufficient to evaluate each $\oint\limits_{C_i} f(z)dz$. Now assuming that we can expand f in powers of $z-z_i$ near $z=z_i$, then by Theorem 8 we have

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z-z_i)^k = \sum_{k=1}^{\infty} c_{-k}(z-z_i)^{-k} + \sum_{k=0}^{\infty} c_k(z-z_i)^k$$

where each c_k represents Cauchy's Integral formula. Then we use the fact that if $k \neq -1$, then $\int\limits_{C_i} c_n(z-z_i)^k dz = 0$. If $k=-1$, then $\int\limits_{C_i} c_k(z-z_i)^k dz = 2c_k\pi i$. Then it follows that only $\oint\limits_{C_i} \frac{dz}{z-z_i} = 2\pi i$ has non-zero integral among all powers of $(z-z_i)^k$. Then by Theorem 9 we can expand the integral as the following.

$$\oint\limits_{C_i} f(z)dz = \oint\limits_{C_i} \sum_{k=-\infty}^{\infty} c_k(z-z_i)^k dz = \sum_{k=-\infty}^{\infty}\oint\limits_{C_i} c_k(z-z_i)^k dz = c_{-1}\oint\limits_{C_i}\frac{dz}{z-z_i}dz = c_{-1}2\pi i$$

Now we define c_{-1} ,the coefficient of $\frac{1}{z-z_i}$ in the Laurent expansion of f about z_i, to be the residue of f at z_i, denoted as $R(f;z_i)=R_i$. Then by considering each C_i we can rewrite our result as

$$\oint\limits_{C} f(z)dz = \sum_{i=1}^{n}\oint\limits_{C_i} f(z)dz = 2\pi i\sum_{i=1}^{n} R(f;z_i)$$

Eg.11. Consider $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$. We want to evaluate $\int\limits_{-\infty}^{\infty}\frac{1}{1+x^2}dx$. By expanding $f(z)$ as a geometric series, we have

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

converges for $|z|<1$ and diverges for $|z|>1$. We observe that there are singularities when $z=i,-i$. Then the radius of convergence $R=1$. We calculate the residue of $\frac{1}{1+z^2}$ at $z=i,-i$. At $z=i$ we have

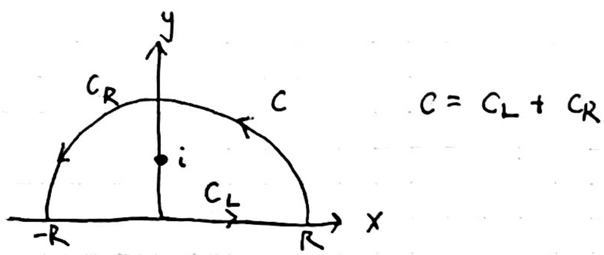
$$f(z)(z-i) = \frac{1}{z+i} = \frac{1}{2i} \neq 1$$

$$f(z)(z-i)^2 = \frac{z-i}{z+i} = 0$$
$$f(z)(z-i)^n = 0$$

Thus we can conclude that $f(z)$ has a pole of order 1 at $z = i$ and

$$R(f;i) = \lim_{z\to i} f(z)(z-i) = \frac{1}{2i}$$

To evaluate $\int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ we use Residue Theorem. Let C be the simply closed curve consisting of $[-R, R]$ on the real axis and be the circle of radius R above the x-axis. Then $C = C_L + C_R$, where C_L is parameterized by $z(t) = x = t$ for $-R \le t \le R$ and C_R is parameterized by $z(t) = Re^{it}$ for $0 \le t \le \pi \implies dz = iRe^{it}dt$. Note that this is a semi-circle containing only i.



Then

$$\int\limits_C \frac{1}{1+z^2}dz = 2\pi i R(f;i) = 2\pi i \frac{1}{2i} = \pi$$

Now consider

$$\lim_{R\to\infty} \int\limits_{-R}^{R} \frac{dx}{1+z^2}dz = \int\limits_{-\infty}^{\infty} \frac{dx}{1+z^2}dz$$

We can write

$$|\int\limits_0^{\pi} \frac{iRe^{it}}{1+R^2e^{i2t}}dt| \le \int\limits_0^{\pi} \frac{iRe^{it}}{|1+R^2e^{i2t}|}dt \le \int\limits_0^{\pi} \frac{R}{R^2+1}dt \le \pi\frac{R}{R^2-1}$$

Then it follows that

$$\lim_{R\to\infty} |\int\limits_0^{\pi} \frac{iRe^{it}}{1+R^2e^{i2t}}dt| \le \lim_{R\to\infty} \pi\frac{R}{R^2-1}$$

and

$$\lim_{R\to\infty} \pi\frac{R}{R^2-1} = 0 \implies \lim_{R\to\infty} |\int\limits_0^{\pi} \frac{iRe^{it}}{1+R^2e^{i2t}}dt| = 0$$

Then combining everything we have

$$\begin{aligned}\int\limits_C \frac{1}{1+x^2}dx = \int\limits_{C_L} \frac{1}{1+x^2}dx + \int\limits_{C_R} \frac{1}{1+x^2}dx &= \lim_{R\to\infty} \int\limits_{-R}^{R} \frac{1}{1+x^2}dx + \lim_{R\to\infty} \int\limits_0^{\pi} \frac{iRe^{it}}{1+R^2e^{i2t}}dt \\ &= \lim_{R\to\infty} \int\limits_{-R}^{R} \frac{1}{1+x^2}dx\end{aligned} \tag{5}$$

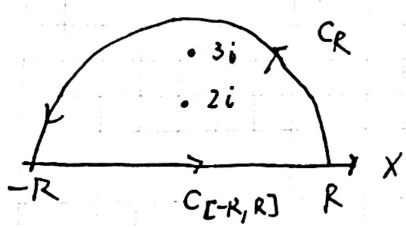
Thus

$$\int_C \frac{1}{1+x^2}dx = \pi \implies \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}dx = \pi$$

and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \pi$$

Eg.12. We want to evaluate $I = \int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}dx$.
Let $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{z^2}{(z-3i)(z+3i)(z-2i)^2(z+2i)^2}$. We have simple poles at $z = 3i, -3i$ and double poles at $z = 2i, -2i$. We find the residue $R(f;3i) = \lim_{z\to 3i} f(z)(z-3i) = \frac{(3i)^2}{(3i+3i)(3i-2i)^2(3i+2i)^2} = \frac{-3}{50i}$. We can write $g(z) = (z-2i)^2 f(z) = \frac{z^2}{(z^2+9)(z+2i)^2}$ and find the residue at $z = 2i$ to be $R(f;2i) = g'(2i) = \frac{-13i}{200}$.
Let C_R be upper half of the circle at radius R and $C_{[-R,R]}$ be the real line below in the interval $[-R,R]$.



Then by Residue Theorem we have

$$\begin{aligned} \int_{C_{[-R,R]}+C_R} f(z)dz &= 2\pi i[R(f;2i) + R(f;3i)] \\ &= 2\pi i(\frac{-3}{50i} + \frac{-13i}{200}) \\ &= \frac{\pi}{100} \end{aligned} \tag{6}$$

We also observe that

$$\begin{aligned} |f(z)| &= |\frac{z^2}{(z^2+9)(z^2+4)^2}| \\ &\leq \frac{R^2}{(R^2-9)(R^2-4)^2} \qquad (R > 9) \\ &\leq \frac{1}{4R^4} \end{aligned} \tag{7}$$

and that $|C_R| = 2\pi R$. Thus we have

$$|\int_{C_R} f(z)dz| \leq 2\pi R\frac{1}{4R^4} = \frac{\pi}{2R^3}$$

Then as $R \to \infty$, $|\int_{C_R} f(z)dz| = 0$. Hence it follows that

$$\frac{\pi}{100} = \lim_{R\to\infty} \int_C f(z)dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{C_R} f(z)dz \implies \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \frac{\pi}{100}$$

Thus

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{100} \implies I = \int_{0}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}dx = \frac{\pi}{200}$$

Eg.13. We want to evaluate $I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4}+\sin\theta}$. We can write

$$\frac{5}{4} + \sin\theta = \frac{5}{4} + \frac{z-\frac{1}{z}}{2i} = \frac{1}{4iz}(2z^2+5iz-2)$$

$$I = \int_{C_0} \frac{4dz}{2z^2+5iz-2} = \int_{C_0} \frac{2dz}{(z+2i)(z+\frac{i}{2})}$$

where C_0 is the unit circle. Then by Residue Theorem we have $I = 2\pi i R(f; \frac{-i}{2})$ where

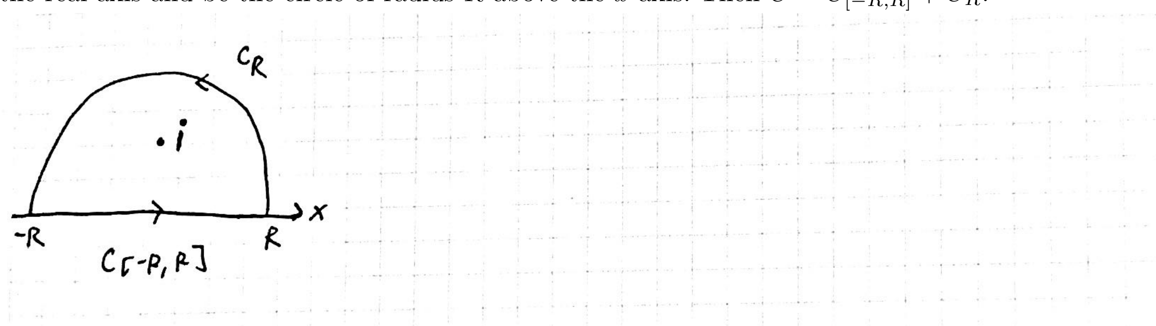
$$R(f; \frac{-i}{2}) = \lim_{z\to\frac{i}{2}} f(z)(z+\frac{i}{2}) = \frac{4}{3i}$$

Thus

$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4}+\sin\theta} = 2\pi i\frac{4}{3i} = \frac{8\pi}{3}$$

Eg.14. We want to evaluate $I = \int_0^{\infty} \frac{\cos(x)}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}dx$.

We set $\cos(x) = \mathrm{Re}(e^{ix})$ so that $I = \frac{1}{2}\,\mathrm{Re}(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx)$. Let C be the simply closed curve consisting of $[-R, R]$ on the real axis and be the circle of radius R above the x-axis. Then $C = C_{[-R,R]} + C_R$.



We consider $f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{-y}e^{ix}}{z^2+1} \to 0$ on the curve C_R. Thus we conclude that

$$\begin{aligned}\int_{C_{[-R,R]}+C_R} f(z)dz &= \int_{C_{[-R,R]}} f(z)dz + \int_{C_R} f(z)dz \\ &= 2\pi i R(f; i) \\ &= 2\pi i \lim_{z\to i} f(z)(z-i) \\ &= \frac{\pi}{e}\end{aligned} \tag{8}$$

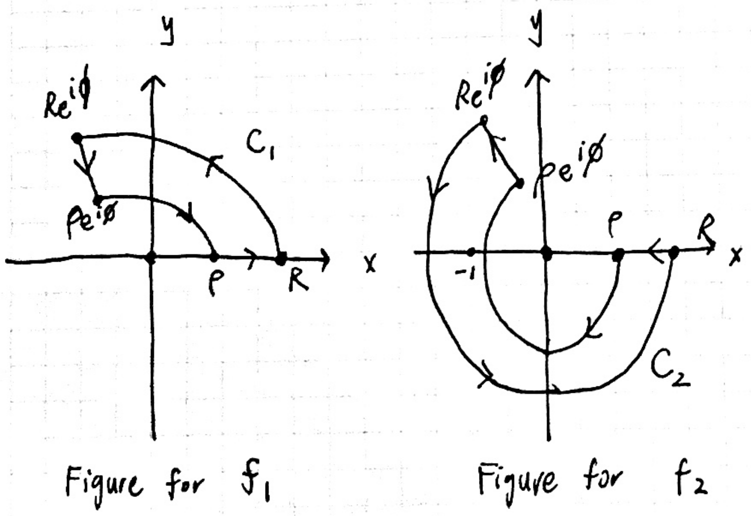
Thus

$$\lim_{R\to\infty} \int_{C_{[-R,R]}+C_R} f(z)dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx + 0 = \frac{\pi}{e}$$

Hence

$$I = \frac{1}{2}\,\mathrm{Re}\Big(\int\limits_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx\Big) = \frac{\pi}{2e}$$

Eg.15. We want to evaluate $I = \int\limits_0^{\infty} \frac{x^{-a}}{x+1}dx$ for $0 < a < 1$. This is a real integral involving branch points and branch cuts. We consider two integral $\int\limits_{C_1} f_1(z)dz$ and $\int\limits_{C_2} f_2(z)dz$, where $f_1(z) = \frac{z^{-a}}{z+1}$ for $|z| > 0$ and $\frac{-\pi}{2} < \arg z < \frac{3\pi}{2}$, and $f_2(z) = \frac{z^{-a}}{z+1}$ for $|z| > 0$ and $\frac{\pi}{2} < \arg z < \frac{5\pi}{2}$. Note that C_1 and C_2 are simply closed contours as shown in the following picture. In the picture $\rho < 1 < R$ and the angle ϕ is $\frac{\pi}{2} < \phi < \pi$.



Since f_1 is analytic on C_1, it follows that $\int\limits_{C_1} f_1(z)dz = 0$. Note that f_2 is analytic on C_2 except for the simple pole at the interior point $z = -1$. Then in the definition of f_2,

$$z^{-a} = e^{-a(\log(|z|)+i\arg z}$$

where $\frac{\pi}{2} < \arg z < \frac{5\pi}{2}$ and we can calculate the residue of f_2 at $z = -1$ as

$$\lim_{z\to -1}(z+1)f_2(z) = \lim_{z\to -1} z^{-a} = e^{-a\pi i}$$

Thus

$$\int\limits_{C_2} f_2(z)dz = 2\pi i e^{-a\pi i}$$

Since $f_1(z) = f_2(z)$ when $\arg(z) = \phi$, it follows that

$$\int\limits_{C_1} f_1(z)dz + \int\limits_{C_2} f_2(z)dz = \int\limits_{\rho}^{R} f_1(x)dx - \int\limits_{\rho}^{R} f_2(x)dx + \int\limits_{\Gamma_1} f_1(z)dz + \int\limits_{\Gamma_2} f_2(z)dz + \int\limits_{\gamma_1} f_1(z)dz + \int\limits_{\gamma_2} f_2(z)dz \quad (9)$$

where Γ_k is the large circular arc and γ_k is the small circular arc of the simply closed contour of C_1 and C_2 in the figure. Since

$$|f_k(z)| = \frac{z^{-a}}{z+1} \le \frac{R^{-a}}{R-1}$$

and since Γ_k are part of the circle where the circumference is $2\pi R$, it follows that

$$|\int\limits_{\Gamma_k} f_k(z)dz| \leq \frac{R^{-a}}{R-1} 2\pi R$$

Thus we have

$$\lim_{R\to\infty} \int\limits_{\Gamma_k} f_k(z)dz = 0$$

When z is on γ_k for $k = 1, 2$, we have

$$|f_k(z)| = |\frac{z^{-a}}{z+1}| \leq \frac{\rho^{-a}}{1-\rho}$$

Therefore

$$|\int\limits_{\gamma_k} f_k(z)dz| \leq \frac{\rho^{-a}}{1-\rho} 2\pi\rho$$

and

$$\lim_{\rho\to 0} \int\limits_{\gamma_k} f_k(z)dz = 0$$

Then combining everything together we have

$$\lim_{\rho\to 0} \lim_{R\to\infty} [\int\limits_{\rho}^{R} f_1(x)dx - \int\limits_{\rho}^{R} f_2(x)dx] = 2\pi i e^{-a\pi i}$$

Since

$$\begin{aligned} \int\limits_{\rho}^{R} f_1(x)dx - \int\limits_{\rho}^{R} f_2(x)dx &= \int\limits_{\rho}^{R} \frac{1}{x+1}(e^{-a\log x} - e^{-a(\log x + 2\pi i)})dx \\ &= \int\limits_{\rho}^{R} \frac{x^{-a}}{x+1}(1 - e^{-2\pi a i})dx \end{aligned} \tag{10}$$

Then by taking the limits it follows that

$$\lim_{\rho\to 0} \lim_{R\to\infty} \int\limits_{\rho}^{R} \frac{x^{-a}}{x+1} dx = \frac{2\pi i e^{-a\pi i}}{1 - e^{-2a\pi i}}$$

which is equivalent to

$$I = \int\limits_{0}^{\infty} \frac{x^{-a}}{x+1} = \frac{\pi}{\sin(a\pi)} \quad (0 < a < 1)$$